KØBENHAVNS UNIVERSITET
INSTITUT FOR NATURFAGENES DIDAKTIK

Danish upper secondary students' apprehensions of the equal sign

- A study based on external didactic transposition and diagnostic testing

Aurora Olden Aglen

Speciale – Kandidat i Matematik

Vejledere: Britta Eyrich Jessen &
Mikkel Willum Johansen

IND's studenterserie nr. 110, 2023

INSTITUT FOR NATURFAGENES DIDAKTIK, www.ind.ku.dk
Alle publikationer fra IND er tilgængelige via hjemmesiden.

IND's studenterserie

70. Jakob Holm: The Implementation of Inquiry-based Teaching (2019)
71. Louise Uglebjerg: A Study and Research Path (2019)
72. Anders Tørring Kolding & Jonas Tarp Jørgensen: Physical Activity in the PULSE Exhibit (2019)
73. Simon Arent Vedel: Teaching the Formula of Centripetal Acceleration (2019)
74. Aputsiaq Bent Simonsen: Basic Science Course (NV) (2019)
75. Svenning Helth Møller: Peer-feedback (2019)
76. Lars Hansen & Lisbeth Birch Jensen: Feedbackformater på Mulernes Legatskole (2019)
77. Kirsi Inkeri Pakkanen: Autobiographical narratives with focus on science (2019)
78. Niels Jacob Jensen: Engineering i naturen og på naturskolen (2019)
79. Yvonne Herguth Nygaard: Diskursanalyse af litteraturen og hos lærer i forbindelse med brugen af eksterne læringsmiljø, med en underviser tilknyttet (2019)
80. Trine Jørgensen: Medborgerskab i naturfagsundervisningen på KBHSYD (2019)
81. Morten Terp Randrup: Dannelse i Fysik C (2019)
82. Thomas Mellergaard Amby: Undersøgelses baseret naturfagsundervisning og science writing heuristic (2019)
83. Freja Elbro: Important prerequisites to understanding the definition of limit (2019)
84. Mathilde Sexauer Bloch Kloster: Inquiry-Based Science Education (IBSE) (2019)
85. Casper Borup Frandsen: Undersøgelsesbaseret undervisning i idrætsundervisningen på gymnasieskolen (2019)
86. Vibeke Ankjer Vestermarken: An Inquiry Based Introduction to Binomial Distributions (2019)
87. Jesper Jul Jensen: Formativ evaluering og faglige samspil i almen studieforberedelse (2020)
88. Karen A. Voigt: Assessing Student Conceptions with Network Theory - Investigating Student Conceptions in the Force Concept Inventory Using MAMCR (2020)
89. Julie Hougaard Overgaard: Using virtual experiments as a preparation for large scale facility experiments (2020)
90. Maria Anagnostou: Trigonometry in upper secondary school context: identities and functions (2020)
91. Henry James Evans: How Do Different Framings Of Climate Change Affect Pro-environmental Behaviour? (2020)
92. Mette Jensen: Study and Research Paths in Discrete Mathematics (2020)
93. Jesper Hansen: Effekten og brugen af narrative læringsspil og simuleringer i gymnasiet (2020)
94. Mie Haumann Petersen: Bilingual student performance in the context of probability and statistics teaching in Danish High schools (2020)
95. Caroline Woergaard Gram: "Super Yeast" - The motivational potential of an inquiry-based experimental exercise (2021)
96. Lone Brun Jakobsen: Kan man hjælpe elevers forståelse af naturvidenskab ved at lade dem formulere sig om et naturvidenskabeligt emne i et andet fag? (2021)
97. Maibritt Oksen og Morten Kjøller Hegelund: Styrkelse af motivation gennem Webinar og Green Screen (2021)
98. Søren Bystrup Jacobsen: Peer feedback: Fra modstand til mestring? (2021)
99. Bente Guldbrandsen: Er der nogen, som har spurgt en fysiklærer? (2021)
100. Iben Vernegren Christensen: Bingoplader i kemiundervisningen – en metode til styrkelse af den faglige samtale? (2021)
101. Claus Axel Frimann Kristinson Bang: Probability, Combinatorics, and Lesson Study in Danish High School (2021)
102. Derya Diana Cosan: A Diagnostic Test for Danish Middle School Arithmetics (2021)
103. Kasper Rytter Falster Dethlefsen: Formativt potentiale og udbytte i Structured Assessment Dialogue (2021)
104. Nicole Jonasen: A diagnostic study on functions (2021)
105. Trine Nørgaard Christensen: Organisatorisk læring på teknisk eux (2021)
106. Simon Funch: Åben Skole som indgang til tværfagligt samarbejde (2022)
107. Hans-Christian Borggreen Keller: Stem som interdisciplinær undervisningsform (2022)
108. Marie-Louise Krarup, Jakob Holm Jakobsen, Michelle Kyk & Malene Hermann Jensen: Implementering af STEM i grundskolen (2022)
109. Anja Rousing Lauridsen & Jonas Traczyk Jensen: Grundskoleelevers oplevelse af SSI-undervisning i en STEM-kontekst. (2022)
110. **Aurora Olden Aglen: Danish upper secondary students' apprehensions of the equal sign (2023)**

IND's studenterserie *omfatter kandidatspecialer, bachelorprojekter og masterafhandlinger skrevet ved eller i tilknytning til Institut for Naturfagenes Didaktik. Disse drejer sig ofte om uddannelsesfaglige problemstillinger, der har interesse også uden for universitetets mure. De publiceres derfor i elektronisk form, naturligvis under forudsætning af samtykke fra forfatterne. Det er tale om studenterarbejder, og ikke endelige forskningspublikationer.*
Se hele serien på: www.ind.ku.dk/publikationer/studenterserien/

UNIVERSITY OF COPENHAGEN
DEPARTMENT OF SCIENCE EDUCATION

Master thesis

Aurora Olden Aglen

Danish upper secondary students' apprehensions of the equal sign

A study based on external didactic transposition and diagnostic testing

Main Supervisor: Britta Eyrich Jessen

Co-supervisor: Mikkel Willum Johansen

Submitted on: December 20th, 2021

Abstract

This thesis aims to demonstrate Danish upper secondary students' understanding of the equal sign motivated by its strong affiliation to school algebra. We have used the methods of didactic transposition and diagnostic test to answer the following two research questions:
RQ1: Based on external didactic transposition, how do the scholarly meanings of the equal sign come into view in knowledge to be taught in Danish upper secondary school?
RQ2: Which challenges related to the external didactic transposition of the equal sign can be detected through a diagnostic test for Danish students in upper secondary school?
We refer to the six meanings of the equal sign proposed by Prediger (2010) in the analysis; the meaning of operational, symmetric arithmetic identity, formal equivalence, conditional equation characterizing unknowns, contextual identity in formula and specification. The external transposition was the reference in constructing the diagnostic test in which we wished to find the internal didactic transposition of *learned knowledge*. We show that it is challenging to construct a diagnostic test of the equal sign as it is an entity treated implicitly in *knowledge to be taught.* The test results demonstrate challenges concerning the meanings of the equal sign, especially what regards contextual identity and formal equivalence. Moreover, we have shown how some of the challenges link to the analyzed knowledge to be taught. However, contrary to prior studies, we show that most students do not favor an operational approach alone. On the other hand, the test show difficulties in distinguishing between misconceptions of the equal sign and school algebra. We conclude that this indeed stresses the relevance of becoming more explicit with the equal sign's many disguises.

Acknowledgments

In doing this dissertation, I have several people I want to thank.

First, a big thank you goes out to the students and teachers who wanted to support and participate in the study. Without you, I would not be able to raise this dissertation.

Second, thank you to my primary supervisor Britta, for being good at capturing my ideas and helping me keep the focus on the right things at the right time. I appreciate the discussions we have had about this project. Also, a big thank you to my co-supervisor Mikkel, to whom I first presented the idea of the equal sign. Writing this thesis was very meaningful when the motivation came from within, and my idea was acknowledged.

Thirdly, I would like to thank the people I have shared office and study frustrations with. So, thank you to Vera, who was the one person in the office since day one. It has been meaningful to hear your personal experience of writing your own thesis and discussing the findings in the textbooks that did not involve mathematics but apprehensions of the Danish language. Thank you to Alexander as well, whom I saw in the office in the beginning. You helped me understand the notions of the equal sign from a physicists' view when I stumbled upon questionable formulations in the books. Furthermore, thank you to my former roomie and good friend Signe for sharing frustrations of the thesis and working hard next to me at the Black Diamond. It has been helpful to keep the motivation up, knowing that you were writing your thesis just like me.

Lastly, a big thank you to family and friends who have supported me and kept me going.

Table of Contents

List of Tables

1. Introduction

1.1 Background

Former studies have proposed several indications of challenges in the transition between arithmetic and algebra in middle school, among others we mention in a Danish context the studies of Cosan (2021) and Poulsen (2015). They illustrate algebra's significant role in the current Danish school system. Understanding algebra is essential to understanding other concepts, such as equality (Knuth, Stephens, McNeil., & Alibali, 2006). However, research shows that students' understanding of the equal sign is defective. Furthermore, as it appears universally throughout mathematics education, this assesses its high relevance (e.g., (Kieran, 1981) (Falkner, Levi, & Carpenter, 1999) (Theis, 2005; Prediger, 2010) (Knuth, Stephens, McNeil., & Alibali, 2006) (Molina, Desarrollo del pensamiento relacional y comprensión del signo por alumnos de tercero de educación primaria, 2006) (McNeil, et al., 2006), (Prediger, 2010), (Stephens, et al., 2013) (Vincent, Bardini, Pierce, & Pearn, 2015) (Faulkner, Walkowiak, Cain, & Lee, 2016)).

The main discussion in many studies is the one regarding an operational approach instead of a relational approach. Kieran (1981) mentions that although the equal sign is to show equivalence, it is not always interpreted as this. A study by Ginsburg (1977) in Kieran (1981) about addition and subtraction detected that both + and = are understood by elementary school students as actions to be performed, i.e., "the equal sign means what it adds up to" when describing $3 + 4 =$, or saying "3 and 5 make 8". In addition, they showed that younger school children want to change the equation $= 3 + 4$ and say $4 + 3 =$, as the former is written "backward" (Kieran, 1981, p. 318). Hence, former data shows that students of compulsory school have an operational approach, especially in the early grades. One can see the operational strategy as linked to arithmetic and relational as linked to algebra. The former gives an answer such that the equal sign indicates that one has found the answer, whereas the latter represents a relationship of equivalence (Vincent, Bardini, Pierce, & Pearn, 2015).

The challenges of the equal sign in connection to school algebra have been known for decades. However, not many studies have done research on students of older grades than middle school and seeing the equal sign from more perspectives than just a binary one. Thus, this dissertation wants to investigate how we can see challenges of the equal sign among Danish students who have entered upper secondary school. Do they have the same challenges, or can the challenges here be different?

To assert some more precise research questions, we will first need to introduce the theory of didactic transposition, which will help find the notions of the equal sign in a teaching context. This will then be the baseline to which we can construct a diagnostic test that wishes to uncover the students' challenges of the equal sign.

1.2 Structure of the thesis

The thesis consists of two research questions, which each have its methodologies and analyses. Section 2 introduces the theory of didactic transposition before introducing the two research questions that we will examine. In section 3, the methodology of RQ1 will be presented, which describes the considerations to what material we have used. Subsequently, we give the analysis of RQ1 in section 4. This will function as the theory to which we can construct a diagnostic test in answering RQ2. The methodology of RQ2 appears subsequently in section 5, which describes the participants in the study, the theory of diagnostic tests, the pilot test, the final test, and the process of analyzing the test results. In section 6, we give the results of RQ2. Lastly, we provide the discussion of RQ1 and RQ2 in section 7 before we propose the conclusion in section 8 answering the two research questions.

Throughout this thesis, I have made the translations from Danish, both what regards the knowledge to be taught in section 4, describing the test items in section 5, and presenting the students' answers in section 6.

2. Didactic transposition

The theory of didactic transposition was presented for the first time by Yves Chevallard in 1980 at a summer school, which he published in his *La transposition didactique. Du savoir savant au savoir enseigné* [Didactic transposition. From scholarly knowledge to taught knowledge] (Bosch & Gascón, 2006). The process to which Chevallard refers as the didactic transposition is the one that describes what transformations an object undergoes from it is constructed, applied, selected, and formed to be taught until it is actually taught in a given educational institution. Didactic transposition today has been generalized into institutional transposition, i.e., how knowledge is transposed from one social institution to another. One, therefore, considers that knowledge is a changing reality taking place in social institutions. It is from this that the anthropological theory of didactic arose (Chevallard & Bosch, 2014). When wanting to transpose a body of knowledge, one needs to carry out transpositive work to make the given knowledge "teachable".

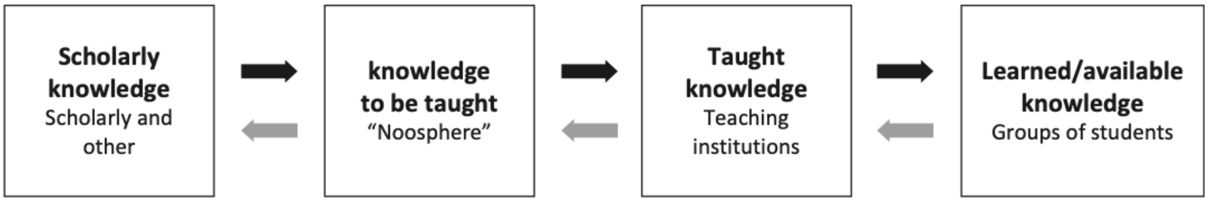


Figure 1. The process of didactic transposition (Chevallard & Bosch, 2014)

There are different participants to this. One divides the transposition into three steps as illustrated in Figure 1. *Scholarly knowledge* refers to the knowledge proposed by the "scholars," which is produced in scholarly institutions such as universities. The *knowledge to be taught* is what one occasionally denounces as 'the noosphere,' i.e., the educational system, which includes curriculum designers, among others (Chevallard & Bosch, 2014). In practice, the product of knowledge to be taught is what we find in curricula, which one refers to in Denmark as official programs and ministerial guidelines. In addition, textbooks and other didactic materials constitute this part.

Taught knowledge relates to the teachers in the classrooms and what they actually teach. The last step of the transposition is related to what can be seen as the end of the didactic process, namely what is actually learned by the students (Bosch & Gascón, 2006), i.e. the *learned knowledge*. One often refers to scholarly knowledge and knowledge to be taught as the external didactic transposition, and the taught and learned knowledge as the internal didactic transposition. Based on this view, one can say that "phenomena of didactic transposition are at the very core of any didactic problem (Bosch & Gascón, 2006, p. 58).

The transpositive work makes it possible to teach, but on the other hand, also creates a lot of limitations. After the transposition process, the school might have lost the reasoning behind the knowledge to be taught, i.e., in mathematics, we might ask ourselves what triangles or limits of functions are suitable for (Bosch & Gascón, 2006). Nonetheless, the transpositive work can also sometimes improve the organization of knowledge and make it more understandable and structured than the original scholarly knowledge (Chevallard & Bosch, 2014). In textbooks, some mathematical concepts are more explicitly stated than others - for example, the concept of a function compared to the concept of the equal sign. Consequently, some mathematical concepts are more explicit than others (Lundberg & Kilhamn, 2016).

2.1 Example of external didactic transposition

The following will consider the external didactic transposition, i.e., the step from scholarly knowledge to knowledge to be taught. We will illustrate this through an example concerning the concept of a function. Considering the scholarly knowledge of functions, one can at university encounter the definition of a function given as

> Let A, B be non-empty sets. A relation f between A and B is called a map or a function from A into B (and we write $f\colon A \to B$), if the following two properties are satisfied:
> 1. For all $x \in A$ there exists $y \in B$ such that xfy (or $(x, y) \in f$)
> 2. If xfy_1 and xfy_2 (or $(x, y_1) \in f$ and $(x, y_2) \in f$), then $y_1 = y_2$.
>
> (Lützen, 2019, p. 134)

To consider the possible gap between the scholarly knowledge and knowledge to be taught, we look at a textbook from the upper secondary level. Here we find the following two definitions of a function:

> Def 1: A function is a correlation between two sizes x and y, that satisfies the following:
> For every value of x there is exactly one corresponding value of y.
> If this is the case, one says that y is a function of x. If the function is called f, we write $y = f(x)$.
> (Carstensen, Frandsen, Lorenzen, & Madsen, 2019, p. 11)

> Def 2: The set of numbers in which the independent variable x can vary is called the definition set of the function and is denoted as $\mathrm{Dm}(f)$.
> The set of numbers in which we find the values of the function, is called the value set of the function and is denoted as $\mathrm{Vm}(f)$.
> (Carstensen, Frandsen, Lorenzen, & Madsen, 2019, p. 11)

There are several things to point out here. First, the notion of a relation does not exist in the knowledge to be taught. Instead, one merely considers "sizes" and "values", terms that are undefined in the textbook itself, and thus presumably considers the terms as part of everyday language. However, a set is denounced in Def 2, namely the set of numbers. In the scholarly knowledge, on the other hand, the notion of a "relation" and "set" are explicitly defined (see Chap 7 and 10 in (Lützen, 2019)). Moreover, the set is arbitrary in this definition. Secondly, the property of existence is described less explicitly in knowledge to be taught i.e., that there exists a corresponding value of *y* if there exists a value *x*. Instead, the definition starts out by stating the property of uniqueness as proposed in 2., i.e., there is a unique value *y* for each value of *x,* and then stating if that is the case, we can call it a function. Thirdly, we note that stating that $y = f(x)$ may confuse in the cases where *f* is not a linear function, as this no longer (strictly) maintains a one-to-one correspondence. Hence, there might be several values of *x* that correspond to the same value *y,* e.g., $\mathrm{y} = \mathrm{f}(x_1) = \mathrm{f}(x_2) = \mathrm{f}(x_3)$... and so on.

To conclude, we see clear gaps between scholarly knowledge and knowledge to be taught in terms of the concept of a function. However, one could argue that defining a "relation" and "arbitrary sets" is unnecessary in upper secondary school to understand the methods introduced here. Thus, this gap is not to be seen necessarily as a critique, but instead making us know that a concept in upper secondary school will need to be contextualized and made teachable for the students. We note that generally, transpositions already made like this one can be of varying quality. Thus, it is always relevant to relate to these objectively so that we can understand and investigate them and determine to what extent they can represent a challenge for teaching and learning – or even lead to misinterpretations.

2.2 Research questions

Based on the theory of didactic transposition, we are now able to phrase the two research questions we pursue to investigate:

RQ1: Based on external didactic transposition, how do the scholarly meanings of the equal sign come into view in knowledge to be taught in Danish upper secondary school?

RQ2: Which challenges related to the external didactic transposition of the equal sign can be detected through a diagnostic test for Danish students in upper secondary school?

Thus, to answer RQ2, we will first need to analyze the external didactic transposition of the equal sign to establish the theory of meanings of the equal sign before constructing the diagnostic test of RQ2. Hence, the two questions are interdependent.

3. Methodology for RQ1

The material used for scholarly knowledge concerns both the scholarly knowledge of mathematics and didactics of mathematics. Regarding the former, we will consider the history of the equal sign and the mathematical definition of the equal sign today. Concerning the scholarly knowledge of didactics of mathematics, this will include the notion of operational vs. relational approach and a different model that identifies further ideas of the equal sign. We advocate that these two sides of scholarly knowledge are equally necessary for helping us see the various facets of the equal sign.

The materials chosen as the knowledge to be taught have taken outset in what type of school the diagnostic test will be handed out to, which is a general upper secondary school (STX) and business upper secondary school (HHX). Firstly, the official programs and ministerial guidelines are included. The curriculum describes the content of the subject mathematics, and the ministerial guidelines serve as the intentions of the curriculum. The textbooks are an interpretation of the formulations of the curriculum (Danish Ministry of Education, Matematik A/B/C, stx Vejledning, 2020, p. 2). As we have students from both STX and HHX, we will revise guidelines from both. Furthermore, the participants of the diagnostic test have just finished their introductory period which is a three-month program before they are to choose their study line, which typically is either a specialization of languages, natural sciences, or social sciences. Thus, what regards textbooks, three books have been analyzed; one textbook covering the introductory period of STX and two general textbooks; one of STX and HHX. When choosing study line In Denmark, there are three levels of mathematics - A, B and C - where A is the highest level. The general textbooks cover mathematics B-level which is the most common level. Thirdly, exemplary screening tests of the introductory period will be analyzed, as they take outset in what the students should have learned at the time when the diagnostic test is given to them. The screening tests that have been analyzed are example test sets as it was not possible to gain access to the official screening tests that have been given. The screening tests consist of three example sets, one of 2017 and two of 2018 which is worked out by the Danish Ministry of Education. Altogether, these materials will show a fair representation of the knowledge to be taught as they are based on official programs, tests and textbooks based on the revised guidelines from 2017 of which the participants of the test are under.

4. Analysis of RQ1

4.1 Analysis I: Scholarly knowledge

4.1.1 History of the equal sign

The equal sign "=" that we use today was first introduced by the British mathematician Robert Recorde in 1557, just about one year before his death. Although he had a background in medicine, he did several mathematical works within algebra, arithmetic, geometry, and astronomy. He had a great sense of pedagogy and wanted to create a dialogue with the reader by giving explanations for each step in an introduced technique (Katz, 2009). In his algebra book *The Whetstone of Witte*, Recorde gave the equal sign that we use today. The book's content is not considered as highly original as he had most of it from other sources. However, his work was to be taught to a whole generation of English scientists, and the equal sign to many more (Katz, 2009).
He argues for the need for the equality sign by saying the following

> To avoid the tedious repetition of these words—is equal to—I will set as I do often in work use, a pair of parallels, or gemow [twin] lines of one length, thus =, because no 2 things can be more equal. (Katz, 2009, p. 396)

Before and after Recorde, other mathematicians used different types of equal signs. It was first in the 16th and 17th centuries that mathematics and algebra were beginning to create the symbols that we use today (Katz, 2009). Mathematicians before this era also did not have common notation. For instance, François Viète's notation for x^2 in 1591 was "*A quadr*." where *A* represented the unknown number (Cajori, 1993, p. 345), and the same goes with the equal sign. Words such as *eaequales, esgale, faciunt*, and *gleich* are all previous notations for equality. However, there were also people before Recorde who used symbols instead of words, e.g., 'eq' was used by Pérez de Moya en 1558, three horizontal lines by Ghaligai in 1552, and the oldest mathematical document that one knows of - the Rhind. Papyrus (1650 BC.) - the symbol is a beetle that means 'becomes' (Molina, Castro, & Castro, Historia del signo igual, 2007).

The original look of Recorde's equal sign had two parallel lines of the same length, which was longer than the one we know today. Other writers such as Weigel in the late 17th century and Swedenborg in the early 18th century used a shorter sign (Molina, Castro, & Castro, Historia del signo igual, 2007). Although Recorde published his work in 1557, it was not implemented and acknowledged among others before much later. The biggest rival of Recorde is likely Rene Descartes, who introduced his equal sign in 1637, much similar to an open infinity sign $\propto$. It was probably thanks to, among others, Leibniz and Newton, who used Recorde's equal sign that made it generally accepted (Cajori, 1993).

One has given various meanings to the equal sign throughout history. For instance, Viète used it to indicate that two numbers were not equal, whereas others used it to denote decimals (e.g., 102=857 denoted 102.857). Moreover, some used it to separate numbers, Descartes used it as *plus ou moins* $\pm$, and a fifth use denotes parallel lines (Cajori, 1993). Several of the previous uses of the equal sign we consider incorrect today. We can find yet another example of this in the "The Colombian Arithmetician" from 1811, where one used the equal sign between different operations like this: $1 + 6, = 7, \quad \times 6 = 42, \;\; /2 = 21$ (Cajori, 1993, p. 307)

Today as well, the equal sign has several uses, some of which are more acceptable than others. Nonetheless, there is not one pure definition of the equal sign in the mathematical world, most likely because it has been used in so many different fields and contexts. However, the notion of an equivalence relation, of which the equal sign can be correctly defined mathematically, will be described in the next section.

4.1.2 The equal sign as an equivalence relation

Although history has shown that the equal sign has had many different looks in terms of symbols and other meanings, there is a sector in which we can describe the equal sign mathematically, namely within the notion of relations, where the equal sign is an example of an equivalence relation. One can define an equivalence relation as the following:

> The relation = on an arbitrary set *A* is an equivalence relation to which it implies that it is reflexive, symmetric, and transitive:
>
> 1) Reflexivity: $\forall a \in A: a = a$

2) Symmetry: $a = b \Longrightarrow b = a$
3) Transitivity: if $a = b$ and $b = c \Longrightarrow a = c$

(Lützen, 2019, p. 120, 124)

Thus, the property of reflexivity expresses that every number is equal to itself; symmetry states that the equal sign works like a mirror – we can read it from both directions; and the property of transitivity affirms that if two numbers are equal to a third number, then the two numbers are equal. As we see below, these three properties of the equal sign have inspired how we define the equal sign in didactics of mathematics.

4.1.3 The equal sign in didactics of mathematics

The big question regarding the relational vs. operational approach is how this way of interpreting the equal sign is developed. Is it because of cognitive limitations, or is it the teaching or textbooks they had? Among others, this has been investigated by McNeil et al. (2006). Students in middle school (ages 11-14) have, according to psychology, fewer cognitive limitations than in elementary school. Therefore, one would think they have a unique understanding of the equal sign. Studies provided by Baroody and Ginsburg (1983) in Kieran (1981) suggested that age alone cannot be the reason behind the operational view of the sign. To solve standard operation-equals-answer equations (i.e., $9 - 3 = 6$), one does not need to understand the symbol of equality (Kieran, 1981). Students can, for example solving the equation $2x + 3 = 11$ without a relational view by a guess and check approach (Vincent, Bardini, Pierce, & Pearn, 2015). The claim of McNeil et al. (2006) is that performing these arithmetic operations consequently creates an association of the equal sign with those operational procedures.

Additionally, there is not a lot of instructional time, if any, given by the teacher or the books explicitly on the equal sign. A possible resolution from the study is that instead of always presenting the equal sign in standard contexts - that is, the operation-equals-answer - teachers could try and show it in non-standard contexts. One cannot expect students in middle school to create new ways of thinking if we only expose them to a limited range of contexts. Hence, although their cognitive abilities are mature enough, they will not necessarily adopt a relational view of the equal sign (Kieran, 1981). A consequence of seeing the equal sign as operational is that students use it to mean "then I did this" and write strings of false equalities, where each step represents a step in a multi-step calculation. For instance, when computing $5 \cdot (13 + 27)$, many students write $13 +$

$27 = 40 \cdot 5 = 200$, as they would do on their calculator (Vincent, Bardini, Pierce, & Pearn, 2015). The string of equalities shows that the student understands the problem; thus, the difficulty is at the symbolic level (Kieran, 1981), which refers to understanding algebraic notation. We denote the string of equalities as string operations in this thesis, to assess its connection to the operational misconception that it represents.

We will in this dissertation extend the meanings of the equal sign as proposed by (Prediger, 2010). Her model has taken outset from the earlier studies that distinguish between an operational approach and the relational approach. The model consists of six meanings of the equal sign, which all satisfy the notion of an equivalence relation. Thus, one can say that they are all equally accurate and valid meanings of the equal sign. We suggest that the binary approach of operational and relational is too small and that the categorization of Prediger captures more nuances of the role played by the equal sign.

The first meaning relates to the operational meaning as we know it from former studies. What concerns the relational meaning, Prediger distinguishes between four types: *symmetric arithmetic, formal equivalence, conditional equation characterizing unknowns, and contextual identities in formula*. What goes for all of them is that they focus on the symmetric use of the equal sign. (Prediger, 2010, s. 81). Regarding symmetric arithmetic identity, it is an "extension" of the symmetry property in an equivalence relation. "Arithmetic" is included in the name to distinguish it from the cases involving one or several *variables x.* A variable *x* is here either a placeholder, an unknown, a meaningless symbol, or a changing quantity (Prediger, 2010). The symmetric property induces commutativity when we consider compositions of two elements a, b in an arbitrary set *A*. Furthermore, numerical identities such as $10^2 - 9^2 = 19$ – which are easy to compute the one way, but not the opposite – are also considered symmetric arithmetic identities (Prediger, 2010, p. 81). Regarding string operations, we will categorize them here as a misconception of symmetric arithmetic.

In reference to formal equivalences, these are equations that always are true, that is, true for arbitrary variables. Thus, it concerns equations of algebraic terms, contrary to symmetric arithmetic. E.g., we can write $(x - 2)(x + 3) = x^2 + \mathrm{x} - 6$ (Prediger, 2010). In Poulsen (2015), we find some categories which can help us better understand what constitutes formal equivalence. We have identified the *laws of distributivity, associativity, and commutativity* of field theory (Poulsen, 2015) i.e., $5x + 5 = 5(x + 1)$ which is due to the multiplicative law of distributivity. Furthermore, formal equivalence is when *simplifying* an expression (Poulsen, 2015) e.g., $(2x + 1) - 5 = 2x - 4$. Lastly, *rewriting* an expression (Poulsen, 2015) refers to the identities of mathematical objects such as logarithms, exponents, and fractions e.g., we can rewrite $\frac{a}{b} + \frac{b}{a}$ into $\frac{a^2+b^2}{ab}$. In other words, one can see formal equivalence as applying calculation rules of a specific mathematical object. Notice that all three ways of interpreting formal equivalence can also be used for symmetric arithmetic. The only difference between the two is whether we consider precise numbers or variables.

What concerns *conditional equation characterizing unknownss for unknowns*, they represent equations that are true for a particular *condition,* as the name indicates. Even if $(x - 2)(x + 3) = x^2 + x - 6$ and the equation $x^2 = \ - \ x \ + 6$ are symbolically near each other; the latter does not apply to all *x*. Thus, it represents an equation of unknowns which we find by solving the equation (Prediger, 2010, p. 82). We will simplify the term in this thesis by only referring to it as the meaning of conditional equation.

Regarding *contextual identity in formula*, this is to be understood as typical or 'known' formulas, such as Pythagoras theorem $c^2 = a^2 + b^2$. These types of equations are general statements, but as it does not apply to all *a, b*, it is named contextual identity in formula since the equal sign is valid only in a specific context (Prediger, 2010). Prediger does not define explicitly her notion of formula. Thus, we will in this thesis develop the meaning of contextual identity to make it apply to any expression that is true for some numbers and demands some assumptions. We will refer to contextual identity in formula as simply a contextual identity. Notice that the difference between the conditional equation and the contextual identity then is that the former can be solved, contrary to contextual identities, which can contain several unknowns and not be solved.

In addition to the operational and relational category, there is a third category that Prediger names specification. It refers to the cases where the equal sign is used as defining identities. From an epistemological point of view, it is to distinguish definitions and propositions from each other (Prediger, 2010). Thus, for example, we can write $\mathrm{f}(x) = 4x + 5$, which means that $\mathrm{f}(x)$ is a name for the function defined as $4x + 5$.

4.2 Analysis II: Knowledge to be taught

4.2.1 Ministerial guidelines

Regarding the curriculum and guidelines of STX, they do not mention the equal sign explicitly anywhere. However, we can find statements in which we can interpret the meanings of the equal sign. The curriculum consists of disciplinary goals and content goals. One of the disciplinary goals is "to operate with numbers and representations of numbers (…)," to which we can interpret the meaning of the operational sign. Moreover, it states "to handle formulas (…)" (Danish Ministry of Education, Bilag 112, Matematik B – stx, august 2017, 2017, p. 1) which it does not further explain, but what we still can somehow identify as contextual identity. Additionally, they mention "calculation of percentage and annuities (…) interest rate formula" (Danish Ministry of Education, Bilag 112, Matematik B – stx, 2017, p. 2). From the ministerial guidelines, the minimum requirements include "to know term designations (words and symbols) and the meaning of concepts". Here one can see the equal sign and the meaning of equality as a subset of this. However, this is not explicitly mentioned as such. Moreover, regarding formulas and functions, they mention "rewriting and reducing formulas and expression by paper/pencil (…)", which can be categorized as the meaning of formal equivalence and thus partly symmetric arithmetic. The meaning of conditional equation can be identified under the headline "equation solving," where it says, "to algebraically solve equations by paper/pencil" (Danish Ministry of Education, Matematik A/B/C, stx Vejledning, 2020, p. 16). Considering the meaning of specification, this does not appear anywhere except for the subchapter of functions in the guidelines where they state to "clarify the difference between a rule and an equation" (Danish Ministry of Education, Matematik A/B/C, stx Vejledning, 2020, p. 10), for instance distinguishing between $f(x) = 2x + 1$ and $2x + 1 = 5$. Thus, all the meanings of the equal sign can somehow be interpreted from the guidelines and curriculum, nonetheless solely implicitly. One can also examine the mentioned goals and definitions without necessarily having an equal sign in mind.

In the official program of HHX, the equal sign is not explicitly mentioned either. However, one of the disciplinary goals is "to handle formulas, including translating between mathematical symbol language and colloquial spoken or written language and symbol language for the solution of problems with mathematical content". We identify this as a meaning of contextual identity in handling formulas and, parallel, the meaning of the conditional equation when they refer to "solution of problems". More precisely, we see in the content goals that one of these are "fundamental computation skills, calculation of percentage and indices, (…), reduction, (…)" to which we can interpret "computation skills" as meaning of the operational sign and "reduction" as a meaning of formal equivalence and symmetric arithmetic. Moreover, the content goal of "equation solving; analytically, graphically and by the help of IT" is implicitly stating the meaning of conditional equation (Danish Ministry of Education, Bilag 41, Matematik B - hhx, 2017, p. 1). Nonetheless, it is difficult to identify the meaning of specification. Concerning the ministerial guidelines of HHX, they mention in the introduction "the mathematical content competencies" and "the competence of ways of thinking" in which they exemplify with "(…), what an equal sign means, and where they are used" (Danish Ministry of Education, Matematik A/B/C, hhx Vejledning, 2021, p. 5). Nevertheless, this merely serves as an example which they do not dig deeper into.

4.2.2 Screening test exercises

The screening test is the final test after the introductory period. The exercises are primarily about linear functions for the table, graph, formula, and linguistic description representation forms. Hence, tasks only involving table, graph, or linguistical description representations, are of no interest if they do not include the fourth type of representation, namely formula. With the latter, we encounter the use of the equal sign in the task and the solutions. Thus, the examples we give are tasks of this form.

Type of exercise: Investigate whether a point P(a,b) is on the line of the linear function f(x):

This example is exercise 3a from the example screening test set 3 of 2017:

> Two linear functions are determined by $f(x) = -0{,}5x + 7$ and $g(x) = 2x + 10{,}4$
> Investigate whether the point P(4,5) lies on the graph of f

(Danish Ministry of Education, Matematik Screening studentereksamen vejlædende opgavesæt 3, STX-MAT-GRUNDFORLØB, 2017)

The task itself displays a meaning of specification as the functions are simply defined as $f(x)$ and $g(x)$ and thus do not represent an equation as such, but more of naming two objects, i.e. the two linear functions. Furthermore, the meaning of the operational sign will be helpful in the task since one is to compute $f(4)$, which indicates that one will set $x = 4$ (specification) everywhere in the expression, obtaining the following: $f(4) = -0{,}5 \cdot 4 + 7 = -2 + 7 = 5$. Thus, the last two equal signs function as operational signs.

Type of exercise: Given intersection in x-axis and y-axis, find the formula of f

In exercise 4 from the same set, one is to find a rule of *f*,

> The graph of a linear function f intersects the first axis in x=5 and the second axis in y=10.
>
> a) Decide a rule of *f*.

(Danish Ministry of Education, Matematik Screening studentereksamen vejlædende opgavesæt 3, STX-MAT-GRUNDFORLØB, 2017)

The following table shows what meanings of the equal sign appear when we have solved the problem:

Solution	**Meaning of the equal sign**
$f(5) = 0$	Specification
$f(0) = 10$	Specification
Since we have the assumption that it is a linear function, we know that we can write *f(x)* as $f(x) = ax + b$	Contextual identity
$f(0) = a \cdot x + b = a \cdot x + 10$	Specification
$\mathrm{b} = 10$	Conditional equation
$f(5) = a \cdot 5 + 10$	Specification
$f(5) = 0 \Longrightarrow 0 = a \cdot 5 + 10$	Specification
$-5\mathrm{a} = 10$	Conditional equation
$\frac{-5a}{5} = \frac{10}{5}$	Conditional equation
$\mathrm{a} = -2$	Operational
$f(x) = ax + b;\ a = -2;\ b = 10$ $\Leftrightarrow f(x) = -2x + 10$	Specification

Table 1. Meanings of the equal sign in finding the rule of f

We see from this solution that the meaning of specification appears quite frequently alongside conditional equation, but we also observe the meaning of operational and contextual identity.

Type of exercise: Solve a linear one-variable equation of the form f(x)=y or f(x)=g(x)

The rules for two linear functions *f* and *g* are given by

$$f(x) = 2x + 1$$
$$g(x) = kx - 3$$

where *k* is a constant.

a) *Set* $k = -2$ and determine the first coordinate in the intersection between the graphs of f and g.

(Danish Ministry of Education, Matematik Screening studentereksamen vejlædende opgavesæt 5, STX-MAT-GRUNDFORLØB, 2018, p. 3)

Solution of the task	**Meaning of the equal sign**
$f(x) = 2x + 1 \qquad g(x) = kx - 3$	Specification
$k = -2$	Specification
$g(x) = -2x - 3$	Specification
$g(x) = f(x)$	Conditional equation
$-2x - 3 = 2x + 1$	Specification
$-4x = 4$	Conditional equation
$x = \frac{4}{-4}$	Conditional equation
$x = -1$	Operational sign

Table 2. Meanings of the equal sign - solving exercise 7 in example screening test 5, 2018

The first three lines of the solution are simply statements of what is $f(x)$, $g(x)$ and *k*, and the meaning of the equal sign thus is a specification. When setting up the equation and solving it, the meaning of the equal sign is that of a conditional equation. Finally, when calculating the value of $\mathrm{x} = \frac{4}{-4}$ makes $\mathrm{x} = -1$, this shows an operational sign.

Type of exercise: Explain the steps of an equation

In this type of exercise, the student is to explain what has happened in each step in the equation solving.

The statements of the task	**Meaning of the equal sign**

$3x = 2 \cdot \left(\frac{1}{2} - x\right) - 6$	Conditional equation
$3x = 1 - 2x - 6$	Formal equivalence
$3x = -2x - 5$	Formal equivalence
$5x = -5$	Conditional equation
$x = \frac{-5}{5}$	Conditional equation
$x = -1$	Operational sign

Table 3. Meanings of the equal sign in exercise 4 of example screening test 5 2018

(Danish Ministry of Education, Matematik Screening studentereksamen vejlædende opgavesæt 5, STX-MAT-GRUNDFORLØB, 2018, p. 2)

We observe that formal equivalence appears although we are solving an equation. We observe formal equivalence as $2 \cdot \left(\frac{1}{2} - x\right) - 6 = 1 - 2\text{x} - 6$, which is due to the law of distributivity of multiplication. Further on, $1 - 2\text{x} - 6 = -2\text{x} - 5$ which represents a reduction. This shows that several meanings of the equal sign can be at stake when solving an equation.

The statements of the task	**Meaning of equal sign**
$a = \frac{y_2 - y_1}{x_2 - x_1}$	Contextual identity
$a = \frac{6 - (-2)}{7 - 3}$	Specification
$a = \frac{8}{4}$	Operational sign
$a = 2$	Operational sign

Table 4. Meanings of the equal sign in exercise 5 of example screening test 6 2018

(Danish Ministry of Education, Matematik Screening studentereksamen vejlædende opgavesæt 6, STX-MAT-GRUNDFORLØB, 2018, p. 2)
We see a similar pattern to that of the former exercise, although the first step in this one is a contextual identity, as it occurs from some assumptions stated in the task.

We detected throughout several of the tasks from the screening tests that they follow a typical pattern in their solution that we will now describe. Firstly, a function is defined, and thus the meaning of specification is the usual starting point, expressing the mathematical object in the task. Secondly, one is to solve an equation, and hence the meaning of a conditional equation with an unknown appears. In finding the solution, one isolates the unknown, and here the meaning of formal equivalence may sometimes appear in some of the steps. Lastly, one calculates the value of the unknown. Thus, one often finalizes these tasks with the operational meaning of the sign. Hence, one could argue that many of the exercises create a result-oriented approach as the meaning of the operational sign is what the solution always comes down to. Accordingly, one needs an understanding of computing just as much as a relational meaning. The meaning of contextual identity does not occur as frequently throughout the tasks of the analyzed screening tests. It appears in the exercises of linear regression or when using a formula related to linear functions. What regards symmetric arithmetic, we did not detect this. However, this is following the ministerial guidelines who proposes it quite vaguely.

4.2.3 Textbooks

The first book of the analysis is a book of the introductory period of STX, *MAT STX grundforløb* (2017). There are three chapters in the book; linear models, mathematical modeling, and number and letter-computation before the book's final part containing tasks. We will refer to the book as the "book of the introductory period". The second book of the analysis is a general book of mathematics B-level of STX, *MAT B1 STX* (2019). The book consists of the topics on functions, [square] roots and powers, annuity, vectors, statistics, and classical geometry. We will refer to the book as the "book of STX". Regarding the third and last book, this is a general book of mathematics B-level of HHX *Matemat10k* (2019). We will refer to the book as the "book of HHX". The book chapters include the subjects of functions, annuities, statistics, and differential calculus.
Overall, the books show a diverse set of topics, and as they are from 2017 and later, they are all under the updated ministerial guidelines of 2017. We will begin the analysis with the cases in which the books present some questionable uses of the equal sign.

Questionable uses of the equal sign

In chapter 1 of MAT STX grundforløb, we encounter in exercise 6 a linear function where *y* is to denote a numerical value in *cm*.

$$y = 6{,}31 \cdot 12 + 79{,}8 = 155{,}5 \text{ cm}$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 17)

The calculation itself is correct but concluding that the *number* we compute equals a *number in cm* shows inconsistency, as this does not maintain the equilibrium of the equal sign. We suggest that this is to be categorized as false formal equivalence, as the terms on each side of the equal sign are no longer equivalent; the left-hand side represents a number. In contrast, the right-hand side represents a number with the unit *cm*. We find a similar example in the textbook of STX,

A trip of 35 km costs $15 + 35 \cdot 17 = 610$ kr

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 10)

This is the same type of example, only this time, we have more text included alongside the equation and the second unit of km, complicating the equation even more. Thus, the equation is included as part of a non-mathematical sentence, concluding with a number in *kr* [Danish currency]. Lastly, we give a similar example from the book of HHX:

$$\mathrm{E}(X) = \mathrm{n} \cdot p = 125 \cdot 0{,}32 = 40 \text{ people}$$

(Axelsen & Dalsgaard, 2019, p. 182)

Conclusively, this shows a misconception of the equal sign, more specifically regarding the meaning of formal equivalence, as maintaining equivalent terms on both sides of the equal sign fails. By either removing all units or making sure that each term has the correct unit, one would balance the equation, thus satisfying the formal equivalence and the relational meaning of the equal sign. Although the text in the given examples explains that it is about cm, kr, and people, one cannot, from a mathematical point of view – mix this non-mathematical language into the mathematical one, i.e., the equation, without stating the units explicitly in the equation. Incorporating non-mathematical notation with mathematical will undoubtedly strengthen the students' difficulties if they are to fully understand the concept of the equal sign or any other symbol for that matter.

We will conclude this section of questionable uses by calculation of percentage, which we have shown is explicitly stated in the official programs of both STX and HHX. In the book of HHX, it gives the following example, Thomas makes 30000kr. a month and gets a pay raise of 3,2%. His new monthly salary is thus:

$$30000 + 3{,}2\% = 30000 \cdot 1{,}032 = 30960$$

(Axelsen & Dalsgaard, 2019, p. 39)

The first equal sign is mathematically invalid as

$$30000 + 3{,}2\% = 30000 + \frac{3{,}2}{100} = 30000 + 0{,}032 = 30000{,}032 \neq 30960$$

where we used the definition of % of chapter 3 of the book:

Percent comes from Latin, pro cent, and means per hundred, i.e. $\frac{1}{100}$ and is written %.

(Axelsen & Dalsgaard, 2019, p. 37)

Moreover, we find a similar misapplication in the book of STX,

$$\frac{7}{100} \cdot 100 = 7\%,$$

that is precisely the difference in the indices: 107 – 100 = 7 percentage points.

(…) 107-103 = 4 percentage points, whereas the percentage-wise increase is

$$\frac{4}{103} \cdot 100 \approx 3{,}9\%$$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 276)

Like the former examples, one mixes standard language with mathematical notation – the books speak of percentages but do not use the %-sign on both sides of the equal sign. Subsequently, the equations do not hold mathematically but merely serve as part of the written text in the examples. Consequently, they do not represent any of the six meanings of the equal sign, and we must thus consider them as deceptive.

Operational meaning

In the very introduction of the last chapter regarding "Number and letter-computation" in the book of the introductory period, we are given some examples as to what one can consider "elementary mathematics":

$$3 + 2 = 5$$

$$2a + a = 3a$$

$$3x + 2x = 5x$$

$$4 \cdot 5y = 20y$$

$$2p \cdot 3p = 6p^2$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 68)

All six examples given in the book show an operational approach. None of them are written as, for example

$$5 = 3 + 2$$

$$2a + a = a + a + a$$

$$4 \cdot 5y = 2 \cdot (2 \cdot 5)y$$

$$5x = 2 \cdot (x + x) + x$$

$$2p \cdot 3p = 2 \cdot \ 3p^2$$

Thus, the book itself implicitly shows an operational meaning of the equal sign, as it does not promote the meaning of symmetric arithmetic identity or formal equivalence in these examples.

There is another example in the book in which this view of the equal sign again comes into sight. It is in relation with calculations of fractions, where it's stated the following:

$$\frac{42}{60} = \frac{2 \cdot 3 \cdot 7}{2 \cdot 2 \cdot 3 \cdot 5} = \frac{7}{2 \cdot 5} = \frac{7}{10}$$

(Carstensen, Frandsen, Lorenzen, & Jørgensen, 2017, p. 75)

The calculation is yet another example of an operational meaning of the sign, which declares how one can go from one fraction and simply reduce it straight-forward. This creates in its turn a result-oriented approach as one does not encounter examples in which one does the opposite.

Considering the general books of STX and HHX at the mathematics B level, the topics and problems are more diverse. In general, we find similarities to the screening test regarding the book of the introductory period; it is with linear functions that we find the operational sign occur, i.e., when one is to compute a value of $f(x)$. However, we also see it when computing annuities or the length of vectors. Like the screening tests, an operational sign turns out to be helpful in most exercises, also within equation solving when one is to deduce the value of the found x.

Specification

We will give the first examples concerning specification from the book of the introductory period, but they are similar to what we found in the books of STX and HHX.

$$y = f(x) = -x + 5$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 108)

This expression shows the meaning of specification occurring twice. First, one defines *y* as $f(x)$, which then is defined by $-x + 5$. However, in the subsequent exercise, the book has, for some unclear reason removed *y*, and simply stated:

$$f(x) = -\frac{2}{5}x + \frac{3}{5}$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 108)

Although both functions are of the same form – they are linear (and single variable). Moreover, we encounter a third example where the book defines a line as:

$$m: 3x - 4 = 2$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 120)

Here *m* defines what the line is. Consequently, we do not find a complete consistency in the difference between using "=" and ":" when asserting a line. In this regard, we have one more example we would like to point out,

Dissolve numerator and denominator in factors and reduce the fraction

$$1.\ z = \frac{4x^2 - 9}{4x^2 + 9 - 12x}$$

(Carstensen, Frandsen, Lorenzen, & Jørgensen, 2017, p. 154)

Here, the equal sign serves as a meaning of specification as *z* is the name of the fraction. Accordingly, one does not need to use *z* to solve this task. Thus, we again encounter some inconsistencies of notation in the book, which are not incorrect uses of the equal sign. However, it is highly contributing to confuse when something is an equation and when something is just a definition, as the guidelines of STX mentions: "clarify the difference between a rule and an equation" (Danish Ministry of Education, Matematik A/B/C, stx Vejledning, 2020). In other words, in what contexts is the meaning of the equal sign a conditional equation or a specification?

We end this section of the meaning of specification, by giving some examples from the books of STX and HHX that we encountered. Regarding the book of STX, we find the meaning of specification in other contexts than just functions and general equations. We also find it when naming terms in statistics, e.g.,

$$\mu = \mathrm{E}(X)$$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 285),

Which mere states that we define the expectation value $\mathrm{E(X)}$ as μ.

Moreover, we find specification concerning geometry when asserting length, e.g., in a figure that illustrates a triangle ΔABC, the side AB is defined as c, BC as $\mathrm{a} = 7$ and AC as $\mathrm{b} = 10$ (Carstensen, Frandsen, Lorenzen, & Madsen, 2019, p. 247).

In the book of HHX, there is no geometry included. However, the topics of functions and statistics are essential, especially regarding the meaning of specification. Hence, we give one example from here similar to that of STX, namely defining variance, Var (X)

$$\sigma^2 = \mathrm{Var}\ (X) = \sum (x - \mu)^2 \cdot f(x)$$

(Axelsen & Dalsgaard, 2019, p. 173)

Overall, we have detected the meaning of specification appears in the books as definitions of objects within functions and statistics, alongside asserting lengths in a geometrical context (only STX).

The relational meanings of the equal sign

We finish the analysis of the textbooks by considering how the relational meanings of the equal sign come into view in the textbooks. The section will end with an example from one of the books in which we display how the different meanings of the equal sign come into view in the same exercise.

Symmetric arithmetic

The meaning of symmetric arithmetic is vaguely mentioned in the guidelines of STX as "Rewriting and reducing formulas and *expression* by paper/pencil (…)". However, we found some examples of it in the books about historical mathematics, such as Fermat's little theorem:

$$8^4 - 1 = 4095 = 5 \cdot 819$$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 51)

Here we see an operational meaning in the first equal sign, followed by the symmetric arithmetic identity. Furthermore, in the book of the introductory period, we find Goldbach's conjecture

> We have, for example, that
>
> $$6 = 3 + 3, \;\; 98 = 37 + 61 = 79 + 19, \quad 112 = 3 + 109 = 5 + 107$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 103)

Where each equal sign here represents symmetric arithmetic identity. In the calculation of annuities, we also find symmetric arithmetic:

> (…) after the first year, the balance is calculated as:
>
> $$2100 = 2000 + 100 = 1 \cdot 2000 + 0{,}05 \cdot 2000 = (1 + 0{,}05) \cdot 2000 = 1{,}05 \cdot 2000$$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 76)

However, they do not make up a significant part of the books. We did not find any example of symmetric arithmetic in the book of HHX. This is most likely because one sees symmetric arithmetic identities in a more generalized context alongside algebraic expressions, hence in the shape of formal equivalence.

Formal equivalence

Regarding formal equivalence, it typically arises as identities of vectors, exponents, logarithms, square roots, and rewriting expressions in solving equations. For instance, For non-negative numbers *a* and *b*, we have that

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 44)

This identity represents a *rewriting* of the expression which applies for all *a* and *b* thus, the definition of formal equivalence. Moreover, one of the identities of a logarithm is that,

For the functions log and ln, we have for all positive numbers *a* and *b* (…), the following rules:

1. $\log(ab) = \log a + \log b$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 68)

This also shows a rewriting of the expression for any *a* and *b*, hence a formal equivalence. Furthermore, the meaning of formal equivalence occurs when using the binomial theorems of a square in *reduction* of polynomials:

$$\frac{x^2+6x+9}{x^2-9} = \frac{(x+3)^2}{(x+3)(x-3)} = \frac{(x+3)(x+3)}{(x+3)(x-3)} = \frac{x+3}{x-3}, \mathrm{x} \neq \pm 3$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 81)

Notice that one has made use of both the identities $(a+b)^2$ and $(a+b)(a-b)$. First, one uses the *law of distributivity of multiplication* to deduce the binomial formula. Second, one *rewrites* the expressions in the numerator and denominator, allowing us to *simplify* or *reduce* the expression. We finish the examples of formal equivalence with an example of the book of HHX, which regards identities of probabilities (and implicitly set theory):

For events in a sample space *U*, there are the following computation rules:

(…)

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

(Axelsen & Dalsgaard, 2019, p. 165)

This is yet another example of a rewriting of an expression. We also note that formal equivalence appears in solving equations parallel to what we portrayed in the section of the screening tests.

Conditional equation

The meaning of conditional equation is probably one of the most frequently appearing meanings of the equal sign in mathematics in upper secondary school because of the focus on algebra. In general, we find it in the context of solving equations of different types of functions, or just expressions in general:

$$2(3x-1)=x-4(2-x) \Leftrightarrow 6x-2=x-8+4x$$
$$\Leftrightarrow 6x-2=5x-8 \Leftrightarrow x=-6$$

(Carstensen , Frandsen, Lorenzen, & Jørgensen, 2017, p. 24)

We see from this example that the first equation represents the meaning of the conditional equation, whereas, in the following two equations, the equal sign has the meaning of formal equivalence. We see this as no changes have been made on the left-hand side, whereas we have rewritten the right-hand side as $x-8+x=5x-8$ which is true for all x. In the final equation, we see the meaning of the operational sign. Thus, although the equation, to begin with, represents the equal sign as a conditional equation, it will often (almost always) have several meanings appearing throughout the solving, which we will show at the end of the section.

Contextual identity

Considering the meaning of contextual identity, this is identified in the books predominantly within geometry, such as finding the area, sum of angles, or applying Pythagoras theorem – but can also be seen with the calculation of annuities and differential calculus. The first example we give is about the topic of annuities, in the book of STX:

> The formula of interest (the formula of capital) calculates the balance on a saving that begins with the value K_0, and that for each term becomes increased with interest by a fixed interest rate *r*. The balance after *n* terms is given by the following formula:
>
> $$K_n = K_0 \cdot (1+r)^n,$$
>
> where K_0 is the start capital, r is the interest rate, n is the number of terms, and K_n is the end capital (the balance) after those *n* terms.

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 77)

The equation given includes some assumptions of several variables, i.e., K_n, K_0, r, and n, thus the equation holds for some numbers, but not in general like a formal equivalence. Furthermore, it cannot be solved like an equation with an unknown as the equation involves more than one unknown. Thus, we see that this represents a contextual identity. We found a similar example within the same topic in the book of HHX,

> For the exponential function $f(x) = b \cdot a^x$, the relative gain is constant. For each time x increases by 1, then the relative gain is
> $$r = a - 1$$

(Axelsen & Dalsgaard, 2019, p. 45)

This again shows contextual identity as we have given some assumptions for an equation with several unknowns. We give Pythagoras' theorem as a third example of contextual identity.

> In a right-angled ΔABC, where $\angle C = 90°$, we have that
> $$a^2 + b^2 = c^2$$

(Carstensen , Frandsen, Lorenzen, & Madsen, 2019, p. 252)

We observe here that the context of using Pythagoras's theorem refers to the assumption of having a right-angled triangle. Said differently, if this equation holds, then the triangle must be right-angled. However, finding *a, b,* and *c* where it does not hold does not disprove the equation, as it has solutions. It merely shows that it does not generally fit like a formal equivalence – and nor can we solve it as we have several variables.

Last, we give an example related to differential calculus, which is given in the book of HHX:

> Given a differentiable function f. The equation for the tangent of the graph, that touches x_0, has the equation
> $$y = f'(x_0) \cdot (x - x_0) + f(x_0)$$

(Axelsen & Dalsgaard, 2019, p. 132)

To use the stated equation, we need the assumption that f is differentiable. Hence, we note that contextual identity represents equations whose solutions exist. Accordingly, a contextual identity stated alone is true and false at the same time if no assumption is given. We note that it has been challenging to detect contextual identities in the textbooks, especially what concerns finding identities that the students know at the time when they are given the test.

Determining n in A_n	**Explanations of the book**	**Meanings of the equal sign**
$A_n = y \cdot \frac{(1+r)^n - 1}{r}$	None	Contextual identity
$100000 = 2000 \cdot \frac{1{,}0033^n - 1}{0{,}0033}$	the numbers are inserted	Specification
$100000 \cdot 0{,}0033 \ = \ 2000 \cdot (1{,}0033^n - 1)$	Multiply with the interest	Conditional equation
$\frac{100000 \cdot 0{,}0033}{2000} = 1{,}0033^n - 1$	Divide with interest and capital repayment [y]	Conditional equation
$1{,}0033^n = \ \frac{100000 \cdot 0{,}0033}{2000} + 1$	Add 1	Conditional equation
$1{,}0033^n = 1{,}165$	Compute the right-hand side	Operational meaning
$\ln{(1{,}0033^n)} = \ \ \ln(1{,}165)$	Use the logarithm	Formal equivalence
$\mathrm{n} \cdot \ln(1{,}0033) = \ln(1{,}165)$	Use the computation rule $\ln(a^b) = \mathrm{b} \cdot \ln(a)$	Formal equivalence
$\mathrm{n} = \frac{\ln(1{,}0033)}{\ln(1{,}165)}$	Divide by $ln(1{,}0033)$	Conditional equation
$\mathrm{n} = 46{,}36$	Compute the right-hand side	Operational

Table 5. Meanings of the equal sign in example 14 in (Axelsen & Dalsgaard, 2019, p. 72)

Before summarizing the knowledge to be taught, we show an example of an equation solving, in which we see every meaning of the equal sign appear, except symmetric arithmetic. The example was found in the book of HHX and concerns the topic of annuities. The book itself has made explanations for each step, which we have decoded into the meanings of the equal sign:

We see that the structure resembles that of the screening test exercises analyzed. What concerns the operational meaning, this comes into view when *computing* during the equation, and in the end, when having found the unknown n. Regarding the relational meanings, formal equivalence occurs when *rewriting* using identities of the mathematical object in consideration, which in this case are calculation rules of the logarithm. Moreover, the conditional equation appears when *solving the equation* and thus finding the unknown n. Contextual identity appears when *using a specific formula* in a particular context with assumptions, so to construct the equation with the unknown – in this case, the formula of an annuity. What regards the meaning of symmetric arithmetic identity, this remains hidden. However, it is mentioned vaguely in official programs, so it is not transposed to knowledge to be taught at the same level as the other ones. Lastly, the meaning of specification comes into view when *inserting* the precise values given in the exercise, namely the assertations of y, r andA_n. We finish the analysis of knowledge to be taught by summarizing our findings in the table below.

Meanings of the equal sign	**Ministerial guidelines**	**Screening tests**	**Textbooks**
Operational	Refers to the operation of numbers and computation skills	When calculating the value of an unknown *x* at the end of the equation solving; computing a value of *f(x)* for a specific *x*	Computing function values, annuities, length of vectors, in general, every solution of an equation ends with finding an expression of *x* one is to compute
Symmetric arithmetic	Vaguely described as rewriting and reducing expressions (by paper/pencil)	None	Deducing the formula of annuity concerning historical mathematics
Formal equivalence	Similar to symmetric arithmetic: Concerning rewriting and reducing expressions and formulas (by paper/pencil)	In steps of equation solving	Rewriting polynomials, calculation rules/identities of, e.g., vectors, logarithms, exponents, can occur in steps of equation solving

Conditional equation	Refers to solving equations algebraically and to understand the balance principle when finding the unknown; by making repetitive applications of opposite operations	When solving an equation with an unknown, either in the context of a linear function or when explaining an equation solving in steps	With functions, finding *x* when you have a function *f(x)* and a given y-value or vice versa, or generally solving any equation with an unknown
Contextual identity	Vaguely described as "to handle formulas", calculation of percentage and interest are specified	Concerning linear functions and identities of the slope coefficient *a* and constant value *b*	Formulas; concerning geometry (sum of angles, Pythagoras), formulas of annuities/interest rate, differential calculus; line of a tangent
Specification	Shortly mentioned as to be able to distinguish between a rule (of a function) and an equation	Naming linear functions at the beginning of the task, when finding the rule for a linear function *f*	Naming functions, terms in statistics and probabilities, vectors, geometry, e.g., when asserting length of a square, defining (probability) sets

Table 6. Illustration of the analysis of knowledge to be taught in reference to the meanings of the equal sign in Danish upper secondary school

5. Methodology for RQ2

5.1 Participants

There were four classes altogether of 77 participants in the diagnostic test. 37 of the students attended a business upper secondary school (HHX), and 40 at the general upper secondary school (STX). The schools were located in the capital region of Denmark and were selected based on contacts to teachers we had from before. Although we had a third HHX class who had said yes, the students here never issued their declaration of consent; thus, we could not use the data in the analysis. However, we suggest that two classes of each school provide a representative distribution of the two groups. We wanted this segment of students as the test to say something about the transition from compulsory school to upper secondary and whether misconceptions can continue even after many years of mathematics teaching. When the students were given the test, they had just finished up their introductory period. We argue that testing them in the introductory period is valuable since it creates a greater diversity in the student segment, as they have not chosen their specialization in mathematics yet. We will in the next section introduce more thoroughly the method of diagnostic tests and the definition of a misconception.

5.2 Diagnostic tests

Diagnostic teaching sprung out from seeing learning in the view of *constructivism* which is a way of seeing learning as three steps: acting and obtaining some experience, the measure of reflection, and lastly, the stage of learning (Brekke, 2002). Before diagnosing, we need the notion of concept structure which is relevant in understanding the case of the equal sign. It refers to the fact that no mathematical concept stands alone with one single idea. They often consist of complex networks of ideas, of which the equal sign is indeed an excellent example. In this regard, in the analysis of RQ1, we have presented six different meanings of the equal sign. There is seemingly a strong tradition of giving the students repetitive tasks of facts and skills to make greater sense of a concept in mathematics education. Thus, the claim is that one can first achieve a great and functional concept when different concept structures are all included in the teaching (Brekke, 2002).

It is essential to understand the notion of a *misconception* and partial concepts in developing diagnostic tasks. These are highly important as one sees diagnostic tasks with the development of concepts. What concepts does the student have so far, and how do the possible misconceptions challenge the process of obtaining a solid concept structure? Brekke defines misconceptions as having "incomplete thoughts of a concept" (Brekke, 2002, p. 10). It is crucial to distinguish misconceptions from mistakes, as the latter can be coincidental due to not reading the task correctly. Misconceptions, on the other hand, are not coincidental. When having a misconception, one perceives one specific idea consistently throughout the mathematical work (Brekke, 2002). Brekke gives an example of learning multiplication. He declares that if one only gets one type of task concerning multiplication, one will develop a small thinking model of what the concept of multiplication is. Moreover, misconceptions occur when students cannot distinguish the concept of multiplication and the multiplication algorithm (Brekke, 2002) – in our case, we consider the concept of the equal sign versus the algorithm "operations equal answer". However, one cannot entirely avoid misconceptions as they are part of the children's normal development. From a constructivist point of view, the students make new ideas by interpreting their prior experience (Brekke, 2002).

One can give diagnostic tasks at any point in the classroom as they do not relate to a specific teaching sequence. Therefore, the tasks may contain exercises that the students have not necessarily seen before. One must inform the students that one will use the test differently from other ordinary tests. The primary purpose of the test is to discover what thoughts they have about different concepts, get to know the complexities related to these concepts, and help the teacher plan the instruction. The test is not supposed to evaluate the students from high to low. Instead, one of its primary functions is to identify and point out misconceptions even if there has not been any explicit teaching on the concept (Brekke, 2002). They should be presented both orally and written. One should question the students how they solved the tasks in some of the problems. The more written material one can get, the more valuable information one can receive about the students' strategies and ideas of the concepts and misconceptions (Brekke, 2002).
Moreover, one should avoid asking questions where the students can obtain a correct answer even if they have wrong ideas to the concept. Diagnostic tests should be regarded as a tool that helps give knowledge about the students' way of thinking and how many students share the different kinds of ideas in the classroom. Finally, one should highlight that each task in a diagnostic test is associated with a specific problem area within the concept investigated (Brekke, 2002).

5.3 Constructing the diagnostic test

Considering the misconceptions from former studies and the meanings of the equal sign introduced in RQ1, we constructed the diagnostic test. In addition, the work of Darr (2003) inspired us with suggestions in what alternative ways one can show the equal sign. The article suggests, among other things, to include true and false number sentences, varying our representations of equations, and to create new names for numbers (Darr, 2003). Moreover, we also seized inspiration from the detection test presented in Jankvist & Niss (2018). As the main topic in the introductory period is linear functions, some examples from this were implemented, along with basic arithmetic, single variable, linear equations, and formulas that the students are supposed to know from before, e.g., the binomial formula and the area of a rectangle. We constructed the test in four parts; A, B, C, and D. Part A was primarily based on the former studies in RQ1 of operational and relational. It consisted of fill-in tasks where the students alongside were to describe their strategies in brief.

We suggest that one first needs to make sure they can solve some exercises procedurally before asking for justification and an explanation. Furthermore, we regard symmetric arithmetic as the one most related to arithmetic. Hence, Part A serves as the least theoretical part in which we have included the most symmetric arithmetic. Accordingly, part B was made such that the students were to determine whether a statement was true or not and justify their given answer. Thus, there were more tasks here, and most of them related to the other meanings of the equal sign.
The B-tasks were thus both based on the former studies, alongside the findings in the textbook and screening tests. Furthermore, part C concerned pure problem solving where the student was to give a calculation followed by a conclusion, which thus was a mix of both procedure and reasoning. Finally, Part D was purely reflection tasks that urged the students to reflect and describe their ways of thinking the equal sign. Since part C and D are considered more theoretical and open tasks, there are fewer test items than in part A and B. Open text boxes was included in each problem to understand the students' ways of thinking. Below we have summarized the purpose behind each part of the test.

Parts of the test	**Purpose**
Part A (A1, A2, A3, A4, A5, A6, A7, A8, A9, A10, A11)	Fill-in tasks: Examine if the students can insert numbers in equations such that the equal sign maintains equilibrium
Part B (B1, B2, B3, B4, B5, B6, B7, B8, B9, B10, B11, B12, B13, B14, B15, B16, 17)	True/false sentences: Examine if the students can validate an equation and give an argument for it
Part C (C1, C2, C3, C4, C5)	Problem-solving: Examine how the students use the equal sign in solving an exercise
Part D (D1, D2, D3)	Reflection: Examine the students' understanding of the equal sign and equality

Table 7. Overview of the purpose behind the four parts of the test

5.4 Pilot test

We did a pilot test on ten individuals who had passed upper secondary school and were at the university level but whose backgrounds were non-mathematical. The overall results from the pilot test showed that there were many exercises of part A and Part B concerning symmetric arithmetic that all had high success rates. Initially, all these tasks had a focus on operations regarding addition. Thus, it was revised to change some of the tasks with subtraction instead. Nonetheless, we decided not to include multiplication and division as this could create too many different actions in the task. Hence, if a student was not to solve some parts of the test, it would be difficult to determine whether it was due to a weaker ability to multiply instead of adding and vice versa. In other words, it could create too many factors in the test we would make it more perplexing to conclude the one thing or the other - was the students' answers wrong because of a common misconception of the equal sign, or merely because she was not strong in computing in general? In addition to this, we rephrased some of the exercises as they were not clear enough for several of the participants. For instance, the first edition of exercise B13 was the following

(Pilot test) B13:

$3x + 2 = 13 + 5x$
and
$-2x - 6 = 5$
Is the same equation

And was rephrased into

B13:

The equation
$3x + 2 = 13 + 5x$
And the equation
$-2x - 6 = 5$
is the same equation

The rephrasing was made to make it more straightforward for the student that each expression is an equation to entirely understand the statement given in the task. Some of the exercises in the pilot test were quite similar and only differed by minor details. Similar strategies in two different exercises suggested that the exercises were identical. Thus, it was concluded to remove some of the symmetric arithmetic tasks in part A and replace them with a task of the conditional equation, formal equivalence, contextual identity, and specification. Thus, in the revision of the pilot test, we also ensured that each meaning of the equal sign was included in the parts of A, B, and C. Regarding part D, this was seeing the equal sign from the binary approach of relational vs. operational. Hence, there was a coherence in that each meaning of the equal sign was represented in each part to see whether the students just used the meanings of them without being able to justify (as in Part B and C) and for them to explain by own words in Part D what they think of the equal sign.

All four parts are constructed so that they all serve their purpose. Hence, the student must give answers in all four to make a most valid sample. If the students were not to make it through the whole test, we would not get an answer from part D, where they will explain their thoughts about the equal sign. Thus, if we were only receiving responses from the first parts of the test, it would be challenging to conclude whether they overall have misconceptions and what categories of the equal sign they do or do not identify.

One of the tasks, which was more challenging to construct than others, was the one concerning string operations in B4. The pilot test showed that many participants declared this as false, which is a satisfying result. However, one could argue that this task is too "arranged"; thus, an additional spring operation exercise, B, was made with three adjustments.

B16:

A student has solved the following task: A function is given by $f(x) = 4x - 5$. Find $f(2)$. Determine whether the solution is true or false. The student's solution: "I find $f(2)$ by inserting 2 in the equation: $f(x) = 4 \cdot 2 = 8 - 5 = 3$."

Firstly, a context in terms of a task formulation was constructed. Secondly, a calculation in several steps was given. As there are several steps in this new calculation, the student is forced to think it through slightly more than the original version. Thirdly, there are two mistakes given in the task and not merely one. The first is regarding the mix of notation between $f(x)$ and $f(2)$, which we identify as a false meaning of specification. The second concerns the original topic of string operations and thus incorrect symmetric arithmetic.

B15:

A student solved the following equation.
$8x + 2 = 18$
$8x + 2 - 2 = 18$
$8x = 18 - 2$
$\frac{8x}{8} = 16$
$x = \frac{16}{8} = 2$
Determine whether the calculation is correct or not

This task was initially part of part C. However, we saw from the pilot test that the majority correctly explained where it goes wrong in the equation. Thus, to make the task a slightly bit more challenging, we moved it to part B to force the student to justify herself whether it was true or false in the first place. We will now describe more in detail how the different meanings of the equal sign appear in the final test.

5.5 Final test

The misconceptions concerning operational meaning have been incorporated in the test as the meaning of symmetric arithmetic since we argue that understanding symmetric arithmetic, one will need to use the relational approach instead of the operational. Thus, if one answers the exercise correctly, one has shown symmetric arithmetic, hence a relational approach. Moreover, what concerns string operations, are also regarded as false symmetric arithmetic. Accordingly, we suggest that searching for symmetric arithmetic comes into view when using the scholarly knowledge of the relational vs. operational approach. We did not find a substantial amount of symmetric arithmetic in knowledge to be taught which makes it interesting to see whether the students can solve the exercises. What concerns the questionable use of units, this is categorized as false formal equivalence. Below, we have summarized how the meanings of the equal sign appear in the test.

Meanings of the equal sign	**Test items**
Symmetric arithmetic	A1, A2, A3, A4, A5, A6, A7, B1, B2, B3, B4, B8, B16, C1, C5
Formal equivalence	A9, A10, B5, B7, B8, B11, B13, C2, C3, C4, C5
Conditional equation	A8, B7, B13, B14, B15, C5

Contextual identity	B9, B10, B17, C3, C4
Specification	A11, B6, B12, B16, C4

Table 8. Summary of the meanings of the equal sign as they appear in the diagnostic test.

Notice that there are more test items with symmetric arithmetic and formal equivalence. This is due to the misconceptions in string operations and misuse of units that apply to the two, respectively. Thus, these meanings are both represented in the misconception form and its true meanings. We also note that there are fewer meanings of specification and contextual identity. We reason that our findings of the two in RQ1 portray knowledge to be taught, which has not been introduced when the students are given the test. Thus, we had fewer examples of exercises that would be meaningful to offer at this time. What regards the conditional equation, we also did not want to make too many tasks with this as it is strongly linked to school algebra.

Even though diagnostic tasks are to investigate one concept within specific problem areas, this turns out to be challenging what regards the equal sign as the different meanings of the equal sign can all appear within the same problem area, as we have shown in RQ1. Moreover, the more open the exercises become in part B, and C, the more meanings of the equal sign can appear in the same exercise depending on what strategy the student uses. We argue that it is challenging to avoid these overlaps in some of the tasks. However, we claim that even though they overlap, we can still get an overview of what meanings of the equal sign are more challenging than others by looking at the whole test gathered.

5.5.1 Symmetric arithmetic

We will now describe the link between each meaning of the equal sign to each part of the test. What regards symmetric arithmetic, we ask in Part A to fill in the number(s) that are missing, e.g.,

A6:

8 – ___ = 6 – ___

We will see whether the student will take the operational approach, which presupposes that the student will just add the numbers in the equation and set the missing numbers equal to that. We then test in Part B whether the student can validate the equality of symmetric arithmetic to be true. One of the symmetric arithmetic tasks in part B looks like this:

B2:

$$3 + 2 = 5 = 8 - 4 + 1$$

We need the symmetric arithmetic identity to justify that since 3+2 is equal to 5 and 5 is equal to 8-4+1, the statement is true - assessing the transitivity of the equal sign. There is also a problem in part B which can represent both symmetric arithmetic and formal equivalence,

B8:

$$\sqrt{4} = \frac{4}{\sqrt{4}}$$

The symmetric arithmetic appears in the rewriting of the number $\sqrt{4} = 2$ and then deducing that $\frac{4}{\sqrt{4}} = 2 = \sqrt{4}$, hence the statement is true. The formal equivalence occurs if one rewrites the expression by using the identity of the square given as $\sqrt{n} \cdot \sqrt{n} = n$ for all real numbers n. Thus,

$$\sqrt{4} = \sqrt{4} \iff \sqrt{4} \cdot \sqrt{4} = \sqrt{4} \cdot \sqrt{4} \iff \sqrt{4} = \frac{4}{\sqrt{4}}$$

Continuing in Part C, we ask the student the following:

C1:

Add 1200 to 30, subtract 10, and multiply that by 2. What do you get? Show your calculation.

The problem-solving of Part C is to see whether the student will satisfy the notion of symmetric arithmetic, or the opposite, use string operations in the calculation. The latter will thus show a misconception and that the student has not adopted to symmetric arithmetic identity.

5.5.2 Conditional equation

When testing conditional equation for unknowns in A, we give the following equation

A8:

$$2x + 4 = 2$$
$$x = \_\_$$

Being able to fill in the blank with 2 shows that one has understood the meaning of the equal sign in a conditional equation for an unknown. Continuing in Part B, one of the conditional equation-items looks like this:

B15:

A student solved the following equation.

$$8x + 2 = 18$$
$$8x + 2 - 2 = 18$$
$$8x = 18 - 2$$
$$\frac{8x}{8} = 16$$
$$x = \frac{16}{8} = 2$$

Determine whether the calculation is correct or not.

Thus, in this task of the conditional equation, the student is given someone's solution to finding an unknown. Hence, instead of asking the student to solve it, she is now asked to consider a solution's validity and argues why. In Part C, we then give a task where the student is given an equation with an unknown she is to find:

C2:

For what values of x is $7x = 8x - x$?

This exercise both uses formal equivalence and conditional equation of an unknown. The former is since the expression is a rewriting, whereas the latter determines for what values of *x* the expression is true. One will answer partially correct if one gives a finite number of values of *x*. We emphasize that we were not interested in making too many tasks related to the pure conditional equation, as it is strongly linked to school algebra. Thus, we acknowledge that it is challenging to seek this meaning of the equal sign, as failing in these exercises might show difficulties regarding school algebra than that of the equal sign.

5.5.3 Contextual identity

What regards contextual identity, we had not found sufficient material knowledge to be taught, and we thus failed in constructing a meaningful task in part A. In part B, we give

B9:

The area of a rectangle = $l \cdot b$
Where *l* denotes the length and *b* the width of the rectangle

We made B9 to see if the students accept a valid contextual identity with the necessary assumptions of *l* and *b* denoting the length and width of a rectangle, i.e. a right-angled square. We note that this formula is not from the analyzed knowledge to be taught but is expected to be known from compulsory school. Thus, the subsequent exercise is Pythagoras' theorem which we detected in the book, although now without giving the necessary assumptions of the theorem:

B10:

$c^2 = a^2 + b^2$
Where a, b and c are natural numbers
(1, 2, 3, 4, …)

The equation of a contextual identity is true and false when we do not give assumptions, in this case not assuming a right-angled triangle. Thus, this exercise can reveal several aspects. If the student states true or false, this will be partially correct as she acknowledges there are solutions. However, this is not the whole story, and it only demonstrates the meaning of symmetric arithmetic in showing this. Moreover, if the students declare this equation to be Pythagoras' theorem, it will show us that they do not strictly consider assumptions of equations. Accordingly, partially correct and incorrect will reveal a challenge in understanding the meaning of contextual identity.

B17:

$5x = y$
x and y are natural numbers $(1, 2, 3, 4)$

B17 is not a known formula from knowledge to be taught like Pythagoras and was thus included as an alternative contextual identity. Will the student accept this just as much as the former two? Regarding part C and the contextual identity, we constructed C3 and C5.

C3:

If we for instance have the equation $$\frac{(x-2)\cdot(x+2)}{x-2} = x+2$$ Is this always true?

C3 includes several meanings interdependently. First, it represents formal equivalence because we can see that we can rewrite the same product of $(x+2)(x-2)$ on both sides of the sign. Second, it represents the meaning of the conditional equation of the same arguments as in C2. Third, it characterizes contextual identity, as we need the assumption of $x \neq 2$ to say that it is always true.

5.5.4 Formal equivalence

We give the meaning of formal equivalence in two exercises of part A,

A9:

$9x + 6 =$ ____ $\cdot (3x+2)$

This represents a rewriting of an expression, which uses the law of distributivity of multiplication.

A10:

10cm + 2cm + 3kr = ___

This task item represents a false meaning of formal equivalence in which we want the student to add three terms together and see if they will give the correct units in the answer. In part B, as one of our findings in RQ1, we have constructed a task of calculating percentage to see if the student will accept the textbook as such.

B5:

$0{,}20 \cdot 100 = 20\%$

This is like A10, a false formal equivalence in which we have non-equivalent terms of units that is, a number a is set equal to the number a in %. Moreover, we have included one of the identities of the binomial formulas,

B10:

$$2ab + a^2 + b^2 = (a + b)^2$$

a and *b* are natural numbers (1, 2, 3, 4, ….)

We have rewritten the theorem so that the terms in the equation are reordered a bit to see if this will affect the way the student will interpret the equation. In addition to these exercises of formal equivalence in B, we also made an equation mixing non-mathematical language in the equation and misuse of units, as we found in RQ1:

B7:

$$\begin{gathered} 120\text{kr} + 1 \text{ travel card} \\ = 120\text{kr} + 80 \\ = 200\text{kr} \end{gathered}$$

There are two mistakes included in this exercise. Firstly, we have not defined a *travel card* in the exercise. Thus, we cannot deduce that 1 *travel card* must be equal to 80, as this is not a given definition. Secondly, 120kr+80 makes 120kr+80 and *not* 200kr, as they do not have the same unit; thus, it is considered a false formal equivalence. In part C, formal equivalence appears in all but C1. C2 and C3 involve more school algebra (see the section on symmetric arithmetic and contextual identity), whereas C4 and C5 can be solved arithmetically.

C5:

$\text{2 theater tickets, and 1 popcorn of 30kr, cost 120kr in total. How much does one theater ticket cost?}$

C4 and C5 will be graded partially correct if the student does not apply the correct units in the calculation and conclusion in maintaining equivalent terms and thus the formal equivalence.

5.5.5 Specification

In part A, we gave the meaning of specification the task of A11, which we got inspired from the textbook using double definitions.

A11:

You are informed that
$a = y = 5x - 3$
and that
$y = 22.$
a + 10 = ___

The specification states that *a* is defined as *y,* which is defined as $5x - 3$, which we can also recognize as the transitivity of the equal sign. Thus, the student will need this relation to find *a+10*. Therefore, if the student answers 32, she shows how to use specifications procedurally. Continuing in part B, we assert *x* as 4 cm,

B6:

$$4 \text{ cm} = x$$

We made this to see whether the student will answer differently in this, compared to meanings of specification that we encountered in the textbook, such as defining a linear function

B12:

$$f(x) = 3x + 2$$

If the student answers B6 as false, and B12 as true, we will have found that they have not fully adapted to the specification meaning as they do not recognize that $4 \text{ cm} = x$ is just as valid as the notation $f(x) = 3x + 2$, as they both satisfy the equivalence relation. We follow up on this in Part C,

C4:

$$In\ a\ rectangle\ the\ length\ is\ l = 2cm\ and\ width\ b = 3cm. What\ is\ the\ area\ of\ the\ rectangle?$$

Here we assert two values in a geometrical context to see if the students distinguish between being given a meaningful context compared to a random assertation.

What regards part D, we wanted to check the research from before regarding many younger students having an operational approach. Is this still the case in upper secondary school? And is the student able to both explain by words and give mathematical examples of when to use the equal sign without giving misconceptions related to the operational? In D3, we ask for the students' argument of two graphs being equal. This was inspired by Molina (2006) and was to see if the students define equality from an arithmetical, algebraic, or geometric/graphic point of view.

5.6 Process of analyzing the test results

A mixed-method approach was adopted when analyzing the test results, i.e., we both made a quantitative analysis and qualitative analysis. First, we made a quantitative analysis in the shape of a 0/1 analysis where 1 was defined as correct answer and 0 as the complementary. We suggest that the test items with a success rate greater than 50% were not relevant to dig deeper into, as this indicates that the majority mastered what the task item was looking for. Second, we considered a substantial difference between answering incorrect, partially correct, and leaving the exercise unanswered. Hence, we made a further distinction of the 0's in the test items with a success rate lower than 50% and subsequently made a discourse analysis. This was to recognize the students' way of thinking better. Accordingly, when grading the tests, the students' argumentation for each exercise was noted down and categorized for each new strategy identified to give examples of correct, partially correct, and incorrect arguments. We assembled this in a document and classified the test items as either of the three or correct. Moreover, we have analyzed the results of the HHX-classes and STX-classes separately as we rejected the following zero-hypothesis:

> H_0: there is no difference between the number of correct answers of the students of HHX and STX

Which we rejected with a significance level of 5%.

6. Results of RQ2

6.1 The A-tasks

Success %	**A1**	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A8**	**A9**	**A10**	**A11**
HHX	100	97,3	100	100	100	83,8	91,9	67,6	75,7	81,1	62,2
STX	97,5	100	97,5	90	97,5	87,5	95	75	60	67,5	52,5

Table 9. Results from the quantitative (0/1) analysis of the A-tasks

In all the A-tasks, HHX and STX had an average success rate greater than 50% for each test item. This must imply that generally, they have the procedures to answer exercises concerning symmetric arithmetic (which most of the tasks here were an example of). In addition, the majority master conditional equation (A8), formal equivalence (A9, A10) and specification (A11). However, we can see a decreasing percentage towards the right in the table, which might be due to the last tasks concerning other meanings than the symmetric arithmetic. They concern school algebra as letters and symbols are involved. The test items A1 throughout A7 only concerned numbers – moreover, they were small numbers, indicating that the students might have used a guess and try-strategy when solving the tasks. However, even with guess and try, one still needs to be able to validate whether the answer given is correct. Based on this, we will not look further into these results as they are generally satisfactory.

6.2 The B-tasks

Success	**B1**	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**	**B11**	**B12**	**B13**	**B14**	**B15**	**B16**	**B17**
HHX	100	100	64,9	97,3	18,9	45,9	24,3	56,8	91,9	16,2	29,7	64,9	56,8	27,0	18,9	8,1	18,9
STX	100	95	62,5	82,5	15	37,5	15	65	80	7,5	22,5	72,5	47,5	30	7,5	10	10

Table 10. Results from the quantitative (0/1) analysis of the B-tasks

The table shows that HHX and STX also show a success rate greater than 50% of symmetric arithmetic in the first tasks until B5. As B1-B4 were symmetric arithmetic tasks parallel to some of the symmetric arithmetic tasks in A, this is adequate as the success rate in the A-tasks also measured more than 50%. Additionally, the results count more than 50% in the task of formal equivalence of B8, the contextual identity of B9, and the specification task of B12. We emphasize that the complementary of these success rates constitute partially correct, incorrect, and unanswered. We will now present the discourse analysis of the tasks with less than 50% success to better understand the students' way of thinking.

B5:

$$0{,}20 \cdot 100 = 20\%$$

Rate %	**Partially correct**	**Incorrect**	**Unanswered**
HHX	10,8	70,3	0
STX	17,5	60	7,5

Table 11. Results from the discourse analysis of the 0's of B6

B5 deals with a problem we did not encounter in the A-tasks, namely computing with percentage. We can see that most HHX and STX students have answered the task incorrectly. We will now give a couple of examples of what kind of arguments the students gave when giving an incorrect answer.

Incorrect answers:

> % means 'out of 100' where these two statements are equal to each other when one converts to %
> (Student answer B5)

This statement from a student declares that she partly knows the meaning of the %-sign. However, she implies that one will convert to % even though our task does not state this anywhere. There is an expectation from the student's side that it must be true, which can also be related to the books they read, which also showed examples of questionable percentage calculations. This task can connect to a contextual identity of the equal sign. The student makes a context and assumption for herself that if we calculate with percentages, it must be correct – although we do not state this assumption in the task. However, we will regard it here as missing the formal equivalence.

> Because if one multiplies 100 with $0{,}2$ it gives 20. and therefore it is the same as 20%
> (Student answer B5)

This type of argument was proposed by several. The first deduction is correct; however, the students lead to a wrong conclusion, hence what one can call an inconsistent statement. Thus, although the student knows $0{,}2 \cdot 100 = 20$, they still accept that one has added a %-sign as if it serves as a symbol.

> $0{,}20$ is the same as 20%, as $1{,}00$ is 100%. By multiplying by 100 one rewrites decimal numbers to percent
> (Student answer B5)

This statement is similar to the former argument in which the student states 0,20 is the same as 20%. What's more striking about this given argument is that the student makes the point that to convert to %, one multiplies by 100. Indeed, one might convert to % on the calculator by first simply multiplying by 100, then adding the %-sign afterward. Nonetheless, the student has learned how to transform a number to a percentage, which is to multiply by 100, but not including the %-sign, although she first stated that 1,00 is 100 *percent*. Thus, this is also an inconsistent argument like the former. One could argue that the reasoning behind the statement is missing. The student knows that one is to compute by 100 to obtain percent but has not understood when the %-sign is included or not. Hence, this also shows a lack of the meaning of contextual identity. The student makes extra assumptions in the form of a percentage formula. However, this is similar to the findings in RQ1.

Correct answers:

> We have not been told that we are converting into percent
>
> (Student answer B5)

This argument does not relate to the numbers stated directly but simply says that one has not noted the context of converting into a percent in the task. Thus the statement is false, which shows that the student has learned to pay attention to what assumptions we must put forward before looking at the calculation. Therefore, this student shows a meaning of contextual identity in this reasoning.

$$1{,}00 = 100\%$$
$$0{,}20 = 20\%$$
$$0{,}20 \cdot 100 = 2000\%$$

> (Student answer B5)

Here, we see an arithmetic argument, which shows that the student has learned that 100% is equals 1,00 and knows how to use this in the calculation. She shows formal equivalence as the unit is correctly used. Because of this, one might explain some of the wrong answers as the students not having mastered school algebra since they have not been able to put forward this "simple" calculation argument.

Partially correct answers:

> It is a number 20. It is also 20% of 100
>
> (Student answer B5)

A handful of students also stated the statement as both true and false (B. These students declare both sides of the story, namely that both 20 and 20% are correct answers, which is interesting because they state that a calculation can give two answers, which one could see as a not well-defined function. Consequently, it is reasonable to comprehend this way of thinking as a perception of the %-sign as a meaningless symbol and not a number.

General observations

Many students perceive percentage as a manipulative symbol rather than the numerical value defined as the rational number $x\,\% = \frac{x}{100}$. Therefore, many of the arguments go according to this way of thinking. Various students announce that $0{,}2 = 20\%$, but still, conclude that the statement must be correct. Hence, they use a valid argument but finalize with a conclusion inconsistent with it. Do these results imply a misconception with the equal sign? Not necessarily, as it might involve a deception concerning computation with percentage.
On the other hand, one could argue that since the incorrect argument often is that $0{,}2 = 20\%$, the students have chosen to ignore the factor of 100, and therefore do not operate strictly with the equal sign. Moreover, one could see this as a result-oriented approach, namely that $0{,}20 \cdot 100 = 20$, which could indicate that 20=20% in some given context, and thus be correct. Consequently, the students make additional assumptions than the task gives, which shows the challenge of the contextual identity meaning. Furthermore, they have not fully adopted the meaning of formal equivalence. However, this goes hand in hand with what we identified in one of the textbooks analyzed in RQ1. Thus, the students might have learned what the textbook told them.

B6:

$$4\,cm = x$$

Rate %	**Partially correct**	**Incorrect**	**Unanswered**
HHX	21,6	29,7	2,7
STX	20	30	12,5

Table 12. Results from the discourse analysis of the 0's of B6

We observe that approximately 1 out of 5 declared it both true and false (B), which we categorized as partially correct. The percentage rate of those who answered incorrectly is almost 10% higher.

Thus, we will now consider some of the most common incorrect and partially correct answers to better understand these students' thinking.

Incorrect answers

Missing a context

> x can vary after situation. In one situation it may well be 4cm, whereas in another [it] is a different number
> (Student answer B6)

The student demands a context to assert that x can be equal to 4 cm. Hence, it seems like the meaning of the contextual identity exists. Still, the student does not acknowledge the meaning of specification, i.e., that one uses the equal sign to name mathematical objects. Thus, she concedes that x can be a variable, but not that x can be assigned to one specific value like 4 cm.

x is not a variable

> Not true because we don't have a unit behind x
> (Student answer B6)

This statement displays a misconception related more to school algebra than the equal sign. x is just a placeholder for any quantity and its corresponding unit. Without a unit, there is no context or relation that we can say something about. Faulkner et al. highlight the importance of unit as an *attribute*:" In one situation we might want to know if things are equal in weight, but in another situation, if they are equal in length. Two objects, then, can be equal in one numerical attribute but not in another" (Faulkner, Walkowiak, Cain, & Lee, 2016, p. 12). In one way, the student has understood the consistency of units but not understood that equations will always need an attribute to which we can relate the numerical value, in this case, the shape of length. Based on this, one could argue that the student understands the meaning of a conditional equation, but not the meaning of specification.

Correct answers

x can vary

x is a variable it can be just anything it wants

(Student answer B6)

This argument is parallel to the incorrect 1). Thus, where some students argue that *because* x is a variable, it cannot be one precise value, other students use the same idea to say that *hence*, the sentence is true. This argument shows a specification meaning as the student declares that one can define x as anything one wants.

No more information

Since there is no more information it is correct [it is] an ascertainment

(Student answer B6)

This idea is a clear indicator that the student acknowledges the meaning of specification. She declares it valid as nothing else is stated – it is just a proclamation, hence true.

Arithmetic

x is unknown, so hence it actually says 4cm = 4cm
(Student answer B6)

This argument uses symmetric arithmetic in arguing, as the student uses the information of $x =$ 4cm to deduce that 4cm = 4cm, thus true. In that way, the student does not directly acknowledge the meaning of specification but indirectly obtains this realization by using other meanings of the equal sign.

Partially correct answers

It [is] just an ascertainment

(Student answer B6)

This student proclaims that because the expression is just an ascertainment, it cannot be true solely, thus partly true. Hence, the meaning of specification is not completely acknowledged.

It depends on what x is

(Student answer B6)

This argument can be seen to be inconsistent as the student says it depends on what x is (which is $4cm$, after all). This may indicate that she needs a meaningful context that is not attained algebraically but possibly with a different representation such as an illustration or a written text.

General observations

What's striking about this task is that in specification task A11, the success rate for HHX and STX are respectively 62% and 52%. Why do so many define it as incorrect or partially incorrect in B6? The argument is often about missing a context; hence they see the equal sign as a meaning of contextual identity instead of identifying it as a meaning of specification. Regarding the specification task of B12, the success rate here is 64,9% (HHX) and 72,5% (STX). Here the students do not question the context as much because they most likely have met the object of $f(x)$ several times during the introductory period. Thus, many stated that the expression was true because they saw it as a function, although it is not specified explicitly with words. It is simply a naming of an object and thus a specification.

B7:

$$\begin{aligned} 120kr + 1\ travel\ card \\ = 120kr + 80 \\ = 200kr \end{aligned}$$

B7	**Partially correct**	**Incorrect**	**Unanswered**
HHX	18,9	56,8	0
STX	12,5	60	12,5

Table 13. Results from the discourse analysis of the 0's of B7

The partially correct answers are pretty similar to those incorrect, so we will only consider the incorrect ones here.

Incorrect answers

Adding assumptions

If travel card costs 80 kr it is true. Then I add.
(Student answer B7)

The student makes assumptions and creates a context on her own. Thus, one could say that the meaning of contextual identity is not fully adapted since one needs to follow the assumptions given to deduce the given result. This assumption is interesting in two ways; firstly, it shows a result-oriented approach from the students' side as the argument is "since the rest fits, a travel card must be 80 kr" and secondly, ignoring the misuse of units as the second equation says $+80$ and not $+80\text{kr}$, thus not using the meaning of formal equivalence. However, we also detected this type of misuse of *kr* in the external didactic transposition.

Result-oriented argument

$$120 + 80 = 200$$

(Student answer B7)

This way of answering the task also shows a desire to obtain the correct calculation of the numbers given, not relating to which context the numbers appear and where they come from, like the attribute mentioned above proposed by Faulkner et al. (2016). Since this task has two equal signs given, it could be possible to believe that the students who answered arithmetically simply focused on the second equation, thus not on the assumptions of the task.

Correct answers:

Unknown price

> I use only my bike (with basket) so [I] do not really know what a travel card costs, therefore it is not necessarily correct that a travel card costs 80 kr
>
> (Student answer B7)

The student uses the meaning of specification when stating that we don't know what a travel card costs and, based on this, declares it as false. She consequently denies a result-oriented approach.

Different units

> It does not say anywhere that travel card is 80 it also doesn't say that it's kr. if that is ok then it is correct
>
> (Student answer B7)

This student uses more arguments than the one above, namely that neither is the value of travel card known nor can this supposed amount of 80 be added to 120kr. Thus, this student both shows a meaning of specification but also the meaning of formal equivalence.

General observations

One can point at this task as a type of contextual identity task because we see from the students who answered incorrect that they accept the not-given context entirely. They do not consider under what restrictions these equations can hold, i.e., if it was defined from the beginning that 1 travel card = 80 (the meaning of specification). Secondly, the meaning of formal equivalence in ensuring consistency in the use of units is also not present in the incorrect answers. The latter is quite interesting; overall, there was an excellent success rate in operating with units correctly in A10 (81,1% among HHX and 67,5 in STX), where there was a unit behind each term, whereas in this one it is the term 120kr vs. the term 80. Thus, it can also be the case that the student has assumed that *kr* was just missing behind 80 and that they, therefore, added it themselves.

B10:

$$c^2 = a^2 + b^2$$

Where *a, b* and *c* are natural numbers (1, 2, 3, 4, …)

Rate %	Partially correct	Incorrect	Unanswered
HHX	62,2	5,4	16,2
STX	65	2,5	25

Table 14. Results from the discourse analysis of the 0's of B10

We observe that the percentage of incorrect is strictly less than 10% for both HHX and STX and will therefore focus on the rate of partially correct, which measures 62,2% for HHX and 65% for STX. As B10 is both true and false (B), we have graded the student as partially correct when stating 1) True or 2) False.

Partially correct answers

True

> It is the Pythagoras theorem. c^2 has the same value as a^2 and b^2 when one computes with right-angled triangles

(Student answer B7)

This argument shows that the student makes some extra assumptions that are not given in the task, namely that the equation goes for right-angled triangles. Thus, the meaning of contextual identity is on trial as the student is to consider that the equality only holds in some particular cases. However, the student declares the equation as true because she made the extra assumptions to make it work. It is likely, the textbook has presented this equality as Pythagoras theorem, which is the only context the student has seen. Thus, the student makes herself believe that the task is similar to the former experience and therefore that it holds the same assumptions.

If c^2 is 4 and $a^2 + b^2$ is 2 then it will be the same
(Student answer B7)

This argument implies that the student finds proof in a statement if one can find an example in which it fits. It is unclear whether the student knows of this right-angled triangle in advance or if she came up with the example herself. Either way, this way of arguing shows a lack of meaning of contextual identity. Although something is true for some numbers, this is not a general formal equivalence which one can apply to any number. It fits in a specific context, and hence under some given assumptions, which we have not provided in the task. Thus, it demands the meaning of contextual identity, which is here not adapted by the student.

False

The equation only works on specific triangles
(Student answer B7)

The argument proposed here is indeed true, and here it seems like the student has not been sure of what it takes for something to be true or false. Although something is not always true, it can be true in some cases. Thus, the student shows a meaning of symmetric arithmetic (pointing at specific cases, i.e. numbers) but does not entirely master that of contextual identity. Hence, like the students who answered true in this task, it goes for both parties that they have not adapted the meaning of contextual identity. Understanding the meaning of contextual identity implies that one acknowledges that equalities can be true for some numbers and other numbers not.

Not all numbers can give a proper answer to c^2.

$$a^2 = 2 \qquad b^2 = 3$$

$$a^2 = 2 \cdot 2 = 4 \qquad b^2 = 3 \cdot 3 = 9$$

There is not a whole number that is multiplied by itself that becomes $4 + 9 = 13$

(Student answer B7)

This example is a parallel argument to those who answered true, using an example of where it was valid. Therefore, they function as the same argument for both cases as the student uses an example to declare it true or false, depending on what the example showed. This implies again, that the student does not master the meaning of contextual identity.

Correct answers

It will be true for specific numbers, but not other

(Student answer B7)

This argument shows a meaning of contextual identity as the student states that for specific numbers - what we can categorize as something within a given context - the equality is true. Similarly, the equation is not true for other particular numbers, i.e., other contexts.

However, some students answered correctly in this task that gave an argument that was not as solid:

They are not always natural numbers, it can also be decimal numbers

(Student answer B7)

Here the student is correct that it is not always natural numbers – it is not even all natural numbers. Thus, the argumentation here is a little bit looser. Nonetheless, the student still seems to have adopted a partly meaning of contextual identity by considering other cases.

General observations

The true arguments are either 1) referring to Pythagoras theorem 2) showing the existence of the equality, whereas false arguments are 1) showing non-existence with the equality 2) saying that it only works on specific triangles. Hence, one can argue that for both sides, true and false, some of the students might have misunderstood the fact when something is true or false. Their way of proving is in the form of examples and cannot be generalized. Therefore, the presented results imply that the students do not entirely master the meaning of contextual identity. Although they have strategies in the A-tasks, using a contextual identity of a function in A11, they do not consider when an equation is true or not. This is similar to B5, which also had a success rate of less than 50% (with a low rate of unanswered). Many students did not consider the context in which the task was given.

B11:

$2ab + a^2 + b^2 = (a + b)^2$ a and b are natural numbers (1, 2, 3, 4, ….)

Rate %	Partially correct	Incorrect	Unanswered
HHX	2,7	43,2	24,3
STX	2,5	47,5	27,5

Table 15. Results from the discourse analysis of the 0's of B11

We defined the partially correct as when the student declared true and false (B). It is strictly less than 10% for both STX and HHX; hence we will not look further into it. The unanswered rate is approximately ¼ among HHX and STX and can indicate that the students have misunderstood the exercise. However, it is remarkable that 43,2% of HHX and 47,5% of STX had answered it incorrectly, which is when they defined the statement as false.

Incorrect answers: 2ab should not be there:

This type of argument constituted the great majority of the incorrect answers; thus, this is the one that we will comment on.

> Not the same. [I] would never be able to create 2ab on the right-hand side hence it would not be the same value. It says after all $a^2 + b^2$ on both sides only expressed differently
>
> (Student answer B11)

> $(a + b)^2$ gives $a^2 + b^2$ which is not $2ab + a^2 + b^2$
>
> (Student answer B11)

There is not equilibrium, $2ab$ is missing on the right side and $\mathrm{a}^2 + \mathrm{b}^2$ is not the same as $(\mathrm{a} + \mathrm{b})^2$
(Student answer B11)

What is interesting about this argumentation is that it expresses a relational meaning. There is a misunderstanding of the notation of $a^2 + b^2$ vs. $(a + b)^2$ as the students claim they are the same. Thus, they argue that $2ab$ should not be on the left-hand side as it is missing on the right-hand side, hence not maintaining equilibrium. This argument was not foreseen in the apriori analysis and is therefore surprising to find. The intention of the task, which was to detect whether the students understand the meaning of formal equivalence, turned out to catch whether they have a relational approach at all – which these answers indicate that these students have. Hence, we also have detected challenges related to their competencies in school algebra.

Correct answers

Algebraic

It is true as we multiply the parenthesis

$$(a + b)^2 \;= (a + b) \cdot (a + b) = a^2 + ab + b^2 + ab = 2ab + a^2 + b^2$$

(Student answer B11)

This way of proving the statement true shows the student's understanding of formal equivalence. Moreover, she also masters school algebra very well.

Existence

$$\text{If } \mathrm{a} = 1 \text{ and } \mathrm{b} \;= 2$$

$$2 \cdot (1 \cdot 2) + 1 + 4 = 9$$

$$(1 + 2)^2 = 9$$

(Student answer B11)

This argument is less abstract and is no actual proof of why the sentence is generally correct. It is the same way of confirming and declining the expression in B10. Thus, there is an absence of meaning of formal equivalence, which was what we wanted to detect with this test item.

General observations

Firstly, it is remarkable that few students have tried to show existence by inserting some arbitrary numbers. Perhaps this way of solving a task is not familiar to them. Secondly, many of the arguments that have made the students answer incorrectly indicate that the student did not understand the notation of $(a + b)^2$ in parenthesis vs. $a^2 + b^2$ without parenthesis. Many argue that the equation thus is not balanced – a relational argument that unfortunately leads to the wrong conclusion. Therefore, we cannot determine that this high rate of incorrect answers is due to a misconception of the equal sign. Instead, it can be just as much about not having understood the school algebra, which unintentionally is luring in this task.

B13:

> The equation
> $2x + 3 = x + 7$
> and the equation
> $4x + 6 = 2x + 14$
> is the same equation

Rate %	Partially correct	Incorrect	Unanswered
HHX	5,4	24,3	13,5
STX	7,5	20	25

Table 16. Results from the discourse analysis of the 0's of B13

B13 is the only test item in which STX and HHX differ in having a > 50% success rate or not. HHX scored just above 50% by 56,8% and STX just below 47,5%. Thus, we will also look deeper into this test item. Since the rate of partially correct is shallow for both schools and the arguments are similar to those that answered correct and incorrect, we refer the interested reader to look in the appendix for these answers.

Incorrect answers: Misunderstanding the question

> The equation $4x + 6 = 2x + 14$ is twice as great as the equation $2x + 3 = x + 7$ (Student answer B13)

This answer clarifies a misunderstanding of what it means for two equations to be the same. The student has detected that the two equations differ by multiplication of 2. From a relational meaning of the equal sign - if all the terms of the first equation have been multiplied by 2 - this indicates that the equation is balanced, and thus no change has been made regarding the solution of the equation. Therefore, this way of answering this task does, to some extent, imply that the student has not fully adopted a relational meaning of the sign, which can be related to having challenges with school algebra.

> The equation has the same construction but does not give the same result as all numbers here have the doubled value (however one could also argue that it is the same equation but different places in the calculation)
> (Student answer B13)

This argument also acknowledges that the difference between the two equations is the multiplication of two as the student says, "the equation has the same construction" and that all the numbers have the doubled value. However, the conclusion from the student is that it implies that the "result" – what we can translate as the solution of the equation – is not the same. Moreover, she acknowledges (in parenthesis), that this could be the same equation but in two different stages of the solving. Hence, this answer also shows that the student might have misunderstood the question's phrasing or do not master the school algebra part of the task.

$$x = 4$$
$$2x = 8$$

> It is not the same. I calculated both equations.
> (Student answer B13)

This third argument also relates somewhat to the other two presented. The student has gathered terms for both equations and deduced that $x = 4$ and $2x = 8$, which is correct but unfortunately leads to a wrong conclusion. Thus, the student seems to know how to use the meaning of conditional equation in finding these two solutions but failing to see that $2\text{x} = 8$ also must imply that $x = 4$ and vice versa. Hence, it can be the case that she oversaw this last step or misunderstood the question.

Correct answers

Multiplication of 2/formal equivalence:

> It gives the same because the change is on the whole of it
> (Student answer B13)

This argument is relational, which relates to the two equations being equal as they only differ by a factor, or what the student here refers to as "change". The change of factor 2 is on the whole of it, and therefore it gives the same. Hence, one can see this argument as a formal equivalence; the expression has been rewritten and is therefore identical to the original one.

Same solution/conditional equation:

> The equation itself is after all not the same, but the result is the same
> (Student answer B13)

This correct argument represents a different strategy in which the student says the solutions of the equations are the same and hence must be accurate. Consequently, she shows a meaning of conditional equation to deduce this, which indicates that there are different meanings of the equal sign that one can use to argue correctly in B13.

General observations

The incorrect answers show us that many students have noticed the correct observation, either acknowledging that; 1) the solutions are the same – conditional equation or 2) the second equation is twice as big as the first equation – formal equivalence. Thus, many students have somewhat misunderstood the question, as they can see the connection between the two but do not conclude that indicates they are the same equation. Again, this can show a not fully developed concept of equations, which is highly related to school algebra. Moreover, it can be just as much about understanding mathematical phrasing, then that of equations and understanding the equal sign.

B14:

The equation $3x + 2 = 13 + 5x$ And the equation $-2x - 6 = 5$ is the same equation

Rate %	Partially correct	Incorrect	Unanswered
HHX	0	51,3	21,6
STX	2,5	25	42,5

Table 17. Results from the discourse analysis of the 0's of B14

B13 and B14 are similar test items, although B14 is slightly more complicated as we have made several changes between going from the first equation to the second (and conversely). Accordingly, we see that the incorrect rate in this one for HHX went from 24,3% to 51,3% - more than doubled – whereas for STX, it went from 20 to 25%. Furthermore, unanswered rates for both schools are higher than B13 (from 13,5% to 21,6% for HHX and from 25% to 42,5% for STX). What is more attention-grabbing is the difference in rates of incorrect and unanswered between HHX and STX; they are almost opposite each other. While 51,3% has answered incorrect and 21,6% unanswered in HHX, the same categories measure 25% to 42,5%, respectively, for STX. Nonetheless, the partially correct rate was meager for both parties in B13, and the same goes for B14. Since the test item has several similarities to the previous one, we will see some of the same arguments, which we thus will not comment on as thoroughly as the previous one.

Incorrect answers

Misunderstanding the question

> Nope. there are 2 *x'es* in the upper [first] equation and much higher values
> (Student answer B14)

This arithmetical argument does not use school algebra as the student argues that there are 2x'es in the upper. Additionally, she declares that there are much higher values in the first and thus concludes that the equations are not the same. We also saw this among the incorrect answers in B13; hence they are of the same character.

Inconsistent argument

> It does not seem right since one cannot do the same things on both sides
> (Student answer B14)

This argument is somewhat more challenging to interpret fully. We cannot be sure of what "things" cannot be done the same on both sides. However, we can partly deduce from this statement that the student does not fully master the justification of why one can go from one equation to another. We find the justification in the relational meaning of the equal sign, namely that one *must* do the same operations to maintain equality. Thus, this answer might imply that the student does not master school algebra.

$$-11 = 2x$$
$$-2x = 11$$

> I calculated it. It did not give the same
>
> (Student answer B14)

This is similar to one of the incorrect answers in B13, where the student also proposed a correct solution with an erroneous conclusion. The student does, to some extent, show the meaning of the conditional equation but concludes that the equations cannot be the same before having isolated *x* in the two.

Correct answers: Same solution

$$x = -2/11 \; and \; x = -2/11$$

> (Student answer B14)
>
> The equation has the same result, as we only are further ahead in the calculation
>
> (Student answer B14)

Both answers represent the same type, so we will comment on them together.
In B13, it was possible to state two correct arguments: either formal equivalence (multiplication by 2) and conditional equation (the solutions are the same). Hence, this test item was made additionally to B13 to extricate the ones that succeeded to see that there was a factor of 2 i.e. the meaning of formal equivalence, vs. finding the solutions by equation solving, i.e.. the meaning of conditional equation. The latter is the only possibility of the two (in addition to guessing and trying) in solving B14. Therefore, these correct answers show that the student must master school algebra to some extent to give the correct answer (T).

General observations

Fewer students assert that the equations have the same solutions in B14 compared to B13 (per success rate in the two). Thus this can explain why there is a higher rate of incorrect answers among STX and HHX. B14 is not as straightforward as B13, in which the difference between the two equations was the multiplication of 2. Like B13, this test item might have caused misunderstandings among the students. Therefore, it is problematic to conclude whether a relational approach exists or not among these students. However, there is evidence that the incorrect arguments the students have proposed show challenges related to school algebra.

B15:

A student solved the following equation.

$8x + 2 = 18$

$8x + 2 - 2 = 18$

$8x = 18 - 2$

$\frac{8x}{8} = 16$

$x = \frac{16}{8} = 2$

Determine whether the calculation is correct or not

Rate %	Partially correct	Incorrect	Unanswered
HHX	2,7	59,4	18,9
STX	2,5	42,5	47,5

Table 18. Results from the discourse analysis of the 0's of B15

We observe that the partially correct rate is meager and will not look deeper into those answers. Regarding STX, there is an equal distribution between the amount incorrect and unanswered, in contrast to HHX, where the incorrect rate measures 59,4% and the unanswered 18,9%.

Incorrect answers

Result-oriented

The calculation is correct

$$8 \cdot 2 = 16$$

(Student answer B14)

This argument appeared among several participants and shows that the student prefers looking for the correct result instead of the mathematical procedure. Hence, one can categorize it as a result-oriented approach, which can imply an operational meaning of the equal sign.

Inconsistent argument

> The calculation does everything on both sides
> (Student answer B14)

This argument is partly inconsistent since the student declares that the calculation does everything on both sides - which is only partially true as it is not done consistently for each equation. Thus, one could argue that the argument tends toward a more result-oriented approach and hence an operational meaning. Nevertheless, it also has a relational meaning when stating "both sides", which might imply that the student sees the equal sign partly from both perspectives.

Suggesting alternative (correct) solution

$$8x + 2 = 18$$
$$8x = 16$$
$$\frac{8x}{8} = \frac{16}{8}$$
$$x = 2$$

> It is correct would have just written it down a little differently
> (Student answer B14)

This is the third type of typical argument proposed in which the student writes down her proper solution, which indeed is true, as it satisfies the relational meaning of the equal sign. However, the student concludes that the statement must be true, implying that she solved the equation to see if one would obtain the same result. Thus, this shows a relational meaning but is also partly operational as the conclusion from the student's side is that the original statement must be true.

Correct answers

> x ends up with the correct value but = is not satisfied in lines 2, 3 and 4

(Student answer B14)

This argument is more a statement from the student, even if it is true. The student does not justify *why* this is not satisfied. Hence, we are not aware of the meaning of the equal sign, but we have reason to believe it is relational since the student rejects the operational meaning.

> Because he inserts -2 for no reason I would say that it is incorrect!
> (Student answer B14)

This argument also affirms that something happened in line 2 "for no reason"; thus, it cannot be correct. Similar to the other correct views presented, the student yields reasoning or justification in solving the equation. Therefore, it displays a more relational approach, in this case as a conditional equation – contrary to an operational one in which the correct result is the only important thing.

General observations

In several cases where people state B15 as true, the students argue that the value of $x = 2$ does apply to the equation, and therefore, the equation is true. Thus, there seems to be a tendency for a result-oriented understanding of the task and hence an operational meaning. A calculation is declared as true if the answer is correct, which is parallel to making string operations; if the computations are not completely strict, we look beyond this and merely consider the result. If the result is correct, the rest will do either way. However, based on the high percentage of unanswered among STX, especially (47,5%), one could also argue that this type of exercise is not typical in the students' books. Thus, they did not know how to relate or understand this test item properly. Therefore, based on this test item, we cannot fully conclude that they only have an operational meaning of the equal sign. Nonetheless, the incorrect answers can be due to a result-oriented approach.

B16:

A student has solved the following task:
A function is given by $f(x) = 4x - 5$.
Find $f(2)$. Determine whether the solution is true or false.
The student's solution:
"I find $f(2)$ by inserting 2 in the equation:
$f(x) = 4 \cdot 2 = 8 - 5 = 3$."

Rate %	Partially correct	Incorrect	Unanswered
HHX	10,8	56,8	24,3
STX	2,5	40	47,5

Table 19. Results from the discourse analysis of the 0's of B16

Before noting the results in the table, we remind the reader that this test item is also similar to B13, B14, and B15. It is false like B15 was – in this case, due to string operations. Concerning partially correct, this is when the student has answered both true and false (B). However, the success rate was highest among STX; 10% succeeded vs. 8% among HHX. However, the rate is shallow, and therefore it is considered to reflect whether the design of the task itself might be the challenge. Indeed, one could argue that the formulation "Determine whether the solution is true or false" is misleading as one can interpret this as only confirming or declining the given answer. The answer "$f(2) = 3$" is correct, and it is equitable to think that the students might have misunderstood the question, hence explaining the high incorrect rate of 56,8% and 40% for HHX and STX, respectively. We consider some examples of the students' answers:

Incorrect answers

Result-oriented

> As $f(2)$ is equal to the result. We hold on to the previous equal sign
> (Student answer B16)

This is parallel to the answer of B15, in which the student shows a result-oriented approach and does not relate to the given procedure in the task. Furthermore, it is challenging to comprehend what the student implies as we cannot determine what the "previous equal sign" and the implication of "hold on to is".

Suggesting alternative (correct) solution

$$f(2) = 4 \cdot 2 - 5$$
$$f(2) = 8 - 5$$
$$f(2) = 3$$

(Student answer B14)

This incorrect answer can also be related to the one in B15. The student shows her proper solution, which is correct but concludes that the statement is true. Hence, she shows both the equal sign's operational and relational meaning.

Correct answers

Direct argument:

$f(2)$ gets the correct value but = is not satisfied

(Student answer B16)

The student declares both the operational and relational meaning but concludes based on the latter that the statement is false.

Referring to string operations:

$f(2)$ is 3[,] in the equation the students missed a -5 after $4 \cdot 2$

(Student answer B16)

This student makes aware of the string operation but does not comment on the mix of notation between $f(x)$ and $f(2)$. Thus, she shows a meaning of symmetric arithmetic, but not ultimately the meaning of specification, and why $f(x) \neq f(2)$.

General observations

In a future test, one should consider rephrasing as one can interpret the statement as only considering whether the answer is correct. However, one could argue that this test item is closely related to B16, B15, and B14. They all show a student's solution and whether the calculation is correct. What concerns the high rates of incorrect answers, this might indicate that the students have a greater focus on whether the result one has inferred is the correct answer rather than the process of arriving at the correct result. In this way, one could argue that it shows a result-oriented approach, consequently producing an operational view of the equal sign. There is another interesting aspect about the way the students answered here. Several students show their solution to it, which is mathematically correct and therefore different from the one proposed in the question. However, they still respond to the test item as true (T), supporting a result-oriented approach. Lastly, what considers the distribution of partially correct, incorrect, and unanswered of B15 and B16 seems to go very much hand in hand, both what regards the distribution of each category and within the two groups (STX and HHX). Hence, this can explain the resemblances of both the correct and incorrect arguments proposed in the two test items.

B17:

$$5x = y$$
$$x \text{ and } y \text{ are natural numbers } (1, 2, 3, 4)$$

Rate %	**Partially correct**	**Incorrect**	**Unanswered**
HHX	43,2	2,7	35,1
STX	42,5	0	47,5

Table 20. Results from the discourse analysis of the 0's of B17

The relatively high unanswered rates among HHX and STX here can be due to the restricted time frame and some students who did not make it through the whole test. The rate of incorrect is superbly low – 43,2% of HHX and 42,5% of STX have answered partially correct; that is, either stated it true (T) or false (F), as the correct answer is both true and false (B).

Partially correct

True

$$5x = y$$
$$x = 5$$
$$5 \cdot 5 = 25$$
$$y = 25$$

(Student answer B17)

This existence argument made the student declare the statement as true, which is a parallel argument to what one saw in B10. Based on this "proof by example", the student asserts that the statement is true in general, thus showing a meaning of symmetric arithmetic but not of contextual identity.

> *y* is going to be 5 times as big as *x*
>
> $$5 \cdot 5 = 25$$
>
> (Student answer B17)

The student announces that *y* has an accurate restriction but still concludes the expression to be true. Thus, this can imply a misunderstanding of the question or a misconception of when something is correct in general or only sometimes correct – the same challenge for the previous example, i.e., understanding contextual identity.

> *y* and *x* are unknown, so it [the expression] is correct
>
> (Student answer B17)

This student uses the meaning of a conditional equation to state that it is true so that the argument goes, "assume *x* and *y* are unknown and that $5x = y$, then the expression is solvable", which is true. Nonetheless, the student does not master the meaning of contextual identity as she does not consider the restrictions of what domain of numbers in which the equation is true.

False

> *y* depends on what *x* is
>
> (Student answer B17)

This student has understood the expression partially correct from the opposite point of view. She has started the assumption the other way around by saying that we can find *y* that satisfies $5x$, but this will depend on what *y* we choose. Therefore, there are cases where it is not valid (take, for example, $y = 13$). Thus, this student declines the statement by considering a counterexample, although not explicitly expressing what this could be. However, we can tell from this argument that the student has adapted to the meaning of contextual identity, although not having conceded that it implies the expression is both true and false.

> 5 times a number does not give the number one multiplies by
> (Student answer B17)

This argument is puzzling as the student has not distinguished between x and y. Certainly, the argument is valid, implying in a mathematical language that $\nexists x\colon 5 \cdot x = x$ [unless $x = 0$]. In this way, one could interpret the statement as the student stressing the property of reflexivity. Nonetheless, we cannot be definite in that the student has read x and y as the same symbol. Alternatively, she has merely misjudged the question.

Correct answers

5-table

> It can partly add up but only when y adds up in the 5-table
> (Student answer B17)

This argument demonstrates that the student has adapted to a meaning of contextual identity when asserting that the expression is sometimes true – that is, for multiples of 5. Thus, one could argue that it is difficult for a student to state that it is true without the meaning of contextual identity. This is also parallel to the correct arguments in B10.

Example

> It depends on what one inserts. But if one inserts 2 and 10 it fits
> (Student answer B17)

This argument is less general, but the student still expresses the meaning of contextual identity by affirming that it is valid for 2, but that it depends on what numbers, i.e., context one considers.

General observations

B17 is parallel to B10 as it promotes the meaning of contextual identity. Thus, we recognize some of the similar correct arguments given in B10. However, the rate of partially correct measures above 60% for B10 is a bit higher, most likely because many could recognize it as Pythagoras theorem. On the other hand, this expression is not a general theorem from the textbooks as such - and can explain the relatively lower success rate in addition to the factor of restricted time frame.

6.3 The C-tasks

Success %	C1	C2	C3	C4	C5
HHX	40,6	29,8	8,1	10,8	29,8
STX	50	17,5	0	7,5	17,5

Table 21. Results from the quantitative 0/1 analysis of the C-tasks

Overall, the success rate in all the C-tasks rates 50% or lower for each problem for both HHX and STX. Time frame is undoubtedly one of the main factors causing that. Secondly, these test items are more open. They request a calculation and answer from the student, contrary to parts A and B, where either the students were to fill out the missing number(s) or validate whether a given statement was true. The C-tasks are text tasks and therefore demand a more significant part of reasoning and explanation while at the same time coming up with an answer.

C1

Add 1200 to 30, subtract 10, and multiply that by 2.
What do you get? Show your calculation.

Rate %	Partially correct	Incorrect	Unanswered
HHX	35,1	8,1	16,2
STX	15	12,5	22,5

Table 22. Results from the discourse analysis of the 0's of C1

We defined partially correct as when the answer was correct, but the calculation had one or several string operations. Thus, we made this test item to see if string operations exist among the students. We observe a higher rate of string operations among HHX vs. STX, specifically 35,1% vs. 15% among the STX students. However, among the two groups, roughly 1 out of 4 students did not respond.

Incorrect: string operations

$$1200 + 30 = 1230 - 10 = 1220 \cdot 2 = 2440$$

(Student answer C1)

We note that the answer achieved by the student is correct. However, this answer was categorized as representing the misconception of a string operation. The student computes step by step, the different calculations and sets a "="-sign for each step. Thus, symmetric arithmetic is not satisfied. Hence, it seems like the student uses the equal sign to divide the task into several parts and connect each step by using the sign.

Correct

$$1200 + 30 = 1230$$

$$1230 - 10 = 1220$$

$$1220 \cdot 2 = 2440$$

(Student answer C1)

This answer is categorized as correct as the student gives satisfactorily separate equations for each step. The student makes a new line and computes for each step before concluding that the answer of the task is 2440. This demonstrates that the student in this setting preserves equilibrium and thus uses the equal sign relationally.

C2

For what values of x is $7x = 8x - x$?

Rate %	**Partially correct**	**Incorrect**	**Unanswered**

HHX	18,9	2,7	48,6
STX	12,5	7,5	62,5

Table 23. Results from the discourse analysis of the 0's of C2

Partially correct was here defined as showing existence. The biggest group in this test item is the one that goes unanswered – both for HHX and STX. One of the most likely reasons for this was the restricted time frame. Hence, it is difficult to conclude whether this group of students would be able to answer it correct or partially correct. Moreover, there is an exceptionally higher rate of partially correct than correct of the HHX students. However, there is not the same difference between partially correct and incorrect for the STX students. They either 1) stated it was true in general, 2) left the exercise unanswered, or 3) responded incorrectly. Another reason for this exceptionally high unanswered rate can also be the phrasing of the question, and that it can be peculiar and thus puzzling for the student. This low success rate of C2 can show a somewhat challenging concept of school algebra. Therefore, it is not straightforward to detect it as a misconception of the equal sign, but rather that a significant share of the students does not fully master school algebra.

Incorrect: non-traceable

$$7x = 8x - x$$

$$7x = 7x$$

(Student answer C2)

The incorrect arguments are all non-traceable, which is when the student responded without giving a conclusion. Thus, one could also argue that they might as well be categorized as unanswered, as we are not entirely sure of what to make of it. Their statement is true, but it does not tell us what the student thinks of the final answer.

Correct: all values

All values $8x - x$ gives 7x so it says $7x = 7x$

(Student answer C2)

This is a correct argument as the student validates the meaning of formal equivalence and affirms that it is valid for all values of x. This way of resonating shows an understanding of school algebra.

Partially correct: existence

$$x = 3 \text{ as } 7 \cdot 3 = 21$$

$$8 \cdot 3 - 3 = 21$$

(Student answer C2)

This argument does not relate to school algebra as theoretically, as the student does not make use of the formal equivalence. Instead, she inserts a value for *x* and confirms that it is true. Thus, the conclusion from the student is that it is valid for this random set value rather than all numbers.

C3:

If we for instance have the equation

$$\frac{(x-2)\cdot(x+2)}{x-2} = x+2$$

Is this always true?

Rate %	**Partially correct**	**Incorrect**	**Unanswered**
HHX	16,2	24,3	51,4
STX	10	25	65

Table 24. Results from the discourse analysis of the 0's of C3

0 students answered C3 correctly among the STX students. This percentage was equal to 8,1% among the HHX students. Furthermore, we notice from the table that 51,4% and 65% of HHX and STX respectively did not answer the task, which again can refer to similar reasons to the results of C3. As the test item refers to formal equivalence when one needs to consider the numerical restrictions, it is a particular case of formal equivalence. The meaning of contextual identity also plays a role. Firstly, the student might not have understood the phrasing correctly. Secondly, if the students are not strong in fractions, this can make the task too complicated to solve. Thirdly, the task is related to school algebra, implying that it will be challenging to solve without a fundamental understanding of that.

Incorrect

Formal equivalence:

Yes because one can remove $(\mathrm{x}-2)$

(Student answer C2)

This argument follows the meaning of formal equivalence as the student declares that one is allowed to remove $(x-2)$, i.e., rewrite the expression. However, the student has not considered that there are restrictions in terms of the domain of numbers. One can remove a denominator if one has assured that it is not equal to 0. Thus, it is not sufficient even if one masters the relational meaning in terms of the formal equivalence. One also needs to understand the meaning of contextual identity.

Existence

$$(x-2)\cdot(x+2) \qquad \frac{x^2-4}{x-2}=x+2$$

$$x^2+2x-2x-4$$

$$x^2-4$$

Always true

(Student answer C3)

As we have seen in previous test items with the meaning of contextual identity, the argument of existence also appears among some students. The inconsistency of this argument is that the question of the task was whether the equation *is always* to be regarded as true. The student has responded with one specific example to answer the question generally. On the other hand, one could argue that the student had taken an 'arbitrary' number and saw that it fit instead of the strategy in finding a counterexample. Nonetheless, the given approach shows that this student does not identify the meaning of contextual identity.

Correct

If it is true once it will always be, if you don't change on something

(Student answer C3)

No, not as long as one divides

(Student answer C3)

No because there are other numbers that you cannot use

(Student answer C3)

These three answers are the only ones that we detected as correct. One could argue that they should not be, as they do not justify specifically the case of what happens when $x = 2$. However, these arguments do mention that "if you don't change on anything", "divides", and that there are "other numbers that you cannot use", they have affirmed that there are cases in which an equality can be true, and other times where it is not. They have thus implicitly referred to the meaning of contextual identity.

C4:

In a rectangle the length is $l = 2$ cm and width $b = 3$ cm. What is the area of the rectangle?

Rate %	Partially correct	Incorrect	Unanswered
HHX	54,1	5,4	29,7
STX	55	2,5	35

Table 25. Results from the discourse analysis of the 0's of C4

The rate of incorrect is meager both for HHX and STX. Although almost 1/3 goes unanswered for both groups, the biggest group is the one that has answered partially correct. One was put into this category if one had replied either 6 or 6 cm, whereas we defined correct as 6cm^2. The numbers in the test item are small and easy to compute in this sense, which can be one of the reasons for the high rate of partially correct. Moreover, the formula of a rectangle is a formula that they are supposed to know from compulsory school and that we mentioned in part B. However, answering 6 or 6 cm shows like B7, that the students do not consider the correct use of units in finding the answer. Thus, this supports the former findings again in that the students have not fully developed the meaning of formal equivalence and that a result-oriented approach is desirable.

Regarding the meaning of specification in this task, we recall that the former test items of B6 and B12 measured 45,9/37,5 (HHX/STX) and 64,9/72,5 (HHX/STX), respectively, in success rates. What we see in B6 is that statement $4\text{cm} = x$ has a relatively lower success rate than that of B12. This can be since B12 is a standard function, which they have worked a tremendous amount of time on in the introductory period as it represents a linear function. The former is introduced less explicitly, and thus several students declared that they were missing a context. This supports the findings in C4, as we see the students indeed use the information given, namely that l = 2cm and b = 3cm without questioning it. This is perhaps because of the context in which it is introduced; we are looking at a rectangle and thus at a precise context.

C5:

2 theater tickets, and 1 popcorn of 30kr, cost 120kr in total. How much does one theater ticket cost?

Rate %	**Partially correct**	**Incorrect**	**Unanswered**
HHX	35,1	8,1	27,0
STX	32,5	15	35

Table 26. Results from the discourse analysis of the 0's of C5

Similar to C4, the success rates and partially correct rates added together make out roughly the same rate, both for HHX and STX. The incorrect rates are strictly less than 50% among both groups, so this is not considerably high. However, the partially correct and unanswered rate almost goes hand in hand as the distribution is almost the same for both STX and HHX. What regards partially correct was defined as the answer 45, and the correct answer defined as 45 kr. Hence, many have the same way as in C4 given the right number but have not put the correct unit on, thus not showing the meaning of formal equivalence. However, we note that many show the meaning of the conditional equation when correctly solving the task. This can be due to the tasks consisting of small numbers and that it is possible to solve them without school algebra. We show an example of each method that we identified.

Correct

Arithmetical solution: symmetric arithmetic

$$120 - 30 = 90$$

$$90kr/2 = 45\text{kr}$$

(Student answer C5)

Algebraic solution: conditional equation/formal equivalence

$$2x + 30 = 120$$

$$2x + 30 - 30 = 120 - 30$$

$$2x = 90$$

$$2x/2 = 90/2 = 45$$

$$45\text{kr}$$

(Student answer C5)

Partially correct: String operations

$$120 - 30 = 90/2 = 45 \text{ kr}$$

(Student answer C5)

General observations of the C-tasks

In practice, the C-tasks were the very last test items for some, as we told the students to go to part D when there were 10 minutes left. This was because categorizing what ways the students see the equal sign was prioritized. Therefore, time frame could be the reason for the low success rates. However, we do see that the unanswered rate of C4 and C5 are lower than for C2 and C3, which might be since it does not demand the use of school algebra and arithmetical strategies can also be used.

6.4 The D-tasks

D1

In what ways can we see "=" as?

When categorizing the answers given in D1, we first made a 0/1 analysis. All answers including a relational argument corresponded to 1, whereas either operational or non-traceable answers were graded as 0. Subsequently, we categorized how many were operational and non-traceable to see how many students one can deduce shares a pure operational view. Although a student gave an operational answer, this was not identified as operational when they also had given a relational argument. The following table shows the distribution of relational, operational, and non-traceable answers. Non-traceable is when 1) we could detect the given answer as neither operational nor relational or 2) unanswered.

Rate %	HHX	STX
Relational	73,1	65
Operational	13,5	25
Non-traceable	13,5	10

Table 27. Results of relational vs. operational arguments

For HHX and STX, the rate of relational answers measures more significant than 50%, which is a clear indicator that we cannot say that the students have great misconceptions of the equal sign. Indeed, they are more towards a relational approach, even if it is not 100% relational. This will be elaborated further in the discussion. We will now give some examples of arguments for each category.

Operational

> It means what the answer gives
> (Student answer D1)

> A solution to a calculation
> (Student answer D1)

> I understand it as the last sign before the final result
> (Student answer D1)

These answers reveal that the student affirms the operational meaning, meaning that they perceive the equal sign as an operator of a 'do something' signal. However, one cannot solely conclude their view is merely operational as this might be just as much about a lack of expression of mathematical vocabulary.

Relational

> Both numbers by an equal sign are the same
> (Student answer D1)

> As a result or maybe a mirror in equations as both sides give the same
> (Student answer D1)

The equal sign is just like a weight, there must be made equilibrium
(Student answer D1)

When something is the same so it says exactly the same on both sides
(Student answer D1)

We categorized an argument as relational if it included a relational phrasing such as "is the same" or "gives the same" or referring to equilibrium. Both for HHX and STX, more than 50% gave a relational argument, which indicates that the students at the beginning of upper secondary school share a relational view of the equal sign.

Non-traceable

As a comparison
(Student answer D1)

That the one side = the other
(Student answer D1)

For me it gives me an explanation to why something is as it is, but it can also be a help when one is to compute equations
(Student answer D1)

These arguments neither show an operational view nor a relational, and we can thus not conclude what idea the students have of the equal sign.

D2

Give an example of when to use “=” and an example where one cannot use it. Remember to elucidate your examples.

Rate %	**Correct**	**Partially correct**	**Incorrect**	**Unanswered**
HHX	70,3	13,5	5,4	10,8
STX	55	22,5	5	17,5

Table 28. Results of the quantitative analysis of D2

Partially correct is when the student only gave one valid example, whereas incorrect is defined as when both examples offered were wrong. We note that 83,5% HHX and 77,5% of STX have shown one or two correct examples, thus, we will not look further into this as the rate is strictly greater than 50% for both groups. The specific examples can be found in the appendix for the interested reader. The examples given varied in the character of and can be categorized into three main categories 1) comparing two numbers (e.g., $0 = 0$ and $1 \neq 0$) 2) comparing two equations (e.g., $2 + 2 = 4$ and $2 + 2 \neq 5$) 3), text answers ("for result or equation. cannot stand alone"). Thus, based on this test item, one cannot conclude whether the students show an operational or relational meaning.

D3

Is h and p equal? Justify why/why not.

In categorizing these answers, we chose the categories arithmetic, geometric and algebraic as proposed by (Molina, Desarrollo del pensamiento relacional y comprensión del signo por alumnos de tercero de educación primaria, 2006, p. 103-105). In addition, we also added 'uses more than one argument' and 'unanswered' as two additional categories. The table below shows the distribution of each argument.

Argument	**Rate %**
Arithmetic	3/77=4%
Geometric/graphic	32/77 = 42%
Algebraic	18/77=23%
Uses more than one argument	16/77=21%
No argument/Unanswered	8/77=10%

Table 29. Results of analysis of equivalence argument given

Arithmetic

> Yes because they are the same length
> (Student answer D3)

Geometric/graphic

They have the same slope, but if you extend it they won't intersect the same place so they are not entirely equal each other
(Student answer D3)

They are because they are parallel
(Student answer D3)

No, as they don't start in the same point
(Student answer D3

Algebraic

No because they don't have the same equation
(Student answer D3)

Yes, the slope coefficient is the same, they just start different places
(Student answer D3)

D3 was included as one of the test items to overview how the students see equality. Thus, this test item did not look for correct and incorrect answers since there is no precise answer to it. This test item is more related to what kinds of arguments the students use to say that something is equal or not. We observe a meager rate of pure arithmetical ideas, most likely because the students learned about linear functions in the introductory period, both HHX and STX. Thus, not so surprisingly, we find that more than 42% (STX and HHX altogether) argue with a graphic argument. This can also be because the representation form given is graphical.

7. Discussion

7.1 Reviewing the results of RQ2 in relation to RQ1

Regarding symmetric arithmetic, it was hidden in knowledge to be taught. However, the success rates in parts A and B indicates that most students mastered it. Thus, we believe that this relational meaning of the equal sign is supported. However, the high success rate can also be due to small numbers, or that symmetric arithmetic is not as strongly linked to school algebra as such.

Nonetheless, one still needs to know when guessing the answer to validate whether it is true. Thus, if the student had an operational approach, we expected that they just added the numbers in the equation and set it equal to the missing number instead of looking for equilibrium. However, they have shown a relational meaning different from former studies presented in RQ1.

Furthermore, the meaning of formal equivalence is challenging for the students. In part A, both rewriting an expression and adding units of the same kind had a high success rate. However, in part B, only B8 had a success rate greater than 50%. Moreover, formal equivalence fails in the calculation of percentage (B5), mixed notation (B7), binomial formula (B11), and what regards Part C (C2-C5). Nonetheless, the binomial formula (B11), C2, and C3 can demonstrate challenges to school algebra. Concerning the tasks of misuse of units (B5, B7, C4, C5), the students' answers go hand in hand with the mathematical notation detected in RQ1. Thus, we can argue that the students have adapted the textbook methods and hence learned the knowledge to be taught.

Considering conditional equation, this was difficult to detect fully, as we did not make too many tasks with this category. Thus, the findings here can be partly misrepresentative. We see that many students show a result-oriented approach when arguing whether one has solved an equation with an unknown correctly or not (B15); they favor the result instead of the method, which is supported in the tasks of formal equivalence as well and symmetric arithmetic in B16. In addition, determining whether the two equations are the same (B13 and B14) turns out to be difficult, as the phrasing itself has seemed to cause some misunderstandings. This can also be due to challenges related to school algebra.

Regarding contextual identities, these are not mastered very well, although knowledge to be taught makes correct use of assumptions when stating theorems. Moreover, it is challenging to detect directly from the tasks affiliated with it, as there were fewer of them. Nonetheless, we have tendencies that the student does not make a difference in solving a task with an assumption (B9, C4) and without (B10). If they have met an equation before, they will act upon it the same way, even if we do not present them with the same assumptions. One could also see this with the tasks of formal equivalence with different units – even if the tasks do not give assumptions of calculating with percentage or what the price of a travel card is defined as, the students might have seen similar arguments in the textbooks. Accordingly, it seems like a focus on restrictions and assumptions may be lacking. This problem of contextual identity is also highlighted in Schou & Bikner-Ahsbahs (2021) where they declare that "formula is seldom questioned or defined in an instructional practice, nor in written materials" and how the equal sign is closely related to this problematic (Schou & Bikner-Ahsbahs, 2021, p. 1). We confirm these challenges in detecting few contextual identities in the knowledge to be taught.

The meaning of specification is likewise a perplexing one. However, there were not many tasks affiliated with it to conclude something fully. Nonetheless, a tendency we observed is that the students accept the meaning if they have encountered the object before, i.e., a (linear) function (A11, B12) or asserting lengths of a geometrical figure (C4) which were objects encountered in RQ1. On the other hand, if they only are given a 'random' statement (B6), they cannot relate and hence not be sure of the information.

Part D reveals that most students have ideas of the relational approach, which goes against former studies. However, the presented challenges of the different meanings of the equal sign indicate that it is not black or white. Thus, we suggest a continuum between the relational and operational approaches. The arguments we are given in Part D advocate that the majority are on their way onto a more relational point of view but still maintaining an understanding of the operational.

7.2 The external didactic transposition of RQ1

In a future project, one could consider removing the factor of HHX and STX and simply examine one of the two. However, the schools we established contact to happened to be both HHX and STX, and as we wanted to hand out the tests to a sample of a reasonable size, we suggest that we had four classes (two from each school) instead of two (of the same school). In addition to the screening tests, one could have included exam exercises from compulsory school to see what the students are supposed to know from before. However, it was not possible to retrieve this data because of restricted access. Regarding the textbooks analyzed, one could have included a book of the introductory period of HHX as well. Nonetheless, there was no access to these in either the public or university libraries at the time.

However, this analysis was not to say something about the frequency of meanings of the equal sign, but *how* we can see them come into view in knowledge to be taught. Thus, we have shown in the analysis of RQ1 how the scholarly meanings of the equal sign are transposed onto knowledge to be taught by giving concrete examples of how they can come into view. However, what concerns symmetric arithmetic, this was not easy to find. Nonetheless, as the primary focus in upper secondary school is algebra, one could argue that symmetric arithmetic is extended to being that of formal equivalence. Our detection of the latter can give an understanding of the former. For future work, it could be interesting to consider an analysis in which one detected the frequencies of the meanings of the equal sign – do some meanings transpire more occasionally than others? If yes, is there causality between what meanings occur most frequently, and is this meaning understood better than the others?

We have also shown in the analysis of RQ1 the hybrid scholarly knowledge that exists between the scholars of mathematics and scholars of didactics. The history of the equal sign shows that its meaning has developed throughout time. Thus, when history itself is complicated, there is maybe no wonder why we are still discussing its connotation today. Some of the uses we consider incorrect and a misconception today had a different epistemic value in the beginning. Regarding the scholars of didactics of mathematics, one can see that the model of Prediger has been valuable in detecting the nuances more than what an operational/relational approach would be able to. However, it turned out to be challenging when we noticed questionable uses in knowledge to be taught, and we were to categorize them within the six meanings.

Regarding these doubtful uses, one could argue that the textbooks presenting them have intentionally wanted to simplify mathematics by linking it to a linguistic description. Nonetheless, as much as this can be helpful for some as there is less to read, others may reproduce the resolutions. Consequently, the students do not understand the "conventions" and thus do not grasp computation of, e.g. percentage, which after all, is part of the official programs for HHX and STX.

7.3 Limitations of the diagnostic test in RQ2

As mentioned in RQ1, in a future project, one could consider only HHX or STX. Furthermore, one could have included more schools in different parts of Denmark to make the sample more representative of a general STX or HHX context. However, we argue that since the classes were in the introductory period, the sample's variation in terms of the student segment was good. Furthermore, we had a sample of 77 students, which we suggest is a fair volume. More tasks in the test could create a bigger picture of the understandings of the equal sign. However, this would require demanding more time from the teachers. It was challenging to schedule the date of the tests, as it was going to happen before the introductory period stopped – but after the screening test – to make it fit with the time we were making out this dissertation. When we gave the students the test, they had not met topics such as probability theory, annuities, and differential calculus which could give rise to more aspects, as we saw in RQ1. On the other hand, we also wanted to test the students at the beginning of upper secondary school, as this is a good mark between the transition between arithmetic and school algebra. Thus, we suggest that the findings in the test are still valid in helping us see what challenges may exist in understanding the meanings of the equal sign.

Concerning the diagnostic test method, it turns out to be essential considering to what degree we can identify the given test as a truly diagnostic one. Regarding diagnostic tasks, one should avoid asking questions where the students can answer correctly, even if they have wrong ideas to the concept, which we cannot be entirely sure of. However, we have still detected students' way of thinking when doing the discourse analysis of the tasks with less than 50% success rate. Nonetheless, we failed to construct tasks associated with only one specific problem area: one unique problem linked to one unique meaning of the equal sign. We found this challenging as the equal sign is one specific object perceived from various perspectives. Even within one unique problem area, the equal sign's different meanings can appear at the one and the same time.

Thus, on the one hand, we can say that we have observed how the students see the equal sign in different cases. On the other hand, we cannot say that we have been able to cover clear misconceptions, but only tendencies. However, the findings in the results of RQ2 still provide an understanding of how the equal sign can shed light on challenges, particularly regarding those linked to school algebra. Accordingly, we have shown the challenge of making a diagnostic test on one specific object used in many settings.

7.4 The categorization of incorrect answers in the diagnostic test

Some tasks in the diagnostic test had a relatively high number of students that left the problem unanswered. We point out the last tasks of B; B14, B15, and B16 in addition to C2-C5. Here, more than 30% left the problem unanswered. On the one hand, one could argue that this indicates that the statements of the tasks were unfamiliar or not understood by the students. Alternatively, it might show challenges to school algebra as the tasks are highly related to it.

Nevertheless, we observe that the STX group makes out a more significant rate of unanswered in each of the mentioned tasks than those from HHX. One possible explanation is that one uses fewer arguments and reasoning in mathematics in HHX, as the central aspect is applied mathematics concerning economics. Thus, an answer is preferable against no answer at all. For STX, the standard focus is theoretical mathematics with more reasoning. Hence the students may not show the same need for an answer if they do not have the arguments for it. What regards partially correct, we see that this rate is relatively high for the test items in Part B in the category both true and false (B10 and B17). This can be since stating either true (T) or false (F) was categorized as partially correct. However, the likelihood of answering (T), (F), or (B) is, in theory, equal; thus, the same probability goes for each task. Nonetheless, one could argue that given that the answer is both true and false, it indirectly creates a higher probability of answering correctly than given the task was either T or F because of an assumption that a problem typically has one answer. What regards the C-tasks, these also showed higher rates of partially correct. In C1, it was due to string operations, whereas C4 and C5 pointed out misuse of units when graded partially correct. The latter follows the findings of RQ1, and we can thus say that the students' answers here satisfy the notation observed in the textbooks.

7.5 The equal sign as an independent object

We would like to finalize the discussion by considering the validity of the equal sign as an independent object. The diagnostic test has shown us challenges in establishing tasks that completely isolate other aspects of mathematics, mainly the question of school algebra. From the findings in RQ1, we saw how the equal sign is implicitly described in the knowledge to be taught. Thus, it is harder to investigate the equal sign thoroughly as it represents a core subject matter that appears in so many different contexts and generally all over mathematics – both in upper secondary school and compulsory school. The following study is different from the former ones by various means. Firstly, few studies have used diagnostic tests but rather investigation during classroom teaching or other tests than diagnostic ones. Secondly, most have taken outset in younger grades of the compulsory school where the students have not been introduced to the topic of algebra to the same extent as a student in upper secondary school. Thirdly, many former studies have only considered the binary approach of operational vs. relational.

We have in this dissertation mentioned the challenge that school algebra is linked to several test items and how it makes it hard to give clear conclusions about what is challenging. However, in which direction can we see the causality between understanding the equal sign correctly and school algebra? Does school algebra make us understand the equal sign, or does the equal sign make us understand school algebra? One could argue that there is an interaction between the two, just like the constructivist point of view declares - our former experiences make us understand new concepts. Thus, examples of when to use the equal sign correctly and vice versa with school algebra could help improve the understanding of the other. Therefore, we endorse to be more explicit with the equal sign and talk about the elephant in the room.

8. Conclusion

We will now conclude by answering our two research questions:

RQ1: Based on external didactic transposition, how do the scholarly meanings of the equal sign come into view in knowledge to be taught in Danish upper secondary school?
RQ2: Which challenges related to the external didactic transposition of the equal sign can be detected through a diagnostic test for Danish students in upper secondary school?

Regarding RQ1, the external didactic transposition we have compiled in this dissertation has demonstrated how the scholarly meanings of the equal sign come into view in knowledge to be taught. The equal sign is not mentioned explicitly in official programs. However, all the six meanings proposed by Prediger (2010) have come into view in the analyzed knowledge to be taught, although some meanings are more evident than others. Symmetric arithmetic and contextual identity were more challenging to detect, whereas the conditional equation is what most often appears in knowledge to be taught. Moreover, we have shown that several of the meanings of the equal sign often come into view together in the same problem, often concerning equation solving, which supports the findings of Prediger (2010). Additionally, the analysis has revealed some added meanings to the equal sign in knowledge to be taught, which is not considered scholarly knowledge, but doubtful uses of the equal sign.

The findings from the external didactic transposition made it possible to construct a diagnostic test and detect the internal didactic transposition of the learned knowledge, which answers RQ2. Specifically, we have revealed different kinds of challenges related to the meanings of the equal sign, some of which relate to the questionable uses found in knowledge to be taught. Especially what concerns units and assumptions in equations turn out to be more challenging, which denotes the meaning of formal equivalence and contextual identity, respectively. Although the meaning of symmetric arithmetic is challenging to identify in the knowledge to be taught, it shows a high success rate. We can thus not proclaim it an apparent misconception. Concerning the meaning of a conditional equation, this is highly linked to school algebra, indicating that it may be the latter that is challenging for the students. Regarding the meaning of specification, the students acknowledge this meaning when they encounter objects that we had detected in knowledge to be taught.

Regarding the operational vs. operational approach, the diagnostic test has shown that most students give a relational argument when defining the meaning of the equal sign. Additionally, the overall results of the test suggest that one should see the students' view of relational and operational as more of a continuum rather than discrete views.

Furthermore, this paper demonstrates the experiment of making a diagnostic test of the equal sign as an independent entity. Some of the tasks with low success rates may imply misconceptions of school algebra just as much as the equal sign. Nonetheless, this shows the complexities and the interdependence of the two. Thus, we advocate the importance of the equal sign alongside school algebra to make their interdependence clearer.

References

Axelsen, R., & Dalsgaard, O. (2019). *Matema10k - Matematik for hhx C+B-niveau, 1st edition.* Frederiksberg: Frydenlund.

Bosch, M., & Gascón, J. (2006). Twenty-Five Years of the Didactic Transposition. *CMI Bulletin No. 58*, pp. 51-65.

Brekke, G. (2002). *Introduksjon til diagnostisk undervisning i matematikk - Kartlegging av matematikkforståelse.* Oslo: Læringssenteret.

Cajori, F. (1993). *A History of Mathematical Notation.* New York: Dover Publications.

Carstensen, J., Frandsen, J., Lorenzen, E. W., & Jørgensen, M. (2017). *MAT stx grundforløb.* Aarhus: Systime.

Carstensen, J., Frandsen, J., Lorenzen, E. W., & Madsen, A. L. (2019). *MAT B1 stx, 5th edition.* Aarhus: Systime.

Chevallard, Y., & Bosch, M. (2014). Didactic Transposition in Mathematics Education. *Encyclopedia of Mathematics Education*, 170-174.

Cosan, D. D. (2021). *A Diagnostic Test for Danish Middle School Arithmetics. Master thesis.* Copenhagen: Department of Science Education, University of Copenhagen.

Danish Ministry of Education. (2017, August). *Bilag 112, Matematik B – stx.* Retrieved from https://www.uvm.dk/gymnasiale-uddannelser/fag-og-laereplaner/laereplaner-2017/stx-laereplaner-2017

Danish Ministry of Education. (2017, August). *Bilag 41, Matematik B - hhx.* Retrieved from https://www.uvm.dk/gymnasiale-uddannelser/fag-og-laereplaner/laereplaner-2017/hhx-laereplaner-2017

Danish Ministry of Education. (2017). *Matematik Screening studentereksamen vejlædende opgavesæt 3, STX-MAT-GRUNDFORLØB.* Unpublished.

Danish Ministry of Education. (2018). *Matematik Screening studentereksamen vejlædende opgavesæt 5, STX-MAT-GRUNDFORLØB.* Unpublished.

Danish Ministry of Education. (2018). *Matematik Screening studentereksamen vejlædende opgavesæt 6, STX-MAT-GRUNDFORLØB.* Unpublished.

Danish Ministry of Education. (2020, February). *Matematik A/B/C, stx Vejledning*. Retrieved from https://www.uvm.dk/gymnasiale-uddannelser/fag-og-laereplaner/laereplaner-2017/stx-laereplaner-2017

Danish Ministry of Education. (2021, June). *Matematik A/B/C, hhx Vejledning.* Retrieved from https://www.uvm.dk/gymnasiale-uddannelser/fag-og-laereplaner/laereplaner-2017/hhx-laereplaner-2017

Darr, C. (2003). THE MEANING OF “EQUALS”. *PROFESSIONAL DEVELOPMENT*, 4-7.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999, December). Children’s Understanding of Equality: A Foundation for Algebra. *TEACHING CHILDREN MATHEMATICS*, 232-236.

Faulkner, V., Walkowiak, T., Cain, C., & Lee, C. (2016). Equality matters The critical implications of precisely defining equality. *Australian Primary Mathematics Classroom 21 (4)*, 11-15.

Jankvist, U., & Niss, M. (2018). The notion and role of ”detection tests” in the Danish upper secondary ”maths counsellor” programme. *Proceedings of CERME10*, 3825-3832.

Katz, V. (2009). Chap. 12: Algebra in the Renaissance. In *A History of Mathematics An Introduction, 3rd edition* (pp. 383-420). Boston: Pearson Education Limited.

Kieran, C. (1981). CONCEPTS ASSOCIATED WITH THE EQUALITY SYMBOL. *Educational studies in Mathematics no. 12*, 317–326.

Knuth, E. J., Alibali, M., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2008, May). The Importance of Equal Sign Understanding in the Middle Grades. *Mathematics Teaching in the Middle School Vol. 13, No. 9*, 514-519.

Knuth, E. J., Stephens, A. C., McNeil., N. M., & Alibali, M. W. (2006, July). Does Understanding the Equal Sign Matter? Evidence from Solving Equations. *Journal for Research in Mathematics Education Vol. 37, No. 4*, 297-312.

Lützen, J. (2019). *Diskrete Matematiske Metoder, 2nd edition.* Copenhagen: Department of Mathematical Sciences, University of Copenhagen.

Lundberg, A., & Kilhamn, C. (2016). Transposition of Knowledge: Encountering Proportionality in an Algebra Task. *International Journal of Science and Mathematics Education*.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-School Students' Understanding of the Equal Sign: The Books They Read Can't Help. *COGNITION AND INSTRUCTION 24(3)*, 367–385.

Molina, M. (2006). *Desarrollo del pensamiento relacional y comprensión del signo por alumnos de tercero de educación primaria.* Granada: Universidad de Granada.

Molina, M., Castro, E., & Castro, E. (2007). Historia del signo igual. In M. Guzmán, *Humanidades y Ciencias. Aspectos Disciplinares y Didácticos. Homenaje a la Profesora Ana Vilches Benavides* (pp. 249-261). Granada: Editorial Atrio.

Poulsen, C. S. (2015). *Basic algebra in the transition from lower secondary school to high school. Master thesis.* Copenhagen: Department of Science Education, University of Copenhagen.

Prediger, S. (2010). How to develop mathematics for teaching and for understanding: the case of meanings of the equal sign. *Journal of Mathematics Teacher Education vol. 13*, 73-93.

Schou, M. H., & Bikner-Ahsbahs, A. (2021, October). Unpacking hidden views: seven ways to treat your formula. *Educational Studies in Mathematics*.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *The Journal of Mathematical Behavior 32*, pp. 173-182.

Theis, L. (2005, Nov.). L'Apprentissage Du Signe = : Un Obstacle Cognitif Important. *For the Learning of Mathematics Vol. 25, No. 3*, 7-12.

Vincent, J., Bardini, C., Pierce, R., & Pearn, C. (2015). Misuse of the equals sign: An entrenched practice from early primary years to tertiary mathematics. *Australian Senior Mathematics Journal vol. 29*, 31-39.

Appendices

Appendix A: Pilot test

A. Udfyld de tomme felter med passende tal så at ligheden er gyldig, og forklar kort din fremgangsmåde. 1-3 sætninger max.

	Opgave	Fremgangsmåde
1)	9 + 4 = __ + 4	
2)	12 = __ + __ + __	
3)	5 + 8 = __	
4)	8 + 6 = __ + 8	
5)	2 = ___	
6)	12 + __ = 7 + ___	
7)	2 + 8 + 6 = 8 + __	
8)	25 + __ = 5 + 40	

B. Afgør om udsagnet er sandt eller falskt, og giv en kort begrundelse.

	Udsagn	Sandt (S) eller falskt (F)	Begrundelse
1)	$7 = 7$		
2)	$9 = 3 + 6$		
3)	$3 + 2 = 5 = 1 + 1 + 3$		
4)	$7 + 5 = 12 + 3 = 15$		

5)	$8 + 6 = 7 + 7$		
6)	$2 + 9 = 2 + 4 + 5$		
7)	$0,20 \cdot 100 = 20\%$		
8)	4 cm = x		

9)	1200 kr + 1 rejsekort = 1200 kr + 80 = 1280 kr		
10)	$\sqrt{2} = \frac{2}{\sqrt{2}}$		
11)	Arealet af et rektangel $= l \cdot b$ Hvor l betegner længden og b bredden af rektanglet		
12)	$c^2 = a^2 + b^2$ Hvor a, b og c er naturlige tal (1, 2, 3, 4, ….)		
13)	$c^n = a^n + b^n$ Hvor a, b, c og n er naturlige tal (1, 2, 3, 4, ….)		

14)	$(a+b)^2 = a^2 + 2ab + b^2$ Hvor a og b er naturlige tal (1, 2, 3, 4, ….)		
15)	F(x) = 3x + 2		
16)	3x + 2 = 13 + 5x og -2x – 6 = 5 Er den samme ligning.		
17)	10x + 40 = 10x + 20		

C. Løs følgende tekstopgaver. Husk en kort konklusion efter udregninger.

	Opgave	**Svar**
1)	Læg 1200 sammen med 30, træk derefter 10 fra, og gang derefter med 2. Hvad får du så? Husk at vise udregning.	

2)	For hvilke værdier af x gælder der, at 3x – x = 2x?	
3)	Hvis f(x) = y = 5x - 3 og y = 22, hvad er f(x) + 10 lig med?	
4)	Den følgende ligning er løst og svaret som er fundet, er korrekt. Der er alligevel to steder hvor der er en fejl. Kan du udpege dem og forklare hvorfor det er forkert? $8x + 5 = 12$ $8x + 5 - 5 = 12$ $8x = 12 - 5$ $\frac{8x}{8} = 7$ $x = \frac{7}{8}$	
5)	Kan en lighed stoppe med at være sand? Hvis vi fx har ligningen $x^2 = -1$, kan den stoppe med at være sand? Er det nogle gange den ikke gælder?	

D. Reflektionsopgaver.

1) Du får en opgave hvor der er givet at f(x) = y = 2x + 1. Hvordan skal man forstå disse to lighedstegn? Forklar med egne ord.

2) Hvad betyder "="-tegnet?

a. Giv et eksempel på hvornår man kan bruge "=", og et eksempel hvor man ikke kan bruge det.

3) Hvis du fik i opgave at bevise at to kaffekander var lig med hinanden, hvordan ville du gøre det? Om kaffekanderne kan det oplyses at de har den samme farve, højde, håndtag, og vægt. Forsøg at opstille en ligning og forklar dine tanker bag.

Appendix B: Final test

Navn: ___________________________________

Ingen hjælpemidler tilladt ☺

A. Udfyld de tomme felter ___ med passende tal så at ligheden er gyldig, og forklar kort din fremgangsmåde (1-3 sætninger)

	Opgave	**Fremgangsmåde**
1)	9 - 4 = ___ - 4	
2)	12 = ___+___+___	
3)	___ = 5 + 8	
4)	3 + 5 = ___ - 2	
5)	2 = ___	
6)	8 - ___ = 6 - ___	
7)	2 + 8 + 6 = 8 + ___	

8)	2x + 4 = 2 x = ___	
9)	9x + 6 = ___ · (3x + 2)	
10)	10cm + 2cm + 3kr = ___	
11)	Du får at vide, at a = y = 5x – 3 og at y = 22. a + 10 = ___	

B. Afgør om udsagnet er sandt (S), falskt (F) eller begge dele (B) og giv en kort begrundelse (1-3 sætninger).

	Udsagn	**Sandt (S) eller falskt (F)**	**Begrundelse**
1)	7 = 7		

2)	$3 + 2 = 5 = 8 - 4 + 1$		
3)	$7 + 5 = 12 + 3 = 15$		
4)	$2 + 9 = 2 + 4 + 5$		
5)	$0,20 \cdot 100 = 20\%$		
6)	4 cm = x		
7)	120 kr + 1 rejsekort = 120 kr + 80 = 200 kr		

8)	$\sqrt{4} = \frac{4}{\sqrt{4}}$		
9)	Arealet af et rektangel $= l \cdot b$ Hvor *l* betegner længden og *b* bredden af rektanglet		
10)	$c^2 = a^2 + b^2$ Hvor a, b og c er naturlige tal (1, 2, 3, 4, …)		
11)	$2ab + a^2 + b^2 = (a + b)^2$ Hvor a og b er naturlige tal (1, 2, 3, 4, …)		
12)	F(x) = 3x + 2		
13)	Ligningen givet ved $2x + 3 = x + 7$, og ligningen givet ved $4x + 6 = 2x + 14$, er den samme ligning.		

14)	Ligningen givet ved $3x + 2 = 13 + 5x$, og ligningen givet ved $-2x - 6 = 5$, er den samme ligning.		
15)	En elev har løst følgende ligning. $8x + 2 = 18$ $8x + 2 - 2 = 18$ $8x = 18 - 2$ $\frac{8x}{8} = 16$ $x = \frac{16}{8} = 2$ Afgør om udregningen er sand eller falsk.		
16)	En elev har løst følgende opgave. *En funktion er givet ved* $f(x) = 4x - 5$. *Find* $f(2)$. Afgør om løsningen er sand eller falsk: *Jeg finder f (2) ved at indsætte 2 i ligningen:* $f(x) = 4 \cdot 2 = 8 - 5 = 3$ *Så f (2) = 3.*		
17)	$5x = y$ Hvor x og y er naturlige tal (1, 2, 3, 4, …)		

C. Besvar følgende tekstopgaver. Husk at forklare din tankegang (1-3 sætninger).

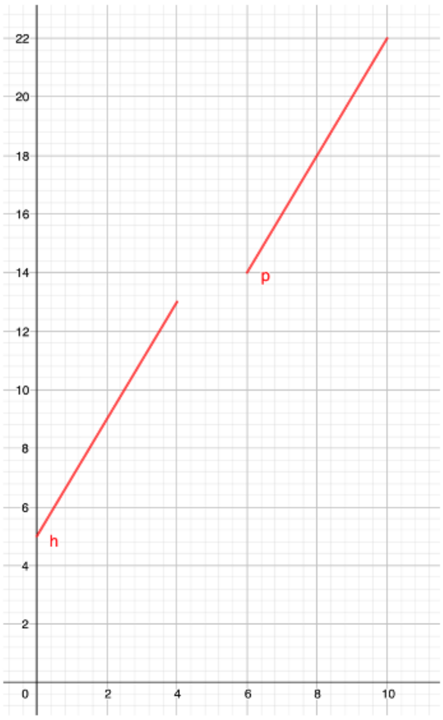
	Opgave	Svar
1)	Læg 1200 sammen med 30, træk derefter 10 fra, og gang derefter med 2. Hvad får du så? Husk at vise udregning.	
2)	For hvilke værdier af x gælder der, at 7x = 8x - x?	
3)	Hvis vi fx har ligningen $\frac{(x-2)\cdot(x+2)}{x-2} = x + 2$ Er den altid sand?	
4)	I et rektangel er længden l = 2cm og bredden b = 3cm. Hvad er arealet af rektanglet?	
5)	2 biografbilletter, og 1 popcorn til 30kr, koster 120kr til sammen. Hvor meget koster en biografbillet?	

D. Reflektionsopgaver

1) På hvilke måder kan man forstå "="-tegnet?

2) Giv et eksempel på hvornår man kan bruge "=", og et eksempel hvor man ikke kan bruge det. Husk at forklare dine eksempler.

3) Er *h* og *p* lig med hinanden? Begrund hvorfor/hvorfor ikke.



Svar:

Appendix C: Declaration of consent template

København, 8. oktober 2021

Samtykke til dataindsamling til specialeprojekt

I forbindelse med mit specialeprojekt i matematikdidaktik skal jeg indsamle data. Mit projekt er tilknyttet Institut for Naturfagenes Didaktik ved Københavns Universitet. Projektet har til formål at undersøge hvordan lighedstegnet optræder i skolematematikken og hvordan det bliver implementeret i lærebøgerne, samt hvad det gør ved danske elevers forståelse. Hertil har jeg lavet en diagnostisk test som undersøger elevernes forståelse af brugen af lighedstegnet. Specialet vil blive publiceret i studenterserien på ind.ku.dk. Data fra testen vil blive brugt i helt anonymiseret form.

Dit barns klasse er udvalgt til projektet. Dato for den diagnostiske test er: __________

Testbesvarelserne vil blive anonymiseret og tjener intet andet formål end empiri til analysen i specialet. Testen vil ikke blive brugt til vurdering af den enkelte elevs matematikkundskaber eller øvrige bedømmelse ift. karaktergivning. Data vil blive gemt og behandlet i henhold til gældende regler for datasikkerhed ved Københavns Universitet. Det er helt frivilligt om man vil deltage i projektet.

Har I spørgsmål, er I velkomne til at henvende jer til mig.

Aurora Olden Aglen
Tlf.: 60212914
e-mail: lsd223@alumni.ku.dk

Denne del klippes af og afleveres til matematiklæreren senest: __________

Med min underskrift giver jeg samtykke til at jeg/mit barn deltager i den diagnostiske test som er en del af Aurora Olden Aglens specialeprojektet ved Københavns Universitet.

Jeg bekræfter desuden, at jeg er tilstrækkeligt informeret om projektets formål, og at min deltagelse er frivillig.

Jeg kan til enhver tid tilbagetrække mit samtykke, dog ikke når data er anonymiseret. Henvendelse skal ske skriftligt til lsd223@alumni.ku.dk.

Ved at underskrive denne samtykkeerklæring accepterer du, at Københavns Universitet opbevarer og behandler de personoplysninger, du/dit barn giver os. Personoplysningerne slettes ved specialets udgivelse ultimo december 2021.

Er eleven ikke myndig, skal forældre(myndighedsindehaveren)/værgen skrive under.

Barnets navn: ____

Forælders navn: ____

Sted og dato: ______________

Underskrift:____

Appendix D: Excel-Analysis (0/1)

A-tasks

	Elev	A-opgaverne										
		1	2	3	4	5	6	7	8	9	10	11
HHX1	1	1	1	1	1	1	1	1	0	0	0	1
	2	1	0	1	1	1	1	1	0	1	1	0
	3	1	1	1	1	1	0	0	0	1	1	0
	4	1	1	1	1	1	0	1	1	0	1	1
	5	1	1	1	1	1	1	1	1	1	1	0
	6	1	1	1	1	1	0	1	0	0	1	0
	7	1	1	1	1	1	0	1	1	1	1	1
	8	1	1	1	1	1	1	1	1	1	1	1
	9	1	1	1	1	1	1	1	0	1	0	0
	10	1	1	1	1	1	1	1	0	1	1	1
	11	1	1	1	1	1	1	1	1	1	1	1
	12	1	1	1	1	1	0	0	1	1	0	0
	13	1	1	1	1	1	0	1	0	0	0	0
	14	1	1	1	1	1	1	1	1	0	1	0
	15	1	1	1	1	1	1	1	1	1	1	1
	16	1	1	1	1	1	1	1	0	1	0	0
	17	1	1	1	1	1	1	1	0	1	1	0
	18	1	1	1	1	1	1	1	1	0	1	1
	19	1	1	1	1	1	1	0	1	1	1	1
	20	1	1	1	1	1	1	1	1	0	0	1
HHX2	43	1	1	1	1	1	1	1	1	1	1	1
	44	1	1	1	1	1	1	1	1	1	1	1
	45	1	1	1	1	1	1	1	1	1	1	1
	46	1	1	1	1	1	1	1	0	0	1	0
	47	1	1	1	1	1	1	1	1	1	1	1
	48	1	1	1	1	1	1	1	1	1	1	1
	49	1	1	1	1	1	1	1	0	1	0	0
	50	1	1	1	1	1	1	1	1	1	1	1
	51	1	1	1	1	1	1	1	1	1	1	1
	52	1	1	1	1	1	1	1	1	1	1	1
	53	1	1	1	1	1	1	1	1	1	1	1
	54	1	1	1	1	1	1	1	1	0	1	1
	55	1	1	1	1	1	1	1	1	1	1	1
	56	1	1	1	1	1	1	1	0	1	1	0
	57	1	1	1	1	1	1	1	1	1	1	1
	58	1	1	1	1	1	1	1	1	1	1	1
	59	1	1	1	1	1	1	1	1	1	1	0
HHX success (%)		100	97,2973	100	100	100	83,78378	91,89189	67,56757	75,67568	81,08108	62,16216
STX2	60	1	1	1	1	1	1	1	1	0	0	0
	61	1	1	1	1	1	1	1	0	0	1	0
	62	1	1	1	1	1	0	1	0	0	0	0
	63	1	1	1	1	1	1	1	1	1	1	0
	64	1	1	1	1	1	1	1	1	1	1	1
	65	1	1	1	1	1	1	1	1	0	0	0
	66	1	1	1	0	1	1	1	1	1	1	1
	67	1	1	1	1	1	1	1	0	1	0	0
	68	1	1	1	1	1	0	1	1	1	1	0
	69	1	1	1	1	1	1	1	0	0	0	0
	70	1	1	1	1	1	1	1	1	1	1	1
	71	0	1	0	0	1	0	0	0	0	1	0
	72	1	1	1	1	1	1	1	0	1	1	0
	73	1	1	1	1	1	1	1	0	0	1	1
	74	1	1	1	1	1	1	1	0	0	0	0
	75	1	1	1	1	1	1	1	1	1	1	1
	76	1	1	1	1	1	1	1	1	1	1	1
	77	1	1	1	0	1	1	1	1	0	0	0
STX1	21	1	1	1	1	1	1	1	1	1	1	1
	22	1	1	1	1	1	1	1	1	1	1	1
	23	1	1	1	1	1	1	0	1	1	1	1
	24	1	1	1	1	1	1	1	1	1	1	1
	25	1	1	1	1	1	1	1	1	1	1	1
	26	1	1	1	1	1	1	1	1	1	0	0
	27	1	1	1	1	1	1	1	1	0	1	0
	28	1	1	1	1	1	1	1	1	1	1	1
	29	1	1	1	1	1	1	1	1	1	0	0
	30	1	1	1	1	0	1	1	1	1	1	0
	31	1	1	1	1	1	1	1	1	1	1	1
	32	1	1	1	1	1	1	1	1	0	1	0
	33	1	1	1	1	1	0	1	1	0	1	0
	34	1	1	1	1	1	1	1	1	1	1	1
	35	1	1	1	1	1	1	1	0	0	0	1
	36	1	1	1	1	1	1	1	0	0	0	0
	37	1	1	1	1	1	1	1	1	0	0	1
	38	1	1	1	1	1	1	1	1	1	1	1
	39	1	1	1	0	1	0	1	1	1	0	1
	40	1	1	1	1	1	1	1	1	0	1	1
	41	1	1	1	1	1	1	1	1	1	1	1
	42	1	1	1	1	1	1	1	1	1	1	1
success rate %		97,5	100	97,5	90	97,5	87,5	95	75	60	67,5	52,5

B-tasks

		B-opgaverne																
	Elev	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
HHX1	1	1	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0
	2	1	1	1	1	0	1	1	1	1	0	0	1	0	0	0	0	0
	3	1	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0
	4	1	1	1	1	1	0	1	0	1	0	0	1	0	0	0	1	1
	5	1	1	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0
	6	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	7	1	1	1	1	0	1	1	1	1	0	1	1	1	0	0	0	0
	8	1	1	1	1	0	1	0	1	1	0	0	1	0	1	0	1	0
	9	1	1	0	1	0	1	0	0	1	0	0	1	0	0	1	0	0
	10	1	1	1	1	0	1	0	1	1	1	1	0	1	1	0	0	1
	11	1	1	1	1	0	1	1	1	1	0	1	1	0	0	0	0	0
	12	1	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
	13	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
	14	1	1	0	1	0	0	0	1	1	1	0	0	0	0	0	0	1
	15	1	1	0	1	1	0	1	0	1	0	0	0	1	0	1	1	0
	16	1	1	0	1	0	1	0	1	1	1	1	1	1	0	1	0	0
	17	1	1	0	1	0	1	0	1	1	1	0	1	0	0	1	0	1
	18	1	1	1	1	0	0	0	0	1	1	0	0	1	0	0	0	0
	19	1	1	1	1	1	0	0	1	1	0	1	1	0	0	0	0	1
	20	1	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	0
HHX2	43	1	1	1	1	0	1	0	1	1	0	0	1	1	0	0	0	0
	44	1	1	1	1	0	1	0	1	1	0	0	1	1	0	0	0	0
	45	1	1	1	1	0	1	1	0	1	0	0	1	1	1	0	0	0
	46	1	1	0	1	0	1	0	1	1	0	0	1	1	0	1	0	0
	47	1	1	1	1	1	0	0	0	1	0	0	0	1	0	0	0	0
	48	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	0	0
	49	1	1	0	1	0	0	0	0	1	0	0	1	1	0	1	0	0
	50	1	1	1	1	1	0	0	1	1	1	1	0	0	1	0	0	0
	51	1	1	1	1	0	0	0	0	1	0	1	0	1	1	0	0	1
	52	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	0	0
	53	1	1	0	1	1	1	0	0	1	0	0	1	1	1	0	0	0
	54	1	1	1	1	0	1	0	1	1	0	1	1	0	0	0	0	0
	55	1	1	1	1	0	0	0	1	1	0	0	1	1	1	0	0	0
	56	1	1	0	1	0	0	0	0	1	0	0	1	1	0	0	0	0
	57	1	1	1	1	0	0	1	1	1	0	1	1	1	0	0	0	1
	58	1	1	0	1	0	0	0	0	1	0	0	1	1	0	1	0	0
	59	1	1	1	1	0	1	0	1	1	0	0	1	1	1	0	0	0
success rate %		100	100	64,8648649	97,2972973	18,9189189	45,9459459	24,3243243	56,7567568	91,8918919	16,2162162	29,7297297	64,8648649	56,7567568	27,027027	18,9189189	8,10810811	18,9189189
STX2	60	1	1	0	1	0	0	0	1	1	0	0	1	1	0	1	0	0
	61	1	1	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1
	62	1	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
	63	1	1	1	1	0	1	0	1	1	1	1	1	0	0	0	0	0
	64	1	1	0	1	0	0	1	0	1	0	1	1	0	0	0	0	0
	65	1	1	1	1	0	0	0	1	1	0	0	1	1	0	0	1	0
	66	1	1	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0
	67	1	1	1	1	0	0	0	1	1	0	0	1	0	0	1	1	0
	68	1	1	0	1	0	1	0	0	1	0	0	1	0	0	0	0	0
	69	1	1	1	1	0	0	0	0	1	0	0	0	1	0	0	0	0
	70	1	1	1	1	0	0	0	0	1	0	0	1	1	1	0	1	0
	71	1	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0
	72	1	1	0	1	0	1	1	1	1	0	0	1	1	1	0	1	0
	73	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	74	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	75	1	1	1	1	0	0	0	0	1	0	1	1	1	0	0	0	0
	76	1	1	1	1	0	1	0	1	1	0	1	1	1	0	0	0	0
	77	1	1	0	1	0	1	0	0	1	0	1	1	1	1	0	0	0
STX1	21	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	1
	22	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	1
	23	1	1	1	1	1	1	1	1	1	0	0	1	1	1	0	0	0
	24	1	1	1	0	1	0	0	1	1	0	1	1	1	1	0	0	0
	25	1	1	1	1	1	1	0	1	1	1	0	1	1	1	0	0	0
	26	1	1	1	1	0	1	0	0	1	0	0	1	1	1	0	0	0
	27	1	1	1	1	0	1	0	1	1	0	0	1	1	0	0	0	0
	28	1	1	1	1	0	0	1	1	1	0	0	1	0	0	1	0	1
	29	1	1	0	1	1	0	0	1	1	0	0	1	0	0	0	0	0
	30	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0
	31	1	1	0	1	0	0	0	1	1	0	0	1	0	0	0	0	0
	32	1	1	1	1	0	1	0	1	1	0	0	1	1	0	0	0	0
	33	1	1	0	0	0	1	0	1	1	0	0	1	1	0	0	0	0
	34	1	1	1	1	0	1	0	1	1	0	0	1	1	0	0	0	0
	35	1	1	1	1	0	1	0	1	1	1	0	1	0	0	0	0	0
	36	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	37	1	1	1	1	0	1	0	1	1	0	0	1	0	0	0	0	0
	38	1	1	0	1	0	0	0	1	1	0	0	0	0	1	0	0	0
	39	1	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
	40	1	1	1	1	1	0	1	1	1	0	1	0	1	1	0	0	0
	41	1	1	1	1	1	0	1	1	1	0	1	0	0	0	0	0	0
	42	1	1	1	1	0	0	0	0	1	0	0	1	0	0	0	0	0
		40																
success stx %		100	95	62,5	82,5	15	37,5	15	65	80	7,5	22,5	72,5	47,5	30	7,5	10	10

C-tasks

		C-opgaverne				
	Elev	**1**	**2**	**3**	**4**	**5**
HHX1	**1**	0	0	0	0	0
	2	1	0	0	0	0
	3	1	0	0	0	0
	4	0	0	0	0	1
	5	1	0	0	0	0
	6	0	0	0	0	0
	7	0	0	0	0	0
	8	0	0	0	0	1
	9	1	0	0	0	1
	10	1	1	0	0	1
	11	0	1	0	0	0
	12	1	0	0	0	1
	13	0	0	0	0	0
	14	0	0	0	0	0
	15	1	1	1	0	1
	16	1	1	0	0	0
	17	1	1	0	1	0
	18	0	0	0	0	0
	19	0	1	1	0	0
	20	0	0	0	0	1
HHX2	**43**	1	0	0	0	1
	44	0	0	0	0	0
	45	0	0	0	0	0
	46	0	0	0	0	0
	47	0	1	0	0	1
	48	0	0	0	0	0
	49	1	0	0	0	0
	50	0	0	0	0	0
	51	0	1	0	1	0
	52	1	0	0	0	0
	53	0	1	0	0	1
	54	1	1	0	0	0
	55	1	0	0	0	0
	56	0	0	0	1	1
	57	0	0	1	0	0
	58	0	1	0	1	0
	59	1	0	0	0	0
HHX success %		40,54054054	29,72973	8,1081081	10,810811	29,72973

	Elev	C-opgaverne				
STX2	**60**	1	0	0	0	1
	61	0	0	0	0	0
	62	1	0	0	0	0
	63	1	1	0	0	0
	64	1	0	0	0	0
	65	0	0	0	0	0
	66	1	0	0	0	0
	67	1	0	0	0	0
	68	0	0	0	0	0
	69	0	0	0	0	0
	70	0	0	0	0	1
	71	0	0	0	0	0
	72	0	0	0	0	1
	73	1	0	0	0	0
	74	1	0	0	0	0
	75	0	0	0	0	0
	76	0	1	0	0	0
	77	1	0	0	0	0
STX1	**21**	1	0	0	1	0
	22	1	0	0	0	1
	23	1	1	0	1	0
	24	0	1	0	0	1
	25	1	1	0	0	0
	26	1	1	0	0	0
	27	0	0	0	0	0
	28	1	1	0	0	0
	29	0	0	0	0	0
	30	0	0	0	0	0
	31	0	0	0	0	0
	32	0	0	0	0	0
	33	0	0	0	0	0
	34	1	0	0	0	1
	35	1	0	0	0	0
	36	1	0	0	0	0
	37	1	0	0	0	1
	38	0	0	0	0	0
	39	0	0	0	0	0
	40	0	0	0	0	0
	41	0	0	0	0	0
	42	1	0	0	1	0
success rate %		50	17,5	0	7,5	17,5

Appendix E: Excel-Analysis of the 0's

B-tasks

		B-opgaverne																
	Elev	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
HHX1	1	1	1	1	1	3	3	3	3	1	4	4	4	4	4	3	3	4
	2	1	1	1	1	3	1	1	1	1	4	4	1	4	4	4	4	4
	3	1	1	1	1	3	3	3	1	1	3	3	4	3	4	4	4	4
	4	1	1	1	1	1	3	1	3	1	3	3	1	3	3	3	1	1
	5	1	1	1	1	1	2	3	3	3	2	4	1	3	3	4	3	2
	6	1	1	3	2	3	3	2	4	4	4	4	4	4	4	4	4	4
	7	1	1	1	1	3	1	1	1	1	2	1	1	1	3	3	3	2
	8	1	1	1	1	3	1	3	1	1	2	3	1	3	1	3	1	2
	9	1	1	3	1	3	1	3	3	1	2	3	1	3	3	1	3	2
	10	1	1	1	1	3	1	2	1	1	1	1	2	1	1	3	3	1
	11	1	1	1	1	3	1	1	1	1	2	1	1	3	3	3	3	2
	12	1	1	3	1	3	3	3	3	1	4	3	4	4	3	3	3	3
	13	1	1	3	1	3	4	3	1	2	4	4	4	4	4	4	4	4
	14	1	1	2	1	3	2	2	1	1	1	4	4	2	3	2	2	1
	15	1	1	2	1	1	3	1	3	1	2	3	2	1	3	1	1	2
	16	1	1	2	1	3	1	3	1	1	1	1	1	1	3	1	3	2
	17	1	1	2	1	2	1	3	1	1	1	2	1	2	3	1	3	1
	18	1	1	1	1	3	2	2	3	1	1	3	2	1	4	4	4	4
	19	1	1	1	1	1	2	3	1	1	2	1	1	3	3	3	3	1
	20	1	1	1	1	3	2	3	3	1	2	3	3	1	3	3	3	2
HHX2	43	1	1	1	1	3	1	3	1	1	2	3	1	1	3	3	3	2
	44	1	1	1	1	2	1	3	1	1	2	3	1	1	3	3	4	2
	45	1	1	1	1	2	1	1	3	1	2	3	1	1	1	3	2	2
	46	1	1	3	1	3	1	3	1	1	2	4	1	1	4	1	4	4
	47	1	1	1	1	1	2	3	3	1	2	3	2	1	3	3	3	2
	48	1	1	1	1	3	1	1	1	1	2	1	1	1	1	3	3	4
	49	1	1	3	1	3	3	3	3	1	2	3	1	1	3	1	3	4
	50	1	1	1	1	1	2	3	1	1	1	1	2	3	1	3	4	4
	51	1	1	1	1	2	2	2	3	1	2	1	2	1	1	3	2	1
	52	1	1	1	1	3	1	1	1	1	2	1	1	1	1	3	3	4
	53	1	1	2	1	1	1	2	3	1	2	3	1	1	1	3	3	2
	54	1	1	1	1	3	1	3	1	1	2	1	1	3	4	4	4	4
	55	1	1	1	1	3	3	3	1	1	2	4	1	1	1	3	3	2
	56	1	1	3	1	3	3	3	3	1	2	3	1	1	3	3	3	2
	57	1	1	1	1	3	3	1	1	1	2	1	1	1	3	3	3	1
	58	1	1	3	1	3	3	2	4	1	4	4	1	1	3	1	2	4
	59	1	1	1	1	3	1	3	1	1	2	3	1	1	1	3	3	2
Antal elever som har svaret rigtigt på opgaven:		37	37	24	36	7	17	9	21	34	6	11	24	21	10	7	3	7
Antal elever som har svaret delvist rigtigt på opgaven:		0	0	5	1	4	8	7	0	1	23	1	6	2	0	1	4	16
Antal elever som har svaret forkert på opgaven:		0	0	8	0	26	11	21	14	1	2	16	1	9	19	22	21	1
Antal elever som ikke har besvaret opgaven:		0	0	0	0	0	1	0	2	1	6	9	6	5	8	7	9	13
Procentdel som har svaret rigtigt på opgaven:		100	100	64,864865	97,297297	18,918919	45,945946	24,324324	56,756757	91,891892	16,216216	29,72973	64,864865	56,756757	27,027027	18,918919	8,1081081	18,918919
Procentdel som har svaret delvist rigtigt på opgaven:		0	0	13,513514	2,7027027	10,810811	21,621622	18,918919	0	2,7027027	62,162162	2,7027027	16,216216	5,4054054	0	2,7027027	10,810811	43,243243
Procentdel som har svaret forkert på opgaven:		0	0	21,621622	0	70,27027	29,72973	56,756757	37,837838	2,7027027	5,4054054	43,243243	2,7027027	24,324324	51,351351	59,459459	56,756757	2,7027027
Procentdel som ikke har besvaret opgaven:		0	0	0	0	0	2,7027027	0	5,4054054	2,7027027	16,216216	24,324324	16,216216	13,513514	21,621622	18,918919	24,324324	35,135135

		B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11	B12	B13	B14	B15	B16	B17
STX1	21	1	1	1	1	3	2	2	1	1	2	3	2	1	1	3	3	1
	22	1	1	1	1	3	2	2	1	1	2	3	2	1	1	3	3	1
	23	1	1	1	1	1	1	1	1	1	2	3	1	1	1	3	2	2
	24	1	1	1	2	1	2	3	1	1	2	1	1	1	1	3	3	2
	25	1	1	1	1	1	1	3	1	1	1	3	1	1	1	3	3	2
	26	1	1	1	1	3	1	3	3	1	2	3	1	1	1	3	3	2
	27	1	1	1	1	2	1	3	1	1	2	3	1	1	3	4	4	4
	28	1	1	1	1	2	2	1	1	1	2	3	1	2	3	1	3	1
	29	1	1	3	1	1	4	3	1	1	2	3	1	3	4	4	4	4
	30	1	1	1	2	4	3	4	1	2	3	4	4	4	4	4	4	4
	31	1	1	2	1	2	2	3	1	1	2	3	1	2	4	4	4	4
	32	1	1	1	1	2	1	3	1	1	2	3	1	1	4	4	4	4
	33	1	1	3	3	3	1	3	1	1	2	3	1	1	4	4	4	4
	34	1	1	1	1	3	1	3	1	1	2	3	1	1	2	3	3	2
	35	1	1	1	1	3	1	3	1	1	1	4	1	4	4	3	3	2
	36	1	1	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	37	1	1	1	1	2	1	4	1	1	4	4	1	3	4	4	4	4
	38	1	1	2	1	2	2	2	1	1	2	3	2	3	1	3	3	4
	39	1	1	3	3	3	3	4	1	2	4	1	3	3	3	3	3	2
	40	1	1	1	1	1	3	1	1	1	2	1	3	1	1	4	4	4
	41	1	1	1	1	1	4	1	1	1	2	1	4	4	4	4	4	4
	42	1	1	1	1	2	3	3	3	1	2	4	1	4	4	3	4	4
STX2	60	1	1	2	1	3	4	3	1	1	4	4	1	1	4	1	4	4
	61	1	1	3	1	3	2	3	3	2	4	4	1	3	1	2	3	1
	62	1	1	3	3	3	3	3	3	1	2	2	1	3	3	3	3	2
	63	1	1	1	1	3	1	3	1	1	1	1	1	3	3	3	3	2
	64	1	1	3	1	3	3	1	3	1	2	1	1	2	3	4	4	4
	65	1	1	1	1	3	3	3	1	1	2	3	1	1	3	3	1	2
	66	1	1	1	1	3	3	3	3	3	2	3	1	4	4	4	3	2
	67	1	1	1	1	3	3	2	1	1	2	3	1	3	3	1	1	2
	68	1	1	3	1	3	1	2	4	1	4	4	1	4	4	4	4	4
	69	1	1	1	1	3	3	3	4	1	4	3	4	1	4	4	4	4
	70	1	1	1	1	3	3	3	3	1	2	3	1	1	1	3	1	2
	71	1	4	4	4	3	1	4	1	4	4	4	1	4	4	4	4	4
	72	1	1	3	1	3	1	1	1	1	2	3	1	1	1	3	1	2
	73	1	1	1	1	3	2	3	4	2	4	4	4	4	4	4	4	4
	74	1	4	4	1	4	4	3	4	4	4	4	4	4	4	4	4	4
	75	1	1	1	1	3	3	3	3	1	2	1	1	1	3	4	3	2
	76	1	1	1	1	3	1	3	1	1	2	1	1	1	3	3	3	2
	77	1	1	3	1	3	1	3	4	1	2	1	1	1	1	4	4	2
Antal elever som har svaret rigtigt på opgaven:		40	38	25	33	6	15	6	26	32	3	9	29	19	12	3	4	4
Antal elever som har svaret delvist rigtigt på opgaven:		0	0	3	2	7	8	5	0	4	26	1	3	3	1	1	1	17
Antal elever som har svaret forkert på opgaven:		0	0	10	3	24	12	24	8	1	1	19	2	8	10	17	16	0
Antal elever som ikke har besvaret opgaven:		0	2	2	2	3	5	5	6	3	10	11	6	10	17	19	19	19
Procentdel som har svaret rigtigt på opgaven:		100	95	62,5	82,5	15	37,5	15	65	80	7,5	22,5	72,5	47,5	30	7,5	10	10
Procentdel som har svaret delvist rigtigt på opgaven:		0	0	7,5	5	17,5	20	12,5	0	10	65	2,5	7,5	7,5	2,5	2,5	2,5	42,5
Procentdel som har svaret forkert på opgaven:		0	0	25	7,5	60	30	60	20	2,5	2,5	47,5	5	20	25	42,5	40	0
Procentdel som ikke har besvaret opgaven:		0	5	5	5	7,5	12,5	12,5	15	7,5	25	27,5	15	25	42,5	47,5	47,5	47,5

C-tasks

STX2	**60**	1	4	2	2	1
	61	3	3	2	2	2
	62	1	4	3	3	3
	63	1	1	2	2	2
	64	1	4	4	4	4
	65	4	4	4	2	4
	66	1	4	4	2	2
	67	1	3	2	2	3
	68	4	4	4	4	4
	69	2	4	4	2	4
	70	2	2	4	2	1
	71	3	4	4	4	2
	72	2	2	4	2	1
	73	1	4	4	2	2
	74	1	4	4	4	2
	75	2	2	4	2	3
	76	2	1	3	2	2
	77	1	4	4	2	3
STX1	**21**	1	2	3	1	3
	22	1	2	3	2	1
	23	1	1	3	1	2
	24	2	1	3	2	1
	25	1	1	3	2	2
	26	1	1	3	2	2
	27	4	4	4	4	4
	28	1	1	3	2	3
	29	4	4	4	4	4
	30	4	4	4	4	4
	31	4	4	4	4	4
	32	4	4	4	4	4
	33	3	4	4	4	4
	34	1	4	3	2	1
	35	1	4	4	2	2
	36	1	4	4	4	4
	37	1	4	4	2	1
	38	3	4	4	4	4
	39	3	4	4	2	2
	40	4	4	4	4	4
	41	4	4	4	4	4
	42	1	3	4	1	2
Antal elever som har svaret rigtigt på opgaven:		20	7	0	3	7
Antal elever som har svaret delvist rigtigt på opgaven:		6	5	4	22	13
Antal elever som har svaret forkert på opgaven:		5	3	10	1	6
Antal elever som ikke har besvaret opgaven:		9	25	26	14	14
Procentdel som har svaret rigtigt på opgaven:		50	17,5	0	7,5	17,5
Procentdel som har svaret delvist rigtigt på opgaven:		15	12,5	10	55	32,5
Procentdel som har svaret forkert på opgaven:		12,5	7,5	25	2,5	15
Procentdel som ikke har besvaret opgaven:		22,5	62,5	65	35	35

		C-opgaverne				
	Elev	**1**	**2**	**3**	**4**	**5**
HHX1	**1**	2	4	4	2	2
	2	1	4	4	4	4
	3	1	4	4	2	2
	4	2	2	3	2	1
	5	1	4	4	3	3
	6	4	4	4	4	4
	7	4	4	4	4	4
	8	2	3	4	2	1
	9	1	2	3	2	1
	10	1	1	3	2	1
	11	2	1	2	2	2
	12	1	4	2	2	1
	13	2	4	4	2	2
	14	2	2	4	3	3
	15	1	1	1	2	1
	16	1	1	4	2	2
	17	1	1	3	1	2
	18	4	4	4	4	4
	19	2	1	1	2	2
	20	3	4	2	2	1
HHX2	**43**	1	2	2	2	1
	44	2	4	4	4	4
	45	4	4	4	4	4
	46	4	4	4	4	4
	47	2	1	3	2	1
	48	2	4	4	4	4
	49	1	4	3	4	2
	50	4	4	4	4	4
	51	2	1	3	1	2
	52	1	4	4	4	4
	53	2	1	3	2	1
	54	1	1	4	2	3
	55	1	2	2	2	2
	56	3	4	3	1	1
	57	3	2	1	2	2
	58	2	1	4	1	2
	59	1	2	2	2	2
Antal elever som har svaret rigtigt på opgaven:		15	11	3	4	11
Antal elever som har svaret delvist rigtigt på opgaven:		13	7	6	20	13
Antal elever som har svaret forkert på opgaven:		3	1	9	2	3
Antal elever som ikke har besvaret opgaven:		6	18	19	11	10
Procentdel som har svaret rigtigt på opgaven:		40,54054	29,72973	8,108108	10,81081	29,72973
Procentdel som har svaret delvist rigtigt på opgaven:		35,13514	18,91892	16,21622	54,05405	35,13514
Procentdel som har svaret forkert på opgaven:		8,108108	2,702703	24,32432	5,405405	8,108108
Procentdel som ikke har besvaret opgaven:		16,21622	48,64865	51,35135	29,72973	27,02703

D-tasks

	Elev	D-opgaverne				Relational					
		1	2	3							
HHX1	1		1			1	Relational with arithmetic reasoning				:= 1
	2		1			1	Relational with arithmetic reasoning				
	3		4	4		4	comparison				:=4
	4		1			1	relational				
	5		2			4					
	6		4			2	operational and relational				:=2
	7		1			1	relational				
	8		4			3	operational				:=3
	9		4			3	operational				
	13		1			2	operational and relational				
	11		2			1	relational				
	12		1			2	relational and operational				
	13		3			3	operational				
	14		1			2	operational and relational with arithmetic reasoning				
	15		1			1	relational with arithmetic reasoning				
	16		1			1	relational				
	17		1			1	relational with arithmetic reasoning				
	18		1			1	relational with arithmetic reasoning				
	19		1			4	non-traceable				
	20		2			3	operational				
HHX2	43		1			1	relational				
	44		1			2	operational and relational				
	45		1			1	relational				
	46		1			2	operational and relational				
	47		1			4	non-traceable				
	48		1			1	relational				
	49		2			2	operational and relational				
	50		1			1	relational with arithmetic reasoning				
	51		1			1	relational with arithmetic reasoning				
	52		1			1	relational				
	53		1			2	operational and relational				
	54		1			1	relational				
	55		1			1	relational				
	56		1			4	non-traceable				
	57		2			1	relational with arithmetic reasoning				
	58		3			3	operational				
	59		1			1	relational				
Antal elever som har svaret korrekt				26							
Antal elever som har svaret delvis korrekt				5							
Antal elever som har svaret forkert				2							
Antal elever som har givet ubesvaret				4							
Procentandel elever som har svaret korrekt				70,27027							
Procentandel elever som har svaret delvis korrekt				13,513514		HHX					
Procentandel elever som har svaret forkert				5,4054054							
Procentandel elever som har givet ubesvaret				10,810811							
Antal elever som har svaret "relational"						19					
Antal elever som har svaret "relational/operational"						8					
Antal elever som har svaret "operational"						5					
Antal elever som har givet 'non-traceable'/unanswered						5					
Procentandel som har svaret "relational"						51,351351			72,9% mentions the relational view		
Procentandel som har svaret "relational/operational"						21,621622					
Procentandel som har svaret "operational"						13,513514					
Procentandel som har givet 'non-traceable'/unanswered						13,513514					

STX2	**60**	1		2	operational and relational	
	61	2		4	non-traceable	
	62	2		3	operational	
	63	2		2	operational and relational	
	64	1		1	relational	
	65	2		2	operational and relational	
	66	2		4	non-traceable	
	67	2		2	operational and relational	
	68	2		3	operational	
	69	4		3	operational	
	73	4		1	relational	
	71	2		3	operational	
	72	4		2	operational and relational	
	73	4		4	non-traceable	
	74	1		2	operational and relational	
	75	1		2	operational and relational	
	76	4		1	relational	
	77	3		3	operational	
STX1	**21**	1		1	Relational with arithmetic reasoning	
	22	1		2	operational and relational	
	23	1		2	operational and relational	
	24	1		1	relational	
	25	1		1	relational	
	26	1		1	relational	
	27	2		3	operational	
	28	1		1	relational	
	29	1		2	operational and relational	
	33	1		3	operational	
	31	1		1	relational	
	32	1		1	relational	
	33	4		3	operational	
	34	1		1	relational	
	35	1		1	relational	
	36	4		4	non-traceable	
	37	1		2	operational and relational	
	38	3		1	relational	
	39	1		3	operational	
	43	1		1	relational	
	41	1		1	relational	
	42	1		3	operational	
Antal elever som har svaret korrekt			22			
Antal elever som har svaret delvis korrekt			9			
Antal elever som har svaret forkert			2			
Antal elever som har givet ubesvaret			7			
Procentandel elever som har svaret korrekt			55			
Procentandel elever som har svaret delvis korrekt			22,5			
Procentandel elever som har svaret forkert			5			
Procentandel elever som har givet ubesvaret			17,5			
Antal elever som har svaret "relational"				15		
Antal elever som har svaret "relational/operational"				11		
Antal elever som har svaret "operational"				10		
Antal elever som har givet 'non-traceable'/unanswered				4		
Procentandel som har svaret "relational"				37,5		65% mentions the relational view
Procentandel som har svaret "relational/operational"				27,5		
Procentandel som har svaret "operational"				25		
Procentandel som har givet 'non-traceable'/unanswered				10		

Appendix F: Comments from the test

A – tasks

A1:

Symmetry:
“I look at both sides and can see that it is the same that is subtracted”
“left and right is the same, that is “=””
“for the equality to be valid I wrote 9, so that it stood the same on both sides of the equal sign”
“because 9-4 and 4-9 that is the same result.” – misconception of negatives
“I have chosen to set 9 because it only tells us what is on the one side is also on the other side”
“I look on the other side of the equal sign and find out is missing”
“\sqrt 81 \\ it is the same”
“because what says on the one side is the same as the other [side]”

Guess and try/trial and error:
“I subtracted 9-4 and found a number that -4 [subtracting 4, ed.] made it the same”

reflexivity
“that is because 9-4 is the same as 9-4” > 2

operational
“because I would think it should have the same result, and hence stand “equal” each other”
“9-4 makes 5 and therefore I write that on both sides of the equal sign”

Conditional equation
“it is 9 as it is going to give 5 on both sides of the equal sign. 9-4 makes 5 and -4 is going to have 9 for it also to be 5”
“there are no unknowns, so I calculate ‘9-4’ (on the left-hand side), and it gives 5. Then I add 5 to 4 (on the right-hand side)=9”

misconception
“9-4=5-4 \\ I calculated”

A2:

> 16 using division of three
“one could also have done it differently but I just did it this way”

Other combinations
“9+1+2”
“5+5+2” > 2 “2+5+5” > 1
“6+2+4” “4+6+2”
“3+5+4” “3+4+5” {5,4,3}
“1+1+10” “10+1+1” > 2
“3+3+6” {6,3,3}

A3:
13 : here there was not so many other options
“this is much like a normal calculation only with the answer first. so I just sat 5+8 together and got 13” – answer equals operation
“13=5+8 is the same \\ 5+8=5+8 is also the same”

misconception
“3=5+8 \\ I calculated”

A4:
misconception:
“3+5=8-2 because 3+5 makes 8”

A5:
- “both sides of the equal sign must be the same”
“I wrote 2 because 2 has the same value as 2”
“1+1 gives 2 \\ 2=2 is 2” > 1
“Have just inserted 2, because 2 is only equal to 2”
“2 is only equal to 2”

A6:
Different combinations of numbers/trial and error
8-2 = 6-0 > 2
8-3=6-1 (> 17)
8-4=6-2 (> 21)
8-6=6-4 > 6
8-7=6-5 > 0
8-10=6-8
8-2=6-0
8-8=6-6 > 4
8-10=6-8 > 3
8- -6 = 6 - -8 > 0

There is a tendency that the lowest numbers added are also the combinations which appear most often

Difference of two-argument
“8-4=6-2 \\ 8 is 2 higher than 6 so I just added the number by 2 on the left-hand side” > 2
“I inserted 7 in both places and afterwards I subtracted 2 from the right-hand side as 8 is 2 greater than 6”

“first took 6-1 since it was the easiest and calculated afterwards 8-x=5”

“I don’t think one can, it can’t give the same”

not distinguishing negative numbers from positives:
“8-6=6-8. \\ it gets repeated”
“8-3=6-11 \\ all of it I wanted to be 5 so set that number in for it to be 5 on both sides”
“8-6=6-8 \\ again, as the two calculations in principle will give the same no matter what way it goes” – symmetric argument, but rejecting the difference between plus sign and minus sign
“8-6=6-8 I guessed”

misconception
“8-2=6 and hereafter I think it does not matter what 6 gets subtracted by but [I] have inserted 2”
“8-2=6-2=4 \\ in order to obtain 6 we must minus by 2”
“8-2=6-2 \\ don’t know what I’m doing”

A7:

-24 “I added all the number son the one side of “=” and wrote it on the other” (this student answersed the other A’s correctly) > 1

A8:

conditional equation solving
-“I moved 2 on the other side and added 4 and divided by 2x with 6 and got 3”

guess and try:
-string operation, but correct result: “2*-1 = -2+4=2”

false
“if one minuses with 4 one finds x” – the student ignored the term 2 in front of the x
“if one adds 4 to -2 it gives 2”
x=1
“there is not added any more x’es, so therefore x has always been 1”

A9:

setting x=1 > 3
-“I sat x=0 so it gave 15, then I added 3 with 2 which gave 5, then I found out how many times 5 goes up in 15.” – most likely the student meant x=1 here, and then it makes sense.

$$\text{"x} = 1\backslash\backslash\backslash\backslash\backslash left(3x + 2\backslash right) = 5$$
$$9 + 6 = 15 = 3 * 5\text{"}$$

using the value of x detected in the previous task
-after deducing that x= -1 in the previous task, the student refers to this in the next one and has chosen __ = -3: “I set x=-1 in x’s place”

A10:

separating units
-“I started to add 10 cm and 2 cm since they have the same letters and that made 12, and since kr and cm is not the same I just sat a plus between them”
“kr and cm cannot be put together”
“cm and kr is not the same it is like a and b. so I just add cm together and let kr be.”

“put those of the same kind [type, ed.] together”
“you can’t put a length and a money value together”
“replaced cm and kr with a and b”
“it can be reduced, but not enough. They can’t be put together”
“I would say that one cannot mix two things that has nothing to do with each other. fx one can also not say 2a+7b and put them together that would give (2a+7b=2a+7b)”
“cm is its own unit together with kr”
“I see that it will only see it in whole units and not mixed together”
“There are different terms. cm+cm is okay, but cm+kr is not okay”

“12 cm \\ Uhm?”

false:
“12=3x”
“because 10cm+2cm+3cm gives 15” – maybe (s)he didn’t look at it correct
“18\\ a 2-kr and a 1-kr put next to each other measure 6 cm”
“15cm \\ just plussed the numbers and wrote it in cm”

A11:

-wrong answer, but correct thinking:
“when a is the same as y and then I could see that it was going to be 10y”
“I know that a=y so I added y with 10” – this student answered A10 correctly, separating the two units, but does not argue the same way when there is a unit vs a solo number
“a+10 = a= 10” – the same student declared in A10 that __=12=3x, separating the two different units from each other from both sides, but not changing the sign.

transitivity argument:
“think that when a=y and y=22 that a also is 22” > 1
“a=y
y=22
a=22
22+10=32”

confusing x with y:

1. “I compute x=25 and then must have a=x\\ a+10=35 \\ 25+10=35”

2. “$5x - 3 = 22$

$5x = 25\ (+3)$
$x = 5$

One sets it up as an equation and the result of the equation is 5 – so <u>a=5 \\</u>
a+10<u>=15”</u>

3. S33: setting x=1, then calculation the value of the line, adding 22, making a mistake and saying that it’s 34 instead of 24. Then lastly concluding that a+10 → 34+10=44
4. “so I calculate *a* in order to set it in the calculations and calculate it”

5. “a+10=32\\

22=5x-3 find x x=5 → afterwards one plusses 22+10= result /32

$$5 * 5 - 3 = 22 \Leftrightarrow y = 22 \quad a = 22 \Leftrightarrow 22 + 10 = 32$$

6. "$a = 22 = 5x - 3$
   $5 * 5 = 25 - 3 = 22$
   a=22=5*5-3
   a+10=32
   22+10=32"
7. "a+10=15/ because when one multiplies 5*5 one obtains (25)-3 which then gives 22"
8. [the student started out stating that $5x - 3 + 10 = 22$ $(y + 10 = 22 = y \Leftrightarrow 0 = 10)$]
   "5x-3+10=22
   5x.3+3+10-10=22+3-10
   5x=15 x=3

   didn't quite understand it"
9. [the student starts out stating that $a = 5x - 3 + 22 \Longrightarrow a = y + y = 2y = 44$]
   "5x-3+22. y=5x-3 and 22 (+) so
   a=5x-3+22+10=5x+29 or
   5x-3 +3 = 22 +3
   $\frac{5x}{5} = \frac{25}{5} = [\Leftrightarrow, ed.]\ x = 5$
   $a + 10 = 5x + \frac{29}{x} = 5$"

misconception
22=2x
a+10=12x
[the student has most likely said: "a=y=5x-3; 5x-3=2x, then y=22=2x, therefore a+10=y+10=2x+10=12x", thus the property of transitivity is accepted by the student. However there is a misconception regarding what terms can be added together. This is also seen in B10 where S62 declares the result is 15]

B – tasks

B1:

"true because it is not a calculation but simply just a statement"
"7 apples are just as many as 7 pears"
"because 7 computed with nothing is just 7"

reflexivity
"there is equilibrium in the expression as 7 is the same value"
"7 and 7 is the same"
"7 has the value 7 therefore it is correct"

“If you have 7 apples then you also can eat 7 apples”
symmetry: “because both sides of an equal sign is the same”

B2

> 18: everything makes 5
– same arithmetical argument comparing numerical values → same goes for 4
“all the sides are equal”

B3

– string operation:

false:
“(…) so the equal sign is not equal” [så lighedstegnet er ikke lige]
“12 \neq 15”
“if it was written differently it would be correct” #explainingbutnotexplaining
“12=15=15”
“as only two out of three terms give the same value”
“7 is 2 greater than 5 so it becomes 12 and 12+3=15”
“there is not equilibrium”
“because I would assume that a number cannot be = with another number if we don’t get any other numbers stated”
“it says that 7+5 gives the same as 12+3 \\ 7+5=12 \\ 12+3=15”
“7+5 does not have the same value as 12+3”
true:

1. “both the first part gives the correct result and the second part does as well”
2. “adds up”
3. “it is a correct calculation”
4. “7+5 makes 12 and then +3 makes 15” > 1
5. “7 and 5 give after all 12, but because it says 12+3 it could also be 15, which it is equal to”
6. “I have calculated that 7+5=12+3 gives 15 so therefore it is true”
7. “they both give the same as the number says = afterwards”
8. “7+5=12+3= 15
9. 7+5=12
   12+3=15
   hence true”

B4:

S6: B “first part with 2+9=2 is not correct, but if one splits them up as: 2+9 and 2+4+5, it gives the same result which is 11”
correct answer but wrong explanation (string operation) “11=5+4=9+2”
“as 9+2 never can give 2, and so 4+5 never can give 9”
“mistake in the calculation as 2+9 does not give 2.”

B5

– percentage:

True:

-“% means 100. Which makes that when you multiply 0,20*100 it makes 20%”
-“0,20 and 20% is the same. Therefore true”
-“0,20*100=20 if one multiplies 0,2 with something one obtains 20%”
-“it is just 20% rewritten to decimal”
-“because if you multiply with a comma [decimal, ed.] number you get %”
-“0,20 one writes one per cent and one always multiplies percent with 100 so it gives 20%”
-“because if one multiplies 100 with 0,2 it gives 20. and therefore it is the same as 20%”
-“I saw that we were to multiply by 100, so there I moved the comma 2 times to the right”
-“because you move the comma twice” > 3
-“% means ‘out of 100’ there these two statements are equal each other when one converts to %”
-“0,20 is 20% written as a decimal, if one multiplies it by 100 one gets it in percent”
-“$0{,}20 * 100 = 20{,}0$
$20{,}0 = 20\%$
20% is the same as 0,20*100”
-“if you multiply 0,20*100 then it gives 20%. Therefore it is true.”
“0,20*100=20
20=20%”
“yes because 0,20*100 makes 20% \\ when one multiplies decimal numbers by 100 one typically finds %”

the argument is inconsistent with the answer T
-“0,20*100=20”
-“0,20 100 times becomes 20”
-“as 0,20*100=20 so it is correct”
-“one can say 20% of 100=0,20”
“0,20*100 is 20 and it can also be in %”
“0,20 is the same as 20%, as 1,00 is 100%. By multiplying by 100 one rewrites decimal numbers to percent”
“0,2*100=20%, which is true, as 0,2*100 gives 20% \\ 20%=2%”
“0,2*100=20 therefore it is 20%”
“20%=0,2”

False:
“doesn’t have to be percent”
“100*0,2 is 20 and not 0,2 that is 20%”
“that is how I would calculate %, but the task says nothing about % \\ 0,2*100=20”
“20 is 2000%”
“we have not been told that we’re converting into percent”
“0,20 is 20%. so 20,0 is 200%”
“1,00=100%
0,20=20%
0,20*100=2000%”
“0,20=20%”

t/f:
“it gives 20, if one wants it to be % one can [do, ed.] that” percent
“it is a number 20. It is also 20% of 100”

"if you have a number * with 0,20 you will obtain 20% of this number \\ the calculation is incorrect it would give 20 but it is 20% of 100"

B6

– defining x=4cm:

true
"true since x can be anything"
"x is a variable it can be just anything it wants"
"x=everything" #meta
"x can very well be =4. if that is the value set"
"because x kan be everything it is just a deputy [substitute, ed.]"
"since there is no more information it is correct \\ an ascertainment [konstatering, ed.]"
"x is unknown and can be everything. in this case known as 4"
"in an occasion this could be true"
"it depends on the context. x can definitely be 4 cm"
"suppose one can use it no matter what side x is sitting [placed, ed]. it could also be named: x=4 cm"
"x is 4cm"
"x=4cm \\ 4cm=4cm"
"i.e. that x corresponds to 4 cm fx. if one [has] a calculation that was \\ 8cm-x, and knew that x was 4cm."
"if x is 4cm which we don't know"
"4cm can very well be x because x is variable and can be everything"
"x is unknown, so hence it actually says 4cm=4cm"
"x=4cm it is the value of x"
"I don't really get to know anything else, so x must be equal to 4cm"
"because x can be everything it is not wrong"
"if it says: $\frac{2x}{2} = \frac{8cm}{2} = [\Leftrightarrow instead\ of\ =]$ x = 4cm \\ that's how I thought" – string operation
"4cm could be termed x in a task"
"because x is always x it may very well say a number"
"it is just true"
"if one says 1+3cm or 2+2cm it is true enough"

t/f:
"because x does not have a value so one decides oneself what value it should have"
"it [is, ed.] just an ascertainment"
"because one can write it as unknown (x) or as a number (4 cm)."
"it depends on what x is"
"well 4cm may well be =x in an equation but 4cm is not equal x outside of context"
"as we don't know the value of x we can neither confirm or decline whether 4cm=x or not"
"x is just an unknown. x may well be 4cm, but it is not always correct"

/: "there is nothing that explains what x is"

the student mixes units (cm, kr, kg) with unknowns (a, x, \theta, banana,)
"false. cm and x are two different"

“4 cm cannot be the same as x”
“4cm is not = x”

false:
“x can vary after situation. in one situation it may well be 4cm, whereas in another [it] is a different number”
“4 cm is not unknown”
inconsistent argument "4cm=4cm"
“not true because we don’t have a unit behind x”
“does not make sense in my head”
“false because it is not supposed to say cm.”
“can be different numbers, not necessarily 4” – acknowledging that x is a variable, but not accepting that this can be true is not contradicting the fact that it also can be other things (at the same time)
“x is not 4”
“it does not have a context”
“because x can be all numbers, not necessarily 4”

B7

– travel card

assuming rejsekort is 80kr > 10
– this makes the student make some assumption for herself. They do not state the problem with units (a number + a unit with a number)
“ because rejsekort is 80”
“if rejsekort costs 80kr it is true. then I add.”
“depends on the price of the rejsekort”
“a rejsekort costs 80kr. That fits well with 200kr”
“but only when it is an adult rejsekort youth rejsekort costs 50kr”
“it doesn’t say anything about the effect of the rejsekort so it is just numbers we can follow”
“80kr=1 rejsekort”
“this is probably true, 1 rejsekort costs 80kr?”
“not sure but would say true as the rejsekort only is =x”
“120kr we don’t know what is, but since it says +80, 80 must be 1 rejsekort. 120+80 makes 200kr”

arithmetic argument,
“120+80=200” – > 8: not considering the assumptions, simply calculating

False
don’t know the price
“I use only my bike (with basket) so don’t really know what a rejsekort costs, therefore it is not necessarily correct that a rejsekort costs 80kr”
different units
“it does not say anywhere that rejsekort is 80 it also doesn’t say that it’s kr. if that is ok then it is correct”
“if 1 rejsekort corresponds to 80 kr. but it doesn’t say kr behind. so it must be false”
“there are three different units, so it cannot give 200kr” > 3
“we don’t know what a rejsekort costs”

“1 rejsekort can never become kr”
“we don’t know how much many kroner [Danish Currency] a rejsekort [costs]”

B8:

Arithmetic argument/comparing numbers:
“if it was the same it would have said \sqrt 4 instead of 4/\sqrt 4”
“\sqrt 4 = 4^2”
“\frac{4}{\sqrt{4}} gives more than \sqrt4”
“it. says 2=1 and that is not correct”
““\frac{4}{\sqrt{4}} becomes far less than \sqrt 4”
“16=4/16 is not the same” – relational argument

Algebraic statement
“\sqrt4=2
4/2=2
It is the same if you write \sqrt{x} or \frac{x}{\sqrt {x}}”

false:
a student declared \sqrt4=2 but then said that 4 divided by 2 makes 0,5. > 1

B9

“it is the correct formula” > 1
“since l and b are placeholders”
t/f: “they [the length *l* and the widt *b*] have probably a different symbol, but one may very well use them”
“just facts”
“prior knowledge”
B: “we don’t know what it leads to”

B10:

Pythagoras theorem?

Referring to a formula () > 14
“Pythagoras thm only the other way around” – first of all, it’s the reference to thm whose assumptions are not included in the task. Second of all, he or she declares that it is “turned around”
“opposite pythagoras”
“it is Pythagoras, so of course it can be natural numbers”
“a, b and c are placeholders and hence the expression can be true”
“it is the Pythagoras theorem. c^2 has the same value as a^2 and b^2 when one computes with right-angled triangles”
“suppose it is the normal/natural numbers those that include or actually are Pythagoras theorem”
“it is used to find a length of a side in a triangle” – not emphasizing right-angled triangle
“Pythagoras theorem. There will always be equilibrium on both sides of the sign”
“made a drawing of a right-angled triangle and stating the letters” – constructing assumptions which are not mentioned in the task
“$a^2 + b^2 = c^2$
this is how you find an unknown side in a right-angled triangle

$5^2 + 5^2 \ = \ 50 \ \ \sqrt{} \ 50 = c^2$ ”
“the formula is used for right-angled triangles a^2+b^2=c^2”
“ [showing a drawing of the squares specified]the square of a+ the square of b = the square of c”

Existence argument

1) (S)

$$c^2 = 5^2 = 25$$
$$a^2 = 4^2 = 16$$
$$b^2 = 3^2 = 9$$
$$25 = 16 + 9 \quad 16 + 9 = 25$$

2) equal to the one above
3) (F) $c^2 = a^2 + b^2$ $\quad$ $5^2 = 1^2 + 2^2$
   a=1
   b=1 $\quad$ $25 = 1 + 4 = 5$
   c=5
   1*1 =1
   2*2=4
   1+4=5
4) one finds an unknown by finding the two others
5) if c^2 is 4 and $a^2 + b^2$ is 2 then it will be the same

t/f:
“it depends on what one is going to use it for”
“they are not always natural numbers, it can also be decimal numbers”
“It will be true for specific numbers, but not other”
“one can write both numbers and letters” - misconception

false:
“the equation only works on specific triangles “
“it doesn’t add up. if a, b and c are whole numbers”
“not all numbers can give a proper answer to c^2.
a^2=2 b^2 = 3
a^2=2*2=4 b^2=3*3=9
There is not a whole number that is multiplied by itself that becomes 4+9=13”
“there is not included c on both sides” – relational argument
“it can change depending on the length of the triangle”

B11

– the binomial formula:

2ab should not be there/not distinguishing between parentheses >12

-“2ab is not natural [number, ed.]”
-“I would say that it should be $2\text{ab} + (a+b)^2 = 2\text{ab}(a+b)^2$ if it was correct”
-“not the same. Would never be able to create 2ab on the right-hand side hence it would not be the same value. It says after all $a^2 + b^2$ on both sides only expressed differently” – mixing notation between $(a+b)^2$ with parenthesis and $a^2 + b^2$
$-$“$(a+b)^2$ gives $a^2 + b^2$ which is not $2\text{ab} + a^2 + b^2$”
-“2 times a and b is not the same as a*a and b*b”
-“we’re missing the firs term, that is 2ab. $(a+b)^2 = \text{a} * a + \text{b} * b$”
-“those 2ab have disappeared”
-“there is not equilibrium, 2ab is missing on the right side and a^2+b^2 is not the same as (a+b)^2” – contradictory argument, as (s)he first announces that the 2ab term is missing, but at the same time that the remaining terms are not equal – which indeed is true.
-“there is missing 2ab on the right-hand side”
-“if “2ab” not was included it could have been correct. I think…”
-“$a^2 + b^2 = (a+b)^2$
2ab makes it unequal because the value becomes too great/big/large”
-“2ab is 2ab where $a^2 = \text{a} * a$ so it is false as $(a+b)^2 = a^2 + b^2$”
“it is (a+b)*(a+b) that gives 2ab+a^2+b^2”

referring to the square formula
“there Pythagoras theorem is not included in the same way, as there is neither a use of c^2”

False calculation
“if a was 2 and b was 3 the terms would not give the same”

$$2 * 2 * 3 + 2^2 + 3^2 = (a+b)^2$$
$$12 + 4 + 27 = (2+3)^2$$
$$43 = 5^2$$

algebraic
-“if one computes $(a+b)^2$ it gives the same”
-correct conclusion, but wrong explanation:

$$2\text{ab} + a^2 + b^2 = (a+b)^2 = \text{a}b^2 + a^2 + b^2$$

-it is true as we multiply the parenthesis
$(a+b)^2 \ = (a+b) * (a+b) = a^2 + \text{ab} + b^2 + \text{ab} = 2\text{ab} + a^2 + b^2$ > 1
“multiply in ( ) [the parenthesis]” > 1 [STX2]

existence
-“2*(2*2)+2^2+2^2=(2+2)^2=8 \\ = 8+4+4=(4)^2 \\ = 16=16”
-“ja:
a is 3
b is 4

2*7+9+16=7*7
49=49
can’t explain why?”

-“if a=1 and b =2
2*(1*2)+1+4=9
(1+2)^2=9”

B12

– defining f(x):
notion of a function > 9:
“f(x)=ax+b// a function must have ax and b// a= b=+2”
“linear functions”
graphical representation (S4)
“function” > 1 said this
“F(x) can also vary”
“that is how a function can look like yes”
“it is a linear function”
“linear formula”
“it is F(x)=ax+b with numbers inserted. the results on each side are therefore equal”
“the formula of a linear line”
“slope of 3 \\ intersects in b on the y-axis”
“it is just a rule. 3x=a \\2a=b”
“it is a completely ordinary function rule”

notion of an unknown:
-“F(x) is unknown so it may well fit”
“an equation”
“F(x) gives the value in a rule and as x is unknown the equation is true”
“it is a rule, which means that f(x)=3x+2 → the value”
“it’s an equation”
“linear equation”
Specification
“F(x) is specified 3x+2”
“yes, it is after all just a rule that is specified and can be computed”
“F(x)=3x+2 is just a rule”
“it is a random equation”

Balance
“equivalent value [ligeværdi]”
“one may well write it as an equation.”

existence
f(x)=3x+2
x=4
3*4+2
12+2=14
f(4)=3*4+2=14

both/neither:
“if one assumes x and f(x) is placeholder”
“we don’t get other information so we don’t know whether it is correct”

“it is a linear rule for a linear function but it is neither correct nor incorrect”
“it can be correct”
“it can’t be correct or incorrect”

false:
“not anything capital F it needs to be little”
“depends on function”

B13

– multiplying an equation by two:
False: misunderstanding the question
S4: “not at all. all the values in the lower [second, ed.] equation is the double” – confusing the question. But correctly stating that it is multiplied by 2. > 3
“they are not equal” –
“the equation 4x+6=2x+14 is twice as great as the equation 2x+3=x+7”
“there is by no means equilibrium between the two equations and therefore the expression is false”
“the numbers have been enlarged”
“the equation has the same construction but does not give the same result as all numbers here have the doubled value (however one could also argue for that it is the same equation but different places in the calculation)”
“the other [second equation] is way too big”
“x=4 \\ 2x=8 \\ it is not the same. I calculated both of the equations.”

True:
S7: “it gives the same because the change is on the whole of it” > 0
“[*showing that both solutions are the same* > 8] the equation itself is after all not the same, but the result is the same”
“Multiplying first equation with 2” > 9
“each term is just multiplied by two in the lower [second equation], and that has no change on x”

t/f:
“it is not the same equation, but you end up with the same result”
“it is not the same equation but one can use the same numbers on them both 4”
“nr 2 is twice as big as no 1 but x will always be the same”

B14:

false
S4: “nope. there are 2 x’es in the upper [first, ed.] equation and much higher values”
“false because one of them will be negative”
“doesn’t seem right since one cannot do the same things on both sides”
“you can’t do the same on both sides”
“not the same amount of x’es in each equation”
“not the same equation because completely different numbers are included”
“-11=2x\\ -2x=11 \\ I calculated it. It did not give the same”

True.

“x=-2/11 and x=-2/11” > 2
“the equation has the same result, as we only are further ahead in the calculation”

B15:
false:
“the student began to multiply, where one was to divide”
“x ends up with the correct value but = is not satisfied in line 2, 3 and 4”
“8x+2=18 and 8x+2-2-18 never can give the same” > 1
false-false: “the student has put 8x and 2 together. They are not the same unit, one cannot [do, ed.] that”
“because he inserts -2 for no reason I would say that it is incorrect!”
“correct in the beginning but then it goes wrong”
“8x=18-2
8x/8 \\ that [is] incorrect, because 18-2=16 and not 8”
“strange calculation”

true:
“because he has computed it correctly”
“the calculation does everything on both sides”
“the calculation is correct \\ 8*2=16”
“$8x + 2 = 18$
$8x = 16$
$\frac{8x}{8} = \frac{16}{8}$
$x = 2$
it is correct would’ve just written it down a little differently” > 1
“it is a correct way of calculating it”
“it is s [true] but the calculation is confusing as the student does not do the same in each term”
“8x+2=18
8x=16
x=2”
“all the calculations give the result if x=8” – inconsistent argument

B16:
false
“f(2) gets the correct value but = is not satisfied”
“f(2) is 3 \\ in the equation the students missed. a -5 after 4*2”
“she resolves x in that calculation”

true:
“as f(2) is equal to the result. we hold on to the previous equal sign”
“$f(2) = 4x - 5$
$f(2) = 4 * 2 - 5$
$f(2) = 8 - 5$
$f(2) = 3$ “ > 2
“the student has computed correctly”

t/f: “the calculation is incorrect, but the answer is correct”

B17:

B V t/f:
“it might as well be true but it might as well be false”
“it depends on what one inserts. but if one inserts 2 and 10 it fits”
“can be correct, e.g.
x=1 y=5 or
x=3 y=15”
“it can partly add up but only when y adds up in the 5-table”

true
- “y need only be 5 times as great like 5 and 25” – stating there is a restriction, but declaring it as true. Can also be because one didn’t understand what the task asked for the student to proclaim.
“5 is the natural number”
“y is a variable it can be all things”
“x and y is placeholder and one can well make it be true”
“correct y can be anything”
“y and x is unknown, so it [the equation, ed.] is correct”
“it may well be. i.e. if y=10 and x=2”
“x can be everything thus 5*x may well constitute y”
-“5x=y
x=5
5*5=25
y=25”
“y is going to be 5 times as big as x
5*5=25”
“x and y is not the same so it can absolutely be correct”
“5x=y
x+x+x+x+x=y
5 times smaller than y”
“as you can decide the numbers yourself”
“because x and y [are] regular numbers so If one inserts the correct numbers it is correct”
“one may well insert numbers.”
“y is just a term for other numbers if one can put it that way”

false
“5 times a number does not give the number one multiplies by”
“5x=x and not y”
“x and y are not the same” – valid argument as it implies that they need to have a one-to-one correspondence in order to make the mapping bijective
“as it does not make any sense”
“as x y can have different numbers”
“y depends on what x is” – notion of a (linear) function

C – tasks

C1:

false/string operation:

a. 1200+30=1230-10=1220*2=2440 (> 2)

b.

$$(1200 + 30 - 10) * 2$$
$$1230 - 10 = 1220 * 2$$
$$\underline{1440}$$

What we see here is partly a string operation, but not 100%. This might imply that the student is aware of the meaning of the equal sign, but still uses it as way of noting what computation (s)he has been doing

c. 1200+30=1230-10
1220*2=2440 >1

d. 1200
30
1230 -10
10
1220 * 2 = 2440

Student 3 answered C1 correct not using string units, but these appeared in C5.

C2:

All values: > 12
"all values 8x-x gives 7x so it says 7x=7x"
"~~all~~ even numbers"
"true for 1,2,3,4,…"
"if x=7 it says 49=56-7 \\ x can be all natural numbers"
"that you can do with all numbers right?"
"all values. it says 7x=7x. x Is the same value for all x" – reflexivity
"in these cases I typically tend to guessing game [trial and error] and use my sense \\ but all cases. since x is the same value"
"1,2,3,4,5 \\ I result [conclude] that it is all natural numbers"
"$x = [-\infty; \infty]$"
"7x=7x
x=x"
"all positive, real numbers?"
"don't understand but believe all x-values"

Existence (0,5 pts): > 7
"7=8-1// 1"
"2"
"1"
"x=2
7*2=14 and 8*2-2=14"

“x=3 as 7*3=21 \\ 8*3-3=21” > 1
“all but 1”
“x=1”

non-traceable
-“7x=8x-x
7x=7x”
-“8x would like to isolate x”
-“uhm…the value is that we minus by 1x”
“there is multiplication sign between [x and the number] og one can set – in front of x”
“Idk [I don’t know]? 14x?”

C3
–division by zero:

correct answer
“if it is true once it will always be, if you don’t change on something”
“no, not as long as one divides”
“no because there are other numbers that you cannot use”
incorrect answer
1. “yes because one can remove (x-2)” > 3
2. “I think I can equalize so: [showing calculation] Thus I think it is always true” – equalizing: > 1
3. “yes, one can probably solve all equations”
4. “it depends on whether x is negative or positive”
5. [inserts for x=3 and computes the following:]
“(x-2)=-1
(x+2)=3
-1*3=-3
x-2=-1

-3/-1=3

x+2=3

it is not always true, as x can amend” – bland argument.

6. “$(x-2)*(x+2)$ $\frac{x^2-4}{x-2} = \mathrm{x}+2$
$x^2 + 2\mathrm{x} - 2\mathrm{x} - 4 \iff$
$x^2 - 4$
Always true”
7. S19: showing what happens when inserting x=2
8. “$\frac{(4-2)*(4+2)}{4-2} = 4 + 2$ yes”

C5:

arithmetical technique
"120-30=90/2=45kr"
"120
 -30
90/2 = 45"

-many students with arithmetic solution > 6 – note down if there is anyone with algebraic. (check the ones with full score?)

Algebraic solution: > 4
2x+30=120
2x+30-30=120-30
2x=90
2x/2=90/2 = 45

Student pattern:

1) in one of the questionnaires, the student answered B5, B16 as correct, but B3 as wrong. He or she rejects pure string operations, but it seems like the case is different when one operates with units as well. In C1, C4 and C5, we see string operations and declaring the area without a unit on. (Student 1)

D – tasks:

D1:

operational
-S6: "as a result"
"a break in a calculation that splits up the calculations"
"that it is a sign that asks for a conclusion of the calculation"
"that it is the sum of something that one is going to put something together to get the sum of something"
"a solution to a calculation"
"that = stands there to separate between the calculation and the result"
"when one is going to compare something or get the final answer"
"that one can compute how to understand the task, that one can rarely without =, as they can be put different places in the task and give different answers"
"that the sign symbolizes an "equality" on both sides of the sign. Or it can symbolize a result. Depends on what context in which it appears"
"that is what shows what something gives or should give"
"I understand it as the last sign before the final result"
"either both sides are going to give the same, or that the answer is…"
"it is a sign that shows result on the right-hand side or equation"
"equilibrium, result"
"the way to a result or when something is/is going to be the same"
"a result, something it can become"
"when one has a result, or when something is to be compared"
"it means what the answer gives"

non-traceable: > 1
-can occur when the student is not good in formulating him or herself
“as a comparison” – using the notion of a sort of relation.
“equals (er lig med)”
“for me it gives me an explanation to why something is as it is, but it can also be a help when one is to compute equations”
“that something is equal with the other \\ that one is in the process of a calculation”
“it is the sign that decides whether a result is correct”
“that the one side = the other”
“that the one side = the other |=) does not matter”

operational and relational → relational
-“both numbers by an equal sign is the same” (Relational with arithmetic reasoning)
-S5: “every possible sum and the same as”
“= can both mean that one is to solve a calculation. But it can also mean that the value on the two sides are equal”
“one can understand it as the value is the same on both sides of the sign, which makes placeholders possible. However one can also understand it exactly as so that placeholders not are legal”
“that something is just as big as something else”
“that something is the same on both sides \\ that something is to give this”
“= is the same as, ‘is’ or ‘gives’ ”
“it is used when something has the same value on both sides of it. something can look different, but one can use computation rules”
“it can be used to show what the calculation gives but it can also be used to show that 2 things give the same”
“the same as and equal to”
“I feel that it can be understood in 2 ways \\it is that it is an equality or if one ends ones calculation”
“the sign means that e.g. two numbers have the same value, and that there is equal distribution of values on both sides”
“as a result or maybe a mirror in equations as both sides give the same”
“it simply means that it’s going to be the same amount or letters/things/words etc on both sides”

pure relational
S6: “as in that it is the same on both sides of the equal sign”
“it means the same as but one can also see it as a weight that always is to be straight [lige]”
-“that there should be just as much on both sides” > 1
“the same as”
“one can understand it as a weight where both sides are going to be equally heavy \\ both sides is going look alike”
“that the numbers or equations on each side of the ‘=’-sign are the same”
“that the values or statements have the same value or meaning even if they stand on their own side”
“as “has/have the same valuea as” \\ what is on each side is that can that is be replaced with each other”
“that it gives the same on both sides fx 5+2=5+2”
“it is like the same thing, it means the same or is just as much”
“when something is the same \\ so it says exactly the same on both sides”

“when two or more things/numbers are the same”
“that it should be the same on both sides”
“the same”
“= is the same as, which means that it is going to give the same on both sides of the “=”sign.”
“the equal sign is just like a weight, there must be made equilibrium”
“the quantity on each side is the same, when one calculates it”

D2:
-comparing numbers (> 12).
“4=4 because it is the same number, f(8=16) because it is not the same number”
“0=0, because 0 is 0 \\ 0=1, because 0 is not 1”
“can 2=2 \\ cannot 2=3 \\ in examples, where the values or the statements have the same value or meaning”
“a=a it is the same \\ 5=2 it Is not the same”
“2=2 if I have 2 apples and you have 2 apples then we have exactly equal amount \\ 2=3 if I have 2 apples and you have 3 apples then we do not have equal amount and therefore not the same”
“1=1 it is the same \\ 1=2 it is not the same”
“when one is to add amounts together: purchases/bills etc.”

comparing two equations:
“2+2=1+3, here one can use it because the value is the same\\ 1+3=4+2, here one cannot, because the values are different”
-“fx between an equation and the result”
“7+3=10 one can use it because it describes what the calculation gives”
“5+3=8 but if you make a calculation one must use if and only if \Longleftrightarrow \\ 2+3+5 5+2++3 \Longleftrightarrow 5+5 = 10.”
“20kr=10kr+10kr but not 20=10
-“4+1=5”
-“14=7*2. Here one can, as it is the same value \\ 20=17/2 v. It does not end up with the same value x ”
“2+2=4 because 2+2 makes 4 \\ 2*3=4 because it does not give 4”
“7+7=14 v \\ =7+7=14%”
“2+1=3 here there is the same value on both sides of the equal sign \\ 3*2=3^2 here the power states that 3 is to be multiplied with itself: 3*3, which is not the same as 3*2 ”
“2+3=6-1, may well be used as it is the same value on both sides \\ 12=5+5, cannot be used as there are different values”
-“2+2=4 due to 2+2 gives 4 \\ 2+2=5 due to 2+2 not = 5”
“4+4 =8 \\ 2+5=+8+9+7 \\ you use it as a completion or to link two things to each other not just for fun”
“one can use it like this: 9+10=19 \\ one cannot use it like this: 9-10=19”
“2+2=4 \\ 2+2=10 \\ one of them is correct that is equal each other whereas the other one does not make sense”
“In an equation, that can be solved, = can be used e.g. 2x=4 where x=2 \\ not in 3x=4 where x=2, then it becomes 3x ≠ 4”
“5+3=8 there is equilibrium and it might as well have said 8=8 \\ 11=10+5 there is not equilibrium and it really says that 11 is the same as 15”

"1+2=2+1 – give the same \\ 5+1=10+10 – both sides do not give the same \\ [example not to use the equal sign]I don't think I can think of that"
"1+1=2 \\ f(x)=ax+b \\ l*b=areal" – operational, specification and contextual identity
"x*2=0"
"5+5=10 is an example on how one can use =. But if one writes 5+5=12 then here one has used it incorrectly"

both comparing numbers and equations
-"2*5=10 5=5 \\ 2*5=25 5=7"
-"7=3 \\ 7+3=10"
-"fx 5+2=5+2   7=7 because it gives the same on both sides \\ fx $5 \neq 7.\ 10 + 2 \neq 5$ becauuse it is not the same on both sides"
-"2%=0,02 means that 2% is the same as 0,02 \\ one can fx not use = when the result is incorrect like 5=7+9"
-"2+2=4 S \\ 6=50 F"

"2+4 = 6	5+3 = 9
6=6	8= 9
the values on both sides are equal	the values are not equal, and the sig can therefore not be used"

-"4+2=2+4 = it becomes the same \\ 4+2=6 when it does not become the same instead one could write 4+2 > 6" – inconsistent argument
-"5+2=7 the result \\ x=2=4 cannot be written 2 times"

misconception:
-"5+5=10 because it is the sum of 5+5 // 7=7 it is not the sum of something it just to write the same when it is the same number" – same student responded correct in B1: "it gives an equal number on both sides"
-"one can use it by fx. 2+2=4 because there the calculation gives a result \\ and fx one cannot use it by 2=2=2=4 because there should be used the correct designations"
"5+3=8 → shows a result v \\ 8=3 → shows neither a result nor any equality values"
"I=Sara (I am Sara and therefore one can say that I and Sara is the same) \\ I = Mona (I am not Mona, and therefore one cannot use the sign here)"
"5+3= \\ because my calculation is done and I want [an] answer"

distinguishing mathematical and non-mathematical language
"2=2 is true \\ 2=banana is incorrect"

alt:
"a solution to something \\ you cannot use it as a plus sign"
"one can use = when one finds a result, but one cannot use it eg. when one goes from stop to step in an equation where one is to use \Longleftrightarrow"
"when one finishes an equation e.g. then one writes x= and one cannot use = if one is not sure of the result then one can use ~ which means approximately"
"one can use it when one is going to make something add up \\ impossible to compare something unspecific"
"in equation and algebra, don't know when one cannot use it"

“for result or equation. cannot stand alone”
“= when one has a calculation \\ = cannot be used by incomprehensible text” > 1 [STX2]
“in an equation \\ in a fraction”

D3:

- do have in mind that the task is questioned graphically, and therefore it is highly likely that a. geometric argument is to be expected from several participants

arithmetic argument
-[altså] they have the same length…
-“yes because they are the same length” > 1

geometric/graphic
1.“they have the same hældning, but if you extend it they won’t intersect the same place so they are not completely equal each other”
2. “no they are not evenly close there is some difference”
3. “not equal to each other// from the axis and out x=1,5 on h axis, a x=2 on p”
4. “no they do not have the same intersection on the y-axis”
5. “no because they cannot be connected with a straight line”
6. “no – they have the same slope coefficient but they intersect they y-axis two different places”
7. “no, since they don’t both have the same slope coefficient and intersection”
8. “no, because the line is not the same. It is two different lines”
9. “no because they don’t intersect the second-axis [y-akis, ed.]”
10. “they have the same slope but not the same intersection \\ therefore they are not equal each other”
11. “no because they are not parallel” – contradictive statement. Does this imply that the student would accept soon as (s)he acknowledged that they in fact are parallel?
12. “no? they seem to have different slope coefficients and the lines lay different places and have to different intersection points in the y-axis”
13. “yes because they cover the same amount of space and they are parallel with each other”
14.” they are not connected but they are parallel and they take up just as much space so they are equal each other” ~ 13.
15. “they probably are not as they are not parallel and do not stop on the same line”
16. “they probably have the same length but they are not parallel so therefore I would say they are not equal each other”
17. “of course, they are parallel to each other”
18. “they are not \\ they don’t have the same intersection but same slope”
19. “the slope on h and p is the same, but the intersection on the y-axis is not the same. therefore p and h are not the same”
20.”no it is not the same line in the coordinate system they will never hit each other”
21. “they are not equal each other, as they are parallel, and no intersection have”
22. “no, although they have the same slope coefficient, but not the same values, so they are not equal each other”
23. “they are immediately [at first sight] equal in length and have the same slope, so I suppose so.”
24. “they don’t intersect the same place on the y-axis, and does not have the same slope so no”
25. “no they don’t have the same intersection on the y-axis”
26. “no since their slope not is the same” – inconsistent argument

27 “no, as they don’t start in the same point”
28. “they are not equal each other. The reason for them not being that is because, that both of them will intersect the y-axis through the y-axis in different points”
29. “no because the coordinates do not give the same”
30. “no, they lay each their own place”
31. “no because if one was to finish the line, it would not touch each other in the middle.”
32. “they are because they are parallel”

algebraic
1. “they do not have the same b value so they are not equal to each other”
2. “no, they have the same a (hældning) but not the same b (skæring) and they both have different points where they stop”
3.“yes, the slope coefficient is the same, they just start different places”
4.“the b-value is not identical and therefore they are no = each other even if the a-value is the same \\ so one might say both because one can both argue for and against, but I would say no” – algebraic
5.” $h = 4x + 5$
$p = 4x + (6{,}14)$??
both functions have the same slope coefficients don’t know if that means that they are equal each other”
6. “they don’t have the same b-value, only the same a so no they are not equal each other”
7. “h=ax+b
5x+3
p=ax+b
14x+2
8. “both \\ the x-values are the same but the y-values are different”
9. “no they don’t have the same rule”
10. “no because they don’t have the same equation”
11. “h and p are not equal each other. They don’t have the same slope coefficient”
12. they are not the same because h has a slope of 3 and p has a slope of 2.”
13. “no they have different points = different rules”
14. ”h(x)=ax+5 \\ p(x)=2x+ca.2 \\ they are not because their b is not the same cannot directly read hs a-value but it looks like it also has one on ca. 2”
15. “they have the same slope, but not the same b-value or Dm(f) [definition set] they are not equal each other”
16. “yes because they have the same slope coefficient”
17. “not the same slope co[efficient]…”
18. “they have the same slope so they are the same”

Uses more than 1 argument:
1.“no they are not alike because even if they increase by the same they do not intersect the y-axis. the same place therefore they are only parallel.” – algebraic/geo
2. “no they don’t intersect in the same points but they have the same length so in one way one could say they were equal each other” – geometric/arithmetic
3.“the equations themselves for the lines are different since they are to different places in the coordinate system. But the slope and length is the same. So if one put them on top of each other they would be equal\\in the coordinate system I wouldn’t say they were, but outside they are” - geometric/arithmetic/algebraic

4. “no, because then they would intersect each other. Their value would never be the same” - algebraic/arithmetic/geometric
5. “both \\ they are equal each other relative to length and slope \\ but they are not equal each other relative to y-value” – arithmetic/algebraic
6. “I wouldn’t say they are equal each other, because they have different function rules. But one can say that they are parallel equal each other” – geometric/algebraic
7. “same length and Ax – not the same B thus, not the same function, thus \neq” - algebraic/arithmetic
8. “they are not. The lines don’t intersect each other. They are not equal in any point. even if they have the same slope coefficient”
9. “they are parallel, but not equal” – geometric/unknown
10. “the slope coefficients are equal, and the pieces are also the same length. However they do not lie on the same place, and “p” does not intersect the y-axis on the picture. Their y-value would also be different, if “p” intersected the axis.” algebraic/arithmetic/graphic
11. “they have the same slope coefficient but not the same position. if position is disregarded they are equal, if position is regarded they are not equal \\
f(x)=a*x+ b
is the same   is not the same”
12.” No, as they don’t have the same values. However the slope is the same”
13. “Same slope, same length. Not the same initial value” – geometric/arithmetic
14. “Yes, equally high only two different places. They have the same slope” – algebraic/graphic
15. “yes, because they are parallel and have the same slope”
16. “yes, equally high, only two different places”