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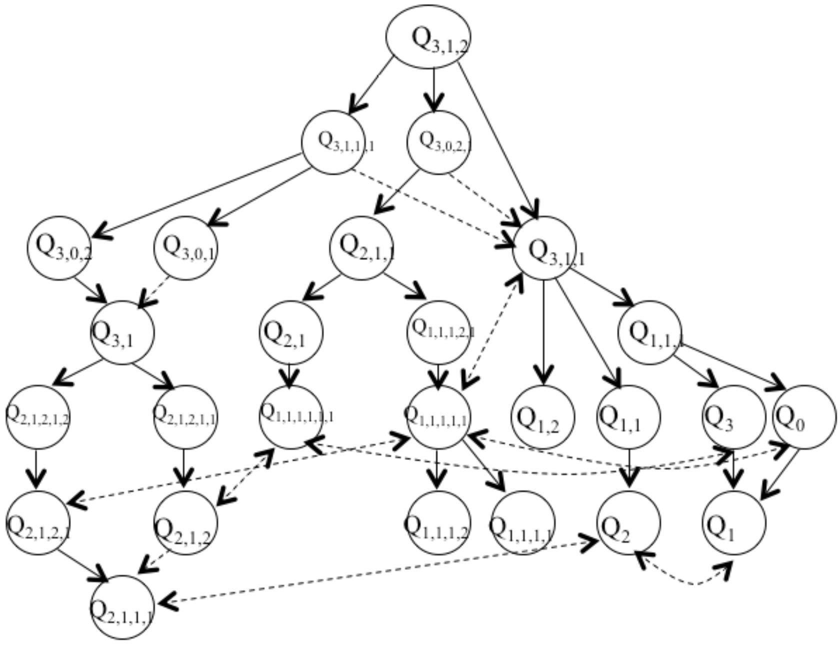

PhD Thesis

Britta Eyrich Jessen

Study and Research Paths at Upper Secondary Mathematics Education

- a Praxeological and Explorative study

Supervisor: Carl Winsløw

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Study and Research Paths at Upper Secondary Mathematics Education - a Praxeological and Explorative study

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Britta Eyrich Jessen
to
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Preface

About

1. Duration: January 2012 - December 2016
2. Research Institution: Department of Science Education, University of Copenhagen.
3. Keywords: The Anthropological Theory of Didactics, Study and Research Paths, Upper Secondary Mathematics Education, Interdisciplinary Teaching, Problem Posing, Mathematical Modelling, High Stake Exams.

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The dissertation at hand was prepared at the Department of Science Education, University of Copenhagen, (DSE, UCPH) as partial fulfilment of the requirements for the degree of Philosophiae Doctor (PhD) in the subject of didactics of mathematics.

Britta Eyrich Jessen
December 2016

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Study and Research Paths at Upper Secondary Mathematics Education - a Praxeological and Explorative study

Abstract

In didactics of mathematics, researchers have for decades been interested in how to teach students to pose questions and solve problems. Several approaches rely on the idea, that students learn mathematics, when they are engaged in activities similar to research mathematicians. This PhD project touch upon these ideas from the perspective offered by the Anthropological Theory of Didactics (ATD). Within ATD, teaching is proposed to be designed as Study and Research Paths (SRP). This thesis investigates how SRP's support the students' learning of mathematics in a bidisciplinary context involving mathematics and biology and in everyday teaching. Through the case studies, it is explored what didactical tools are needed to realise the theoretical potentials of SRP based teaching. Fruthermore, it is explored what the teaching of mathematical modelling can gain from SRP as a design tool. Finally, the project explores the viability of SRP based teaching in an institutional context as upper secondary mathematics in Denmark. The case studies showed promosing results with respect to engage students in autonomous questioning of the knowledge domains in the bidisciplinary design. However, a more precise analysis of the biological part of the design might have strengthened some students' outcomes. Sequences of SRA's seem to be a promising way of designing everyday teaching exploiting the potentials of SRP. In both case studies, certain media have been incorporated in the designs. In the

SRA study, students were suggested media as guidance for the study and research process. Additionally, students were required to present their preliminary answers to their fellow students. These presentations served as media for fellow students, who engaged in autonomous questioning of mathematical details beyond the scope of teaching at this level. With respect to mathematical modelling, links and gaps were identified between scholarly knowledge and knowledge to be taught in secondary school. It is suggested that SRP based teaching can bridge parts of the identified gaps. Finally, it is found that in order for SRP to be a viable alternative to prevailing mathematics teaching, the establishment of paradidactic infrastructures are needed to support teachers professional development. Furthermore, the restriction of high stakes written exams should be considered by ATD researchers. It is argued how the case studies directly relate to the exam formats for the teaching, which they were prepared for. But the largest amount of exam backwash stems from written exams. Therefore written exams should be designed to promote SRP based teaching, if possible.

Dissertation advisor: Prof. Carl Winsløw Britta Eyrich Jessen

Study and Research Paths at Upper Secondary Mathematics Education - a Praxeological and Explorative study

Resume

Indenfor matematikdidaktisk forskning eksisterer der en vedvarende idé om, at elever kan lære matematik ved at engagere sig i aktiviteter, der påforskellig vis har ligheder med en matematikers forskningsaktiviteter. Derfor er spørgsmål som, hvordan lærer man elever, at stille faglige spørgsmål og at løse matematiske problemer. Fra det perspektiv som den Antropologiske Teori for Didaktik (ATD) tilbyder, berøres de ovenfor nævnte spørgsmål. ATD tilbyder et designværktøj, der kan adressere disse spørgsmål. Designværktøjet kaldes pådansk for Studie- og Forskningsforløb (SFF). Formålet med denne afhandling er at undersøge, hvordan SFF baseret undervisning kan understøtte elevernes læring af matematik gennem tværfaglige såvel som monofaglige design. Herunder undersøges det, hvilke didaktiske værktøjer, der kan være behov for, for at realisere det fulde teoretiske potentiale i SFF. Derudover undersøges det, hvilke elementer af matematisk modellering, der kan understøttes af SFF baseret undervisning. Endelig undersøges det i hvilket omfang SFF vil være et reelt alternativ til mere almindelig undervisning i et skolesystem som det danske. To design studier er blevet gennemført og afprøvet, hvilke viser mulighederne for at fåeleverne til at udforske de vidensområder SFF ligger indenfor. Det tværfaglige design tyder pået behov for yderligere præcision i den biologiske analyse, der kan sikre alle elever et stort fagligt udbytte. Designet af Studie- og Forksnings Aktiviteter

(SFA) tyder på, at denne model for almindelig undervisning udfolder væsentlige dele af potentialerne for SFF. I begge designs har konkrete medier været tænkt med i udformningen. I SFA designet blev eleverne tilbudt et ”ressource rum” med foreslåede medier, der guidede eleverne i deres studie- og forskningsprocesser. Eleverne skulle ydermere dele deres foreløbige svar med resten af eleverne. Disse præsentationer kom til at fungere som yderligere medier for de øvrige elever, der engageret spurgte ind til detaljer i præsentationerne. Dette førte eleverne ud i at arbejde med matematiske detaljer, pået højere matematisk niveau end det, der blev undervist på. Med hensyn til matematisk modellering, såblev der identificeret aspekter af modellering i videnskabelig sammenhæng som ikke indfanges af styringsdokumenter og andre rammer for undervisningen pågymnasieniveau. Flere af disse elementer kan dog realiseres gennem SFF baseret undervisning. Endelig undersøges mulighederne for at implementere SFF baseret undervisning i bredere forstand i dansk gymnasiekontekst, hvilket synes at kræve opbygningen af nogle rammer for udviklingen af lærernes egen professionspraksis i form af paradidaktiske infrastrukturer. Ydermere problematiseres afsluttende eksameners indflydelse påundervisningen. Der er i afhandlingen redegjort for, hvordan de eksisterende eksamensformer kan spille sammen med SFF baseret undervisning. Den store udfordring ligger dog i den skriftlige eksamen for matematik. Fra ATD perspektiv, bør man arbejde med udviklingen af helt nye typer af eksamensopgaver, der kan understøtte implementeringen af SFF baseret undervisning.

Acknowledgments

In order to complete this PhD project, I have been fortunate enough to gain a number of peoples goodwill and cooperation. When I was teaching at Frederiksborg Gymnasium & HF, my colleagues were very helpful around the experimentation of my two teaching designs, offering me adequate classes for my research, providing additional textbooks and inspiring conversations. Some colleagues were more involved than others. Svend Erik Nielsen and Claus Gudum Faaborg, thank you for playing along and letting me test my ideas in our joined classes – and to answer all my questions with respect to biology. To Jesper Bjørnholt, a very heartfelt thank you for observing and transcribing the notes from the second case study, I am great full for the interest you have taken in my project.

The realisation of this PhD project, especially relies on the cooperation from all the students taught in the case studies: 1.a 2013/14, 2.v 2011/12 and 1.z 2013/2014. You all took the challenge seriously and were honest and straight forward with respect to feedback on the teaching – and best of all, it has been a joy to teach you mathematics, though it might not be your favorite discipline!

For many reasons, it has been a pleasure being part of the Department of Science Education. Tinne Hoff Kjeldsen from MATH, UCPH, who I co-authored two papers with, used to be part of DSE as well. I have enjoyed our conversations about history of mathematics, modelling and many other things. At the department, I have been so fortunate to be part of a very nice group of PhD students, always keen on sharing experiences, good as bad ones. The breakfast meetings and laughs about how we sometimes feel like we stumble through our studies – it has been very good to share this with you! And to my fellow PhD students in didactics of mathematics: Klaus, Dyana, Jacob, Louise, Zetra, Yukiko and Ignasi; it has been great fun traveling with you and most enlightening discussions of didactics of mathematics – I hope we will continue to do so, in some way, for many years to come!

A very special thank you to Carl Winsløw, my supervisor. We have had many – and sometimes long – conversations revolving around secondary mathematics education and notions and details of ATD. Thank you for supporting the realisation of my ideas and to keep posing the questions that moved my work forward and challenged the boundaries of what I felt, I was capable of doing and thinking. It has been a pleasure to be supervised by you.

Last but not least a very special thank you to my small family. My mother almost moved in to our house and took care of the domestics during the last month of writings. My sons have been perfect distraction during the last month, when a break was needed, every day asking me to play a game or watch the children's Christmas calendar on TV. And to my dearest husband, who has taken care of the kids, appointments, gave me technical support throughout the writing period and just knew when the moral support was needed the most – I simply do not know what to do without you!

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Papers Included in this Thesis

- Paper I: **How can Study and Research Paths Contribute to the Teaching of Mathematics in an Interdisciplinary Setting?**
  Jessen B. E.
  Annales de didactique et des sciences cognitives. 2014, 19, pp. 199-224.

- Paper II: **How to Generate Autonomous Questioning in Secondary Mathematics Teaching?**
  Jessen B. E.
  Submitted to: Recherches en didactique des mathématiques

- Paper III: **An Analysis of the Relation Between Mathematical Modelling in Scientific Contexts and Upper Secondary Education**
  Jessen B. E. & Kjeldsen, T. H.
  Submitted to: Educational Studies in Mathematics

- Paper IV: **Modelling: from Theory to Practice**
  Jessen B. E., Kjeldsen, T. H. & Winsløw, C.
  In K. Krainer & N. Vondravá (Eds.) *Proceedings of the ninth Congress of European Research in Mathematics Education*, Feb. 2015. Prague, Czech Republic, pp. 876-882.

- Paper V: **Questioning the World by Questioning the Exam**
  Jessen B. E.
  Submitted to: Educaçao Matemática Pesquissa

- Paper VI: **The Collective Aspect of Implementing Study and Research Paths – the Danish Case**
  Jessen B. E.
  In M. Achiam & C. Winsløw. (Eds.) *Educational design in math and science: The collective aspect: Peer-reviewed papers from a doctoral course at the University of Copenhagen.* IND's Skriftserie, Vol. 46. Nov. 2015. Copenhagen, Denmark, pp. 40-46.

Other Papers by Author Referred in this Thesis

- **Hansen, B & Winsløw, C (2011)**. Research and study course diagrams as an analytic tool: The case of bi-disciplinary projects combining mathematics and history.

- **Jessen, B. E. (2015)**. What role do study and research activities play in the relation between research and education?

- **Jessen, B. E., Holm, C., & Winsløw, C. (2015)**. Matematikudredningen: Udredning af den gymnasiale matematiks rolle og udviklingsbehov.

- **Jessen, B. E., Holm, C. & Winsløw, C. (In press)**. MatematikBroen – Fra grundskole til Gymnasium.

Abbreviations

A	ATD	The Anthropological Theory of the Didactics
C	CAS	Computer Algebra System
D	DE	Didactical Engineering
	DSE	Department of Science Education
F	FG	Frederiksborg Gymnasium & HF
H	hf	Danish abbreviation for preparatory exam (for higher education)
	hhx	Danish abbreviation for mercantile upper secondary education
	htx	Danish abbreviation for technical upper secondary education
I	IBME	Inquiry-Based Mathematics Educaiton
	ICME	the International Congress on Mathematical Education
	IND	Institut for Naturfagenes Didaktik, Danish for DSE
M	MATH	Department of Mathematical Sciences at UCPH
	MCT	Mathematical Competence Theory
P	PPW	Problem based Project Work
R	RME	Realistic Mathematics Education
S	SFA	Danish abbreviation for SRA
	SFF	Danish abbreviation for SRP
	SRA	Study and Research Activities
	SRP	Study and Research Paths
	STX	Danish abbreviation for general academic upper secondary education
T	TDS	Theory of Didactic Situations
U	UCPH	University of Copenhagen

Part I

Introduction and Background

1 Introduction

At ICME10 in Denmark 2004 I was working as technical assistant, invited to do so by my bachelor supervisor. At that time I had no knowledge of the wide range of research in the area of mathematics education, but was aiming at a carreer as upper secondary teacher. At ICME I had my first encounter with French didactics – a hyped and crowded lecture by Guy Brousseau on the Theory of Didactic Situaitons (TDS). I did not understand the lecture, however I became curious of what the fuss was all about. This led to my first meeting with Carl Winsløw who became my supervisor for several projects.

Carl persuaded me into a master thesis within the Anthropological Theory of Didactics (ATD), analysing bidisciplinary students' projects from upper secondary education bearing on topics where the students should combine methods from mathematics and history (Hansen, 2009). In the beginning I found this theoretical framework unnecessarily cumbersome, but after months of struggle, I was fascinated by the mathematical detail of content analysis it offered in terms of praxeology and Study and Research Paths. This led to my first encounter with the warm and welcomming community of ATD researchers at the third international ATD conference in Sant Hilari Sacalm (Girona), Spain, 2010, where I presented a conference paper based on the work of my master thesis (Hansen & Winsløw, 2011).

Listening to all the presentations, especially the ones reporting on experimentations with Study and Research Paths (or Courses as it was mostly called back then) made me question the feasibility of implementing SRP's for upper secondary teaching of mathematics under the constraints and conditions of everyday teaching. That is how the idea of my PhD project took form.

1.1 PhD Project Objective

The objective of this PhD project, as it was formulated in my application for enrollment as PhD student at the University of Copenhagen, is the following:

> "the systematic design and exploration of SRP as a design tool applied in the teaching of mathematics at level B and C at STX, partly through development and test of concrete designs, partly through theoretical and methodological clarification of the design tools' potentials in the institutional context"

The STX is the Danish abbreviation for general upper secondary education. What is meant by the levels (such as B and C) of mathematics education in the quote is explained in section 1.3. The point here is that the designs were meant for lower achieving students and those who do not take much interest in mathematics classes. In the application it is stated that the designs of the PhD project should seek to engage students in mathematical activity and thereby improve the students' gains from the teaching. These objectives should be met by teaching designs based on SRP and the outcomes (written and orally) should be analysed by use of methods from ATD. The designs developed as part of this PhD project should be based on questions which are real to the students, in the sense that they can relate to them. Moreover the questions and designs should address the potential of cross disciplinary activities in Danish upper secondary education (cf. section 1.3), while others would involve questions posed in a purely mathematical context with no requirement for knowledge from other disciplines. Because of the nature of the generating questions to be designed, involving both pure and applied mathematics, the notion of mathematical modelling should be addressed in the PhD project. And as a last objective, the potentials of the design tools' viability in the institutional setting of the PhD project should be clarified.

1.1.1 PhD Project Overview

The six paper of this thesis address these objectives as initially formulated in the application with different approaches and notions from ATD. The PhD project covers design and implementation of two different teaching designs implemented in level B and C mathematics classes respectively. The analysis of students' achievements from these interventions is based on praxeological analyses, identifying questions and answers dealt with by the students and how these interrelate. These studies are presented in paper I and II. In paper I, the notion of SRP in bidisciplinary context is explored, and the analysis is represented as tree diagrams. The teaching design addressing both mathematics and biology concerns the dosing of pain killers and their effect in the body as a function of time. In paper II the herbartian schema is employed to emphasise the role played by the dialectics of media and milieu, and how the other students became media of each others. The design is developed for teaching mathematics (in a monodisciplinary setting) and regards a specific type of savings account, but students are not asked to investigate knowledge on banking systems to engage in the generating question, but the main challenge for students is to develop their models of exponential growth in the functional setting required at upper secondary school. In short, the outcomes of both studies are promising with respect to what can be gained from SRP based teaching experiments. But the main results in these papers probably lie in the ideas proposed in regard to designing milieus where study processes become as important as research processes – and thus, to a larger extent than in many other experiments with SRPs, allows the researcher to exploit the potentials of the dialectics between the two processes.

Paper IV addresses the differences in the notions of mathematical modelling which result from two different approaches to didactical design (SRP from ATD and Problem based Project Work in the context of Mathematical Competence Theory) and how the theories, which the design formats stem from, are reflected in the teaching which these formats can lead to. It is suggested that the students' conception of what mathematical modelling is, will be strongly affected by the theoretical framework forming the basis of the teaching design. Pursuing these ideas further, the notion of didactic transposition from ATD is employed in paper III to study the relation between mathematical modelling in the scholarly context and mathematical modelling in the knowledge to be taught in upper secondary education. There are identified close relations as well as gaps between the two. In sections 1.5 it is discussed what part of the gaps might be bridged by SRP based teaching, drawing on the analysis conducted in paper I and IV.

In paper V we address a question that seems slightly overlooked in the ATD context: what could and should be the relation between SRP-based teaching and high stake exams? Applying the notion of levels of didactic codetermination from ATD, the regulation performed by high stake exams on the activities taking place in the classrooms are discussed. The two empirical studies reported on in paper I and paper II both address the existing examination formats for the teaching, and in particular for carrying out SRP's in the two contexts. Experiences from these first two case studies are drawn upon in paper V. The last paper gives a brief and rather explorative approach to an analysis of current structures of teacher collaboration and the possibility of engaging teachers in SRP based teaching practice. This paper draws on the herbartian schema and the notion of paradidactic infrastructures. The explorative element of this paper refers to the use of herbartian schema to distinguish between different ways of teacher collaboration.

This brief overview of the PhD also gives a rough idea of the research questions addressed, but before an actual formulation of the research questions (which is given in section 1.4), I will briefly describe the conditions and context in which I have worked as a PhD-student, and also give a systematic introduction to the notions of ATD already mentioned above and employed in the rest of the thesis.

1.1.2 Conditions and personal context of the work

This PhD project has been conducted with a part time status during the period January 2012 - December 2016. Since 2009 I have been employed as upper secondary teacher in mathematics and physics at Frederiksborg Gymnasium & HF, Hillerød, Denmark (FG). When I got enrolled as a PhD-student, I kept between 70 and 100 % of my position at FG. The actual degree of employment has been negotiated each year. The empirical investigations all took place in my own classrooms at FG. Hence, I was the usual teacher of these classes, and the study lines where I experimented my designs were determined by the classes I was given by FG. The school did fulfill my request to have mathematics classes at level C and B as my primary obligation. In the period December 2012-October 2013 I was on maternity leave.

At the University of Copenhagen I was hired as PhD student and as a research assistant in the same period. Together with my PhD studies, I taught the introduction on didactics for the mathematics students (DidG, mandatory for prospective upper secondary school teachers), I have for two years taught tutorials on ATD in the advanced course on didactics of mathematics (DidMatV), and I supervised a bachelor project and co-supervised a master thesis.

Furthermore, I have supervised upper secondary mathematics teachers in a project on transition from lower to upper secondary school; the supervision concerned their teaching designs and materials. This project is described in the final report by Ebbensgaard, Jacobsen & Ulriksen (2014).

From January 1st 2015, I have been fully employed by UCPH with a part of my time designated for my PhD project. The remaining time, I worked with different projects related to upper secondary mathematics education and teaching. During the first half of 2015 I engaged in an evaluation of mathematics teaching at upper secondary level together with Carl Winsløw and Christine Holm. The evaluation led to a status report named "Matematikudredningen" (Jessen, Holm & Winsløw, 2015), published and funded by the Danish Ministry of Education. During the spring of 2016, I was course responsible for the course DidG (mentioned above). In the period of spring 2015 untill November 2016 I have been coordinating and reporting on the Math Brigde project funded by the A. P. Møller Foundation. The Math Bridge Project strives to ease or improve the transition from lower to upper secondary mathematics education for the students. This is done by DSE offering workshops to a group of experienced mathematics teachers in both institutional settings, who subsequently designed and delivered workshops and teaching materials for lower secondary mathematics teachers. The workshops held by DSE covered notions such as praxeology and SRP (Jessen, Holm & Winsløw, In press). It was in relation to the Math Bridge project, that I co-supervised the master thesis of Poulsen (2015) on praxeological analysis of school algebra in the transition from lower to upper secondary mathematics education. Hence, all my activities during the last five years revolve around the early stages of upper secondary mathematic education.

1.2 The Anthropological Theory of Didactics

The theoretical frame of this thesis will be presented in this section and as mentioned, the thesis is based on the Anthropological Theory of Didactics (ATD), in particular its notions of Study Research Paths (SRP) and praxeology. The development of ATD was initiated by Yves Chevallard in terms of the didactic transposition which offer an epistemological approach to the relation between research mathematics and the teaching of mathematics in different institutional contexts. The didactic transposition can be represented as in figure 1.2.1 where scholarly knowledge is transposed into the school system and as such take another form than it had in the scholarly sphere. This first step is called the external didactic transposition.

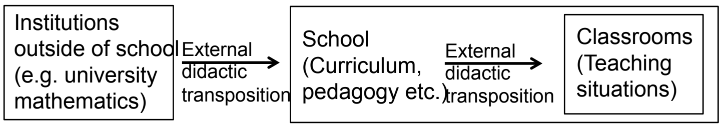

Figure 1.2.1: The didactic transposition and how knowledge is transposed from where it is produced outside of school systems to the teaching in an institutional setting

The second step is the internal didactic transposition, from the school system's formulation of knowledge in curriculum, pedagogy, and other circumstances affecting the knowledge, to the knowledge of the teaching situations in the classrooms (Chevallard, 1985). During the last decades the didactic transposition has been manifested as a tool in didactics of mathematics research and in other research fields as well (Gericke, 2009; Mortensen, 2010; Ricardo & Pietrocola, 2011). This has lead to formulation of further details and extensions of the above diagram and its uses in formulations of epistemological reference models (Bosch & Gascón, 2006 & 2014), but before going into further detail of the didactic transposition and how it has been employed in this PhD project, other notions of ATD must be defined. The core elements of the project is Study and Research Paths (SRP) and praxeology, which we now proceed to introduce.

1.2.1 The notion of praxeology

In ATD, several tools have been developed to analyse mathematical practice as a result of human activity, using the notion of *praxeology* (Winsløw, 2011). The notion of praxeology enables a precise description of mathematics in terms of tasks, mathematical techniques and the articulation of how the latter solves the former. At different levels praxeologies can be said to be linked and through these links, knowledge fields can be modelled from an anthropological point of view. The notion praxeology can be used to describe all human activity, while in this thesis emphasis is on mathematical praxeologies.

A praxeology consists of a praxis block and a logos block. Praxis represents the practical form of knowledge underlying actions, including actions as basic as those taken to greet a friend, by shaking hands or cheek kissing when you meet in the street. Logos is (loosely speaking) the discourse and the reasoning behind the action, which can be rooted in traditions and culture, as in the example of how to greet another person. The two blocks are not separate, but intertwined in the sense that "praxis thus entails logos which in turn backs up praxis" (Chevallard, 2006, p. 23).

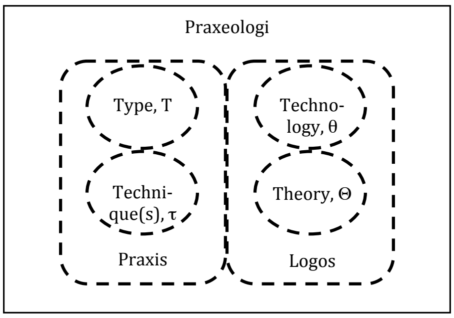

Figure 1.2.2: A diagram showing the components of a mathematical praxoelogy: type of task, technique(s), technology and theory (Hansen, 2009, p. 48)

Figure 1.2.2 show a diagrammatic form the components of a praxeology. In a praxeology, praxis is divided into type of task, T, and technique(s), τ, which solves the type of task. However, for many tasks a number of techniques will be involved in the solution. Table 1.2.1 shows an example of a praxeology concerned with finding local extrema of cubic polynomials. This type of task can be solved

Type of task	Find the extrema of a third degree polynomial written on the form $f(x) = x^3 + ax^2 + bx + c$
Techniques	Find the derivative, $f'(x)$, solve the equation $f'(x) = 0$ with respect to x and determine whether the slope is positive or negative in the intervals between the x-values, which solves the equation $f'(x) = 0$.
Technology	The discourse regarding identifying an extremum of the curve as points where the slope of the curve is equal to zero, how to find the derivative of a polynomial, the interpretation of the sign of f' and to solve a quadratic equation.
Theory	Differential calculus

Table 1.2.1: The praxeology of finding extrema of a third degree polynomial, identifying type, techniques, technology an theory.

in a number of different ways with or without CAS tools, which means that several techniques can be deployed. Here a CAS tool is a Computer Algebra System, which solves equations and carries out symbolic manipulations. Often these systems come with a graphic and spread sheet environment as well (Maple, TI Nspire, Geogebra etc.).

A commonly available solution for Danish high shool students, to the task described in table 1.2.1, would be to find the derivative $f'(x)$, solve the equation $f'(x) = 0$ with respect to x, using the formula $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$, where the derivative is written as $f'(x) = ax^2 + bx + c$. The formula will give zero, one or two solutions. Depending on the number of solutions, x-values are chosen in order to examine the value of f' below and above the solutions to the equation $f'(x) = 0$; based on the signs of f', it is decided which (if any) are (local) extrema, and the values of f at these are computed. Hence, this solution to the type of task requires many technical elements – to find the derivative alone requires a technique such as $(x^a)' = ax^{a-1}$. It is often the case that the technique to solve a type of task consists of a sequence of more elementary techniques (applying to subtasks), as seen in the example of table 1.2.1. Such techniques are denoted τ_i and the set of techniques solving the task: $\{\tau_1, ..., \tau_n\}$ where $n \in \mathbb{N}$. If we let $P(\tau_i)$ denote the subtask solved by the technique τ_i, then $P(\tau_i) \subseteq T$ (Chevallard, 1999, pp. 224). In this way the techniques listed in table 1.2.1 can be regarded as one technique. At the same time it is worth noting that the presented set of techniques, which composed the general technique solving the type of task presented in table 1.2.1, will not always be the most straight forward strategy. For instance, if the polynomial is of form $ax^3 + bx + c$ (no quadratic term), the identification of critical points can be done more easily. This implies that the notion of praxeology and the modelling of (mathematical) activities is a dynamic model, which can capture minor differences in the activity being analysed or when mapping the potentials of different tasks posed to students. The idea that each type of task can be divided into subtasks, each of which are solved by a subset of techniques, is in particular used in this PhD project when analyzing students work with generating questions.

The logos block consists of a technology, θ, and a theory, Θ. In the context of ATD and praxeologies, the notion of technology refers to the discourse about the techniques, which solves the given type of task (Bosch et al, 2005, p. 5). Hence in the example of table 1.2.1 an example of technology could be an explanation of how the extrema are identified based on the solutions to $f'(x) = 0$, invoking how the derivative of a function indicates the slope of the graph of a function in a particular point. Hence, wherever the slope is equal to zero, a curve has an extremal point. Thus, the technology is the explanation of how the techniques solve the type of task. Most often in school contexts the technology is quite informal, or the formal details are only roughly sketched. In the example of extrema of a third degree polynomial, students would usually not explicitly formulate the technology. The theory is a more formal level of justification of technologies and is often applying to several praxeologies. The example of table 1.2.1 belongs to differential calculus as it is taught in grade 11 in Denmark, and in general praxeologies are strongly conditioned by the institutional context.

The technology further carries the potential of developing new techniques. Let $[P/L]$ denote a praxeology, which contains a type of task which is "new" to some study community. In some cases one can employ and extend an already developed praxeology, $[\Pi/\Lambda]$, and adapt it in order to solve the new type of task, which can be represented as $[\Pi^*/\Lambda^*] = [P/L]$. A very simple example could be students who are able to calculate the area of a rectangle and the circumference of a circle, they could then extend and adapt this to develop the technique of determine the area of the surface of a cylinder (being open in both ends). To find the surface area of the cylinder requires the development of a technology, which justifies how the combination of the two techniques solve the task. In such cases the technology will be used to argue how and why the adapted version of the uses of certain techniques solve the new type of task (Chevallard, 2006, p. 7). This way of developing praxeologies also applies to the description of work of research mathematicians, while much of current teaching is based on the transmission of solution strategies from the teacher to the students (see further in section 1.2.2). Though the above kind of praxeological extension can be found in students' autonomous work as well, it ought to be promoted further, if students should be given the potential of developing the rationale behind the techniques they use for solving tasks. In paper I (and to some extent paper II) it is difficult to determine what part of the students' answers are developed through the above described process or if the answres have been obtained by mimicking procedures found in works of others (textbooks, group work, web pages etc.).

Praxeologies often occur in smaller or larger clusters, called punctual, local and regional praxeologies. A punctual praxeology is similar to the example referred above, with one type of task. If several types of tasks share the same technology, they form a local praxeology, $[T_i, \tau_i, \theta, \Theta]$. Some of the techniques and explanations in table 1.2.1 apply to the type of tasks determining extrema for any polynomial. The praxeologies, which share the same theory is called regional praxeologies (Barbé, Bosch, Espinoza & Gascón, 2005). In the referred example the theory would, as described, as a minimum contain some defination of differential quotients, and a theorem about extrema occurring only at critical points (under suitable conditions). Praxeologies made up by several theories may form entire domains, such as Calculus to which the example of table 1.2.1 belongs. It should be noted that what constitutes "praxis" and "logos" is specific to the institution, where the praxeology lives. What in table 1.2.1 is regarded as the theoretical justifications of the praxeology in the teaching institution of upper secondary education could play the role of tasks at the university level, when it is regarded as a teaching institution. Hence at university level the proof of a theorem concerning extremal points could be a task posed to students (see further elaboration of this in (Winsløw, 2008)).

The fine grained analysis rendered possible by modelling based on praxeologies is to some extent illustrated by the analyses presented in paper I and paper II. In these papers, the questions students work with are identified. The final questions of the paths refer, more or less, to point praxeologies. In section 1.2.5 it is further discussed how the notion of praxeology can be used to describe mathematical modelling.

1.2.2 Study and Research Paths

Study and Research Paths (SRP) were introduced as the design tool for teaching within the paradigm, which Chevallard calls *Questioning the world* (Chevallard, 2004 & 2015). The name has changed since its first formulation. In French the design tool is called *Parcours d'Etude et de Recherche (PER)*. The word parcours has been translated into programs, courses and most recently paths, which is used to emphasize that the "routes" to an answer is an important stake of research based on this format. Consensus seems to be reached to use the word "path" in English.

In general, a significant amount of mathematics teaching consists of the transmission of pieces of knowledge in the sense that the teacher presents a technique, how it can be used to solve problems (by presenting a couple of examples) and hereafter the students are supposed to solve similar problems, mimicking the teachers examples (Bosch & Winsløw, 2016, p. 335; Schoenfeld, 1988, p. 161). This way of teaching is compared to visiting tourist sights in the sense that a piece of mathematics (a formula, a method, etc.)

> ”[...] is approached as a monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raisons d'être*, now or in the past.” (Chevallard, 2015, p. 175).

In the alternative teaching paradigm, suggested by Chevallard, questions play the central role in the development of a herbartian attitude within the learner in the sense that:

> ”[...] this receptive attitude towards yet unanswered questions and unsolved problems, which is normally the scientist's attitude in his field of research and should become the citizen's in every domain of activity.” (Chevallard, 2015, p. 177)

In this thesis emphasis is put on the domain of mathematics and bidisciplinary teaching or how some kind of herbartian attitude towards the content of the teaching might be developed through SRP based teaching. When first introduced, Chevallard proposed SRP for teaching, which could be cross disciplinary in its nature to address a (at the time) new form of student work at secondary education i France (Chevallard, 2004, p. 7, Winsløw, Mercier, Matheron, 2013, p. 269), but the idea has since been experimented in other contexts as well.

The idea of SRP is that students are presented with a generating question, Q_0, which have a strong generating power in the sense that it is capable of sustaining a study and research process. In ATD, what is meant by study is the activity of a student when studying the works of others, e.g. reading the textbook in detail in order to employ the praxeologies it presents. But in ATD media cover all kinds of presentations of previous work of others: textbooks, newspapers, podcasts, videos, webpages and other channels for disseminating knowledge (Winsløw, Matheron & Mercier, 2013). Research refers to the activity of students when they adapt their praxeologies to solve new problems (as exemplified in section 1.2.1) – in particular when applying newly *studied* praxeologies in the development of an answer to a question (Winsløw et al., 2013). In ATD, learning can be described as the development of (coherent) praxeologies, which are obtained through the deconstruction and reconstruction of knowledge (Barquero, Bosch & Gascón, 2013, p. 334). The reconstruction is an activity needed when studying the work of others, identifying new techniques, which can be used for reconstruction of new answers in terms of new praxeologies.

When students are posed a generating question, they are supposed to engage in study and research processes in order to answer the question. It must not be possible to answer Q_0 without engaging in a study and research process (Chevallard, 2006). Barquero, Bosch and Gascón (2007, p. 2058) argue that the process has similarities with iterative processes in modelling activities in the sense, that if students have understood the question, they are assumed to construct preliminary hypotheses of answers, which are not warranted or sufficient and therefore call for further study and research until a new hypothetical answer can be tested. In this successive process some praxeological organization, built by the student, will expandand if the direction pursued by the student is relevant, her knowledge about the question grows. The insufficient answers typically lead to derived questions. Returning to the type of task of table 1.2.1; to find the extremal points of a polynomial, (Q_0), students need to be able to answer the question, what is the derivative of an expression of the form x^a? This question is derived from $Q0$ in the sense that it is asked to solve $Q0$, and it may in itself require a study process. In order to track the paths or traces of questions and answers the derived questions are represented by indices such as $Q_{i,j}$ (where $Q_{i,j}$ is derived from Q_i).

The results of SRP's are depicted, in our work, with so-called *tree diagrams*; an example is represented in figure 1.2.3. In the diagram each circle represents a

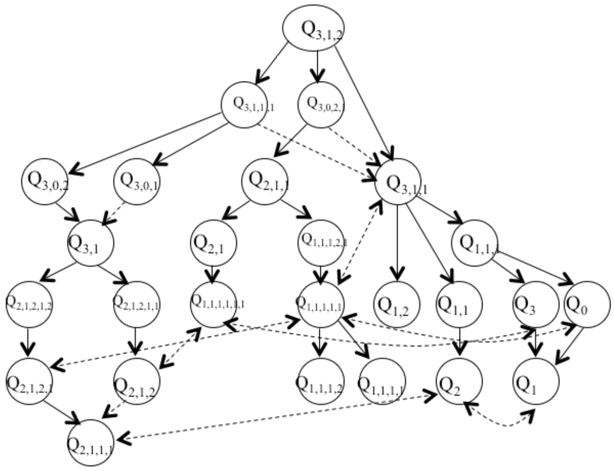

Figure 1.2.3: The figure show an example of a tree diagram depicting the a priori analysis of one of the generating questions of paper I

question raised (implicitly or explicitly) and answered by the students. In figure 1.2.3 the questions are formulated by the researcher as possible paths, which the students might pursue in their study and research processes. The arrows indicate possible relations between the questions – or praxeologies – which can be found in the discourse of the answers. One could mirror the diagram of questions into one covering the technology of the praxeologies which provide answers, but they are left out here for readability of the diagram, though implicitly assumed. The tree diagram is explicitly used in paper I and paper II to show the potential of generating questions and the outcome of the students engagement with these questions.

1.2.3 The herbartian schema

Another more recent development of ATD is the herbartian schema, which has been introduced to further represent crucial details of study and research activities (Chevallard, 2008; Kidron et al., 2014). What can be shown by the herbartian schema is the relation between the didactic system, S, (formed by a group of students, X, who are assisted by a group, Y, in the study of question, Q)

and the didactic milieu, M, in the development of a personal answer, $A^{\heartsuit}$:

$$[S(X;Y;Q) \to M] \hookrightarrow A^{\heartsuit}$$

The didactic milieu consist of answers known to students, $A_i^{\diamondsuit}$, (while not really answers to Q) as well as works, O_i, which "can be theories, experiments, questions, brought into the milieu M by members of $X \cup Y$" (Kidron et al., 2014 p. 157). Consequently, the milieu can be described as the set:

$$M = \{A_1^{\diamondsuit}, A_2^{\diamondsuit}, ..., A_k^{\diamondsuit}, O_{k+1}, ..., O_m\}$$

The answers, $A_i^{\diamondsuit}$, can also be drawn from resources including fellow students and teachers. In the case of students being the resource of an answer, they provide elements of their already developed praxeologies. When Chevallard introduced the herbartian schema he emphasized, in teaching activities, the slightly forgotten role played by media in inquiry processes where: "examining books and other sources of information [...] plays a key role in the discovery of already existing answers" (Chevallard 2008, p. 4). The design in Paper II explicitly incorporates media (as in textbooks, vodcast etc.) with a key role in the design, but also other students' oral presentation of preliminary work, is of crucial importance in this design, in order to serve as media for fellow students.

The notion of being "herbarian" or having a herbartian attitude towards the world is according to Chevallard (2015) another way of describing the teaching paradigm the society should aim at. To what extent this attitude is evoked in the students is analysed using the herbartian schema in paper II. The explicit listing of the didactic system, answers $A_i^{\diamondsuit}$, and other works O_j allows the analysis of the study and research process to be explicated, for instance to record the role of different members of the study community, as they study or bring into the milieu the different questions and answers. This enables the analysis of to what extent to which students engage in autonomous questioning.

In paper VI, the herbartian schema is used to investigate the possibilities existing today for professional development of the mathematics teachers' practice. In this context, professional development is interpreted as teachers' learning related to a concrete didactic challenge, such as: if, when and how to teach students the concept of limits, when teatching differential calculus in grade 11? This is a didactic question, Q, which the teacher can explore through a study and research process, alone or with others. To what extent teachers involve themselves (or are given the possibility of doing so) in such processes is discussed in the paper, considering in particular the potential to disseminate and implement SRP based teaching in an educational system like the Danish system.

1.2.4 Study and Research Activities

The ambition of teaching through generating questions and SRP is certainly a well argued objective for the teaching of mathematics. However most teaching takes place in school systems with curricula being of a rather monumentalistc nature, even if they are formulated in terms of competencies (Chevallard, 2006 & Rasmussen, 2016). The Danish school system offers a considerable amount of liberty with respect to how teaching is conducted and how the curriculum goals are met – although this may not be that different from other countries. What might be slightly special are the requirements, in upper secondary school, to engage students in bidisciplinary activities which are subsequently evaluated in high stake exams. These bidisciplinary activities are favorable conditions for SRP designed teaching, as argued in paper I.

What still remains a challenge is to cover all elements of the curriculum and thereby fulfill the responsibility of a teacher (towards the institution and not least towards the students, whose exam results from upper secondary school are decisve for their future opportunities) – to secure that all students have had adequate opportunity to "encounter the monuments of curriculum". To address this challenge, the notion of Study and Research Activities has been employed. A SRA is similar to SRP as it emerges from a generating question, though the openness of the question is limited compared to a generating question of a SRP.

The generating question of a SRA aims to initiate a study and research process, which covers a certain amount of answers or monuments from the curriculum. According to Chevallard (2004, p. 6; 2006, p. 18), in the teaching solely based on SRA there is a risk of students not developing the full rationale of the developed techniques, and teaching based on SRA will lack in motivation for the questions posed and treated by the students. More recent developments of ATD consider SRA as a branch of a SRP playing the role of developing a certain answer, A_i needed in the construction of an answer to the generating question Q_0 of the SRP (Barquero & Bosch, 2016, p. 262). A first attempt to elaborate on SRA as being a branch of a SRP or an independent teaching situation is given by Barquero, Serrano and Ruiz-Munzón (2016, p. 3). They define three different type of SRA's ranging from the ones described by Chevallard as having no or little potential of students developing rationales of the pre-established answers the SRA aims at studying, to minor study and research activities where students engage in de- and reconstruction of knowledge. The latter carries the stronger potential for student development of coherent praxeologies and can be included as a branch or path in a SRP. If teaching is based on a number of SRA's being interrelated as in order to cover a praxeological organisation, the SRA's can be represented with diagrams sharing traits with the tree diagrams (Barquero et al., 2016, p. 6). In this PhD project, in paper II, the figure 1.2.4 has been used to visualise how the SRA's $(Q_1 - Q_4)$ are connected and support the study of Q_0: What characterises an exponential function and where can it be applied? Q_0 was not explicitly posed as part of the teaching, but appears later on, in the preparation of the oral exam. This is elaborated in paper II.

1.2.5 A perspective on modelling

Modelling is not as such the core objective to be studied in this PhD project. However, modelling seems to be both a side track and the underlying substance in the designs and empirical studies conducted, and therefore we will discuss it in more detail in this section. Different approaches to inquiry also involve different

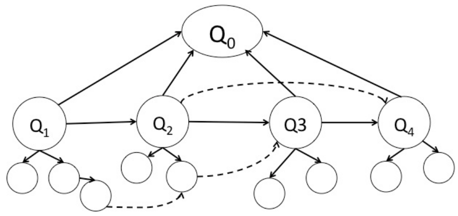

Figure 1.2.4: The figure show an example of a tree diagram depicting the SRA's experimented and analysed in paper II

ideas about modelling (See Artigue & Blomhøj, 2013). Some approaches insist on distinguishing between intra- and extra mathematical fields and that mathematical modelling takes place in a translation between the two (e.g. Blum & Borromeo-Ferri, 2009).

In ATD modelling is related to the dynamics of questions and answers, where the modelling of a question within some praxeology may give rise to a more or less complete answer. García, Gascón, Ruiz Higueras & Bosch (2006, p. 232) formulates it as: " [...] mathematical activity is essentially a modelling activity in itself". When Chevallard explains how answers to questions can be developed as adjusted praxeologies where developed techniques are used in a different context to create a new praxeology ($[\Pi^*/\Lambda^*] = [P/L]$); this is an example of modelling one praxeology with(in) another one. In this sense the students are doing modelling when they do mathematics. As it is questions, which drive the process, there is no simple or given distinction between the intra- and extra-mathematical in ATD. This was especially addressed in paper I where biological praxelogies were developed along with praxeologies on differential equations. García et al. (2006, p. 233) considered that the developed praxeologies are normaly of an increasing complexity. However, when a branch of a SRP address a certain answer needed to carry through an argument – as in a SRA – it seems that it will not always lead to the development of the full rationale of the praxeological organization needed. An example can be found in paper II, where certain details on the nature of logarithms could in principle still need development, which might be addressed in another SRA or SRP, if at all. In fact, if the study and research processes should run freely and the students mainly be guided by the dialectics of questions and answers, it seems impossible to secure or controle what increasing complexity would be developed.

With respect to modelling, paper III employs the didactic transposition, as introduced in section 1.2.6, in order to investigate the nature of the relation between scholarly modelling and the activities promoted by the noosphere. Paper IV compares two approaches to the teaching of mathematical modelling at upper secondary level and what conceptions of modelling are promoted in each approach – and also to what extent these conceptions stem from the didactical theories on which the designs were based. In section 1.5 it will be discussed what can be gained through SRP based teaching with respect to mathematical modelling as a preparation of students for higher education in fields where mathematics and modelling play a significant role.

1.2.6 The ecological dimension

As explained in section1.1 the experimentations with SRP and SRA in this project have been conducted under the constraints and conditions which currently prevail in upper upper secondary mathematics education in Denmark. It is an objective of this thesis to clarify the potentials of SRP based teaching in this and similar contexts; in order to make it transparent what contexts are sufficiently similar to make our results relevant to them, we will say more about the context of our research in this section, and how it can be analysed from an ecological point of view. For this analysis the notions of *didactic transposition* and *the scale of levels of codetermination* have been employed. Even if the two empirical investigations of SRP and SRA design presented in Paper I and II do not contain a full and explicitly detailed reference epistemological model, they do involve an analysis of the external transposition of knowledge, as a crucial element of the didactic design. The relation between scholarly knowledge and school knowledge on the notion of exponents of real numbers, when dealing with exponential functions is discussed in (Jessen, 2015). The paper is not included this thesis, since it does not explicitly address the reaserch questions. However, since this thesis has study and research processes as its core focus, the papers do not include the full preliminary didactic transposition analyses of the designs. On the other hand, when addressing the conception of modeling, didactic transposition offers a lens to study elements of modelling activities where some of these are not given much attention in school contexts today.

A schematic presentation of the didactic transposition, as it has been used more recently, is given in figure 1.2.5. When carrying out didactic transposition analysis, one must detach oneself from institutional perspectives, which is a core element of ATD methodology. Bosch argues, that this "implies an important enlargement of the empirical unit of analysis considered [...] attention is first put on what is learnt and taught, its "nature (what it is made of), "origin" (where it comes from) and "function" (what is it for)." (Bosch, 2015, pp. 53-54). Over time, schools may more or less completely forget the original rationales of the taught knowledge and then have lead to the monumentalistic teaching paradigm, characterized in section 1.2.2. Therefore we need to remind ourselves "the questions that motivated the creation of this knowledge" Bosch and Gascón (2006, p. 53). To investigate this rationale, knowledge must be studied in the different institutional settings affecting the teaching as pointed out in figure 1.2.5. Knowledge to be taught should be understood as in producing institutions,

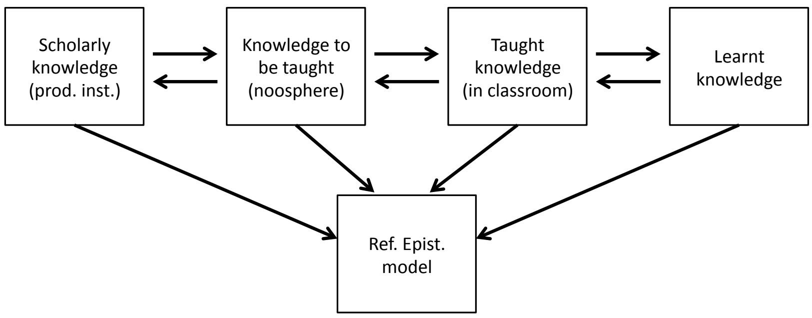

Figure 1.2.5: A more recent representation of the didactic transposition after (Bosch & Gascón, 2014, p. 71).

where knowledge is of a nature, that guarantees and legitimates the teaching of it and can be used to develop new knowledge as well as to organize its latest developments (Bosch & Gascón, 2006). In this PhD project, this approach has been used to investigate mathematical modelling through historic cases. Mathematical modelling is part of the official justfication of teaching mathematics, and one of the objectives of the teaching in itself (Danish Ministry of Education, 2013c & 2013d). But what is the nature of mathematical modelling, its origin and function outside of school contexts? This is addressed in paper III.

Knowledge to be taught, largely determined by noosphere (agents around the school instution), produces the first transposition of knowldge from producing institutions into the school system. According to Bosch & Gascón sources indicating this first transposition of knowledge cover "official programs, textbooks, recommendations to teachers, didactic material etc." (2014, p. 71) and the people who produce this transposed knowledge are "the people who "think" and make decisions about educational processes, such as curriculum developers, policy makers, mathematics advisors, associations for teachers, educational researchers etc." (Bosch, 2015, p. 55). In paper III we investigate the written documentation, which teachers might consult when designing the taught knowledge: Curriculum, the upper secondary school act, the guidelines for teachers, three textbooks and the high stake written exam exercises. To investigate the internal didactic transposition (from the above mentioned sources to taught knowledge) would have strengthened the study. But the purpose of this study was to investigate the linkage between mathematical modelling in the two institutionally contexts in order to discuss what can be gained from SRP based teaching on a meta level. In section 1.5, we discuss some of the shortcomings of the external didactic transposition we have been able to point out, and to what extent these can be overcome by SRP-based teaching.

An important method to clarify the potentials and viability of SRP, is to study and analyze the conditions and constraints affecting the experimented teaching. For this purpose the scale of levels of didactic codetermination can be usefull.

The scale consists of eight levels, which each conditions the activities realized in classrooms, as shown in figure 1.2.6. The notion of scale of levels of didactic

Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline

↔ Domain ↔ Sector ↔ Theme ↔ Question

Figure 1.2.6: The scale of levels of didactic codetermination after (Barquero, Bosch & Gascón, 2013, p. 15).

codetermination was introduced by Chevallard (2002), where he argues how the levels of the scale affect or restrict teachers' practice and in particular how they are able to organize the taught knowledge. The notion was used by (Barbé et al., 2005, p. 258) to identify where the restrictions on upper secondary teaching of limits of function stem from, which relates to the thematic confinement of the teacher, sometimes called "thematic autism". Recently Barquero, Bosch and Gascón (2013) investigate the ecology of teaching mathematical modelling at tertiary level with similar findings with respect to the thematic confinement. Hence, this seem to be a general restriction of the design of teaching of mathematics. Winsløw (2011) describes how the lowest levels of the scale correspond to point praxeologies and hereby what is taught in the classroom and outline the application of the scale in order to spot else wise invisible restrictions on classroom activities.

In paper V, the scale of levels of didactic codetermination is used to illustrate how teaching is affected by high stake written exams, and where the restrictions on written exams come from – which is not just of importance to the Danish school system. These constraints do not favor SRP-based teaching and therefore ATD researchers need to might need to engage in innovative design work in order to propose potentially viable ways for SRP to relate to standard high stake mathematics exams. Without such innovative research, there is little chanche that the teaching paradigm of questioning the world could become the dominant of mathematics classrooms.

1.3 The context of the study

The context of the study of this PhD project is general upper secondary mathematics in Denmark. In order to understand the context of the experimented SRP and SRA, this section will provide a more detailed presentation of the context, which has not been possible to find space for in the papers. Some elements of the context might be unusually favorable towards SRP based teaching while others might not be that different from from conditions to be found in other countries. Either way, a clarification is assumed to help the reader of this thesis judge the extent to which the results found in the Danish context can be translated to some particular other one.

In Denmark upper secondary education is divided into four different tracks: the general academic (stx), the technical (htx), the mercantile (htx) and the preparatory exam (hf). The abbreviations are the Danish ones. The preparatory exam lasts two years and all the others are three years long. All tracks have common disciplines and bidisciplinary course elements, which are assessed through an oral exam, and moreover through a bidisciplinary written "project" which carries a high weight in the computation of the final, high stakes grade of the baccalaureat. All tracks are divided into a number of *study lines*. A study line is defined (at stx, htx and hhx) by three disciplines at certain levels, which the students study together with the mandatory disciplines. Which disciplines are mandatory depends on and reflects the profile of the track. The following presentation concerns the general track, though to a large extent mathematics is similar from one track to another and therefore the results are likely to apply to the other tracks as well.

1.3.1 General academic track, stx

In the general academic track (stx), there exist more than 200 different study lines (Nielsen, 2008), which by the upcoming reform will be reduced to around 20 different lines (Danish Ministry of Education, 2016). Disciplines can be taught at three levels, A, B and C, A being the highest. Disciplines at level A, B and C correspond to studies for 3, 2 and 1 year, respectively. Mathematics can be chosen at all three levels. Danish and History are always taught at level A and cannot be study line disciplines. In a study line, there must be a discipline at level A. A very popular study line is English A, Social Sciences A and Mathematics B, which prepare students for higher education in social sciences. An example of a study line for students having a special interest in humanities is English A, French A and Social Sciences B. This study line will have Mathematics C as a mandatory discipline. The study of Paper II is conducted in such a class. A study line, which prepares student for higher education in natural sciences is: Mathematics A, Physics B and Chemistry B. In the last year of stx students have the liberty to choose extra disciplines to specialise in. In the case of the latter study line, many students would study physics or chemistry at level A.

Currently there exist a number of study lines preparing students for later college education that prepare for health care occupations (nurse, physiotherapist etc.) or for higher education in fields like medicine or pharmacology. Such a study line could be defined by the disciplines: Biology A, Mathematics B and Social sciences B. The study of Paper I was conducted in such a class, which was reflected in the choice of generating question concerning the dosing of pain killers.

The preparation of the new reform, expected to be initiated summer 2017, has the objective that almost no students should be able to take only level C mathematics (Danish Ministry of Education, 2016, p. 25). Presently the school system is facing a huge challenge with a large number of students attending level B, but who arrive to upper secondary education with highly varying and often very limited prerequisites and interests with respect to mathematics (Jessen, Holm & Winsløw, 2015, p. 28). Further every year somewhere between 20% and 30 % of the students fail the written exam at this level (they still pass the baccalaureate, as this is based on an elaborate average of grades in all disciplines). It should be mentioned that the best achieving students at level B choose mathematics A as their specialization during the last year and therefore do not attend the written exam at level B – Nevertheless, the high failure rate is a source of great policy concern. And in general the teaching of mathematics (especially at level B) is (didactically) a source of controversy and political concern.

1.3.2 Mathematics in stx

As mentioned in section 1.2.6 the final high stakes exams affects the teaching in profound ways, which is why a more detailed account of the exam system will be given here. In Denmark all upper secondary education, exams in single disciplines are on draw, meaning that the Ministry of Education, name the students from each scool who are supposed to attend a certain exam – except from a bidisciplinary oral exam and a bidisciplinary written report, which are both mandatory. After describing mathematics curriculum and exams briefly, a more presentation of the bidisciplinary exams will be given.

Level C mathematics

At level C mathematics, the final exam is an oral examination, where a certain share of the questions, which students draw from, at the beginning of the exam, should be based on *thematic projects*. These project are short written reports prepared by the students during the year, on specific topics. Usually the student is required to present a piece of knowledge and show some of its applications (Danish Ministry of Education, 2010b). When the thematic projects were introduced, they were inspired by the work of Grønbæk, Misfeldt and Winsløw (2009), who developed a new format for the oral exam at the introduction course on real analysis at the University of Copenhagen with the purpose of changing students' learning of theoretical material in this particular area of tertiary mathematics. What was observed was, that "it seems that the circumstances and traditions governing university mathematics teaching make it difficult to assess more than the use of standard techniques or the passive knowledge of textbook material" (Grønbæk, Misfeldt and Winsløw, 2009, p. 85). The assessment format of thematic projects was developed to change assessment from students memorizing theorems and proofs from the textbook at the oral exam, to instead more autonomous student productions at the theoretical level (involving formal definitions, theorems and proofs). The thematic projects at level C are designed by the teacher of each class and vary in form and content.

A glimpse of the monuments of curriculum for level C mathematics is: basic geometry and trigonometric functions applied to calculations of angles, side length and area of triangles, descriptive statistics including box plot and cumulative sum curves, and the simple relations between variables x and y ($y = a \cdot x + b$, $y = b \cdot a^x$ and $y = b \cdot x^a$, where a and b are coefficients), regression conducted with a spread sheet, and some elements of the history of mathematics (Danish Ministry of Education, 2013d).

Level B mathematics

Level B mathematics focuses more on developing a rationale behind the techniques applying to types of tasks, which are already studied in level C mathematics. The curriculum also includes new contents: logarithms, polynomials, statistical tests (e.g. χ^2-test), differential calculus and integral calculus. The level B mathematics is assessed by a written exam and an oral exam, both on draw. The written exam consist of one hour with six exercises, pen and paper and in direct continuation three hours with full allowance of computer and CAS-tool – and a number of exercises whose solutions require such tools(Danish Ministry of Education, 2013c). The students are allowed to use home pages online, which has been employed in the teaching, but not to conduct new web searches nor to communicate with others – which of course is hard to check and regulate by the schools. The tasks are mostly exercises which can be solved by standard techniques and procedures (Jessen, Holm & Winsløw, 2015, p. 43). Examples and a further description of the exercises are provided in paper III. There seems to be a tendency of teachers being more concerned about preparing student for the written than the oral exam, and a reason for this might be found in table 1.3.1, since it is much more likely for students to be drawn to the written exam. The oral exam has the same regulations with respect to thematic projects as level C. But at level B, students are often requiered to prepare small proofs or arguments to be shown at the board during examination. At both levels, the students know the formulation of the questions from the beginning of the exam period, they have preparation time from after they draw their specific question until entering the exam. At level C the exam last 24 minutes and at level B it last 30 minutes.

Level	Oral exam	Written exam
A	21%	101 %
B	30%	74%
C	20%	-

Table 1.3.1: The percentages of students drawn to oral and written exams respectively in 2014 mathematics at the three different levels in Denmark (Jessen, Holm & Winsløw, 2015, p. 53)

In table 1.3.1 it is indicated that more than 100 % of the students attend written exam at level A mathematics. The reason why this can be true is that these numbers count those who take mathematics A as a supplementary course after graduating from upper secondary education, together with ordinary level A students (Jessen, Holm & Winsløw, 2015, p. 54).

1.3.3 Bidisciplinary course elements

As mentioned, the bidisciplinary exams are mandatory. The written exam is a 15-20 pages report addressing a set of question, where the answers require the students to combine two disciplines. The disciplines should be two of those defining the study line the student belong to or an A level discipline from the study line combined with a discipline at least at level B – often Danish or History is chosen. During the fall of the third year, each student choose the two disciplines and a special area of interest, which is discussed with the teachers of the disciplines. Based on these conversations, the teachers (together) formulate the questions which the student must answer. The students are supposed to study the topic beforehand and from the time when the questions are handed out, the students have two weeks without teaching to write the report answering the posed questions. The report is graded by one of the teachers and an external examiner, which taken together represent both disciplines (Danish Ministry of Education, 2013a). The students are prepared for this high stake exam, through the preparation of a minor report combining History and Danish just before the summer break in the first year and a 10-15 pages report combining two study line disciplines after New Year the second year of upper secondary education. For the second year project, the two disciplines are decided by school management and the topic is chosen by the involved teachers. The students are given two days without teaching to finalize their reports; all other work on this project should be carried out after school hours. The study reported upon in paper I is an example of a second year project.

The bidisciplinary oral exam is a synopsis exam, which is conducted after New Year at the third year of upper secondary education. Ministry of Education announces a theme under which students are supposed to formulate a case study or problematic they will investigate (Danish Ministry of Education, 2013b). In 2016 the theme was "Limits" (but the Danish word can be translated into borders or boundaries as well), in 2005 the theme was: "Communication - possibilities and limitations" and again the word communications can have all sorts of interpretations as biological communication and nerve physiology; in mathematics it could be related to computers, in physics it could be related to electrodynamics and of course the humanities had a large variety of approaches to that question. All topics have been relatively open in order for all disciplines to be able to address the theme. Throughout the three years, students will have had a number of thematic weeks where different themes such as climate change, scientific fraud or colonisation issues – big questions, which can involve going across several disciplines. Together with concrete cases to be studied, students are taught general methodology of natural sciences, social sciences and humanities, how to write a synopsis, how to avoid plagiarism, find valuable references etc. In these synopses, students are encouraged to work across faculties in the sense of approaching cases using elements from different faculties, e.g. biology and psychology. After students have written their synopses in the spring, they are defending them at an oral exam just before graduation, where they are supposed to add further perspective on the case or problematic they have studied.

The local regulations at my school, regarding the second year reports, were that students should rewrite their second year reports into a synopsis and give the problematic a further perspective. These synopses were defended at an oral ”rehearsal examination” before the summer break between second and third year. There are examples given in paper I on the results of this student work.

1.4 Research Questions and Methodology

After presenting the theoretical framework of this PhD project as well as the context of the study, the scene is set to provide a more precise formulation of the objectives of the project. As referred to in section 1.1, the general objective of this project, stated in my application for the PhD project, was three fold: the project was supposed to design mathematics teaching for the lower levels of upper secondary mathematics, the project should explicitly address application of mathematics and its cooperation with other disciplines and finally it should be clarified to what extent SRP based teaching is viable in this context. This has led to the following research questionst (RQ):

1. How can SRP be said to support students' development of praxeologies in more open bidisciplinary settings as well as in a more constrained ordinary classroom setting? What possible didactical tools are needed for realizing as much of the potentials of the generating questions as possible?

2. Based on the ATD notion of mathematical modelling, what can be captured and gained from real modelling activities, when teaching mathematical modelling through SRP?

3. To what extent is SRP and SRA based teaching a viable alternative to the prevailing teaching paradigm of upper secondary education?

Below in section 1.5 it will be elaborated further how the papers address these research questions, but in short RQ1 is explicitly addressed by paper I and II, RQ2 is addressed by paper III and IV in a discussion of paper I, comparing with other experimentations of teaching of mathematical modelling, and finally is RQ3 mainly addressed by paper V and VI, where paper V draws on results from paper I and paper II.

The methodology of this PhD project is based on the traditions of didactical engineering (DE) and didactics as a design research (Artigue, 2009; Barquero &

Bosch, 2015). This approach is based on qualitative research methods and it originates in the tradition of French didactics as it strongly emerged in the 1980s (Artigue, 2009, p. 11). As such ATD represents a development and a slightly different take on DE compared to Brousseau's (1997) Theory of Didactic Situations (TDS). The ATD approach to DE can be represented as in figure 1.4.1 and is conducted through the four phases: 1) Preliminary analyses, 2) Design and a priori analysis, 3) Implementation, observation and data collection and 4) A posteriori analysis, validation and development.

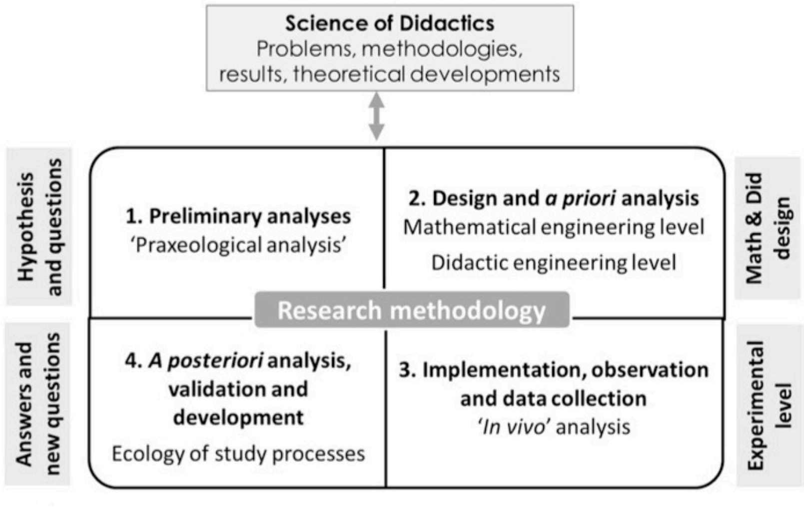

Figure 1.4.1: The phases of didactical engineering research methodology within ATD (Barquero & Bosch, 2015, p. 263).

When and how each step of the methodology is carried out differs from one study to another. In this section a presentation of the methodology within ATD as employed in this PhD project will be given.

The first two papers, which directly report on the realization of two teaching designs, mainly present the second and the fourth category analysis in the sense of validating the nature of developed praxeologies among the students in the two cases. The preliminary analyses rely on consulting ministerial guidelines, available media, and possible connections between the disciplines. Based on the preliminary analyses, the a priori analyses of the generating questions were conducted and the design adjusted, which led to the first tree diagrams.

In paper I the regulations from the school which forbid the teacher to teach in the topic after handing out the generating question to the students, prevented the implementation from being strongly influenced by the teacher. Main data from this study are students written reports and synopses. The praxeological analysis of this experiment is based on the methodology developed and presented in (Hansen, 2009; Hansen & Winsløw, 2011). The text is divided into minor sections which address a certain issue relating to the generating question. Detailed analysis of students' texts determines what questions (or type of taks) are answered and how in each section. In sections with a mathematical content, these can be analysed as praxeologies, as described in section 1.2.1. In sections with a biological content or with a mixture of disciplinary praxeologies, sections were analysed as pairs (Q_i, A_i). The discourse further indicated the linkages between different praxeologies indicated in the tree diagrams.

Furthermore data was collected from the correspondence between the teachers and the students. This was done by requiring that all questions were posed in writing on email – it turned out that they could also be answered by email. This data has been used to identify how the students worked together and what questions their study and research processes had led to. To gain insights regarding the students' hypotheses throughout the processes, the students were asked to write down what knowledge they drew on in their reports, and what new knowledge needed to study further. This was written down when the generating question was handed out, half way through the period before, the days of writing and just before the final writing.

In the study reported on in paper II, it was a harder task to stick to the role of objective observer, since the study involved classroom teaching by the author. But throughout the preliminary analysis and the design and a priori analysis, this was taken into consideration. The design consists of an epistemological part which takes students previous teaching and presumed knowledge into account, as well as media for the students to study. Additionally, strict regulation for the classroom interactions were designed: groups formation based on students assumed praxeoogical equipment, timeframe of 5-7 minutes, required presentations including written arguments at the board, and certain proposed media. Together with the generating questions, these elements secured students engagement in the study and research process. If groups did not know where to start, they were suggested to study media proposed by the teacher. If groups were in doubt, whether their work was correct, they were asked to present it anyway, in order to get the other groups' feedback. In this sense, the guidance needed by the students was provided by media and fellow students (which again was based on some media). When questions raised concerning mathematical notions, not covered by the suggested media, additional media was found by fellow students. In the transcribtion of the classroom interaction the responses of the teacher can seem kind of rude, when refraining from answering directly, but suggesting the students to further study. However, as shown in paper II it worked well in order to engage students in the intended processes.

In the classroom teaching, it was recognized how difficult it would be to collect data through video recording. Therefore the group work was audio recorded, but these data did not provide substantial extra knowledge regarding the students' interaction with the generating questions. For the group presentations, observation notes were taken and pictures of the students written presentations at the whiteboards. The observation was carried out by a colleague from the school. He taught the class in another discipline, but is also a mathematics teacher. So, he knew most of the students and he knew the content, which the class was working with. He wrote down the oral part of the conversation, the written part, and if students made pointing gestures. Since the presentations were preliminary and students only had short time for preparing, they did not say much, which made the note taking feasible. The data from this study could have been collected theough video recording as well, though the advantage of the chosen method was, that if words was missed out students was asked to repeat. Therefore data from this study was detailed observation notes.

In order to validate the two designs and their potentials of engaging students in developing coherent praxeological organizations and autonomous questioning, tree diagrams (a described in section 1.2.2) have been widely used, where a priori and a posteriori analyses are compared, as described as part of the DE methodology by Artigue (2009, p. 11). Furthermore, the herbartian schema is employed in the analysis of paper II to highlight the role of media in generating students autonomous questioning, which is emphasized as part of the DE methodology within ATD by Barquero & Bosch (2015, p. 261).

Still these experimentations of SRA and SRP as design tools are small scale and serves at most as ”proof of existence” in the sense of pointing to dynamics of processes which can be furthered by the didactical design and enlightened by the praxeological analysis. How and why is argued in the next section. The analyses carried out in paper III-VI are all pointing forward and are not examples of didactic engineering or design research. The methodology of paper III has similarities with the preliminary analyses of figure 1.4.1, as it is a version of the external didactic transposition of mathematical modelling for upper secondary mathematics education. Paper IV points out five variables used to compare and discuss problem based project work and SRP. The methodology of paper V is based on literature reviews regarding the role of exam, the formats of collaboration and the paradidactic infrastructures.

More details on methodological considerations are given in each paper.

1.5 Discussion of papers

To facilitate the coherence of the discussion of the papers and their relations to the research questions, each research question is treated separately below. In the first sections, the discussion is heavily drawing on the elements from ATD presented above. Finally, at the end of this section the this PhD project will be discussed in relation to mathematics education research in a broader perspective.

1.5.1 How praxeologies are developed through realized SRP

The first research question relates to the realization of SRP in at least two different settings, namely the bidisciplinary projects of paper I and the everyday classroom teaching of paper II. In paper I, the following research question is addressed: "how the bidisciplinary setting can help developing mathematical knowledge – and more concretely, in how a SRP combining mathematics with a discipline like biology could support the learning of mathematics." (Jessen, 2014, p. 200). This is answered through a praxeological analysis of the students reports leading to tree diagrams of the students' reports. This resulted in the identification of (implicitly) posed questions of a mixed nature. Some questions treated were purely mathematical or purely biological, while others went beyond traditionally boundaries between the disciplines. An example of a mixture could be, when students use a mathematical model to argue for the dosing of a drug and the biological implications in the body. This relates to a questions such as: why are patients only allowed to take a dose of paracetamol every four hours? This analysis led to tree diagrams, where two of them are presented in the paper in varying detail.

The paper leads to a central problem involved in SRP design, when posing these open generating questions and at the same time wanting students to explore certain paths or elements of a praxeological organization. Some students ignored what seemed obvious paths to others. The reasons for this could be, that students were not used to be posed open questions and to question the media they read or studied. Normally, students might suppose they can simply find and apply answers in proposed media to accomplish their assignments. The students are not used to study the piece of knowledge further, to develop further praxeological equipment. This observation can be related to the fact, that SRP based teaching relies on a changed didactic contract (in the sense of Brousseau, 1997), to which students do not necessarily adjust or adapt to easily. This was also found recently by Rasmussen (2016) in the context of pre-service teacher education. To seriously study and know how to employ media in a productive way, as in de- and reconstruction of praxeologies, will not come immediately to students, who have had no or very limited experience with open questions as in SRP.

It is suggested in Paper I (Jessen, 2014, p 219) that the need for further study of a mathematical (in some cases also biological) piece of knowledge relates to the lack of precise biological analysis and how the biological questions deserve an answer drawing on mathematics. To create well functioning generating questions, the teachers need to study each others' fields of expertise in order to design and analyze potentials of generating question across disciplinary boundaries (as they are found in classic school institutions). This was partly fulfilled in the study, as the didactical researcher as well as the biology teacher engaged in minor study and research activities in order to formulate the generating question and conduct the a priori analysis. Still, the lack of a clear identification of how to analyse a biological praxeologies and the dynamic use of it became a weakness of the a priori analysis. Mortensen (2011) has made initial analyses of such praxeologies, but the study of this PhD project could probably have been strengthened by involving a biologist with didactical background in the design phase, in order for praxeologies to become a powerful tool in the conducted analysis. This might have led to improved formulation of the generating question, which could have supported the study and research processes of all the students further.

The question of how to design and organize the study and research processes can be inspired by the students who participated in the bidisciplinary study of paper I. As part of study I, the teachers had an email correspondence with the students who wanted to ask questions to the teachers, in order to keep track of the communication between students and teachers. These revealed that those students who performed well in the reports had formed their own study groups. The productive study groups posed very interesting questions based on their discussions, and pursued answers to these as well. Most of their questions to the teachers regarded whether it was reasonable to follow the tracks they found interesting – and it was. The paths or tracks they found interesting were inspired by media they found, shared and studied together. The idea of study groups and the sharing of findings align with other experimentations of realizations of SRP. In other experiments, the use of student ”conferences” or other sharings of ideas have been employed (e.g. see (Rasmussen, 2016),(Barquero et al. 2013), (Serrano et al., 2010) and (Thrane, 2009)). Based on the experiences of the first paper and these other references, the idea to design an orchestration of students' engagement with a generating question, where study processes and media were a core element, emerged. In the study of paper I it was not an option to arrange sharing sessions as described in the cited papers, due to the constraints in the particular context of paper I. It is an open question, whether students need sharing sessions when writing their autonomous bidisciplinary reports, if they had adjusted to SRP based teaching and generating questions during everyday teaching. It could be interesting to study what and if supervision could still be helpful for such students.

The question of how to orchestrate students' study and research processes was more directly addressed in the second experimentation, which is presented in paper II. The design explicitly seeks to exploit the positive potentials of the group dynamics detected in the first experimentation. The study was guided by the research question: ”How can a changed didactic contract support the management of autonomous questioning in a Study and Research Activity?” (Jessen, submitted, p 3). Before conducting this study, a ”pilot design” on a sequence of SRA's on linear functions ($f(x) = a \cdot x + b$) was experimented. The purpose was to introduce the changed didactic contract in the teaching in the field of linear functions. Linear functions is a topic of lower secondary education as well, hence it was suspected that most students had already (to some extent) developed the intended praxeologies but might need to consolidate them - and indeed, all students needed to adapt their praxeological equipment on linear functions to the more formal theory blocks of upper secondary education being more formal. In other words, this pilot study supported the students' development of technology and theory, which is also a main reason to revisit linear functions at first year of upper secondary mathematics.

When initiating the SRA's of the design of paper II regarding the notion of exponential functions ($f(x) = b \cdot a^x$) focus could be put on the possibility of students' engagement with the generating questions and the genuine and explicit questioning of knowledge as catalyst for developing new praxeologies. The experimentation is a "proof of existence" of students' autonomous questioning, and several elements of the teaching design seem to have had a positive effect with respect to furhter the students' autonomous questioning.

What is interesting to discuss based on this paper and pursued in further research is the role played by media, which Chevallard (2008) points out as being crucial and slightly forgotten in other approaches to inquiry based teaching activities. In the study of paper II students are explicitly suggested to consult media, which have not yet read or encountered. The media suggested are of varying nature: paper textbook, textbook as i-book (which means that the book is an electronic file, with links to supplementary materials such as extra projects, further data and applets, which can be found online, through to book) and youtube videos from a Danish counterpart to the Kahn Academy. The reason for these choices was to offer students media, which they actually would be able to study. The underlying assumption was that if students found at least one of the presentations of knowledge "understandable", then they were more likely to engage in de- and reconstruction of knowledge into answers for the generating questions. Recalling the milieu, as it was defined in the herbartian schema presented in section 1.2.3, the design explicitly offers a number of works, O_i's. And from the data of this study it is obvious that some students consult other works that were not given to them by the teacher, or even known to her. So media came to play an important part in the students' paths. The different technologies, which the students met through the media, clearly supported the development of their own rationales of the developed techniques.

Equally interesting in the study of paper II is how student groups became media to be studied by fellow cstudents. All groups provided preliminary answers to the generating questions. Some groups adapted techniques from media and applied them. Other groups relied mainly on research, based on their previously developed answers. As all groups were required to present their answers orally *and* in writing on the board, in order that all answers to be considered by everyone at the same time, it was possible for groups to address similarities and differences in each others' answers. This can be regarded as an important ressource for the development of the technological part of the praxeologies, which the employed techniques belong to. Several of these examples were implicitly given and mainly shown in their thematic projects, but as paper II shows, there were few very explicit examples as well. Students did occasionally question other groups' rationale for employed techniques and, as shown in paper II, used to build or develop new answers among the students who questioned the rationale. In this respect, the answer of one group, $A_j^\diamondsuit$, became explicit – including elements of the technology of $A_j^\diamondsuit$.

Forcing students to present preliminary work (and the teacher to refrain from validation) seems to move students' focus from right and wrong answers to the study each others' works as another important media. The imperfect versions of students' answers naturally call for being questioned. Some questions might ask for elaboration, where a formula was found or if the presented formula or sketch of a situation fits what was said about the situations, during the presentation. These question might be naive or less interesting, but they seem to support the creation of a didactic contract or social regulations for the teaching where questions are welcome. Furthermore, those questions seemed to promote more advanced mathematical questions or commentaries. An example of such more profound questions relate to some students' use of logarithms in an argument, and how and what these logarithms were made of (Jessen, submitted, p 17). In this the students explicitly question a preferred answer $A^\heartsuit$ of another group. Hence, what is a developed answer of one group becomes a media which is deconstructed by another group.

The dynamics described in paper II using the herbartian schema might have been furthered by the formation of groups. In the study of paper I, students who performed equally well formed their own study groups for the bidisciplinary projects. In paper II, the group dynamics were initiated as part of the design. The idea is based on how new praxeologies can evolve from already developed ones, the idea described as $[\Pi^*/\Lambda^*] = [P/L]$ in section 1.2.1. To exploit this idea, students need to be in groups, where they to some extent share the same praxeological equipment, even if there may be personal variations. If not, there exist a risk that one student brings in an answer, which becomes a "monument to visit" for the other group members. Therefore the class was divided into groups performing equally well in the teaching before the experimentation. Students appreciated this as they could participate on equal terms. Students explained, how they felt obliged to actively engage in the study and research process – the group actually needed them. In other words the group dynamics forced students to engage in the process.

1.5.2 The conception of modelling and SRP

The second research question is mainly addressed by the third and fourth paper, though all the first four papers involve mathematical modelling in the ATD sense, cf. section 1.2.5. Before elaborating on what paper I and paper II can offer RQ2, paper III will be discussed. In Paper III the work was guided by the question: "What relation – if any – is there between mathematical modelling in professional scientific contexts and in mathematics teaching at upper secondary level?" (Jessen & Kjeldsen, submitted, p. 2) and is pursued by an analysis of the external didactic transposition (cf. figure 1.2.1). The work with this question was driven by our curiosity towards the role played by mathematical modelling in the teaching of mathematics, since application and modelling of other fields often quoted to justify the teaching of mathematics at upper secondary level, preparing students for higher education in natural sciences and social sciences and economy (Danish Ministry of Education, 2013c; 2013d). Similar arguments can be found in other countries (Blum & Borromeo-Ferri, 2007, p. 47). Frejd and Bergsten (2016) initiated the study of the didactic transposition of a broad perspective on mathematical modelling analyzing modelling at work places as representing scholarly knowledge of producing institutions (including academia) and concluded that these activities were rather remote and specific compared to secondary education with its the teaching of broad aspects of the modelling process as well as the study of concrete, preexisting examples. Concretely, the modellers carried out the same small subproccess of the entire modelling activity conducted at her wokplace, using very advanced mathematics. Therefore workplaces cannot serve directly as inspiration for school activities. But what knowledge is then supposed to be transposed into teachable knowledge in school systems? And if it is not specific mathematical content which is to be taught, what is it then?

In paper III we took the approach to study what could justify mathematical modelling in school context, based on episodes from the scientific context from a history of mathematics perspective. Doing so, gave the advantage that examples of mathematical modelling of other domains or disciplines took place not that long ago though, still being relatively simple in its nature in order to analyze, what elements, processes and ideas went into the modelling of phenomena in those research fields. Processes involved in the analysed cases covered using analogies from other knowledge fields, creativity, development of new mathematical knowledge, praxeologies one might say, and discussions of epistemic value of the models. What is deemed valuable by a biologist or economist might not be the same as what the mathematician or physicist find valuable.

To a large extent, these crucial traits of real scientific modelling activities do not appear in the curriculum or other official documents. The curriculum does to some extent reflect the elements of what was found in the historic cases, but the ministerial guidelines (complementing the curriculum and largely read by teachers) suggest a more algorithmic approach to the process. This is amplified in the textbooks, where chapters addressing modelling were of a monumentalistic nature; also there were no real modelling potentials in the exercises attached to these chapters – the exercises can be solved by mimicking the examples of the textbooks. To be fair to the authors of the textbooks, the books contain suggestions for more inductive approaches and larger projects in supplementary chapters and appendices. It is argued in paper III, that these parts are disregarded, since the books state that main chapters ”cover” the curriculum. Furthermore, the written exam exercises with an apparent modelling focus, are actually more like mathematics tasks requiring one or two techniques, masked as a modelling task with a story from the real world. We cannot argue that students do not get any experience with modelling tasks in the actual teaching, however the noosphere does not seem to nurture the traits of the activity identified in the scholarly sphere of the historic cases, on the contrary there seems to be a gap between the two which might in fact obstruct the preparation of students to pursue higher education in disciplines employing a certain amount of mathematics and mathematical modelling. Finally, to develop critical citizenship requires a more serious work with modeling activities if students should become able to question the creation and others' use of mathematical models in general.

This leaves the question of how to teach students the elements of modelling activity, which seem to be overlooked in the analysed materials from the noosphere. This question is implicitly addressed by paper IV, which treats the following research question: What differences, if any, does it make for the design of new teaching practices, whether the theoretical control apparatus comes from Mathematical Competence Theory (MCT) or ATD? In particular, are there differences between uses of the design formats Problem based Project Work (PPW) and SRP which can be related directly to the different theoretical backgrounds' notions of modelling found within MCT and ATD. This is investigated, comparing the experimentation of paper I with a similar case study by (Blomhøj & Kjeldsen, 2006) conducted by two teachers from the same upper secondary school (FG). Their teaching design was developed at a course for in-service teachers at Roskilde Unisversity on problem based project work (PPW), mathematical competence theory (MCT) and the corresponding modelling cycle (cf. figure 1.5.1) In this framework the modelling cycle is regarded as a reflection tool enlightening the subprocesses of mathematical modelling activity represented in the figure. As indicated by the arrows, modellers can move around, forth and back, following the arrows of the diagram in order to complete a modelling problem.

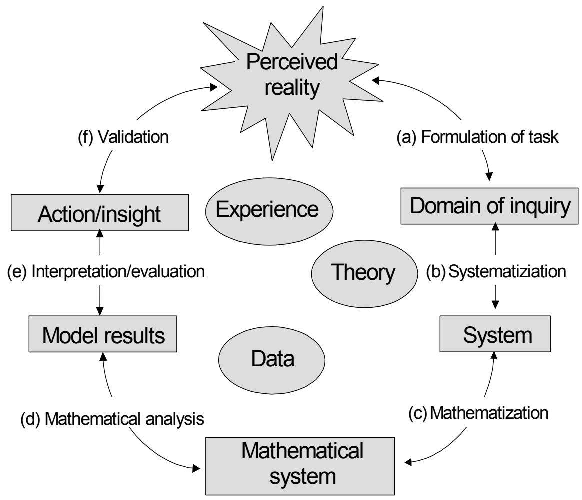

Figure 1.5.1: The modelling cycle from Mathematical Competence Theory as it is represented in (Blomhøj & Kjeldsen, 2006, p.166).

The teaching conducted concerned the dosing of asthma medicine and was combining mathematics and chemistry (Blomhøj & Kjeldsen, 2006). The two teaching designs were compared based on a number of parameters: practical meaning of "modelling", goals of the activity, organisation in time, distribution of roles among students and teacher(s) and adaptation to local constraints and conditions (Jessen, Kjeldsen & Winsløw, 2015, p. 877). It is concluded that in

MCT students are taught the elements or phases of the modelling cycle in order to reflect upon their own activities. In ATD the process is guided by a generating question and students' autonomous questioning of the knowledge related to the generating question. In the MCT approach one distinguishes between "inside" and "outside" of mathematics, which is not the case in ATD. The MCT approach might help students to relate to metaissues, such as the identified discussions of epistemic value of a model. In the ATD approach it is sufficient if the model created can provide some validated answers, which rely on the de- and reconstruction of knowledge. Conversely the ATD approach seem to support the creative and non-algorithmic approach to modelling activities as found in the historic cases. From the point of view of media being crucial in inquiry approaches to the conduct of scientific activities, ATD incorporate this explicitly. The MCT case relied on teachers introduction of relevant notions, techniques and data.

Paper III should not be regarded as a proposal of teaching mathematical modelling through historic cases. But in cases, where there might exist sources, which can enlighten the emergence or development of real model, and which can serve as a partial answer to a (generating) question, the historic approach could be an interesting element in the teaching of mathematical modelling and specifically as a source of media in SRP based teaching. However, this is not investigated further in this PhD project.

1.5.3 The viability of SRP in the Danish school system

As it is argued in paper V, it has been known for decades now that classroom activities in mathematics, to a large extent, are constrained and conditioned by high stakes exam, if students' outcome is evaluated in that way (Schoenfeld, 1988). The work presented in paper V, was guided by the research question: What is needed from school systems in general and exams in particular for this new paradigm of "questioning the world" to be a viable alternative to the paradigm of "visiting monuments"?

In paper V, the scale of levels of codetermination is employed to analyse why and how high stake exams affects the teaching as reported and why exams should be considered in the context of SRP and a changed teaching paradigm. A considerable amount of restrictions on what a written exam in mathematics should be, seem to stem from the level of society. It is not only the school system itself, but the public in large hold opinions on the question (e.g. see Webb, 1992), which matters to political stakeholders with respect to, what a mathematical exit exam is and should be. This affects the noosphere and the school system in the design of the exam exercises. Hence, if the public is resistant to reforms, it can be hard to change the exam.

Despite the detected resistance, Swan and Burkhardt (2012) and Silver (2013) and Suurtamm (2016) propose to exploit the backwash effect of exams in a positive manner. They argue that if teaching should have problem posing or in this case, students' autonomous questioning as a key component of the teaching, it should be reflected in the exam as well. But there exist no significant experiments with methods to do so; indeed experiments with high stakes exams are not easy to set up, for obvious reasons.

Paper V points out how SRP and SRA based teaching might relate to the current exam formats, which exist for the teaching designed and reported in paper I and II. To further students' interest in the study of fellow students' preliminary answers in the SRA teaching, the format of thematic projects (see Grønbæk, Misfeldt & Winsløw, 2009), preparing students for a possible oral exam, worked well. To write the best possible thematic project, the students needed to be able to do more than mimicking the answers provided by other groups or media. It forced all groups to pay attention and formulate precise questions to the mathematical content. The construction of the oral exam engaged students in the dialectics of study and research processes – and the autonomous questioning of the students were reflected in the oral exam, which went very well.

The same was found in the written reports of paper I. But the students who had difficulties engaging in productive study and research processes, would probably gain further from a structuring of the work in study groups and the sharing of preliminary answers. This could not be done in the concrete context, but under other conditions it could be considered. As such the format of a ”scientific report” serves as another way to organize a written exam, though it does not offer the strict measures of individual control which are traditionally attached to standard written exams in mathematics. To meet those restrictions or to convince the higher levels of codetermination to adapt to other formats requires further design work – and a broader dissemination of the results of experiments to people outside the ATD community.

Another approach to investigate the question of viability is what mechanisms and support of teachers' professional development exist currently in the educational system, to realize a transition from the paradigm of visiting monuments to questioning the world. During the last couple of years, experimentations have been made with the design of in-service teacher courses on how to teach mathematical modelling from the perspective of ATD and through SRP design (Barquero, Bosch & Romo, 2015). In-service courses are definitely an important arena to investigate ideas on how to enable SRP becoming a viable alternative for current teaching designs.Though, in this paper the approach is not course activities for teacher, but the following research question: what constraints and conditions exist in the implementation of SRP based teaching and the paradigm of questioning the world from the collective perspective on the teaching of mathematics? (Jessen, 2016, p. 41)

Paper VI employs the herbartian schema to represent different constellations of collaborations among teachers and discusses to what extent there already exist structures in the Danish school system, which could support the transition from one teaching paradigm to another. The different kinds of collaboration are found in Jessen, Holm & Winsløw (2015) who asked all upper secondary mathematics teachers about their habits and possibilities for professional development and inspiration with respect to solve teaching tasks. The herbartian schema emphasizes whether the collaboration results in genuinely new answers or if teachers simply adopt each othes' answers. The main point, and constraint found, is the lack of paradidactic infrastructures to reflect upon the teaching before and after it is conducted together with peers. If this is not facilitated, how can the outcomes be validated or designs improved further? In the current situation professional development is initiated by the individual teacher asking colleagues for opinions and ideas for specific teacher tasks. There does not seem to be much room for the teachers to de- and reconstruct studied answers to improve their own teaching designs. Combined with findings of Barquero, Bosch and Romo, who found that teachers had difficulties to stick with the ideas of SRP when realizing their own SRP designs, it seems crucial to construct the paradidactic infrastructures with time designated for the development of SRP based teaching and reflect upon it afterwards. Another limitation emerging from the survey (Jessen et al., 2015), is that the Danish mathematics teachers have only very limited education in domains as content didactics (as in German stofdidaktik) and didactics of mathematics, which is likely to constrain their capacities for designing SRPs and orchestrate them in the classroom.

1.5.4 A broader perspective

This PhD project is in all aspects conducted within the Anthropological Theory of Didactics, however questions very close to the questions addressed in this thesis have been formulated by others in other theoretical constructs within mathematics education as a research field. As formulated by Kilpatrick (2014), posing problems and assist others in solving them is as old as mathematics itself. And the idea that students would gain from teaching being akin to the activity of mathematical researchers has existed since before the establishment of mathematics education as a research field.

In 1945 Polya published *How to Solve It*, which is still referred and deemed seminal in mathematics education (Arigue & Blomhøj, 2013, p. 802). Schoenfeld (1992, p. 352) emphasizes Polya's formulation of heuristic competences as being crucial in problem solving activities. What Polya described in his book was the working mode or attitude, the tools and approaches to a mathematical problem familiar to mathematicians. Schoenfeld further argues for a distinction between problems and exercises in the sense that exercises can be solved by known solution strategies, whereas problem solving activity requires more independent work. In this light generating questions could be regarded as problems and the heuristic competences could be descriptions of what is reached through study and research processes. Schoenfeldt describes the activity of problem solving as a combination of algorithmic and non-algorithmic procedures, existing knowledge and tools (1985, p. 15). But there is no explicit mentioning of the role played by not yet studied materials as textbooks etc. Futhermore the renewed interest and research put in problem solving in the 70s and 80s did not answer the question of how to teach it: "in a compilation of dissertation studies in problem solving over the 1970s, indicated that the teaching of problem-solving strategies was "promising" but had yet to pan out [...] problem solving strategies are both problem and student-specific" (Schoenfeld, 1992, p. 353). Thus it was an open question how to teach students heuristic competences and problem solving in general.

Throughout the 1980s the idea that problem solving would gain from students being engaged in problem posing activities grew stronger. Ellerton concludes on the study of engaging pre-service mathematics teacher students in problem posing activities, that: "For too long, successful problem solving has been lauded as the goal; the time has come for problem posing to be given a prominent but natural place in mathematics curricula and classrooms" (2013, p. 90). In the introduction, in the same special issue of Educational Studies of Mathematics, Ellerton reasons together with Singer and Cai, why problem posing should have this dominant role: "Problem posing improves students' problem-solving skills , attitudes, and confidence in mathematics, and contribute to a broader understanding of mathematical concepts and the development of mathematical thinking" (Singer, Ellerton & Cai, 2013, p. 2). Hence, it is believed that problem posing improves problem solving competences. Approaches to teach problem posing are discussed by Bosch and Winsløw (2016) in the context of presenting SRP as another approach to give students' autonomous problem development a prominent role in the teaching of mathematics. Bosch and Winsløw provide an theoretical analysis of how teaching can nurture and promote students autonomous question formulation as part of the teaching; paper II is to some extent an example of how it has been done in reality.

However, the study of paper II does not solve the problem of how to teach students problem solving and for students to develop heuristic competences, but regard challenges from problem solving and problem posing literature from the perspective of ATD.

As pointed out by Winsløw (2011), ATD shares the idea of regarding mathematics as being a human activity with Freudenthal and the tradition of Realistic Mathematics Education (RME) (Gravemeijer & Terwel, 2000). Both in ATD and RME we find the idea of teaching revolving around problems or questions being real to the students and to study the paths or trajectories initiated by teachers for students to follow (Barquero, Bosch & Gascón, 2007, p. 2052), (Heuvel-Panhuizen, 2000, p. 5). However, the underlying theoretical approaches are different, in the sense that RME has a phenomenological approach (Gravemeijer & Terwel, 2000) and ATD an epistemological theory. Therefore, what is really shared is the idea that learning of mathematics can take place in situations designed to offer students to engage in the role of inquirer, similar to research activities. But this is shared with a number of theoretical constructs within mathematics education research, where a number of these are presented and discussed by Artigue and Blomhøj (2013) in their attempt to conceptualize Inquiry Based Mathematics Education (IBME). Together with the emphasis of inquiry activities the approaches to inquiry, seem to relate to mathematical modelling activities. The role and conception of modelling in this PhD project has already been discussed. What is explicitly made different in the case studies of this PhD project compared other approaches to IBME and modelling, is the employment of media. The designs explicitly involve media which are suggested in order to secure that students are offered the full potentials of the generating questions. ATD stresses that the study process is as important as the research or inquiry process when de- and reconstructing knowledge into answers. To quote Sir Isaac Newton: "If I have I have seen further than others, it is by standing on shoulders of giants" (Katz, 1993). Newton certainly did not have teaching and learning processes in mind formulating this, but indeed building on the works of others is a fundamental part of scholarly work, and therefore it seems crucial to incorporate in inquiry based approaches to teaching, as argued by Chevallard (2008). In this thesis concrete examples of how this can be done are provided.

1.6 Conclusion

This PhD project investigate the potentials of SRP based teaching for upper secondary mathematics education. The following research questions have been studied:

1. How can SRP be said to support students' development of praxeologies in more open bidisciplinary settings as well as in a more constrained ordinary classroom setting? What possible didactical tools are needed for realizing as much of the potentials of the generating questions as possible?

2. Based on the ATD notion of mathematical modelling, what can be captured and gained from real modelling activities, when teaching mathematical modelling through SRP?

3. To what extent is SRP and SRA based teaching a viable alternative to the prevailing teaching paradigm of upper secondary education?

The study of RQ1 showed that a SRP involving two disciplines as mathematics and biology certainly can promote students autonomous questioning of the involved knowledge domains. This was guided by some few first derived questions posed by the teachers. Especially students who, on their own initiative, formed study groups, shared media, shared preliminary answers and derived further questions together, wrote good adequate reports as answers for the generating questions. However, other students did not see the same potentials. It is suggested that a deeper praxeological analyses of both disciplines would strengthen the design and all the students' performances.

Similarly, good results were found in the second case study. Under the restrictions of everyday teaching SRA seem to be a promising tool for the design of teaching, which engage students in the dialectic between study and research processes. The requirement, that all students had to present their preliminary answers, promoted the students questioning of the knowledge domain. Together with the guidance offered by proposing the students to study concrete examples of media, furthered their study processes. The explicit and detailed analysis of students interactions using th herbartian schema and the explicit guidance of the study processes through proposed media are the novelties provided in this thesis.

RQ2 was explored though two minor studies. The external didactic transposition of mathematical modelling showed links and gaps between the scholarly notion of modelling and the notion found in sources produced by the noosphere. The comparative study of teaching designs indicates that part of the gap can be bridged by SRP based teaching. When students' modelling activites are guided by the dialectics of questions and answers, the activities become more akin to the processes found in the historic cases on scholarly knowledge on mathematical modelling.

In order to explore RQ3, the herbartian schema is used to analyse the collaboration of teachers and to what extent this collaboration can suport the teachers' development of their teaching practice. In this study, what teachers do to develop their own practice is analysed as if the teachers were engaged in a SRP, where the the generating question is of a didactical nature. The study shows, that there does not exist paradidactic infrastructures, where teachers can plan and reflect upon their practice in order to develop their practice further. This is needed, if teachers in general should be teaching through SRP's.

Finally, a litterature review is carried out on the backwash of high stakes exam and its restriction on the teaching of mathematics. This is a point which not yet have been considered by ATD research, but it is evident that the restriction of high stakes exams affect the viability of SRP based teaching and the realisation of changing the prevailing teaching paradigm. When everyday teaching is designed as sequences of SRA's and teaching is evaluated in an oral exam, thematic projects (as basis for the oral exam) seem to work well. SRP turned out to be an adequate design tool for the bidisciplinary projects and the high stakes exam format, they represent. What is left to consider, is how can the backwash of high stakes exam be approached from the point of view of ATD? Can high stakes exam be designed to favor teaching based on SRP or SRA's?

1.7 Perspectives

As raised under the exploration of RQ3, the consideration of high stakes exam would be a natural continuation of the research initiated in this PhD project. In the litterature of mathematics task design it has been proposed to promote changes of teaching through changes in the high stakes exam. For natural reasons the high stakes exam is not and object to be experimented with. Therefore detailed and innovative design studies should be conducted. It is still an open question how and if students autonomous questioning could or should be part of such task designs. It has been argued that SRP based teaching support students development of coherent praxeologies. This includes rationales of techniques and not simple to use them, when told to. Is this coherence and rationale, elements which can be evaluated by tasks suitable for high stakes written exams? And what can praxeological analysis offer such taks designs and investigations?

The restriction performed by high stake written exam, only represents one constraint with respect to the change of prevailing teaching paradigm. Other studies indicate challenges with respect to teachers' implementation of their own SRP designs after participation in an in-service teacher course. Teachers tend to return to their usual practices when they return to their classrooms with their SRP designs. Therefore it can be argued, that the teachers need systems for detailed planning, maybe joined implementation and reflection upon new teaching designs and other didactical tools. It could be interesting to design a research project involving both course activities for teachers as well as the creation of sustainable paradidactic infrastructures. Would it be possible to identify similar students' engagement in study and research processes as found in this PhD project?

Finally, the incorporation of media in the SRP and SRA designs of this thesis resulted in students actually studying the works of others. Students even started to search for further adequate media. This definitly strengthen the dialectics between media and milieu and improved the students' performances in reports, projects and the oral exam. It could be interesting to investigate to what extent media should be explicitly proposed to students, if they already have adapted to the changed didactic contract of SRP based teaching? If students to some extent have gained a herbartian attitude towards education, would the strict orchestration of classroom activities still be needed?

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Part II

How can study and research paths contribute to the teaching of mathematics in an interdisciplinary setting?

2

Paper I: Bidisciplinary SRP

This paper reports on the design and implementation of a SRP combining mathematics and biology. The generating question, analysed in this paper, covers the dosing of pain killers, their effect in the body and its modeling through differential equations. The teaching was finalized in written reports, which have been analysed through discourse analysis leading to praxeological analysis identifying what derived questions each student address and what answer is given. The paper mainly addresses research question 1, regarding what praxeologies students might develop from a SRP and what didactical tools can support the development of these. Due to the bidisciplinary nature of the design, this paper also provide knowledge on research question 2, and what can be gained from SRP based teaching with respect to scientific approaches to mathematical modelling. The paper has been published in *ANNALES de DIDACTIQUE et de SCIENCES COGNITIVES* in 2014.

Britta Eyrich JESSEN

HOW CAN STUDY AND RESEARCH PATHS CONTRIBUTE TO THE TEACHING OF MATHEMATICS IN AN INTERDISCIPLINARY SETTING?

Abstract. This study investigates the perspectives of using study and research paths (SRP) as a design tool for bidisciplinary work at upper secondary level. This study is using a special kind of diagrams both as tool for SRP design and as a tool to analyse the actual SRP realised with students. Specifically I present the design and realisation of a SRP combining mathematics and biology. The results point to advantages of the SRP approach in terms of the way bidisciplinary work is organised, but also challenges in relation to the design process. As for the last point, the test of the designs raises the question to what degree of detail is it necessary to know the practice and theory of both disciplines in order to formulate questions that help students to develop the intended praxeologies, and also for the weak students to discover the need of mathematics for solving problems in other disciplines.

Key words. Upper secondary level; bidisciplinary work; Study and Research Paths.

Résumé. Comment les Parcours d'Etude et de Recherche peuvent-ils contribuer à l'enseignement des mathématiques dans un contexte interdisciplinaire ? Cette étude examine les perspectives d'utilisation des Parcours d'Étude et de Recherche (PER) comme outil de conception pour du travail bidisciplinaire au niveau secondaire supérieur. Cette étude utilise un type spécial de schémas comme outil à la fois pour la conception de PER et pour analyser le PER réellement réalisé avec les élèves. Plus précisément, je présente la conception et la réalisation d'un PER combinant mathématiques et biologie. Les résultats montrent les avantages de l'approche PER en termes d'organisation du travail bidisciplinaire, mais signalent aussi les conditions à remplir pour la conception. En ce qui concerne le dernier point, le test des réalisations soulève la question du niveau de détail auquel il est nécessaire de connaître la pratique et la théorie des deux disciplines, afin de formuler des questions qui aident les élèves à développer les praxéologies voulues, et aussi permettent aux élèves faibles de découvrir le besoin de mathématiques pour résoudre des problèmes d'autres disciplines.

INTRODUCTION

This study presents the results of testing the design tool called *Study and Research Paths* (SRP) at upper secondary level. The basic idea of a SRP is to organise students' approach to a field of knowledge through meaningful and challenging questions. I describe this tool in more detail in the theory section. SRP has been tested in both monodisciplinary settings (e.g. see Winsløw, Matheron & Mercier, in press) and in bidisciplinary settings (Barquero, Bosch & Gascón, 2007; Thrane,

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2009). The SRP developed by Barquero, Bosch and Gascón (2007) concerned the growth of a population of geese on an isolated island; it does not require students to develop substantial knowledge on population biology, but the motivating problem is clearly extra-mathematical. Thrane (2009) experimented a series of SRP concerning analysis of curves in different sport activities actually, which involve students knowledge of how to perform different sport activities, and the students were supposed to use the mathematical analysis improving their own performance in these activities. In this sense the latter seems to be integrating the two concrete school disciplines more than the first one. This study explores the use of SRP in a bidisciplinary setting combining mathematics and biology where the two disciplines are given equal importance. I am particularly interested in how the bidisciplinary setting can help developing mathematical knowledge – and more concretely, in how a SRP combining mathematics with a discipline like biology could support the learning of mathematics. This is not a new idea and similar ones are presented in (Davison, Miller & Metheny, 1995) and (Czemiak, Weber, Sandmann & Ahern, 1999). What this study offers is a thorough analysis of the students detailed outcomes in terms of presented praxeologies, which illustrates the disciplines and their possible connections regulated by the potentials and limitations of ATD and study and research paths.

This paper is a natural continuation of previous work (Hansen & Winsløw, 2011 and Hansen, 2009), which presented a method to use SRP for analysing bidisciplinary written assignments combining mathematics and history. The study revealed severe challenges for creating bidisciplinary projects, that are well functioning both from the viewpoint of students and teachers. The reason for the identified shortcomings were not just caused by the manifest distance between mathematics and history as disciplines, but also by the fact that the teachers' formulation of the assignments were often leading to a parallel structure in the students' work where the two disciplines were not interacting at all. This was clear already from an *a priori* analysis of the assignments. How the a priori analysis is carried out will be elaborated in the section on methodology.

Context of the study

The institutional frame for the experiments with SRP presented in this paper, was general high school (upper secondary level) in Denmark. In this context, a certain amount of time and lessons are devoted to bidisciplinary work. There are many formal regulations of the bidisciplinary work, which acted as constraints and conditions for the testing of the SRP. The most important condition for our experiment was that the SRP should combine mathematics and biology and that the students should write a bidisciplinary report at the conclusion of their work. The report described in this experiment should prepare the students for writing an autonomous report combining to disciplines (called the "study line project"), which represents a high stake exam at the end of high school, and so it is heavily regulated. Similar the report and this experiment was highly regulated; I give some details in order to allow the reader to grasp the setting of our design. After handing in the report, the students get feedback on their writing from the teachers and they rewrote their report as a 3 pages synopsis to be defended in an oral exam months later. The students must do all work on the first version of the report along with their mandatory classes; after six weeks they get two days off for the final writing. They are allowed to write the report in groups of two students. Each student must hand in at most 10 pages.

After handing in the reports, students should have some kind of evaluation of their work. The rules require that students get a grade for their reports along with comments. These comments must reflect what is expected of the student in their "study line project". Therefore a sheet of comments was created for each student. The comments were formulated with explicit reference to the ministerial guidelines for grading study line projects. This means that the students would get comments from both teachers on the following sentences:

> "To what extent are the questions answered? To what extent does the report fulfil the ministerial aims of the biology teaching? To what extent does the report fulfil the ministerial aims of the mathematics teaching? To what extent are the sections of the assignment mutually coherent? Is the use of notes and citations in the text appropriate? Is the list of references satisfying? What is the overall impression of the assignment?"

Based on the comments, they rewrote their report to the synopsis – a paper containing introduction, research questions, answers to these, conclusion and a section putting the problematique in a broader perspective – used at the oral exam.

On the side of the teachers, none of them have an academic background in both mathematics and biology. The biology teacher is an experienced teacher of biology and geography. He is involved with didactic developments in Danish high school, but not a researcher and without any experience teaching SRP. The mathematics teacher has some years experience in teaching mathematics and physics in Danish High school. She is also a researcher in the field of didactics of mathematics and the author of this paper. Both teachers are the everyday teachers of the class in biology and mathematics respectively. The research part was only conducted by the mathematics teacher, which is reflected in the analysis. The choice of disciplines depends on the disciplines the class specialises in. Therefore it is not likely both disciplines are in the academic background of one teacher. The experiment included the entire class of 25 students.

Theory

The theoretical framework of this study is the anthropological theory (ATD). The key notion is *study and research path,* which is used as a design tool as well as for analysing the outcomes of the student reports, and which we now proceed to explain in more detail.

The notion of SRP was presented by Chevallard around 2004[1] and he describes it as based on what he calls a *generating question*, which will be denoted Q_0. This question must be so strong[2] that students can derive new questions $Q_i´$ from it – here, each index *i* represent a branch of inquiry. The answers to the derived questions add up to an answer for the original question Q_0 (Chevallard, 2006, p. 28). Another requirement for the generating question Q_0 formulated by Barquero, Bosch & Gascón, is that it must be "of real interest to the students ("alive") […]" (2007, p. 3). The research and study process leads to tree diagram of pairs (Q_i, A_i) of question and answers (Barquero, Bosch et Gascón, 2007 and Hansen & Winsløw, 2010), such as the example shown in figure 3 – for simplicity the answers to each question (arising from praxeologies developed by the students) are left out of the diagram.

The notion of inquiry can be interpreted as in inquiry-based mathematics education (IBME), which has been conceptualized by Artique and Blomhøj (2013). As they argue "ATD is also a theoretical frame whose design perspective seems especially adapted to IBME" (Artigue & Blomhøj, 2013, p. 806), and further discusses the potentials and limitations regarding the inquiry reflecting the choice of study and research activity or programme as they call it. The strong link between study and research paths and inquiry-based learning is addressed in (Winsløw, Matheron & Mercier, 2013), although they stress the importance of the study process, which cannot be discarded from the inquiry process.

We now return to our context to explain how SRP fit with the conditions for the bidisciplinary work leading to a synopsis for the oral exam. The students are supposed to get training in applying existing knowledge. In terms of ATD this means activating existing *praxeologies,* a term which indicates a complex system of practical and theoretical knowledge (Chevallard, 1999). The students knew a little on first order differential equations and human physiology, including the nervous system. They are supposed to apply their knowledge in new contexts and

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1 However there has been made different suggestions for the translation of *parcours d'étude et de recherché*. In this paper I have chose to use *study and research paths*.

2 A strong question means that students are able to understand it but unable to deliver a complete answer before studying works of others and use these answers in the formulation of an answer to the generating question.

hopefully get a wider picture of both fields. In terms of ATD this is to develop new (mathematical, biological or bidisciplinary) praxeologies from the existing ones (Barquero, Bosch & Gascón, 2007, pp. 9; Hansen, 2009, p. 53). Another requirement for the assignment is that the students should gain experience with searching for information and resources for answering the assignment questions – and also, where possible, develop answers on their own. This is consistent with what Chevallard calls the dialectic of media and milieu (2006, p. 9) where the student on the one hand is studying existing "works", and at the same time is exploring a problem (in this case, mathematical modelling of the distribution of a drug). It is important to point out the necessity and delicacy of this dialectics (Winsløw, 2011, p. 129): a SRP must include *both* study (of works) and *research* (on problems). The students are supposed to do this since the answers were not directly available in the textbooks. On the contrary the students must study the works of others (the textbooks, new materials from library, internet and likewise), and they have to deconstruct this knowledge, combine this with existing praxeologies in order to develop new praxeologies as answers to questions – formulated by themselves or the assignment questions.

The teaching design

The starting point for testing SRP in the bidisciplinary setting was to formulate a generating question fulfilling the conditions set by the school regulations.

The design was created on the basis of a teaching material for mathematics at upper secondary level, published by Technical University of Denmark. The material deals with the function of painkillers in the body and its' modelling by differential equations (Jónsdottir et al., 2009). The reason for choosing this material as inspiration for the generating question is that many of the students involved in the experiment were interested in biology and wanted to work in the health care system later on. Hence the teachers assumed that these students would find a problem on the dosing of medicine relevant and interesting. This might not give students a better mastery of their immediately lived world but it could help them relate their school knowledge to real uses which, in the end, could fulfil the higher goal of a better mastery of their lived worlds.

Based on the material, the generating question was formulated. It starts by questioning how one of the most common drugs used in households can relieve patients from their pain, how the functioning can be described from a mathematical perspective and how that description can be used to design a correct dosing. The full formulation is shown below:

> Q_0: How can a patient be relieved from his pain by painkillers like paracetamol – how does deposit medication work and how can this be modelled mathematically? Q_1: Explain the biological functioning and consequences of

> taking paracetamol orally versus taking it intravenously. Q_2: Create a mathematical model using differential equations that illustrates the two processes and solve the equations in the general case. Q_3: Give a concrete example, where the patient is relieved from pain and estimate from your own model how often paracetamol has to be dosed – which parameters (absorption, elimination factor, bioavailability) are important to be aware of? $Q_{3,1}$: Does it make any difference whether the dose is given oral or intravenously? Use your models while giving your answer. (translated from Danish)

Notice that some of the derived questions are already given along with the generating question in order to guide the inquiry of the students (Chevallard, 2012, p. 11). It is crucial for the SRP to be successful that the students gets some guidance and are not left alone with a too open and overwhelming question. In this setting, the regulation of students' and teachers' work further necessitates that some of the "guiding" is provided from the outset. It should be possible for the students to see, from the outset, that their praxeological equipment in biology and mathematics can help them answer the generating question, and the given derived questions serve this purpose, asking for more specific cases to guide and delimit the student inquiry.

The formulation of the questions was followed by an a priori analysis before handing out the assignment. This a priori analysis will be presented in section on results.

Methodology

To carry out an *a priori* analysis means to explore what derived questions and answers could occur from the particular formulation of Q_0, i.e. what possible paths the students could follow based on their expected praxeological equipment and available media; concretely, a complete "tree" of derived questions and answers is produced. Figure 1 and 2 show the diagrams of the a priori analysis for the SRP considered in this study. In this case, the *a priori* analysis led to minor corrections of the design before it was tried out with students.

The school does not allow the use of lessons for guidance or classroom debate on the progress of the students work. Therefore, other ways to keep track of the students' work with the SRP were developed. To record the students' first thoughts on the generating question, they were asked to provide their spontaneous answer to the question in writing immediately after reading it. Two and four weeks later the students were asked to answer the following questions:

> What is your answer to the generating question right now? What have you done to answer the question? What are you planning to do next in order to come up with more fulfilled answers?

The teachers were only allowed to answer questions from the students after class. These conditions for guidance made it hard to track the exact progress of each student. Therefore the students were told only to ask questions if they gave them in writing by e-mail before meeting the teachers. It actually turned out that most questions could also be answered by e-mail. Examples of questions are given in the section on results.

Because of the little data available from the students working process, it is the outcome of the students' writings which is the main evidence of their study and research process. The reports were analysed as SRP, using the method developed earlier (cf. Hansen, 2009, pp. 60) and which I now describe. While reading the reports every small section was identified with the (derived) question it treats. An example could be "how to model a one-compartment system when knowing the diffusion of the drug from the vein alone relies on the elimination factor?" This can be answered by the praxeology of "setting up a first order differential equation from given conditions". This is a praxeology on mathematical modelling using differential equations. In this way the entire report was split up in small pieces of questions and answers (Q_i, A_i). The organisation and relation between praxeologies can be depicted by tree diagrams (see figure 3). The relations were identified from the way the student referred to or drew on previously presented praxeologies ie. sections or part of sections. When it comes to the parts of the reports consisting of pure biology, the praxeologies were only identified as a question and the answer given by the student – that is, I did not model or analyse biological praxeologies in detail, due to lack of knowledge in the field of biology.

The analysis of the students' reports was compared to the a priori analysis. The comparison of the diagrams showed to what extent the students had developed the intended praxeologies and maybe some unexpected ones. At the same time the diagrams show to what extent the two disciplines were incorporated and combined in the report and solutions. This helps to answer the crucial question: Does the formulation of the generating question function as a bidisciplinary task and do the student use and combine both disciplines while answering the assignment?

For the last part of the project the students were told to continue to ask questions by e-mail while rewriting their reports. The synopses were handed in electronically and during the oral exam written notes were taken. From this the new praxeologies were identified even though the synopsis format is not suitable for a thorough praxeological tree diagram analysis. Through these steps of analysis the results of the design and the students activities can be presented.

Results

As expected, there was a great diversity in the students' reports. Some students worked thoroughly with the questions and were able to formulate derived questions themselves – even explicitly. Others were not able to see the use of mathematics in the assignments and tried to answer the exact questions formally, without further inquiry. This was expected as the class was not particularly "strong" (mathematically and academically) – but many of them were hard working and for them the study phase seemed very enriching. They clearly developed new mathematical praxeologies during their work with the SRP, as will be explained in detail in this section

The analysis of the formulation of the assignments gives the tree diagram figure 1 which shows the connections between the generating question and derived ones.

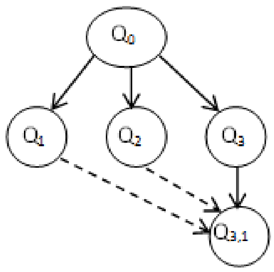

Figure 1: Tree diagram showing the formulation of the assignment

The dotted line indicates that question number $Q_{3,1}$ draws on the knowledge worked out as answers to questions Q_1 and Q_2. The solid lines indicates that the questions are derived questions in the sense described by Chevallard (2006); in short, derived questions are natural prolonging of the former in order to achieve a more detailed inquiry. The tree diagram in figure 1 is part of the *a priori* analysis of the assignment. To get a more complete picture of the potentials of the SRP design, a full *a priori* analysis was made. This analysis is presented in the tree diagram of figure 2. Question numbers refer to the same as those in figure 1. The rest of the questions are derived questions, which are the questions students are intended to work with in this particular SRP. The answers to those questions are the praxeologies the students are supposed to develop in the field of differential equations and nerve physiology in relation to the diffusion of a drug in the body. The lines connecting the questions have the same interpretation as in figure 1.

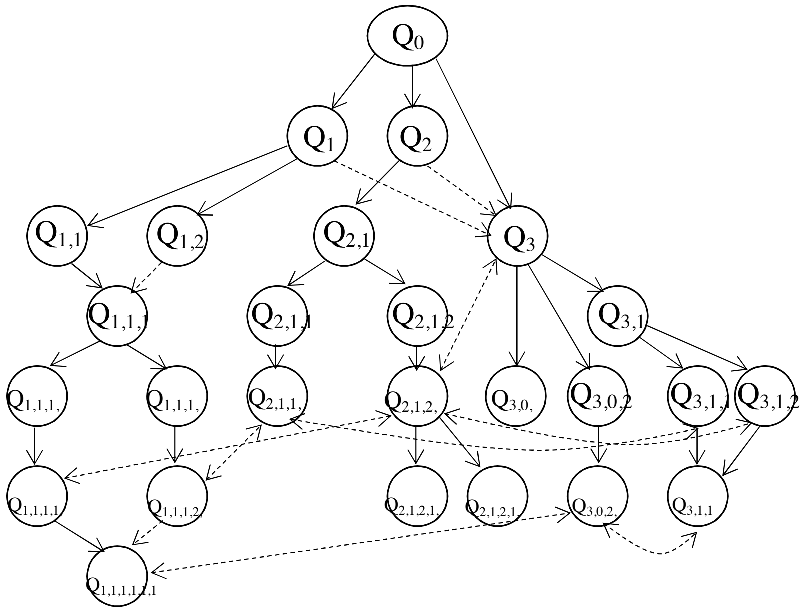

Figure 2: Tree diagram of the a priori analysis of the assignment. See the text below for the contents of each question.

The questions formulated by the author[3], only having an academic background in mathematics, during the a priori analysis is the following where question numbers corresponds to those of figure 2. The questions representing the expected praxeologies are written in italic (these are not handed out to the students):

Q_0: How can a patient be relieved from pain, using a drug like paracetamol? How does deposit medication work and how can this be modelled mathematically?

Q_1: Explain the biological functioning and consequences of taking paracetamol orally versus intravenously.

$Q_{1,1}$: What is the biological mechanism underlying the concept of pain?

$Q_{1,2}$: What kind of drug is paracetamol?

$Q_{1,1,1}$: How does paracetamol function in the body?

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[3] As mentioned earlier, this research was conducted by the author, who is the mathematics teacher and a didactic researcher. This means that the perspective of the questions is considered only from the standpoint of the mathematician. The *a priori* analysis would look differently, if it was carried out by others, with a different academic background.

$Q_{1,1,1,1}$: How does paracetamol function when it is dosed orally?
$Q_{1,1,1,1,1}$: How is paracetamol transported from the stomach to the vein biochemically seen?
$Q_{1,1,1,1,1,1}$: How long does it take from the drug is injected in the vein till a person is relieved from his pain?
$Q_{1,1,1,2}$: How does paracetamol function when it is dosed intravenously?
$Q_{1,1,1,2,1}$: What is the biochemical functioning of paracetamol in the vein?

Q_2: Set up a mathematical model using differential equations that illustrates the two processes and solve them in the general case.
$Q_{2,1}$: What is a differential equation?
$Q_{2,1,1}$: What can the differential equation $y' = ky$ model and what is the general solution?
$Q_{2,1,1,1}$: How do we model a one compartment system modelled using the elimination factor?
$Q_{2,1,2}$: What can the differential equation $y'(t) = c_1 z(t) - c_2 y(t)$ model and what is the general solution?
$Q_{2,1,2,1}$: How can we model the effects of the absorption using differential equations?
$Q_{2,1,2,2}$: How can we model the effects of the bioavailability using differential equations?

Q_3: Give a concrete example, where the patient is relieved from pain and estimate from your own model how often paracetamol has to be dosed – which parameters (absorptivity, elimination factor, bioavailability) are important to notice?
$Q_{3,0,1}$: What numbers can be put on the relevant notions and what do they tell?
$Q_{3,0,2}$: How can we model multiple dosing using the existing models?
$Q_{3,0,2,1}$: How often must the doses be given in order for the patient not to feel any pain?
$Q_{3,1}$: Does it make any difference whether the dose is given oral or intravenously? Use your models to support your answer.
$Q_{3,1,1}$: What does the model of multiple dosing look like in the case of intravenous dosing?
$Q_{3,1,2}$: What does the model of multiple dosing look like in the case of oral dosing?
$Q_{3,1,1,1}$: What differences appear while comparing the graphic presentation of the two functions of multiple dosing?

The diagram of figure 2 is satisfactory in terms of the requirements for the design, as it shows several paths for the students to pursue, with possibilities for the students to work interdisciplinarily, to activate their initial praxeological equipment, and potentially to develop new praxeologies in the field of differential equations and nervous physiology.

Results of the students writings

I will now present the outcomes of this teaching design. I will do so by presenting a well written report richly unfolding the intended praxeologies. After this I present a report written by a weak student only poorly unfolding the potentials of the design and finally I give some of the outcomes of the synopsis and oral exam.

As mentioned some of the students were able to realise these potentials and wrote mathematically rich and substantially bi-disciplinary reports. Figure 3 shows a tree diagram of the analysis of one of these reports in its final state. We can say a little about the process of the author of this report from what she wrote as spontaneous and intermediate responses to the generating question. Just after seeing the question, she noted that she needed to know something about the dosing of the drug in relation to the weight of a given person. She calls it the "strength" of the drug. And she needs to know something about how long the drug stays in the body, and refers to what she calls "the half-life of the drug"[4]. This she planned to use to find out how to relieve a patient from pain for a longer time period. This indicates that she believed from the start that the model involves an exponential function, without knowing anything else about this question.

Two weeks later (when again asked for her ideas on the generating question), this student also wants to know more about how paracetamol is functioning biologically, and she indicates that she needs more knowledge on mathematical modelling. This is what she is planning to study the next weeks. This indicates that she is narrowing down to more specific questions for her to answer.

The notion model or modeling in the students writings probably refer to the one the student encounters in her textbook and official documents for Danish high school, which is somehow close to the notion in mathematical competence theory (see Niss et al., 2002 and Blum & Fermi, 2009, p 46). However the approach to modeling in ATD is that it is the development of praxeologies in two domains answering a generating question.

In the text below several technical terms are used. They are translations of the notions the student used. Many of them comes from the biological field being modelled and therefore will not be explained further. As to differ questions formulated by the student from those she has adopted from the assignment handed out, the students' questions and formulations are put into squared brackets.

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4 She knows this notion from previous work on exponential function and from radio activity.

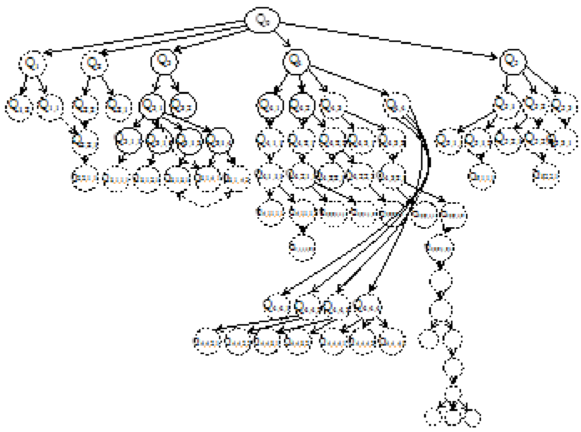

Figure 3: Tree diagram of the analysis of a handed in report. Question contents are detailed in the text.

This student actually formulated a number of derived questions in her report and used them as headings, e.g.: “Q_1: How is pain registered? Q_2 How does paracetamol relieve pain (pharmaco dynamic)? Q_4: How can the dosing be modelled mathematically based on the biological knowledge?” (Appendix A) Other headings were not phrased as questions but were simply a word as “Absorption”. Derived questions not phrased as above are identified through further discourse analysis of the text. Examples are $Q_{2,1}$: How does paracetamol relieve pain relative to the amount of dose? And $Q_{2,2}$: How does paracetamol relieve diffuse pain. The student relates her answer to this question to $Q_{1,1}$: How are diffuse pains registered and what are diffuse pains? This is indicated in figure 3 by a dotted line. I will now give a short review of this report, for an extensive list of the questions the student treats see Appendix A.

The student starts by posing and answering the questions: “How is pain registered?” (Q_1) and divides this into the treatment of what are diffuse pains and how they are registered as well as what are diffuse pains and how they are registered ($Q_{1,1}$ and $Q_{1,2}$). Then she poses the question: How does paracetamol relieve pain (pharmacodynamic)? (Q_2). This is dealt with through questioning how paracetamol relieves pain relative to the amount of dose, how it relieves diffuse pain, what effect the drug has on the nervous system and what is known about the drug in general ($Q_{2,1}$, $Q_{2,2}$, $Q_{2,1,1}$ and $Q_{2,1,1,1}$).

After this the student poses the question: How is paracetamol transported through the body (pharmacokinetics)? (Q_3). This is investigated through the study of how drug is transported in the case of orally dosing, how drug is transported in the case of intravenous dosing. The former is further explored by showing how the drug is absorbed in the body, how this process runs in the small intestine and how the drug is distributed in the body, which leads to a description of the biochemical conditions and mechanisms that are relevant for this problem, hence how substances are transported through cell membranes ($Q_{3,1}$, $Q_{3,2}$, $Q_{3,1,1,1}$, $Q_{3,1,2}$, $Q_{3,1,2,1}$ and $Q_{3,1,2,1,1}$). Finally the metabolism of paracetamol including the chemical reactions occurring and the elimination with the role of the kidneys and timescale of the process is presented ($Q_{3,1,3}$, $Q_{3,1,3,1}$, $Q_{3,1,4}$, $Q_{3,1,4,1}$ and $Q_{3,1,4,2}$). These questions and answers all represent pure biological praxeologies, which are found relevant in order to model the processes of dosing paracetamol a long with the discussion of how should be used….

Next the student poses question number Q_4: "How can the dosing be modelled mathematically based on the biological knowledge?". She finds the answer by looking at the form of the model in the case of intravenous dosing, how the proportionality between added amount of paracetamol and elimination can be modelled, what can be described by the equation $\frac{dA}{dt} = -K.A$, the biological interpretation of –K with respect to the former treated questions ($Q_{4,1}$, $Q_{4,1,1}$, $Q_{4,1,1,1}$ and $Q_{4,1,1,1,1}$). These are all bidisciplinary praxeologies where the student alternates between using the established biological praxeologies in the construction and justification of a first order differential equation – a mathematical object. She ends this section by finding the complete solution using a CAS tool and showing by hand, that this solution actually solves the equation ($Q_{4,1,1,1,2}$ and $Q_{4,1,1,1,2,1}$). These two questions are identified as pure mathematical.

After treating the more simple case she looks at the oral case and performs the same praxeologies though taking into account that she needs to treat the two compartments separately and combine these results in one equation describing the entire system ($Q_{4,2}$, $Q_{4,2,1}$, $Q_{4,2,1,1,}$ $Q_{4,2,1,1,2}$ and $Q_{4,2,2}$). She further argues how the added amount of paracetamol can be described by the solution to the differential equation of the stomach compartment and how the model incorporates the bioavailability ($Q_{4,2,2,1}$ and $Q_{4,2,2,2}$). Again, this is denoted bidisciplinary praxeologies. The student investigates what can be described by the equation: $\frac{dA}{dt} = A.K_a.A^{stomach} - K.A$, finds the solution and argues that the model solves the equation as in the simple case ($Q_{4,2,2,2,1}$, $Q_{4,2,2,2,1,1}$ and $Q_{4,2,2,2,1,1,1}$).

The student uses the two models to form functions describing the concentration of paracetamol in the blood, she gives all parameters numerical values and discusses both the mathematical and the biological interpretation of K_a>K ($Q_{4,3}$, $Q_{4,3,1}$, $Q_{4,3,2}$,

$Q_{4,3,2,1}$ and $Q_{4,3,2,1,1}$). Finally she discusses the knowledge of the numeric versions of the functions and its graphical representation. From these representations she discusses the long term effects, high amount dosing and how patients can be relieved from their pain through multiple dosing and how this can be carried out repeated dosing with constant amount of paracetamol ($Q_{4,3,2,1,2}$, $Q_{4,3,2,1,2,1}$, $Q_{4,3,2,1,2,1,1}$, $Q_{4,3,2,1,2,1,1,1}$ and $Q_{4,3,2,1,2,1,1,1,1}$). The method of these praxeologies is mainly mathematical but constantly links to her knowledge in the biological field and she concludes on the biological issues from the mathematical models. Hence These praxeologies are regarded bidisciplinary. The student further notes that multiple dosing leads to a concentration alternating around a mean called steady state. She uses the mathematical models to determine steady state level and whereas the patient feel a constant relieve of pain when maximum recommended dose is given every 4 and 6 hours (the two standard time intervals) ($Q_{4,3,2,1,2,1,1,1,2}$, $Q_{4,3,2,1,2,1,1,1,2,1}$, $Q_{4,3,2,1,2,1,1,1,2,1,1}$, $Q_{4,3,2,1,2,1,1,1,2,1,1,1}$, $Q_{4,3,2,1,2,1,1,1,2,1,1,2}$ and $Q_{4,3,2,1,2,1,1,1,2,1,1,3}$). As the praxeologies just mentioned these are regarded bidsciplinary for the same reasons.

In the end the student compares the two ways of dosing the drug with respect the type of pain it is supposed to relieve. This is done by a comparing the concentration profiles, discussing similarities and differences (Q_5, $Q_{5,1}$, $Q_{5,1,1}$, $Q_{5,1,2}$ and $Q_{5,1,1,1}$). These praxeologies are likewise bidsciplinary since biological results are based on mathematical models treated by mathematical tools. The treatment of Q_5 ends in a further investigation of intravenous dosing, as to what kind of situations and what kind of lack in health condition among patients calls for this kind of dosing ($Q_{5,2}$, $Q_{5,2,1,1}$, $Q_{5,2,1,2}$ and $Q_{5,2,1,2,1}$). These praxeologies are mainly biological. They discusses some of the results showed in the graphical representations of the concentration function, but it is only treated in a biological context. The last two biological praxeologies performed are examining the relation between concentration functions and the recommendations on the painkiller packages and further discusses whether the functions implies a change of recommendations ($Q_{5,3}$ and $Q_{5,3,1}$).

After this the students returns to the models and functions she has created discussing the limitations of these ($Q_{4,4}$ – the choice of numbering reflects praxeologies relation the rest of the SRP and not the chronology of the report). She starts by discussing in general terms the meaning of modelling the real world, then she turns to biological conditions effecting absorption, bioavailability and the pharmacokinetics in general due to the patient being pregnant, a child or elderly. This is supported by listing the consequences of taking other drugs, eating, vomiting or having diarrhea while taking paracetamol ($Q_{4,4,1}$, $Q_{4,4,2}$, $Q_{4,4,2,1}$, $Q_{4,4,2,2}$, $Q_{4,4,3}$, $Q_{4,4,3,1}$, $Q_{4,4,3,2}$, $Q_{4,4,4}$, $Q_{4,4,4,1}$, $Q_{4,4,4,2}$ and $Q_{4,4,4,3}$). These praxeologies are mainly biological though they are all used in a critique of the models created by the student.

As indicated, the path starting from Q_4 is mainly treating the mathematical organisation. The answers are constantly referring to the biologically field which is being modelled. Still the student uses pure mathematically praxeologies such as $Q_{4,1,1,1,2}$ and $Q_{4,1,1,1,2,1}$. These praxeologies are examples of intended mathematical praxeologies which the student has developed working with this specific SRP.

Comparing figure 2 and 3 it is obvious that the student has followed most of the intended path and even added necessary details in order to answer the question in a satisfying manor. The student also adds branches not intended such as $Q_{4,3,2,1,2,1,1,1,2}$, where she treats the notion of steady state concentration both mathematically and biologically.

The student asks three questions during the writing process and they concerns her critique of the mathematical models – she lists 3 points and asks if they are reasonable – the notion of deposit medication and how it is interpreted and finally she asks if she can put her mathematical calculations in appendix due to many pages of text. This means that her study of the sources is done without help from the teachers and the tree diagram is showing her working process with the SRP. This diagram and others like it (based on other student reports) show that it is possible to create bidisciplinary assignments on the basis of SRP that function well for some students.

Rich outcomes were found in other reports as well. Students normally having difficulties working on the theoretical level engaged themselves in the SRP and managed to develop arguments on how to model the transportation of a drug in the vein. One student explains that when you are modelling the change of the amount of drug in the vein, differential equations are suitable since they model how fast something changes. In a particular case she needs to know how much drug is added and how fast it eliminates from the vein. From this she presents the model, with the factors representing added and eliminated amount of drug. This student is normally quick at solving simple standard tasks, but she rarely argues precisely at the theoretical level. The reason for the change in the setting of the SRP could be that the student consulted classmates and was inspired by their work. Another reason could be that the entire assignment makes it obvious for her that she needs to justify her model explicitly – it is not possible to answer the questions "mechanically".

The students having difficulties to engage seriously with the SRP were those who generally find mathematics and biology hard. Some of those students did not find the topic interesting. They were able to solve simple questions involving simple praxeologies. Some of them did not succeed to combine mathematical and biological praxeologies, these students mainly referring the source (Jonsdottír et al., 2009) and some textbooks on the biological topic. When they were supposed to interpret the models, they would invent two persons in order to compare the amount of drug in the bodies – comparing a child and an adult, and, ignoring that the biological factors are different from children to adults. This shows that they were merely able to study the handouts based on separate praxeologies already developed during mathematics and biology classes. They did not develop the intended new praxeologies and so they were only able to solve simple tasks in the field of differential equations and human physiology. An example of a tree diagram of a report handed in by one of the weak students is shown in Figure 4.

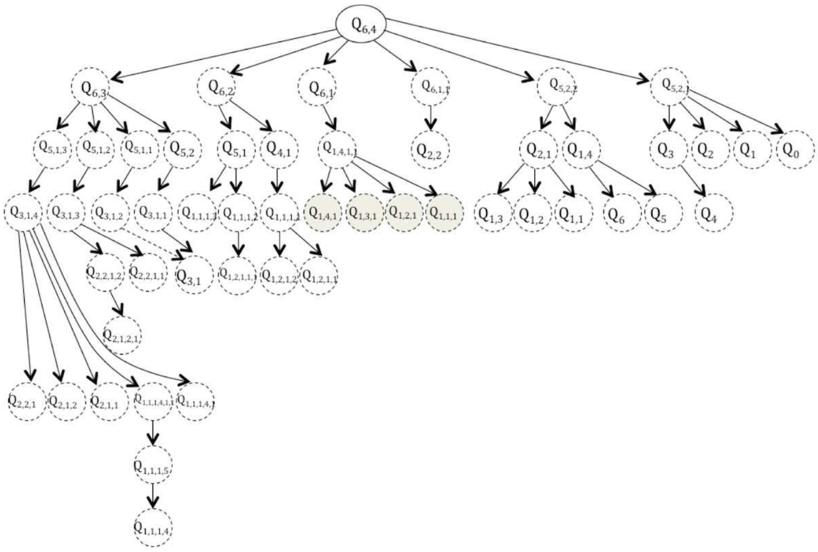

Figure 4: Tree diagram of the analysis of a report handed in by a weak student. An outline of the questions is presented in the text

The diagram of Figure 4 shows that the student spread her attention on many different directions (subquestions) but none of the questions are treated thoroughly or connected to others. The path starting from Q_6 is the only one which involves mathematics. The student presents some equations for calculating the amount of drug in the vein of an "average person", the maximum concentration of drug in the case of intravenous dosing, the time it takes to reach maximum concentration and finally an equation of the steady state concentration. She does not mention differential equations at all or how to deduce the models from them. This implies that the student has not developed the intended mathematical praxeologies. The same goes for biology. The presentation of the biological answers is very superficial and the text only cites sources in general terms. The praxeologies coloured grey in figure 4 actually short versions of question number $Q_{1,1}$, $Q_{1,2}$, $Q_{1,3}$,

$Q_{1,4}$ – nothing new is added to the text book presentation of the notions of absorption, distribution, metabolism and elimination. Only once the student combines two answers ($Q_{1,3,1}$ and $Q_{1,4,1,1}$ on how the kidneys contribute to the elimination of the drug). The rest of the report had a parallel structure, which indicates that the student was not able to combine the different answers to subquestions. The teacher proposed, as an explanation, that this student is often doing her work in last minute, and so she did not see how much effort she had to invest to properly answer the questions. This suggests that working with SRP requires adaptation through more than one experience, at least for some students.

Other students simply were not able to see the relevance of mathematics in the response to the generating question. An example is a student who answer the question on modelling by citing the handout without commenting or using the model. This indicates that the student only sees this question as a way to add mathematics to the report or project, but not as something necessary from the theoretical point of view. She spends several pages on drug development[5] and obviously finds this interesting. Maybe she did not have enough time for the mathematical part because she spent her time on what she found most interesting. This student usually was able to combine simple praxeologies but was not theoretically strong. This supports the hypothesis that the student did not see the need of mathematics to answer the generating question. How to deal with this concern will be discussed later.

Outcomes of the synopses and oral exam

To begin with, the focus was put on the reports, but interesting findings occurred during the students' work with synopses and at the oral exam. The students who did well on the reports were still performing well in the synopsis and at the oral exam. Some students who made acceptable reports were able to improve their work after the written feedback. As mentioned earlier I did not get equivalently systematic evidence from this part of the students' work. The findings presented below are therefore simple and tentative descriptions of student work in this phase.

One of the most interesting observations occurred with a student who had made a nice report using differential equations and explaining them using knowledge from biology. When asked to place the case of using paracetamol in a broader context she did an Internet search and found articles written by Danish researchers on the use of the drug during pregnancy. The article discussed whether there was a significant amount of degeneration of the genitals of baby boys when the mothers had taken paracetamol during pregnancy. The result was not clear and in fact there

5 As part of the teaching the class visited Faculty of Pharmaceutical Sciences at University of Copenhagen to learn about drug development research and how drugs are distributed and functioning in the body.

is no recommendation against the drug during pregnancy today. The article gave the numbers of women tested, the expected percentage of degenerations and the actual number of boys born with these problems. The researchers used a statistical test with level of significance at 5 %. The student did not know the particular type test therefor she performed a χ^2-test instead, which gave a p-value just above the level of significance. She used this in a discussion of the recommendations on whether the drug should be available outside pharmacies. She further referred to articles found in journals and on the Internet. This study was very surprising for both teachers. They did not know the relation to pregnancy nor had χ^2-test been part of the intended mathematical praxeologies for the SRP, but it was a tool the student knew from classes and put to use in a new context. This is a nice example of a potential of SRP: "that the contents learnt [...] have not been planned in advance" (Chevallard, 2012, p. 7).

The student who did the report represented in Figure 3 continued her work on the effect of paracetamol in the brain and the nervous system. She was able to explain how new ideas could be modelled and tested, as she focused on the problematique of mentally ill people whose abuse of paracetamol cause long-term damages. She discussed this in relation to question of the drug being sold legally outside pharmacies.

The results mentioned above from the oral exam are examples of students combining the study of works of others combined with an autonomous treatment of results. In this sense a more general aim for the SRP was reached. On the other hand, the students who handed in poorer reports were not able to improve for the synopsis and did not perform well at the oral exam either. There remains, thus, a considerable challenge in making this SRP successful for all students.

Discussion

Many students engaged in a real study process, to find answers on their own rather than just citing the works of others, which on the other hand seems to be the pitfall for other students. The real world problem seems to motivate the students for an inquiry where they can use and combine their previous knowledge and experience from both mathematics and biology.

The SRP enables most students to make the two disciplines interact. As already said it is crucial to choose a strong generating question that engages the students to develop the intended praxeologies, and the quality of this choice could secure the possibility of actual interdisciplinary work. This means that a thorough *a priori* analysis must be the starting point of all bidisciplinary SRP designs since the interaction between disciplines is clearly not obvious or automatic.

But there are still issues to deal with if SRP should be successful for all students. The interplay between the two disciplines was weak or absent in the work of some students. These students fail to see the need of one discipline (primarily mathematics) or were not able to realise it in the given setting. Probably it requires more directions, by way of concise questions in both disciplines, to secure that students develop new intended praxeologies. This was seen in the report written by the student focusing on drug development as well as the report depicted in Figure 4. The big question is how to detect and treat these obstacles while creating the design. This relates to the *a priori* analysis of the SRP designs and to a more theoretical study of the possible interplays of mathematics and biology. Is it sufficient that two teachers (representing each discipline) formulate the design? or is it necessary for the teachers to do an analysis of the didactic transposition (e.g. see Bosch & Gascón, 2006, pp. 55) of the interplay of the involved scientific disciplines in order to identify interdisciplinary praxeologies combining the school disciplines? What are the scientific interactions between biology and mathematics and how can they be transposed to interactions between the secondary school subjects? To identify bidisciplinary praxeologies and what questions they answer we need to know more about what a biological praxeology is (and more generally, what are praxeologies in the natural sciences). This is formulated by Mortensen (2011) and Madsen & Winsløw (2009) but in other contexts.

Another approach to bidisciplinarity is found by Hansen (2009, p. 35) who suggests that what constitutes a discipline (as well as interdisciplinary praxeologies) is the methods of the disciplines used in the particular praxeology together with the objects of knowledge. This means that in order to formulate more concise questions, it is needed to identify the methods of mathematics and biology respectively as well as the relevant objects of knowledge. From this one can form the didactic transposition of the bidisciplinary knowledge, which can be used in a reference model for the SRP while carrying out the a priori analysis. In this way one might be able to create the more concise bidisciplinary questions which seem to be needed by some students.

The general hypothesis is that after identifying the possible interdisciplinary praxeologies, one will be able to formulate more exact questions which allow students to see the need of combining the two disciplines, and to develop more precise and complete answers. Also, by focusing on the interplay between the disciplines we might be able to make the students develop new monodisciplinary (e.g. mathematical) praxeologies.

Another concern regarding the students who wrote the poor reports is if the generating question hinders their engagement. It is obvious it is almost impossible to find generating questions which everybody finds equally exiting. Maybe the question seemed too vague compared to what they are used to. This obstacle can be handled using SRP in every day teaching so the students know the concept and what is required of them.

Knowing how students generate these new or derived questions would certainly be another way to overcome this challenge formulating good generating questions. It is an open problem in ATD. It is assumed that if posed a generating question within reach of the students existing praxeological equipment they are able to consult relevant medias – or in this study they know, that they need to study more advanced differential equations or exponential models. Therefor they consult medias on these topics and from the media pose new more concrete questions. It is assumed if the generating question is more guided in order to secure the student develop certain praxeologies, some of the potentials of the design and inquiry process disappear. This is also discussed in relation to inquiry in (Artigue & Blomhøj, 2013, 806). Further study in this matter could be interesting to pursue.

Some of the difficulties among the weak students might have been avoided, if the external conditions and constraints had been different. In the study of Barquero, Bosch and Gascón (2007) and Thrane (2009) the procedure of carrying out the SRP is that students share their findings. They present their findings and discuss academically what path tends to be the most promising one, and then everybody follows it. These sequences secure that no one remains stuck, with no ideas of how to progress. There are several reasons for organising the SRP process this way. When the student argues that one praxeology is a better or more general solution to a certain task they learn the scope and limitations of each praxeology, which helps them developing the intended knowledge.

The reason for not creating these sessions during the testing of the teaching design was that the requirements set by the institutional frame prescribed that the project should not use mathematics or biology lessons for the work. The students were supposed to work autonomously or in groups of two – not as a whole class together. This condition makes sense since they are supposed to get training for their final autonomous project. But the students did actually meet after classes to discuss their findings. This process seemed fruitful. Still some of the students who needed it the most did not attend. Because of this one could argue for a loosening of the constraints so that it is allowed for the teacher to organise such sessions and to guide the debate. If the students engaged themselves in this process one could argue that they still work autonomously – just in a more collective manner.

A final point: for the bidisciplinary assignments to function, the teachers must engage themselves in what could be called a bidisciplinary SRP for themselves as well. It is not evident that both teachers know the knowledge field of the other discipline. Therefore, in order to form questions concerning the interplay between the disciplines, the teachers must study a certain amount of the other discipline. For an academically trained person, this task is reasonable and crucial for the SRP to function as bidisciplinary assignment. The gained knowledge should be used to perform the a priori analysis and reveal the possibilities and limitations of the two disciplines in treating a given problematique or generating question.

Conclusion

The experiment and open issues with the SRP design showed clear evidence for the advantages of using SRP as a model for designing bidisciplinary assignments. The *a priori* analysis secures that the possible paths of inquiry are connected in the sense that the disciplines are interacting – not just in theory but also in reality. The reports the students handed in substantiated this finding since most students actually pursued the intended paths and even identified new directions, corresponding to substantial new derived questions. The students even succeeded in giving more detailed arguments and rich mathematical sections of their reports. Still the format of an academic-like autonomous written report is a difficult task for the students, therefor it is suggested that students encounter these types of reports more often in order to deliver rich and detailed documentation for their inquiry process, which these SRP's represents.

The experiment also showed that the teachers must be prepared to engage themselves in a SRP as well. For the teacher to carry out the *a priori* analysis she must cross disciplinary boundaries in order to see possibilities and pitfalls in the SRP design. The teachers must do the inquiry of the bidisciplinary field before formulating the assignment. Though it should be noted that boundaries between mathematics and biology are historical and evolving constructions that do not have to be taken from granted outside school institutions – nor in the praxeological analysis done here in the case of mathematical questions extended to biological phenomena treated in the SRP.

Moreover the tree diagrams shows to be a strong tool for depicting the praxeologies presented in the reports as the result of the discourse analysis. This diagram compared with the one from the a priori analysis gives a more clear view to what extend the intended praxeologies are present in students work. Concretely the two presented tree diagrams show two very different reports. It could be a question for further study to what extend the tree diagrams can be direct indicators for the richness of students writings.

The experiment suggests that some of the conditions for carrying out this particular design were not to the advantage of all students. The fact that almost all work on the SRP was placed outside school, and the lack of debate on particular paths to take during the inquiry, were problematic to some students. On the other hand many students were successful in engaging themselves with the SRP.

The experiment finally revealed questions for further inquiry. It is still an unresolved task to formulate bidisciplinary questions which all students see as such. Moreover the notion of bidisciplinary praxeology needs further exploration in terms of how to define and identify them, and in terms of their role for students' success with monodisciplinary praxeologies. Further it is suggested that in order to carry out a sufficient a priori analysis it would be enriching to formulate an reference epistemological model as described in the didactic transposition. It is supposed that this could enlighten some disconnection regarding the students inability to see the full need of mathematics in their answer to the generating question.

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Appendix A

The entire report treats the questions listed chronologically in Appendix A. The question numbers refer to those of figure 3:

Q_1: How is pain registered?
$Q_{1,1}$: “What are diffuse pains and how are they registered?”
$Q_{1,2}$: What are acute pains and how are they registered?
Q_2: How does paracetamol relieve pain (pharmacodynamic)?
$Q_{2,1}$: How does paracetamol relieve pain relative to the amount of dose?
$Q_{2,2}$: How does paracetamol relieve diffuse pain?
$Q_{2,1,1}$:What is known about the effect of paracetamol on the nervous system?
$Q_{2,1,1,1}$: What is known about paracetamol in general?
Q_3: How is paracetamol transported through the body (pharmacokinetics)?
$Q_{3,1}$: How is the drug transported in the case of orally dosing?
$Q_{3,2}$: How is the drug transported in the case of intravenous dosing?
$Q_{3,1,1}$: How is the drug absorbed in the body?
$Q_{3,1,1,1}$ How does this process function in the small intestine?
$Q_{3,1,2}$: How is the drug distributed in the body?
$Q_{3,1,2,1}$: What biochemical conditions and mechanisms are relevant for this process?
$Q_{3,1,2,1,1}$: How are substances transported through cell membranes?
$Q_{3,1,3}$: How is paracetamol metabolized?
$Q_{3,1,3,1}$: Which chemical reactions occur during the metabolism of paracetamol?
$Q_{3,1,4}$: How is paracetamol eliminated in the body?
$Q_{3,1,4,1}$: What is the role of the kidneys, with respect to the metabolites?
$Q_{3,1,4,2}$: What is the timescale or half-life of paracetamol in the body?
Q_4: How can the dosing be modelled mathematically based on the biological knowledge? $Q_{4,1}$: What does the model look like in the case of intravenous dosing?
$Q_{4,1,1}$: How can the proportionality between added amount of paracetamol and the elimination be modeled?
$Q_{4,1,1,1}$: What is described in by the equation $\frac{dA}{dt} = -k \cdot A$?
$Q_{4,1,1,1,1}$: What is the biological interpretation of –k?
$Q_{4,1,1,1,2}$: What is the complete solution to the differential equation?
$Q_{4,1,1,1,2,1}$: How can one check the validity of a given solution?
$Q_{4,2}$: What does the model look like in the case of oral dosing with a two-compartment system?
$Q_{4,2,1}$: How can the stomach compartment be modeled?
$Q_{4,2,1,1}$: What is described by the equation $\frac{dA^{mave}}{dt} = -k_a \cdot A^{mave}$?
$Q_{4,2,1,1,1}$: What is described by $-k_a$?

$Q_{4,2,1,1,2}$: What is the solution to the differential equation?
$Q_{4,2,2}$: How can the dosing be modeled from the perspective of the vein compartment?
$Q_{4,2,2,1}$: How can it be argued that the added amount of paracetamol can be described by the solution to the differential equation of the stomach compartment?
$Q_{4,2,2,2}$: How is the bioavailability incorporated in the model?
$Q_{4,2,2,2,1}$: What is described by the equation: $\frac{dA}{dt} = F \cdot k_a \cdot A^{mave} - k \cdot A$?
$Q_{4,2,2,2,1,1}$: What is the complete solution to the differential equation?
$Q_{4,2,2,2,1,1,1}$: How can one check the validity of a given solution?
$Q_{4,3}$: How can the concentration of paracetamol in the blood be modeled?
$Q_{4,3,1}$: How does this look in the case of intravenous dosing?
$Q_{4,3,2}$: How does this look in the case of oral dosing?
$Q_{4,3,2,1}$: What numbers are reasonable for the constants: K, F, F_a and V?
$Q_{4,3,2,1,1}$: What is the biological interpretation of K_a>K?
$Q_{4,3,2,1,2}$: What function describes the concentration?
$Q_{4,3,2,1,2,1}$: How does the function look graphically?
$Q_{4,3,2,1,2,1,1}$: How can the concentration be interpreted in relation to longtime effect and high amount of paracetamol?
$Q_{4,3,2,1,2,1,1,1}$: How can a patient be relieved from his pain due to multiple dosing?
$Q_{4,3,2,1,2,1,1,1,1}$: How can this be carried out sequentially with constant amount of paracetamol?
$Q_{4,3,2,1,2,1,1,1,2}$: What is steady state?
$Q_{4,3,2,1,2,1,1,1,2,1}$: When and how is this state reached?
$Q_{4,3,2,1,2,1,1,1,2,1,1}$: What is concentration at steady state?
$Q_{4,3,2,1,2,1,1,1,2,1,1,1}$: In which cases are the amount of dose 1000mg every 4 hours?
$Q_{4,3,2,1,2,1,1,1,2,1,1,2}$: In which cases are the amount of dose 1000mg every 6 hours?
$Q_{4,3,2,1,2,1,1,1,2,1,1,3}$: Which function is modeling multiple dosing?
Q_5: When and why is orally and intravenous dosing used respectively?
$Q_{5,1}$: How can the two concentration profiles be compared?
$Q_{5,1,1}$: When do the two profiles reach their maximum concentrations?
$Q_{5,1,2}$: When does the effect of paracetamol die out?
$Q_{5,1,1,1}$: When is there an effective difference between the two forms of dosing?
$Q_{5,2}$: When is intravenous dosing preferable?
$Q_{5,2,1,1}$: In which cases will time be the determining factor for choosing intravenous dosing?
$Q_{5,2,1,2}$: Under what health conditions are the intravenous dosing preferable?
$Q_{5,2,1,2,1}$: What kind of conditions of the stomach makes the intravenous dosing preferable?
$Q_{5,3}$: What is the dosing profiles telling about the dosing of paracetamol compared to the recommendations on the painkiller packages?

$Q_{5,3,1}$: How should paracetamol be dosed according to the profiles?
$Q_{4,4}$: What biological factors are disregarded in the mathematic models?
$Q_{4,4,1}$: What is the relation between a (mathematical) model and the real world?
$Q_{4,4,2}$: What other biological factors affect the absorption K_a?
$Q_{4,4,2,1}$: What effect causes other drugs taken a long with paracetamol?
$Q_{4,4,2,2}$: What effects are caused by eating while taking paracetamol?
$Q_{4,4,3}$: What factors can effect the bioavailability?
$Q_{4,4,3,1}$: What are the consequences of vomiting?
$Q_{4,4,3,2}$: What are the consequences of diarrhea?
$Q_{4,4,4}$: What other factors affect the pharmacokinetics?
$Q_{4,4,4,1}$: What effects are caused by pregnancy?
$Q_{4,4,4,2}$: What effects are due to the person being a child?
$Q_{4,4,4,3}$: What effects are due to the person being elderly?

Part III

How to generate autonomous questioning in secondary mathematics teaching?

3

Paper II: SRA and autonomous Questioning

This paper presents the second design and its experimentation. The design is a sequence of SRA regarding exponential functions. The analysis of the paper focus on the students work with the doubling time, where interesting potentials with respect to exploit students as media in the study and research processes became explicit. The design changed the didactic contract, structure the group formation, explicitly proposed media for the students to study and was restricted by the students preparation of thematic projects for the potential oral exam. The paper address research question 1, where the employed didactical tools seem to create favorable conditions for students development of coherent praxeologies. The paper was submitted October 2016 to: *Recherches en Didactique des Mathématiques*.

HOW TO GENERATE AUTONOMOUS QUESTIONING IN SECONDARY MATHEMATICS TEACHING ?

Britta Eyrich Jessen

HOW TO GENERATE AUTONOMOUS QUESTIONING IN SECONDARY MATHEMATICS TEACHING?

Abstract – In mathematics education it is still a major challenge to find ways to nurture students' posing and pursuing their own questions in order to learn mathematics. During the last three decades problem posing has been explored through different approaches and in empirical studies. This paper presents the result of an empirical study, where teaching was designed and conducted based on The Anthropological Theory of Didactic. It show how a changed didactic contract and teacher posed open questions can support students' autonomous questioning, leading to the development of knowledge among students that goes beyond curriculum requirements among students.

Key words: problem-posing, the anthropological theory of didactics, study and research activities, upper secondary mathematics, exponential functions.

¿CÓMO GENERAR CUESTIONAMIENTO AUTÓNOMA EN LA ENSEÑANZA DE MATEMÁTICAS DE SECUNDARIA?

Resumen – En el campo de la educación matemática sigue siendo un gran desafío hacer que los estudiantes planteen y busquen respuestas a sus propias preguntas con el fin de aprender matemáticas. Durante las últimas tres décadas, la formulación de problemas se ha estudiado a través de diferentes enfoques y en estudios empíricos. En este artículo se presenta el resultado de un estudio empírico donde el proceso de enseñanza se diseñó y llevó a cabo en el marco de la Teoría Antropológica de lo Didáctico. Se muestra cómo un cambio en el contrato didáctico y docente basado en preguntas abiertas puede servir de base para el cuestionamiento autónomo de los estudiantes. Este nuevo planteamiento lleva al desarrollo de conocimientos entre los estudiantes que van más allá de los requisitos curriculares.

Palabras-claves: formulatión de problemas, teoría antropológica de lo didáctico, matemáticas de secundaria, función exponencial,

COMMENT GENERER DES QUESTIONS AUTONOMES DANS L'ENSEIGNEMENT DES MATHEMATIQUES AU LYCEE ?

Résumé – Dans le domaine de l'enseignement des mathématiques reste un défi majeur nourrir les élèves à soulever et de chercher des réponses à leurs propres questions afin d'apprendre les mathématiques. Au cours des trois dernières décennies, la formulation du problème a été étudié grâce à différentes approches et études empiriques. Cet article décrit les résultats d'une étude empirique où le processus d'enseignement a été désigné et réalisé dans le cadre de la Théorie Anthropologique du Didactique est présenté. Il démontré comment un changement dans le contrat didactique et enseignants ont posé des questions ouvertes peuvent soutenir l'interrogatoire autonome des élèves. Cette nouvelle approche conduit au développement des connaissances chez les élèves qui vont au-delà des exigences du programme d'études.

Mots-Clés : formulation de problèmes, théorie anthropologique du didactique, activités d'étude et de recherche, mathematiques au lycée, fonction exponentielle.

1. INTRODUCTION

A practical and broad characterisation of research is "posing questions and searching for answers". In educational research efforts have been made to engage students in more research like activities (Artigue & Blomhøj, 2013, p. 797). Since the 1980s mathematics educators have had an interest in the study of how we can nurture or promote students to pose questions and formulate answers to these. Some of the reasons for this research interest were formulated by Singer, Ellerton & Cai (2013, p 2):

"Problem posing improves students' problem-solving skills, attitudes, and confidence in mathematics, and contributes to a broader understanding of mathematical concepts and the development of mathematical thinking". The potentials of problem posing have led to formulations, in curriculum and educational standards, on requirements for students to pose and treat mathematical problems (e.g. Ministry of Education of Denmark, 2013; NCTM, 2000, p. 335). When inviting students to engage in activities similar to researchers an inquiry approach is often chosen. A common feature in inquiry approaches, which stems from the work of Dewey, is the students' formulation of questions (Artigue & Blomhøj, 2013, p. 800). The design tool employed in this study support inquiry based teaching, yet inquiry is not the core interest of this paper. The interest and emphasis will be put on students' autonomous formulation of questions, and on contributing to a new direction in problem posing research: to develop frameworks and structures to guide the problem posing experience (Liljedahl et al., 2016).

Research on students' question formulation range from how students can pose questions based on a number of informations given to them, to how students create problems similar to newly solved problems, on to the description of a phenomena based on which problems should be formulated (Bosch & Winsløw, 2015, pp. 371). Despite good intentions and much research, most teaching can still be characterised as transmission of syntheses – that is the cultivated version of knowledge as it is presented in most textbooks. The problems and questions students are supposed to work on, are put forward by teachers, which give students little experience of exploring a piece of knowledge autonomously (Bosch & Winsløw, 2015, pp. 362). This is not in alignment with the needs of the students in real life. Referring to Kilpatrick and his argument that in real life most problems must be posed by the person who solves the problem (Kilpatrick, 1987, p 124), Bosch and Winsløw state that: "Still, the challenge remain: is it feasible and desirable to have students take a more active role in identifying or formulating the questions they work in, and thus make their activity more akin to what Kilpatrick considers the situation "in real life outside school"?" (Bosch & Winsløw, 2015, p. 344). To address the question of feasibility, Bosch and Winsløw suggest to develop study and research paths (SRP) based on the Anthropological Theory of Didactics (ATD). This is the motivation of this paper, which treats the following research question:

How can a changed didactic contract support the management of autonomous questioning in a Study and Research Activity?

Autonomous questioning refers to situations where students pose questions to an answer they have studied or been presented. In this context an answer can be a piece of knowledge, a technique or a theorem. Autonomous questioning does not cover questions such as "will you please repeat?" or "please explain?". The goal is students raising questions, which address a mathematical notion, technique or phenomenon, thus it generates a confirmed mathematical study process for the students posing the question. In this sense, the autonomous questioning relates to Kilpatricks' claim on who formulate problems in real life.

2. THEORETICAL BACKGROUND

The study of this paper is based on notions from ATD, which will be presented in this section. SRP was introduced by Yves Chevallard (2006b & 2015) as a tool for designing autonomous transdisciplinary student work in upper secondary education in France (Winsløw, Matheron & Mercier, 2013, p 269). A SRP is initiated when a group of students begin the study of a generating question Q_0. In a teaching context the teacher formulates the generating question in advance. The question should be strong enough to guide an exploration of a knowledge domain. Students should understand the question but not be able to answer it, unless they engage in a study and research process. This process is supposed to be driven by initial hypothesis of an answer, which is incomplete and therefore lead to new, derived questions Q_i (Chevallard, 2015, p. 179). In order to answer the derived questions, the students are supposed to study media to gain new knowledge. Media are the works of others, like textbooks, webpages, podcasts and other materials produced in order to disseminate (mathematical) knowledge (Kidron, Artigue, Bosch, Dreyfus & Haspekian, 2014, p. 158). The students are supposed to deconstruct the new knowledge and reconstruct it as answer to a question they work on. In the reconstruction process, the students are supposed to draw on previously acquired knowledge and combine it with the knowledge from studied media. The process of reconstruction of knowledge is characterised as research, which takes place in a milieu. The milieu in ATD is defined as works to study, previous acquired knowledge and the question considered (Kidron et al., 2014). Students' knowledge construction is then concieced as the result of the dialectics between study and research processes (Winsløw et al., 2013, p. 269). The study and research process leads to a number of paths, detours and dead-ends in the process of developing a coherent answer for the generating Q_0 (see Bosch & Winsløw, 2016, p. 350). The numbering of the derived questions indicates the relation between the derived questions and the paths they belong to.

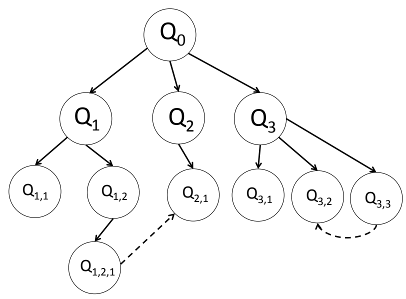

Figure 1. The tree diagram showing the derived questions and their internal relation.

Figure 1 shows a "tree diagram" depicting paths (or branches) of questions. It is possible for questions from different branches to relate, as between $Q_{1,2,1}$ and $Q_{2,1}$ in figure 1. This means that the answer of $Q_{1,2,1}$ draws on the same technique (e.g. technique for solving equations) as applied in the answer of $Q_{2,1}$. This type of representation has been used for depicting a priori analyses of teaching designs, and give a sense of the generative power of Q_0 - possible paths and derived questions. The posteriori analysis of realised paths and questions can also be depicted as tree diagrams, showing the realised questions of the process.

The herbartian schema captures the role played by media and milieu in the process of constructing knowledge. In this study, the herbartian schema explicate how students posed questions based on the study of answers provided by their classmates.

The herbartian schema shows the interaction between a didactic system and a milieu, which leads to the development of a personal answer, $A^\heartsuit$, to Q_0:

$$[S(X,Y,Q) \curvearrowright M] \hookrightarrow A^\heartsuit$$

Here, S represents the didactic system consisting of a group, X, of students, a group, Y, of people assisting the students (this can simply be one teacher), and the question, Q, they study together. The hart is only used for specific answers provided by a group. Generalised versions of answers are not given a hart in the following analysis. The study of Q is conducted as an interaction with a milieu M. A study process is based on students' previously acquired knowledge, which is regarded as previously developed answers, A°, and may appear as elements of the milieu. The available media is denoted O_i. These can be suggested by *Y* or autonomously be drawn upon by *X*. Accordingly the milieu can be written as the set: $M=\{A_1^{\circ},A_2^{\circ},\ldots,A_n^{\circ},O_m,\ldots,O_{k+1},Q\}$ (see Bosch & Winsløw, 2016, p. 31; Kidron et al., 2014, p 157). This analysis can be applied for any kind of teaching, where a group of students work on a question, assisted by a teacher.

3 EMPIRICAL DEVELOPMENTS – AND SRA

During the last decade, empirical studies have been conducted showing the potential of SRP's as a design tool for teaching, often as supplement to more common forms of classroom teaching. García and Higueras (2005) used a SRP at secondary level for the purpose of teaching proportional relationships and functional relationships, through a generating question on savings plans for an end-of-year trip. Barquero, Bosch and Gascón (2013) studied several iterations of a workshop on modelling attached to a mathematics course in the first year of an engineering programme. The purpose was to support the development of raison d´être for course content and to relate the different elements of the content. Serrano, Bosch and Gascón (2010) studied the tutorials of a mathematics course at first year university studies of economics. The problem to be solved during tutorials concerned a report on sales forecast for a private company and was based on mathematical modelling with one variable calculus.

Recent research seeks to explore the potential of designing course activities not anchored in traditional lectures and other common classroom activities. Jessen (2014) designed a bidisciplinary project for upper secondary education, where students were supposed to write individual papers combining mathematics and biology. In order to guide the study process, a few derived questions were posed together with the generating question. Students were guided through email correspondence based on questions posed by the students. Rasmussen designed a course element of an interdisciplinary course combining mathematics and science called "Health – risk or chance". Rasmussen focused his design on supporting the autonomy of the students by methods called 'selective picking', side questions and student diaries (Rasmussen, 2016). Most recently Florensa, Bosch and Gascón have designed an engineering course on elasticity of materials, where questions for weekly status reports handed in by the students guide the study and research process (Florensa et al., 2016, p. 7). Otaki, Miyakawa and Hamanaka (2016) designed three lessons of proving activities as SRP drawing on Internet search as media for further study.

All of these studies share the aim of students' pursuing a multitude of paths and posing what can be characterised as "implicit questions". In their account for students' work, researchers interpret elements of reports or diaries as signs of interest for further study or research but not necessarily as explicit questions (Rasmussen, 2016, p. 167), (Jessen, 2014, p 205). In the study of Rasmussen, students were asked to pose questions for the teachers of the course, which they then would like teachers to give a presentation of at the beginning of the following session (Rasmussen, 2016, p. 166). These studies substantiate the advantages in students' construction of knowledge through questioning and relate to the first of three issues raised by Bosch and Winsløw with respect to the practical realisation of sustainable questioning:

"What are the didactic and mathematical infrastructures (and resources), as well as the associated knowledge, required for the design, monitoring and evaluation of sustainable study and research processes?" (Bosch & Winsløw, 2015, p. 33).

Rasmussen addresses this point explicitly by employing "selective picking" and the use of "side questions" (Rasmussen, 2016). Further, he discusses SRP-based teaching in a course frame with a "monumentalistic curriculum" and a changed didactic contract. The SRP took up 20% of the course activities in a pre-service teacher education course (Rasmussen, 2016, p. 161). In the case study by Jessen, no content-based curriculum existed for the projects but the knowledge acquired by the students should relate to the content of the curriculum for mathematics and biology separately (Jessen, 2014, p. 203).

In order to explore the potentials mentioned above in a classroom where specific curriculum requirements must be met, we propose to consider Study and Research Activities (SRA) as a design tool.

3.1 The notion of SRA

It is a challenge to design teaching based on generative questions when current curriculum structures are "monumentalistic" and to some extent list a number of works to visit (Rasmussen, 2016; Chevallard, 2006). In ATD, SRA are described as a practical organisation of the study of works described in curriculum. In the case of teaching based on SRA, Chevallard points to the risk of atomising the mathematical knowledge, which could easily lack the rationale of the developed techniques and the motivation for the questions posed to the students (Chevallard, 2006a, p. 18). Barquero and Bosch regard SRA as a special branch of a SRP focusing on a certain answer A_{k°. Whenever the teaching is based on a generative question, Barquero & Bosch argue that: "what then appears is a sequence of linked study and research activities called study and research paths (SRP)" (Barquero & Bosch, 2015, p. 261). There is no clear line between SRP and SRA but it can be said that together, SRP and SRA provide tools for describing teaching and learning processes ranging from transmission to inquiry based approaches (Barquero & Bosch, 2015, p. 262). Barquero, Serano and Ruiz-Munzón identify three types of SRA's starting with SRA disseminating a pre-established answer to the type of SRA, which engage students in "search, de- and re-construction of external answers and objects according to the new SRP needs" (Barquero, Serrano & Ruiz-Munzón, 2016, p. 3). The common feature of these descriptions of SRA's is that they support students' development of answers being similar to the answer intended by the teacher: $A^{\blacktriangledown} \sim A_{Y^\circ}$.

The tree diagrams have been used for illustrating sequences of SRA's and their interrelation. It can be depicted as in figure 2, which Barquero, Serrano and Ruiz-Munzón used to illustrate a sequence of SRA's on modelling different functions and sequences.

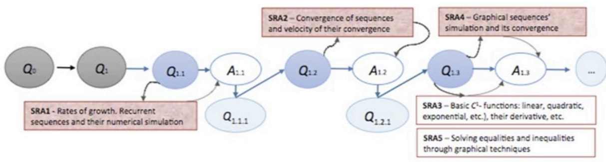

Figure 2: The picture of a sequence of SRA indicating how these are interrelated and the derived questions they generate (Barquero et al., 2016, p. 6).

4 THE MANAGEMENT OF STUDY AND RESEARCH PROCESSES

Previously Schoenfeld has characterised traditional mathematics teaching as: rules presented, explained and rehearsed (1988, p. 161). The kind of teaching where students expect the teacher to present a rule, provide an example and then ask them to rehearse the use of the rule on a number of examples, is still dominant. Chevallard has characterised this prevailing teaching paradigm as visiting monuments (Chevallard, 2015), meaning that students are presented some rule, which they are supposed to appreciate through its use in textbook examples and exercises. The didactic contract of this teaching leaves the responsibility of presenting rules and reasons to the teacher.

Brousseau characterises the didactic contract as: “These (specific) habits of the teacher are expected by the student and the behaviour of the student is expected by the teacher; this is the didactical contract” (Brousseau, 1997, p. 225). Mutual but different expectations and interpretations of teachers’ and students’ activities in classrooms will always be in play. At upper secondary level the students have gained experience with the school system and therefore they have an expectation regarding their own and their teacher’s roles in the classroom. In the management of the sequence of SRA’s in the present study, the roles of teacher and students were changed.

In the SRP and SRA as we have seen above, much initiative lies with the students since they are the ones who must engage in study and research processes, pose derived questions, and select and study media. Bosch and Winsløw report that in experiments with SRP, it is frequently observed that the students are resistant to accept the new didactic contract, but also that teachers have had a tendency to revert to the prevailing didactic contract (Bosch & Winsløw, 2015, p 369). Rasmussen identified a group of derived questions as a “residual group”. This group of questions was interpreted as a possible “metadidactic resistance to the changed didactic contract” (Rasmussen, 2016, p. 169). In the experiment of this paper, the change of the didactic contract was made explicit through requirements for the students. In particular, students were expected to formulate questions and answers and share it with the class.

5. CONTEXT OF THE STUDY

The teaching experiment was conducted at upper secondary level in Denmark. The class had their main elective focus in humanities (languages) and only took one year of mathematics. Hence they were expected to be less keen on mathematics lessons. The class had 24 students aged 15-16. Mathematics at this level is evaluated in a high stake oral exam where student present written thematic projects on various mathematical topics. Here a thematic project is a synopsis like format prepared for oral exams covering a number of non-standard mathematical questions (see further description by Grønbæk, Misfeldt

& Winsløw (2010)). The sequence of SRA's prepared the student to write a synopsis on exponential functions.

The author was both the ordinary mathematics teacher of the class as well as a didactic researcher. The experiment was conducted with the author as the teacher and another mathematics teacher who observed the lessons. The observing teacher taught the class in another discipline. Hence he knew the students and they knew him. The field notes, taken by the observing teacher, include the dialogue from students' presentation of preliminary answer at the whiteboard, what they wrote, and pointing gestures. In addition pictures were taken of the boards to support the notes.

It is worth to notice that the class had already completed a sequence of SRA's on linear functions. This means that students were familiar with the changed didactic contract: they were familiar with the requirement of providing an answer for the posed question and to present preliminary versions at the whiteboard.

The class was divided into eight groups where the members of one group performed equally in mathematics.

6. THE KNOWLEDGE TO BE TAUGHT

It is stated in the curriculum that: "students should be able to […] use relations between variables for the purpose of modelling data, predict how the modelled system evolves, and be able to discuss how good the model fits the system". Furthermore they should work with "equations describing […] exponential relations between variables […]" (Danish Ministry of Education, 2013). The curriculum is supported by guidelines, which suggest that the teaching of exponential functions should build on arithmetic calculations with exponents and how to interpret expressions such as $\sqrt{2}$ or 10^{100} (Danish Ministry of Education, 2010, p. 5). However official documents leaves out what rationale could or should be developed for the notion of exponents and exponential functions. Textbooks at this level do not explain the expression a^x, where $x \subset \mathbb{R}$, except from pointing out that it can easily be found using a calculator as if the exponent is a natural number (Jessen, 2015, p. 72). Textbooks disregard that the expression cannot be interpreted as the product of finite number of *a*'s (Winsløw, 2013, p. 5). Hence, the textbook does not elaborate on a rationale either.

The aim of the sequence of SRA's was for the students to know and be able to use the notion of exponential relation between variables

y and x as in: $y = b \cdot a^x$, a, b being constants of real values. Further students were supposed to be able to find the expression of the exponential function passing through two points on its curve, (x_1, y_1) and (x_2, y_2), using the formulas: $a = \sqrt[x_2 - x_1]{\frac{y_2}{y_1}}$ and $b = \frac{y_1}{a^{x_1}}$. Finally students must be able to find the doubling time of the growth using the formula: $T_2 = \frac{\log(2)}{\log(a)}$. These formulas represent the "monuments" of the curriculum for exponential growth at this level (Danish Ministry of Education, 2010, p. 5) and the intended answers of the teaching. Students were supposed to acquire the use of them but need not to know the rationale of these answers, or what questions initially led to these answers. In contrast to curriculum, the sequence of SRA's aimed at students developing answers through work on questions, which cover these techniques along with some reasoning behind the techniques. The analysis of this paper will focus on the students work with doubling time.

7. PRESENTATION OF THE TEACHING DESIGN

The experiment explored a sequence of four SRA's on exponential growth. The SRA's are linked together and have the potentials for students to develop the above monuments as coherent knowledge. An overarching Q_0 capturing the sequence of SRA's is the following:

Q_0: What characterises an exponential function and where can it be applied?

This question was answered in the students' thematic projects. To guide the study of Q_0, students were posed the following questions, which were studied jointly by the groups in class:

Q_1: Grandparents starts a saving account for their newborn grandchild by putting 5,000 dkr into an account at an annual rate of interest of 2.5%. Bank regulations say that the balance may not exceed 50,000 dkr. Will that be a problem?

Q_2: The neighbours have a similar savings account for their child and initially they put 5000 dkr into his account. After 10 years the balance has increased to 5947.22dkr. How much money can the neighbours' kid withdraw?

Q_3: If the children only are allowed to withdraw their money when the amount is doubled, how long should they wait?

Through their work with these questions, students will gradually expand their knowledge on exponential growth, and they need to use knowledge from former SRA's in the construction of answers for the next question. The last question, Q_4, concerned regression carried out using spreadsheets.

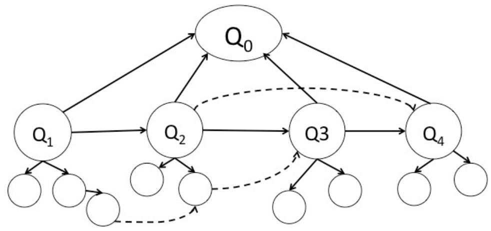

Figure 3: The relation between questions in the sequence of SRA's, which constitute the teaching of exponential function.

The reason for putting Q_0 above the others is that each SRA individually should generate a subanswer to Q_0. At the same time the questions are related as indicated by the horizontal arrows. The blank circles indicate potential derived questions and how they can be related to other derived questions. In the presentation below we focus on Q_3, since this was the question, which most clearly led students to pose questions autonomously and beyond what was expected.

For each SRA, students were guided in their work by media explicitly proposed by the teacher. The media included certain pages in a textbook, online pages and video clips. The division of the class based on their previous achievements was done to secure that students existing answers were similar. This is important because of the intention of students' construction of answers based on their existing knowledge and the de- and reconstruction of studied knowledge. If one student performed better than the remaining group, there is a risk he presents the others with monuments to visit and hinder their learning potential. After 5-7 minutes, the groups were asked to present their work in fields occupying 1/8 of the whiteboard. The groups were not allowed to erase anything, as they wrote and drew their preliminary ideas.

7.1 The a priori analysis of the SRA

The a priori analysis of Q_3 is shown in figure 4. The numbers in the circles correspond to those of the questions listed below the figure.

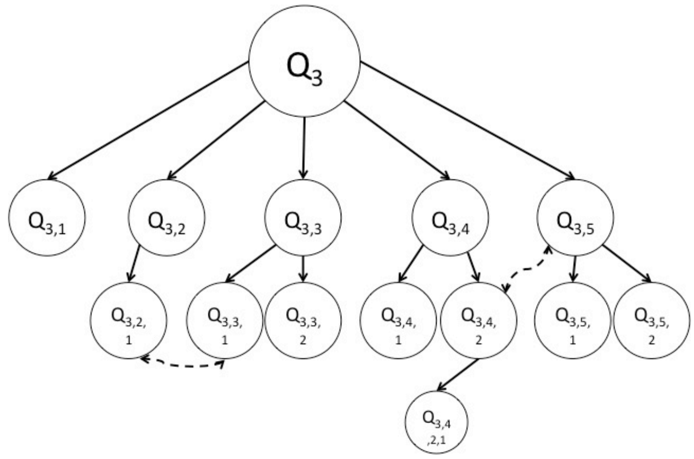

Figure 4: The a priori analysis showing the possible paths to an answer to Q_3.

The questions below are formulated by the author as a priori analysis of Q_3.

$Q_{3,1}$: How can we solve the problem by "trial and error" meaning, calculate the value of y for different values of x in the expression: y=5,000·1.025^x until we get y = 10,000?

$Q_{3,2}$: How many times must the initial amount of money be multiplied by the factor 1.025 to exceed 10,000?

$Q_{3,2,1}$: How can we answer the question if the calculations from the above question are plotted in a coordinate system, the graph drawn and the x-value corresponding to y = 10,000 found graphically?

$Q_{3,3}$: How can we find the solution by drawing the graph, which show the relation $y = b{\cdot}a^x$ in a coordinate system?

$Q_{3,3,1}$: How can the above strategy be done with pen and paper?

$Q_{3,3,2}$: How can it be done with a computer program? (CAS or spreadsheet)

$Q_{3,4}$: How can we solve the equation 10,000 = 5,000·1.025^x with respect to x?

$Q_{3,4,1}$: How can the equation be solved by a CAS-tool (Maple, Geogebra, TI Nspire , etc.)?

$Q_{3,4,2}$: How can we solve the equation 2 = 1.025^x with respect to x?

$Q_{3,4,2,1}$: How can we solve the equation by "trial and error" with different values of x?

$Q_{3,4,2,2}$: How can the identity $\log(a^x) = x{\cdot}\log(a)$ solve the equation?

$Q_{3,5}$: How can the formula $T_2 = \log(2) / \log(a)$ solve the problem?

$Q_{3,5,1}$: How can it simply be used as an algorithm?
$Q_{3,5,2}$: How can the formula be justified?

There is a clear relation between $Q_{3,4,2,2}$ and $Q_{3,5,2}$ in the sense that the technique of $Q_{3,4,2,2}$ is part of the reasoning needed to answer $Q_{3,5,2}$. The questions could reasonably be pursued further, however to pose these two questions is already beyond the scope of the curriculum and the ministerial guidelines. Other of these potential questions represent strategies to solve the problem at hand in more or less precise ways, which was assumed to be possible paths for the students to follow with their former achievements in mind.

8 METHODOLOGY

The posteriori analysis is based on the field notes and pictures of the whiteboards. These data were analysed with respect to what questions and answers they provide with respect to the question Q_3. Some derived questions were explicitly, and others implicitly posed. An example of an implicit question was when a group started to isolate x in the equation $10{,}000 = 5{,}000 \cdot 1.025^x$. They wanted to answer how long it took before the balance was doubled, counted from the starting point where $x = 0$. It means they had asked the question: “how can we solve this equation with respect to x, which represents the time it takes to double the balance?”.

An explicitly formulated question was: “why does that happen?” pointing to another group’s answer, where the group used a specific mathematical rule on logarithms. In this example the question was rephrased as: “why does this specific rule solve the problem?”.

Diagram 5 shows the results of the posteriori analysis. Grey circles mark the autonomous questions posed by students. The sense in which these represent autonomous questioning will be explained below.

9 RESULTS

Two cases will be presented where students pursued their own questions in the study and research processes, as examples of autonomous questioning. In the first case students study the nature of logarithms. In the second case students investigate certain “time constants” based on that knowledge regarding logarithms.

The class raised a number of initial questions from the top of their heads before they studied the proposed media and conducted any research:

$Q_{3,0,1}$:"what figures do we have from earlier? 5,000 and 2.5%?"

$Q_{3,0,2}$:"Can we use the $K_n = K_0\ (1+r)^n$ formula and isolate n?"

$Q_{3,0,3}$:"How do you take the n^{th} root using nSpire [the CAS-tool]"? And "how can you solve this problem [pointing towards the equation $1.025^x \cdot 5000 = 10{,}000$]?"

The numbering indicates that it was preliminary questions. All groups got the same reply form the teacher: the advice to study the proposed media, denoted by O_1, O_2, O_3. The media is a classic textbook for this level of mathematics (Clausen et al, 2010, p. 72-74), and two YouTube videos from a Danish website similar to Khan Academy run by high school teachers (Clausen & Clausen, 2014). The questions indicate that students start to pose questions immediately. Moreover, the questions were all addressing the mathematical content of the question and therefore they constitute autonomous questioning with respect to Q_3.

Below the answers of the groups are numbered as in the a priori analysis above, showing paths realised by the students. The tree diagram of the a posteriori analysis is presented in figure 5.

$Q_{3,4}$: How can we solve the equation $10{,}000 = 5{,}000 \cdot 1.025^x$ with respect to x?

$Q_{3,4,2}$: How can we solve the equation $2 = 1.025^x$ with respect to x?

$Q_{3,4,2,2}$: How can the identity $\log(a^x) = x \cdot \log(a)$ help to solve the equation?

$Q_{3,4,2,2,1}$: How can we prove the identity $\log(a^x\ ) = \mathrm{x} \cdot \log\ (a)$?

$Q_{3,4,2,2,1,1}$: What defines the logarithm?

$Q_{3,5}$: How can the formula $T_2 = \log(2) / \log(a)$ solve the problem?

$Q_{3,5,1}$: How can it simply be used as an algorithm?

$Q_{3,5,1,1}$: How can we use the formula if the doubling happens twice?

$Q_{3,5,1,1,1}$: Why is it 28 years every time, when the interests are increasing?

$Q_{3,5,2}$: How can the formula be justified?

$Q_{3,6}$: Can the two doubling time constants be described as one, T_4?

$Q_{3,6,1}$: How can we "prove" a formula of T_4?

$Q_{3,6,2}$: Can we deduce a T_8, which describe the time required for all the savings to be increased by a factor 8?

Group 1 followed the branch of $Q_{3,4}$. The answer they presented ended by posing $Q_{3,4,2}$. They used previously acquired knowledge and they translated the concrete problem into an equation, which led them to the expression $2 = 1.025^x$. One can argue that this branch stems from the immediate plenum question $Q_{3,0,3}$. This is shown in figure 5 below.

The next four groups did not answer the derived question, $Q_{3,4,2}$, but presented their synthesis. They gave various versions of the following answer:

$A_{3,5}$: The doubling time is given by the formula:

$T_2 = \log(2) / \log(a)$

In our case $a = 1.025$, therefore $T_2 = 28.07$.

These answers clearly drew on the study of suggested media since the formula was presented in them. At this level of mathematics, the answer represents what the students were expected to do. In the written media students were encouraged not to worry about the meaning of "log(a)" apart from considering it as a button on their calculators. Group number 2-5 followed this path.

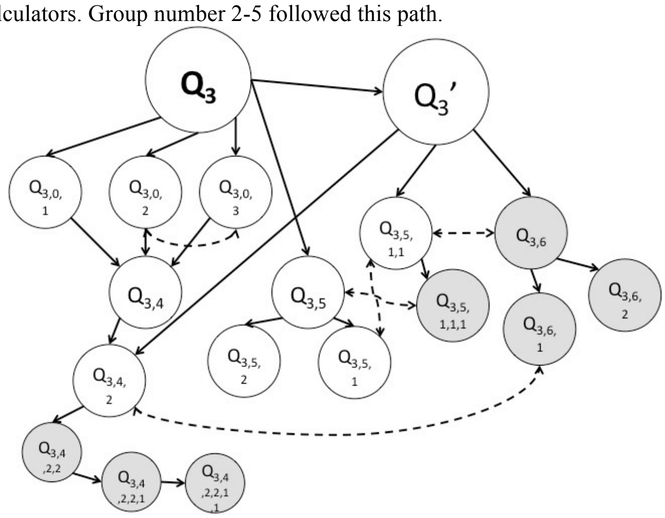

Figure 5. The result of the analysis of the third SRA. The grey circles indicate students autonomous questioning.

9.1 First example of autonomous questioning

Group 6 started where group 1 posed their question on how to solve $1.025^x = 2$ with respect to x. The group continued by writing:

$$x \cdot \log(1.025) = \log(2)$$

$$(x \cdot \log(1.025)) / (\log(1.025)) = (\log(2)) / (\log(1.025))$$

x=28.07

Hence the group gave an answer, $A_{3,4,2,2}^{\heartsuit}$, based on using a rule on logarithms. Group 7 yet another version of $A_{3,5}$. The last group claimed they had used the same technique as in $A_{3,5}$, but instead of formulating their answer right away they ask: "[…] but why does that happen?" [the group pointed towards group 6's use of logarithms]. Hence, they questioned $A_{3,4,2,2}$, by posing $Q_{3,4,2,2,1}$.

Group 6 replied: "we used the rule: $\log(a^x) = x \cdot \log(a)$. Isn't it like if $f(x) = 10^x$ then the opposite is $f(x) = \log(x)$?".

The class was unfamiliar with the concept of inverse functions but still some students dared to share a vague idea about it, being like "the opposite operation of addition is subtraction". Moreover, group 6's answer to group 8 can be interpreted as a new question, $Q_{3,4,2,2,1}$: Is the rule $\log(a^x) = \mathrm{x} \cdot \log(a)$ based on the fact that, $y = \log(x) \Leftrightarrow 10^y = x$?

To give a full answer to $Q_{3,4,2,2,1}$, one needs to question how $\log(a^x) = x \cdot \log(a)$ can be justified, which represents $Q_{3,4,2,2,1,1}$. The textbook's answer to this is based on rules for calculating with exponentials as $(a^p)^q = a^{pq}$ (Clausen et al,. 2010). This last technique was never explicitly mentioned and no explicit answer for $Q_{3,4,2,2,1}$ was provided. We note that, the textbook disregard to differ between cases where a and b are natural numbers and when they are real numbers.

Simultaneously with the teacher, group 6 looked up logarithms in the textbook. Both group and teacher suggested this as a media to study by the other groups. These pages could lead to an expansion of the milieu. The episode represents a branch in the SRA, which exemplifies autonomous questioning. The teacher did not expect students at this level to study logarithms as more than a calculator button. This branch is marked in figure 5 by grey circles, and starts with $Q_{3,4,2,2}$.

9.2 Autonomous questioning analysed with herbartian schema

Using the herbartian schema, to describe group 6's answer to Q_3 looks as below:

$$[S(X_6, y, Q_{3,4,2,2}) \curvearrowright M] \hookrightarrow A_{3,4,2,2}^{\heartsuit}$$

Group 6 developed their answer by exploring the milieu: $M = \{A_1^{\diamond}, A_2^{\diamond}, \ldots, A_n^{\diamond}, O_1, O_2, O_3, Q_{3,4,2,2}\}$. The group must have studied more media than those suggested by the teacher, since they knew the relation between logarithms and exponential functions – or a member had picked it up earlier, in another context. The Internet is flooded by webpages offering tutorials and guidance of varying quality of these topics.

Group 8 adressed the answer of group 6 explicitly by raising $Q_{3,4,2,2,1}$. This gave rise to a joint study of the two groups, X_6 and X_8.

Groups 8 provided the question based on their shallow study of $A_{3,4,2,2}^{\heartsuit}$. Group 6 sought to answer $Q_{3,4,2,2,1}$ based on media familiar to them and their acquired knowledge. This can be described as:

$$[S(X_8, X_6, y, Q_{3,4,2,2,1}) \curvearrowright M] \hookrightarrow A_{3,4,2,2,1}^{\heartsuit}$$

Group 6 initiated the formulation of an incomplete answer and suggested further media. The definition of logarithms as the inverse function of $y = 10^x$, was a known answer of group 6, but a work to study for group 8.

Through the answer group 8 provided for Q_3, it was clear that they had studied the suggested media and applied the formula given, $T_2 = \log(2) / \log(a)$. In the textbook no justification of the formula was provided. Whether group 8 linked the answer of group 6 to the answer they had studied, is unclear from data. But when the group was introduced to $A_{3,4,2,2,1}$ they started to formulate questions. And group 6 was eager to help construct or develop answers.

Hence, it became explicit that responsibility had changed regarding who supported the study and research process, who posed and answered questions and who delivered media for further study. The changed didactic contract led students to present and study each other's answers, which induced an autonomous questioning and the development of reasoning related to the rule of calculating the doubling time. It is worth noticing that the responsibility taken by the students does not mean the teacher is not needed. The teacher's role is to set a scene with potentials for autonomous study and research processes for the students. Further the students use the teacher to validate their vague ideas on inverse functions as well as this choice of media on logarithms.

9.3 Further study and research on doubling time

Since most of the class gave versions of $A_{3,5}$ as their answer, the teacher wanted to know what idea the students had about the doubling time. Therefore she asked a derived question to the whole class, Q_3': "How long will it take if the money stays in the account until another doubling of the balance occur. When will that be?"

Most groups answered that it takes another doubling time, hence the total time can be calculated as: $2 \cdot T_2$. This represents $A_{3,5,1,1}$. But others were in doubt, asking $Q_{3,5,1,1,1}$: "Why is it 28 years every time, when the accrued interest is increasing?" Another group answers, $A_{3,5,1,1}^{\heartsuit}$: "When the rate of growth (a) is the same, the time for doubling is the same". This relates back to $Q_{3,5}$. It is unclear how the group came up with this answer. The textbook (Clausen et al., 2010) has a graphic representation of the doubling time, which indicates that the doubling time is the same regardless where you look at the graph. This could have inspired the answering.

The first group argued that finding the time it takes for the balance to increase by a factor 4, is equivalent to solving an equation they reduced to: $1.025^x=4$. This is similar to their work with Q_3 and is indicated by the arrow from Q_3' unto $Q_{3,4,2}$ in figure 5. This group did not study the answers of other groups. The last two groups continued the work of group 1. They argued for a “double doubling time” constant: $T_4 = \log(4) / \log(a)$. Their argument was built on the equation: $1.025^x = 4$. It can be interpreted as they raise the questions:

$Q_{3,6}$: Can the “two doubling times” be described as one, T_4?

Q_3,6,1: How can we “prove” a formula of T_4?

The groups deconstructed and reconstructed the answer of group 6 using the rule: $\log(a^x) = x \cdot \log(a)$. It was an example of autonomous questioning since the students raised derived questions based on their study of answer $A_{3,4,2,2}$. The two groups used group 6’s answers in the sense of de- and reconstructing the knowledge into an argument, which led to $A_{3,6,1}{}^{\heartsuit}$. Group 8 made further hypothesis about an “8-time constant”, T_8. Yet, the claim was never investigated further or formalised by the students. It is depicted as $Q_{3,6,2}$ in figure 5.

The investigation of constants T_4 and T_8 was certainly beyond the scope of the ministerial guidelines and does not represent core mathematical content for upper secondary mathematics. Nevertheless, in this study it spurs students’ mathematical curiosity to do so. Students formulated and solved problems without being directly required. The questions on logarithms and time constants were surprising outcomes of the study.

DISCUSSION

The two episodes described above suggest the feasibility of students take an active role in identifying and formulating the questions they work with, as discussed by Bosch and Winsløw (2015). Though autonomous questioning was realised, the sustainability can be questioned. The study realised the potential of continued formulation of questions and answers from the students. Although, from a pure mathematical perspective, one could wish for more. Why did the students not question none-integer exponents? What are they? How does the calculator find the decimal number representing 3^{π}? As argued earlier, these questions are beyond the curriculum and the media treat exponents as something natural, not to be questioned. This indicates, if the studied works treat notions as something not to be questioned, it might limit the students' initiatives regarding problem posing. When students studied other groups' answers it seemed "legal" to question it. It was natural to question the use of logarithms and pursue this further when a vague answer using inverse functions did not satisfy the other groups. This aligns with recent results of Otaki, Miyakawa and Hamanaka who reports that SRPs makes it easier for students to formulate "why-questions" and pursue these (Otaki et al., 2016, p. 17).

Hence the answer to the research question of this paper is that explicit requirements to share preliminary answers for an open question supports students' autonomous questioning. Moreover the changed didactic contract seemed to reinforce the milieu in order to promote students formulation of questions and pursuing them.

The strict time frame performed a constraint securing no group presented a perfect answer. Other students could always question elements of the other groups' answers. And no group would waste a lot of time on questions they could not overcome. Group 1 kept encountering equations with exponential notions, which they were not able to solve. They did not use the rule: $\log(a^x) = x{\cdot}\log(a)$, as other groups did. For this group the time frame might have been too strict. However their final thematic project employed the rule. Whether they studied the works of the others based on their notes is impossible to determine, but in the end they were able to present a coherent answer to Q_0.

In order to address the question raised by Bosch and Winsløw (2015) on the mathematical and didactic infrastructures needed to realise sustainable study and research processes, this study has realised some key potentials of SRP's regarding students problem posing and development of answers. With respect to didactical infrastructures, the planning of the lesson – including the a priori analysis and choice of appropriate media – it takes more time than preparing common classroom activities. Similarly, the four SRA's took three lessons of 95 minutes to complete. It is worth noticing that the class did not need the teacher to institutionalise the intended knowledge. After the SRA's, students solved standard exercises and performed better than similar classes at the oral exam, measured on the grades given. Here the thematic projects were coherent and reflected the questions and answers, which were presented at the whiteboards, including the use of logarithms.

In order to monitor the work of students, the requirement to use the whiteboard functioned well. In the beginning (the SRA’s on Q_1 and Q_2) the teacher initiated the students’ study of the other groups’ answers. The teacher explicitly asked what similarities and differences the class could find between the presented answers. This led to discussions on such topics as notation but also to studying the relation between multiplying by the factor 1.025 ten times versus calculating $f(10)$ when $f(x) = 5000 \cdot 1.025^x$. This might not be a ground shaking mathematical discussion but it gradually expanded their techniques and autonomy for solving problems about exponential expressions.

The disadvantage of the rigid requirement of all groups presenting sometimes similar answers were the time consumed and that it became boring to attend. For a longer study alternative, configurations of use of boards might be needed.

While designing the SRA’s, the mathematical infrastructures were taken into account and the interrelation between the SRA’s as made explicit through the story on grandparents. Through the a priori analysis the possibility of reinvestment of techniques and strategies was successfully aimed for. This might strengthen the students’ inclination to use a developed answer in their subsequent study and research process. For Q_1 it should be possible to answer the question based on previously acquired knowledge. Hence the question opens for the possibility of answering the question based on research activity. However the problem is much more directly approached if the students study the suggested media. This idea was continued in all the questions of the SRA.

CONCLUDING REMARKS AND FURTHER PERSPECTIVES

Whether the SRA designed and studied in this paper is akin to what Kilpatrick described as the situation in real life outside school, is hard to determine. But it seems evident that it suggests new approaches to problem posing in teaching. When the students answer Q_3' by employing rules on logarithms which have never been presented by the teacher, it indicates a significant development of problem solving skills. This is not just solving problems similar to presented examples by imitating a procedure. Students in this study reused procedures of other students, but after questioning the nature of the rule. This supports the development of "a broader understanding of mathematical concepts and the development of mathematical thinking" (Singer, Cai & Ellerton, 2013, p. 2). Based on the study of this paper, it seems crucial for the problem posing that students develop questions and answers in a genuine study and research process – where students point to gaps or inconsistencies, which they need to mend. When this happens, the students have the possibility of developing coherent mathematical knowledge within a prescribed area. This result of the study supports that problem posing should be considered a part of inquiry approaches to teaching (as in Artigue & Blomhøj, 2013) rather than some special activity conducted apart from standard classroom activities as listed by Bosch & Winsløw (2015, p. 371).

Based on the thematic projects, we confirm that no group simply adopted the rules of the textbooks. The groups' answers were interrelated and they gave reasons for their solution methods. In that sense the result of the taught sequence of SRA's avoided a common tendency to atomise the mathematical knowledge. In that light, the SRA taught under the condition and restrictions of strict time frame, being group based, with required sharing of answers (spoken and written), teacher proposed media and finalised in a thematic project, seemed fruitful for curriculum bound teaching.

The external constraints of this experiment might be stronger than in previous experimental studies on SRP's. A full SRP certainly has advantages regarding the potential of students' learning e.g. regarding the dialectics of media and milieu. But as the study of Jessen (2014) and Otaki, Miyakawa and Hamanake (2016) shows, the students might realise (also qualitative) very different paths. Rasmussen (2015) planned only 20% of the course activities as SRP because of the challenges in securing the students' acquaintance with the "monuments of curriculum". The other studies were not core course activities. In light of this more empirical work exploring potentials of generating questions and how to manage these in the ordinary classroom must be conducted. Furthermore setting up longitudinal studies where students can adapt to the changed contract through sequences of SRA's combined with full SRP's, would be interesting for the study of students problem posing. In this sense some elements of the teaching could be close to a given curriculum, while other parts could be in depth studies of pieces of mathematical knowledge with students' problem posing as a core element.

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Part IV

An analysis of the relation between mathematical modelling in scientific contexts and upper secondary education

4

Paper III: Modelling and didactic transposition

This paper investigates the relation between mathematical modelling in a scientific context and in the educational system. A historic context is chosen for the scientific approach to modelling in order to be able to characterize modelling activity in the emergence of new cross disciplinary research fields. Compared to modelling activities described in the educational system, similarities and gaps are identified. The paper address research question 2 describing what characterizes mathematical modelling as scholarly knowledge. The paper was submitted December 2016 to: *Educational Studies in Mathematics*.

An analysis of the relation between mathematical modelling in scientific contexts and upper secondary teaching

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Abstract

Mathematical modelling and application has been agreed as a justification for the teaching of mathematics at several levels in educational systems across the world – and as a core element of the teaching itself. Several theoretical constructs have been developed, which give approaches on how to design and develop modelling activities for the teaching of mathematics, however it still represents a challenge. This paper addresses the relation between the mathematical modelling in scientific contexts and regulations of modelling activities in educational contexts as they are described in ministerial guidelines, textbooks and exam exercises as part of the didactic transposition of mathematical modelling. We use our analysis to discuss implications of this relation for what modelling can be in an educational context, and to what extend the current situation prepares students for higher education in natural sciences involving mathematics.

Introduction

The notions of mathematical model and modelling took form and entered the discourse of mathematics during the 20th century with the increasing mathematization of a variety of fields such as e.g. economics, climate science, biology and medicine. The mathematization of these fields has initiated new scientific practices and new sub-disciplines have emerged where problems and new knowledge are gained through mathematical modelling. During the past decades, this development has been reflected in the mathematics curriculums in many countries as justification for mathematics (Blum et al. 2007). In Denmark, it is stated for upper secondary mathematics that:

> […] students should gain insight into how mathematics can contribute to the understanding, formulation and treatment of problems in different fields of knowledge […] Hereby, the students should become capable of relating to others' use of mathematics as well as gain sufficient competencies in order to pursue higher education which involves mathematics (Danish Ministry of Education, 2013).

The teaching and learning of mathematical modelling is by now an established research area with theoretical developments, various approaches and empirical studies (e.g., Ärlebäck & Doerr 2015; Barquero et al, 2013; Blomhøj & Kjeldsen 2010; Blum & Borromeo-Ferri 2009). Recently, papers have been published where the relation between modelling in school and authentic practices of modelling is investigated (e.g. see Biehler et al. 2015; Frejd & Bergsten, 2016). In the theory of didactic transposition, it is assumed that knowledge of mathematical modelling has a pre-existence outside of school, and that modelling in school is shaped from or has diffused from scholarly knowledge within the field of knowledge production through transposition processes, see fig. 1 (Bosch, 2006). Frejd and Bergsten (2016) have investigated modelling as a professional task in the workplace as represent of knowledge production. They point out that there are major differences between modelling in the workplaces and in school. They indicate that the influence from scholarly knowledge on school mathematics is weak (2016, p. 12). This concern was raised a decade ago by Jablonka (2007, p. 196). She discussed the issues of recontextualisation and authenticity of mathematical modelling at school. She pointed out that the recontextualisation "causes a transformation of the unmediated discourses found in out-of-school practices", such that the original practice is invisible in many modelling problems that have been designed for teaching modelling in school. She found, that this results in an implicit pedagogy that influences students' thinking about mathematical modelling. According to Jablonka "it would be more sensible to introduce a meta-discourse on the nature of mathematics that helps to differentiate between different practices of using mathematics" (p. 196).

In the present paper we address these issues of the relation between mathematical modelling in scientific contexts and modelling in upper secondary school mathematics. Our analyses have been directed by the question:

> What relation – if any – is there between mathematical modelling in professional scientific contexts and in mathematics teaching at upper secondary level?

To investigate the relation, we analyse elements of the didactic transposition of mathematical modelling in different institutional contexts (see fig. 1).

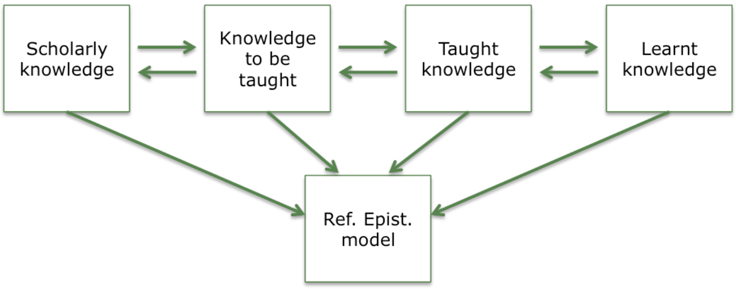

Figure 1: The didactic transposition of knowledge in a school system (see (Bosch & Gascón, 2014, p. 70)).

To characterize possible sources of this transposition we need to investigate how mathematical modelling is practiced and used outside of school. Frejd and Bergsten (2016) point towards a lack of investigations of the *scholarly knowledge* of mathematical modelling and they conceive of their empirical investigation as a contribution to a description of scholarly knowledge. They conducted interviews with nine practitioners of mathematical modelling in research and private companies regarding the kind of activities such as communicate, describe, simulate, predict and construct, that goes into their job as mathematical modellers. Our paper seeks to complement their work, by analysing authentic modelling episodes from the 20th century in biology and economics with respect to modelling strategies, modelling as a scientific practice and discussions of what constitute valuable knowledge across the disciplinary boundaries. Our approach represents another way of gaining insights into mathematical modelling practices, and as such, it constitutes a contribution to the scholarly knowledge from another angle. Our analysis is not complete and it can only capture traits of scholarly knowledge of uses of mathematical modelling in scientific context.

Our analysis is divided into two parts: first we present the analysis of the historical cases, and second, we analyse: ministerial guidelines for mathematics at upper secondary level in Denmark, three wide spread textbooks and national high stake exam exercises from the period 2010-2014. In the discussion, we compare our analyses of mathematical modelling in these first two institutional settings in the didactic transposition in order to address our research question. Further we point out perspectives regarding the educational relevance. Previously, Garcia and Higueras (2005) and Bolea, Bosch and Gascón (2004) have conducted didactic transpositions for modelling activities with focus on specific elements of mathematical content, whereas our analysis aims at more general perspectives of modelling processes.

Methodology

Investigations of historical examples of the mathematization of other research fields can shed light on practices when applying mathematics and on the entanglements of mathematics with methods of other research fields (Dalmedico, 2001). They can be used to gain insights into parts of the scholarly knowledge of mathematical modelling. With respect to processes of didactic transposition, Bosch and Gascón (2006, p. 53) discuss how "school loses […] the questions that motivated the creation of the knowledge". Historical cases can be used to explore and discuss the relation between what motivated the creation of mathematical modelling outside of school and mathematical modelling as part of the teaching of mathematics in school. As pointed out by Jablonka (2007, p. 199) "To distinguish and further analyse different mathematical practices (including their history) helps clarifying the "relationship between applications and modelling and the world we live in"". We have used this approach to bring out meta-issues of applications of mathematics by analysing authentic historical texts from the scientific literature.

We have chosen historical cases that exhibit features of mathematical modelling in the production of knowledge in biology and economics, which are areas that only recently have been subject to mathematization. We are using examples from the beginning of these developments because they are authentic and simple enough that it is possible to highlight essential features of modelling that also prevail today and make them objects of upper secondary mathematics students' reflections. In contrast, as e.g. Frejd and Bergsten's (2016, p. 31) study showed, much of professional modellers' work "is based on knowledge and experiences reaching far beyond what can be found in a secondary mathematics classrooms". We analyse the cases with respect to what knowledge was gained and what reactions from scientists within these other fields of research were provoked. Hence, we highlight discussions between researchers from different disciplines about the epistemological status of the use of mathematical modelling. This is an important aspect, because, as phrased by Jablonka (2007, pp. 197): "A universal description of the process of mathematical modelling falls short of the varying methodological standards, criteria for validation and evaluation that are relevant in different contexts […] The criteria used […] in defining what is considered a solution are external to mathematics […]". To be specific, we look at the content matter of two examples from mathematical biology (Vito Volterra's work on the Lotka-Volterra predator-prey model and Nicolas Rashevsky's early work on cell division) and one example from mathematical economics (John von Neumann's model of general economic equilibrium). These cases have been chosen because they represent early attempts of the migration of mathematics into new fields of inquiry in which mathematical modelling by now play a significant role as a scientific practice. The cases are accessible for non-specialists and they share relevant features with modern practices. The author's work was subject to discussions and criticism from researchers in the target discipline, which gives us insights into issues from both sides. The examples show the interaction with disciplines that are not mathematical sciences and the episodes are able to exhibit barriers of understanding, differences in approaches to knowledge production and discussions on the epistemic value of modelling results. These issues are important in the teaching of mathematics preparing students for higher education.

When exploring knowledge to be taught we follow Bosch and Gascón (2014, p. 71) who state that: "“knowledge to be taught” can be accessed through official programs, textbooks, recommendations to teachers, didactic materials, etc." In Denmark the teaching of mathematics is regulated by ministerial guidelines in form of curriculum and the corresponding instructions. Mathematics at upper secondary level is assessed by high stake exams and therefore the written exam exercises further regulate the teaching of mathematics. We have analysed all written exam exercises in the period of 2010-2014 with respect to modelling tasks. Finally, we have analysed three wide spread textbooks (Carstensen, Frandsen & Studsgaard, 2006), (Clausen, Schomacker & Tolnø, 2011), (Grøn, Felsager, Bruun & Lyndrup, 2012). The books are authored by in-service mathematics teachers with a considerable teaching experience. They are written on the basis of the ministerial guidelines in order to cover the content and intentions of these. In the preface of Carstensen et al. (2006, p 5) it is stated that the chapters of the book cover curriculum. However, the book further provides a number of appendices, which can be used as supplementary material. These appendices cover, among others, applications and existing models (e.g. statistical models as Hardy-Weinberg on genotypes). Clausen et al. (2011, p. 5) explain in their preface, that the core content of curriculum is presented in the first chapters, and that they recommend an inquiry approach for these topics and exercises. Materials for inquiry approaches are placed in the latter chapters of the book. These covers explorative exercises, which are carried out in a Computer Algebra System (CAS) tool (e.g. Maple, Geogebra, TI Nspire etc.). Grøn et al. (2012, p. 1), is an interactive book, in the sense that it is electronic available with applets and supplementary material on an online platform to which students or schools are required to buy access. The authors explain that every chapter begins with an introduction to the topic showing applications of the notions of the chapter. All chapters end by a number of projects, which further explore the applications of the notions at stake. As it is optional if the teachers use the supplementary material, our focus has been on the main chapters. Upper secondary education in Denmark covers bidisciplinary course elements and projects as well. These elements of secondary education represent good frameworks for modelling activities as discussed in (Jessen, Kjeldsen & Winsløw, 2015), but mathematics is not required to be part of these activities, why they are disregarded in this paper.

Analysis I: Scholarly knowledge of mathematical modelling activities

Our three examples give insights into different approaches to using mathematics and modelling in scientific practices, from taking point of departure in observation of concrete phenomena to highly abstract theoretical considerations. They illustrate, that there exists different strategies, that modelling demands creativity, that it draws on analogies and import concepts from other fields and that new mathematics might need to be developed during the process. They show that scientists from different disciplines may disagree on the epistemic value of model results.

Our first example is John von Neumann's (1937) paper "Über ein ökonomische Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes". Historians of economics have coined this paper "the single most important article in mathematical economics" (Weintraub, 1983, p. 13). It was translated into English in 1945 with a change in title: "A Model of General Economic Equilibrium" which clearly indicates that by 1945, his paper was conceived of as dealing with modelling in economics. Von Neumann considered a situation where n goods (G_1, …, G_n) are produced by m processes (P_1, …, P_m), and he asked the question: "Which processes will be used (as "profitable") and what prices of the goods will obtain?" He mathematized the problem as a system of six linear inequalities:

$$x_i \geq 0 \qquad (1)$$

$$y_j \geq 0 \qquad (2)$$

$$\sum_{i=1}^{m} x_i > 0 \qquad (3)$$

$$\sum_{j=1}^{n} y_j > 0 \qquad (4)$$

$$\alpha \sum_{i=1}^{m} a_{ij} x_i \leq \sum_{i=1}^{m} b_{ij} x_i, \text{ for all } j \qquad (5)$$

where $y_j = 0$ if '<' holds.

$$\beta \sum_{j=1}^{n} a_{ij} y_j \geq \sum_{j=1}^{n} b_{ij} y_j, \quad \text{for all } i \qquad (6)$$

where $x_i = 0$ if '>' holds. Here a_{ij} (expressed in some unit) denotes the amount of G_j that is consumed in the process P_i, and b_{ij} denotes the quantity of G_j that is produced by the process P_i. The intensities of the processes are $x_1, \ldots, x_m$ while $y_1, \ldots, y_n$ represent the prices of the goods. α is the expansion factor, and β is the interest factor.[1] This mathematization turned the question of existence of economic equilibrium into the question of existence of a solution to the above system of inequalities. Von Neumann's result in the paper was to prove that a solution exists, and for this he needed an extension of Brouwer's fixpoint theorem, which he proved in the paper.

Von Neumann's model is a theoretical construct, which is based on an abstract structuring of economic "reality", and it is this abstract construct that he analysed and investigated. It illustrates Skovsmose's (1990, pp. 769) point that "Every mathematical model must be based on a specific interpretation of reality […] a model is not a model of reality as such." Von Neumann's model was not evaluated against a reality outside of mathematics, but against internal mathematical consistency, namely the existence of a solution. Von Neumann's proof is a so-called existence proof. He did not construct a solution. The model can only be said to describe an economic reality in as far as it shares some significant features of this reality. The relationship between reality and model is limited to the structuring of the economic reality on which the model was based. This was addressed by the English economist David G. Champernowne (1945), who wrote a review of von Neumann's paper, when it appeared in English. He criticized the (lacking) relationship between reality and model:

> "Approaching these questions as a mathematician, Dr. Neumann places emphasis on rather different aspects of the problem than would an economist.[…] The paper is logically complete […]. But at the same time this process of abstraction inevitable made many of his conclusions inapplicable to the real world […] the reader may begin to wonder in what way the model has interesting relevance to conditions in the real world." (Champernowne, 1945, p. 10-12)

[1] For a detailed historical analyses of von Neumann's paper see (Kjeldsen, 2001)

Champernowne ends his review with a warning saying that “utmost caution is needed in drawing from them [von Neumann’s results] any conclusions about the determination of prices, production or the rate of interest in the real world.” (1945, p. 15). This warning illustrates that it is not clear what counts as a solution, when mathematics is used in other fields of inquiry – it depends on the context. It relies on the disciplinary lens used, especially when new modes of inquiry are under development. Despite Champernowne’s critic, von Neumann’s model has played a significant role in the development of theoretical economics (see e.g. Dore et al. (1989)) and is an example of models as elements of scientific theories. The case illustrates that how modelling and models are perceived depend on the recipient’s conception of the purpose of the model – e.g. models are judged with respect to whether they are used for development of (economic) theory or to solve concrete (economic) problems in practice.

Our second example is Vito Volterra’s work beginning in the mid 1920’s on what is now known as the Lotka-Volterra equations of predator-prey systems. Volterra’s work rose from an observation of a concrete phenomenon of the increase of the percentage of predator fish in the catch in the Upper Adriatic during WWI due to a decrease in fishing. It puzzled the biologist Umberto D’Ancona that the reduced fishing seemed to be more favourable for the predator fish than for the prey, and he asked Volterra if he could explain this phenomenon.

Volterra approached the problem in accordance with methods of classical mechanics, when friction from the environment is neglected.[2] He disregarded external causes and focused on the internal ones. It is not obvious how methods of mechanics can be transferred to the predator-prey system and used to grasp the underlying mechanism in the biological system. Volterra commented this by saying:

> […] on account of its extreme complexity the question might not lend itself to a mathematical treatment, and that on the contrary mathematical methods, being too delicate, might emphasize some peculiarities and obscure some essentials of the question. To guard against this danger we must start from the hypotheses, even though they be rough and simple, and give some scheme for the phenomenon. (Volterra 1928, quoted from Knuuttila and Loettgers (2016))

He constructed a hypothetical system that only took the predatory and fertility of the co-existing species into account. He assumed that prey and predator increases and decreases continuously in order to be able to apply the mathematics of differential calculus, that the system is homogenous, that the birth rate of the prey is constant (i.e. the population grows exponentially), and that the number of predators decreases exponentially if there is no prey it can feed on. In order to describe the mechanism of predation, he followed a mechanical analogy. He assumed that the probability of a prey and a predator meet is proportional to the product of the numbers of the two species. In order to justify this he compared the system to molecules in gasses – the number of collisions between particles of two gasses is proportional to the product of their densities. He assumed that the interaction between two competing species (predator and prey) occurs at random as with collisions between molecules in a perfect gas. If we let x denote the number of prey, y the number of predators and t is time, these assumptions led Volterra to the equations which are now known as the Lotka-Volterra equations:

[2] Our description of Volterra’s derivation of the Lotka-Volterra equations is based on Knuuttila and Loetters (2016), and Israel (1993).

$$\frac{dx}{dt} = Ax - Bxy$$
$$\frac{dy}{dt} = Cxy - Dy$$

In Volterra's discussions with D'Ancona, Volterra was concerned with the relation between the empirical data and the mathematical system, whereas D'Ancona's views were in accordance with a more abstract mathematical modelling approach. D'Ancano was not convinced that his empirical data of the fisheries could validate Volterra's theory. However, the quote below taken from a letter D'Ancano wrote to Volterra, shows that this did not make D'Ancano reject Volterra's biomathematics. In D'Ancano's opinion Volterra's theory did not need to be supported by empirical data – it had value in itself:

> My observations [of the fisheries in the Upper Adriatic] could be interpreted in the sense of your theory, but this fact is not absolutely unquestionable: it is only an interpretation. …You should not think that my intention is to undervalue the experimental research supporting your theories, but I think that it is necessary to be very caution in accepting as demonstrations these experimental researches. If we accept these results without caution we run the risk of seeing them disproved by facts. Your theory is completely untouched by this question. It lay on purely logical foundations and agrees with many well-known facts. Therefore it is a well-founded working hypothesis from which one could develop interesting researches and which stands up even if it is not supported by empirical proofs." (D'Ancano to Volterra 1935, quoted from (Israel, 1993, p. 504)).

In Volterra's approach, methods and theories from other fields (including mathematics) guided his construction of the equations from the very beginning. Another interesting aspect of this case is that D'Ancona's letter shows that it can make sense and lead to new insights to investigate a mathematical model that is derived from a concrete phenomenon even if it cannot in detail be confirmed by data. The system was criticised for not taking predator's adaption to new conditions into account (Knuuttila & Loettgers, 2016, p.13). This critique might relate to the crossing of disciplinary boundaries, discussions and disagreements about the relationship between reality and model.

Our last example is Nicolas Rashevsky's attempt to derive a physico-mathematical explanation of cell division. He was a pioneer in mathematical biology (Abraham, 2004), and even though his approach to explaining cell-division failed, his work shows ways of how scientists work, explore and think about the use of mathematics to gain insights into phenomena outside of mathematics. Rashevsky made an analogy of cell division to the physical phenomenon of droplets. He conceptualized a cell as a liquid system that takes in some substances from the surrounding medium and transforms them. He explained his ideas to biologists at a symposium for quantitative biology in 1934 where he got into a debate with the biologists. His talk and the following discussion are published in (Rashevsky, 1934). In the introduction Rashevsky carefully explained his scientific views, his methodology and his presumptions to the audience:

> Unless we postulate some factors unknown to the inorganic physical world […], it is simply a logical necessity, free of any hypothesis, that some physical force or forces must be active within the cell to produce a division of the latter into two or more smaller cells. […] If however we entertain the hope of finding a consistent explanation of biological phenomena in

> terms of physics and chemistry, this explanation must of necessity be of such a nature as the explanation of the various physical phenomena. It must follow logically and mathematically from a set of well defined general principles. (Rashevsky, 1934, p. 188).

The quote above illustrates that Rashevsky was searching for fundamental causes of a biological phenomena and he was looking for a physical force. He argued that in order not to assume some independent mechanism "we must take some such general phenomena [that occur in all cells] and investigate its mathematical consequences". He chose metabolism and made an analogy to a physical liquid system. The approach was in accordance with mechanics. He made an analysis of the forces produced in a liquid system due to a gradient of concentration produced by metabolism from which he derived a mathematical expression of the forces produced in a cell. He made a further idealization to homogenous and spherical cell. Such a cell is, as he explained:

> […] highly idealized and rather remote from actual, very complex cells. The result will therefore hardly apply with any degree of precision to actual cells. But they will give us a general qualitative picture of various possible phenomena […]. (Rashevsky, 1934, p. 191)

For these idealized cells, Rashevsky calculated the free energy and deduced that when the cell reaches a certain size a division of the cell will result in a decrease of the free energy of the system. He concluded that his investigations had established "the *necessary* but not the *sufficient* conditions for spontaneous division." (p. 192). His approach is theoretical and hypothetical addressing causal relationships and analysing these. He could not determine the cause of cell division. He could point out a *possible* causal relationship.

The biologists raised critical questions to Rashevsky's modelling about the things he had left out (Rashevsky 1934, pp. 195). They wanted to know which was the nearest example in nature to this theoretical case. They questioned Rashevsky's explanation as a general solution, since "a spherical cell isn't the commonest form of a cell", as the biologist Davenport emphasized.

Notice, how knowledge claims, to some extend, need to be negotiated or agreed upon between researchers from the different fields in order to be taken seriously in the target domains. It is especially clear in this last case where Rashevsky defended his simplification of the cell's shape by emphasising that this systematization allowed him to investigate the liquid system and study the mechanisms, which could be responsible for the division. This was acceptable conduct within the physico-mathematical paradigm in which Rashevsky was working. Still, the biologists found it problematic to accept as contributing to the knowledge of cell division. Modelling as a scientific practice seems to require a certain degree of knowledge and insights into the involved disciplines and a willingness to cross boundaries from both sides.

Our analyses of the historical cases concretise features of mathematical modelling and its function in the production of scientific knowledge. Conceptual systems are being modelled, and which relation to reality they have depends on the assumptions and analogies lying underneath the conceptualisation. Gunawerdena (2014), a mathematical biologist, distinguishes between two kinds of modelling strategies in the current literature: Reverse modelling and forward modelling. Reverse modelling begins with data, whereas in forward modelling one starts with known or suspected causal relationships. In Gunawardena's terminology, Rashevsky was employing a forward approach. Gunawerwardena also emphasizes the important difference between "models […] which are based on phenomenology and guesswork, and models based on fundamental physics" (p. 4), emphasising the different epistemological status of such models. The former models have no predictive power. However, they may support reflections, and here simplicity is often useful, even though the models do not describe reality. The Lotka-Volterra model is such an example. It is simple and non-linear, and as such it is an important resource for studying non-linear systems' behaviour (Knuuttila & Loettgers, 2016). Another point is that, as we saw in the case of Volterra's approach; the mathematics used might influence possible assumptions in the hypothetical systems. In Volterra's case the mathematical representation, techniques and concepts were transferred as a whole from mechanics to population biology. In other situations new mathematics needs to be developed, as in the case of von Neumann. Moreover, the 'success' of a model does not necessarily depend on the models' relationship to reality – von Neumann's model became very important for the development of theory in economics. All three historical cases show that scientists did not agree on the epistemic value of the knowledge results and that the disagreements might be caused by different opinions regarding methods and assumptions made in the process of modelling. This aspect is important, but often neglected in discussions of knowledge claims based on modelling. When model results turn out to be "wrong", not coinciding with real-world behaviour, it is rarely due to mathematical flaws, but the assumptions made and the limitations of the model.

Finally, we want to point towards the creativity and 'messiness' involved in modelling, which are exhibited in all three cases. Yates at al. (1968) formulated it in the proceedings of the III Systems Symposium in System Theory and Biology as (p. 142):

> Modeling is an aspect of human psychology; it is an art, and no general rules or guide lines apply to all cases. The development of a model depends upon a mixture of experiment, data analysis, intuition and synthesis, that may be blended in seemingly unmethodical fashion.

We will not go into details of their paper, only noting that the group recognises the role that the choice of mathematics plays in the early stages of the modelling process and how it influences the formulation of a model that can serve as a working hypothesis. Further, Yates et al. remark that in "attempt to codify a strategy for modelling, […]. We confess, though, that the philosophy is retrospective; the unfolding of the model may not have followed exactly the logical ordering of our activities as we present them here." (p. 142). Hence, the idea of "models" for modelling should be regarded as reflection tools, and not perceived as recipes. The modelling process is more diverse than what can be described with a "model".

The various approaches to modelling and discussions of the epistemic value of knowledge gained from modelling might be equal important for teaching as from a scientific point of view, when teaching is supposed to prepare students for higher education involving mathematics and to be able to reflect upon others' use of models. To what extend the features captured by our analyses are reflected in the educational system will be discussed below.

Analysis II: Knowledge to be taught

This section aims at describing the knowledge to be taught as the institutional frame of the noosphere (Bosch & Gascón, 2014) in terms of official regulations, exam exercises and textbooks.

Official regulations of mathematical modelling

The ministerial instructions claim that mathematics covers a number of methods for modelling and curriculum requires that students should become able to:

> “apply simple functions in modelling data, carry out simulations and extrapolations, be critical towards the idealistic element of models and their limitations […] demonstrate knowledge upon application of mathematics […] in the treatment of more complex problems” and students should work with “principle qualities in mathematical modelling” (Danish Ministry of Education, 2013).

Danish curriculum is inspired by the work of Mogens Niss and colleagues, who formulated eight competencies that together span what it takes to be competent in mathematics (Niss & Jensen, 2011). Accordingly, the curriculum is formulated in terms of competencies (disciplinary goals) and content. The latter covers different types of functions, statistical models, elements of trigonometry etc. Modelling is part of both content and competences that students should acquire. The ministerial guidelines state that modelling covers development of models and that modelling activities cover a number of elements such as the ability to: “delimit the problem and formulate it in words, translate the problem from words to mathematical language, using numerical values” (Danish Ministry of Education, 2010, p. 28). Further, students should be able to use models and translate the mathematical analysis back into the language of the problem and finally validate the result and carry out a critical analysis. The attempts to outline the modelling process by the ministry conflicts with Yates and colleagues (1968), who argued that no general rule applies to the process.

The written exam exercises

As it has been stated several times during the last century backwash of exam on classroom activities is dominant (Suurtamm et al., 2016). Therefore we have carried out an analysis of all written exam exercises in the period of 2010-2014. The exercises were categorised by what kind of mathematics was needed to solve the problem. Examples could be trigonometry, calculating obtuse angles using the cosine relations, statistics and χ^2- test of significant differences in voters’ preferences. Or the exercise could state a relation between age and weight of certain species of fish modelled by a power function, questioning the weight of a fish of a certain length. The type of exercises explicitly claiming to describe phenomena in reality has been characterised as modelling exercises. The exercises often begins with a story explaining a relation in words, claiming that a given function describes the relation between two variables or list a dataset for two variables, where students are expected to carry out the required regression (using their CAS-tool).

An example is exercise 7, from the written exam on May 25th 2012. The students are given a table showing the amount of dollars spent on lobbyism in US Congress during the time period 1999-2009. Students are told that the amount of money spent can be described by an exponential function ($f(t) = b \cdot a^t$) and that they should find the formula describing the development of the investment over time. The students are required to find the doubling period of the function and to discuss the model’s correctness based on the fact that in 2010, 2.61 billion $ was spent on lobbyism. The exercise could equally well be formulated as three purely mathematical questions: carry out exponential regression (with full allowance of a computer), find the doubling period (use the formula from the textbook), criticize the model and its limitations by stating that the model does not fit the real value outside the domain of the dataset. Other exercises from the period 2010-2014 offer the students a model and all the questions ask the students to carry out mathematical manipulation of the expression. This finding aligns with results found by Frejd (2013, p. 432).

Textbooks and mathematical modelling

In Denmark, textbooks are written by experienced in-service teachers. Several of the textbook authors are also designing the written exam exercises on behalf of the Ministry of Education. The first book of our analysis, Carstensen et al. (2006), has no explicit chapter on mathematical modelling, but it has a chapter called “growth models” (pp. 109). The chapter covers different types of functions such as linear, exponential and power functions, but not the word “model”. Examples are given, e.g. the development of U.S. gross national product of each year in a period of 18 years and how the development can be described carrying out exponential regression. Further examples on logistic growth are treated but no reflections upon the modelled reality. Focus is only on mathematical implications, which can be drawn from the formulas alone. In the end of the chapter Newton’s cooling law is stated verbally and the authors argue how it can be described mathematically and show how to solve the corresponding differential equation. They discuss the meaning of time going to infinity in the model and claim that the limit fits with “our” experience. It is further claimed that the rate of change of temperature over time is proportional to the difference between the temperature of the object and of the surroundings, which might not be “an everyday experience”. The authors treat the model purely mathematically. In this sense there is no discussion of the epistemic value of the model. The introduction of the functions is en presentation of the syntheses of modelling activity, not the activity in itself.

All the exercises relate to the cooling example as in the following one:

“When cooling a liquid the temperature of the liquid is given by $f(t) = 70 \cdot 0.75^t + 19$, t is time in minutes and $f(t)$ describes the temperature of the liquid. When is $f(t) = 19$?” (Carstensen et al., 2006)

If we look at the exercise without knowing the examples provided by the book, different strategies and techniques could be applied. In this appearance it is a word problem since it can be regarded as “a task for which no routine method of solution is available and which therefore requires the activation of (meta) cognitive strategies” (Vershaffel et al., 2014, p. 642). If students know Newton’s Cooling Law it can be used to deduce an answer. More likely students will argue what happens when $t \to \infty$ from a graphic representation or based on arithmetic – or mimic the example in the book.

The appendices of Carstensen et al. (2006) cover statistical models, models on mortgage and a headline called “Mathematical models”. This last part covers five modelling tasks where conditions of the “real world” are described. Most of the models are based on geometric diagrams where models are built on trigonometric functions. The students are invited to be “co-constructors” by solving concrete mathematical tasks as: “Use the Pythagorean Theorem to argue that $KB^2 = r^2 - (r-h)^2 = 2rh - h^2$, where all variables and KB denote line segments” (Carstensen et al., 2006, p. 290). Hence, the models pre-exist in the examples. The students are invited to reconstruct the model by following the steps indicated by the posed tasks.

The second textbook in our analysis has a chapter on modelling (Clausen et al., 2012, pp. 84). The chapter begins with a narrative of applications of mathematics in other disciplines and a short presentation of a modelling process is given. The introduction to the notion of modelling is of a philosophical nature stating that “Mankind’s desire to understand his environment serves two main purposes, firstly to explain, perhaps satisfying man’s curiosity, and then to use the knowledge to advantage” (John D. Donaldson in (Clausen et al., 2012, p. 87)). A brief picture of the use of models for quantitative and qualitative answers is given. A general process for modelling is outlined, as: identify patterns, choose state variables, obtain relationships connecting the variables, obtain mathematical solution and compare the solution with the physical situation (Clausen et al., 2012, p. 89). It is later stated that a model can have a limited domain but be useful for predictions. The process is depicted as shown in figure 2.

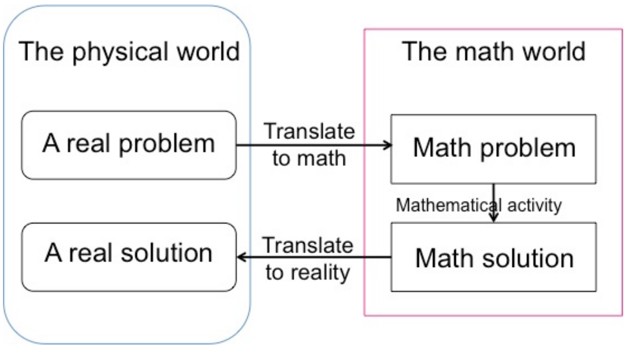

Figure 2: The modelling process as it is depicted in (Clausen, 2012, p. 95) as a translation from real world to mathematics and back.

This description of the modelling process holds for some activity but not all. The von Neumann's work did not take point of departure in a real world situation and economists warned others to adopt the model. Volterra's inspiration to model the relation between prey and predator fishes draws on physics and not the system itself. The textbook gives one picture of what modelling can be and leaves others out. The examples of the book cover: folding a piece of paper, explaining the assumptions made during a modelling process on optimal speed limits on main roads (unfortunately the example leads to unrealistic results), the maximum size of a barge in a canal and finally an example developing a logistic growth model of bacteria is presented. After these "model constructions" the book provides a number of examples. These are all of the same kind as in (Carstensen et al., 2006) in the sense that the models are already there and the students are only meant to manipulate mathematical formula.

The very last section of the chapter is called "More on models" and lists what can be modelled, where models are used in financial an political decision making and discuss if we can trust models – but no concrete examples are provided. The exercises connected to this chapter are similar to the example provided above from (Carstensen et al., 2012).

Our last textbook is the interactive book mentioned in the methodology section. Chapter 1 in the book is named "Mathematical modelling – optimisation problems and functions" (Grøn et al., 2013, p. 40). The chapter begins by summarising a previous book's focus on variables and that emphasis will be on representations: formula, data points and graphs. The notion of function is introduced through a historic introduction to optics and colours of the rainbow, including theory from physics, leading to different mathematical models of the scatter of light. The degree of openness of the examples provided in the beginning is discussed, but the examples showing students how to answer real world problems, are based on pre-existing models in terms of formulas.

The chapter ends with a list of different projects, which carry titles such as "The square as an optimal figure", "The rainbow as a geometric entity", "Optimisation in geometry – experiments and proofs" and "Optimisation problems in geometry – an experimental approach" (Grøn et al. 2013, p. 54). The projects are interesting mathematical problems with pre-existing formulas to manipulate – as in the two other textbooks. Though, the authors do have four chapters under construction presenting collaborations with physics, chemistry, biology and social sciences, which could lead to interesting modelling projects.

The book summarises mathematical modelling praxis as (our translation):

1. Problem formulation (delimit problem, what do we know and what do we need?)

2. Analysis and mathematical description (e.g. identify variables, relations (if you encounter expressions with roots, choose other relations))
3. Mathematical solution to problem (apply relevant mathematical methods)
4. Interpret the results (translate the mathematical result into natural language and relate to the real world problem (Grøn et al., 2013, p. 64)

This is similar to figure 2 and both seem to draw on an algorithmic approach to modelling described in the ministerial guidelines. Hence, there is no mentioning of discussions of the epistemic value of a model or of strategies drawing on analogies, which were key elements in the episodes of the modelling by Rashevsky and Volterra.

Concluding remarks

As our analysis show, there are similarities and differences in the appearance of mathematical modelling in the historic cases and in the noosphere. In both contexts laws and relations of natural sciences or economics are presented with mathematical formulas. In both contexts manipulations of these formulas are supposed to generate knowledge of the relation being modelled. However the main point of our analyses is that: core concepts of modelling processes are missing out or disregarded in the noosphere's treatment of modelling and application of mathematics as a mere add on to mathematics as a teaching discipline. The use of analogies, creativity and "trial and error"-approaches are left out as well as the discussion of the epistemic value of the models with respect to the knowledge field being modelled. This might hinder students' engagement in genuine discussions of value and limitations of models and hinder students to learn how models are reflection tools for professionals in many knowledge fields. There is a risk that it leaves the models without meaning – the meaning, which is used to support the justification of mathematics as teaching discipline. The questions, which led to the models, have become invisible.

As pointed out by Frejd and Bergsten, most modern uses of modelling are not fit for teaching in secondary school, but maybe historic cases can offer good cases to study forward and reverse modelling? To expose students to real world situations of how scientists have applied modelling in attempts to understand phenomena that were not well understood at the time, and to support students' reflections with respect to disagreements about the value of modelling results?[3] Maybe the teaching of (mathematical) modelling needs to be reconsidered from a didactic design perspective? To teach students the elements of modelling processes revealed in the historic contexts require for students to engage in similar processes, where the questions (posed by both teachers *and* constructors – students) are guiding the process. We have earlier discussed and compared two approaches to the teaching of modelling (Jessen, Kjeldsen & Winsløw, 2015), leading to different student conceptions of modelling, though not covering all revealed aspects. It seems crucial to develop teachers' but equally important the noosphere's knowledge on didactic transposition of mathematical modelling, if the teaching should prepare students for higher education involving mathematics and critical citizenship.

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[3] See (Kjeldsen, forthcoming) for an example of how the case of Rashevsky has been used in teaching.

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Part V

Modelling: from theory to practice

5

Paper IV: Comparing modelling designs

This paper compare and discuss the similarities and differences in the teaching of mathematical modelling through two similar case studies. One case is based on Problem based Project Work (PPW) and Mathematical Competence Theory (MCT), the other case study is the one presented in paper I. The two cases are compared through five variables chosen by the authors. It is concluded that the perspective on mathematical modelling stemming from the theories might affect the conception of modelling, gained by the students. This addresses research question 2 and what is gained from SRP based teaching with respect to mathematical modelling. This paper was published in the *Proceedings of the Ninth Congress of the European Society for Mathematics Education*, Feb. 2015.

Modelling: From theory to practice

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Modelling: From theory to practice

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Mathematical Competence Theory and the Anthropological Theory of the Didactic each offer different frameworks for the analysis and design of "modelling" as a central component of mathematics teaching. Based on two comparable cases from each research programme, we investigate how these differences appear in concrete design work, and what their practical consequences may be.

Keywords: Mathematical modelling, mathematical competence theory, anthropological theory of didactics, bidisciplinarity.

WHAT IS MODELLING AND DOES IT MATTER?

The fact that primary and secondary school students all over the world study a subject called "mathematics", with relatively similar contents and methods, is intrinsically linked to certain assumptions about the *relevance* if not *necessity* of this subject for every citizen in modern society. The formulation of these assumptions change over time and they are of course the object of constant debates, but an invariant common contention seems to be the *utility* of what is taught *in the actual or future lives of students*, or at least *its roles outside of school mathematics*.

One formulation, which has gained importance in the mathematics education community over the past 30 years, is based on the notion of *mathematical modelling*, defined roughly as "translations between reality and mathematics" (Blum & Borromeo-Ferri, 2009, p. 45). More complex descriptions of the modelling process, usually in the form of a *modelling cycle* (e.g., ibid, p. 46) have become commonly known and used in research into the ways in which these translations can appear in the school subject. It is a common assumption among researchers within this line of research that students' experience with all steps of the modelling cycle is essential to the justification of school mathematics in society (ibid, p. 46). In particular, Niss and colleagues (2002) proposed to consider *modelling competence* – the students' capacity to carry out mathematical modelling – as one of eight universal competence goals for the teaching of mathematics, linked to other goals equally defined in terms of competences. Their mathematical competence theory (MCT) thus integrates and develops earlier work on mathematical modelling, as an educational activity and goal, in a comprehensive framework for the analysis and design of school mathematics in a broad sense.

Another perspective on modelling stems from inquiries into the nature of mathematics as a school subject: how it is related to the science called mathematics, and more generally to "mathematical practices" appearing in society outside school? The anthropological theory of the didactical (ATD) emerged from the notion of *didactic transposition* (Chevallard, 1985) according to which school mathematics is a cultural set of practices and knowings which are inseparable from the institutions (schools) in which they are taught and learnt. In this theoretical framework, "mathematics" and "reality" are not *a priori* defined or distinguished; all human activity and knowledge is described in terms of *praxeologies* (Chevallard, 1999). Modelling has a wider meaning in this framework, as the elaboration of praxeologies in one domain in view of studying one or more *questions* in another domain. The school institution refers to this as "intra-mathematical modelling" when both domains are recognized as belonging to school mathematics, e.g. if school algebraic praxeologies are elaborated to study a question from school geometry (García, Gascón, Ruíz-Higueras, & Bosch, 2006). In ATD, modelling thus serves to create meaningful links between otherwise separate praxeological domains, whether or not these are considered as belonging to school mathematics or not.

The two theories are related to specific *design formats* which are often used for the design of teaching that involves modelling (cf. Miyakawa & Winsløw, 2009, for the distinction of theory and design format). In

MCT, it is *problem oriented project work* (PPW), in which students are to develop their competences while experiencing some or all steps in a modelling cycle (Blomhøj & Kjeldsen, 2006). In ATD, it is *study and research paths* (SRP), departing from one or more questions; the further development is sometimes represented with a tree like "map" of derived questions and praxeologies which students did construct while working with the questions (Barquero, Bosch, & Gascón, 2008; Jessen, 2014).

At this point, we have only hinted at some of the differences between two perspectives on modelling. The research question which interests us is a theoretical, but also quite practical, one: *What differences, if any, does it make for the design of new teaching practices, whether the theoretical control apparatus comes from MCT or ATD? In particular, are there differences between uses of the design formats PPW and SRP which can be related directly to the different theoretical notions of modelling found within MCT and ATD?*

We shall take an inductive approach to this question: we first present two cases of design of modelling activities for students in Danish upper secondary school, constructed from each of the two perspectives but otherwise similar in contents. Then we analyse the differences in view of providing tentative answers to the research question. To prepare that analysis, the presentations of the cases focus on the following variable features of modelling activities:

(V1) *Practical meaning of "modelling"* in the activity, as described by the authors

(V2) *Goals for the activity* (e.g. for student learning) *and their assessment*

(V3) *Organisation in time* of the activity

(V4) *Distribution of roles among students and teacher(s)*, in particular the way in which student autonomy is controlled (limited, furthered, differentiated, etc.)

(V5) *Adaptation to local conditions and constraints* (features of the activity which result from these adaptations, including choices made for (2)–(4)).

The case presentations given below are based on more extensive studies (Jessen, 2014; Blomhøj & Kjeldsen, 2006). V5 is further treated in these papers.

CASE 1: A STUDY AND RESEARCH PATH

The first case we will present comes from an experiment with study and research paths (SRP) in the context of Danish high school students' study line reports written in the second year of high school (a study line report is a bidisciplinary report students write in the second year as a preparation for the bidisciplinary "study line project", which is a high stake final exam in upper secondary school in Denmark, cf. Jessen, 2014, p. 2). The reports are about 15 pages long accounts of an autonomous work done by one or two students, within 6 weeks and with very limited access to help from teachers (V3). The study line of the students determines what disciplines are to be involved in the report. Before the 6 week period, the teachers formulate a set of questions for the students to work on (V4). For the study line of the experiment, the theme should combine the disciplines *mathematics* and *biology* with equal weight. These circumstances are constraints (V5) which affect the concrete design and in particular the variables V1-V4.

The aim for the study and research path (V2) was for students to develop new praxeologies in the domains of nervous physiology and differential equations by working with a certain generating question, given by the teacher together with some supplementary questions to ensure the involvement of both disciplines:

Q_0: How can a patient be relieved from his pain by painkillers like paracetamol – how does deposit medication work and how can we model this mathematically? Q_1: Explain the biological functioning and consequences of taking paracetamol orally versus taking it intravenously. Q_2: Create a mathematical model using differential equations that illustrates the two processes and solve the equations in the general case. Q_3: Give a concrete example, where the patient is relieved from pain and estimate from your own model how often paracetamol has to be dosed – which parameters (absorption, elimination factor, bioavailability) are important to be aware of? $Q_{3,1}$: Does it make any difference whether the dose is given oral or intravenously? Use your models while giving your answer. (*Translation of the original questions in Danish*)

Notice that in ATD, *modelling* means the elaboration of praxeologies in the two domains – done by students in view of answering the generating question (V1). However, in the assignment, "mathematical model" refers to a more restricted sense, which is closer to the notion of model found in MCT and, at least in outline, is the one found in official documents and text books for Danish high school.

The above assignment is based on a generating question Q_0 which the students can immediately understand, but not answer. In general, a generating question should be so strong, that it is necessary for the student to formulate derived questions Q_i, each representing a branch of inquiry, in order to answer Q_0. The answers R_i to the derived questions adds up to a final answer of Q_0 (Chevallard, 2012, p. 6). At the same time it is purposed that the generating question must be "alive" in the sense that students should be able to relate the question to things they perceive as interesting and real. These aims were deliberately pursued by the teaching design, knowing that several students in the class wanted to study medicine or similar after graduating.

The derived questions formulated by the teachers serve as supports for the students' study process (V4). In general, it is crucial that the students are not left with "big questions" that are unrelated to their praxeological equipment (Chevallard, 2012, p. 11); the relation to praxeologies from specific disciplines must be ensured. This was even more crucial in our context since no teaching activity was accompanying the SRP work of the students. Some students met after classes and formed their own working groups discussing strategies for answering the questions. The teachers were allowed to answer questions during the six weeks, and in order to keep track of the students working progress, the exchange of questions and answers was only permitted in writing (V4). For the same reason, students were asked to provide their immediate answer to the generating question Q_0 when it was handed out (without the derived questions Q_1–$Q_{3,1}$). After that, the entire assignment was given to them. After two weeks, and again two weeks later, the students were asked to answer the following questions in writing:

> What is your answer to the generating question right now? What have you done to answer the question? What are you planning to do next in order to come up with more fulfilled answers?

We cannot go into all the details of the analysis of this SRP, neither before nor after the experience (the latter being analyses of students' reports, cf. Jessen, 2014). However we notice that to construct the "mathematical model" asked for in Q_0, students must somehow examine the relationship between the amount of drug given, and the distribution of the drug in the body. How the pain is cured and how the drug is eliminated must be answered by praxeologies from the domain of physiology. The latter leads to consider that the pain is relieved in relation to how often the drug is given, the size of the body and the pain perception. Thus, the progressive development of a mathematical praxeology (involving tasks, which can be solved using techniques available to the students, e.g., CAS-based solution methods for differential equations) is closely articulated to the development of a biological praxeology. The modelling process in terms of ATD is not a question of following certain steps, it is an individual process where the students uses their praxeological equipment to investigate domains, form new questions, answer them with existing or new praxeologies unfolding the disciplinary organisation at stake (V1).

The intermediate answers from the students showed a variety in their working progress, which reflected different praxeological equipment among the students. Some students responded the first time, that they needed to know the half-life of the painkillers this indicates, that the students suspect, that there is a time dependence in the model, and that the model includes an exponential function. During mathematic classes they have seen that exponential equations are part of the solution to many differential equations. This implied, that they were trying to relate the generating question to the newly developed praxeologies in mathematics. Also they studied relevant medias since they were able to formulate relevant search topics. The students formulated derived questions such as the following: Q_1: How is pain registered? Q_2: How does paracetamol relieve pain (pharmaco dynamic)? Q_4: How can the dosing be modelled mathematically based on the biological knowledge? (Jessen, 2014, p. 11). The entire analysis shows that the students are constantly narrowing down their inquiry, by alternately studying the questions through physiology and differential equations.

The teacher involved was sure that for some students the generating question would not suffice to develop a reasonable model. It was for this reason that a part of the derived questions was handed out before the independent work of the students. Some of the students would otherwise not have been able to develop new praxeologies in the intended domains. With these more precise questions, they were able to identify relevant media (web-pages etc.) and although some of them uncritically adopted models constructed by others, they were all able to make use of them for simple calculations (e.g. of the amount of drug in the vein of a patient) (Jessen, 2014). Thus their modelling of the intended praxeologies was not as richly developed as in the previous case.

CASE 2: A PROBLEM ORIENTED PROJECT ON ASTHMA MEDICINE

Our second case presents a PPW on mathematical modelling related to the administration of asthma medicine. In MCT modelling competency is defined (V1) as

> A person's insightful readiness to autonomously carry through all aspects of a mathematical modelling process in a certain context and to reflect on the modelling process and the use of the model (Blomhøj & Jensen, 2003, p. 127).

The key words are *autonomy*, *modelling process*, *reflections*. PPW is particularly well suited to foster students' autonomous participation in the modelling process (Blomhøj & Kjeldsen, 2011). The goal for students' learning (V2) in MCT is to develop and/or enhance their competency.

A mathematical modelling process can be depicted analytically as a cycle consisting of six sub-processes (ibid., p. 387). Concrete modelling activities, like the case presented here, may have a variety of more specific goals for students' learning (V2) in order to adapt to local conditions and constraints (V5).

In a PPW, students work in teams with a problem for a longer period of time to produce a product representing the team's solution (V2+V3). The central idea is that the problem should function as the "guiding star" for all decisions made by the students in the sense, that all decisions should be justified by their contribution to the solution of the problem. This provides the students with (parts of) the responsibility of directing the project. It is crucial that the students are involved in (most of) the decisions taken in the modelling process and become involved in reflections upon the different steps in the modelling cycle. PPW opens for a distribution of roles among students and teacher(s) that makes it possible to direct the students' autonomy e.g. through specific requirements to the product of the project (V2+V4). PPW has the potential to foster in the students all the key elements in developing modelling competency which makes this format an obvious pedagogical choice in MCT.

The asthma project was designed by two teachers for first year students in mathematics in high school. The students were to: 1) work more independently than usually over a longer period (ten mathematics lessons of 1.5 hour each and a similar amount of homework); 2) develop new theory by working with modelling within a subject area (exponential growth) they hadn't worked with before; 3) work with a more complex and authentic problem for which they did not possess a standard method or technique such that the modelling, the mathematization, the interpretation of the results and the reflections about the modelling process and the use of the model became part of the project; 4) analyse a set of data in order to build a mathematical model; 5) use familiar concepts such as graphs and equations for functions in a concrete context; 6) develop their mathematical communication skills; 7) use ICT throughout the project. (V2)

These aims were achieved through a strict organization of time (V3) and a setup that allowed for and supported the students' autonomy (V4). The teachers divided the project into four phases (Figure 1). The teachers controlled phase 1–3, and the students controlled phase 4. The aim of the first three phases was to prepare the students for their independent work in phase 4. In phase 4, the teachers took on the role of

1. (1.5 module) Presentation of the problem (see Fig 2). Excel course, IT for project management, social contract. Crash course in the modeling process.
2. (1 module) Problem formulation Phase and decision.
3. (3 modules) Work with a set of four exercises related to the project in phase 4.
4. (4 modules) Working with the actual project.

Figure 1: The four phases of the design. 1 module corresponds to a 90-minute lesson (Blomhøj & Kjeldsen, 2006)

consultants (V4) that the students could ask for advice on specific problems.

In phase 1, the teachers introduced the students to a cyclic representation of the modelling process. The teachers used the process to inform the students about the various elements in mathematical modelling within MCT, and they asked the students to be aware of and to explain where in the modelling process they were at any given stage in their work. Hereby, the teachers made sure that the students became engaged in posing the modelling problem, constructing the model, solving the mathematical system and suggesting solutions to the problem (V2, V3 & V4/V5). In phase 2, the teachers trained the students' competence in posing mathematical modelling problems through discussions in the class room guided by the teachers (V1/V5).

The problem from phase 2 was given to all students with some data (Figure 2). The exercises in phase 3 were not included in the students' independent work. They served as inspiration and illustrated the level of mathematics, communication and documentation expected in phase 4. The product of the project work was a report, handed in by each group after phase 4 (V4/V5). The teachers formulated a set of requirements for the report to direct the students' autonomy in phase 4.

ANALYTIC COMPARISON OF THE CASES

A synthetic presentation and comparison of the two cases can be achieved using the five variables identified in the first section and indicated as they are "filled" by the above presentations (see Figure 3).

Despite evident similarities between Q0 in case 1, and the problem (Figure 2) underlying case 2, the contexts and constraints are quite different: in case 1, the students must work independently most of the time, and have to combine the two major disciplines (mathematics and biology) of their study line; while in case 2, the work is done as part of the regular teaching of one discipline (mathematics). In the Danish regulations for high school, *mathematical modelling* more or less understood as in MCT forms part of the competency goals for mathematics as a discipline (Niss et al., 2002); the bidisciplinarity required for study line projects is a more diffuse and general principle for the study line projects while in the case of mathematics, it is also often associated with the same notion of mathematical modelling. Despite these differences coming from the contexts, some more principal differences arising from the theoretical background of the two cases can also be identified.

Differences coming from the design formats

The variables V2–V4 are clearly shaped by the design formats. In PPW, everything begins with a *problem* defined in more or less commonly accessible terms, which should then be sharpened and translated into mathematical terms, in order to allow for applications of relevant mathematical machinery, either known in advance or developed through the project work. The PPW in itself does not suggest explicit structuring

Asthma patients' problems with exhalations may be alleviated medically by increasing the concentration of the drug theophylline in the blood of the patient. If the concentration of theophylline is below 5 mg / L it has hardly any positive effect. If the concentration is above 20 mg / L it has toxic effects. The problem is to administer medication such that the concentration of theophylline stays within a certain range in which the medicine is effective, say a concentration between 5 and 15 mg / L. The substance is excreted from the body through the kidneys; hence the amount of the substance in the blood will drop with time, which means that the patient will suffer, unless you "fill up" periodically. At the hospital where the patient is hospitalized you try, for the sake of the daily organization of work and to reduce the risk of errors, to schedule the medication so the patient is supplemented with an equal dose, D mg, with equal intervals of time, T hours. A doctor is examining how to choose D and T so that the concentration of theophylline remains within the range of 5- 15mg / L. On a patient, he has measured how the concentration of the substance decreases with time following an injection of 60 mg of the drug.

h	0	2	4	6	8	10	12	14	16	18
mg/L	10.0	7	5.0	3.5	2.5	1.9	1.3	0.9	0.6	0.5

Figure 2: The problem and the data (Blomhøj & Kjeldsen, 2006)

and requirements regarding the students' work besides the fact that the problem should be formulated in such a way that it can function as a guide. The formulation of the problem is part of PPW. Hence, it is left to the teacher to set the "scene" for the students' work within the given context, depending on his or her learning goals. A SRP begins with a *question* which, like the problem in PPW, is too open to allow for immediate, complete answers. In order to proceed, students need to work with subquestions arising from supplementary assumptions, suggested by the original question or by some first, intuitive hypotheses or answers. Both design formats leave the teachers with tools for *directing* the students work: in PPW, the structuring can allow students more or less autonomy depending on how the teacher choose to structure the work, and through specific requirements for the product – in this case a report - the students should deliver (Blomhøj & Kjeldsen, 2006, p. 168), while in SRP, the teacher may supply students with some derived questions to start with, some specific media to study, etc. (Winsløw, Matheron, & Mercier, 2013, pp. 271–282). In both cases, an initial planning may be adjusted to the work of the students, with the tree diagram of the SRP and the learning goals and (parts of) the modelling cycle as the main tools for control of these adjustments of the initial design.

Differences coming from the theories

MCT assumes a clear and evident boundary between mathematical and extra-mathematical phenomena, which implies (through the processes of problem formulation, demarcation of a domain of inquiry, and systematization), the construction of an object to be modelled. This object is then translated into a mathematical representation, which in daily work is also often referred to as *the model*. The preparation and conduct of the PPW can thus be structured according to the movements from the problem to the mathematical domain, and back – with an explicit notion of being "outside" and "inside" mathematics. ATD, on the other hand, is based on a general theory of human practice and knowledge, in which the organisation of praxeologies into disciplines is merely an institutional construction; the boundaries of what is called "mathematical" are not universal but contingent.

In MCT, it is part and parcel of mathematics teaching to develop students' explicit knowledge and experience of how mathematics (as a universal entity) *applies* to problems outside of that domain. In ATD, praxeologies are simply answers to questions which have been developed sufficiently to allow students to find culturally established answers through media or through research based on praxeologies they are familiar with; the main feature of modelling to experience is the development of praxeologies through this dynamics of study and research, independently of institutional classifications into disciplines of the praxeologies.

These theoretical differences have an impact on practice. In PPW based on MCT the disciplinary contents are in principle subordinate to the problem. The chief purpose is to reach a satisfactory solution to the problem through realisation of (specific features of) the

	Case 1: study and research path	Case 2: problem oriented project work
V1	Starting from a big question Q_0, develop derived questions and praxeologies which can answer these and in the end, at least partially, Q_0. Didactic theory is not taught.	Starting from a problem *P outside* mathematics, reformulate it as a mathematical problem, treat this, and evaluate solution relative to *P*. The modelling process is explicitly taught.
V2	Develop specific bidisciplinary praxeologies as answers to Q_0.	Modelling competency through phases of modelling of data and problem.
V3	Six weeks of independent work (individually or in pairs) based on Q_0 and some derived questions, with encouragement to search for media.	Project team work for ten 90-minute modules and similar amount of homework, structured by phases of modelling as shown in Figure 1.
V4	Teachers deliver Q_0 and some derived questions; students do study and research on these, with very limited access to teachers, to prepare their study line reports.	Teachers structure the work of teams according to the phases, with most autonomy required in the last phase (once mathematical formulation and expectations are established).
V5	Regulations of study line reports (combining math and biology)	Aims for regular mathematics lessons, which include mathematical modelling.

Figure 3: Syntheses of didactic variables as set by the two cases

mathematical modelling process including choosing disciplinary theory relevant for solving the problem. The mathematical content brought into play will depend on the mathematical competencies and knowledge of the modellers and their abilities to expand these. In the ATD approach to modelling, a more or less strongly directed SRP can be planned based on a priori analysis of its potential to realise certain institutionally defined disciplinary praxeologies as answers to the initial question. This could make the ATD approach to modelling implemented through SRP more attractive in institutional contexts where the disciplinary focus is strongly constrained. On the other hand, as we have argued and illustrated, the choice of design has theoretically determined consequences for the kinds and qualities of mathematical modelling activity, which students get to engage in. For further investigation one might analyse the activity students carry out in the classroom (how are answered produced and validated, etc.) and to what extend are the students able to solve other modelling problems in the future.

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Part VI

Questioning the world by questioning the exam

6

Paper V: Exam and SRP

In this paper is the restriction performed by high stake exams are discussed in relation to the implementation of SRP based teaching and the paradigm of questioning the world. To identify where the restrictions stem from the discussion employ the scale of levels of codetermination. The teaching designs of paper I and paper II explicitly addresses the exam format of the teaching they are designed for. Potentials of these formats are discussed in relation to SRP based teaching. Still the analysis points the need for ATD (and SRP designs) to consider high stake written mathematics exams. The paper address research question 3 and the viability of SRP in a school system as th Danish one. A shorter version of the paper was presented at the Fifth International Conference on the Anthropological Theory of Didactic, Jan. 2016, Castro Urdiales, Spain. The paper was submitted Sept. 2016 to: *Educaçao Matemática Pesquissa*.

Questioning the world by questioning the exam

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Resumen. En este trabajo se analiza el papel del examen y su impacto en la enseñanza de las matemáticas utilizando el concepto de niveles de cogestión de la teoría antropológica de la didáctica. Se argumenta que un debate del examen en relación con líneas de estudios e investigación es decisivo para tener éxito en el cambio de la paradigma de enseñanza actual de "visitando momentos" a "cuestionando el mundo".

Résumé. Ce document examine le rôle de l'examen et de son impact sur l'enseignement des mathématiques à l'utilisation de la notion de niveaux de codétermination de la théorie anthropologique du didactique. On fait valoir qu'une discussion de l'examen dans le parcours d'étude et de recherche est crucial pour réussir à changer le paradigme de l'enseignement actuel de «la visite des œuvres» à «questionnement du monde».

Abstract. This paper discusses the role of exam and its impact on teaching of mathematics using the notion of levels of codetermination from the anthropological theory of the didactics. It is argued that a discussion of exam in relation to study and research paths is crucial in order to succeed in changing the current teaching paradigm from "visiting monuments" to "questioning the world".

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El paradigma del cuestionamiento del mundo en la investigación y en la enseñanza (pp. xx-yy)
V congreso internacional de la TAD (Castro Urdiales, 26-30 enero 2016)
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Editorial, año

1. Introdiction

Researchers in mathematics education have argued for decades that common classroom activities in mathematics lack in potentials to teach students more coherent mathematical knowledge, autonomous mathematical thinking and creativity based on problem posing activities (e.g. Singer, Ellerton & Cai, 2013). Common teaching activities rather lead students to imitate the teachers' or textbooks' solution strategies for standard exercises (Bergqvist, 2004). Schoenfeld (1988) characterises common practice of mathematics teaching as activities where a rule is presented by the teacher, to be subsequently rehearsed by the students in standard exercises. Within the Anthropological Theory of the Didactics (ATD) this kind of teaching has been characterised as "visiting monuments" (Chevallard, 2015). The students are presented with a monument (e.g. the Pythagorean Theorem, the division algorithm etc.) and how it can be used in order to solve exercises. As such, mathematics teaching is similar to tourist excursion with a guide who explain or highlights the specific features of the monuments in order for the visitors to appreciate it. This monumentalistic teaching is often aiming at preparing students for a high stake exam, which raises the attention to the monuments but often have a negative influences such as surface oriented strategies developed to cope with external demands. When final exams consist of standard exercises which the students encounter progressively and passively in the teaching, the exam defines the monuments of the teaching – often more than the official curriculum and even textbooks. The result of this kind of teaching is problematic in the sense that "students are almost reduced to mere spectators" (Chevallard, 2015, p. 3).

In ATD, another teaching paradigm has been proposed: the paradigm of "questioning the world". The idea is that teaching situations should be based on a generating question that leads students to engage in posing questions and develop answers for these questions. The answers should be developed through a deconstruction and reconstruction of knowledge based on students' praxeological equipment, search and investigation of media, etc. Principles for designing this kind of teaching have been named study and research courses or lately study and research paths (SRP) (Chevallard, 2006 & 2015). The potentials of the dynamics between students' research and study processes have been discussed and experimented in a wide range of contexts (see (García & Hiugeras, 2005), (Serano, Bosch & Gascón, 2010), (Barquero, Bosch & Gascón, 2013), (Jessen, 2014 & 2015), (Rasmussen, 2016), (Florensa, Bosch & Gascón, 2016). The experiments reported on in these papers show the potentials of using SRP as design tool. Several designs address modelling explicitly, usually involving different kinds of functions. The goals of the experiments range from modelling with linear and exponential functions or with differential equations, engaging in bidisciplinary work, developing teachers' professional knowledge, learning statistics and even improving engineering education. However not much research discusses the relation between SRP and high stakes examinations. For some of the above mentioned studies, it might not be relevant in the context. Nevertheless, if the paradigm of questioning the world should become a viable alternative to the paradigm of visiting monuments in contexts where high stakes examinations form part of the institutional constraints, we need to address how they could work with the new paradigm.

In fact, the two studies by Jessen (referenced above) relate to assessment in the context of exam situations as they are prescribed by existing constraints and condition of Danish secondary mathematics teaching. Below it is pointed out what can be gained from these experiments, keeping in mind that students and teachers tend to return to the prevailing teaching paradigm even after a successful implementation of SRP based teaching, as pointed out by Barquero & Bosch (2015, p. 268).

We note here that in the general mathematics education literature, there is no shortage of papers which document the impact exam and assessment has on teaching (e.g. see Webb, 1992). The present paper intends to analyse, from such a broader perspective, the need for addressing the relationship between the teaching paradigm of "questioning the world", SRP based teaching and different forms of high stakes exams. Specfically the paper addresses the research question:

> What is needed from school systems in general and exams in particular for this new paradigm of “questioning the world” to be a viable alternative to the paradigm of “visiting monuments”?

The mechanisms of “backwash” of exam requirements on teaching and learning practices are analysed using the notion of levels of codetermination from ATD.

2. Why discuss exams at all?

In the ATD context there is not much mentioning of exams as part of a changed teaching paradigm. However the scale of levels of codetermination defined can be used to indicate where potential constraints and conditions for the teaching stems from. We recall that the scale consists of the following levels:

civilisation – society – school – pedagogy – discipline –
domain – sector – theme – subject

(Chevallard, 2002). As we will see in this section, the higher levels of codetermination affect the exit examinations and through them, the classroom activities and teaching paradigms. In this section, a minor literature review is presented, which indicates that exams must be taken into account if a change of paradigm should become reality.

2.1. A long term challenge

For the last 200 years or so, school examinations have been regarded alternatively as encouragement, grading and “outright corruption in allocation of scarce opportunities” (Cheng & Curtis, 2004, p. 7). Ever since written exams were introduced in Boston schools in 1845, it has been discussed what these tests are measuring and if they are objective in some sense. Throughout the 20th century, psychologically founded studies have investigated effects and results of testing (Romberg, 1987). In the 1980s and 90s more research was done on the impact of exams on the teaching and learning of mathematics. Schoenfeld reports on a study where a class was well taught, but the students still lacked justification of the techniques they used for solving problems: “By virtue of obtaining the correct answer, the students indicated that they had mastered the procedures of the discipline. However they had clearly not mastered the underlying substance" (Schoendfeld, 1988, p. 148). The study shows that the teacher kept lectures to a minimum and time was spent on students working with problems closely related to curriculum. It is further noted that the

> "examination is the primary measure of both teacher and student success […]" and further that "examinations were well established and quite consistent from year to year. Thus the amount of attention to give to each topic, and way to teach it (for "mastery" as measured by the exam), were essentially prescribed" (Schoendfeld, 1988, p. 153).

This is an example of the exam defining *de facto* the monuments of the teaching and thereby the actual contents of the teaching itself. The incentive for the teacher to let the test affect the teaching is clear; the teacher's performance is often measured by the students' results. Another study by Webb (1992) shows that "large-scale assessment has an influence on what is taught in classroom […]" (p. 678). The examination becomes a crucial constraint of the teaching since both students and teachers are measured based on the scores from the exam. Webb discussed purposes of assessment, and regarding final exams he points out that:

> "[the exam] is to provide information to decision makers, including those within the educational system, governmental policy makers, and others. At this level, assessment results are used by parents, administrators, school boards, and taxpayers as the basis of judgements about the effectiveness of the educational program in general and, in some cases about the relative skill and ability of individual teachers […]" (Webb, 1992, p. 663).

This was repeated by Suurtamm and colleagues (2016) in the commission for the topic group discussion of assessment at ICME13. Suurtamm et al. (2016) emphasise that the exam is most certainly a strong part of the constraints put on teaching from civilisation, society, school (the higher levels of co-determination). Webb points out the role (and constraints) played by society by claiming that "public has, in large part, been resistant to reform efforts to expand assessment beyond norm referenced testing […]" (Webb, 1992, p. 664). Hence, these influences of society in general might cause challenges for changes at the level of pedagogy and be at the root of the resistance reported by Barquero & Bosch (2015).

The literature referred above mainly discusses the situation of national exit exams, however the backwash effect of exams are not restricted to mathematics education nor restricted to national mathematics examinations (Cheng & Curtis, 2004; Bergqvist, 2007). Bergqvist investigated what kind of reasoning is needed for good performances at the exam of an introduction courses on calculus at the beginning of undergraduate programmes at four Swedish universities. It is argued that emphasis is put on imitative reasoning and the use of algorithmic procedures in exam exercises, and this can prevent students from learning relational reasoning. Reasons for this are primarily the impact of assessment on teaching and that relational reasoning focused instruction is time-consuming, according to teachers (Bergqvist, 2007).

In the Danish context similar findings have been reported: "it seems that the circumstances and traditions governing university mathematics teaching make it difficult to assess more than the use of standard techniques or the passive knowledge of textbook material" (Grønbæk, Misfeldt & Winsløw, 2009, p. 85). Grønbæk et al. is referring to written as well as oral exams in an introduction course on real analysis at the University of Copenhagen. The study of Grønbæk, Misfeldt and Winsløw investigated the possibilities of altering assessment and exam situations in order to "minimize regrettable effects of assessment, while retaining a visible incitement for students to meet necessary work requirements as well as a credible declaration of the results of this work" (Grønbæk et al., 2009, p. 85). They further argue that if they want to improve quality and scope of students' work, they need to alter the exam, the frames of the teaching and the role of the teacher (Grønbæk et al., 2009, p. 91). The course was changed by experimenting with a new design of the oral exam, based on "thematic projects". A thematic project is a written synopsis covering mathematical tasks, which are relatively open. Some of the tasks were more complex and theoretical than standard exercises (Grønbæk et al., 2009, p. 93). This format of oral exams has been introduced in upper secondary mathematics in Denmark, and below we will explain how this can function as assessment format in relation to SRP or SRA based teaching.

2.2. Questioning the world, SRP and test design

When Chevallard (2015) proposed the teaching paradigm of questioning the world, he emphasised the role of questions to make the teaching of mathematics truly educational and adapted to present-day society. The aim of the paradigm of questioning the world should be for students – and citizens – to become *herbartian*, which means having a "receptive attitude towards yet unanswered questions and unsolved problems, which is normally the scientist's attitude in his field of research" (2015, p. 178). In this sense the paradigm relates to problem posing, with respect to which Silver (2013) argues that:

> "Not only do we need more attention to issues of reliability and validity in the measures we use to assess the impact of mathematics instruction infused with problem-posing activities, but we also need to explore ways in which problem-posing tasks might be used as assessments of desired mathematics learning outcomes." (Silver, 2013, p. 161).

Both issues remain largely open in the mathematics education research literature; to address them from a task design perspective might promote a change of teaching paradigm putting questions and problem posing in front. Furthermore, Suurtamm and colleagues (2016) argue that the backwash effect of high stake exams is not uniquely negative: "if countries have nationally organised exit examinations these may drive (or hinder) reforms [of mathematics teaching]" (Surtamm et al., 2016, p 20). Further, Swan and Burkhardt (2012) suggest that "teaching to the test" can have positive outcomes as long as the test items are designed to align with and nurture the intended reform of teaching. In terms of the scale of levels of codetermination, if we want to change the dominant teaching paradigm – hence make changes at the level of pedagogy – we need to take into account the higher levels of codeterminations (school, society and civilisation), including the preferences situated at these levels for evaluating the outcomes of mathematics teaching through high stake exams. In doing so we need to explicate how these high stake exams should be designed in order to promote the change of teaching paradigm.

And, of course, be clear about what is it that high stake exams should promote.

3. SRP and current exams

In order to discuss the paradigm of questioning the world and SRP in relation to examination, a more detailed presentation seems appropriate. The paradigm of questioning the world is proposed through teaching designed as Study and Research Paths (SRP). A SRP is initiated when a group X (of students) starts to study a generating question Q_0, assisted by another group Y (one or more teachers) (Chevallard 2006, 2015). A generating question is formulated in a form so that students understand the question, but are unable to answer the question without engaging in a study and research process. During the study and research process, the two processes are entangled and not necessarily possible to separate. The study process is characterised as students' study of *works* created by others (textbooks, papers, online videos etc.) – including other students. The knowledge studied is deconstructed and reconstructed as an answer to the question in what is characterised as the research process. The de- and reconstruction of knowledge is carried out on the basis of the students' praxeological equipment – their previously acquired knowledge – which might be combined and related to create new knowledge. The process often requires that students pose derived, perhaps more manageable, questions Q_i (Barquero & Bosch, 2015). When studying works in order to answer Q_0, students supposedly formulate questions addressing the nature of minor rules or notions, which need further study in order to develop a coherent answer to Q_0. As such the question Q_0 generates numerous paths and side-tracks in the process of developing an answer to Q_0 (Bosch & Winsløw, 2016). Hence, the learning process of a SRP is characterised as a dialectic between research and study (Winsløw Matheron et Mercier, 2013), which the design of a strongly generative Q_0 supports.

It is evident that this way of teaching might lead to relatively different SRP's, meaning students develop different praxeologies and therefore different mathematical knowledge from working with a SRP. Moreover this calls for ideas regarding orchestration and guidance of the study and research process. The guidance have been approached by didactic researchers in ATD with different means (see (Barquero et al, 2013; Serrano et al, 2010; Rasmussen, 2015; Florensa et al. 2015; Jessen 2014 & 2015). When individual or group findings are shared, to some extent it can be expected that students develop similar but not identical answers. This is a challenge in the sense of national exit examinations with exercises testing certain techniques which are the main monuments of curriculum. In this situation, common classroom activities and imitative reasoning might prepare the students equally well or even better for the examination exercises. This leads to the question if there exist exam formats where SRP prepare students better, which are as reliable as existing formats, and which test curriculum in more satisfactory ways. Below we discuss these parameters in relation to the studies conducted by Jessen (2014 & 2015). But before we can discuss these cases we give a short account of the context of these.

3.1. The context of the two studies

In Denmark, upper secondary mathematics education is divided into three levels. Level C takes one year and is intended for students having their main academic interest in humanities or social sciences. Level B takes two years and most students attend this. Level A lasts all three years of upper secondary school and is required for higher education in natural sciences, engineering, economy etc.. Students' mathematical knowledge (specified in terms of competencies) is tested in three different types of assessment depending on the level of mathematics the students are studying.

The most common type is written exams which students sit for 4 or 5 hours in a gym hall, solving a number of exam exercises (Danish Ministry of Education, 2013). The first hour is dedicated to exercises to be solved with just pen and paper. During the remaining time there is a full allowance of calculators and computers, including Computer Algebra Systems (e.g. Geogebra, TI Nspire, wordmath, Maple etc.). It is not allowed to communicate with others during examination period, but students are allowed to re-visit webpages which have been used prior the exam by the students. This causes a variety of technical challenges including student fraud, which are pointed out in a Danish report on upper secondary mathematics teaching (Jessen, Holm & Winsløw, 2015), but which will not be elaborated further here.

The second form is oral exams. Students draw a question, which they are familiar with and have prepared at home. They are given 24 or 30 minutes for preparation and examination. Questions often require students to present a notion, rule or theorem and how it can be applied in contexts outside mathematics. For levels B and A, students are expected to provide some proof or justification of the rules or theorems they present. A significant amount of the questions should be based on "thematic projects" – the format designed for the calculus course mentioned earlier and presented in (Grønbæk et al., 2009). The ministry of Education was inspired by Grønbæk and colleagues and required the format used in secondary education. However the format was aliened to most secondary teachers and represents a challenge for most teachers to design.

The last examination type is bidisciplinary projects, where students combine two disciplines they have studied at A- or B-level. The teachers of the two disciplines formulate a problem or set of questions, which the student must answer employing knowledge from the two disciplines. The answer must take the form of a 15-20 pages report, which is evaluated and given a grade by an external evaluator. This exam counts twice in the total score of the students' performance and is therefore very important to their final grade from upper secondary education. It is the individual student who decides if mathematics should be one of the disciplines in their bidisciplinary project.

All exams in Denmark are on draw, which means that each spring the Ministry of education randomly determines which students are attending what exams. Hence, students can graduate without attending any mathematics examination – and others are tested both written and orally. Below are the rates of students from each level attending the oral and written examinations in the academic year 2013/2014:

Level of mathematics (A high)	Percentage of students attending oral exam	Percentage attending written exam
A	21 %	101 %
B	30 %	74 %
C	20 %	-

Table 1. The table shows how many percents of students at each level is attending oral and written exam respectively at each level in Danish upper secondary education (see (Jessen et al, 2015, p. 37).

The reason why more than 100% of the students attend the written exam at level A is that students can attend this exam even if they did not attend upper secondary education, but attended upgrade courses for higher education. It is evident that the probability of attending written exam is much higher than the oral exam. Jessen et al. (2015) reports how this affects the activities in the classroom, according to both teachers and students: large part of the teaching aims at students rehearsing their ability to answer the written exam exercises.

4. SRP as design for bidisciplinary written projects

The high stakes bidisciplinary written exam in Denmark runs around Christmas in the last year of upper secondary. Students choose two disciplines from their study line. Upper secondary education is organised around study lines with a number of common disciplines and three disciplines defining the study line, e.g. Biology, Mathematics and Social science. Students choosing this particular study line often choose higher education in the field of health and medical sciences.

The bidisciplinary exam results in a written report, which the students produce during two weeks without teaching. Together with the teachers of the two disciplines, the student finds an area of interest and the teachers formulate a problem or some questions the student must answer in the 15-20 pages report. The problem should cover both disciplines and the student is supposed to write a report combining the disciplines in a meaningful way.

The first case study we want to refer to was conducted in a study line as the one described above. During the second year the students wrote a minor bidisciplinary report (10-15 pages) on a problem formulated by the biology teacher and the mathematics teacher of the class. The students had two days without teaching and were expected to carry out most of the work after school hours. This second year report served as rehearsal for the high stake exam in the third year.

In 2012, a rehearsal problem was designed in terms of a generating question covering mathematics and biology (Jessen, 2014). The topic was how to administer painkillers in order to relief a patient from his pain. This requires developing and activating praxeologies from the fields of differential equations and nerve physiology. Students formed minor study groups posing questions, sharing different media and discussing each others' hypotheses for answers to derived questions (Jessen, 2014, p. 218). The teachers followed the process through email correspondences with the students. Students were asked to pose questions on email so the teachers could keep track of the process. This gave insight into the sharing and the group dynamics of the class during the study and research process. Further it forced the students to be precise in formulating the challenges they encountered and the formulations of the ideas they were working on. SRPs proved to be a suitable model for the activities and the a priori analysis secured the connections between disciplines. Moreover students engaged themselves in an inquiry of the fields involved, leading to rich answers regarding the generating questions. However some students did not see the need of mathematics in the project. A reason could be insufficient details in the initial questions handed out to the students. It was concluded that SRPs and the design principles were recommendable for this kind of exam.

As implication for the design of SRP related examinations, one may thus recommend more open projects, which require students to pose derived questions and engage in study and research processes, as input to the second issue formulated by Silver (2013), regarding test designs including problem posing and reflecting the classroom activity – under the assumption that students were taught mathematics through SRP's beforehand. However, it is hard to control if written reports were formulated by others than the stated author – practices of organised fraud were recently reported on in a documentary broadcasted by Danish national television. As a result, it has been proposed that the study line project should be defended at an oral exam, as with university dissertations. An alternative would be to require that students did the work in a gym hall, which would limit the study process, and in view of time constraints imposed by that, students might not be asked equally broad generating questions. For sure this type of examination requires further elaboration in design and orchestration before being fully applicable for large-scale mathematics examinations. Also, Swan & Burkhardt (2012) argue that mathematical processes might not require time-consuming project examinations. We will discuss a concrete alternative below.

5. SRA based teaching and oral exam

Jessen (2015) used study and research activities (SRA) at level C mathematics (cf. above) to teach the basics of exponential functions. The study shows potentials regarding students' problem posing and development of answers to these. But also students' reading of mathematical texts and their engagement in the study of different media was furthered. Moreover the study shows potentials in supporting the students' de- and reconstruction of mathematical knowledge in autonomous working processes. The teaching was based on a sequence of generating questions like those of a SRP, however there was an specific target mathematical organisation for the students to develop, corresponding to the "monuments" of curriculum (for further details see Jessen, 2015 & Jessen, to appear). The design had similarities with the sequence of SRA's presented by Barquero, Serrano & Ruiz-Munzón (2016). The first generating question leading to the first SRA was the following:

> Q_1: Grandparents start a savings account for their new-born grandchild by putting 5,000 dkr into an account of annual rate of interest of 2.5%. Bank regulations say that the balance may not exceed 50,000 dkr. Will this be a problem?

The groups chose different solution strategies ranging from repeated multiplication with the factor 1.025, to graphic representations and to formulate the question as an equation, which the students needed to study new techniques in order to solve. The groups were only given 5-7 minutes for each question before presenting preliminary results. This assured the classroom discussion on strategies, which encouraged the students to pose derived questions, discuss mathematics and justify their ideas.

The orchestration in the classroom was relatively strict. The class was divided into groups of three students with similar praxeological equipment, to the extent this could be estimated based on their previous performances. The groups were required to present preliminary answers to Q_0 and their own derived questions both orally and in writing at their groups' field of the whiteboard. They were not allowed to erase anything. Within this relatively tight framework, the students did share their work and questioned each other's strategies, and in fact they developed more coherent answers to the generating questions than those expected and found in curriculum and standard textbook materials (Jessen, to appear). The sequence of SRA's prepared the students for producing a thematic project, on which the oral exam was based. As we shall now see, this created a clear alignment between the in-class working format and the format of the final exam.

5.1. Thematic project exam and its link to SRA's

The thematic project on exponential functions, which the students answered as homework, were based on the following questions:

> Q_a: Give a short presentation of the notion of exponential function. Apply the notion of rate of growth in your presentation.
>
> Q_b: Given two points on a graph of exponential growth, which can be described by the expression $y = b \cdot a^x$, show how you can find the rate of growth a and the graphs intersection with the y-axis, b.
>
> Q_c: Show how to model exponential growth based on a word problem and how it can be used for forecasts.
>
> Q_d : What information is contained in the doubling time of an exponential function and how to calculate it ?
>
> Q_e : What is exponential regression ? Give an example where you have used it and discuss the choices you made.

The students were explicitly encouraged to draw on their answers for the generating questions from the sequence of SRA's. The students wrote the thematic projects in the same groups as those they worked in during the SRA's. They knew the thematic project would be the basis of one or two questions at the oral exam. Therefore, the students genuinely engaged in studying the answers of the other groups during class sessions. Everybody wanted to perform at their best at a potential examination, and this generated an external motivation for the students to be able to justify the rules they might have to present at the exam. This can be a reason for the observed autonomous questionings by students during classes as reported in (Jessen, to appear). The draft projects were handed in to the teacher. The teacher gave comments on the projects, as in what parts were well written and well justified and what parts could need further elaboration. Based on the comments, the groups revised their projects before the oral exam.

This is similar to the management of the thematic projects described by Grønbæk, Misfeldt and Winsløw (2009). In the university course, students handed in six reports (4-5 pages) covering different topics. At the oral exam, students drew one report to present, instead of a theorem from the textbook as in the more usual practice. The reports were worked out as group work. For each project, a date was fixed for handing in to the tutors, in order to get feedback on the reports. Tutors pointed out what was good and which parts needed elaboration. Lectures were partly dedicated to the topics of the projects, and the lecturer was also answering questions about them during lectures (Grønbæk et al., 2009).

Even if only 20% of the level C mathematics students attended the oral exam in 2014, according to table 1, it turned out to be 70 % of the class where the sequence of SRAs had been experimented. The entire curriculum on functions was taught on the basis of SRA sequences, and finalised in thematic projects. At the exam the students performed slightly above average, and better than previous classes in the same study line. The good grades were mainly given to students who drew questions relating to the notion of functions. This indicates that students were better prepared for the oral examination through the teaching based on open questions. At first, the external examiner expressed a great deal of concern regarding the SRA teaching, which he found “crazy”. After the last student had left the examination room he admitted, that it might not be so crazy after all. The reason for mentioning this episode is that this kind of resistance represents an obstacle for changing the teaching paradigm. Even if enthusiastic teachers wish to change the way he teach, it might meet resistance from other parts of the educational system, including colleagues and external examiners.

6. Discussion

The paradigm of questioning the world introduced through teaching designs based on SRP and SRA certainly shows potentials for teaching students not only the use of techniques, but also elements of the rationale of the techniques. Worth noticing is that the need for justification comes from the students, when they are trying to develop coherent answers rather than justification as another monument to visit (Jessen, to appear). Despite the good potentials and already documented results (e.g. in higher education), it is still a challenge to persuade teachers (who rarely have the motivations of a didactical researcher) to continue teaching based on SRP's (Barquero & Bosch, 2015).

The literature on large scale assessment and exit examinations indicates massively how assessment affects the teaching paradigm, which is why this paper investigates the possibilities of designing stronger links between SRP or SRA based teaching and examinations. One can argue that the teaching paradigm will not change by just producing good examples of its realisation. An infrastructure of tools and regulations may have to complement the good examples.

The study of Jessen (2014) indicates that SRP is a strong tool regarding design and evaluation of the students' performances in project based examinations. The high stakes bidisciplinary written reports might be a rare case of examination, outside Denmark. However the study supports the suggestion of using larger projects for assessment in mathematics education. Similar ideas have been proposed by other researchers (e.g. see (Niss, 1993); (Frejd, 2013); (Suurtamm et al., 2012)).

Others, like Swan and Burkhardt "contest the assumption that tasks assessing mathematical processes need to be of project length" (2012, p. 31). They argue that high stakes exam exercises testing processes can be designed appropriate for written exams lasting e.g. four hours. Bearing in mind that project examinations never really have gained ground in mathematics education, the idea of assessing the result of SRP based teaching in yet another SRP might not convince actors and decision makers at the higher levels of codetermination on how to design exit examinations.

With respect to oral exams, the thematic projects are promising formats which deserve further experimentation and design. It could be an interesting endeavor to cover all teaching for the one year mathematics through sequences of SRA's, with all exam exercises then relying on thematic projects. The study by Jessen (2015 & to appear) exemplifies the idea that the change of examination drive the change of teaching, as proposed by Suurtamm et al. (2016), in the sense that writing the best possible thematic project delivered yet another incentive for students to engage in the SRA's.

This directly relates to the research question of this paper and what is needed from school systems and exam regulations, in order for the paradigm of questioning the world to be a viable alternative to the paradigm of visiting monuments. One answer could be to let examination rely only on oral examinations based on thematic projects. In the real analysis course redesigned by Grønbæk and colleagues (2009) the students performance was measured by the oral exam as well as by a more traditional written exam and the two performances were given one grade. This could be another solution, to retain the attention to basic written skills. Here we must point out that a strong constraint on national examinations is the cost of the exam; it would be very costly to require all students to attend oral exams, and it is far from the current situation (see table 1).

This leads to the idea of designing test items for written exams, which are aligned with SRP and SRA based teaching. For teaching based on sequences of SRAs, the students are supposed to develop certain praxeological organisations which could serve as a basis for design of test items. Following the design presented by Jessen (2015), students could be tested reasonably by standard exam exercises regarding exponential functions. A classic example is exercises of the form: given a function describing the relation between the age and weight of a fish, how old is a fish of a given weight? How old is the fish when it has doubled its weight? A fish of a certain age had this weight, how does that fit the model? What are the limitations of the model?

The paradigm of questioning the world might prepare students to answer this exercise, but the exercise can hardly drive a change of teaching paradigm into questioning the world. In a teaching paradigm aiming at students developing raison d'être of the techniques they employ, justification should be a core element of the test items. For this purpose development of exercises as the following is needed:

Given the equality: $(3y - 6x) : 5 = 3$, argue if it represents a linear relation between the variables y and x?

This type of task requires that the students apply basic algebra techniques to manipulate the expression. Further, the students must know what is meant by a linear relation between the variables. Hence, the exercise tests a larger praxeological organisation, rather than isolated techniques. There is a need for developing a large variety of these types of tasks aiming at students' mathematical processes and justifications, as argued by Swan and Burkhardt (2012). Moreover, the praxeological analysis of ATD could become a strong tool for assessment design, in the sense that the explicit praxeological references models could guide what connections between point praxeologies should appear in the test, and what is actually tested in different type of tasks. Praxeological analysis of tasks has been developed in detail by Chaachoua (2010), who conducted detailed analysis of type of tasks and techniques in order to develop an applet in terms of a computer algebra system aiming at students solution strategies. This approach to praxeological analysis might also be useful for the design of more advanced test items.

In relation to SRP based teaching, the a priori analysis of possible paths and side-tracks including the final point praxeologies students could also serve to develop more advanced items than the types of task usually covered in the written exam exercises.

Redesign of exam exercises might not be an easy path to changing the teaching paradigm. The studies of Romberg (1987), Schoenfeldt (1988), Webb (1992), Cheng and Curtis (2004) and Suurtamm et al., (2016) all indicate that radical changes of the exam is not easy, due to the influence of policy makers, societal stake holders and parents, who all hold strong views regarding the nature of mathematics exams. Representatives of the higher levels of codetermination need to be taken into account when promoting a changed teaching paradigm.

Additionally, the teachers should be prepared for changes through professional development. Barquero, Bosch and Romo (2015) and Rasmussen (2015) present two different approaches to introducing and assisting teachers in designing SRP based teaching. Barquero et al. (2015, p. 813) have developed online in-service courses, and documented some of the barriers for teachers to actually change the way they teach. Rasmussen (2015) developed part of a pre-service teachers' course as an SRP, and encountered resistance from some students, who challenged the changed didactic contract of SRP based teaching, compared to more common classroom activities. Prevailing didactic contracts install in students, including future teachers, some very solid beliefs about what mathematics teaching should be and how its outcomes must be assessed.

Further, if the paradigm of questioning the world should be anchored in the school systems, teachers need support at the lower levels of codetermination: discipline, domain, sector, theme and subject. The teachers need frames for professional development and tools on how to design and manage everyday SRP based teaching. This calls for paradidactic infrastructures. Here the Japanese tradition of lesson study, joined preparation and reflection upon shared teaching experiences, could serve as inspiration (for further details in paradidactic infrastructures see (Winsløw, 2012)).

7. Concluding remarks

It seems evident that to change the teaching paradigm of school mathematics requires changes at many levels of the school system and at the higher levels of codetermination. However, leaving aside the more political dimension of the question, there are intriguing needs for didactic researchers to pursue a number of design problems. This paper mainly offers indications of ideas on how to implement SRP and SRA based teaching in upper secondary schools.

There is, however, no doubt that ATD researchers should address more intensely the backwash effects of exams on the classroom activities, by designing and experimenting modified forms of high stakes examinations. The aim of this work should be to design examination formats which can further and adequately assess teaching according to the paradigm to questioning the world. This endeavour incudes solving open questions on how to assess problem posing activities, to capture and value students' mathematical thinking, their creativity and study processes.

The format for oral exams offered by the notion of thematic projects seem promising and can drive the study and research process of teaching based sequences of SRAs. Much more open is the question regarding alternatives to or variations of standard written exams. There are potentials and shortcomings in written project as examination format. Maybe the answer to this question lies within new and more advanced uses of praxeological analysis.

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Part VII

The collective aspect of implementing study and research paths – the Danish case

7

Paper VI: Paradidactical infrastructures

First experiences with in-service teacher courses on the design and implementation of SRP based teaching indicates that teachers might need more than a course to base their teaching on SRP designs. Therefor this paper studies current systems for collaboration of teachers and analyze the nature of development of their teaching practice. To investigate what systems exit for professional development of teachers and their reflection upon their own practice, the notion of paradidactical infrastructures is employed. To study the nature of collaboration and development of own practice, teachers own indications with respect to this matter (see Jessen, Holm & Winsløw, 2015) is analysed employing the herbartian schema. This paper address research question 3 by pointing to the need of paradidactical structures.This paper was published Nov. 2016 in: *Educational design in math and science: The collective aspect: Peer-reviewed papers from a doctoral course at the University of Copenhagen.*

The collective aspect of implementing study and research paths – the Danish case

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For more than 10 years papers have been published showing the potentials of designing teaching based on Study and Research Paths (SRP) – a notion developed as part of the Anthropological Theory of the Didactics (ATD). SRP relates to the proposed teaching paradigm of Yves Chevallard: "Questioning the world". In this paper it is being argued that teachers collaboration in implementing reforms or changes of a dominant teaching paradigms is an important element in making these initiatives real alternatives to existing situation. Elements of current collaboration are analysed and the analysis points towards teachers need of a common language to discuss and articulate design and evaluation of or reflection upon teaching.

The Anthropological point of view

It has been proposed by Yves Chevallard that it is time for change of teaching paradigm. He characterises the current teaching paradigm as "visiting monuments" and proposes a new one called "questioning the world" (Chevallard, 2015). The paradigm of visiting monuments refers to teaching where a theorem or other pieces of mathematical knowledge is presented, it is shown how to use it for solving exercises and maybe proved. Students are expected to learn and appreciate the piece of knowledge. However the piece of knowledge does not necessarily answer a question being of any interest of the student who is supposed to learn it. This might end up with mathematics being perceived as a list of monuments to visit, a number of techniques to solve exercises – but without a raison d'être and without relations between the monuments (Chevallard, 2015, pp. 175).

On the contrary, Chevallard proposes, that teaching should be based on living questions, which leads students into a study and research process developing answers. Study and research path (SRP) is a design tool where teaching is based on an open question, which is supposed to initiate a study and research process for one student or among several students. The idea is that teachers pose a generating question, which the students understand but cannot answer immediately. The students have to study different media (textbook, video, internet, teacher etc.) decompose and reconstruct the new knowledge in a research process in order to formulate an answer for the generating question. The research process takes place in a milieu consisting of old knowledge, knowledge offered by media, pen and paper, calculator etc. A strong generating question Q_0 will lead to derived questions Q', Q'' and so forth (Jessen, 2014). The answers to the derived questions add up to an answer for the generating question Q_0.

The process and dynamics of SRP can be described using the herbartian schema:

$$[S(X;Y;Q_0) \to M] \hookrightarrow A^{\heartsuit}$$

Where the system S consist of a group of students X interested in studying the question Q_0 assisted by a group Y. In doing so the system interacts with a milieu M developing their personal answer, $A^{\heartsuit}$. The milieu $M = \{A_1^{\diamond}, \dots, A_n^{\diamond}, O_{n+1}, \dots, O_m, Q_{m+1}, \dots, Q_p\}$ consists of p elements where $A_i^{\diamond}$ represents existing answers within the X, O_j are the works of others which are being used in the study process and Q_k represents the questions raised during the study an research process. The processes described by the herbartian schema further the development of the raison d'être of the developed knowledge, which can be characterised as praxeologies or mathematical organisations (Kidron et al., 2014 p. 157). Chevallard formulates the aim of the new teaching paradigm as: "the new didactic paradigm wants the future as well as the full-blown citizen to become Herbartian" (Chevallard, 2015, p. 178), meaning a person who questions the world and develops answers based on existing knowledge, the works of others and who deconstruct and reconstruct all this into knew answers. Regardless how interesting this sounds, it is evident that the change of teaching paradigm will not happen easily. Therefor the research question of this paper is:

What constraints and conditions exist in the implementation of SRP based teaching and the paradigm of questioning the world from the collective perspective on the teaching of mathematics?

Moreover it will be discussed what infrastructures exist in the teaching system of upper secondary mathematics in Denmark today? What elements support or hinder the diffusion of the paradigm of questioning the world and SRP based teaching? It is evident that a first hindrance is that almost no Danish high school teachers are familiar with ATD or SRP. But even if they were, the analysis of this paper points towards challenges.

In the analysis the herbartian schema is applied, partly to teachers own descriptions of their professional development (electronic survey with more than 1000 respondents across the country), as it is described in a recent report giving a status on mathematics teaching at upper secondary level in Denmark conducted by Jessen, Holm and Winsløw (2015) and partly on the ministerial regulation for bidisciplinary work at this level. Hence, the basis of the following discussions are the previous work of myself and colleagues on the teaching of mathematics and other disciplines at the high school level, i.e. ((Jessen, Holm & Winsløw, 2015), (Jessen, 2015), (Jessen, 2014) and ministerial guidelines). Further the analysis engages the notion of paradidactic infrastructures by Carl Winsløw (2012). The paradidactic infrastructures are "everything which conditions and constraints the PS [Paradidactic System] in its different phases and in the interplay between phases" (Winsløw, 2012, p. 293). The phases of the PS is the pre-didactic system (PrS), the didactic observation system (DoS) and the post-didactic system (PoS), which relates respectively to the planning, the observation and the evaluation of the didactic system (DS). The didactic system is defined as a group of people studying some objects or organisations using some artefacts in doing so (Winsløw, 2012, p. 292). The PS runs parallel with the DS and in some institutional settings not much attention is paid to the Prs and the PoS. This is not the case in Japan where lesson study is a formalised structure of the elements of the paradidactic system. Lesson study in Japan functions as means for professional development (Winsløw, 2012, p. 295). Initiatives have been made with respect to implementing lesson study in the Danish school system, however these efforts do not cover the upper secondary level.

The Danish context

In Denmark a major reform of upper secondary education took place in 2005. The education was divided into a large number study lines (more than 200 see (Ministry of Education, 2013c)). The students must choose one before entering upper secondary education. A study line consists of three disciplines at a certain level (there exist three levels, C to A, A highest). An example of a study line could be biology A, mathematics B and social science B or mathematics A, Physics B and chemistry B. Students have the freedom of choosing extra disciplines (psychology or philosophy etc.) or they can use this liberty to raise the level of one of the disciplines in the study line. Further there are a number of compulsory disciplines as native language, English and History among others. Moreover students follow general study preparation which is evaluated in an oral, high stake exam based on a bidisciplinary synopsis treating a case linked to the topic of the year determined by the ministry of education (Winsløw, 2012, p. 299 & Ministry of Education (2013d)). On top of this the students attend another high stake exam, the study line project, which is a bidisciplinary written report of 15-20 pages students hand in a half year prior graduation (see further in (Jessen, 2014) and (Hansen & Winsløw, 2011)).

Regulations of the teaching of mathematics at each of the levels A, B and C is stated in curriculum and elaborated in documents called ministerial guidelines. In addition, the written exam represents a strong constraint on the teaching as it has been pointed out in (Jessen et al., 2015, pp. 13 & Jessen, 2016). During the first half a year at high school students attend two crossdisciplinary subjects: general introduction to natural sciences and general introduction to language structures (our translations). The first subject must be taught by at least two teachers representing minimum two disciplines within the natural sciences introducing students to different methods across natural sciences – mainly focusing on different ways to work in laboratories and inquiry based. The other discipline introduces students to commonalities with respect to language, an equivalent to the former focusing on grammar and methods from humanities.

Previous experienced study and research

In 2012 Jessen designed study line projects based on SRP and handed out generating questions instead of the usual problem formulations (a list of questions students should answer). The result of this study is to be found in Jessen (2014). It is concluded that SRP is a suitable tool for designing questions for the study line projects, but Jessen also points out some of the challenges in designing these collaboratively (Jessen, 2014). It is here worth noticing that in Denmark most teachers have a minor in one discipline and a major in another (often linked as mathematic and physics) and teach both disciplines in high school. However it is still reported to be a challenge to go across disciplines and collaborate with teachers of a third discipline and connect these (EVA, 2015) – teachers are not trained in this.

In 2014 Jessen explored the potentials of SRP as design format. In this study every day teaching at level C and B was designed around generating questions in mathematics (Jessen, 2015). In this context the questions were designed so it lead to development of praxeologies and mathematical organisations given in curriculum among the students. This means in terms of herbartian schema that students develop answers closer the teachers answer: $A^{\heartsuit} \approx A_{y}^{\diamond}$. These designs are called study and research activities (SRA). This teaching did not explicitly require collaboration with other teachers however an examiner at oral exam expressed serious concerns regarding the teaching. The teacher did not find the format of using open questions suitable for students not being fond of or gifted in mathematics. The exam was a success and the examiner would like to know more about the teaching. However this scepticism is a constraint with respect to implementation of a new paradigm for teaching.

Collaboration in bidisciplinary activities

The above mentioned bidisciplinary exams naturally requires collaboration of some kind between teachers representing different disciplines – and sometimes different faculties as the humanities and natural sciences. The two introduction courses to natural sciences and humanities also require a certain amount of collaboration or at least coordination of shared topic for each class to work with. The reform further require from teachers to "tone" their disciplines, meaning that it should be visible for students how elements of the content of each discipline can support the study of main disciplines or the other way around. Hence in mathematics classes in a language study line it is suggested to let students study topics of history of mathematics in original language or draw anthropological studies as examples when discussing descriptive statistics. Even though the reform heavily relies on successful collaboration between teachers there seem to lack knowledge on how to do this productively. As described by Winsløw (2012) most schools have committees arranging general study preparation (and the other bidisciplinary elements as well). Their work, focus on delegation of teaching tasks and responsibilities instead of actual collaboration. There is not much focus on content knowledge from the involved disciplines – or at least it is expected that the respective teachers plan and design the teaching individually (Winsløw, 2012, p 299). Likewise the bidisciplinary written reports are often planned individually in parallel, which is reflected in the questions handed out to the students and similar in the written reports handed in by the students (Hansen & Winsløw, 2011, p. 687).

Collaboration within mathematics

In general the collaboration among teachers of mathematics at upper secondary level in Denmark happens in informal contexts. In an evaluation of upper secondary mathematics in Denmark Jessen, Holm and Winsløw found that for 88% of the teachers the main forum for discussing the teaching of mathematics is the group of mathematics teachers at the school where they are employed – during lunch. 36 % read the magazine produced and distributed by the mathematics teacher association and 29% points to a closed Facebook group for teachers of mathematics at upper secondary level (Jessen et al. 2015, p. 63). Hence, situation is much like the one described in (Winsløw, 2012, p. 302), where a teacher is arguing that he did not apply for a job of collaboration but for teaching individually. In light of the quote it is positive that a relatively high number of teachers seek professional development and inspiration from colleagues. Nevertheless the report by Jessen, Holm and Winsløw states by quoting a teacher that the quality of what is being shared differs a lot. A teacher formulates it in an interview as: "it is free of charge to discuss Maple commands over lunch compared to discuss how to improve teaching" (A high school teacher in an interview asked about current situation for professional development in (Jessen et al, 2015, p. 63)). To some extend the classroom is perceived as private property – and no one but the teacher is to discuss or comment on what is going on in there. However two teachers in the report by Jessen, Holm and Winsløw mention joint preparation as a source for development and another teacher gets his inspiration from “two teacher arrangements” (in total the system employ more than 2000 mathematics teachers at upper secondary level). The latter is a lesson where one teacher plans the lesson and the other one participate in order to assist students while solving exercises or to guide them in project work. The second teacher does not take part in the planning of the lesson. Usually, no pre-didactic nor post-didactic system (in the sense of (Winsløw, 2012, p. 292)) are related to this kind of activity.

This means, that to some extend teachers are given “external infrastructures” by school management meaning time to meet, suggestions from committee to teachers on topic, materials and so forth. However the “internal infrastructures” are missing real collaboration in the sense that teachers still plan bidisciplinary didactic systems in parallel ending up with situations as described in (Winsløw, 2012, p. 301, figure 15.2). In this paper we will discuss this situation further applying the herbartian schema in order to be more explicit about the challenges in collaboration in terms of ATD.

In Denmark many schools facilitates professional development of different teacher groups. Close to half of the teachers answering the electronic survey in the report by Jessen, Holm and Winsløw have “mathematics teachers group meetings” for all mathematics teachers at their school. The meetings can be organised around workshops with topics as how to lower the rate of students failing mathematics B or how to improve students’ ability to write mathematical text? The workshops are often a sharing of best practices, sometimes combined with an invited speaker giving an introduction to the topic. Afterwards it is the individual teachers responsibility to implement the new knowledge in his classroom. This is another example of existence of some external infrastructures for professional development however it does not seem to affect teaching much, when the internal structures are missing.

Herbartian analysis of Danish collaboration

In this section we describe the collective dimension of teachers work in terms of herbartian schema meaning identifying what questions are raised, what answers are consulted and who brought them in and what works are shared?

As mentioned above, most teachers get inspiration for improving their teaching over lunches with colleagues. This means teacher y_1 raises an open question e.g. Q_0: How do you introduce differential calculus in your second year high school classes? The study of this question is supported by the group of mathematics teachers $Y = \{y_2, \dots, y_n\}$. The teacher y_1 plays the role of student x, but could also take part of the answer development, bringing in his or her own existing answer $A_1°$. But often the sharing will be the sharing of materials $\{O_1°, \dots, O_m°\}$ combined with some answers in terms of didactic praxeologies $\{A_1°, \dots, A_n°\}$. In this case the development of the didactic praxeology of introducing differential calculus can be described as this schema:

$$[S(y_1;Y;Q_0) \rightarrow \{A_1°, \dots, A_n°; O_1°, \dots, O_m°\}] \hookrightarrow A^{♥}$$

Teacher y_1’s answer to Q_0 presumably will be closely related to a preferred answer given by one of the teachers e.g.: $A^{♥} \approx A_i°_{y3}$. Teachers take over materials from others revise it a little according to their initial praxeologies but it is not decomposed and reconstructed in the sense of a rich study and research process even if the teacher experiment the new $A^{♥}$ in the classroom. This could be characterized as cooperation in the sense that teachers are not engaging in a study and research process improving and developing a new answer to a generating question they share as part of their professional work. If teachers joined together in such a process bringing in existing answers and works of others the schema would look like this:

$$[S(y_1, \dots, y_n; \emptyset; Q_0) \rightarrow \{A_1°, \dots, A_n°; O_1°, \dots, O_m°, Q_{m+1}, \dots Q_p\}] \hookrightarrow A^{♥}$$

In this the teachers will share the same answer to Q_0. But further a research process in this context would to some extend require the test of ideas meaning classroom interventions and observations or what was characterised as didactic observations systems (DoS). It is worth noticing that this is not part of the infrastructures offered by management. In the report by Jessen, Holm and Winsløw teachers mentions the magazine distributed by the mathematics teacher association as a mean to professional development. It adds to the milieu M of the study of Q_0 but it is up to each teacher whether it acts as a monument to visit or it is studied and incorporated in the teachers practice and development of new $A^\heartsuit$. The mentioned Facebook group mainly functions as a sharing of teaching material why this collective element is characterised as the first cooperation schema and equals lunch talks.

Looking at the planning of bidisciplinary works as study line projects, general study preparation or the general introduction to methods of natural sciences there is not much shared development of an answer to a Q_0. Take the example of general study preparation: teachers of two or three disciplines are told by school management to carry out one thematic week where the teachers find a case to study (e.g. global warming) and then they must cover some of the learning objectives for general study preparation. Examples of these objectives are "to write a synopsis", "to find and study suitable media at the library" or "be able to discuss what contributions each discipline can bring relative to the methods of the discipline" (Ministry of Education, 2013d). In these situation y_1, y_2 and y_3 often finds a common field to built a case for the students to study based on their $\{A_1^\heartsuit, \ldots, A_n^\heartsuit\}$. These answers, which the teachers have developed, are partly "darlings" related to their own study of the discipline as well as their "professional darlings", meaning their teaching praxeologies with respect certain disciplinary organisations. Each teacher offers to cover a certain part or angle to approach the case. From the perspective of the other teachers this means that, what is brought in is existing praxeologies, which they never decompose or reconstruct, hence they play the role of $A_i^\diamond$'s. Further, they present media for the students to study, meaning they offer some monuments for the other teachers to visit, $O_j^\diamond$'s. Teachers do not often find time to engage in a study and research process trying to do a thorough a priori analysis of the student activity, hence they agree upon a topic and do not cross disciplinary boundaries but hope students will be able to do this.

This means that modelling this teacher work a vague pre-didactic system can be described with the schema below:

$$[S(y_1,y_2,y_3;?;Q_0)\rightarrow\{A_1^\diamond, \ldots,A_n^\diamond;O_1^\diamond,\ldots,O_m^\diamond\}] \hookrightarrow A^\heartsuit$$

However the answer to Q_0, which the teachers share is more an equivalent to the milieu they are planning to offer the students rather than a newly developed teaching praxeology. Even though management form committees for planning the general study preparation they do not offer the setting of a real paradidactic system, meaning the current situation have some vague pre-didactic system, but no didactic observation system and no post-didactic system. This might not be a problem however students report on parallel structured teaching where there is no relation between what they are taught (EVA, 2015, pp. 42), but the critique does not affect the professional development of each teacher much or how the work is planned. This is mainly, because the teachers cannot change this situation themselves.

To sum up there do exist elements of teachers planning teaching or assisting each other in the planning of teaching at upper secondary level in Denmark. However the quality of these activities lack in richness of the media and milieu of the teachers study process of their didactic question and the presence of the raison d'être of the activities is not clear. Teachers are eager to share notes, experiences and teaching materials. And they do plan bidisciplinary teaching together but it is not clear if they actually collaborate, cooperate or simply coordinate and delegate the different lessons between them. And it seems that the activities do not offer the possibility of teachers discussing teacher practices and scopes of different approaches.

The report by Jessen, Holm and Winsløw shows that when mathematics teachers at upper secondary level are asked what kind of in-service teachers courses they would like to have 51% points to "stofdidaktik" or content didactics. This is course activities where teachers are taught how scholarly theory can become "teachable" theory. Courses offering this have been characterised as "capstone courses" and a presentation of such a course and the need of those are given by Winsløw and Grønbæk (2014). The report by Jessen, Holm and Winsløw further shows that 48 % of the teachers wish for courses in applied mathematics and 39 % answer courses in didactics of mathematics (Jessen et al., 2015, pp. 59). It is argued in the report that it is reasonable to assume that this means teachers actually request courses enabling them to reflect upon teaching and improve it. But it also shows that teachers feel a lack in their competences with respect to engage in a full blown paradidactic system as the one described in the Japanese case in (Winsløw, 2012).

Challenges in Introducing SRP in Danish context

Barquero, Bosch and Romo (2015) illustrates how SRP can be introduced to in-service teachers by letting them carry out a SRP-TE designed in order for them to discuss the raison d'être of the mathematical content to be taught but also to teach them useful notions from didactics of mathematics. Emphasis was put on how these notions could solve problems in relation to teachers practice rather than being notions a teacher ought to know (Barquero et al, 2015). However the teachers has a tendency of falling back on old habits planning the SRP for their own classes. Nevertheless the collective aspect seemed to affect some teachers to design activities with a more open character as intended (Barquero et al., pp. 6). From the recent report by Jessen, Holm and Winsløw there are reasons to believe that it will be equally difficult to change the teachers practice into the paradigm of questioning the world by engaging them in a single course activities. In the course activity presented by Barquero, Bosch and Romo there are paradidactic structures supporting the in-service teachers professional development. In the SRP-TE the stages 1-4 constitute the PrS (Q_0: How to teach…?, live a SRP, analysis of lived SRP and design of an SRP). Stage 5 implementing and posteriori analysis of a designed SRP covers the DoS, letting the living of the SRP be the DS. The posteriori analysis points in the direction of PoS, including revised lesson plans (Barquero et al., 2015, p.7). But it is unclear to what extend this is done as a collective work in the sense of the Japanese case presented in (Winsløw, 2012). What is observed by colleagues who in details know the lesson plan, compared to what the teacher registers during the lesson might be slightly different and lead to different reflections about the lesson afterwards.

It could be interesting to design activities for mathematics teachers at upper secondary level in Denmark drawing on the experiences with SRP-TE incorporating the notion of communities of practice to let teachers develop SRP's in collaboration (Gueudet & Trouche, 2012, p. 307), implement them and reflect upon them in groups in the same school as well as in minor scale across schools. Moreover to offer in-service teachers the full paradidactic system in order to develop sustainable professional development opposed to courses where ideas are never truly implemented. Materials, designs, ideas and experiences could afterwards be shared with the community of mathematics teachers at involved schools and further with the association of mathematics teachers at upper secondary level – to some extend in the sense of the of Sésamath (Gueedet & Trouche, 2012, p. 310). It seems crucial in this context still to emphasise the development of a shared "didactic of mathematics language" in order for the teachers to formulate challenges in their practice in precise terms and further to discuss and solve these challenges. Moreover it would be needed in terms of dissemination of designs and their teaching potentials or simply the idea of SRP and questioning the world.

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