PhD Thesis
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A Comparative Study of Danish and Indonesian Pre-service Teachers’ Knowledge of Rational Numbers

This thesis has been submitted to fulfilment of the requirements for the degree of Philosophy Doctor.

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Title and subtitle: A comparative study of Danish and Indonesian pre-service elementary teachers’ knowledge of rational numbers

Topic description: The thesis aims to develop a theoretical and methodological model to study and compare pre-service teachers’ mathematical and didactical knowledge of rational numbers. Five hypothetical teacher tasks (HTTs) are designed, and the praxeological reference models are developed from the HTTs based on the Anthropological Theory of the Didactic. Then, they are applied to investigate and compare Danish and Indonesian pre-service teachers’ individual and collective mathematical and didactical knowledge of rational numbers.

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Abstract

The arithmetic of rational numbers is not only a difficult topic for pupils to learn and for teachers to teach, but mastering this topic is also crucial for pupils to succeed in basic algebra and all of the more advanced topics which depend on algebra. It is a natural assumption that teachers’ knowledge of rational numbers is important to pupils’ success on this and other topics, and indeed much previous research focuses on investigating and comparing teachers’ individual knowledge using a large-scale study and written tests. Meanwhile, the present PhD thesis deals with the issue of studying and comparing teachers’ individual and collective mathematical and didactical knowledge of rational numbers. This study takes an approach from the Anthropological Theory of the Didactic (ATD), especially praxeological analysis, and an explicit model to study teachers’ individual and collective knowledge known as hypothetical teacher task (HTT). Five HTTs covering different aspects of rational numbers, representations, order structure, and arithmetic, were designed based on the didactic research literature on rational numbers. Each HTT provides an opportunity for teachers to show their individual mathematical praxeologies and to share and construct some potential didactical praxeologies during subsequent pair discussion. To enable a systematic and precise analysis of teachers’ mathematical and didactical knowledge, praxeological reference models in the sense of ATD are developed. The empirical case studies were conducted with 31 Danish PSTs and 32 Indonesian pre-service teachers. The studies in particular led to three overall results related to the theoretical and methodological choices, the comparison of Danish and Indonesian pre-service teachers’ praxeologies, and the new notion of praxeological change. As a first result, we argue that HTT could be a promising model to investigate pre-service teachers’ mathematical and didactical knowledge related to a specific teaching situation. Pupils’ possible misconceptions and challenges, as presented in each HTT, provide opportunities for the pre-service teachers to share not only their practical knowledge but also their theoretical explanations, justifications and beliefs. The second result, from the comparative study, is that the Danish pre-service teachers have more solid mathematical and didactical praxeologies for teaching rational numbers than the Indonesian pre-service teachers. The Danish pre-service teachers constructed their didactical praxeologies from pedagogical assumptions on teaching and they also draw on didactical theories developed and disseminated within Western societies. They propose didactical praxeologies based on real life situations and also based on explicit ideas about the mathematical construction involved. On the other hand, the Indonesian pre-service teachers draw their didactical praxeologies from an implicit yet consensual assumption that mathematical techniques should be conveyed through direct instruction and subsequent training in their use. The causes for this basic
idea of direct teaching of mathematics as a set of techniques can be further explained by institutional conditions and constraints. The third result is that praxeological change is an important factor for pre-service teachers to succeed in supporting pupils to understand more subtle properties of rational numbers, such as density. Through this notion, one could describe the change of practices and logos from natural to rational numbers, and investigate to what extent the praxeological change among teachers is needed to deal with pupils’ difficulties on rational numbers. Finally, some perspectives generated from this study are related to the complexity of teachers’ knowledge and teaching as a profession.
denne antagelse om direkte instruktion i matematik, som en samling af teknikker, kan forklares nærmere gennem institutionelle betingelser og begrænsninger. Det tredje resultat er at prakseologisk skift er en vigtig faktor i de lærerstuderendes forudsætninger for at understøtte elevernes forståelse af mere subtile egenskaber ved rationale tal, såsom tæthed. Ved hjælp af begrebet prakseologisk skift, kan man beskrive de forandringer i praksis og logos som knytter sig til bevægelsen fra naturlige til rationale tal, og undersøge i hvilket omfang dette skift er en forudsætning for lærernes håndtering af elevernes vanskeligheder med rationale tal. Endelig er nogle af de perspektiver, som dette studium rejser, knyttet til kompleksiteten af lærerviden og til undervisning som profession.
**List of papers**

This thesis contains the following papers:

Paper I: A framework for a comparative study of pre-service elementary teachers’ knowledge of rational numbers.
- Accepted in *Educação Matemática Pesquisa*
- Introduction to the theoretical and methodological framework to study and compare teachers’ mathematical and didactical knowledge based on the anthropological theory of the didactic (ATD), specifically *praxeology*.

Paper II: A praxeological analysis of pre-service elementary teachers’ knowledge of rational numbers
- Accepted modulo major revisions for *Recherches en didactique des mathématiques*
- Present detailed *praxeological reference models* of rational numbers and apply to the empirical study of Indonesian pre-service elementary teachers’ knowledge of rational numbers.

Paper III: Danish pre-service teachers’ mathematical and didactical knowledge of operations with rational numbers
- Submitted to *Nordic Studies in Mathematics Education*
- An empirical study of Danish pre-service teachers’ mathematical and didactical knowledge and their collaborative work on operations with rational numbers based on ATD, mathematical and didactical praxeologies.

Paper IV: Praxeological change and the density of rational numbers: The case of pre-service teachers in Denmark and Indonesia
- Manuscript to be submitted
- Introduction to the notion of praxeological change developed based on ATD as a framework to study pre-service teachers’ practices and theory, and applied here to support pupils’ learning the density of rational numbers.
Other papers


Abbreviations

ATD  The Anthropological Theory of the Didactic
BAN-PT  Indonesian abbreviation for the Indonesian National Accreditation Agency for Higher Education
CITAD V  French abbreviation for the fifth International Congress of the Anthropological Theory of the Didactic
CK  Content Knowledge
DE  Didactical Engineering
DO  Didactical Organisation(s)
ECTS  European Credit Transfer System
HTT  Hypothetical Teacher Task(s)
KEMDIKBUD  Indonesian abbreviation for the Ministry of Education and Culture of Republic of Indonesia
MKT  Mathematical Knowledge for Teaching
MO  Mathematical Organisation(s)
PCK  Pedagogical Content Knowledge
PRM  Praxeological Reference Model(s)
Prodi PGSD UR  Indonesian abbreviation for the Teacher Education for Elementary School Study Programme University of Riau
PsET  Pre-service Elementary Teacher
PST  Pre-service Teacher
RISTEKDIKTI  Indonesian abbreviation for the Ministry of Research, Technology, and Higher Education
RME  Realistic Mathematics Education
TDS  Theory of the Didactical Situation
TEDS-M  The Teacher Education and Development Study in Mathematics
1 Introduction

The present PhD thesis reflects the essential elements of my academic work over the last three years. It consists of four papers, the first paper has been accepted for publication in a special issue of *Educação Matemática Pesquissa* emerging from the fifth international congress of the anthropological theory of the didactic (CITAD V) in 2016, the second paper has been accepted modulo major revisions for *Recherches en Didactique des Mathématiques*, the third paper is under review, and the last one is about to be submitted to a journal. The order of the papers presented is based on the flow of my study, from designing a framework for a comparative study of teachers’ knowledge (paper I) to comparing Danish and Indonesian pre-service teachers’ (PSTs) knowledge of rational numbers, focusing on specific subjects such as density (paper IV). During my study, I have also written five other papers (paper V to IX) that I do not include in this thesis because paper V reports on a study involving in-service teachers, and the others focus on specific cases that can be regarded as preliminary versions of one of the main papers. The papers presented in this thesis are essentially in the same form as those submitted for the publications. I just adjusted the layout of the papers to fit the format of the thesis.

In the following sections, I will explain the basic motivation for research focusing on rational numbers and teacher knowledge. It is followed by presenting previous study in the area, as well as an explicit outline of the mathematical construction of rational numbers. Then, I present the theoretical and methodological framework, which was developed based on the anthropological theory of the didactic (ATD), and discuss how it is applied to analyse and compare Danish and Indonesian PSTs’ mathematical and didactical knowledge of rational numbers. To close this introduction, I outline the findings from my study, and discuss how the findings provide at least partial answers to the research questions of the whole project. I conclude with some perspectives for further research.

1.1 Motivation of this study

“Teaching and learning fractions has traditionally been one of the most problematic areas in primary school mathematics” (Charalambous & Pitta-Pantazi, 2007, pp.293).

I started my career as a mathematics teacher educator at the department of elementary school teacher education, University of Riau, Indonesia, in 2012. I taught several courses related to mathematics and mathematics education, and one of them was “Learning and teaching mathematics for pupils at the upper grades of elementary school” (grade 4 to 6). The mathematical
and didactical aspect of rational numbers was one of the central topics in that course, and it turned out to be very difficult and challenging for some students to learn. Similar difficulties can be found with in-service teachers. For instance, in an early study (paper V), I asked 50 in-service training teachers who participated in that course to pose real-world problems whose solutions involve multiplication of fractions. Only 22% and 64% of them gave appropriate problems involving, respectively, multiplication of a fraction by a whole number and multiplication of two fractions. These disappointing results led me to the hypothesis that teachers, in many other ways, could have an insufficient relation to the meaning of fractions, and this could also affect their mathematical and didactical knowledge of rational numbers in general. This became my motivation to conduct an in-depth study on teachers’ knowledge of rational numbers.

A certain level of discrepancy between theory and practice in learning and teaching rational numbers makes this topic so difficult to many pupils (Gabriel et al., 2013) and also teachers, as illustrated in particular by paper IV. The pupils are progressively introduced to “objects” such as simple fractions, decimals, percentages, etc., without much motivation or justification. Teachers simply expect pupils to accept these objects as natural parts of the growing universe of numbers they work with; but the mass of rules to remember becomes overwhelming and somewhat irrational, unless a systematic work is begun to build up a theoretical superstructure. Even if this must be informal in elementary school, it should at the same time be reasonably correct. This is the big dilemma, but if it is ignored, it leads to solid obstacles for the pupils to learn more advanced mathematics. As a first evidence of this contention, several empirical studies indicate that pupils’ knowledge of fractions and of division in elementary school is one of the strongest predictors of their overall mathematics achievements in high school (e.g. Siegler et al., 2012), even after statistically controlling for other types of mathematical knowledge (including basic addition, subtraction, and multiplication), general intellectual ability, working memory, family income and education. In fact, the implicit algebraic rules which are required to manage fractions and division are also required to solve many algebra tasks, such as equations, and pupils who failed to pick them up in the more concrete setting may get entirely lost in the more abstract setting of symbolic algebra. While they may try to memorize certain standard procedures, the mass of implicit rules ends up appearing as non-sense to them. This perspective is one of my main motivation to focus on rational numbers.

Pupils’ difficulties with rational numbers have been discussed by numerous scholars since the nineteenth century. One of the most interesting early contributions relies on the idea of teaching elementary mathematics from a “higher standpoint”, as proposed by the famous mathematician Felix Klein (1908, 2016). To Klein, this means letting our design of the elementary teaching be
guided by more advanced mathematics, as a source of insight and for conceiving new connections and paths in the curriculum. But unlike many present-day mathematicians who voice opinions on school mathematics, Klein does certainly not want to simply impose the academic approach in school. On the contrary, he often criticizes the loss of meaning which such “modern” teaching often leads to, as in the setting of fractions:

“The modern presentation is surely purer, but it is also less rich. For, of that which the traditional curriculum supplies as a unit, it gives really only one half: the abstract and logically complete introduction of certain arithmetic concepts, called “fractions”, and of operations with them. But it leaves undiscussed an entirely independent and no less important question: Can one really apply the theoretical doctrine so derived, to the concrete measurable quantities about us? Again one could call this a problem of “applied mathematics”, which admits an entirely independent treatment. To be sure, it is questionable whether such a separation would be desirable pedagogically” (Klein, 2016, pp. 32).

Klein criticised certain modern text books according to which “the fraction \( \frac{a}{b} \) is a symbol, a number-pair with one can operate according to certain rules” (Klein, 2016, pp. 31), but which fail to relate this abstraction to “concrete measurable quantities around us”, including most familiar settings in which pupils encounter fractions. In fact, connections between theory and practice, as well as between the various representations and interpretations which are used in different settings, seem to be major challenges in this area.

Many studies have identified multifaceted informal interpretations or meanings of fractions (part-whole, ratio, operator, quotient, and measure) that the pupils have to learn (e.g., Charalambous & Pitta-Pantazi, 2007; Kieren, 1993). Related to these, Charalambous & Pitta-Pantazi (2007) found that pupils were more successful on the tasks related to the part-whole meaning compared to the others, but this knowledge was not a sufficient condition to support them to deal with the operations of fractions. This indicates that they need to master a wider, coherent set of rules related to fractions to acquire proficiency with the different operations of fractions.

Indeed, some authors point out the dilemma that even a large inventory of informal meanings of fractions do not suffice to give teachers and pupils a complete and coherent picture of what a fraction or a rational number is (Wu, 2014). The metaphorical representation of fractions as parts of a pizza, is cherished by teachers and students a like, but it is not very helpful for fractions which are not between 0 and 1 (Wu, 2014); is does not make sense to multiply or divide two pieces of a pizza, and this metaphor gives no clue of how such operations can be useful to solve problems about speed or ratio. Wu argues that there is a need to also teach the fundamental definitions and
rules governing fractions, even if in a somewhat informal way; and that this should be thoroughly linked to the number line (measurement). This means that one should treat fractions as numbers with explicit properties, and that these have to be understood by pupils as reasonable, using a number line as a natural reference point for fractions.

The background for this proposal is that in many countries all over the world, including Indonesia, the teaching of fractions and decimal numbers focuses mostly on the “mechanism” of carrying out operations (e.g., Sembiring, Hadi & Dolk, 2008), illustrated only in very special cases by informal metaphors such as pizzas and rectangle diagrams. The teachers tend to dictate formulas and procedures to their pupils (Armanto, 2002; Fauzan, 2002; Hadi, 2002). For example, a teacher first presents a mathematical task, such as multiplication of two fractions, and then shows how to solve the task using a standard procedure. If many pupils could not follow that first example, the teacher gives another similar one and solves it with the same procedure. The teachers believe that the pupils can easily understand and apply this technique to solve other mathematical tasks (Armanto, 2002; Sembiring et al., 2008). The Indonesian government, on the other hand, has tried to reform the mathematics curriculum several times, in order to be consistent with certain international trends, such as the importance to support pupils in learning high order thinking skills (The Ministry of Education and Culture of Republic of Indonesia [Kemdikbud], 2014); but the curriculum does not really follow, especially when it comes to the content of elementary mathematics (Mailizar, Alafaleq & Fan, 2014). Moreover, as this thesis will confirm, there is a deep gap between such ideal goals for the mathematical knowledge to be learned by pupils, and the mathematical and didactical knowledge held by the teachers who currently are to support the pupils’ learning.

Many authors in fact point out that teachers’ mathematical and didactical knowledge is one of the main conditions for supporting pupils’ successful learning of mathematics, and particularly of rational numbers (e.g., Lortie-Forgues & Siegler, 2017). Various models to study teachers’ knowledge have been developed based on Shulman’s seminal work of content knowledge (CK) and pedagogical content knowledge (PCK) (Shulman, 1986). Recently, “Mathematical knowledge for teaching” (MKT) has become one of the most well-known models to study teachers’ CK and PCK (Ball, Thames, & Phelps, 2008), with research including quantitative large-scale studies of teachers’ MKT. While large scale studies may evacuate some of the individual variation which is often believed to characterize the teaching profession, teachers’ professional knowledge involves complex links between mathematical and didactical knowledge as well as collaborative forms of practice. Studying teachers’ individual performance on multiple choice items cannot give a complete picture of what teachers knows, can do and can explain in a community of practice.
Therefore, more elaborate epistemological models need to be developed to study teachers’ individual and collective mathematical and didactical knowledge and practice, if one wants to address their professional knowledge more deeply.

In this PhD thesis, I try to do so for the study teachers’ knowledge of rational numbers, using ATD (Chevallard, 1992; 2006). This theory proposes an epistemological modelling tool, the theoretical notion of so-called praxeologies, to study human practice and thinking. Durand-Guerrier, Winsløw and Yoshida (2010), and Winsløw and Durand-Guerrier (2007) have showed how it can be used in a small-scale comparative study of teachers’ mathematical and didactical knowledge of geometry and early algebra. They introduced a proposal, so-called hypothetical teacher task (HTT), in which teachers work in pairs to share their mathematical and didactical praxeologies to address some specific challenges in the teaching of a concrete topic. Using the idea of HTT, I designed a much more comprehensive set of tasks to study teachers’ mathematical and didactical knowledge of rational numbers (Paper I and II). Then, I present how the notion of praxeology becomes a useful epistemological tool to study, and also to compare, Indonesian and Danish PSTs’ individual and collective mathematical and didactical knowledge of rational numbers (Paper II, III, and IV).

1.2 Previous research on teachers’ knowledge of rational numbers

Many studies have been done to investigate teachers’ knowledge of rational numbers (e.g., Chinnappan & Forrester, 2014), including comparative studies (An, Kulm & Wu, 2004) and large-scale surveys (Van Steenbrugge, Lesage, Valcke & Desoete, 2014). Some studies focus on specific contents such as division of fractions (e.g., Ma, 1999) while other include a wider selection of topics.

To study and compare teachers’ knowledge of rational numbers, various frameworks have been developed, mostly based on Shulman’s ideas (1986) of CK and PCK. Depaepe et al., (2015) investigate PSTs’ CK and PCK, as well as the relationship between CK and PCK, and compared the CK and PCK of generalist PSTs for elementary schools and PSTs studying to teach specific subjects. They designed written tests to study CK and PCK independently. The CK items consist of “conceptual” tasks and “doing operations”. One example of a conceptual task consists in asking the PSTs to order a set of given decimal numbers. The PCK items can be roughly classified as “misconception tasks” and “instruction tasks”. The misconception tasks elicit teachers’ interpretation of given (and faulty) pupil answers, and the instruction tasks ask for ideas for teaching rational numbers in a given situation. Quantitative methods are developed to score and analyse the informants’ answers. The findings of this study show some gaps in PSTs’ CK and
PCK, while there is still an overall positive correlation between CK and PCK. More precisely, the study concludes that “CK is a necessary, but not sufficient condition for PCK” (Depaepe et al., 2015, pp. 91). As many similar studies, this study only focuses on PSTs’ individual performance on closed questions which can be used in a written test, and it does not provide a situation where PSTs are able to share their mathematical and didactical reasoning behind the given answers and to show a link between CK and PCK.

Among many comparative studies of teachers’ knowledge of rational numbers, there is a large bulk of studies which try to compare U.S and Chinese teachers (An et al., 2004; Lin, Becker, Ko & Byun, 2013; Ma, 1999; Zhou, Peverly & Xin, 2006). Most of these studies seem to be inspired by Ma’s study of teachers’ knowledge of division of fractions by extending the contents to other aspects of rational numbers. An et al., (2004), for instance, focus their study on comparing the U.S and Chinese teachers’ PCK on rational numbers. The results of this study confirm Ma’s conclusion that Chinese teachers have more sophisticated PCK than the U.S teachers. The Chinese teachers emphasise developing pupils’ procedural and conceptual knowledge, while the U.S teachers emphasise real-life activities and skills, but with a lack of connection between the use of concrete objects and representations on one hand, and on the other, the routine techniques whose justification they, themselves, do not seem to know. In term of the methodology, in addition to studying teachers’ knowledge through written answers, this study also conducted some interviews with selected teachers, to get more insight into individual teachers’ mathematical and didactical knowledge related to the given tasks. However, this study still focuses on teachers’ individual knowledge of rational numbers.

Also in the area of teachers’ knowledge of rational numbers, Johar, Patahuddin and Widjaja (2017) conducted a comparative study of PSTs’ teaching practice in Indonesia and the Netherlands. This study assessed one PST from each country during their teaching practicum with fourth graders, where they were using two fraction tasks designed based on contextual problems. One of the tasks is to find how many kg of beef one person will get if there are 15 kg of beef for 20 people. Then, the pupils were asked to work in small groups to solve that task. It turned out that the Indonesian PST firmly guided the pupils to represent the task using a diagram representation based on the part-whole relationship. The teacher mainly focused on how to divide 15 by 20 without further attention to the meaning behind the numbers. One could speculate that the Indonesian PST was only able to support pupils by explaining fractions as a part-whole relationship, and this meaning is only effective to explain a fraction in the interval [0,1]. By contrast the Dutch PSTs supported the pupils to engage with the situation by reminding the pupils the connection between kg and gram. This lead the pupils to solve the task in different ways, such
as based on ratio. For instance, one group considered that if 1500 gram was for 10 people, so 750 gram was for 20 people.

Summing up, previous studies offer a variety of approaches to study and compare individual teachers’ mathematical and didactical knowledge of rational numbers. The most common method is based on a written-test, but this method may not give us detail information about teachers’ mathematical and didactical thinking and reasoning. Thus we begin from the hypothesis that letting teachers share their mathematical and didactical knowledge will provide us with more insight into what they could and could not explain, both on the practical and theoretical level. Studying teachers’ teaching of rational numbers, such as in ordinary or design-based teaching, and interviewing them subsequently, could also provide a detailed view of their mathematical and didactical knowledge, but it is naturally a very time consuming approach, and also one where the contingency on local conditions (what to teach when, differences of schools and pupils) might conflict with the intention to focus on the teachers’ knowledge.

Considering this challenge, we looked for a model to comprehensively analyse teachers’ individual and collective mathematical and didactical knowledge both in theory and practices. Durrand-Guerrier et al., (2010) and Winsløw and Durrand-Guerrier (2007) proposed a model for a comparative study of teachers’ knowledge of teaching similarity and proportion, and the multiplication of two negative numbers, based on the ATD framework. Constructing HTT and letting teachers work individually and collaboratively to solve those tasks provide an opportunity to know teachers’ mathematical knowledge and how they use it to construct didactical techniques for teaching. In this study, we developed the methodology of HTT to study teachers’ individual and collective knowledge of rational numbers. The results of this study are expected to present more comprehensive teachers’ mathematical and didactical knowledge, both when it comes to concrete practice and to more theoretical positions and beliefs.

1.3 The mathematics of rational numbers at a glance

“God made the integers; all else is the work of man” (Leopold Kronecker (1886) in Weber, 1891/92)

This quote reflects how mathematicians in the late nineteenth century were at last arriving at a satisfactory and coherent theory of how to construct the set of rational numbers from the integers, then real numbers and complex numbers, all based on explicit axioms and (ultimately) set theory. In this section, we just outline the basic ideas for the construction of rational numbers from the integers, and link to elementary school mathematics. It becomes the basis for the theoretical model of praxeology.
If we have the equation $bx = a$ that is defined for any integer $a$ and $b$, with $b \neq 0$, we may not always find the solution of the equation in the set of integers. For instance, if $a = 2$ and $b = 3$, it does not give any answer to $x$. In fact, there is no multiplicative inverse in the set of integers, except for two numbers ($\pm 1$). This mathematically motivates extending the ring of integers. Informally we simply declare $\frac{a}{b}$ (“$a$ divided by $b$”) to be a number (for $b \neq 0$), while identifying symbols of this type in the usual way (reflecting properties of division well known from the integers). It is an **extension** of the integers since, if $b$ equals to 1, the fraction $\frac{a}{b}$ becomes $a$, meaning that every integer is a rational number.

Formally, however, the new numbers are defined as equivalence classes of pairs of integers $(a, b)$ with $b \neq 0$ (Eves, 1990). It goes as follows: first, we define a new relation $\equiv$ on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ by

$$(a, b) \equiv (c, d) \text{ if and only if } ad = bc.$$  

One then proves that $\equiv$ is an equivalence relation, and hence (from the general theory of such) it gives raise to a class division of $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. We denote the equivalence class containing $(a, b)$ as $[a, b]$, and call the corresponding quotient space $\mathbb{Q}$ (and its members, rational numbers). We finally prove that the following identities gives well defined operations on $\mathbb{Q}$:

$$[(a, b) + (c, d)] = [(ad + bc), bd]$$

$$[(a, b) \cdot (c, d)] = [ac, bd]$$

which moreover extend the usual operations on $\mathbb{Z}$ (or more precisely, the “copy” of $\mathbb{Z}$ contained in $\mathbb{Q}$, consisting of all numbers of form $[a, 1]$, with $a \in \mathbb{Z}$).

The set of rational numbers, together with the addition and multiplication operations, is in fact the so-called a quotient field of the integral domain $(\mathbb{Z}, +, \cdot)$. In particular, $\mathbb{Q}$ is a field. This means that $\mathbb{Q}$ possesses all the algebraic properties of $\mathbb{Z}$ and, in addition (and in contract to $\mathbb{Z}$) every element of $\mathbb{Q}$ has a multiplicative inverse. In fact, according to the definitions, $[a, b] \cdot [b, a] = [1, 1]$.

The set of rational numbers are countable, i.e. there is a bijection between this set and $\mathbb{Z}$, thanks to Cantor’s classical diagonal argument. We can extend the order of $\mathbb{Z}$ to $\mathbb{Q}$ by posing

$$[a, b] < [c, d] \text{ if and only if } ad < bc$$  

assuming $b, d > 0$ (we can always choose representatives with positive second coordinate). Again it requires proof that this gives rise to a well defined order relation, with the usual order on the copy of $\mathbb{Z}$ inside $\mathbb{Q}$. The bijection between $\mathbb{Z}$ and $\mathbb{Q}$ is not order preserving, and in fact, the density of $\mathbb{Q}$ shows that it is, also as an ordered space, quite different from $\mathbb{Z}$.
It is clear that this formal construction cannot be realised directly in elementary school – it has to be transposed. On the other hand, it is crucial that teachers understand that “fractions” is a subtle human construction, and that the new (rational) numbers are not simply the fractions, but that to each number there corresponds an infinity of (equivalent) fractions. From the “advanced standpoint”, outlined above, it becomes clearer not only why students struggle with fractions and rational numbers, but also what they, ultimately, struggle with. Gaining knowledge of rational numbers through partial, informal and sometimes erroneous explanation, does not make their struggle easier.

1.4 Theoretical Framework

In this section, the theoretical framework for this PhD thesis is presented. It is based on ATD, particularly the notion of praxeology. The theory serves to design reference models for rational numbers (Paper I and II) and then the models are used to study and compare Indonesian and Danish PSTs’ mathematical and didactical knowledge (Paper II, III, and IV). This study primarily focuses on PSTs’ mathematical and didactical praxeologies related to rational numbers, and these can of course be related to what appears in the more distant, academic-scholarly knowledge (outlined in the previous section), as well as in institutions closer to the primary school. Therefore, we begin to discuss the ATD from the didactic transposition process.

The theory of didactic transposition can be regarded as the corner stone of ATD (Bosch & Gascón, 2006; Chevallard, 1985; Chevallard & Bosch, 2014). The main point of this theory is to recognize that a body of knowledge learned by the pupils at school is originally produced in institutions outside the school, and that its trajectory towards the pupils is often long and opaque. The transposition involves several processes and adaptations, aiming to make the knowledge appropriate for the school environment. In figure 1, Bosch and Gascón (2006) provide a simplified model of the process of didactic transposition. The transformation of knowledge starts from the scholarly knowledge produced by mathematicians or other producers mostly in various societal institutions. In order to make it available for students in the learning institutions, the scholarly knowledge has to be transposed into what is called knowledge to be taught, and it is decided by people in the educational system or the noosphere, such as curriculum designers, textbook developers, government (often a ministry of education), politicians, etc. At this stage, the knowledge to be taught can be different in many respects from its origin and can be disappeared from the rationale (Bosch & Gascón, 2006). The mathematical knowledge to be taught materializes in curricula and textbooks, and then it is further transformed to taught knowledge by teachers within teaching institutions. One can say that the taught knowledge may be influenced by the
teachers’ professional knowledge that they get from teacher education programmes and subsequent professional experiences, and also the teachers’ past experience as pupils; it can be further influenced by local traditions, interaction with colleagues and so on. In the end, the learned knowledge is considered as knowledge acquired by the pupils, and it can be seen as the end of the didactic process. The double sided arrows reflect that there is also a retroaction, for instance, the teachers’ observation of learned knowledge may lead to modifications in the taught knowledge, etc. In some respects, the model is (as any model) a simplification of reality, in particular it does explicitate the often long history of didactic transposition – each of the boxes has its own intricate history and internal development, besides interacting with the others.

**Figure 1:** The didactic transposition process in mathematics education (Bosch & Gascón, 2006; Chevallard & Bosch, 2014)

The radical extension of the domain of didactics, which the ATD implies, is to consider as its object to the entire transpository process, from scholarly knowledge to learned knowledge, taking into account how the object of knowledge is subjected to different life conditions—ecologies—within the various institutions involved in the transposition process (Chevallard & Bosch, 2014). Take, for example, the mathematical object of knowledge investigated in this thesis: rational numbers. This knowledge has to be informally introduced to pupils since in the early stage of their elementary school, teachers and curriculum planners struggle—in different institutions, and with different resources— to improve the often poor results in terms of learned knowledge. These (partially independent) processes are certainly not always strongly coordinated with (or even synchronic with) scholarly knowledge (Figure 1). In the taught or learned knowledge, it really depends on several aspects including pupils’ conditions of learning and cognitive faculties, still in what some psychologists term the concrete stage. A major approach is then to embed the concept of rational number in various more or less artificial “real-life situations” and emphasis the meaning of a fraction as a part of a whole (confirmed by paper III). Meanwhile, the knowledge to be taught also depends on guidelines in the curricula, requirements from national tests, the textbooks available, and also—if not primarily—on the scope of teachers’ knowledge. In addition, it may also be influenced by various general factors such as social, cultural, pedagogical, and psychological aspects.

The dialectic between bodies of knowledge appearing in the producing institutions and in the school practices needs to be explicitly described by the researchers in order to understand how
different kinds of knowledge, beliefs and interests affect the process of didactic transposition. To
do so, Chevallard (2006) introduced the notion of praxeology as an epistemological model to
describe human knowledge and practices. The praxeology accommodates both human doing and
thinking, and it is a basic unit in which one can analyse human action into two interrelated
components of *praxis*, i.e. the practical block, on the other hand, of *logos*, the knowledge block.

The praxis or practical block consists of a *type of task* (T) to be investigated or studied, and
*techniques* (τ) used to solve the task. An example of a type of task related to rational numbers is
to add two rational numbers. In a university level in which the knowledge to be taught is produced,
the task proposed to be studied by mathematicians could involve a general definition, the question
of a set being closed under addition, well-definedness, commutativity etc. In contrast, the task
given to elementary school pupils calls only for a technique to solve the given task, sometimes
embedded into a real-life situation which requires substantial work to identify the operation. The
technique depends heavily on the representation of rational numbers in the task, whether presented
as fractions, decimals, or percentages. For instance the mathematical task $\frac{1}{2} + \frac{1}{4} = \cdots$ may be
transformed, by a teacher, to a task such as: “a student has $\frac{1}{2}$ kg of grapes, and then he buys another
$\frac{1}{4}$ kg of grapes, how many kg grapes does he has then?” The didactical task for the teachers or PSTs
can be much more comprehensive because it has to take into account both knowledge to be taught,
students’ previously learned knowledge, and also both mathematical and didactical knowledge of
the teacher himself (for instance see paper III discussing about addition of fractions). Eventually,
for the task of adding two fractions, some possible techniques can be employed by the pupils; in
the case of fractions, the standard technique is to change both fractions into a common denominator
and add the numerators.

The second part of the praxeology is the *logos* or knowledge block that functions to justify
and explain human practices or actions. The knowledge block is composed of a *technology* (Θ) to
explain the techniques and a *theory* (Θ) to justify and otherwise unify several technologies. The
two components may not be always easy to be observed especially in elementary school, because
there is limited space given by the teachers to let pupils develop these components. Barbé, Bosch,
Espinoza, and Gascón (2005) studied and analysed mathematical and didactical practices related
to the teaching of pointwise limits of functions in a Spanish high school context, and they found
that it was not possible to achieve coherent technological discourses during the classroom
interactions; the reasons for such shortcomings may not only lie in insufficient teacher knowledge,
but also in incoherencies with in the knowledge to be taught. In this case, it appeared that the
curriculum emphasized mostly the practical block as algebra of limits along with some elements
of scholarly knowledge of topology of limits emphasized on the theoretical block (like a “definition” of continuity). In the case of learning and teaching addition of fractions, the technology to explain the standard procedure could be based on informal mathematics properties, such as the convenience of changing both fractions to a uniform “unit” before adding them. Although this technology is rather informal, it can be important for the teachers to have it in order to support the pupils to both relate, and distinguish, the procedures for adding two natural numbers and two fractions. The general arithmetic properties (e.g. associative law) of rational numbers belong to the formal theory, and one can say that it is an important part of scholarly knowledge that may not be taught directly to the pupils at school, while teachers and PSTs should have this knowledge to identify and work with the difficulties that the pupils have. Otherwise, the teachers may not know what to do when they find pupils’ misconceptions on this area. Even when teachers do master the technique itself, as in the case for addition, more involved tasks may also find them short on techniques. For instance, some Indonesian PSTs could not explain why 0.5 is greater than 0.45, because they did not have the appropriate technique to solve the task and almost no technology and also theory to construct, let alone justify, the technique (Paper VII).

In most case, a praxeology may not stand alone but it is interrelated and organised in collections. An isolated praxeology which contains just one type of task and techniques is called punctual organisation, but in many cases several techniques are shared a common technology, and this then gives rise to a local organisation. The collections of praxeologies sharing a common theory are called regional organisation. An example of a regional organisation on the case of teaching and learning rational numbers is the arithmetic of rational numbers. While, global organisations refer to the collections of praxeologies constituting a common mathematical domain, such as rational numbers. Some studies may focus on one or more punctual organisations, while others look at more global organisations. In this PhD thesis, only regional organisations (paper II) are covered, and some papers just focus on punctual and local organisations (paper III and IV).

In the study of teachers’ knowledge and also teaching practices, one does not only deal with mathematical praxeologies or organisations (MO) but also with didactical organisations (DO) (Durand-Guerrier et al., 2010; Rodríguez, Bosch & Gascón, 2008). A DO consists of a type of didactical task, didactical techniques, didactical technologies, and didactical theories. An example of a type of didactical task related to rational numbers is to teach pupils to add two fractions. Didactical techniques suggested by teachers can be varied, and they may be much more influenced by teachers’ MO gained from the teacher institutions and teaching experiences. For instance, the Indonesian PSTs tend to directly instruct pupils to master the standard mathematical
technique, and they believe that to foster pupils’ learning of mathematics, the pupils need to mimic the mathematical technique demonstrated by the teachers, to large collections of similar task (mostly discussed in paper II). The principles and virtues of direct instruction can be seen as the didactical theory to justify the technology explaining how to organise didactical processes involving demonstration and drill.

The notion of praxeology proposed by ATD has been widely used by researchers to analyse processes of learning and teaching mathematics. In the context of high school mathematics, Barbé et al., (2005) developed reference models for the case of limits of functions. They show how such reference models become useful tools for the analysis of mathematical and teaching practices. They modelled MO into algebra of limits, as it appears in the Spanish high school curriculum, and the topology of limits. Meanwhile, Wijayanti and Winsløw (2017) developed praxeological reference models (PRM) for various aspects of proportional reasoning. They described possible types of task and techniques that could appear in Indonesian school mathematics textbooks, and then used the references to analyse and compare a number of different textbook series. Drawing on this fundamental idea, we develop PRM on rational numbers for elementary school that are used to analyse PSTs’ individual and collective mathematical and didactical praxeologies. In paper I and paper II we have presented the detailed analysis of types of task and techniques from the HTTs and discuss technological and theoretical discourse for some HTTs.

Chevallard (2006, pp. 23) argues that “human praxeologies are open to change, adaptation, and improvement”. With this in mind, we argue that pupils need to fundamentally change their mathematical praxeologies when passing from natural to rational numbers. Some properties of natural numbers can not be directly transferred to tasks involving rational numbers. For instance, the counting technique to find how many natural numbers there are between two natural numbers needs to be replaced when dealing with rational numbers, due to the density property of the ordered field of rational numbers. Vamvakoussi and Vosniadou (2004) posed some questions to high school students, regarding how many numbers there are between two fractions or between two decimals, e.g., between $\frac{2}{5}$ and $\frac{4}{7}$. Many students fail to give the answer that there are infinitely many number in between the two fractions because they try to extrapolate their praxeology from natural to rational numbers, rather than realizing the need for complete change.

In paper IV, we introduce more generally the didactical idea of praxeological change and apply it to study and compare Indonesian and Danish PSTs’ mathematical and didactical knowledge of the density of rational numbers. Praxeological change is defined as an institutional deconstruction and reconstruction a set of new praxeologies in order to cope with an extended set
of objects or tasks, when the old praxeologies only apply to a part of these and cannot be extended or generalised to apply to them all. The idea of praxeological change can be used to study how pupils and teacher develop their praxeologies in situations where a smooth extension is not possible. The change must affect not only the practical block but also the knowledge block. There are also subtle connections between the change of praxis and logos. For instance, the praxeologies of natural numbers come from the activity of children such as counting objects, while the praxeologies of rational numbers are developed based on several interpretations and meanings of rational numbers such as presenting as a fraction with the meaning of a part and whole relationship. We elaborate more aspects of praxeological change in paper IV.

The ATD also provides a viewpoint to do so-called comparative studies of didactic praxeologies, through levels of didactic codetermination (Artigue & Winsløw, 2010; Chevallard, 2002). The levels consist of nine categories from comparing a minimum unit of human activity, a point praxeology, to a more global unit of civilisation, e.g., Culture (Figure 2). A comparative study can occur at different levels of the didactic codetermination. Some studies may focus on the above levels, such as a comparative study of teaching content in teacher education programmes (e.g., Rasmussen & Bayer, 2014), while others are interested in comparing how specific subject are taught, such as teaching and learning of a small collection of mathematics praxeologies (Paper IV).

![Figure 2](image)

**Figure 2**: Possible levels of comparison of the didactic codetermination (Artigue & Winsløw, 2010).

Durand-Guerrier et al., (2010) have modelled a comparative study of teachers’ knowledge in Denmark, France, and Japan. The study was conducted on the three lower levels (subject, theme, and sector) related to praxeologies. Using two HTTs, they then show how certain systematic differences of teacher education systems among those countries can explain some differences in PSTs’ praxeologies. Using a similar idea, paper IV presents a comparative study of Danish and Indonesian PSTs’ mathematical and didactical knowledge related to the density of rational
numbers. Supported by the other two papers, paper II and paper III, one can proceed to similar comparative analyses of PSTs’ praxeologies on working on the other HTTs.

1.5 The context of the study

The PhD project was conducted in two different teacher education contexts, Denmark and Indonesia. In paper IV, we have discussed the similar and different characteristics of the teacher education programmes, and we try to further elaborate them on this section.

In Denmark, teacher education is decentralised, while to some extent based on national legislation (Rasmussen & Bayer, 2014). The teacher education is known as a professional bachelor’s programme meaning that the programme strongly focuses on professional practices supported by theoretical knowledge and its application (The ministry of higher education and science, 2015). There are two types of teacher education programmes, namely a programme for teaching elementary schools (to teach grade 1-6) and a programme for teaching lower secondary schools (to teach grade 4-10), and these programmes are conducted by seven University Colleges. Each programme is a four-year professional bachelor degree programme in which PSTs have to complete 240 points in terms of the European Credit Transfer System (ECTS). Each teacher education programme consists of four components: 1) Courses relating to the teacher’s basic competence such as pupil learning and development (60-80 ECTS), 2) Teaching subjects such as mathematics (120-140 ECTS), 3 Teaching practice (30 ECTS), and 4) Bachelor project (10-20 ECTS) (Ministry of Higher Education and Science, 2015).

In the Danish teacher education programme, PSTs can select two or three teaching subjects, and they can select one between two mathematics subjects: mathematics for grade 1-6 or mathematics for grade 4-10. There are four courses for each subject, and each course is 10 ECTS. For instance, mathematics for grade 1-6 consists of 1) Learning mathematics, numbers, and processes in arithmetic, 2) Teaching mathematics and geometry, 3) Assessment and stochastics, and 4) Pupils with special needs and mathematical tools (Metropolitan University College, 2018). The significant differences with the courses for mathematics for grade 4-10 concern the contents; for instance, in the latter, the first course includes some elements of school algebra. In relation to the topic of this PhD project, the arithmetic of rational numbers is part of the first course, and it is taught in the first year of the study programme. In addition, PSTs also get some instructions on how fractions or rational numbers can be taught based on two didactical theories, realistic mathematics education and theory of didactical situation (Hansen, Skott & Jess, 2007)
In Indonesia, the teacher education system is also decentralised. In 2015, there are more than 421 public and private teacher education institutions, that offer more than 3000 education programme (The ministry of research, technology, and higher education [Ristekdikti], 2016). In relation to this PhD project, we found that there are 267 teacher education programmes for elementary school (to teach grade 1-6), and 268 mathematics teacher education programmes for lower, upper, and vocational school (The Indonesian national accreditation agency for higher education [BAN-PT ], 2017). Especially in the Riau province, in which this study was conducted, there are 5 elementary and 6 mathematics teacher education programmes. The teacher education programme is also a four-year professional bachelor degree programme (144 credit points in the Indonesian sense), but the subjects to be taught is decided locally, by each study programme. The government only provides a general guideline for all study programme that is called the national qualification framework of Indonesia (Ristekdikti, 2015). So, it just gives general information about what qualification should be obtained in the bachelor programme. For instance, one of the learning outcomes is to have students master the scientific basis and skills in their area, so they can find, understand, explain, and formulate a solution for a given problem. Every study programme has to develop its own curriculum that can vary considerably from one programme to another.

Due to this variance, we only describe the details for the context in which the data were collected, namely the Elementary School Teacher Education programme at the University of Riau. The teacher education programme consists of five main components: 1) General courses such as religion (18 credits), 2) Teacher basic competence courses such as pupil development (10 credits), 3) Courses related to teaching practices and skills (15 credits), 4) Development of education courses including the Bachelor project (12 credits), and 5) Elementary teacher education professional courses (89 credits). The courses related to mathematics and mathematics education (18 credits) mostly belong to the fifth category, namely 1) Fundamental mathematics for elementary schools I, 2) Fundamental mathematics for elementary school II, 3) Mathematics education for the lower grades of elementary school, 4) Mathematics Education for the upper grades of elementary school, 5) Statistics for education, 6) Capita Selecta mathematics (e.g. problem solving and modelling in mathematics) and 7) Realistic mathematics education in Indonesian context (Teacher education for elementary school study programme, University of Riau [Prodi PGSD UR], 2017). Here, PSTs have to select just one of the last two courses. The arithmetic of rational numbers is part of the courses 1) and 4), and these are taught in the first and second year of the study programme respectively. The contents of the courses are entirely focused
on mathematics for elementary school, with topics such as introducing rational numbers through fractions and decimal representations.

1.6 Research Questions

In this section, we present an overview of the main objectives of this PhD study. As mentioned in paper I, the aim of this study is to develop a framework within ATD and then apply it to study and compare Danish and Indonesian PSTs’ mathematical and didactical knowledge of rational numbers. The results of the study are expected to give detailed insight into the extent and qualities of PSTs’ mathematical and didactical knowledge, and how they might use it to support pupils’ learning rational numbers. In connection with these objectives, four research questions for this PhD study have been mentioned in paper I. The last research question related to the levels of didactic codetermination has been adjusted during this project, by focusing more closely on the comparative analysis of PSTs’ mathematical and didactical knowledge at the levels related to praxeologies. In particular, we chose to focus on comparing PSTs’ mathematical and didactical praxeologies deployed while dealing with the HTT about the density of rational numbers (Paper IV), but we could have performed a similar comparative analysis based on paper II and paper III. In addition, the notion of praxeological change is introduced in paper IV, and it becomes a main concern of the last research question. Thus, the four research questions of this study are stated as follows:

RQ1. How can the theoretical and methodological framework of ATD be applied and elaborated to study PSTs’ mathematical and didactical knowledge of rational numbers?

RQ2. How could it be used to study and compare PSTs’ individual and collective mathematical and didactical knowledge of rational numbers?

RQ3. What similarities and differences can be identified between Danish and Indonesian PSTs’ mathematical and didactical knowledge of rational numbers?

RQ4. To what extent is praxeological change needed to address specific challenges in teaching rational numbers to school mathematics, such as questions related to density?

These four research questions have been investigated in paper I to IV that are included in this thesis, but it does not mean that one paper addresses just one research question. In relation to RQ1, paper I presents a general overview of the theoretical and methodological framework to study and compare teachers’ mathematical and didactical knowledge. This is further elaborated by paper II through giving more detailed PRM corresponding to each HTT. Then, the HTTs are applied to study Indonesian PSTs’ mathematical and didactical knowledge of rational numbers. Indeed, paper II provides an answer to RQ1 and also a partial answer to RQ2 and RQ3. Paper III,
which focuses on the Danish case, presents how Danish PSTs share their mathematical knowledge to construct didactical knowledge to teach four operations of rational numbers. This also gives an answer to RQ2 and a partial answer to RQ3. Meanwhile, paper IV presents a comparative study of PSTs’ knowledge from the HTT about the density of rational number and exhibits the detailed praxeological changes needed to support pupils’ learning the density properties of rational numbers. So, this paper addresses RQ3 and RQ4.

1.7 Methodology

This PhD project is based on a qualitative research methodology developed within ATD (Chevallard, 2006). In qualitative approaches, one can explore a problem and develop an understanding of a phenomenon (Creswell, 2012), and capture the dynamic nature of human interactions: to seek trends and patterns over time (Cohen, Manion, & Morrison, 2007). Cohen et al., (2007, pp 461) argues that a qualitative study depends much on interpretation, so there is no unique “correct” way to analyse and present the qualitative data. We need to have some assumptions or even a theory to avoid inconsistent and vague interpretations of the data. Therefore, the ATD plays an important role in modelling mathematical activity to prevent us from studying the phenomenon without any explicit epistemological viewpoint (cf. Rodriguez et al., 2008). The notion of praxeologies supplies an epistemological model to study and analyse teachers’ mathematical and didactical knowledge of rational numbers.

Durrand-Guerrier et al., (2010) and Winsløw and Durrand-Guerrier (2007) have developed a model for mathematical teachers’ knowledge based on ATD together with the methodological discussion how to evaluate such knowledge in practice. They introduce HTT as a method to study teachers’ individual and collective mathematical and didactical knowledge. The HTT is constructed based on what could reasonably appear in school mathematics in most countries. The contents of the mathematical task of each HTT are standard and elementary. For instance, the task such as adding two fractions is a common mathematical task for the upper grade elementary school pupils in both Denmark and Indonesia, and teachers should have sufficient mathematical praxeologies to handle it. Later in the HTT, they use their mathematical knowledge to deal with a didactical task. Following this approach, we design five HTTs about rational numbers that cover some aspects of rational numbers, e.g., representation, order structures and arithmetic of rational numbers. The setting of HTTs, in which PSTs have to share their mathematical and didactical praxeologies spontaneously when dealing with didactical situations, could present rich information about what mathematical and didactical praxeologies they have, and what didactical obstacles would hinder them to succeed in teaching rational numbers.
On Figure 3, we illustrate how the whole process of this PhD project was carried out. It consists of six phases: 1) Preliminary analysis; 2) Designing HTTs and a priori analysis of them, 3) Pilot study, 4) Elaborating detailed PRM, 5) Implementation and data collection and 6) A posteriori analysis. These phases are inspired by the tradition of didactical engineering (DE) (Barquero & Bosch, 2015), but the process in each phase and the aim of the study, indeed, are clearly different from classical DE studies. For instance, the a priori analysis on DE within the ATD is to model an a priori mathematical and didactical design for a classroom mathematics teaching activity. It includes possible paths that may appear from an open problem given to students. Meanwhile, in our study the a priori analysis presents an interpretation of types of task from HTTs given to PSTs, anticipating informant’s practice and theory. That is, we explicitly model possible mathematical and didactical praxeologies that may appear during PSTs’ individual and collaborative work.

**Figure 3:** The phases of didactical research on studying and comparing teachers’ mathematical and didactical knowledge

In the preliminary analysis, the didactical research literature about representations, order structure and arithmetic operations of rational numbers was explored, including research on teaching and learning rational numbers in elementary schools and also research on teachers’ knowledge of rational numbers. The aim of this activity was first to capture what difficulties or obstacles have been investigated by many researches in this area, how the previous studies model
these phenomena (even if informally), and what methods, claims and hypothesised result from analysing the data collected. For instance, the didactic research on rational numbers conducted by Brousseau (1997) becomes one of the main resources in this process. We conducted a detailed review of Brousseau’s studies in order to decide what areas should be investigated for this study, and further details of these preliminary studies have been presented and discussed in paper II. In order to take the contexts of this study into account, we also studied national curriculum and elementary school textbooks and textbooks for teacher training from the two countries. In addition, some international mathematics textbooks for elementary teachers (e.g., Billstein, Libeskind & Lott, 2007) were studied in order to have a broader picture on how rational numbers may be presented and treated with PSTs at this level. At the end, we develop a global model of the didactics of rational numbers that covers the main areas from working with multiple representations, over the order structure, to the four operations. These all appear in the design of the HTTs.

In the second phase, five HTTs on rational numbers are constructed based on our investigation on the didactic research on rational numbers (Paper I and paper II). The HTTs are designed with explicit reference to the literature and based on teaching situations that could appear at elementary schools in both countries, while also reflecting the areas that PSTs encounter in the teacher education programmes. To do so, we first investigate elementary school mathematics textbooks from both countries, identifying the commonalities of the subject to be taught to the pupils, and then study the contents of mathematics and didactics from mathematics courses given to PSTs. We found that the arithmetic of rational numbers belongs to the course of “mathematics education for the upper grades of elementary school” in Indonesia and “Learning mathematics, numbers, and processes in arithmetic” in Denmark. Each HTT is expected to lead teachers to employ their mathematical and didactical knowledge to react appropriately to the given tasks. A mathematical problem is a standard or more complex mathematical task. Also, the mathematical problems retained are based on what may appear in the elementary school mathematics textbooks. The didactical problem asks the informants to handle certain hypothetical difficulties for pupils, as they try to solve the mathematical problem (presented first). So, the mathematical and didactical task are closely related to each other. Each HTT is a non-contextual problem since we expect teachers to propose different didactical ideas to teach pupils, including reconstructing the tasks into contextual situations, when they discuss prospective didactical praxeologies. Initially, a collection of ten HTTs was designed, including an HTT about ratio and proportion. Then we discussed those HTTs with Danish and Indonesian mathematics education researchers and teacher educators, and we decided to choose five HTTs that could represent the main topics of rational numbers learned and taught at elementary school in both countries. As an example, the HTT about
ratio and proportion was excluded from our study because this topic was beyond the elementary school subjects, and it needs to be investigated separately (see Wijayanti, 2017).

After constructing five HTTs, the *a priori* analysis was conducted with every task in each HTT. The main concern of this analysis was to describe the detailed mathematical and didactical techniques which PSTs could be expected to deploy on the tasks, but technological and theoretical discourses are also discussed and presented. We call these *local PRM*, and the local PRM for one HTT has been presented in paper I.

The five HTTs were then pilot tested with eleven Danish PSTs from one teacher college in Copenhagen, Denmark. They were first-year students who were taking a course on numbers and algebra for primary and middle school, and arithmetic of rational numbers is one of the main areas in that course. They had already been presented with this material when the pilot study was conducted. Thus, they may reflect on what they experienced in the institution when they solve the mathematical and didactical task in each HTT. The PSTs worked in pairs, except for one group consisting of three PSTs. This pilot study was conducted in January 2016, and the detailed information and the result from two HTTs of this pilot study have been presented in paper III. In the Indonesian case, a pilot study was also conducted with a pair of freshly graduated students from the teacher education for an elementary school study programme. These trials also aimed to explore the readability of the tasks (some details of formulation were subsequently improved) and to confirm the feasibility of using the HTTs in a larger study.

In the fourth phase, the local PRM was developed based on the data from the pilot study. The main concern of this phase was to present more precise mathematical and didactical techniques and to develop the model of didactical technologies or theories. The detailed local PRM has been presented in paper II.

The main data collection took place in one teacher education institution in Indonesia, and in three teacher colleges in Denmark. Actually, we have tried to manage the data collection from one institution in Denmark in order to be comparable to the Indonesian case, but the small number of students and complicated practical circumstances, such as difficulties to make appointments with the students and to find suitable time to conduct the study with them, etc., forced us to accept the institutional diversity of informants from the Danish case. However, every Danish institution follows the same programme structure, fixed by the earlier mentioned rules from the Ministry of Higher Education and Science (2015). In total, we collected data from twenty Danish PSTs, and all of them worked in pairs. They had all completed the course “Learning mathematics, numbers, and processes in arithmetic”, in which arithmetic of rational numbers and its teaching are taught. On the other hand, a large number of students in one teacher institution in Indonesia, and their
readiness to participate in this study, made it easier to collect the data for the Indonesian case. In total, 32 fourth-year Indonesian PSTs (16 pairs) participated, and all of them were preparing to become elementary school teachers. They had all completed all courses on mathematics and mathematics education. They learn arithmetic of rational numbers and its teaching in the sense of fractions and decimals in the two mathematics courses: “Fundamental mathematics for elementary school I” and “Learning mathematics for the upper grades of elementary school”.

In the last phase, the results were analysed based on the local PRM. The detailed analysis for the Indonesian case is presented in paper II. This paper addresses both theoretical and methodological issues related to the use of HTT to study teachers’ mathematical and didactical knowledge of rational numbers. A more detailed a posteriori analysis is carried out in paper IV, focusing on one specific HTT. The comparative study of teachers’ knowledge is the main point of the analysis, together with the praxeological change needed to succeed with the density task and to support pupils’ doing so as well.

1.8 Outline of Results

This section aims to present the outline results from paper I through IV. We discuss them in relation to the research questions. First, the discussion focuses on answering RQ1 and partly RQ2 about the ATD as the theoretical and methodological underpinning to study teachers’ knowledge of rational numbers, and how it works out with HTT as a concrete model for investigating teachers’ collective knowledge. Then, it is continued by outlining the results from Danish and Indonesian PSTs’ knowledge of rational numbers to answer partly RQ2 and RQ3. Finally, we discuss and answer RQ4 on how PSTs need a change of their praxeologies from natural to rational numbers to succeed on proposing didactical praxeologies of rational numbers.

1.8.1 PRM and HTT as a framework to study teachers’ mathematical and didactical knowledge

Paper I and paper II are dealing with the theoretical and methodological issue of this study. In paper I, we just give a preliminary design of the HTT and PRM, and then it is further elaborated in paper II.

The main point of paper I is to give a brief overview of the whole study from discussing the levels of didactic codetermination, and how they are used to a comparative study of teachers’ knowledge. Artigue and Winsløw (2010) have suggested that the study can be used through horizontal or vertical comparison. Our study focuses on horizontal comparison at the levels concerning mathematical and didactical praxeologies which are directly related to data on teachers’ mathematical and didactical knowledge.
Following a model of the comparative study proposed by Durand-Guerrier et al., (2010), we develop five HTTs related to rational numbers. As mentioned in the methodological section, the HTT is based on teaching and learning situations that might be experienced by teachers during their teaching of mathematics at elementary school. Through studying MOs of rational numbers from punctual to global organisations and also investigating some previous research on rational numbers, such as Brousseau (1997) and the Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2008), we designed five HTTs covered the area of rational numbers as presented in figure 4. For instance, HTT 5 about multiplication and division with decimals is adapted from a TEDS-M item (Tatto et al., 2008), and it is related to the knowledge to be taught or learned knowledge for pupils in both countries in the last year of their elementary school programmes (grade 6).

![Diagram](image)

**Figure 4:** Main areas of rational numbers for designing teachers’ tasks presented in paper II

The diagram shows possible relations among MOs of rational numbers. At the top we have praxeologies directly related to the use and change of representation of rational numbers, which play an important role for all other MOs. For instance, common techniques for execution of arithmetic operations depend entirely on the representation, and it is sometimes convenient to change it. For instance, to calculate $0.45/0.2$, one may find it expedient to change to fractions.

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1 An early version of this model of mathematical organisations of rational numbers is presented in paper I, and it was further developed in paper II
Meanwhile, there are also some connections among local and regional organisations, related to sharing a common technology or theory. For instance, the theory of equivalent representations is used in the other four local MOs, such as to justify the technology above concerning the division of decimals.

An a priori analysis was done for each HTT. In paper I, we describe mathematical and didactical praxeologies which could be generated by HTT 1 about equivalent fractions. Paper II presents a detailed PRM for HTT 5 about multiplication and division with decimals and give an outline of PRM for the other HTTs in the appendix.

In the first step, the reference model is aimed to describe types of mathematical task from each HTT. Some HTTs involve more than one type of mathematical task. For instance, HTT 5 consists of two types of mathematical task: multiplication and division problems (Figure 5). Several possible mathematical techniques can be used to solve each type of task, and we used various resources to identify those techniques, such as mathematics textbooks for elementary schools, teacher education textbooks, and research literature. It is because there is no universal reference model for the knowledge to be studied, and thus it becomes an important endeavour for researchers to develop and validate such a model. Then, we also describe the technology to justify the techniques as well as a possible theory to back and unify the technologies. We try to summarize the PRM of HTT 5 from paper II in table 1.

As a teacher, you ask pupils to compute the following as homework:

\[ a) \ 0.25 \times 8 = \ldots, \ b) \ 8 \div 0.25 = \ldots. \]

At the next meeting in the class, a pupil notices that when he enters \( 0.25 \times 8 \) into a calculator, the answer is smaller than 8, and when he enters \( 8 \div 0.25 \), the answer is greater than 8. He is confused with this answer and thinks that the calculator must be broken.

What can you do to help such pupils understand this result? (discuss in pairs for 8 minutes, use the space below if necessary, and write your ideas to support the discussion)

**Figure 5**: HTT about multiplication and division of decimals presented in paper IV.
### Table 1: A summary of mathematical praxeologies for HTT 5, presented in paper II.

The mathematical techniques presented in table 1 are not given in detail. Some mathematical techniques depend crucially on specific representations as pointed out above. In such cases, there are of course also specific technological-theoretical discourses to justify those techniques. In our experiments, the logos become particularly visible during PSTs’ collective work.

The second step, which seems to be more important but depends crucially on the first, is to describe DOs corresponding to the five HTTs. As mentioned by Barbe et al., (2005) a DO of teachers is related to helping pupils to acquire or deploy an appropriate MO. Like MO, DO also consists of four tuple praxeologies, and they are independently developed and created by individual teachers, but are instead institutional and historical products of the profession (Rodriguez et al., 2008). The didactical task is related to the mathematical task presented on the HTTs. So, six types of didactical task are identified from the five HTTs, as HTT 5 includes two didactical tasks of different types (Paper II). For each task, we can describe some possible didactical techniques, and some possible didactical logos can also be described to justify the

<table>
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<tr>
<th>Type of task</th>
<th>Techniques</th>
<th>Logos</th>
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<tr>
<td>$T_m$: given a decimal number $a$ and an integer $b$, calculate $a \times b$.</td>
<td>- Standard algorithm for multiplication of decimals&lt;br&gt;- Change decimals into fractions, and then apply the standard algorithm for multiplication of fractions&lt;br&gt;- Proportional reasoning&lt;br&gt;- Repeated addition</td>
<td>- Place value and decimal numeration systems&lt;br&gt;- Equivalent representations of rational numbers&lt;br&gt;- Direct proportion&lt;br&gt;- Reasoning based on the multiplication of natural numbers as repeated addition</td>
</tr>
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techniques. In the following table, we give a summary of such a PRM for the didactical task corresponding to $T_m$, and a similar idea can be used to describe DOs corresponding to $T_d$.

<table>
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<tr>
<th>Type of task</th>
<th>Didactical techniques</th>
<th>Logos</th>
</tr>
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| $T_m^*$: help pupils understand why $a \times b$ gives a smaller result than $b$. | - Instruct pupils about one of mathematical techniques for multiplication task described in table 1 (e.g., teach the standard algorithm for multiplication of decimals)  
- Contextualise the mathematical task using a real life situation.  
- Use visual representations such as diagrams to show the size of fractions.  
- Provide pupils with a simple problem and let them link to the task given. | - Direct instruction, reference to school mathematics textbooks  
- Contextual teaching and learning models  
- Teaching and learning using concrete representations or models  
- Learning proceeding from simple to more complex mathematical problems |

Table 2: A summary of didactical praxeologies for the multiplication task in HTT 5 as presented in paper II.

The details of each didactical technique have been presented in paper II. For instance, “instructing pupils about the standard algorithm for multiplication of decimals” reflects that some teachers consider that the pupils should be given the standard general technique to solve any problem related to multiplication of decimals. They propose this technique because it is what they find in many school textbooks or what they have encountered themselves at school, or during teacher education. On the other hand, some didactical techniques are developed based on didactical theories. For instance, when we look at the mathematics textbook for the teacher education programme (*Matematik for lærerstuderende Ypsilon*) (Hansen, Skoot, & Jess, 2007) taken by the Danish PSTs, there were two didactical theories proposed in relation to rational numbers: realistic mathematics education (RME) and the theory of the didactical situations (TDS). These could be the discourse to explain their didactical techniques such as contextualising the problems into a real life situation and to explain based on “pizza” representations. In addition, Winsløw and Durrand-Guerrier (2007) found that the Danish PSTs focus more on the classroom situations, and that their strategy is usually to firstly present the mathematical concepts or rules to be learned, and then engage in finding real life examples to help pupils understand the rules.
1.8.2 A comparative analysis of Danish and Indonesian PSTs’ mathematical and didactical praxeologies

In relation to the comparative study of Danish and Indonesian PSTs’ knowledge of rational numbers, we outline the results from paper II, III, and IV. First, we discuss the result from paper III focusing on Danish PSTs’ mathematical and didactical knowledge of arithmetic operations with rational numbers, while paper II analyses the Indonesian PSTs’ mathematical and didactical knowledge from all 5 HTTs. Through these two papers, we can identify some similarities and differences between the two groups. In addition, paper IV also presents a comparative analysis between Danish and Indonesian PSTs on the density properties of rational numbers, and it links to the idea of praxeological change.

Paper II shows that all Danish PSTs did not have any difficulties to solve the mathematical tasks about addition and subtraction of fractions using the standard technique: change each fraction into a common denominator and then add numerators. When given a didactical situation related to pupils’ answers, such as \( \frac{2}{3} + \frac{1}{2} = \frac{3}{5} \), almost all of them agreed that the pupils inappropriately applied the properties of natural numbers. This means that the pupils recognized a fraction, such as \( \frac{1}{2} \), as merely two numbers that can be operated on separately. During the discussion, four pairs argued that the pupils needed a more solid interpretation of individual fractions. This, for them, relates to the meaning of a fraction as a part of a whole and a fraction as quotient (Charalambous & Pitta-Pantazi, 2007), but they mostly rely on the first meaning when proposing some didactical praxeologies. They mostly suggested teaching pupils through visualizing both tasks into real-life situations, and pizza diagrams become the main attempt in their discussion, especially to the addition task. A didactical technology to justify this didactical technique is based on their belief that the pupils need to be instructed using a real-life situation to grasp the meaning underlying the standard algorithm.

The Danish pairs have more challenges to deal with the HTT about multiplication and division with decimals. The most challenging was to propose didactical praxeologies to the division task. For instance, a PST argued that she knew how to use the calculator to solve the division task, how to read the result, but could not explain to the pupils how such a result could be produced otherwise. She and her partner could not find a real-life situation to illustrate the division task, like what they simply found and explain to the multiplication task. Meanwhile, most of the Danish pairs still considered to use the pizza diagram, but they only used it to show the equivalence of 0.25 and \( \frac{1}{4} \), and to some extend to the multiplication task. Multiplication as repeated addition and division as repeated subtraction become the main mathematical technology underpinning the
didactical praxeologies based on the real-life situation. For instance, one PST suggested posing the problem to divide 8 litres of milk into some 250-millilitre bottles. On the other hand, four Danish pairs also tried to construct didactical praxeologies directly from their mathematical knowledge. One of these was to develop technology based on the notions of variable relation and functions, involving factors and products. The mathematical idea is to recognise that the change of factors will affect the product. This situation can be represented as a linear function of \( f(x) = 8x \), and the mathematical technology is related to the meaning of fractions as a linear mapping (Brousseau, 1997). However, this didactical praxeology was not further elaborated by them due to their theoretical position that it was too abstract for teaching elementary school pupils.

Paper II shows the general results that the Indonesian pairs did not have as many difficulties to solve the HTT on addition and subtraction of fractions, compared to the other four HTTs, but they mostly solved the mathematical tasks using a standard technique and instructed pupils to master that technique. When they did not know or remember any standard technique to solve the mathematical task in the HTT, e.g., \( 8 \div 0.25 \), they could not discuss the tasks, or they might propose a didactical technique that was not substantially related to the mathematical task. For instance, the case of \( S_{2A} \) and \( S_{2B} \) presented in paper II did not discuss any mathematical and didactical praxeologies related to the division task. They almost spent the whole time to discuss several possible praxeologies related to the multiplication task. Even though, at the end of their collaborative work, they actually mentioned that the task of division could be rewritten as a division of fractions, they did not elaborate and present the corresponding mathematical technique.

A few Indonesian pairs tried to elaborate their didactical praxeologies based on a real-life situations or to use diagram representations and number lines, but many of them got into difficulties when they had to explain the details during their collaborative work. For instance, a pair from group 4 tried to represent \( \frac{8}{9} \) using a rectangle representation (HTT 1), but they were not aware that each part should have the same size. In fact, a few of the representations or real-life situations proposed by the Indonesian pairs could not serve to support pupils to learn better what rational numbers are, or for instance to justify the standard algorithms for the arithmetic operations of rational numbers.

From the two papers, we observe that the only similarity between Danish and Indonesian pairs is found in the mathematical techniques used initially to solve the pupils’ mathematical tasks such as addition and subtraction of fractions. They all used the standard technique, and this could be influenced by the characteristic of mathematical tasks presented in the HTT. When comparing to other HTTs, HTT 4 clearly asked PSTs to solve the two mathematical tasks first \( \left( \frac{2}{3} + \frac{1}{2} = \cdots \right) \) and
\[ \frac{4}{7} = \frac{1}{3} = \cdots \] which lead them to present their mathematical techniques. It is not the case for the other HTTs in which they have to analyse and rectify pupils’ mistakes, such as to explain why multiplication with rational numbers and decimals do not always give a greater result. So, they have to consider some technological discourse to engage in this task and propose some possible didactical techniques based on their mathematical knowledge.

A manifest difference between Danish and Indonesian PSTs is related to their didactical praxeologies. The Danish PSTs mostly tried to build their didactical praxeologies based on real-life situations. The pizza diagram seems almost a reflex to them, as they start explaining the meaning of fraction as a part of a whole. They really displayed a strong pedagogical conviction in the situations: that teaching should start from concrete activities that are closed to pupils’ experiences (also found with Danish PSTs in Winsløw & Durand-Guerrier, 2007). On the other hand, the Indonesian PSTs mostly assumed that teaching mathematics consists in direct instruction of a standard technique, and then the pupils have to mimic it to solve to other similar mathematical tasks. In fact, they provide almost no technological justification to the techniques given. One can say that the Indonesian PSTs view of teaching is monumentalistic (Chevallard, 2015): pupils have to learn the standard techniques or algorithms without raisons d’être for the praxeologies encountered.

A further comparison between Danish and Indonesian PSTs is presented in paper IV, focusing on the density properties of rational numbers. The participants for this study were all Danish PSTs from the pilot study and the main data collection, so there were 31 Danish PSTs compared to 32 Indonesian PSTs. The tasks related to the density of rational numbers can be seen as a tricky one because it is not so commonly worked with by the pupils at school, and requires sophisticated knowledge of the order structure of rational numbers to justify the techniques proposed. Since most of the Indonesian PSTs interpreted the type of task as ordering fractions or decimals instead of employing the denseness to the two given fractions \( \frac{2}{5} \) and \( \frac{4}{5} \), and the decimals 0.4 and 0.8, they tended to instruct pupils just based on ordering natural numbers, so almost all of them agreed that there are finite many numbers between the two fractions or decimals. By contrast, the Danish PSTs recognised that the two tasks asked to examine how many numbers there are between two fractions and two decimals which each represent the same two numbers. Supported by their appropriate praxeologies related to equivalent representations of rational numbers, they thus recognised that the two tasks must have the same answer. Meanwhile, unlike with the previous types of task, the Danish PSTs pay less attention to find a real-life situation corresponding to the task. The Danish pairs seem more flexible and open minded when they work with different types.
of task on rational numbers. In contrast, the Indonesian PSTs focus more on instructing pupils directly on the one technique they have themselves which is then also to be remembered by the pupils, without having any meaningful justification. We can conclude that the Danish PSTs have better mathematical and didactical knowledge of the density properties of rational numbers and rational numbers in general than the Indonesian PSTs.

1.8.3 Teachers’ praxeological change and its role on teaching and learning rational numbers

Paper IV introduces the notion of praxeological change and uses it to describe a need for changing mathematical praxis and logos when moving from natural to rational numbers. We propose this notion to point out a limitation of epistemological models of some previous studies on this topic. For instance, conceptual change approach is commonly used to explain pupils and teachers’ development of mathematical concepts (e.g., Vamvakoussi & Vosniadou, 2004), but it does not allow for a detailed analysis of how to study a “complex of notions” and their simultaneous change, let alone the change in discourse and theory which accompany conceptual change. This does not mean that the praxeological change totally differs from what these authors describe as conceptual change, but the praxeological analysis allows for a more complete and detailed analysis of what the change requires.

To illustrate and describe the idea of praxeological change, we analyse and compare Danish and Indonesian PSTs’ knowledge of the density of rational numbers. The findings show that the Danish PSTs were more successful than the Indonesian PSTs in dealing with the tasks on numbers situated between two fractions and between two decimals. This indicates that the Danish PSTs did not rely on their praxeologies related to the order structure of the set of natural numbers, and had at least to some extent realised a praxeological change in order to appreciate the properties of the ordering of the rational numbers. This change can not be separated from the interaction between their practices and theory. For instance, some Danish pairs first did not realise that the fraction task (how many numbers there are between $\frac{2}{5}$ and $\frac{4}{5}$) and the decimal task (how many numbers there are between 0.4 and 0.8) were really about the same numbers, but eventually they did. From this observation, they could build their technological discourse refuting the natural number technique of the hypothetical students and, at least rudimentarily, were able to articulate the density property of rational numbers as well as the resulting infinity of rational numbers in the given interval. By contrast, some Indonesian pairs employed the same technique, converting fractions to decimals or vice versa, thus realising that the students could not be right, but they did not invoke the density property nor arrive at a satisfactory solution to the tasks. They were seemingly unaware of a need to change praxeologies, both practices and theories, from what they knew about the
familiar, ordered sequence of natural numbers, to the situation involving rational numbers. While they were able to do operations with, compare and change representations of such numbers, they thus basically shared with the imaginary pupils a lack of intuition and formal knowledge about these numbers as an ordered collection.

In general, we define the status of PSTs’ praxeological change into three categories (Vamvakoussi & Vosniadou, 2004), which affects also their capacity to support pupils in gaining knowledge about the structure of the rational numbers as a whole: discrete-praxeology (considering the rationals essentially as a variant of the integers, with no knowledge to support the praxeological change), discrete-denseness praxeology (with some intuition of denseness, but still insufficient technical and technological knowledge to support the praxeological change), and denseness praxeology (sufficient knowledge to support praxeological change). These categories aim to clarify to what extent PSTs are aware of praxeological change needed to support pupils learning the density properties of rational numbers, closely related to their own praxeologies. Most of the Indonesian pairs belonged to the first category, as they only consider the order of fractions or decimals as a simple extension of the order of natural numbers. They were not aware that one can always find a fraction between two consecutive fractions by using a technique such as arithmetic mean. A lack of questioning the logos during their collaborative work – otherwise focused on identifying simple techniques, involving just a few individual numbers – is at the root of their difficulties. This could be related to a general didactical theory of Indonesian teachers, that learning and teaching mathematics is mainly a mechanistic process of imparting certain standard procedures to solve corresponding mathematical tasks (Armanto, 2002; Fauzan, 2002; Hadi, 2002). They are not used to question why such a technique is appropriate or not to the given task. Only one Indonesian pair was aware of praxeological change, while almost all Danish pairs were observed to be in the third category. The gap between Danish and Indonesian PSTs’ praxeological change could be argued to result just from their individual mathematical knowledge. This is not only about what they learn at the teacher education institution, but is likely to be affected also by the mathematical knowledge build during their school time, and the resulting belief that learning mathematics is about memorising standard procedures, with the teachers having main responsibilities to clearly and directly impart these to their pupils, without asking them to engage in questioning and critical discourse (Sembiring et al., 2008).

Didactical praxeologies proposed by the Danish and Indonesian PSTs to deal with pupils’ difficulties on the density properties of rational numbers certainly depends crucially on their own praxeological change. The change of praxeologies affects not only on the practical block but also the theoretical one. As an example, we find that some Indonesian pairs suggested to teach pupils
about converting fractions to decimals and vice versa, and some of them also realised that both tasks were the same, but in the end they still considered that there was some finite collection of numbers between the two decimals or between the two fractions, even if they could not identify it. The Danish pairs, who used the same mathematical technique, came to the different conclusion that there were infinitely many rational numbers between the two decimals or between the two fractions. As they mostly did not produce a complete justification, they probably encountered this theoretical fact beforehand. Thus, one may have quite different theory blocks and still master essentially the same techniques.

1.9 Conclusions

The thesis deals with the quest for a theoretical and methodological framework to study teachers’ mathematical and didactical knowledge, specifically on rational numbers, and then investigates how it applies to study and compare Indonesian and Danish PSTs. In a concrete way, this study investigates the four research questions RQ1-RQ4 that have been presented in section 1.6.

RQ1 is investigated through designing five HTTs on rational numbers. The mathematical and didactical tasks embedded in each HTT aim to engage the PSTs in sharing their mathematical and didactical knowledge explicitly, during their collective work. To study this knowledge systematically, ATD supplies the theoretical underpinning to analyse the HTT using PRM. The a priori analysis of types of task and techniques is given in detail to let the a posteriori analysis of actual PSTs’ individual and collective work be as systematic as possible. More generic mathematical and didactical logos was also considered to allow the analysis to capture perspectives that go beyond the individual HTT. In addition, the a priori analysis shows that there are some links among MOs related to rational numbers. This means that new MOs, e.g., arithmetic operations of rational numbers, are built upon other MOs, foremost those related to equivalent representations of rational numbers. However, we found it quite challenging to describe complete mathematical and didactical praxeologies, especially the didactical logos. The latter is much influenced by the institutional dimension on how and where the knowledge appeared and developed. For instance, we might assume a didactical technique such as to teach pupils the addition of fractions through contextualising the task using a real life situation, as suggested by the two groups, but the didactical logos to justify this technique can be varied and elusive, and it depends on various aspects of the above levels of the didactic codetermination. It could in particular be strongly linked to elements of general pedagogy and cognitive theory such as principles of Piagetian constructivism (Winsløw and Durand-Guerrier, 2007).
The study of RQ2 showed that PRM can be used as an important tool to study and compare PSTs’ individual and collective knowledge of rational numbers. The a posteriori analysis of PSTs’ written answers and collaborative work was done through analysing their mathematical techniques to solve the mathematical types of task, and then how they used them to construct didactical techniques. The a posteriori analysis of mathematical and didactical logos from their collaborative work was investigated through carefully interpreting their interaction, and to some extent the logos part explicitly appeared during their discussion, but in many cases, PSTs did not explicitly engage in discussions of the logos part. Perhaps, we should consider for future research to let the researchers ask some questions in order to obtain more clarification of PSTs’ rationales and beliefs in relation to didactical techniques during the discussion, so that the analysis of PST’s didactical logos can become more precise and detailed.

RQ3 was explored through paper II, III and IV in relation to Danish and Indonesian PSTs’ mathematical and didactical praxeologies. In term of similar features, they employ a standard mathematical technique when they are asked to solve a simple mathematical task, such as adding two fractions. But when the type of task is about teaching rational numbers, there are clear differences between the two groups of informants. The Danish PSTs emphasised more the pedagogical aspects of teaching and are also clearly influenced by didactical theories developed in societies close to their own. By contrast, the Indonesian PSTs still rely on traditional teaching and learning models in which the teachers remain the central source of learning, which could be rooted both in the institutions they have frequented, and in their society. As argued by Revina and Leung (2018), the teaching and learning model in Indonesia is influenced by the Javanese culture in which teachers are assumed to be “scholarly people” who know everything about the knowledge to be learned, and the pupils have a role to receive that knowledge without questioning it.

The exploration of RQ4 is addressed through the analysis of the HTT about the density of rational numbers. The reason to choose this HTT is due to the considerable challenge for both pupils and teachers to deal with the theoretical idea of density of rational numbers. Presenting the density task to Indonesian and Danish PSTs, it shows to what extent praxeological change of the teacher is needed to build an appropriate didactical praxeology. PSTs need to re-construct their mathematical knowledge about numbers and probably in more theoretical ways than those required by pupils. Indeed, the required change concerns the both aspects: the practical and the theoretical blocks of praxeologies. An insufficient theoretical link, in teachers’ praxeology, between the order structure of rational numbers and the density property of rational numbers, may severely constrain PSTs ability to realise, even in a collaborative setting, this particular need for praxeological
change. The Indonesian PSTs who only mastered the order structure of rational numbers at a technical level were unable to link the task to the density property.

1.10 Perspectives for future research

Teachers’ knowledge is complex, and to study this knowledge in depth, it is not enough to evaluate whether teachers’ answers to simple items are correct or not. It is not only about mathematical knowledge that can be investigated by using some standard mathematical tasks, or even tasks at the horizon of their knowledge. The format of HTT achieves, at least to a much larger extent, a comprehensive study of teachers’ practical and theoretical knowledge, by placing informants in somewhat simplified situations that mirror difficulties that can appear in real teaching situations. PSTs’ collective production become a central point for the design of HTT that makes them different from other research tools, such as the items commonly used to measure MKT (Ball et al., 2008). Mathematical and didactical praxeologies could be explored during the PSTs collaborative work (although, as mentioned above, this potential be further developed), and also one can observe how the two sets of praxeologies are connected. Of course, time limitation for the discussion could be an obstacle for some pairs to further explore and share their praxeologies, especially the logos part. But in practice, a lack of mathematical knowledge about rational numbers is often a more important reason for their difficulties with constructing the didactical praxeologies asked for in the items.

As teaching is a profession, it could be maintained that teachers should know what their profession know, and they should learn what the profession is able to learn (Chevallard, 2007). Part of that needs to be learned through practice, as in any profession. However, before being entrusted to teach, one could argue that in principle the PSTs should have the necessary mathematical and didactical knowledge, such as for teaching rational numbers, by the end of their preservice study. However, this expectation is far from the reality. especially for the Indonesian PSTs. As we will see in the analysis of some of the HTT, some of them struggle to solve rather basic mathematical tasks about rational numbers, and in that case they may obviously get difficulties to teach students to solve such tasks. This raises some questions that need to be further investigated. The first and most important one is: what mathematical and didactical knowledge about rational numbers can be learned at the teacher institutions? Would it be possible devise better teaching models for teacher education programmes that can support PSTs’ mathematical and didactical praxeologies of rational numbers, and other mathematics subjects? can we use HTT as a tool to develop PSTs’ mathematical and didactical knowledge, while questioning it explicitly and in collaborative settings? and what didactidal aspects should be considered while designing
HTTs for teaching programmes? Those four questions remain open to be studied in future research and development.

Finally, the comparative study conducted in this thesis only focuses on the levels related to praxeologies. We do not discuss much the higher levels of the didactic codetermination that could influence the similarities and differences between the two countries. It could be interesting to design a follow-up research project to address this issue in order to identify the relevant institutional conditions and constraints leading to the observed difficulties with rational numbers. Since most of the informants have probably become teachers at school and now teach rational numbers, it could be interesting to study if and how they learn from their teaching experiences, and if and how the knowledge they acquired in teacher training institutions appears as a support their current function as teachers.

1.11 References


2 A framework for a comparative study of pre-service elementary teachers’ knowledge of rational numbers.

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Abstract
This paper presents a framework of PhD research of the first author about a comparative study of pre-service elementary teachers’ knowledge of rational numbers between Indonesia and Denmark. To obtain the data, the authors design a series of hypothetical teacher tasks (HTT), inspired by a paper of Durand-Guerrier, Winsløw, and Yoshida (2010). Subjects in this research are pre-service elementary teachers from a selection of different University Colleges in Denmark and from the elementary school teacher education study program, Riau University, in Indonesia. The praxeological reference models and the levels of didactic codetermination are used as tools to analyse the result.

Resumen
Este artículo presenta un marco general de un proyecto de investigación doctoral. Se trata de un estudio comparativo sobre el conocimiento acerca de los números racionales en la formación de maestros de enseñanza primaria en Indonesia y Dinamarca. Para obtener los datos los autores a construir una serie de "tareas hipotéticas de enseñanza" (HTT), inspiradas de un artículo de Durand- Guerrier, Winsløw, y Yoshida (2010). Los sujetos de nuestra investigación son estudiantes de una selección de diferentes Colegios Universitarios en Dinamarca y del programa de formación de maestros de primaria de la Universidad de Riau en Indonesia. El análisis utilizará los niveles de codeterminación didácticos y los modelos praxéológicos de referencia.

Résumé
Cet article présente le cadre d’un projet de recherche doctoral. Il s’agit d’une étude comparative sur les connaissances sur les nombres rationnels, chez les étudiants en formation pour enseigner à l’école primaire, en Indonésie et au Danemark. Pour obtenir les données les auteurs construirons une série de « Tâches hypothétiques d’enseignants » (HTT), inspiré par un article de Durand-

1. Introduction

Comparative studies on teaching and learning rational numbers have been done by several researchers. For instance, Li (2014) compared British and Taiwanese pupils’ conceptual and procedural knowledge of rational numbers, more specifically of adding fractions. Taiwanese pupils performed better than British peers because they were more successful to apply algorithms for adding fractions. British pupils had a tendency to add numerators and denominators, respectively. Similar comparisons have been done among other countries around the world. Lai and Murray (2014) compared Hong Kong Chinese and Australian pupils’ understanding of decimal numbers. Even though Hong Kong Chinese pupils performed better than their Australian peers, they had similar misconceptions about decimal numbers, for instance in comparing two decimal numbers, as pupils struggled with the concept of place value.

Comparative studies between Western and East Asian countries have indeed become common in recent years and are often motivated by a desire to understand the background for different performance in international tests such as PISA or TIMSS. We consider specific topics, such as the arithmetic of rational numbers in order to understand, in a fine grained way, the differences at first more coarsely observed. A main motivation to get such an understanding is to assess what factors are important in causing the observed differences. Here, teachers’ knowledge has often been advanced as a key factor.

Some researchers have already conducted studies on elementary teachers’ knowledge of rational numbers. Ma (1999) compared U.S. and Chinese pre-service and in-service elementary teachers’ capability of solving and constructing meaningful problems involving fraction division. She found that U.S. teachers were less successful than Chinese teachers on both kinds of tasks, and most of them did not understand the rationale underlying their calculation and the meaning of division by fractions. Meanwhile, Stacey et al., (2001) investigated Australian pre-service teachers’ knowledge about pupils’ difficulties with decimal numbers. Their result was that pre-service teachers mainly possessed simple content knowledge about decimals. They could notice pupils’ errors with comparing decimal numbers, but they could not explain why these occurred. Both studies have similar approaches to investigate pre-service and in-service elementary teachers’
knowledge about rational numbers through simple tests, based on a cognitive paradigm that focuses on individual knowledge.

This research project takes a different approach, based on the anthropological theory of the didactic (ATD) introduced by Chevallard (1992, 2006, 2007). In this framework, knowledge is considered as institutionally situated, and it is studied through praxeological reference models (PRM). A comparative study of secondary level teacher students’ knowledge was conducted by Winsløw and Durand-Guerrier (2007) and Durand-Guerrier, Winsløw, and Yoshida (2010); our study adopts their notion of hypothetical teacher tasks (HTT) and associated PRM. The mathematical focus of the present study is the order structures and arithmetic of rational numbers. Our aim in this paper is to develop a framework to study pre-service elementary teachers shared mathematical and didactical knowledge about rational numbers. The framework will be applied to a comparative study of pre-service elementary teachers (PsETs) from Indonesia and Denmark. We would like to do this in both countries because one of the researchers comes from Indonesia and is doing his PhD program in Denmark. We hope that this research can address the gap between knowledge development by teachers at universities and their subsequent resources for teaching pupils at schools. The results of this study will also contribute to develop our knowledge about teaching didactical knowledge of rational numbers to PsETs in both countries. To clarify the goals of our research, we formulate specific research questions for the entire PhD program of the first author as follows:

1. How can the ATD function as a framework to study pre-service elementary teachers’ mathematical and didactical knowledge of rational numbers?
2. In particular, how could HTTs be used to study pre-service elementary teachers’ mathematical and didactical knowledge of rational numbers?
3. What similarities and differences can be identified between Danish and Indonesian pre-service elementary teachers’ knowledge of rational numbers?
4. At what levels of didactic codetermination the origin of these differences can be identified?

In this paper, we focus on the first research question by describing mathematical and didactical praxeologies.

2 The ATD and the levels of didactic codetermination

The ATD provides a detailed model of the levels of didactic codetermination which may help to explain the sources of differences in PsETs’ knowledge of rational numbers, as shown in figure 1 (Artigue & Winsløw, 2010; Bosch & Gascón, 2006, 2014). In general, the levels are divided into nine categories. Some educational studies only focus on the levels above discipline, while specific
subject didactical studies, such as didactic of mathematics, mostly concern to the levels at or below the level of the discipline (Bosch & Gascón, 2006, 2014). These levels relate to the teaching of mathematical praxeologies, which requires teachers to pose both mathematical and didactical knowledge. The higher levels are mainly governed by school principals, curriculum developers, and politicians. The first five levels of analysis cover both mathematical organisations (MOs) and didactical organisations (DOs) that can be directly observed in teaching and learning practices, as well as in tests or documents such as textbooks, curriculum, etc. The MOs are linked to the mathematical contents that teachers should teach and, thus, are supposed to be highly competent on. For instance, a teacher who gives a task to pupils such as adding and subtracting two fractions should have a MO that gives beyond one or two techniques for solving the task. The teacher should be able to explain a variety of techniques given, relate them to other tasks, and provide some justifications based on appropriate technological-theoretical discourses. While, the DO is about teaching and learning praxeologies related to the MO. To design a lesson plan to teach addition and subtraction of two fractions is an example of DOs.

Figure 1. The levels of the didactic codetermination.

Level one up to three, subject, theme, and sector, are the main levels that we use to design HTT to investigate PsETs’ knowledge of rational numbers. Those levels are related to PRM. The subject corresponds to a type of task \( T \) and a technique \( \tau \) (see also Artigue & Winsløw, 2010; Winsløw et al., 2014). To assess teachers at the subject level, the HTT include mathematical tasks that are designed to uncover characteristic difficulties among pupils, identified in the research literatures on teaching rational numbers. As an example, the type of task \( T \), for instance, can be e.g., Adding and subtracting of two fractions
to add or subtract two fractions with different denominators. To solve such tasks, a technique (τ) is needed, such as changing each fraction into fractions with a common denominator. Meanwhile, a technology (θ) and a theory (Θ) occur at the level of theme and sector respectively. The explanations of the techniques are contained within a wider technology about operations with fractions (a discourse on how to calculate with). A theory behind that technology contains more or less formal definitions, rules, and proofs which justify the technology. It is developed from the arithmetic of rational numbers.

The next two levels, domain and discipline, refer to more global MOs. Arithmetic is the domain for the school praxeological model of addition and subtraction of fractions. Mathematics is the discipline in a given school institution.

The last four levels are pedagogy, school, society, and civilisation. The pedagogy is proper to school institutions and implemented by teachers as a professional body. Also a school has rules and regulations, for instance concerning the autonomy of teachers, and a school institution is situated with in the rest of society, along with superior institutions such as the Ministry of Education, which has the power to regulate the school through a national curriculum, funding, and national assessment of pupils. As an example, certain systematic differences among the teacher education systems in France, Denmark, and Japan, may be observed through the differences in the teacher students’ performance on the mathematical tasks (cf. Durand-Guerrier et al., 2010). Meanwhile, the civilisation may also influence the teachers’ and pupils’ performance on mathematics, school’s level of autonomy, etc.

The levels of didactic codetermination have been used by Artigue and Winsløw (2010) to compare and analyse studies such as PISA and TIMSS. They showed that the comparative studies could rely on a horizontal comparison (between two contexts at the same level) or on comparing certain vertical relations between different levels with in each context (Figure 2). Differences between two contexts at the same level could be claimed to be caused by other higher level differences. We apply the latter method in our study starting by comparing mathematical and didactical praxeologies, specifically mathematical and didactical techniques (τ). Then, we also investigate some factors that affect the differences among teachers’ praxeologies in the two countries, through comparing factors at higher levels, such as the curriculum and textbooks used by schools and by teacher education institutions.
3. Hypothetical teacher tasks (HTT)

The notion of hypothetical teacher tasks (HTT) first appears in the study of pre-service lower secondary mathematics teachers’ knowledge of teaching similarity and proportion, and the multiplication of two negative numbers (Durand-Guerrier et al., 2010; Winsløw & Durand-Guerrier, 2007). A HTT is constructed so as to introduce a teaching situation which could conceivably appear in a classroom setting, and where teachers would have to invest both mathematical and didactical knowledge, in order to act properly in the situation. The HTT thus initially enables us to study pre-service teachers’ mathematical and didactical knowledge. Each HTT consists of a mathematical and a didactical task. The mathematical task is a standard task given to pupils at schools, but the task is set for teachers within a situation where pupils struggle to find a correct answer. So, the teachers have to provide various mathematical techniques. Meanwhile, the didactical task is a task for teachers to handle in a didactical situation (cf. Brousseau, 1997), and they must suggest some didactical techniques to further pupils’ learning. The didactical tasks are strongly related to the mathematical tasks.

The HTTs developed in this project aim to investigate the knowledge of PsETs about rational numbers, and the teaching of such knowledge. We first study a MO of rational numbers from punctual to global organisations. A punctual organisation contains just one type of task such as to find a fraction which is equivalent to $\frac{3}{4}$ (HTT 1). But, various types of task that employ a common technology (such as the equivalence of two equivalent fractions) are unified as a local organisation of specific themes. Several technologies may be justified by a theory (e.g. a theory of order structures of rational numbers) and a family of praxis sharing one theory is known as a regional organisation. Some regional organisations may be further unified in a global organisation of
specific domains (e.g. rational numbers). In fact, MOs are structured and stratified in mathematical domains or knowledge to be learnt, while in teaching practice, they are often established only at the punctual or local level (Durand-Guerrier et al., 2010).

In order to study PsETs’ knowledge on rational numbers, we designed five HTTs (Figure 3). The first three tasks, HTT 1, HTT 2, and HTT 3 are all linked to the order structures of rational numbers. Techniques related to the equivalence of rational numbers can be applied to solve the type of task of HTT 2, HTT 3, and also HTT 4. Meanwhile, HTT 4 and HTT 5 concern the arithmetic of rational numbers. In HTT 5 the main point is that multiplication of rational numbers cannot, in general, be explained as “repeated addition”.

4. Praxeological reference models (PRM)

To study PsETs’ knowledge in a systematic way, we have constructed a reference model for each HTT, specifying the corresponding mathematical and didactical praxeologies. We focus mostly on the reference model for practical blocks i.e. a type of task and techniques. In this paper, we only describe the detail models of HTT 1, and we assume readers can figure out how it is done to the other four HTTs.
HTT 1 is about equivalent fractions, and the problem given to pairs of PsETs is presented as follows:

A pupil claims that \( \frac{3}{4} = \frac{8}{9} \) because if you add 5 to both the top and the bottom of a fraction, the new fraction must be equal to the original.

a. What do you think about this answer? Please explain! (to be solved individually within 3 minutes).

b. What would you do as a teacher to help the pupils from this case to understand the concept of equal fractions better? (to be discussed and solved in pair within 5 minutes).

Figure 4. HTT 1 about equivalence of fractions.

An a priori analysis of HTT 1 consists of mathematical and didactical praxeologies. A mathematical task given to pupils can be described on the following type:

\[ T_1: \text{given a positive fraction, } \frac{a}{b}, \text{ determine other fractions that are equal to it.} \]

We can describe some possible mathematical techniques to solve the tasks of type \( T_1 \):

\( \tau_{11} \): compute correct equal fractions of \( \frac{a}{b} \) by multiplying/dividing each numerator and denominator by the same positive integer.

\( \tau_{12} \): first represent \( \frac{a}{b} \) in a model such as a rectangle or a circle diagram, then draw another model for \( \frac{a}{b} \) by dividing it into 2, 3, or more parts. Finally, it can be shown that both models generate equal fractions, e.g. as follows:

\[
\begin{array}{c}
\begin{array}{c}
\frac{3}{4} \rightarrow \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8} \\
\frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}
\end{array}
\end{array}
\]

\( \tau_{13} \): first change \( \frac{a}{b} \) into a decimal, then find another fraction that is equal to that decimal.

There are three possible techniques to change fractions into decimals: a division algorithm, specific fractions memorised as decimals (e.g. \( \frac{1}{4} = 0.25 \)), and finally using calculators, computers, or other electronic devices to divide \( a \) by \( b \). Meanwhile, we also predict that some teachers probably use addition and subtraction of numerator and denominator by the same natural number. This technique is called as \( \tau_{14} \) (a minus indicates the technique is incorrect).

\( \tau_{14} \): compute equal/equivalent fractions of \( \frac{a}{b} \) by adding or subtracting the same positive integers to/from the numerator and the denominator, which amounts to the (wrong) claim that \[ \frac{a}{b} = \frac{a \pm n}{b \pm n}. \]
The second type of mathematical task is implicit in question a. as follows:

\( T_2 \): given two positive fractions, \( \frac{a}{b} \) and \( \frac{c}{d} \), decide if they are equal.

There are some possible mathematical techniques to solve such tasks. The first technique is to change both fractions into the same denominator and then compare numerators. We state this technique as \( \tau_{21} \).

\( \tau_{21} \): first change both fractions into an equal denominator and then compare numerators, e.g. \( \frac{3}{4} \neq \frac{8}{9} \).

\( \tau_{22} \): represent both fractions into rectangle or circle diagrams (sometimes called pizza diagrams) and compare their areas or sizes, for instance as follows:

\( \tau_{23} \): change fractions into decimals to show both fractions are not equal (use one of these techniques: a division algorithm, specific fractions memorised as decimals, or using calculators, computers, or other electronic devices), e.g. \( \frac{3}{4} = 0.75 \) and \( \frac{8}{9} = 0.88 \), so \( \frac{3}{4} \neq \frac{8}{9} \).

\( \tau_{24} \): represent both fractions on a number line, and show that the numbers are positioned at different points, for instance as follows:

\( \tau_{25} \): for fraction \( \frac{a}{b} \) and \( \frac{c}{d} \), divide \( c \) by \( a \) and \( d \) by \( b \), or multiply \( a \) by \( d \) and \( b \) by \( c \), when the results are equal, the fractions are equal.

\( \tau_{26} \): for fraction \( \frac{a}{b} \) and \( \frac{c}{d} \), compute \( b - a \) and \( d - c \) or \( c - a \) and \( d - b \), when \( b - a = d - c \) or \( c - a = d - b \) concludes that the fractions are equal.

In general, the fundamental law of fractions (for any fraction \( \frac{a}{b} \) and any integer \( n \neq 0, \frac{n \cdot a}{n \cdot b} \)) and the definition of equivalence of fractions (two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) are equivalent if and only if \( a \cdot d = b \cdot c \)) are the main technological-theoretical justifications for thus mathematical techniques for the tasks of types \( T_1 \) and \( T_2 \), respectively (Long & De Temple, 2003, pp. 351). Multiplicative or proportional reasoning and multiple representations of rational numbers (e.g. fractions,
decimals, percentages, or diagrams) can be the other possible technological-theoretical blocks to justify other mathematical techniques.

The type of didactical task of question b. can be described as follows:  

\( T_1^* \): propose strategies to help pupils to solve a task of type \( T_1 \).

A common didactical technique to solve the task of type \( T_1^* \) is to simply explain one of correct mathematical techniques for the tasks of type \( T_1 \) or even \( T_2 \). For instance, a teacher shows to pupils how to find an equal fraction of \( \frac{3}{4} \) by multiplying 2 to the numerator and to the denominator to get \( \frac{6}{8} \) (\( \tau_{11} \)). This didactical technique is coded \( \tau_{11}^* \). Hence, we get four possible different didactical techniques as \( \tau_{11}^* \), \( \tau_{12}^* \), \( \tau_{13}^* \) and \( \tau_{14x}^* \) corresponding respectively to \( \tau_{11} \), \( \tau_{12} \), \( \tau_{13} \) and \( \tau_{14} \) (adding \( x \) to represent a didactical technique based on an incorrect mathematical technique). Meanwhile, the didactical techniques \( \tau_{21}^* \), \( \tau_{22}^* \), \( \tau_{23}^* \), \( \tau_{24}^* \), \( \tau_{25}^* \), \( \tau_{26}^* \) and \( \tau_{26x}^* \) correspond respectively to \( \tau_{21} \), \( \tau_{22} \), \( \tau_{23} \), \( \tau_{24} \), \( \tau_{25} \) and \( \tau_{26} \). There are also other possible didactical techniques that some of them can be variants of those techniques (coded by adding a letter):

\( \tau_{12a}^* \): represent both fractions into one (two different) number line(s) and show pupils that both fractions stand in the same point.

\( \tau_{15}^* \): present and explain the mathematical task \( T_1 \) into an appropriate contextual or real life situation, e.g. a task related to share pizzas or cakes.

\( \tau_{15x}^* \): present inappropriate contextual or real life situation for the mathematical task \( T_1 \) or suggest to teach pupils through a contextual or real life situation but do not know how to do that. (adding \( x \) to represent an inappropriate didactical technique)

\( \tau_{16}^* \): use a simple fraction such as \( \frac{1}{4} \) and \( \frac{1}{2} \) as a starting point to explain a mathematical technique for the task of type \( T_1 \).

\( \tau_{17}^* \): organize a class discussion of different pupils’ answers.

\( \tau_{22x}^* \): show to pupils that both fractions are not equal through wrong rectangles or circle diagram representations.

\( \tau_{27}^* \): show a counter example to the claim that adding the same numbers to the numerator and denominator gives an equal fraction, because then you should also be able to subtract the same (‘‘going back’’), but \( \frac{3-2}{4-2} \neq \frac{3}{4} \) is obvious.

Actually, the lists of techniques mentioned above are not exclusive. We suppose other possible techniques or even technologies could be suggested by PsETs during their discussion. On the other hands, they might not offer a model for the technological and theoretical discourses upon working with the HTT.
5. A methodological approach to empirical studies

After we designed and analysed the HTTs, we conducted the first empirical study in January 2016, with 11 pre-service teachers (prepared to teach pupils at grade 4 to 10 or approximately age 9 to 15) at the Metropolitan University College (MUC). From February to March 2016, we tried the HTTs with 32 PsETs (prepared to teach grade 1 to 6 or approximately age 6 to 12) from the Elementary School Teacher Education study program, Riau University, Indonesia. Finally, we tested the HTTs from December 2016 to March 2017, with 20 PsETs (also prepared to teach grade 1 to 6 or approximately age 6 to 12) from other three university colleges in Denmark. Most of them worked in pairs except for one group consisting of three pre-service teachers.

During the first data collection, we focus on whether pre-service teachers could understand and solve the HTTs, and what constraints they have when they are working individually and in pairs. In general, they were able to solve all HTTs except for the HTT 5 (see appendix 1) that was really challenging. For instance, when we had a short conversation after the test, a pre-service teacher said that they could solve the mathematical task, but they lacked didactical techniques such as to explain and justify the mathematical techniques to pupils. They were not able to construct an appropriate situation or context related to that mathematical task. It seems that the HTTs are relevant to study teachers’ knowledge since they have various levels of difficulty. The other obstacle was the number of pre-service teachers in a group. Since there were three pre-service teachers in one group, and the group had more difficulties to share their ideas. For instance, when a pre-service teacher shared a technique to solve a task, another pre-service teacher sometimes seemed to dismiss it, by proposing another technique. They easily moved from one technique to another before they had developed a clear idea for the previous technique. So, we decide for the main study that PsETs should work in pairs.

The main data collected from the work of Indonesian and Danish PsETs consist of written answers and video recordings. The written answers can be coded directly based on mathematical praxeologies, specifically the techniques, while the video recordings are transcribed using NVivo version 11.0.0 (1497) computer programming. When we find difficulties to code some texts into a specific technique, we plan to discuss them with other experts in this area. The mathematical and didactical praxeologies discussed by both parties will be compared qualitatively, but we might also consider giving attributed points to the answers, similar to what was done by Durand-Guerrier et al., (2010); 0 point for an inappropriate technique which could not support pupils learning process, 1 point for a reasonable technique which might support pupils learning process but lack of reasoning, and 2 points for an appropriate technique which involves adequate justifications of the
techniques. Then, we will compare the points obtained by pre-service elementary teachers in Indonesia and Denmark. We expect that these results could provide overall trends related to the research questions. Finally, the levels of didactic codetermination will be used to explain similarities and differences between Danish and Indonesian.

6. Summary

In this paper, we have explained how the ATD can be used to study PsETs’ mathematical and didactical knowledge about rational numbers. The idea is to use a specific kind of items, the HTT, and to construct PRM that predict PsETs’ mathematical and didactical techniques when they solve the HTT. In this paper, we presented and analysed HTT 1 in details, as we consider that this suffices to give readers an impression of how such items can be analysed and used. We have mainly presented the analysis at the level of techniques, but in analysing actual PsETs’ work, a more explicit analysis of technology and theory evidenced in that work will be of capital importance. We have only outlined the general tools for such an analysis (e.g. in Figure 3).

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References


Appendix 1: Hypothetical teacher tasks

HTT2 (Comparing decimals)
Fifth-grade pupils are asked to compare the size of 0.5 and 0.45.
Some pupils answer that 0.45 is greater than 0.5, while others say that 0.5 is greater than 0.45.
Analyse the pupils’ answers. Explain your ideas to handle the situation in this class? (to be solved individually in 3 minutes)
How do you use this situation to further the pupils’ learning? (to be discussed and solved in pair within 5 minutes)

HTT3 (Denseness of rational numbers)
You first ask fifth-grade pupils to discuss how many numbers there are between $\frac{2}{5}$ and $\frac{4}{5}$, and how many numbers there are between 0.4 and 0.8.
Then, they say that there is only one number between $\frac{2}{5}$ and $\frac{4}{5}$ namely $\frac{3}{5}$; they also say 3 numbers between 0.4 and 0.8.
How do you interpret this claims? (to be solved individually within 3 minutes)
Explain your ideas to teach these pupils? (to be discussed and solved in pairs within 5)

HTT4 (Addition and subtraction of fractions)
You ask sixth-grade pupils to solve $\frac{2}{3} + \frac{1}{2} = \ldots$, and $\frac{4}{7} - \frac{1}{3} = \ldots$
How do you solve these problems? (to be solved individually within 3 minutes)
You find that many pupils add and subtract fractions in the following way $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$, and $\frac{4}{7} - \frac{1}{3} = \frac{3}{4}$.
How do you interpret the pupils’ methods? (to be solved individually within 3 minutes)
What strategies can you propose to teach these pupils? (to be discussed and solved in pair, 5 minutes)
HTT5 (Multiplication and division of decimals, using calculators)

As a teacher, you ask pupils to compute the following as homework:
a) $0.25 \cdot 8 = \cdots$, b) $8 ÷ 0.25 = \cdots$.

At the next meeting in the class, a pupil notices that when he enters $0.25 \cdot 8$ into a calculator, the answer is smaller than 8, and when he enters $8 ÷ 0.25$, the answer is bigger than 8. He is confused with this answer and thinks that the calculator must be broken.

What can you do to help such pupils understand this result? (discuss in pairs in 8 minutes, use the space below if necessary, and write your ideas to support the discussion)
3 A praxeological analysis of pre-service elementary teachers’ knowledge of rational numbers

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UNE ANALYSE PRAXEOLOGIQUE DE LA CONNAISSANCE DES NOMBRES RATIONNELS DES ENSEIGNANTS ELEMENTAIRES AVANT LEUR SERVICE

Résumé – Des recherches récentes ont étudié les connaissances mathématiques et didactiques individuelles des enseignants à travers des items à choix multiple. Cette méthode est particulièrement pratique pour une étude à grande échelle, mais elle ne permet pas de rendre compte des connaissances mathématiques et didactiques collectives des enseignants, y compris les connaissances qu'ils sont susceptibles de développer ensemble et partager avec des collègues. Pour examiner cette question dans un cas concret, nous utilisons une méthode connue sous le nom de tâches hypothétiques d’enseignants (HTT) pour étudier les connaissances mathématiques et didactiques de futurs enseignants élémentaires (PsET) sur les nombres rationnels. Cette méthode est basée sur la théorie anthropologique du didactique, en particulier la notion de modèle praxéologique de référence. Cette notion est utilisée à étudier les connaissances mathématiques et didactiques de trente-deux PsET indonésiens, par rapport aux nombres rationnels. Grâce à cette méthode, nous avons trouvé que les PsET ont tendance à proposer une instruction directe des techniques mathématiques pour enseigner aux élèves les nombres rationnels, et ils ont souvent du mal à construire des techniques didactiques appropriées en raison de connaissances mathématiques insuffisantes. En outre, nous voyons comment les HTT peuvent être utilisées pour étudier les composantes des connaissances mathématiques et didactiques collectives des futurs enseignants.

Mots clés: connaissances mathématique et didactique, théorie anthropologique du didactique, modèles praxéologiques de référence s, tâches hypothétiques d’enseignants.
Resumen – La investigación reciente ha estudiado el conocimiento matemático y didáctico individual de los docentes a través de elementos de opción múltiple. Este método es especialmente conveniente para un estudio a gran escala, pero no cubre el conocimiento matemático y didáctico colectivo de los docentes, incluido el conocimiento que pueden desarrollar juntos y compartir con otros. Para examinar esta pregunta en un caso concreto, usamos un método conocido como tareas hipotéticas de profesor (HTT) para estudiar el conocimiento matemático y didáctico de los maestros de preescolar (PsET) sobre los números racionales. Este método se basa en la teoría antropológica de lo didáctico, específicamente en los modelos de referencia praxeológicos. Se aplica para estudiar los conocimientos matemáticos y didácticos sobre los números racionales de treinta y dos PsET indonesios. A través de este método, encontramos que, para enseñar a los alumnos los números racionales, los PsET tienden a proponer una instrucción directa de las técnicas matemáticas, y a menudo tiene dificultades para construir técnicas didácticas apropiadas debido a la falta de conocimiento matemático. Además, vemos cómo los HTT se pueden utilizar para estudiar los componentes colectivos del conocimiento matemático y didáctico de los docentes.

Palabras-claves: conocimiento matemático y didáctico, la teoría antropológica de lo didáctico, los modelos de referencia praxeológicos, tareas hipotéticas de docentes.

Abstract
Recent research has studied teachers’ individual mathematical and didactical knowledge through multiple choice items. This method is especially convenient for a large-scale study, but it does not cover teachers’ collective mathematical and didactical knowledge, including knowledge the teachers may develop together and share with others. To examine this question in a concrete case, we use a method known as hypothetical teacher tasks (HTT) to study pre-service elementary teachers’ (PsETs) mathematical and didactical knowledge of rational numbers. This method is based on the anthropological theory of the didactic, specifically praxeological reference models. It is applied to study thirty-two Indonesian PsETs’ mathematical and didactical knowledge of rational numbers. Through this method, we found that the PsETs tend to propose a direct instruction of mathematical techniques to teach pupils about rational numbers, and they often struggle to construct appropriate didactical techniques due to a lack of mathematical knowledge.
In addition, we see how HTT can be used to study collective components of teachers’ mathematical and didactical knowledge.

**Keywords:** mathematical and didactical knowledge, the anthropological theory of the didactic, praxeological reference models, hypothetical teacher tasks.

**Introduction**

What is teachers’ mathematical knowledge? How can we measure this knowledge? These two questions have become essential for many researchers. Some studies have been conducted to provide answers to both questions, including widely cited surveys on Mathematical Knowledge for Teaching (MKT) (Hill, Schilling & Ball, 2004) and Teacher Education and Development Study in Mathematics ([TEDS-M]; Tatto et al., 2008). MKT, as well as TEDS-M, envisages teachers’ mathematical knowledge as individual knowledge composed of mathematical content knowledge and pedagogical content knowledge (Shulman, 1986) that are studied through large-scale surveys. Both approaches aim to give a general picture of teachers’ mathematical knowledge. However, they do not emphasise teachers’ mathematical knowledge and didactical knowledge, knowledge for teaching a particular mathematics topic, as a collective construction which teachers may develop together when faced with a difficult challenge in a learning and teaching situation.

Teachers’ professional knowledge, both mathematical and didactical knowledge, is a major factor to explain learning and teaching phenomena (Chevallard, 2007). We can not investigate the complexity of teachers’ mathematical knowledge only by giving multiple choice questions and answers since teachers’ mathematical knowledge relates to teaching mathematics to pupils. Moreover, teachers frequently talk and share their mathematical knowledge to others to construct appropriate didactical techniques for teaching mathematics. Therefore, the aim of this study is to develop a method to study teachers’ mathematical and didactical knowledge focusing both teachers’ individual and shared knowledge, specifically how teachers carry their individual mathematical knowledge to a discussion for constructing relevant didactical knowledge to teach mathematics. We use *hypothetical teacher tasks* (HTT), designed concisely and recognisably based on a teaching situation which could appear at schools and which request teachers to employ their potential mathematical and didactical knowledge to act appropriately (Durand-Guerrier, Winsløw & Yoshida, 2010; Winsløw & Durand-Guerrier, 2007). We can say that an HTT includes two types of task, mathematical and didactical, that can be analysed based on the *anthropological*...
theory of the didactic (ATD), specifically praxeology (Chevallard, 1992, 2006). More details are given in the section of “method”.

Considering a broader area of mathematical knowledge learned by pupils and taught by teachers, we focus our study on a particular mathematics topic, namely rational numbers. We choose this topic since it is central within the development of pupils’ knowledge of numbers. Pupils often struggle to accept different properties of rational numbers. They tend to inappropriately apply natural number properties to rational number tasks (Van Dooren, Lehtinen & Verschaffel, 2015). In addition, many previous studies showed that pre-service and in-service teachers also struggle to explain this area to pupils (Depaepe et al., 2015; Ma, 1999). For instance, Ma (1999) showed that most U.S teachers found difficulties to explain how the algorithm of fraction division works through a contextual situation. Those phenomena motivate us to study teachers’ mathematical and didactical knowledge of rational numbers through a different setting, collaborative work, that challenges teachers to share their knowledge. Thus, the research questions of this paper are:

a. What can we learn from the use of HTTs to study pre-service elementary teachers’ mathematical and didactical knowledge of rational numbers?

b. What mathematical and didactical knowledge, including mathematical and didactical techniques about rational numbers, can be shared by pre-service elementary teachers (PsETs)?

We now give an overview of the paper. The next section is about relevant research on rational numbers and addresses how we apply them for the design of HTTs. The third section presents a theoretical framework focusing on how ATD, specifically praxeologies, models mathematical and didactical knowledge of rational numbers. In section 4, we present a methodological approach especially about the setting of the case study including the design of HTTs, participants, and procedures of data collection and data analysis. A detailed a priori analysis of HTTs is presented in section 5. We choose to present the detailed mathematical and didactical praxeologies of a task about multiplication and division of decimals. We assume that this full description of a priori analysis will show readers how the ATD, specifically praxeology, is applied to model teachers’ mathematical and didactical knowledge. In section 6, the praxeological reference model (PRM) is used for an a posteriori analysis of teachers’ mathematical and didactical knowledge of rational numbers. Finally, in section 7, more general perspectives are discussed as an approach to study teachers’ mathematical and didactical knowledge, and the characteristics of mathematical and didactical knowledge shared by pre-service elementary teachers.
Didactical research on rational numbers

Many studies in mathematics education show, in some way, that the transition process of pupils’ knowledge of natural numbers to positive rational numbers is a challenge (Brousseau, 1997; Ni & Zhou, 2005; Van Dooren et al., 2015). The features and behaviour of rational numbers are different from natural numbers. Some problems, such as \( a \times 3 = 2 \), have no solutions in the set of natural numbers because no natural number can satisfy that equation (Brousseau, 1997, p.151). It is necessary to build the rational numbers, an extension of the natural number set, to ensure that this and similar equations have a solution.

The first challenge is that rational numbers can be represented in several distinct systems of representation (fractions, decimals, percentages, diagrams, …; cf. Duval (2006) for a general theory of representations in mathematics). Pupils and sometimes even teachers may fail to realise that symbols such as 0.5, 0.50, 50%, \( \frac{2}{4} \), \( \frac{5}{10} \), … are just different representations of the same rational number (Vamvakoussi, Van Dooren & Verschaffel, 2012). Fractions present particular challenges: some pupils view fractions as two independent natural numbers (Depaepe et al., 2015; Liu, Xin & Li, 2011), or otherwise have difficulties to accept a fraction as a genuine number in itself.

Rational numbers are commonly represented as fractions, which are related to several fundamental mathematical ideas, such as measurement, linear mappings, and ratio (Brousseau, Brousseau & Warfield, 2007). In term of measurement, the classical definition of fraction \( \frac{n}{m} \) is that to measure \( \frac{n}{m} \) of a quantity, one divides the unit in \( m \) equal parts and take \( n \) parts. But, this definition has the limitation in providing some practical solution to a measurement problem if one can not produce the subdivision of unit needed for the quantity being measured (Brousseau et al., 2007). An alternative definition of fraction \( \frac{n}{m} \) which is sometimes useful is that to measure \( \frac{n}{m} \) of a quantity using a unit, the number of \( m \) copies of the unit coincides with \( n \) copies of the quantity. This definition is, for instance, useful in the situation of paper thickness (for more details, see Brousseau, Brousseau & Warfield, 2004). Fractions as linear mappings mean that a fraction \( \frac{n}{m} \) can be interpreted as a linear function \( x \mapsto \frac{n}{m} x \). For example, Brousseau et al., (2007) provide an example of the mapping \( G_{kg/ltr} \) as the linear mapping that maps the value 8 ltr to the value 7 kg. The image of the unit 1 ltr is \( \frac{7}{8} \) kg. Fractions as ratios mean that a fraction \( \frac{n}{m} \) defines as a comparison between two quantities, involving natural or rational numbers, with or without unit.

Fractions also have a quotient interpretation, as the result of a division (Charalambos & Pitta-Pantazi, 2007). This can be used to develop the theoretical relation between fractions and
decimals. In practice to learn decimals, one does not need to learn fractions first (Brousseau et al., 2007). However, decimals help one to replace fraction operations with operations that resemble those known from the natural numbers, and the decimals are also very practical to be used for measurement problems. On the other hand, one should be aware about the place value when working with decimals, because it is used to solve problems related to order and arithmetic operations.

Rational numbers (and hence decimals) differ essentially from natural numbers both when it comes to ordering and topological properties (Brousseau et al., 2007). The set of all rational numbers is countable, and the rational numbers can be ordered, but they are densely ordered. This means that between two given rational numbers, one can always find another rational number, therefore there are infinitely many rational numbers between the two rational numbers. Vamvakoussi and Vosniadou (2004) argue that pupils tend to provide inconsistent answers when given a question about how many numbers one can find between two rational numbers. Arithmetic mean is one technique to help pupils think about this problem. Brousseau et al., (2007) introduce a game of trap a fraction in an interval in which the pupils could develop the idea that in each interval there are many fractions.

Another step of learning about rational numbers is to do the basic arithmetic operations. These also provide many challenges for pupils and teachers. New “calculation methods” have to be introduced, and they depend heavily on representations (fractions, decimals, or percentages). Especially if students are taught these methods as meaningless procedures, classical mistakes occur. In the case of addition and subtraction of fractions, for instance, pupils sometimes add or subtract the numerators and the denominators separately. For instance, in the situation of paper thickness (Brousseau et al., 2004), the teacher asks pupils to stick together a sheet with thickness $\frac{10}{50}$ and a sheet with thickness $\frac{40}{100}$. Then, she asks the pupils to figure out the thickness of the result, and most of them agree the result is $\frac{50}{150}$. In the situation, the correct operation emerges from “reducing to the same number of sheets”.

Multiplication and division of rational numbers also cause trouble, for instance, to accept that multiplication of some number A by a rational numbers may give something smaller than A, and that division by a rational number may give a greater number. Pupils and even adults often overgeneralize from natural numbers to rational numbers, such as the meaning of multiplication as repeated addition and division as equal sharing. All of these difficulties also represent challenges for teachers in teaching multiplication and division of rational numbers. To counteract these problems, Brousseau et al., (2004, 2007, 2008, 2009) propose situations where multiplication and
division by rational numbers appear in activities related to measurement and linear mappings. For the case of a product of a rational number and a whole number, they propose an activity to the pupils to find the thickness of several sheets of a type of papers, starting from 3 sheets, then 5, 20, 100, and 120. Meanwhile, to introduce the division of a rational number by a whole number, the pupils are asked to calculate the thickness of one sheet. This activity is challenging for them because they only know the division as defined between two whole numbers, but they succeed to find the answer because this activity is meaningful for them.

The second idea of teaching multiplication and division of rational numbers is based on the linear mappings or functions (Brousseau, 1997). This is introduced through an activity of enlargement and reduction of puzzles. Through this activity, the pupils learn how to find the mappings of fractions and decimals, and it is because there is no restricted of numbers involved in the model to be reproduced. On the other way around, if the pupils are asked to find the image of the reproduction, they come to the concept of division as reciprocal mapping of multiplication. In fact, the inverse property of multiplication has been learned informally. However, it is not the case for division of decimal numbers. The technique used in the division of decimal numbers are facilitated by its similarity to long division in the natural numbers (Brousseau et al., 2009). This algorithm is based on the measuring something using something smaller as the unit. If the task is to divide the divisor is greater than the dividend, or if the dividend is less than two, the pupils will have difficulties to get the meaning and to do the operation.

To sum up, the HTTs are designed based on previous didactical research on specific obstacles (or "misconceptions") which pupils commonly encounter in their work with rational numbers. Moreover, the HTTs concern different areas of school mathematical work with rational numbers, which we will now explain briefly; the areas are linked as shown (by arrows) in figure 1.

![Figure 1](image.png)

**Figure 1.** Main areas of rational numbers for designing of teachers’ tasks.
Rational numbers (\(\mathbb{Q}\)) appear in school mathematics in several different representations (such as decimals or fractions). The use of representations is a crucial challenge and condition for all work with rational numbers. It requires that pupils learn that different representations can “stand for” the same number (equivalence), e.g., \(\frac{3}{4} \); 0.75; \(\frac{9}{12}\). Other main tasks for pupils involve the order rational numbers, such as comparing to given numbers, find numbers in between them, etc. (and ultimately density of rational numbers), and in order to master the arithmetic operations. In fact, much school work with numbers concerns calculations (arithmetic) and the techniques to do the four operations depend crucially on the representation used. Finally, the techniques associated with the complete order on \(\mathbb{Q}\) also depend on the representation used, and the relation between order and arithmetic operations is different from what the pupils know about integers (there are no unique “successors” in \(\mathbb{Q}\), as one symptom of its denseness). These main challenges in the school mathematical treatment of \(\mathbb{Q}\) have been illustrated in figure 1, and the HTTs designed for this projects aim to examine PsETs’ mathematical and didactical knowledge in all five areas (yellow bottom boxes) in figure 1.

**Theoretical framework: ATD through the notion of praxeologies**

The anthropological theory of the didactic (ATD) is a research programme in mathematics education that was introduced by Yves Chevallard in the 1980s (Bosch & Gascón, 2006). The central idea of ATD is to model a human action using a basic unit known as a praxeology (Bosch & Gascón, 2006, 2014; Chevallard, 1992, 2006, 2007; Winsløw, 2015; Winsløw, Barquero, De Vleeschouwer & Hardy, 2014). According to Chevallard, (2006, p. 23)

> A praxeology is, in some way, the basic unit into which one can analyse human action at large. […] What exactly is a praxeology? We can rely on etymology to guide us here – one can analyse any human doing into two main, interrelated components: praxis, i.e. the practical part, on the one hand, and logos, on the other hand. “Logos” is a Greek word which, from pre-Socratic times, has been used steadily to refer to human thinking and reasoning – particularly about the cosmos. Let me represent the “praxis” or practical part by P, and the “logos” or noetic or intellectual part by L, so that a praxeology can be represented by [P/L]. How are P and L interrelated within the praxeology [P/L], and how do they affect one another? The answer draws on one fundamental principle of ATD – the anthropological theory of the didactic –, according to which no human action can exist without being, at least partially, “explained”, made “intelligible”, “justified”, “accounted for”, in whatever style of “reasoning” such an explanation or justification may be cast.
The practical block is made of a type of task \((T)\) and corresponding techniques \((τ)\) to solve the tasks of a given type. In most cases, we can easily observe the practical block of human activities because it directly relates to an action showed by someone to handle a task. The theoretical block is made of a technology \((θ)\) and a theory \((Θ)\). The technology functions to justify the techniques, and the theory constitutes to justify the technology. In many cases, the theoretical block is rather difficult to be observed especially in a case of learning mathematics focusing on finding a correct answer for a certain type of task. Someone in this situation focuses more to the practical than the theoretical level of praxeology.

The praxeologies often occur in larger systems that share some of the same explicit elements i.e. knowledge blocks (Artigue & Winsløw, 2010). A unique type of task and some techniques to deal with the task are categorised as a punctual organisation (praxeology). A technology generating some techniques is called as a local organisation. A regional organisation is produced by coordinating and integrating several local organisations in a common theory, and a global organisation is a link of some regional organisations. For example, the arithmetic of rational numbers is a global organisation in elementary school mathematics, but what is learned and taught can be just a loose collection of punctual or local organisations, such as adding and subtracting two fractions.

To study teachers’ mathematical knowledge is not only about mathematical praxeologies but also about didactical praxeologies. An example of a didactical task which corresponds directly to the mathematical task of adding two fractions, for instance, is to help pupils understand how the standard technique “works”. A teacher may have several didactical techniques to handle this task. It is also related to how a teacher describes, justifies, and organises the teaching and learning in the considered institution (Barbé, Bosch, Espinoza & Gascón, 2005; Rodríguez, Bosch & Gascón, 2008).

The mathematical praxeology actually developed by pupils depends crucially on the didactical praxeology developed by teachers (Winsløw, 2015). It means that a teacher who has comprehensive mathematical and didactical knowledge will help pupils to develop a more flexible praxeology, specifically techniques and technologies. For example, when a teacher finds many pupils get difficulties or often make mistakes to solve a mathematical task, such as adding two fractions, the teacher probably tries to propose other didactical techniques or also technologies to handle this situation.

In this study, we use praxeologies as an analytical tool to analyse PsETs’ mathematical and didactical knowledge of rational numbers. First, we describe some possible mathematical and didactical techniques for the HTTs. Potential logos (technologies and also elements of theory) to
justify the techniques are also presented. The result of this *a priori* analysis is called *praxeological reference models* (PRM). The models are then used to analyse PsETs’ individual and collective work on the HTTs.

**Methods**

As we have introduced earlier that to study teachers’ mathematical and didactical knowledge we design HTTs. A HTT consists of mathematical and didactical tasks. The mathematical task is a task commonly given to pupils at schools and presented in a situation where pupils struggle to find a correct answer or have some misconceptions. The teachers have to analyse this situation and provide some mathematical techniques individually, and then they bring their ideas to the discussion on the corresponding didactical task. So, the didactical task is on how teachers handle the situation and then suggest some didactical techniques for further pupils’ learning. Generally, the mathematical task strongly links to the didactical one.

The five HTTs on rational numbers designed in this study pertain to the mathematical components of knowledge about rational numbers which are based on the didactical research on rational numbers summarised in figure 1. The first HTT is about equivalent fractions. HTT 2 and HTT 3 are respectively about comparing decimals (Putra, 2017) and denseness of rational numbers (Putra, 2016). HTT 4 is about addition and subtraction of fractions, and HTT 5 is about multiplication and division of decimals. For each HTT, it can be analysed regarding mathematical and didactical praxeologies that are used to classify different responses to the tasks.

Five HTTs were tested to 32 fourth year pre-service elementary teachers (PsETs) (approximately 21 year olds) from the teacher education for elementary school study programme from a middle ranked University in Indonesia. They had already completed all courses in mathematics and mathematics education set by the programme. The courses are fundamental mathematics, mathematics education for the lower grades (grade 1 to 3), mathematics education for the upper grades (grade 4 to 6), and designing and developing mathematics teaching and learning. They had also completed 6-month internships at public elementary schools for teaching at least four subjects; mathematics, science, social science, and Indonesian.

The HTTs were originally written in English and subsequently translated into Indonesian by the researcher. To ensure the consistency of HTTs, two Indonesian mathematics education researchers checked the translations. They gave some suggestions related to the wording used in HTTs, for instance, the word of claim is translated into “*menyatakan*” instead of “*mengklaim*” because “*menyatakan*” is more common for Indonesian. The HTTs were also piloted with a pair of recent graduated students from the teacher education programme. They gave some comments.
related to the difficulties of interpreting the tasks, but they agreed that the HTTs related to what they had learnt at the study programme.

The data consist of PsETs’ written answers mostly for the mathematical tasks and video recordings from discussion carried out in pairs of PsETs for the didactical task. For the video recordings, we first wrote transcripts for all groups using the software NVivo computer program. Then, the written answers and video transcripts were coded based on PRM. During the coding process, we also discussed the data (especially questionable points) with two other mathematics education researchers.

**A priori analysis: The case of multiplication and division of decimals**

We now present the HTT about multiplication and division of decimals (HTT 5) in details in order to give readers a detailed insight in how HTTs are constructed and analysed a priori. We choose to present the detailed a priori analysis of this HTT because the multiplication and division task of decimals or rational numbers is known as one of challenging topics for teachers to explain. For instance, most teachers know how to multiply two decimals using the standard multiplication algorithm and know how to explain it, but when pupils ask the teachers what the result means, for instance how a product can be smaller than both factors, unlike in case of multiplication of two natural numbers, teachers may be more challenged. In addition, the multiplication and division tasks are often the last topic taught in elementary school (grade 6). This means that the teachers need to coordinate various knowledge of rational numbers in order to deal with this kind of question.

The a priori analysis does not only focus on the analysis of a type of task and techniques but also some possible technologies and the theory, given to PsETs in a hypothetical situation of teaching in the context of elementary school. The similar method is applied to design and analyse other HTTs (see also the appendix).

The HTT about multiplication and division of decimals is adapted from TEDS-M studies (Tatto et al., 2008). The setting of the task is modified in which PsETs have to discuss in pairs how they handle pupils’ misconception about multiplication and division of rational numbers. This situation gives a chance for them to share both mathematical and didactical knowledge. The HTT given to PsETs is described as follows:
As a teacher, you ask pupils to compute the following as homework:

a) $0.25 \times 8 = \cdots$, b) $8 \div 0.25 = \cdots$.

At the next meeting in the class, a pupil notices that when he enters $0.25 \times 8$ into a calculator, the answer is smaller than 8, and when he enters $8 \div 0.25$, the answer is greater than 8. He is confused with this answer and thinks that the calculator must be broken.

What can you do to help such pupils understand this result? (discuss in pairs for 8 minutes, use the space below if necessary, and write your ideas to support the discussion)

Figure 2. - HTT about Multiplication and division of decimals.

The mathematical tasks given to the pupils in this HTT belong to two types of task, multiplication and division of two given decimals. Some possible mathematical techniques to solve these types of task are described in the following table:

<table>
<thead>
<tr>
<th>$T_m$</th>
<th>$T_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>given a decimal number $a$ and an integer $b$, calculate $a \times b$.</td>
<td>given a decimal number $a$ and an integer $b$, calculate $b \div a$.</td>
</tr>
<tr>
<td>$\tau_{m1}$</td>
<td>$\tau_{d1}$</td>
</tr>
<tr>
<td>the standard multiplication algorithm for decimals.</td>
<td>the standard division algorithm for decimals.</td>
</tr>
<tr>
<td>$\tau_{m2}$</td>
<td>$\tau_{d2}$</td>
</tr>
<tr>
<td>change decimals into fractions and then apply the standard algorithm for multiplication of fractions.</td>
<td>change decimals into fractions and then apply standard algorithm for division of fractions.</td>
</tr>
<tr>
<td>$\tau_{m3}$</td>
<td>$\tau_{d3}$</td>
</tr>
<tr>
<td>first find an integer $c$ such that $c = na$, if $c \times b = m$, then $a \times b = \frac{m}{n}$.</td>
<td>first find an integer $c$ such that $c = na$, if $b \div c = m$, then $b \div a = mn$.</td>
</tr>
<tr>
<td>$\tau_{m4}$</td>
<td>$\tau_{d4}$</td>
</tr>
<tr>
<td>$a \times b = \frac{b + \cdots + b}{a \text{ times}}$</td>
<td>if $b \div a = b - a - \cdots - a$, then $a \div b = n$.</td>
</tr>
</tbody>
</table>

Table 1. - Mathematical praxis to multiplication and division of decimals

From the mathematical technique $\tau_{d1}$, it is possible to derive a mathematical technique involving the division of integers in order to do more flexible calculations. This technique can be written as $\tau_{d11}$: convert $a$ and $b$ into integers by multiplying each number by an appropriate power of ten, and then do a division algorithm for integers; in the end, divide by the powers of ten used first.

Technologies to explain those techniques can be different from one another, but one technology may justify several techniques. The first technology to justify $\tau_{m1}$ and $\tau_{d1}$ is related to the link between the numeration system of decimal and whole numbers ($\theta_1$). The order of the decimal units follows the same order as the power of 10.

The second technology is based on deliberate choices among equivalent representations (decimals, fractions). For example, $0.25$ is equivalent to $\frac{25}{100}$ or $\frac{1}{4}$, so the multiplication task can be rewritten as $\frac{25}{100} \times 8$ or $\frac{1}{4} \times 8$. This kind of technology can be used to justify the mathematical
techniques $\tau_{m2}$ and $\tau_{d2}$. The use of equivalent representations is also used to justify the mathematical technique $\tau_{d11}$. In general, this technology can be written as:

$\theta_2$: explanations based on the fact that decimals and fractions can represent the same rational numbers.

Meanwhile, the technology to justify the techniques $\tau_{m3}$ and $\tau_{d3}$ can be based on ratio and proportion ($\theta_3$). The technology is little bit different for multiplication and division: in the division task, the technology is based on the inverse proportion between the operator and the result. When the divisor becomes bigger, the quotient becomes smaller. For the last techniques, $\tau_{m4}$ and $\tau_{d4}$, the technology ($\theta_4$) is drawn from the basic properties of multiplication and division of two integers, e.g., multiplication as repeated addition and division as repeated subtraction.

Finally, a possible theory to justify those technologies is a fundamental law of multiplication and division of rational numbers.

$\Theta_{md}$: if $x$ and $y$ are any rational numbers, then $x \div y = z$ if and only if $z$ is a unique rational number such that $z \times y = x$. (Billstein et al., 2007, pp.323; Sonnabend, 1997, pp.292).

This theory could coordinate and integrate several local praxeologies, e.g. including multiplication and division of fractions.

The HTT about multiplication and division with decimals also includes didactical tasks:

$T_{m^*}$: help pupils understand why $a \times b$ gives a smaller result than $b$, and $T_{d^*}$: help pupils understand why $b \div a$ gives a greater result than $b$.

Many possible didactical techniques can be applied to solve this kind of task. The most common didactical techniques are direct instruction (with demonstration) of the mathematical techniques proposed to solve the tasks of type $T_m$ and $T_d$. For instance, a PsET may show how to apply the mathematical technique $\tau_{m1}$ to solve one or more tasks of type $T_m$, so the pupils can see the result is smaller than $b$. So, the direct instruction gives one didactical technique for each mathematical technique. For a mathematical technique $\tau_x$, this didactical technique is coded as $\tau_{x^*}$. Then, as shown in table 1 we can associate four didactical techniques for $T_{m^*}$ and five didactical techniques for $T_{d^*}$.

There are also other didactical techniques that could be related to specific mathematical techniques. The first and most common one is to situate the mathematical problem in a contextual or real life situation.

$\tau_{m5^*}$: present the multiplication task embedded into appropriate contextual problem/real life situations that the pupils can experience. ($\tau_{d5^*}$ is the didactical technique for the division task).
In relation to the didactical technique $\tau_{m5}^*$, a teacher, for instance, proposes a situation such as how to find an area of a rectangle within 0.25 m wide and 8 m long. The teacher asks the pupils to draw a rectangle diagram $1 \text{ m} \times 8 \text{ m}$, and then shade the part with an area of $0.25 \text{ m} \times 8 \text{ m}$. So, they can see the relation between $0.25 \text{ m} \times 8 \text{ m}$ and $1 \text{ m} \times 8 \text{ m}$: the shading area is a quarter of the total area. Meanwhile, the didactical idea behind $\tau_{d5}^*$ is based on measurement or repeated subtraction. For instance, a teacher may present a situation such as if you have 8 m of ribbon, how many 0.25 m long pieces of ribbon can you make? From this case, the teacher can use a number line representation. Another example is to use pizza diagrams to show how many quarters of a pizza you have in 8 pizzas.

The next possible didactical technique is described as follows:

$\tau_{m6}^*$: provide pupils with a simpler problem. e.g. $0.5 \times 2$, let them link to the task given, and realise why the result is smaller than that. ($\tau_{d6}^*$ is the didactical technique for the division task)

PsETs may also propose a general didactical technique based on the pedagogical perspective on teaching as a social activity. It leads to ideas such as to organise a classroom discussion, let some pupils present their mathematical techniques, and let students discuss which are better ($\tau_{m7}^*$ and $\tau_{d7}^*$). If some of them are still not able to follow, the teacher could explain it and provide alternative technological discourse.

Some PsETs may outline one of the didactical techniques described above, but without giving detailed mathematical explanations related to that technique. For instance, a pair of PsETs suggests to use the didactical technique $\tau_{d2}^*$, but none of them present any details relating to the mathematical technique $\tau_{d2}$. We denote such a didactical technique by adding $x$, to indicate somewhat incomplete (in some cases even mistaken) didactical technique. Besides, some PsETs may not be able to propose any didactical technique to the didactical task given, and this didactical technique is described as follows:

$\tau_{mx}^*$: explain a mathematical technique that is not directly related to the mathematical task of type $T_m$ or do not present and discuss any mathematical techniques related to the mathematical task of type $T_m$. ($\tau_{dx}^*$ is the didactical technique for the division task)

To each didactical technique, there is a didactical technology to explain and justify it. Some didactical techniques could be explained by one didactical technology. For example, PsETs may argue that they instruct the pupils using the mathematical technique $\tau_{m1}$ and $\tau_{d1}$ because it is how the mathematical textbook they use presents it. According to previous studies (e.g., Wijayanti & Winsløw, 2017) Indonesian mathematics textbooks usually focus on one standard algorithm, demonstrated through many examples and exercises. This could lead to a technological discourse to justify other didactical techniques, such as $\tau_{m2}^*$ and $\tau_{d2}^*$.
Another possible didactical technology to justify $\tau_{m3}^*$ and $\tau_{d3}^*$ is that one can build pupils’ knowledge about multiplication and division of rational numbers from what they have already known about natural numbers. To teach, for instance $0.25 \times 8$, one can start from explaining $25 \times 8$, and then the pupils can see the relation between the two tasks. Meanwhile, a possible didactical technology to justify the didactical techniques such as $\tau_{m5}^*$ and $\tau_{d5}^*$ could be based on the teachers’ pedagogical perspective: teaching mathematics in elementary school should be based on concrete examples or contextual situations. Therefore, there are many possible didactical technologies, also to justify $\tau_{m7}^*$ and $\tau_{d7}^*$ based on an assumption that collaborative work could accelerate pupils learning of mathematics because they share their mathematical praxeologies. We do not list all of the potential didactical technologies or theories like we did for the didactical techniques. In practice PsETs may not propose any coherent technological-theoretical discourse to the given didactical technique, and this can be seen as incomplete praxeologies.

A posteriori analysis of mathematical and didactical praxeologies

The a posteriori analysis is presented in two sections. The first section is a detailed analysis of data from two pairs of PsETs working on the HTT about multiplication and division of decimals. The aim of this presentation is to show how PRM functions to study teachers’ mathematical and didactical knowledge of rational numbers. The second section presents a general analysis of mathematical and didactical praxeologies shared by PsETs from those 5 HTTs.

1. Two case analyses of HTT about multiplication and division of decimals

We present a detailed analysis of how two pairs (S$_{2A}$ and S$_{2B}$; S$_{13A}$ and S$_{13B}$) of PsETs discuss their mathematical and didactical techniques for multiplication and division of decimals, to show how a praxeological analysis is done in this study.

The case of S$_{2A}$ and S$_{2B}$

S$_{2A}$ and S$_{2B}$ started reading the HTT, and they got surprised by the statement “when he enters $0.25 \times 8$ into a calculator, the answer is smaller than 8, and when he enters $8 \div 0.25$, the answer is greater than 8”. Therefore, S$_{2B}$ asked the researcher for permission to use a calculator to check the answers. After getting the answers, 2 for multiplication and 32 for division, they started the discussion in order to solve the didactical task in the HTT. The first didactical technique appeared from the discussion was $\tau_{m4}^*$ implicitly inferred from the following excerpts:

S$_{2A}$: Let’s try, for example, multiplication is repeated addition.
S$_{2B}$: Oh, yes. Multiplication is repeated addition, so it is 8 times 0.25; add 0.25, add 0.25 until 8 times.
$S_{2A}$: Yes.
$S_{2B}$: So, division is repeated subtraction, isn’t it? Why is the result 32?
$S_{2A}$: No, not like that. It is just for multiplication.

$S_{2A}$ and $S_{2B}$ agreed with repeated addition to explain the multiplication task, but not with repeated subtraction for the division task. They were unsure about the meaning of division as repeated subtraction because the result for the division task, $8 \div 0.25$, was greater than 8. It did not match to the meaning of division of natural numbers that they seemed to refer to.

After a while, $S_{2A}$ suggested to change decimals into fractions and continued the discussion as follows:

$S_{2A}$: Well, to help him understand this problem, how do you think if, for example, we change 0.25 into a fraction?
$S_{2B}$: Yes. It is because he directly types 0.25 [in the calculator], so he does not know exactly if he calculates manually, it can be written as a decimal or as a fraction.
$S_{2A}$: But, what happened?
$S_{2B}$: Yes. We can also say to him that it is not possible to display $\frac{25}{100}$ on the calculator. if it is without the calculator, [0.25] can be changed into a fraction to help him calculate easily.

As 0.25 has 2 digits after the decimal point, it means $\frac{25}{100}$ times 8.
$S_{2A}$: Yes. It can be simplified.
$S_{2B}$: Ok. It is simplified and becomes $\frac{1}{4}$ times 8 [and] equals to $\frac{8}{4}$. It means $8 \div 4$, and the result is 2.
$S_{2A}$: How do we explain it to the pupils?
$S_{2B}$: Just directly explain like this (using the standard procedure of multiplication of fractions).
$S_{2B}$: Use the concept of fraction. How is about the division? it is also divided by $\frac{25}{100}$.
$S_{2A}$: This means that 8 divided by this (0.25) is just changed into fractions. The didactical technique $\tau_{m2}^*$ for the multiplication task was clearly discussed and developed by both PsETs. The idea was to show pupils the mathematical technique to convert a decimal into a base-10 fraction and then simplify it into a proper fraction. A technological discourse behind this didactical technique can be based on equivalent representations between fractions and decimals ($\theta_2$) which the calculator could not show, for instance, to explain that 0.25 is equal to $\frac{25}{100}$. They explicitly presented the standard procedure for the multiplication of fractions but not for the division of fractions. They only mentioned that the division task with decimals could be rewritten into fractions, $S_{2A}$ simply wrote $8 \div \frac{1}{4}$ in their worksheet, but she did not complete the division
operation. In fact, we could not see whether they themselves were able to solve the division task using the mathematical technique $\tau_{d2}$ or not. The technological discourse to justify the didactical technique $\tau_{m2^*}$ mainly relegates the example to the algorithm for multiplication of fractions. This justification also seems to be what they might directly instruct to pupils, as an explanation of $\tau_{m2}$.

Then $S_{2A}$ explained another mathematical technique also for the multiplication task:

$S_{2A}$: Like this. For example, if a half of 8 is equal to 4, a quarter of 8 is equal to 2.

$S_{2B}$: A quarter of 8 is equal to 2.

$S_{2A}$: Yes, it is a quarter, isn’t it? 0.25

$S_{2B}$: Yes. Ok, that is our logic. So, how can we explain it to the pupils who do not understand it yet?

$S_{2A}$: Yes. We first explain it based on fractions.

In other words, $S_{2A}$ proposed an alternative mathematical technique, $\tau_{m3}$, for the multiplication task. She explained it through proportional reasoning and multiplication by a fixed fraction as an operator ($\theta_3$). This means that a quarter is considered an operator which decreases the multiplier. However, $S_{2B}$ thought this technique might be difficult to explain to pupils. So, $S_{2A}$ suggested using the didactical technique $\tau_{m2^*}$ before $\tau_{m3^*}$.

$S_{2A}$ and $S_{2B}$ did not speak much about didactical techniques related to the division task. They did not even present the standard algorithm for division of fractions. One can speculate that they could not apply $\tau_{d2}$ to the division, and have little idea on how to explain the result of that division to pupils. Meanwhile, they proposed some didactical techniques for the multiplication task, and implicitly provided some didactical technologies to justify them. For instance, they agreed to teach the pupils the mathematical technique $\tau_{m2}$ because the pupils did not realise the relevant equivalent representations (decimals and fractions) when they used calculators. In addition, the didactical techniques proposed really rely on what mathematical techniques the PsETs used themselves to solve the multiplication task, and they tend to automatically propose direct instruction of one of those techniques. If they did not explain why such a mathematical technique was correct (e.g., to teach division task using $\tau_{d4^*}$), their technology may simply be limited to instructions on how to use the technique.
The case of $S_{13A}$ and $S_{13B}$

$S_{13A}$ and $S_{13B}$ discussed more techniques for the division task than for the multiplication task. $S_{13A}$ started the discussion by asking $S_{13B}$’s idea.

$S_{13A}$: Okay. How do you think? 0.25 means…

$S_{13B}$: Wait a minute. 0.25 is multiplied by 8. 0.25 is less than 1, isn’t it?

$S_{13A}$: Hmm.

$S_{13B}$: Indeed, the result is less than 8.

$S_{13A}$: So, we explain a relation between decimals and fractions, don’t we? Then,

$S_{13B}$: What [do you mean] fractions?

$S_{13A}$: No. 0.25 can be written as a fraction or as a decimal. Later 0.25 can be written into $\frac{1}{4}$, can’t it? 0.25 is equal to $\frac{1}{4}$.

Both PsETs had different perspectives about how they handled the task. $S_{13B}$ described based on a technological discourse for multiplication task of decimals. She gave an argument corresponding to a general mathematical technology, multiplication of a positive integer, for instance $n$ and a decimal less than 1, gives the result is less than $n$. Meanwhile, $S_{13A}$ focused on converting decimals into fractions. $S_{13A}$ gave more attention to the representations of rational numbers than directly solve the decimal tasks.

The discussion was continued, and it was dominated by the attempt of $S_{13A}$ to explain $\tau_{m2}$.

$S_{13A}$: 0.25 is less than 1, and it is equal to a quarter. If 1 is divided by 4, it is equal to 0.25. After that, it is multiplied [by 8] like this example (He writes $\frac{1}{4} \times \frac{8}{3}$ in his worksheet). So, there is a relation between decimals and fractions since both representations have the same value, $\frac{1}{4}$ is equal to 0.25. Because they have the same value, indeed, we can use this fraction to the multiplication task of decimals. So, one multiplied by 8 is equal to 8, and per 4 (He continued solving the task $\frac{1}{4} \times \frac{8}{1} = \frac{8}{4}$).

$S_{13B}$: Hmm

$S_{13A}$: So, 8 divided by 4 is equal to

$S_{13B}$: 2.

$S_{13A}$ emphasised that 0.25 and $\frac{1}{4}$ have the same value or size, and after the change, one can apply the standard procedure of fraction multiplication. The multiplication of fractions also gives the same result as the multiplication of decimals. In fact, we can say that he justified the mathematical technique $\tau_{m2}$ using $\theta_2$ (equivalent fractions and decimals). He continued to propose a similar idea for the division task:
Meanwhile, for the division [task], we explain what is 8 divided by 0.25. We have already known and explained the relation between decimals and fractions; 0.25 is equal to a quarter. If the task is division of fractions, the denominator [of the divider] is moved to the top, so it becomes the multiplier. [The task] becomes like this. (He starts to write the notation of the division task of decimals into the division task of fractions).

...  

So, $\frac{8}{1} \div \frac{1}{4} = \frac{8}{1} \times \frac{4}{1}$, isn’t it? 8 multiplied by 4 is equal to.

S13B: 32.

S13A presented the division algorithm for fractions without explanation (or other technological elements). Meanwhile, S13B had a different idea how to handle this task:

S13B: I think like this, because this is eaaa 0… For the division task, it might be true that the result is more than [8]. 8 is divided by 0.25. This means, from 8. Assuming there are, for example, 8 pieces of cakes (She draws 8 circles). 4, 5, 7, and 8.

S13A: Hmm.

S13B: 0.25 is equal to a quarter. This means 4, 4, 4, 4, 4, 4, 4, and 4 (She divides each circle into 4 pieces). The total is 32.

S13A: 1, 2, 3, 4, 5, 6, and so on (He points and counts each slice of circles). Isn’t it?

S13B: Yes. Then, why the result of $0.25 \times 8$ is less [than 8], because 0.25 is less than 1. How to explain it. If it is multiplied by 1, the result is 8, and it is much greater than this (she points the multiplication task of decimals). [It is] only 0 comma. How to explain it.

S13A: For the concept of division, it is better [to do] like these (He points the circle representations), making drawings like these. However, the multiplication task cannot be drawn like these because it is better to convert decimals into fractions, isn’t it?

S13B tried to link the division task of decimals to diagram representations. She applied the context of dividing cakes (how many quarters in 8 cakes). Through those representations, it can be implicitly inferred that she presented the didactical technique $\tau_{df}^*$ for the division task. However, she did not provide an interpretation of the result; 32 as the total number of quarters of a cake, contained in eight whole cakes, and in fact she could not link this idea for the multiplication task. She was not aware about the relation between the division and the multiplication situation. Instead she tried to explain the mathematical technique $\tau_{m3}$ by proportional reasoning for the multiplicative task, but she was little bit doubtful how to present it to the pupils. On the other hand,
S\textsubscript{13B} supported S\textsubscript{13A}'s mathematical techniques for the division task, but he preferred to explain through fraction representations.

In summary, S\textsubscript{13A} and S\textsubscript{13B} did not entirely agree with each other about the didactical techniques for the multiplication task of type T\textsubscript{m}\textasciitilde and the division task of type T\textsubscript{d}\textasciitilde; their discussion show their collective mathematical and didactical knowledge may advance through the sharing of technology, such as the difficulty of using diagrams to represent multiplication of rational numbers as an argument to pass to equivalent (fraction) representations.

2. Analysis PsETs' mathematical and didactical praxeologies

We now present the mathematical praxeologies given to PsETs during the individual and collaborative work, and then how they are linked to the didactical praxeologies. This analysis is based on the PRM presented in the appendix and applied to the first four HTTs. We also outline the simultaneous analysis of mathematical and didactical praxeologies to HTT 5

Analysis of HTT 1: Equivalent fractions

Twenty-two PsETs gave their judgments to the pupil’s claim, \( \frac{3}{4} = \frac{8}{9} \). Sixteen of them clearly realised that was a wrong answer, and the other tended to state it as somehow improper. Their answers were mostly followed by writing a mathematical technique to find the equal fractions (T\textsubscript{1}). Multiplying nominators and denominators by the same positive integer (\( \tau\textsubscript{11} \)) was suggested by 17 PsETs. In addition, four PsETs presented a mathematical technique to show \( \frac{3}{4} \) and \( \frac{8}{9} \) were not equal (T\textsubscript{1}'). Three of them changed both fractions into decimals by the division algorithm. Meanwhile, one PsET interpreted the mathematical task T\textsubscript{1} as \( \frac{3}{4} + 5 \) and suggested the mathematical technique for addition of fractions.

The frequencies of didactical techniques shared by PsETs are showed in figure 3. We observed that only seven pairs initiated an explicit discussion of didactical techniques while the rest mainly discussed how they themselves could solve the mathematical task; they are then noted for the corresponding "direct instruction" techniques. Four out of 12 pairs who suggested the didactical technique \( \tau\textsubscript{11} \) showed that \( \frac{3}{4} \) and \( \frac{6}{8} \) are equal by changing them into 0.75. They might use this as a justification to show to pupils that the algorithm of equivalent fractions is correct. Five pairs suggested to use a real life or contextual situation, as they focused on showing that \( \frac{3}{4} = \frac{6}{8} \).

They provided pupils with a situation of sharing cakes, and then modelled it into rectangle or circle diagrams (\( \tau\textsubscript{13} \)). For instance, a pair, G\textsubscript{9}, first drew a rectangle, divided it into four equal parts, and shaded three of them. Then, they drew another similar diagram, divided it into 8 equal parts, and
shaded 6 parts. So, they argued that the pupils could see from these drawings that the two fractions, \( \frac{3}{4} \) and \( \frac{6}{8} \), were the same. Meanwhile, a pair, G4, tried to use rectangle diagrams to show that \( \frac{3}{4} \) and \( \frac{8}{9} \) were not the same, but they came to an incorrect representation for \( \frac{8}{9} \) (Figure 4). It seems that they were not aware that the size of each unit on the diagram should be the same.

![Figure 3. - Didactical techniques for HTT 1](image)

![Figure 4. - A rectangle representation for \( \frac{8}{9} \)](image)

We also observed how PsETs support their mathematical and didactical techniques (i.e., corresponding mathematical and didactical technology). To support the didactical technique \( \tau_{11^*} \), S_{11B} from G11, for instance, reasoned that it would likely be consistent with the pupils’ textbook. She reflected on her experiences during a practicum at an elementary school. S_{9B} from G9 said that tasks like finding an equal fraction of \( \frac{3}{4} \) are difficult for pupils. She suggested to use something that pupils can recognise, like cakes, in order to ground the early concept of equal fractions. Meanwhile, PsETs from G11 produced a technological explanation for pupils’ misconception: the pupils were confused between adding or multiplying the numerator and denominator by the same positive integer.
**Analysis of HTT 2: Comparing decimals**

Most PsETs suggested to add 0s to equalise the number of digits after the decimal point (τ₃₃). This mathematical technique was presented by 10 PsETs. The other common mathematical technique was to change decimals into fractions (τ₃₄), but five out of 11 PsETs could not change 0.45 into a fraction.

During the discussion, PsETs suggested more diverse didactical techniques for HTT 2 on comparing decimals than for any of the other HTTs (Figure 5). The most common didactical technique for τ₂ was τ₂₃. PsETs explained that 0.5 could be represented with two decimal digits, as 0.50, and then pupils could easily compare to 0.45. Other common didactical techniques employed number lines (τ₂₇ and τ₂₇ₓ), to provide contextual or real life situations such as to compare two parts of a cake, (τ₂₆ and τ₂₆ₓ), and to change decimals into fractions (τ₂₄ and τ₂₄ₓ). However, ten pairs presented wrong or incomplete didactical techniques. For instance, three pairs suggested pupils to change decimals into fractions, but did not themselves know how to do that for 0.45 (τ₂₄ₓ). A PsET, S₃ₐ, from G3 said

“I am confused. I change them into fractions. From fractions, they can be represented into rectangle diagrams, so we can see them. For instance, we know that 0.5 is equal to a half. If this is 0.45, what fraction is it? Later, it is drawn. From the drawing, pupils can compare, to see which one is greater” (Putra, 2017).

S₃ₐ knew that she could show to pupils which decimal was greater by changing both into fractions, but we observed that neither her nor her partner could actually do that. Another interesting finding is that five pairs suggested to explain to pupils how to change decimals into percentages, but three of them were in fact not able to do so correctly. For example, one PsET, S₁₀ₐ, presented to his partner the mathematical technique of changing decimals into fractions. He changed 0.5 into 500%, and none of them realised the mistake.

**Figure 5.** - Didactical techniques for HTT 2
As for the theoretical block of the mathematical praxeologies produced, four pairs mentioned that 0 digits at the end of a decimal were rarely written, but one could always add them without changing the value of the number represented. Another mathematical discourse was about equivalence classes of rational numbers. They observed that 0.5 could be rewritten as a two-digits decimal, since five-tenths is equal to fifty-hundreds. For instance, \( S_{7B} \) argued that a teacher had to explain first that five-tenths is equivalent to fifty-hundreds. In addition, \( S_{1B} \) from G1 explained that we seldom find, for instance, 0.50 written as such in a computer program and appeared as 0.5. There is also a technological discourse about representing both decimals on a number line. \( S_{1A} \) from G1 agreed that it could help pupils to clearly see the positions of the two decimals. On the other hand, \( S_{3A} \) from G3 considered that representing the two decimals on a number line could lead pupils to some doubts; in fact, her partner put the decimals in incorrect positions.

**Analysis of HTT 3: Denseness of rational numbers**

Based on PsETs’ written answers, only one PsET, \( S_{1A} \) from G1, confidently argued that there are many numbers between two fractions and between two decimals. She provided some mathematical ideas to support her claims. For instance, \( S_{1A} \) wrote

“0.4 and 0.8 can be written as 0.4000… and 0.8000…, so many numbers can be found between 0.4 and 0.8”.

Meanwhile, three PsETs considered that there is a finite number of fractions between two fractions, but there can be many numbers between two decimals. In the end, only one pair, G1, provided both appropriate mathematical and didactical techniques for HTT 3 during their collaborative work.

While discussing HTT 3, a stunning thirteen pairs agreed and maintained that there is only one fraction between \( \frac{2}{5} \) and \( \frac{4}{5} \), namely \( \frac{3}{5} \) (Figure 6). Most of them suggested to teach pupils how to list intermediate fractions, based on ordering fractions with an equal denominator (\( \tau_{35x}^* \)), so the pupils just need to order the numerators. For the denseness of decimals, most pairs agreed that there are only 3 decimals between 0.4 and 0.8. They suggested teaching pupils a similar technique as that for ordering fractions mentioned above, namely listing according to decimal digits (here, tenths: 0.4, 0.5, 0.6, 0.7, 0.8) (\( \tau_{36x}^* \)). Even though seven pairs tried to represent those fractions or those decimals on a number line in order to help pupils figure out how many numbers there are between \( \frac{2}{5} \) and \( \frac{4}{5} \) and between 0.4 and 0.8, but they got the same wrong numbers of fractions and decimals between the given numbers (\( \tau_{37x}^* \)). We observe that they presented the two tasks using different number lines. For instance, \( S_{8A} \) explained to her partner that they could use a number line to represent the fraction task. She drew a line, divided it into some parts, and wrote fifths fractions
below the line (Figure 7). From this drawing, she mentioned that there was only $\frac{3}{5}$ between the two fractions, and her partner totally agreed with this. Meanwhile, when $S_{8A}$ looked at the decimal task, she recognised that it was in tenths. She then drew another line, divided it into ten parts, and wrote the decimal fractions (Figure 7). From this, she agreed that only three numbers lie between 0.4 and 0.8. Her partner, $S_{8B}$, just accepted her explanation without any question or justification. In fact, they could not realise both tasks are the same. Moreover, the number line does not, for them, seem to contain infinitely many decimals or fractions.

Figure 6. - Didactical techniques for HTT 3

Figure 7. – Number line representations

We also find that five pairs presented inconsistent answers in which they agreed on finitely many numbers between the two fractions, but not between the two decimals. They did not realise that the cited fractions and decimals represent the same rational numbers. For example, $S_{13A}$ wrote on his worksheet:
“I think if a pupil answers that there is a number between \(\frac{2}{5}\) and \(\frac{4}{5}\), namely \(\frac{3}{5}\), it means that his understanding of fractions is correct. Meanwhile, if he thinks that there are three numbers between 0.4 and 0.8, it means his understanding of decimals is still not sufficient because decimals are different from fractions. If he thinks after 0.4 there are 0.5, 0.6, 0.7, and then 0.8, it is not correct because there are also 0.51, 0.52, etc.”

S_{13A} does not seem to be aware of other fractions, such as \(\frac{5}{10}\), that can be between \(\frac{2}{5}\) and \(\frac{4}{5}\). He just focused on the fraction with the denominator of 5. His statement “decimals are different from fractions” indicates that he might not fully realise that fractions can be converted into decimals. Meanwhile, when he worked with decimals, he did find many numbers between them because he could add some more digits after the comma. He does not seem to realise that a very similar procedure can be used with fractions, for instance to find numbers between \(\frac{50}{100}\) and \(\frac{60}{100}\).

Actually, there were three pairs who discussed a didactical technique based on changing decimals into fractions or vice versa, but they still agreed that there were finitely many numbers between two decimals and between two fractions (τ_{31x}^*). For example, two PsETs from G5, S_{5A} and S_{5B}, discussed their ideas as follows:

S_{5A}: … So, to explain it (the decimal task) to the pupils, change 0.4 and 0.8 into …
S_{5A} & S_{5B}: fractions. (S_{5A} writes in her worksheet)
S_{5A}: So [0.4 and 0.8] become four-tenths and …(She writes \(\frac{4}{10}\) in her worksheet)
S_{5B}: eight-tenths …( S_{5A} writes \(\frac{8}{10}\) in the worksheet)
S_{5A}: So, we just try to find, how many numbers there are: five-tenths, six-tenths, and seven-tenths (She writes \(\frac{5}{10}\), \(\frac{6}{10}\), and \(\frac{7}{10}\) in her worksheet). It’s just like that. So, to have them equal and to find numbers in between two decimals, just change decimals into fractions. Indeed, it is true between \(\frac{2}{5}\) and \(\frac{4}{5}\) that there is one, and between 0.4 and 0.8 that there are three numbers.

They changed the two decimals into fractions, but they did not realise that \(\frac{2}{5}\) is equal to \(\frac{4}{10}\) and \(\frac{4}{5}\) is equal to \(\frac{8}{10}\). It seems that they do not have appropriate knowledge of equivalence of rational numbers. In another case, we observed that PsETs changed fractions into decimals, and considered that there were three numbers between the two decimals and between the two fractions.

Actually, PsETs did not provide many technological discourses to justify the techniques chosen. One given “justification” for the didactical techniques (τ_{35x}^* and τ_{36x}^*) is that it corresponds to what is presented in the textbooks. Pupils just need to order fractions and decimals based on the order of integers. Meanwhile, those PsETs who recognised that there are many numbers between
two decimals used the same technological discourse as for HTT 2, based on the possibility of adding final 0s digits to a finite decimal number.

**Analysis of HTT 4: Addition and subtraction of fractions**

Thirty PsETs solved the mathematical task of type $T_4$ correctly by changing each fraction into a common denominator and then add and subtract numerators ($\tau_{41}$). One of them also wrote an alternative mathematical technique presenting fractions as rectangles ($\tau_{43}$). When given the task of type $T_4^\#$ (interpreting pupils’ answers of $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$ and $\frac{4}{7} - \frac{1}{3} = \frac{3}{4}$), most PsETs explained that pupils erroneously just added and subtracted nominators and denominators separately ($\tau_{42}^\#$), then they proposed a correct mathematical technique $\tau_{41}$ to teach pupils directly, instead attending to resolving the pupils’ mistake.

During the collaborative work, we observed that all pairs discussed the didactical technique $\tau_{41}^*$ for the didactical task of type $T_4^*$ (Figure 8). Mostly, they suggested the following steps to teach pupils how to add and subtract fractions:

a. Tell pupils that a fraction consists of a numerator and a denominator.

b. To add and subtract fractions, pupils have to change the fractions into a common denominator ($\tau_{41}$) and then add the numerators.

c. The mathematical technique to find a common denominator is to find the least common multiple (LCM) of the denominators or simply use the product of the denominators.

\[ \text{d. Simplify the result.} \]

![Figure 8. - Didactical techniques for HTT 4](image)

However, a few pairs also suggested other didactical techniques. For example, PsETs from G1, S$_{1A}$ and S$_{1B}$, discussed using rectangle models ($\tau_{43}^*$). S$_{1A}$ gave an explanation to S$_{1B}$ on how to add fractions using rectangle models, and she suggested to apply such an explanation to teach pupils (Figure 9). She started by telling the pupils that there was a piece of paper and then it was divided into two equal parts. Then, she represented the paper as a rectangle diagram, divided into
two, and shaded a half of it. To show two-thirds, she did not directly divide the second rectangle into three parts, but she used the least common multiple of 2 and 3. Then, she divided the first rectangle diagram into six parts to show \( \frac{1}{2} = \frac{3}{6} \). After that, she divided the second rectangle into six-parts and shaded two-thirds of the rectangle. We notice that S_{1A} does not really try to use the diagrams to show how the addition of two fractions work, and how the common denominator appears from a need to combine the two diagram representations. It seems that she just uses the diagrams to clarify the standard algorithm for adding fractions.

Figure 9. - Rectangle representations for \( \frac{1}{2} + \frac{2}{3} \).

The most common technological element of the discussion is to insist that making an equal denominator and then add numerators is the only way to add and subtract fractions. Only four pairs tried and succeeded to explain the erroneous pupil solution given in the HTT.

**Analysis of HTT 5: Multiplication and division of decimals**

The most common didactical techniques applied by PsETs for the multiplication and division tasks were \( \tau_{m2}^* \) and \( \tau_{d2}^* \) respectively (Figure 10). They suggested to change decimals into fractions and then apply the algorithm for multiplication and division of fraction. None of them provided any justification of the standard algorithm for fraction division. For instance, S_{11A} mentioned that the divisor had to be inverted because it was the rule to divide fractions. The PsETs did not always suggest similar didactical techniques for multiplication and division tasks. For instance, G_{13} discussed the didactical techniques \( \tau_{d2}^* \) and \( \tau_{d5}^* \) as relevant techniques for the division task, but they could only explain the multiplication task to the pupils using \( \tau_{m2}^* \) (as we saw in the discussion of S_{13A} and S_{13B} in the previous subsection).
In addition, three pairs, G3, G4, and G14, were not able to present any correct mathematical technique for the multiplication and division tasks, and in fact they could not propose any didactical technique to support pupils’ learning (their didactical ideas are categorised as \( \tau_{mx}^* \) and \( \tau_{dx}^* \)), but some of them could still roughly explain why the multiplication gives an answer smaller than 8, and division gives the answer greater than 8. For instance, S\( _{3A} \) from G3 argued that the multiplier [0.25] was smaller than 1, so multiplication (respectively division) by this number yielded a smaller (greater) result.

**Discussion and concluding remarks**

The results presented in section 6 shows how a praxeological analysis is used to study PsETs’ mathematical and didactical knowledge. Especially through the two cases studies of the work of G2 and G13, we see how PsETs share and develop their mathematical and didactical knowledge about multiplication and division of decimals. We would like to emphasise that the a priori analysis of potential mathematical and didactical techniques for the HTTs functions as a tool to help us doing a systematic analysis of PsETs’, as well as the theoretical blocks they produce during the collaborative work.

Now, we would like to discuss and answer the first research question about what we learn from HTTs as a method to study teachers’ mathematical and didactical knowledge. Unlike the multiple choice items used in large scale studies, the HTTs allow us to simultaneously study teachers’ individual techniques and practical and theoretical knowledge, as well as their capacity for sharing and constructing mathematical and didactical techniques and technologies with colleagues. Moreover, we can observe their difficulties in constructing appropriate mathematical and didactical techniques and technologies and the reason behinds their difficulties (e.g. lack of
Another advantage of HTTs over most other methods to study teachers’ knowledge (e.g. MKT or TEDS-M) is that PsETs can develop more than a single technique for the questions, and discuss their advantages. The two case studies show how the PsETs negotiate and compare some possible mathematical and didactical techniques. During the discussion, each PsET tries to convince his or her partner, sometimes through giving a theoretical reason for the didactical techniques proposed. We also see how strongly the didactical techniques depend on mathematical knowledge. For example, if a pair struggles to find a single correct mathematical technique for a pupil task, they tend to propose direct instruction of this technique and ignore the explanation of the pupils’ difficulty, as outlined in the HTT. In fact, the didactical techniques proposed by PsETs are often limited to giving “the correct answer” or “a standard algorithm” without justifying them in a way that would also support pupils’ praxeological change (cf. Putra, 2018), for instance by developing a technology which both supports the correct technique and shows the inadequacy of that proposed by the pupils (according to the HTT formulation).

We also observe some more general characteristics about the PsETs mathematical and didactical knowledge. Many of the participating PsETs seem to assume that the mathematical tasks for pupils are mostly about how to find a correct answer. Pupils have to be taught the correct mathematical technique, often “algorithmic” techniques such as finding equivalent fractions by multiplying/dividing each numerator and denominator by the same positive integer. In fact, the technological discourse given by PsETs mainly consists in providing examples for the application of a preferred technique. This leads them to teach pupils to giving direct instructions about the mathematical techniques they are capable of. Some PsETs suggest to teach pupils using contextual or real life situations, but only few of them are really able to produce concrete and relevant situations (this echoes one of the main observations by Ma, 1999). The PsETs who try to use visual representations such as diagrams, also frequently are unable to do so effectively and correctly, for instance a pair who tried to represent $\frac{8}{9}$ into a rectangle representation. One likely and major influence on PsETs’ mathematical and didactical techniques is how mathematical techniques are presented in textbooks in Indonesia. Wijayanti and Winsløw (2017) showed (using praxeological analysis as in the present paper) that the textbooks tend to present several examples of a given mathematical type of task, with all tasks being solved by the same standard technique, which is mainly explained through the examples. After that, pupils have to use the same technique to solve
other mathematical tasks of the same type, which the textbook also provides (usually in abundance).

However, some PsETs present more varied didactical techniques for some HTTs, such as HTT 2 on comparing decimals. We consider that this broader diversity of techniques is influenced by two factors. The first factor is probably that there is no single algorithm or procedure stated in textbooks or learnt at schools, related to that task. The second one is that the numbers used in HTT2 are easily transformed into other representations. On the other hand, many PsETs do not recognise that two different representations of rational numbers, fractions and decimals, have the same size. Hence, they fall into the trap of the task without noticing the absurdity of having different answers to the same problem, depending on whether the given numbers are represented as fractions or decimals. Indeed, the difficulties of rational numbers are not only a challenge for pupils but also for pre-service teachers (Depape et al., 2015; Ni & Zhou, 2005; Van Dooren et al., 2015).

Concerning the theoretical blocks to justify specific techniques, we also observe that some PsETs refer to their teaching experiences during practicums. They try to remember what they have seen in pupils’ textbooks and assume that those provide model didactical techniques to teach pupils. The didactical discourse given by the PsETs depends on what mathematical techniques they propose. A similar technological discourse can justify two different mathematical techniques (correct and wrong techniques in the same task). For instance, a pair from group A agreed that using a number line helps pupils to recognise the positions of the given rational numbers, meanwhile, a pair from group B conclude that representing the given rational numbers on a number line is not ideal for knowing the positions of the given numbers because pupils themselves might struggle to find the correct positions of those numbers.

As a conclusion on this study, we would like to emphasise that the mathematical knowledge always links to didactical knowledge, and to study teachers’ didactical knowledge any method used has to deal also with mathematical knowledge and how it links to the didactical knowledge. We have demonstrated in what ways HTTs can be used as a method to study both individual and collective forms of teachers’ mathematical and didactical knowledge, and how these relate to each other. After all, teacher knowledge relies (as many other forms of professional knowledge) on intimate links between individual practice and shared knowledge.

In relation to the theoretical and methodological links between HTTs and the use of the ATD for the \textit{a priori} and \textit{a posteriori} analysis, the main point is that the praxeological analysis helps to predict, precisely analyse and further discuss the phenomena which appear during teachers’ individual and collaborative work with the HTTs. As usual, the \textit{a priori} analysis of possible
mathematical and didactical praxeologies helps, in particular, to exhibit what does not appear in the observations (but could, or even should). Moreover, the full potential of the PsETs’ individual work, including technological and theoretical components, usually only appear in the collective part (discussion); moreover in this part, individual mistakes or shortcomings are also sometimes resolved. Shared shortcomings in the PsETs’ technological equipment also become more visible as they discuss possible strategies for helping the fictitious students during their collaborative work, and thus limitations of their mathematical and didactical knowledge beyond simple techniques. In fact, by developing the methodology of this study, we think further information of these kinds may be obtained, for instance, by subsequent questioning focused on the informants’ mathematical techniques, in order for them to share more broadly their mathematical technology and even theory.

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APPENDIX
The appendix only the *a priori* analysis of mathematical and didactical types of task and the associate techniques for HTT 1 to HTT 4. We could not list the technological discourses to justify those techniques due to the limitation of the space, but one can associate them based on how they are explained and described for HTT 5 about multiplication and division with decimals.

HTT 1: Equivalent fractions

You ask fourth grade pupils to find fractions which are equal to \(\frac{3}{4}\).

*A pupil claims that* \(\frac{3}{4} = \frac{8}{9}\) *because if you add 5 to both the top and the bottom of a fraction, the new fraction must be equal to the original.*

What do you think about this answer? Please explain! (*to be solved individually within 3 minutes*).

What would you do as a teacher to help the pupils from this case to understand the concept of equal fractions better? (*to be discussed and solved in pair within 5 minutes*).

The type of mathematical task given to pupils can be described as follows:

\(T_1\): given a positive fraction, \(\frac{a}{b}\), determine other fractions that are equal to it.

Possible mathematical techniques to solve the task of type \(T_1\):

\(\tau_{11}\): multiply/divide each numerator and denominator by the same positive integer, i.e. \(\frac{a}{b} = \frac{na}{nb}\), where \(n\) is a positive integer.

\(\tau_{12}\): change \(\frac{a}{b}\) into a decimal using one of these techniques (division algorithms, specific fractions memorised as decimals, or using calculators, computers, or other electronic devices), and then find other fractions that are equal to the decimal.

The second type of mathematical task is inferred from question a. as follows:

\(T_1^\#\): given two positive fractions, \(\frac{a}{b}\) and \(\frac{c}{d}\), decide if they are equal.

Possible mathematical techniques to solve the task of type \(T_1^\#\) are to change both fractions into an equal denominator and then compare numerators (relates to \(\tau_{11}\)), to change both fractions into decimals (relates to \(\tau_{12}\)), or represent both fractions into diagrams (based on the meaning of a fraction as a part of a whole) and divide each diagram precisely in order to show the different size of shaded areas from the two diagrams as follows:
A type of didactical task can be inferred from question b. as follows:

\( T_1 \): propose strategies to help pupils to solve a task of type \( T_1 \).

Possible didactical techniques to solve the task of type \( T_1 \):

\( \tau_{11} \): teach pupils by simply explaining the mathematical technique \( \tau_{11} \).

Through a similar approach, it can be derived the didactical technique \( \tau_{12} \) (adding \( x \) to indicate the didactical techniques based on the incorrect mathematical techniques). There are also other possible didactical techniques that might be variants of those didactical techniques and coded as follows:

\( \tau_{13} \): present and explain the mathematical task \( T_1 \) into an appropriate contextual or real life situation, e.g. a task related to share pizzas or cakes, and then explain it using diagram representations: first represent \( \frac{a}{b} \) in a model such as a rectangle or circle diagram, then draw another model for \( \frac{a}{b} \) by dividing it into 2, 3, or more parts. Finally, it can be shown that both models represent equal fractions, e.g. as follows:

\[
\begin{align*}
\frac{3}{4} & \quad \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8} \\
\frac{3 \cdot 3}{4 \cdot 3} & = \frac{9}{12}
\end{align*}
\]

The diagrams can also be used to show that the two fractions \( \frac{3}{4} \) and \( \frac{8}{9} \), are not equal.

\( \tau_{13a} \): represent both fractions into one (two different) number line(s) and show pupils that the position of both fractions are the same.

\( \tau_{14} \): use a simple fraction such as \( \frac{1}{4} \) and \( \frac{1}{2} \) as a starting point to explain a mathematical technique for the task of type \( T_{11} \).

\( \tau_{15x} \): propose other inappropriate didactical techniques that are based on incorrect interpretation of the mathematical task.
HTT 2: Comparing decimals

Fifth grade pupils are asked to compare the size of 0.5 and 0.45. Some pupils answer that 0.45 is greater than 0.5, while others say that 0.5 is greater than 0.45.

Analyse the pupils’ answers. Explain your ideas to handle the situation in this class? (to be solved individually in 3 minutes).

How do you use this situation to further the pupils’ learning? (to be discussed and solved in pair within 5 minutes).

The type of mathematical task given to pupils can be described as follows:

$T_2$: given two different decimal numbers, $0 < a < 1$, and $0 < b < 1$, decide if $a > b$ or $a < b$.

Possible mathematical techniques to solve the task of type $T_2$:

$\tau_{21}$: change $a$ and $b$ into integers, multiplying by an appropriate power of ten.

$\tau_{22}$: use lexicographical order/place value order to compare the decimals.

$\tau_{23}$: add 0 digits after the comma to get the same number of digits in both decimals, and then compare them.

$\tau_{24}$: change decimals into fractions with the common denominator and compare the numerators, or change decimals into common fractions and compare them intuitively, e.g., $0.5 = \frac{9}{20}$ that is less than a half.

$\tau_{25}$: subtract $b$ from $a$. If the result is greater than 0, then $a < b$, otherwise $a > b$, or divide $b$ by $a$.

When the result is greater than 0, then $a < b$, otherwise $a > b$.

The type of didactical tasks can be described as follows:

$T_2^*$: given two difference pupils’ answers to the task of type $T_2$, determine what to do as a teacher to facilitate pupils learning.

Possible didactical techniques to solve the task of type $T_2$:

$\tau_{21}^*$: teach pupils by simply explaining the mathematical technique $\tau_{21}$.

Through a similar approach, it can be derived some other didactical techniques namely $\tau_{22}^*$, $\tau_{23}^*$, $\tau_{24}^*$, and $\tau_{25}^*$ respectively from $\tau_{22}$, $\tau_{23}$, $\tau_{24}$, and $\tau_{25}$ (adding $x$ to indicate the didactical technique based on the incorrect mathematical technique). There are also other possible didactical techniques that might be variants of those didactical techniques and coded as follows:

$\tau_{26}^*$: present the mathematical task $T_2$ into an appropriate contextual or real life situation such as to compare the size of two cakes, and this could be followed by presenting them in the
diagrams. The teachers need to divide the diagrams into equal sectors to get the precise representations.

τ27*: design or explain through number lines to show positions of the two decimals. One could explain by relating those numbers into a metric system such as 0.5 m and 0.45 m.

τ28*: provide pupils with other comparing decimal problems, such as giving some common decimal numbers, such as 0.25 and 0.5, to be compared, and use it as a point to explain other tasks.

τ29*: organize a class discussion about the two solutions, ask pupils why they give such answers, and ask others to justify them and realize what is the correct answer.

HTT 3: Denseness of rational numbers

You first ask fifth grade pupils to discuss how many numbers there are between \(\frac{2}{5}\) and \(\frac{4}{5}\), and how many numbers there are between 0.4 and 0.8.

Then, they say that there is only one number between \(\frac{2}{5}\) and \(\frac{4}{5}\) namely \(\frac{3}{5}\); they also say 3 numbers between 0.4 and 0.8.

How do you interpret this claims? (to be solved individually within 3 minutes).

Explain your ideas to teach these pupils? (to be discussed and solved in pairs within 5 minutes).

The mathematical tasks given to pupils can be described as following type:

\[ T_3 : \text{given two different rational numbers, } \frac{a}{b} = \frac{m}{c_1}c_2 \cdots \text{ and } \frac{c}{d} = \frac{n}{d_1}d_2 \cdots, \text{ find how many numbers between } \frac{a}{b} \text{ and } \frac{c}{d}, \text{ and } m, c_1 c_2 \cdots \text{ and } n, d_1 d_2 \cdots. \]

Possible mathematical techniques to solve the task of type \(T_3\):

τ32: change fractions into decimals or vice versa, and show both tasks are the same.

τ32: first show that there is one number between \(x\) and \(y\) (\(x, y\) representing the general terms for rational numbers such as \(\frac{a}{b}\), and \(m, c_1 c_2 \cdots\)). There exists \(z\), so \(x < z < y\), then use this to find \(z_1\) so that \(x < z_1 < z\), continue to \(z_2\) so that \(x < z_2 < z_1\), and etc.

Some techniques to find \(z\) described as follows:

τa: find \(z\) between \(\frac{a}{b}\) and \(\frac{c}{d}\) using a formula: \(\frac{a+c}{b+d}\).

τb: find \(z\) between \(\frac{a}{b}\) and \(\frac{c}{d}\) using a formula: \(\frac{ad+bc}{2bd}\).

τc: if \(b = d\), take a number \(m\) between \(a\) and \(c\), then \(\frac{a}{b} < \frac{m}{d} < \frac{c}{b}\).
τₐ: find \( z \) between two decimals \( m, c_1 c_2 \cdots \) and \( n, d_1 d_2 \cdots \) by considering a number between two numbers after comma.

τₖ: find \( z \) between \( m, c_1 c_2 \cdots \) and \( n, d_1 d_2 \cdots \) using a formula: \( \frac{n.c_1 c_2 \cdots + n.d_1 d_2 \cdots}{2} \).

τ₃₃: find equal/equivalent fractions for \( \frac{a}{b} \) and \( \frac{c}{b} \), and show that the greater denominators, the more fractions with the same denominator can be found between the two denominators, thus there is an infinity of fractions between the two given numbers.

τ₃₄: put 0 digits after the comma and show that there are more decimals can be written into two, three, and many decimal digits.

τ₃₅: change both fractions into the same denominator (in case they have different denominators), and by finding the natural numbers between two numerators, and using these as numerators, we find the numbers between the given fractions.

τ₃₆: consider decimals as natural numbers by omitting commas, and find natural numbers between them through ordering numbers, and put the comma’s back (this will give the numbers in between the given numbers).

The task given to PsETs belong to the following:

\( T_3^\# \): given pupils’ answers about denseness of rational numbers between \( \frac{a}{b} = m, c_1 c_2 \cdots \) and \( \frac{c}{d} = n, d_1 d_2 \cdots \), analyse these answers.

PsETs will at least evaluate the pupils’ answers (\( T_3^\# \)) by stating whether the answers are correct (e.g. they agree with the finite number of numbers between two fractions and two decimals), partly correct (finitely many numbers between two fractions but not between two decimals, or vice versa), or incorrect (infinitely many numbers between two fractions and also between two decimals). If they disagree with pupil’s answers, they may also discuss more systematic praxeological shortcomings behind the answers.

The type of didactical tasks can be described from question b. as follows

\( T_3^* \): given problems and pupils’ responses to the type of task \( T_3 \), determine what ideas as a teacher to teach pupils.

Possible didactical techniques to solve the task of type \( T_3^* \):

τ₃₁*: teach pupils by simply explaining the mathematical technique τ₃₁.

Through a similar way, it can be derived some other didactical techniques namely τ₃₂*, τ₃₃*, τ₃₄*, τ₃₅x* and τ₃₆x* respectively from τ₃₂, τ₃₃, τ₃₄, τ₃₅, and τ₃₆. There are also other possible didactical techniques that might be variants of those didactical techniques and coded as follows:

τ₃₁x*: teach pupils through changing fractions into decimals, but still find finite numbers.
provide pupils some rational numbers that are between two given rational numbers, and ask them to consider whether those rational numbers lie between the two rational numbers. e.g. given pupils a decimal.

design or explain through number lines, and show that one always can find a number between two numbers (τ37\* indicates of using number lines but still consider finite numbers between two rational numbers).

design or explain through an appropriate contextual/real life situation. e.g., there is 1 m rope, and then one can divide it into two halves, and then each half can divide into two quarters, and so on. (τ38\* indicates inappropriate contextual/real life situation).

other inappropriate didactical techniques, e.g., suggesting to organise a classroom discussion but still come to the answer of finite numbers between two rational numbers.

HTT 4: Addition and subtraction of fractions

You ask sixth grade pupils to solve $\frac{2}{3} + \frac{1}{2} = \ldots$ and $\frac{4}{7} - \frac{1}{3} = \ldots$

a. How do you solve these problem? (to be solved individually within 3 minutes)

You find that many pupils add and subtract fractions in the following way $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$ and $\frac{4}{7} - \frac{1}{3} = \frac{3}{4}$.

b. How do you interpret the pupils’ methods? (to be solved individually within 3 minutes)

c. What strategies can you propose to teach these pupils? (to be discussed and solved in pair, 5 minutes)

The task from question a. can be described as following type:

$T_4$: given two positive rational numbers, $\frac{a}{b}, \frac{c}{d}$, determine $\frac{a}{b} + \frac{c}{d} = \ldots$ and $\frac{a}{b} - \frac{c}{d} = \ldots$

The possible mathematical techniques to solve the task of type $T_4$:

τ41: change each fraction into a common denominator and then add and subtract numerators to get the result.

τ42: change fractions into decimals and then add and subtract the decimals.

τ43: represent fractions into circle or rectangle models, then partitioning each fractions into the same units, and finally take one to the other (for addition of fractions) or find the difference (for subtraction).
The task from question b. can be described as following type:

\[ T_4^b \]: give your interpretations of pupils’ incorrect mathematical technique for the type of task \( T_4 \).

This type of task requires teachers to justify the pupils’ technique. Two possible answers given by the PsETs can be described as follows:

\( \tau_{41}^# \): pupils apply the technique for multiplication of fractions.

\( \tau_{42}^# \): pupils consider a fraction as two numbers separate by a line, so they add and subtract both numbers based on their position.

To justify pupils’ answers, PsETs may provide some mathematical techniques as follows:

\( \tau_{4a}^# \): compare \( \frac{a}{b} + \frac{c}{d} \) to \( \frac{a+c}{b+d} \) and \( \frac{a}{b} - \frac{c}{d} \) to \( \frac{a-c}{b-d} \).

\( \tau_{4b}^# \): present into rectangle models.

\( \tau_{4c}^# \): change fractions into decimals. e.g. \( \frac{2}{3} + \frac{1}{2} = 0.\overline{6} + 0.5 > 0.6 = \frac{3}{5} \).

The task from question c. can be described as following type:

\[ T_4^* \]: propose strategies to help pupils solve the task of type \( T_4 \), and to explain and justify the techniques which solve such task.

The possible didactical techniques to solve the task of type \( T_4^* \):

\( \tau_{41}^* \): teach pupils by simply explaining the mathematical technique \( \tau_{41} \).

Through a similar way, it can be derived some other didactical techniques namely \( \tau_{42}^* \), \( \tau_{43}^* \), and \( \tau_{44x}^* \) respectively from \( \tau_{42} \), \( \tau_{43} \), and \( \tau_{44} \). There are also other possible didactical techniques coded as follows:

\( \tau_{44}^* \): provide pupils with some common fractions (e.g. \( \frac{1}{2} + \frac{1}{2} \)) to calculate with, to have them revise their techniques.

\( \tau_{45}^* \): explain to pupils that they may confuse the standard algorithms for multiplication and addition of fractions.

\( \tau_{46}^* \): show pupils that their’ answers are not correct through one of the mathematical techniques \( \tau_{4a}^# \), \( \tau_{4b}^# \), \( \tau_{4c}^# \).
4 Danish pre-service teachers’ mathematical and didactical knowledge of operations with rational numbers

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Abstract
The aim of this study is to investigate Danish pre-service teachers’ (PSTs) mathematical and didactical knowledge of operations with rational numbers. This knowledge is studied through their collaborative activities to certain tasks related to the teaching of operations with rational numbers. Explicit models of the teachers’ mathematical and didactical knowledge are designed based on the anthropological theory of the didactic (ATD) and used to analyse five-groups of PSTs’ collaborative work. The findings show that the Danish PSTs prefer to use contextual or real-life situations in their teaching, but they encounter various challenges to realise general ideas in the context at hand.

Keywords: operations with rational numbers, mathematical and didactical knowledge, anthropological theory of the didactic, collaborative work.

Introduction
Several studies have showed that many pupils struggle to learn operations with rational numbers (e.g. Gabriel et al., 2013). They tend to apply procedural techniques which are often incorrect such as adding fractions by simply adding the numerators and the denominators. It is also the case in Denmark that one-third of all Danish grade nine pupils could not solve a simple addition task of two fractions at the national exam (Danish Ministry of Education, 2011). This leads to serious problems for them to learn more advanced mathematics, especially basic algebra.

Pupils’ difficulties to learn rational numbers are believed to be caused by several factors, and one of them is insufficient teachers’ knowledge (Siegler & Lortie-Forgues, 2017). Ma (1999), for instance, has showed that U.S teachers who merely mastered a procedural technique for the division task of fractions, were unable to construct real-life problems whose solution involves the
same computations. In contrast, Chinese teachers performed better. Meanwhile, there is no evidence how Danish teachers deal with pupils’ difficulties on rational numbers. This motivates the author to focus on Danish student teachers’ knowledge of rational numbers, specifically the arithmetic operations with rational numbers.

The present study investigates pre-service teachers’ (PSTs) collective knowledge of operations with rational numbers. It is different from most large-scale studies focusing on teachers’ individual knowledge, as measured through written tests (Bradshaw, Izsák, Templi, & Jacobsen, 2014; Castro-Rodríguez, Pitta-Pantazi, Rico, & Gómez, 2015; Chinnappan & Forrester, 2014; Depaepe et al., 2015; Jacobsen & Izsák, 2015; Şahin, Gökkurt, & Soylu, 2015; Van Steenbrugge, Lesage, Valcke, & Desoete, 2014). The teachers’ collective knowledge means the knowledge which they could generate and share with colleagues, and which may represent more adequate pictures of the basis of their professional work. Therefore, the design of this study is based on small-scale collaborative work situations. Francisco (2013) showed that “collaborative work provides opportunities for students to critically re-examine how they make claims from facts and also enable them to build on one another’s ideas to construct more sophisticated ways of reasoning”. One can hypothesize that the same will be the case when two or more teachers discuss a professional task for teachers (which includes both a mathematical and a didactical problem, the latter referring to a teaching problem related to the mathematical problem). This idea was initially developed by Durand-Guerrier, Winsløw and Yoshida (2010) and Winsløw and Durand-Guerrier (2007), who experimented some simple examples of what they call hypothetical teacher tasks (HTT). Working with an HTT, pairs of teachers or PSTs are given the opportunity to share and link their mathematical and didactical knowledge. Moreover, as a result of the informants working in pairs, one can gain information not just about the techniques they are able to apply to the task, but also how they describe and justify these techniques. More specifically, the present study investigates the following research questions:

RQ1. What mathematical and didactical knowledge do Danish PSTs possess to address challenging situations in the teaching of rational number operations?

RQ2. How do they produce mathematical and didactical knowledge of operations with rational numbers during their collaborative work? What relations do appear between the two components?

In the following sections, I first review some previous studies on teachers’ knowledge of operations with rational numbers. Then, I introduce the anthropological theory of the didactic (ATD) as a framework to study Danish PSTs’ mathematical and didactical knowledge of operations with rational numbers. The method section presents information about the participants, the design of HTTs, and the data analysis. After that, I present the results from the Danish PSTs’
individual and collaborative work on each HTT. Finally, the discussion tries to answer the two research questions and to provide some perspectives on how ATD is used to study teachers’ knowledge in the setting of collaborative work.

**Teachers’ knowledge of operations with rational numbers**

Operations with rational numbers provide several challenges for teachers (An, Kulm, & Wu, 2004; Ball, 1990; Depaepe et al., 2015; Ma, 1999; Newton, 2008; Van Steenbrugge et al., 2014). Many teachers cannot relate standard computational procedures with appropriate rationales for those procedures (e.g. Van Steenbrugge et al., 2014). One reason for this gap is that PSTs have difficulties to recognise a fraction as a single rational number. Another reason is that rational numbers have several common representations, unlike natural numbers (Van Dooren, Lehtinen, & Verschaffel, 2015). For instance, rational numbers can be represented as fractions, decimals, percentages, or diagrams. Two different representations can be used to serve the same number (Vamvakoussi, Van Dooren, & Verschaffel, 2012), and each representation comes with different methods or algorithms related to doing operations.

In the case of addition and subtraction of fractions, a prevalent mistake among PSTs is to add and subtract numerators and denominators based on their positions (Newton, 2008). They tend to use this technique especially when the task involves two fractions with different denominators. A common method to investigate teachers’ knowledge of adding and subtracting fractions is to ask them to write word problems (Austin, Carbone, & Web, 2011; Dixon et al., 2014). Austin et al., (2011), focusing their study on adding fractions, found an issue related to PSTs’ inability to recognise that uniform units have to be used. For instance, a PST poses a task such as pupil A has \( \frac{1}{2} \) of his pizza and pupil B has \( \frac{3}{4} \) of her pizza, how much pizza do they have together? This word problem looks correct, but the two units (pizzas) may not being the same size and shape. A similar finding is also discussed by Dixon et al., (2014) on subtracting of fractions: when PSTs are asked to pose a word problem, for instance \( \frac{a}{b} - \frac{c}{d} \), they tend to write a problem to represent \( \frac{a}{b} - \frac{c}{d} \times \frac{a}{b} \). They tend to use an incorrect redefinition of the whole, and the fractions involved in the tasks influence PSTs’ performance on writing the word problems.

Previous studies on how teachers deal with multiplication and division of rational numbers also show that teachers may succeed to solve operations of fractions themselves but still be unable to explain the reasons behind the used techniques (Alenazi, 2016; Ball, 1990; Erdem, Gökkurt, Şahin, Başbüyük, & Soylu, 2015; Ma, 1999; Slattery & Fitzmaurice, 2014). Ma (1999) found that the U.S. teachers had various misconceptions about the meaning of division by fractions.
Confounding division by \(\frac{1}{b}\) with division by \(b\) and confounding division by \(\frac{1}{b}\) with multiplication by \(\frac{1}{b}\) were the two common misconceptions when they tried to model the symbolic fraction division task by a word problem. This means that they could not explain the meaning of the task based on several interpretations, such as measurement (or quotitive meaning), partitive meaning, and product and factors. Some of PSTs’ mistakes on multiplication and division of fractions come from their perception that multiplication “makes bigger” and division “make smaller” (Slattery & Fitzmaurice, 2014; Tirosh, 2000).

In the present study, I design two teacher tasks based on known misconceptions and difficulties in teaching operations with rational numbers. The first task is about exhibiting teachers’ mathematical and didactical ideas to handle pupils’ incorrect answers for addition and subtraction of fractions. The second task aims to investigate teachers’ mathematical and didactical knowledge for dealing with pupils’ difficulties on multiplication and division of decimals (multiplication makes bigger and division make smaller). I design the two tasks with two different representations, fractions and decimals, in order to cover different representations of rational numbers.

**Theoretical framework: Mathematical and didactical praxeologies**

Shulman’s (1986) seminal work of content knowledge (CK) and pedagogical content knowledge (PCK) led many researchers to focus on teachers’ knowledge. In mathematics education, for instance, some researchers try to develop a model to study teachers’ CK and PCK, and one of the notable frameworks is *mathematical knowledge for teaching* (MKT) (Ball Thames, & Phelps, 2008). This model, like others, mostly focuses on developing large-scale studies of teachers’ individual knowledge, but does not provide a detailed method to analyse the connection between teachers’ knowledge of theory and practices.

Addressing the need to study teachers’ practical and theoretical knowledge as well as individual and collective knowledge, I decided to study teachers’ knowledge using the anthropological theory of the didactic (ATD) (Chevallard, 1992, 2006, 2007). I chose ATD because it proposes an epistemological model, that is known as a praxeology, to analyse human activities - including producing, teaching, learning, and conducting mathematical activities. The praxeology is a basic unit into which one can analyse human practice and knowledge into two interrelated components: praxis and logos. The praxis or the practical block consists of a type of task (\(T\)) and associated techniques (\(\tau\)) to accomplish each type of task. The logos or the theoretical block is related to human thinking and reasoning. Chevallard (2006) argued that no human doing goes unquestioned in the long run, so one needs a technology (\(\theta\)) to explain and justify the
techniques and a theory (Θ) to explain and unify several technologies; both belong to the theoretical block. So, a praxeology consists of a 4-tuple (T, τ, θ, Θ) in which all components are intimately linked, and this is a useful tool to analyse teachers’ practical and theoretical knowledge together.

The notion of mathematical praxeologies is used to describe teachers’ mathematical knowledge of operations with rational numbers (CK). An example of a mathematical type of task is to add two rational numbers. The techniques applied to solve that task depend on how the rational numbers are represented. If rational numbers are represented as fractions, a mathematical technique can be a standard procedure of addition of fractions (first rename each fraction with the least common multiple, then add numerators, and finally simplify the result). Another possible mathematical technique is to change fractions into decimals and then apply the standard procedure for the addition of decimals. An example of technological discourses related to converting fractions to decimals is that changing a fraction into a decimal does not change the value of the number. Notions such as equivalence classes of representations go into the theory to justify that technology.

To study teachers’ PCK, ATD proposes the notion of didactic or didactical praxeology (Bosch and Gascón, 2014; Rodríguez, Bosch & Gascón, 2008; Winsløw, Barquero, De Vleeshower, & Hardy, 2015). A didactical praxeology is defined as a praxeology aiming at making other (mathematical) praxeologies start living in and being shared within human groups (Bosch and Gascón, 2014). So, the didactical praxeology begins with specific tasks related to help pupils to acquire a mathematical praxeology. A didactical praxeology consists of a type of didactical task, didactical techniques, a didactical technology and theory (Rodríguez et al., 2008; Winsløw et al, 2015). An example of a didactic type of task is to teach pupils how to add two fractions. The didactical techniques used can be influenced by factors such as teachers’ mathematical knowledge or learning experiences. The most common didactical technique is to present directly a common mathematical technique (e.g., an algorithm for adding or subtracting fractions) to the pupils. An example of the technological discourse to justify this didactical technique is the assumption that pupils will learn faster if teachers first present the mathematical technique and then ask pupils to mimic it while solving similar mathematical tasks. A theory to justify this didactical technology is direct instructional teaching and learning (Klahr & Nigam, 2004).
Methods

Participants
The participants were eleven first-year PSTs enrolled in the Bachelor of teacher education programme in a teacher education institution in Denmark. These PSTs volunteered to participate in the study because they desired to learn about rational numbers and were willing to help the researcher to study their learning. They were first-year students who were being prepared to teach pupils from grade 4 to grade 9. During the data collection, they were taking a course about learning numbers and algebra, and the topic of rational numbers is certainly part of the course. This also motivated the decision to use first-year students as participants. Meanwhile, the main aim of this study is not to evaluate the teacher education programme but to find out how PSTs apply their mathematical knowledge about rational numbers in hypothetical teaching situations.

Design of hypothetical teacher tasks (HTT)
The notion of hypothetical teacher tasks (HTT) was firstly introduced by Winsløw and colleagues (Durand-Guerrier, Winsløw, & Yoshida, 2010; Winsløw & Durand-Guerrier, 2007) to study and compare PSTs’ collective mathematical and didactical knowledge in Denmark, France, and Japan. HTTs are supposed to present mathematical and didactical tasks that can be found in common teaching situations. The teachers have to use various parts of their knowledge to act appropriately. The mathematical contents of the situations are both standard and elementary. For instance, adding two fractions is a common type of task that can be found in grade 5 in Danish primary schools (Freil & Kaas, 2007), and dealing with common pupil misconceptions, such as adding the two fractions based on their positions, is the didactical task that teachers will meet during their teaching experience.

The first HTT focuses on addition and subtraction of fractions (Figure 1). This type of task is introduced to Danish pupils in grade 5 and grade 6 (Freil & Kaas, 2006, 2007). HTT 1 consists of three tasks. The PSTs have to solve individually for the first two tasks, and then use their individual mathematical knowledge to propose didactical praxeologies during the collaborative work in the third task.
The second HTT focuses on multiplication and division of decimals (Figure 2). The Danish pupils start to learn it in grade 6 (Freil & Kaas, 2006). The task is adapted from the teacher education and development study in mathematics ([TEDS-M], Tarto et al., 2008), and the mathematical task is integrated with the didactical task. PSTs are expected to share their mathematical and didactical knowledge to handle pupils’ misconception on multiplication and division of decimals simultaneously during their collaborative work.

Figure 2. HTT about multiplication and division of decimals (Putra & Winlsøw, accepted).

**Data collection and data analysis**

The HTTs were originally written in English and subsequently translated into Danish by a master’s student who studied didactics of mathematics. Then, a mathematics educator at a University College and a mathematics education researcher at a University reviewed the translated items. After some revisions, the HTTs were tested in January 2016 with the eleven Danish PSTs.
The PSTs mostly worked in pairs, except for one group consisting of three students, so there are five small groups/pairs\(^2\) of PSTs (coded as S\(_1\) – S\(_5\)). Each pair solved the two HTTs one by one, and the researcher and the master’s student observed and videotaped their collaborative work. The main focus of PSTs’ collaborative work is to discuss the didactical tasks, such as how to handle pupils’ misconceptions and difficulties and how to pose adequate didactical praxeologies.

The data from this study consist of PSTs’ written answers and video recordings. The transcripts from their collaborative work were done by the master’s student. She first transcribed the data into Danish and then translated it into English. Then, the written answers and video transcripts were analysed in terms of mathematical and didactical praxeologies; techniques, technologies, and also possible theories exhibited by the PSTs. The process of analysing data was done by the author, and passages indicating mathematical and didactical praxeologies were read several times. The passages containing questionable points were discussed with the master’s student and one mathematics education researcher who was familiar with the content and context of the study.

Findings

The praxeological analysis of PSTs’ written answers on adding and subtracting of fractions

The eleven Danish PSTs did not have any difficulty to solve the mathematical task for addition and subtraction of fractions (T\(_a\)). All of them gave correct answers using the standard technique; change each fraction into a common denominator and then add numerators (\(\tau_{a1}\)). Three PSTs showed how they found the common denominator.

The answers to the task of interpreting pupils’ answers (T\(_b\)) indicated that all of them recognised the mistakes underlying the pupils’ answers. The pupils just added and subtracted the numerator to the numerator and the denominator to the denominator (\(\tau_{b1}\)). They gave two technologies to explain that technique. Ten PSTs assumed that the pupils inappropriately applied properties of natural numbers to the tasks of adding and subtracting two fractions (\(\theta_{b1}\)). The second technology is that the pupils just employed the standard procedure for multiplication of two fractions (\(\theta_{b2}\)). \(\theta_{b2}\) is only produced by one PST, S\(_{2a}\), who wrote: “the pupils have confused the

\(^2\) The term “pair” is used referring to all groups.
calculation rules for addition and multiplication of fractions, so they operated the numerators and the denominators as the multiplicative rule”.

**The praxeological analysis of PSTs’ collaborative work on adding and subtracting fractions**

All Danish pairs started the discussion by presenting technological discourses to explain $\tau_{a1}$. Besides producing accompanying technologies, four pairs also mentioned that pupils’ misunderstanding the concept of fractions was the main cause for their incorrect answers to the task type $T_a$. The abstract notion of fraction belongs to mathematical theory ($\Theta$) to justify the technologies. Three pairs provided a detailed explanation what they mean by the concept of fractions; a fraction as a part of a whole, and a fraction as a quotient (Charalambous & Pitta-Pantazi, 2007). One example comes from the discussion of group 1, $S_{1a}$ and $S_{1b}$, as follows:

$S_{1b}$: … It seems like they do not fully understand the concept of fractions. They just think that there are two numbers on both sides of the line, so that they just can add and subtract the numbers.

$S_{1a}$: They have not understood the value of the numbers, like $\frac{1}{2}$ and $\frac{2}{3}$. You cannot add those before they have the same value.

…

$S_{1b}$: It is not 1 and 2. It is a half.

$S_{1a}$: Yes. It is the same as 0.5.

The excerpt shows that this pair had the concept of fraction as a number (in the last case, even with an equivalent decimal representation). This means that the pupils need to perceive that a fraction $\frac{a}{b}$ has a value. Mastering this meaning, the pupils will not have any constraint on the size of the fraction, whether the numerator is smaller, equal, or greater than the denominator (Charalambous & Pitta-Pantazi, 2007).

The Danish pairs continued the discussion to formulate some possible didactical techniques for the didactical task of type $T_c$. The concern of the discussion was to teach pupils correct mathematical techniques and also to prove $\tau_{a1}$ incorrect. A common didactical technique was to visualise both problems into concrete examples ($\tau_{a1}$). All pairs suggested to represent the task into a well-known context by the pupils, such as pizzas, but only three groups gave a detailed explanation how they could make use of the context to teach pupils, and almost none of them was really aware of uniform units when proposing a real-world problem. They just mentioned the context and then directly moved to the diagram representations. It is illustrated by a discussion of $S_{5a}$ and $S_{5b}$ as follows:
S₅₅: I will use concrete examples; the chocolate and the pizza. However, the pizza is not easy to divide into 7.
S₅₅: Yes. But [the task is] \( \frac{2}{3} + \frac{1}{2} \). I will say to them “\( \frac{3}{5} \) is smaller than a whole, however we should end up with more than a whole, so your method cannot be right”.
S₅₅: Yes. I think we should
S₅₅: Draw pizza (both draw circles and shade \( \frac{2}{3} \) and \( \frac{1}{2} \))
S₅₅: Then we get relate them - can you remember “learn about pizza” by those Norwegians where they divide pizza? It’s good!
S₅₅: I think that they need to have an understanding of the concept. We could practice rules of fractions with them, although I think they will learn more from the pizza. Afterwards they could help develop the rules of fractions.

…
S₅₅: And then the other task with subtraction... I think we should explain the same.
S₅₅: I would do the same. If a guy wants \( \frac{1}{3} \) of my pizza.
S₅₅: I think I will use some kind of chocolate bar with 7 rows.
S₅₅: Yes. Then we have 4 rows left.
S₅₅: Yes. Then we can have 21 as a division. If you use pizza you need to divide it in 21 slices, which is a lot.

From the excerpt, it can be seen how they chose a context to suggest a didactical technique \( \tau_{c1} \). The decision to choose the context of pizzas or chocolate bars depends on the denominators involved in the tasks. It seems that the pizza representations or diagrams have some limitation to represent fractions with uncommon or bigger denominators. Their preference for teaching pupils using diagrams was supported by their belief that pupils would learn more from that, and grasp the concept of fractions. This argumentation is a kind of technological discourse that was presented among other pairs during their discussion. Their previous experiences in a mathematics course where they had watched a video how Norwegians worked with pizzas also affect their didactical praxeologies, but they did not discuss this in detail. In addition, at the beginning of the discussion S₅₅ provided a technological discourse based on a number sense strategy, using 1 as a benchmark. He used it to argue why the result \( \frac{3}{5} \) was incorrect.

Besides suggesting the didactical praxeologies based on the contextual or real-life situations, the Danish pairs also proposed some alternative didactical techniques. Two of them were to
provide pupils with some common fractions to work with \((\tau_{c2})\), and to change fractions into decimals and then apply the standard procedure for the decimals \((\tau_{c3})\). Since they just mentioned this without giving any detailed explanation, it was not possible to grasp technology-theoretical discourses underlying those didactical techniques. For instance, S\(_{1a}\) said “pupils could try to change [the fractions involved in the addition task] to decimals and then add them and check if [the answer] gives the same result as \(\frac{3}{5}\). They probably realize that it is not the same”. Then, S\(_{1b}\) just agreed with it without considering some challenges that pupils could find when they changed fractions into decimals because three fractions involved in the tasks have periodic infinite decimals \((\text{e.g., } \frac{4}{7} = 0.571428)\).

The case of multiplication and division of decimals
The second HTT consists of two mathematical tasks given to the pupils. They belong to the following types:

- \(T_m\): given a positive decimal number \(a < 1\) and a positive integer \(b\), calculate:
  - \(T_{m1}\): \(a \times b = \cdots, 0 < a < 1\), and \(b\) is a positive integer.
  - \(T_{m2}\): \(b \div a = \cdots, 0 < a < 1\), and \(b\) is a positive integer.

The didactical task presented in the HTT is to ask Danish PSTs’ ideas to help pupils to understand why the results of the mathematical task \(T_m\) contradict to the pupils’ common knowledge of multiplication and division of natural numbers \((T_d)\). To answer this didactical task, the Danish pairs proposed several ideas that can be generalised into two didactical praxeologies; didactical praxeologies based on supposedly “intuitive contexts” and didactical praxeologies constructed directly from their mathematical knowledge. I discuss both praxeologies in detail in the following subsections.

Didactical praxeologies based on supposedly “intuitive contexts”
One of commonalities among five Danish pairs in discussing the second HTT was to want to teach pupils based on real-life situations. They provided several ideas for how to situate the mathematical tasks in intuitive contexts that they assumed to be related to pupils’ experience. The Danish pairs found some challenges, especially on the division task, because they struggled to shift from division with whole numbers to rational numbers. In general, they proposed two didactical techniques that can be described as follows:

- \(\tau_{d1}\): represent the mathematical tasks into contexts that are assumed to be related to what pupils experience in their daily life.
τ_{d2}: use concrete or manipulative objects to support pupils’ learning of multiplication and division with rational numbers less than 1.

Five and four pairs discussed the didactical technique $\tau_{d1}$ for the multiplication and division task respectively, while the didactical technique $\tau_{d2}$ was only discussed by S_1 for the multiplication task and S_2 for the division task but not in detail. They used various contexts, but most of them still considered to begin with a pizza or a cake context to show that 0.25 equals to $\frac{1}{4}$. As an example, S_{2b} began the discussion by arguing that they could use a cake or a pizza to show the pupils that $\frac{1}{4}$ corresponds to 0.25. They successfully applied this context to the multiplication task with decimals, but some of them found it difficult to explain the division task with decimals. For instance, S_{1b} argued “I do not know how to do division in this case. I can calculate it on my pocket calculator and I know how to read the answer on the display”.

In term of mathematical praxeologies, four and three Danish pairs tended to construct the contextual problems based on the meaning of multiplication as repeated addition and division as repeated subtraction (quotative division) respectively. For the multiplication task, 0.25 $\cdot$ 8 was mostly reinterpreted as 8 $\cdot$ 0.25. For instance, S_{4a} said "here is 25 øre and if you have 8 of these coins how many krones do you then have? (1 krone = 100 øre)”. His partners agreed that it was a good example because they could use the repeated addition. It seems that they were really unaware of how to differentiate $0.25 \cdot 8$ from $8 \cdot 0.25$ when modelling the task by a contextual problem. Meanwhile, when posing the didactical techniques $\tau_{d1}$ based on repeated subtraction, they tended to convert decimals into fractions or natural numbers. For instance, S_{5a} said “in the other case we can say that we have 8 litres of milk, and these should be divided into these 250 [ml]”. This kind of didactical technique will lead the pupils to learn not only the division of rational numbers but also the relation among standard units (1 litre = 1000 ml). Alternatively, one group constructed the didactical technique $\tau_{d1}$ for the multiplication task based on proportion and for the division task based on the inverse of multiplication (Alenazi, 2016). S_{1b} proposed an idea using the example of a quarter of a cake. She said “what do you have with 4 quarters of a cake? You have a whole cake. What if you have 2 cakes? Then you have 8 quarters”. For the division task, it is illustrated by the discussion of PSTs from group 3 as follows:

S_{3a}: However, I think it is a good idea to say that as before “we are 4 people and I get 8, how much do we then have together?

S_{3b}: I have been told always to make a story about the case. I got 100 krone or something like that, and it is told that it was my share of the profit. We are 8 people sharing the profit. How great was the profit?
S₃ₐ: Yes.

S₃ₜ: It must have been 12.5[%], so 100 divided by 0.125. This gives in total 800.

The excerpt shows that they tried to construct a contextual problem that was indirectly recognised as the division task. The pupils may solve this problem using the technique for multiplication instead of the division. In fact, a teacher has an important role to make the pupils aware that there is a link between multiplication and division, namely, division is the inverse of multiplication.

In the discussion, all Danish pairs provided justification why they would teach the pupils based on the real-life situations. They considered that the multiplication and division tasks given to the pupils were too abstract, and those tasks needed to be visualised.

S₂₅: That is what I had in mind. I will try to visualise it. Do they say which grade is it?

S₂₅: No. I don’t think so.

S₂₅: No, but even in the higher grades, they find it difficult to see that it is just a number. I like to visualise it.

S₂₅: Absolutely! We have to show them that 0.25 is the same as $\frac{1}{4}$.

In general, this justification can be formed as a following didactical technology:

θ₃₁: the multiplication and division tasks need to be visualised to help pupils understand the value of numbers better, and the result of the operations.

None of Danish PSTs directly mentioned any didactical theory to justified this didactical technology, but the general virtue of teaching mathematics beginning from contextual problems can be seen as an informal didactical theory ($\Theta_{₃₁}$). This is in line with the Piaget’s theory of cognitive development, including that the pupils in the age of 7 to 11 years are still in the concrete operational stage (Piaget, 1964). But this is not referred explicitly by the PSTs.

**Didactical praxeologies constructed directly from their mathematical knowledge**

Four pairs also discussed some possible didactical techniques that were built directly from their mathematical knowledge. This is mainly because the whole number and the decimal involved in the multiplication and division tasks were intuitively recognised by them. In general, the didactical techniques suggested can be described as follows:

τ₃₃: reformulate mathematical sentences for the multiplication or division tasks that can be easily understood by the pupils.

τ₃₄: use pupils’ knowledge of operations with natural numbers to explain operations with rational numbers (from proportion to function, see example given by S₄₅ below).

τ₃₅: convert decimals to fractions and then teach the standard algorithm for multiplication and division of fractions.
τₐ₆: explain the relation between multiplication and division of two rational numbers (if \(a \times b = c\), then \(c \div b = a\) or \(c \div a = b\))

The didactical technique τₐ₃ was discussed by S₁ and S₃ in relation to the multiplication task, and by S₂ in relation to the division task. One example of τₐ₃ related to the multiplication task was to change 0.25 into 25%. The multiplication task given to the pupils was to take 25% of 8. A mathematical theory underlying this didactical technique is the operator interpretation of rational numbers (Charalambous & Pitta-Pantazi, 2007).

Two pairs discussed the didactical technique τₐ₄ especially for the multiplication task. Actually, there was pros and cons when they proposed this didactical technique because they still considered that it was too abstract and not really related to the contextual situation. As an example, it was illustrated by a discussion from S₄.

S₄c: I think we can make a number table.

then he draws a table as follows:

```
x : 8
<table>
<thead>
<tr>
<th>4</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>12</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
```

S₄c: If we ask them to place numbers like 8, 24, and 32 in a row, then they can see that it has to be greater. Then we could make them write numbers in the row below. It could be a half and then we get 4. Afterwards they could make a graph and see if they can see any relations between the numbers and try to explain why division by decimals gives a greater number.

S₄a: I still think that it is too abstract for them. They need to look at pizzas and work with small things.

S₄b: I think they have misunderstood that. I remember that multiplication always gives a greater number and this is the rule they struggle with. They just think that all multiplication gives a greater number. That is the problem.

S₄c: Yes, but not always. When you multiply by something smaller than 1 you get something smaller.

S₄a: That is the same problem as before (addition and subtraction tasks), where they did not understand the numerator and the denominator in a fraction. The rest is just numeracy. I think it is still too abstract for them.

S₄c: I can see that.

From the excerpt, it can be seen that S₄c provided the didactical technique τₐ₄ built upon the concept of relation and function between factors and products. His mathematical idea was to support the
pupils to recognise that the change of factors will affect the products. Letting the pupils draw a graph might put them into a challenge and learn more advanced mathematics, including early algebra and linear functions such as \( f(x) = 8x \). However, his partners considered this didactical technique as too abstract because it was not so in line with the common didactical theory (\( \Theta_{d1} \)).

The didactical technique \( \tau_{d5} \) was only discussed by one pair, S2. They considered this as an alternative didactical technique to teach pupils. They believed that the pupils could understand this didactical technique if they got the meaning of the numerators and the denominators. This was achieved through visualising the tasks into something that the pupils recognise e.g., a pizza (\( \theta_{d1} \)). Since the task was integrated into multiplication and division of decimals, two groups simultaneously discussed the relationship between multiplication and division (\( \tau_{d6} \)). One example from the discussion of group 4 as follows:

\[ S_{4a}: \text{What happens if we take 2 times 3, this is 6, divided by 3 gives 2. (write: } 2 \times 3 = 6, \ 6 \div 3 = 2, \ \text{and puts a ring around the number 2). What happens if we multiply 0.25 by 8, That is 2. And if we divide 2 by 8 (writes: } 0.25 \times 8 = 2 \text{ and } 2 \div 8 = 0.25). \]

\[ S_{4c}: \text{Can you not choose to make the division the other way around?} \]

\[ S_{4a}: \text{Yes. We can do that. 8 multiplied by 0.25 is equal to 2. And 2 divided by 0.25 is equal to 8. (writes: } 8 \times 0.25 = 2 \text{ and } 2 \div 0.25 = 8). \text{ It is the same journey.} \]

\[ S_{4a} \text{ started presenting an example of multiplication of two positive integers, and then he moved to the example of multiplication of a positive integer by a decimal. He tried to show how multiplication and division were linked each other, and it can be a potential didactical technique to explain the inverse relationship of multiplication and division of rational numbers. Meanwhile, a didactical technology mentioned by this group to justify the didactical technique } \tau_{d6} \text{ was that the pupils had to understand how to multiply by rational numbers before learning the division tasks with rational numbers (} \theta_{d2} \text{). This means that learning the division with rational numbers should link to the previous pupils’ praxeologies on multiplication (} \Theta_{d2} \text{). This theoretical idea is further substantiated by the discussion of concrete didactical techniques.} \]

On the other hand, PSTs from group 1 who tried to employ the didactical technique \( \tau_{d6} \) got difficulty to differentiate between dividing by \( \frac{1}{4} \) with dividing by 4. For instance, after realising the relation between multiplication and division and trying to explain the division task, \( S_{1a} \) said “here (referring to the division task) we multiply by \( \frac{1}{4} \). No, we divide…I can not explain it”. So, what they had experienced is similar to the case of the U.S teachers in Ma’s study (1999).
During the discussion, three Danish pairs mentioned that the pupils have to understand that multiplying a natural number with something less than 1 gives a result smaller than that natural number, and one pair also used an opposite argument that dividing a natural number with something less than 1 gives a result bigger than that natural number. This corresponds to a general mathematical theory about multiplication and division with rational numbers that can be stated as follows:

\[ \Theta_m: \text{if } 0 < a < 1 \text{ and } b \text{ is a natural number, then } a \times b < b \text{ and } b \div a > b. \]

In fact, one can say that these pairs have an adequate mathematical theory that can be used to justify their mathematical technologies.

**Discussion and concluding remarks**

This study examined two research questions. RQ1 concerns Danish PSTs’ mathematical and didactical knowledge which they can develop to address challenging situations in the teaching of rational number operations. This knowledge is studied in term of techniques and technological-theoretical discourse. RQ2 concerns how the Danish PSTs produce mathematical and didactical knowledge through collaborative work, and what the relations are between the two components. This was mostly addressed in the context of the second HTT about multiplication and division of decimals, where it turns out to be more challenging for the Danish pairs to propose satisfactory didactical praxeologies.

Regarding RQ1, the results from this study confirm that the Danish PSTs prefer to build didactical praxeologies based on real-life situations. They tend to explain the meaning underlying the mathematical operations involved in the HTTs based on ”pizza representations” of rational numbers, and this context functions well for the first HTT, but not for the second HTT. In fact, pizza representations mainly apply to small rational numbers, especially in the interval [0,1], as it is based on the meaning of a fraction as a part of a whole. That is only one among five informal models of rational numbers (Charalambous & Pitta-Pantazi, 2007), and instructing pupils based on this model will not be sufficient to build more complete semantics of rational numbers (Lamon, 2012). When the Danish PSTs proposed didactical praxeologies for the multiplication and division tasks based on contextual situations, they mostly interpreted the mathematical meaning of the tasks in terms of repeated addition and subtraction, respectively. But some of them are unsecure whether the contextual problems suggested are consistent with the given mathematical task, such as 0.25 × 8. Misinterpreting the operations of rational numbers into unappropriated contextual problems will blur the meaning of the operations (An et al., 2004). Meanwhile, the most common
technological discourse relates to PSTs’ belief that concrete representations, such as pizzas, chocolates, etc., are the best aid to teach operations with rational numbers. One possible reason underlying this technological discourse is their learning experiences either as pupils at schools or as student-teachers at the university college. The latter included, for some of them, a video of Norwegian pupils’ work with pizza representations, used in the didactics of numbers and algebra. Another reason could be a general pedagogical ideology that “mathematics in real-life” solves all problems for pupils.

Answering RQ2, the Danish pairs struggle to produce appropriate didactical praxeologies, especially as they work on the second HTT. They first consider to propose didactical techniques based on real-life situations, and these techniques relate to their mathematical knowledge, mostly based on a model of multiplication as repeated addition, and division as repeated subtraction. But, some of the PSTs mention that they could not find an appropriate didactical technique to explain the division task. This is caused by the state of their mathematical praxeologies, especially the technological-theoretical discourse related to the justification and details of the standard techniques. In general, PSTs need sufficient mathematical knowledge to build and develop pertinent didactical knowledge (Depaepe et al., 2015). Meanwhile, some potential didactical techniques such as τ_d,t, constructed directly from their mathematical knowledge, tend to be neglected because they consider that it is too abstract for the pupils. This finding is in accordance with what has been found by Winsløw and Durand-Guerrier (2007) that “the Danish PSTs pay considerably more attention in their discussions to the pedagogical aspects of the situation. They primarily search for applied or concrete examples and illustrations”.

To sum up, this study reveals a gap between Danish PSTs’ mathematical and didactical knowledge to teach operations with rational numbers. Their tendency to associate mathematical tasks with real-life problems sometimes does not result in appropriate didactical techniques for teaching operations with rational numbers. An insufficient mathematical technological-theoretical discourse for the technique associated with the operations is one of the main factors of their difficulties. At the same time, it must be admitted that this study has some limitations. The first one is related to the small sample of Danish PSTs participating in this study. There were only eleven first-year students, all from the same teacher education institution in Denmark. So, the data from this study cannot be taken as representative for all Danish PSTs. Secondly, the HTTs only focused on two specific types of task on operations with rational numbers. Further research should address these limitations by extending the number of participants from different years of study and several institutions, and by providing a wider range of mathematical and didactical tasks types, that may be leading them to produce broader mathematical and didactical praxeologies. In relation
to the methodology experimented here, the collaborative setting leads PSTs to discuss their mathematical and didactical praxeologies spontaneously. This can help identify what mathematical knowledge the PSTs have individually, and how they use and negotiate this knowledge to build didactical techniques for teaching rational numbers.

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5 Praxeological change and the density of rational numbers: The case of pre-service teachers in Denmark and Indonesia

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Abstract

The present study aims to introduce the notion of praxeological change, developed based on the Anthropological Theory of the Didactic, to describe a necessity of changing mathematical praxeologies when passing from natural to rational numbers. It is applied to study and compare Danish and Indonesian pre-service teachers’ (PSTs) knowledge of the density of rational numbers. They work in pairs to solve and discuss a hypothetical teacher task, which involves both mathematical and didactical tasks, related to the density of rational numbers. The findings highlight significant differences of the mathematical and didactical knowledge which are shared by the Danish and Indonesian PSTs. In particular, the Danish PSTs are more successful than the Indonesian PSTs in proposing didactical techniques. They use the mathematical idea of converting fractions into decimals or vice versa and representing fractions and decimals on the same number line, while the Indonesian pairs tend to suggest pupils to order fractions and decimals based on the ordering properties of natural numbers.

Keywords: praxeological change, anthropological theory of the didactic, hypothetical teacher task, density of rational numbers.

Introduction

An extensive amount of research on teachers’ knowledge about rational numbers exists (An, Kulm, & Wu, 2004; Depaepe et al., 2015; Ma, 1999; Zhou, Peverly, & Xin, 2006). Among the general observation is that teachers may successfully solve a mathematical task dividing one fraction by another, using a standard procedure, and still be unable to explain why it works or what such an operation could be used for; and that this translates into teaching focused on procedures with little meaning. Moreover, the set of rational numbers has properties, such as density, which make it quite different from the set of natural numbers, and which are difficult to understand even for
teachers. For instance, Widjaja, Stacey, and Steinle (2008) found that Indonesian pre-service teachers (PSTs) had severe difficulties with the density of the set of finite decimals. The teachers overgeneralized their knowledge about whole numbers to the case of decimal numbers.

Many previous studies (McMullen, Laakkonen, Hannula-Sormunen & Lehtinen; 2015; Prediger, 2008; Vamvakoussi, Christou, Mertens & Van Dooren, 2011; Vamvakoussi & Vosniadou, 2004, 2010) have argued that there is conceptual change involved in the process of passing from natural to rational numbers. This means that learning rational numbers requires one to change one’s prior conceptions of something, like numbers, in order to be compatible with a new mathematical situation – they cannot simply be adapted but need more fundamental revision. However, the change required to pass to rational numbers is not only about concepts, but also about practices. Vamvakoussi and Vosniadou (2004), for instance, studied students’ knowledge of the structure of the set of rational numbers from the conceptual change approach, and analysed students’ answers in terms of discreteness–density categories of conceptual knowledge. To understand these categories better, and indeed to model students’ knowledge in a more complete way, one needs to have a theoretical model that could describe the change of practice and theory from natural to rational numbers – in other words, respondents’ mastery of theoretical ideas such as density needs to be related with their grasp of practical tasks, like adding or ordering two numbers.

To achieve such a more detailed and complete model of teachers’ knowledge, this study is based on the Anthropological Theory of the Didactic (ATD) (Chevallard, 1992, 2006, 2007), that emphasises mathematical knowledge as a human practice consisting of two related components than can be modelled into a praxeology (the praxis or the practical block and the logos or theoretical block). To be able to use the praxeology to describe this collective change, and circumvent the individual/cognitive ideas related to the term “conceptual change”, we introduce the notion of praxeological change to describe a change of knowledge that concerns not only concepts in isolation but a coherent whole of practice and theory.

In the present study, we introduce the theoretical framework and in particular the new idea of praxeological change that students (and sometimes also pre-service teachers (PSTs)) need to achieve in order to fully master the practices and theory of rational numbers. Concretely, we shall study and compare Danish and Indonesian PSTs’ mathematical and didactical knowledge related to the density of rational numbers, as an indicator of the extent to which they have themselves achieved the praxeological change from natural to rational numbers, and of the extent to which they could support pupils to accomplish this change. One motivation to compare Danish and
Indonesian PSTs on this specific (and somewhat advanced) aspect of primary school mathematics and didactics is that it might help to explain the gap between Danish and Indonesian pupils’ achievements in international surveys like TIMSS and PISA (Mullis, Martin, Foy & Hooper, 2016; OECD, 2016). Indeed, it is generally assumed that pupils’ mathematical achievement is strongly related to their teachers’ mathematical and didactical knowledge for teaching (Hill, Rowan & Ball, 2005).

Praxeological change

The anthropological theory of the didactic (ATD) is a research programme in mathematics education research that has been developed over more than three decades (Chevallard, 1992, 2006, 2007). ATD emphasises and models the institutional dimension of mathematical and didactical activity explicitly (Bosch & Gascón, 2014). Mathematical practices and knowledge are considered as situated in institutions and are modeled in terms of mathematical praxeologies. The practices and knowledge of an institution related to the teaching and learning of mathematical praxeologies are modeled in terms of didactical praxeologies.

The notion of praxeology is in fact assumed to describe any kind of human practices and knowledge (Chevallard, 1999, 2006). The praxeology consists of two interrelated components, namely praxis and logos. The praxis or practical block consists of a type of task (T) and associated techniques (τ), and the logos or the theoretical block consists of a technology (θ) and a theory (Θ). The techniques functions to solve some tasks of a given type. The technology is used to justify the techniques, and then a theory serves to explain and unify a set of technologies.

In a mathematics teaching and learning situation, a teacher is faced with didactic tasks. Bosch and Gascón (2014) have argued that a didactical praxeology is generated by the task of making certain mathematical praxeologies available to students. Teachers’ knowledge is thus modeled in terms of didactical techniques, technology and theory (Barbé, Bosch, Espinoza, & Gascón, 2005; Rodríguez, Bosch & Gascón, 2008), which are by definition closely linked to mathematical praxeologies.

In the context of elementary mathematics, Arithmetic is one of the important domains to be taught by the teachers and learned by the pupils. There are many intriguing mathematical and didactical praxeologies in this domain. For instance, in the early years of elementary education, common mathematical praxeologies are around natural number properties. One of the mathematical tasks is to find how many natural numbers one can find between 1 and 10. A
technique to deal with this task is counting from 1 to 10. A technological discourse justifying this
technique is that 2 is the successor of 1, 3 is the successor of 2, and so on. The principle (or axiom)
for ordering the set of natural numbers is the theory which justifies this technology. A didactical
task could be to plan a learning activity for pupils to find how many numbers there are between 1
and 10. An example of the didactical technique is to let pupils practice with some concrete
materials such as number blocks. A teacher may argue that the concrete materials help pupils to
count the numbers easily from 1 to 10 (a didactic technology), and in fact, principles of using
concrete objects to represent natural numbers can be seen as an (informal) didactical theory to
justify this technology.

Shifting from natural to rational numbers (often represented as fractions and decimals in the
context of elementary school mathematics), pupils need to reconstruct and deconstruct their
mathematical praxeologies. In this process, they have to develop new mathematical practices,
technologies, and theory, as they will not succeed if they just adopt and extend previous
mathematical praxeologies related to natural numbers. This is not only something which is to be
done by the individual pupil, but it is a process in which the praxis and knowledge blocks of all
pupils have to adapt, over a period of time, to a new institutional standard of praxeologies related
to numbers. In general, this process is called as praxeological change; that is an institutional
deconstruction and reconstruction a set of new praxeologies in order to cope with an extended set
of objects or tasks, when the old praxeologies only apply to a part of these and cannot be extended
or generalised to apply to them all. For instance, there is no way to extend the arithmetic founded
on counting techniques to rational numbers. The change concerns not only the technical level but
also the theoretical one: many notions which are important at the early natural number theory, such
as odd and prime numbers, reverse number (e.g., 21 as the reverse of 12), etc., cannot be extended
to the rational numbers.

To give such a more detailed and definite case where praxeological change is needed, one
can consider the mathematical task of finding rational numbers between $\frac{2}{5}$ and $\frac{3}{5}$. Mathematical
techniques to solve such a task of “finding the numbers in between” have to be changed relative
to what was previously done, and they are not just simply an extension of the old mathematical
techniques for natural numbers. Even though one may notice that rational numbers are
topologically discrete, their order structure makes them “dense in itself”. It means that between
two successive rational numbers such as $\frac{2}{5}$ and $\frac{3}{5}$, one can find other rational numbers – in fact,
ininitely many rational numbers. The praxeologies related to arithmetic mean can be used to
prompt pupils to perceive the concept of density (Brousseau, 1997; Vamvakoussi, & Vosniadou, 2004). In fact, between two rational numbers $x$ and $y$, one can find a new rational number $w_0 = \frac{x+y}{2}$, repeat the procedure for the new interval $[x, w_0]$ to find $w_1$ between $x$ and $w_0$, and so on. This mathematical technique leads one to infer that there are infinitely many numbers between two rational numbers. A similar technique to show the density of rational numbers is based on equivalence classes of rational numbers. For instance, to find other rational numbers between $\frac{2}{5}$ and $\frac{3}{5}$, one can prolong fractions to find the equivalent value for both numbers, $\frac{2}{5} = \frac{4}{10}$ and $\frac{3}{5} = \frac{6}{10}$, and then one can see that $\frac{5}{10}$ is between the two rational numbers and repeat this technique until one realises that there are infinitely many rational numbers between the two rational numbers. Yet another possibility is to convert fractions to decimals (or vice versa), for instance, to find 0.5 between 0.4 and 0.6, 0.41 between 0.40 and 0.50, etc. This reinforces the theory of representations of rational numbers and the practice of changing between them. Consequently, the change of theory (for instance on how numbers are represented) needs to be coordinated with the change of practice. The praxeological change from natural to rational numbers can be illustrated on figure 1.

![Figure 1. A model of praxeological change from natural to rational numbers.](image)

The necessity of pupils (mathematical) praxeological change also has consequences for teachers’ didactical praxeologies, as they need to face the didactic task to support pupils’ praxeological change. In practice, teachers in many countries struggle themselves with more advanced sides of the praxeologies involved in the successive construction of numbers (natural $\Rightarrow$ decimal $\Rightarrow$ rational $\Rightarrow$ real). Mathematical and didactical tasks involving the density of rational numbers can be seen as such “advanced sides” and used to study teachers’ praxeologies and praxeological change, because they require the teachers to reconstruct and deconstruct their mathematical and didactical knowledge as explained above.
Background and research questions

There are quite extensive studies on pupils’ and teachers’ knowledge on the density of rational numbers (Christou, 2015; Depaepe et al., 2015; Durkin & Rittle-Johnson, 2015; McMullen et al., 2015; Stacey, Helme, Archer, & Condon, 2001; Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof, Verschaffel, & Van Dooren, 2015; Widjaja et al., 2008). Most studies show that pupils and teachers often over-generalise the discrete nature of natural numbers to rational numbers. Vamvakoussi and Vosniadou (2004) have argued that the idea of discreteness is the fundamental presupposition that constrains pupils’ knowledge of density. The pupils only see a finite number of different numbers between any two given rational numbers. A similar result is also found in the study of Belgian PSTs: they had only developed an intermediate level of understanding the notion of density, and assume that there was a finite number of different numbers between two given rational numbers (Depaepe et al., 2015).

McMullen et al., (2015) have found that very few pupils exhibit well-developed mathematical knowledge about the density of rational numbers after having attended one-year instruction on decimals and fractions. One reason for pupils’ difficulties about the density of rational numbers is that pupils do not get any direct instruction, involving mathematical notions that can be useful to handle this difficult subject. Actually, they learn that fractions can always be further broken down, such as dividing a “pizza” diagram in smaller units, but this is not sufficient for them to grasp the mathematical phenomenon of the density of rational numbers.

Wu (2014) has argued that the learning of fractions (or rational numbers), including the density of rational numbers, cannot be promoted by emphasizing only the pedagogical, cognitive, or some other learning issues because, above all else, the mathematical development of the subject must be accorded a position of primacy. The education community should begin to accept this fact that there is a need for a “good reference model” which is consistent with a formal theory of current mathematics, such as a rational number is a point on a number line (Behr, Lesh, Post & Silver, 1983), and it can be used to support pupils’ understanding the density properties of rational numbers. In addition, it may be important to investigate the official curricula of schools in relation to such a model, and begin to question if the latter is altogether coherent. Even though one can certainly question whether Wu’s idea of teaching formal definitions in elementary schools is realistic, much research (e.g. Ma, 1999) points to the fact that teachers also do not know these well, and share with pupils more the informal – in part misleading – ideas of what fractions (and more formally, rational numbers) are.
A good reference model needs to be developed in order to examine the ways in which pupils and their teachers perceive the set of rational numbers between two given rational numbers. Stacey et al., (2001) have suggested to use linear model representations such as number lines to support pupils to develop the density of rational numbers. They emphasise that the number line representation has a structure where one can see more rational numbers in between two given rational numbers. One also can see that two or more representations of rational numbers can be located in the same position, and this can lead one to grasp the equivalence classes of rational numbers. On the other hand, teachers who do not realise this fact may put fractions and decimals in two different number lines, and locate them like ordering whole numbers on the number lines.

**Teacher education programmes in Denmark and Indonesia**

The study was conducted in two different countries, Denmark and Indonesia, that have both similar and different characteristics of teacher education programmes. The teacher education in Denmark is decentralised, but in accordance with common legislation (Rasmussen & Bayer, 2014). The teacher education programmes for elementary schools (to teach grade 1-6) and lower secondary schools (to teach grade 4-10) are conducted in seven University Colleges. Meanwhile, the teacher education programme for upper secondary schools takes place in universities. The teacher education program is a four-year professional bachelor degree programme (240 European Credit Transfer System [ECTS]). Every PST has to choose at least two main subjects, with three main subjects being the norm (Ministry of Higher Education and Science, 2015). The total ECTS for the main subjects are around 140 ECTS, and mathematics for grade 1 to 6 or grade 4 to 10 is one among several main subjects that can be selected by PSTs. The mathematics/mathematics education courses to be taught in the programmes are decided by Ministry of Higher Education and Science (2015), but the content of the education is decided by each individual university college (Winlsøw & Durand-Gurrier, 2007). There are four courses related to mathematics and mathematics education (40 ECTS/ 16.7%), namely 1) Learning mathematics, numbers, and processes in arithmetic (plus algebra for grade 4 to 10), 2) Teaching mathematics and geometry, 3) Assessment and stochastics, and 4) Pupils with special needs and mathematical tools.

In Indonesia, the teacher education system is also decentralised. The programme for elementary teacher education (to teach grade 1-6) is offered by 267 universities/university colleges, and the mathematics teacher education programme (to teach grade 7-12) is offered by 268 universities/university colleges (The Indonesian national accreditation agency for higher education [BAN-PT], 2017). Specifically, in Riau province where the data for this study are collected, there are 5 elementary and 6 mathematics teacher education programmes. The teacher
education program is a four-year professional bachelor degree programme (144 credit points in Indonesian sense). The teacher education programmes are under the Ministry of Research, Technology, and Higher Education, but there is no a specific guideline for them (Ristekdikti, 2016). So, every programme has to develop its own curriculum and decide the mathematical contents to be taught. For instance, the Elementary School Teacher Education program at the University of Riau, provides 7 courses (18 credits/12.5%), namely 1) Fundamental mathematics for elementary schools I, 2) Fundamental mathematics for elementary school II, 3) Mathematics education for the lower grades of elementary schools, 4) Mathematics Education for the upper grades of elementary schools, 5) Statistics for education, 6) Capita Selecta mathematics (e.g. problem solving and modelling in mathematics) and, and 7) Indonesian realistic mathematics education (Teacher Education for Elementary School Study Programme University of Riau [Prodi PGSD UR], 2017). The last two courses are optional, but PSTs have to choose one of them.

Research Questions

Taking into account the discussion above, this paper will contribute to the knowledge about the nature of teachers’ mathematical and didactical praxeologies related to the density of rational numbers, taking the comparison of Danish and Indonesian PSTs as a case. The second aspect that we emphasize is about their collective work to support pupils’ praxeological change from natural to rational numbers. We elaborate this aspect on the method section. So, this study will address the following research questions:

RQ1: What are the differences between Danish and Indonesian PSTs’ mathematical and didactical praxeologies related to the density of rational numbers?

RQ2: To what extent have they themselves realised the praxeological change?

RQ3: What didactical ideas to support pupils’ praxeological change on the density of rational numbers can Danish and Indonesian PSTs work out collectively?

Methods

This study was based on ATD, specifically praxeologies, through designing hypothetical teacher tasks (HTT) to investigate and compare Danish and Indonesian PSTs’ knowledge on rational numbers. The notion of HTT is firstly introduced by Winsløw and Durand-Guerrier (2007) and Durand-Guerrier, Winsløw, and Yoshida (2010) to investigate how PSTs interact with a teaching situation which could reasonably arise at school. The mathematical contents of the situations are elementary, in the sense that they can be handled by elementary mathematical and didactical
knowledge. In the context of the density of rational numbers, a problem, such as pupils assume that there is only one fraction between $\frac{2}{5}$ and $\frac{4}{5}$, namely $\frac{3}{5}$, can be seen as a potential situation that may occur in the course of teaching. PSTs need to have adequate and relatively well-developed praxeologies on rational numbers in order to propose appropriate didactical praxeologies on the density of rational numbers.

**HTT about the density of rational numbers**

The HTT about the density of rational numbers is one among 5 HTTs designed for a comparative study of PSTs’ knowledge of rational numbers (Putra & Winslòw, in press). All HTTs consist of two kinds of tasks, mathematical and didactical tasks. The HTT about the density of rational number is designed based on the literature, which has been discussed in the previous section, showing that many pupils and also teachers have severe difficulties to grasp the density property of rational numbers. It is set to enable PSTs to work individually for the mathematical task (question a) and then deploy their mathematical knowledge to address the didactical task (question b) during the discussion. The detailed HTT given to PSTs is described as follows:

*You first ask fifth-grade pupils to discuss how many numbers there are between $\frac{2}{5}$ and $\frac{4}{5}$ and how many numbers there are between 0.4 and 0.8.*

*Then, they say that there is only one number between $\frac{2}{5}$ and $\frac{4}{5}$ namely $\frac{3}{5}$; they also say 3 numbers between 0.4 and 0.8.*

**a. How do you interpret this claims? (to be solved individually within 3 minutes).**

**b. Explain your ideas to teach these pupils? (to be discussed and solved in pairs within 5 minutes).**

**Participants**

The participants are 32 Indonesian PSTs (16 pairs) from the Elementary School Teacher Education study program from a university in Riau province, Indonesia and 31 Danish PSTs (14 pairs and one group of three PSTs) from four teacher training colleges in Denmark. All the Indonesian PSTs were fourth-year students and had completed all courses on mathematics and mathematics education. The course about mathematics education for the upper grades of elementary schools was taken in the second year of their study in which teaching and learning rational numbers, in the sense of fractions and decimals, was the main part of that course. Meanwhile, the Danish PSTs were from different years of study, but most of them were third and fourth-year students. Although they had not completed all courses on mathematics and mathematics education, all of them have already got the instruction related to rational numbers. This topic is part of the course about
learning mathematics, numbers, and processes in arithmetic that was given in the first-year study. So, we assume that they may not have a problem to understand the HTT, and they could also use their prior mathematical knowledge about numbers.

**Procedures and data collection**

The HTT about the density of rational numbers was written in English, and then it was translated into Indonesian by the researcher and into Danish by a Danish graduate student in didactics of mathematics. Two Indonesian and two Danish researchers/educators checked the consistency of the Indonesian and Danish translation, respectively. Then, the HTT about the density of rational numbers, along with the other HTTs on rational numbers, was tested to Indonesian and Danish PSTs from January 2016 to March 2017.

The data consists of PSTs’ individual written answers and video recordings from the discussions carried out in pairs. For the video recordings, the researcher did the Indonesian transcription, and the Danish graduate and Ph.D. students in didactics of mathematics did the Danish transcription. The data were mostly transcribed into English in order to let the data be visible and to be discussed with other didactics mathematics researchers in that area.

**Data analysis: an a priori analysis of HTT**

First, an *a priori analysis* of the HTT about the density of rational numbers is presented through describing some possible reference models based on praxeologies, called *praxeological reference models* (PRM). Then, the PRM is used as a broader guideline to an *a posteriori analysis* of actual data from individual and collective work.

Two possible types of task can be modelled from the HTT; 1) a mathematical task (from question a), and 3) a didactical task (from question b). The types of task can be written, respectively, as follow, and the preliminary *a priori* analysis has been presented in Putra (2016; 2017).

**Tₐ:** given two different fictitious pupils’ answers about how many numbers there are between two fractions and between two decimals, then interpret these answers.

**Tₐ:** given the tasks of type Tₐ together with the pupils’ answers to these tasks, identify methods to teach the pupils?

In general, possible mathematical praxeologies can be illustrated based on three categories; namely discreteness, discrete-density, and density (Vamvakoussi, & Vosniadou, 2004). PSTs who apply directly a praxeology for natural numbers to rational numbers are said to be captured in a
discrete praxeology, as an evidence of a missing or incomplete praxeological change. PSTs who consider that the density of rational numbers depends on how such numbers represented are said to have a discrete-denseness praxeology. For instance, they may say that there are infinitely many numbers between two decimals but not between two fractions, arguing in the first case that they could easily add many 0s to the digits after a comma, but in the second case operating only with one denominator. Finally, having a denseness praxeology means that one has successfully accomplished the praxeological change, to the point of mastering density of rational numbers regardless of their representation. One then successfully figures out that there are infinitely many numbers between to different rational numbers, no matter the representations they appear in.

In relation to the task of type T_a, PSTs may provide different interpretations based on their judgment of the pupils’ fictitious given answers. Some PSTs may agree with pupils’ answers, so they consider to apply mathematical techniques using whole number reasoning to rational numbers. On the other hand, some PSTs may realise that mathematical techniques used for natural numbers, such as counting, has to be changed when they work with fractions or decimals. For example, PSTs recognise that the pupils’ answers are incorrect because the pupils just notice the mathematical task is about ordering fractions with a common denominator or decimals with one-digit, so they suggest a mathematical technique such as converting fractions to decimals or vice versa, partitioning the interval, and arithmetic means. Technological discourse underlying the technique, such as converting fractions to decimals, concerns the passage between different representations of rational numbers, including the passage between decimals and fractions, and between equivalent fractions.

PRM for the didactical techniques to the task of type T_b is developed based on how PSTs interpret the pupils given answers. They may consider that their answers are correct, and then they probably instruct pupils using ordering numbers. On the other hand, PSTs who consider pupils’ given answers as incorrect may propose to explain one of the appropriate mathematical techniques, or they may try to represent both tasks into a number line to show that the two given pairs of fractions and decimals have identical value. Meanwhile, a possible technological-theoretical discourse underlying this didactical technique proposed by PSTs is based on their belief that the pupils need a concrete mathematical task which brings out this point.
Findings

The praxeological analysis of Danish and Indonesian PSTs’ written answers

The Danish PSTs’ written answers have more emphasis on technological discourses than on direct judgment for pupils’ answers to the tasks of type $T_a$. Twenty-seven Danish PSTs provided at least one technology behind pupils’ given answers and misconceptions, and the most common didactical technology for giving such answers was that pupils just considered fractions and decimals as number sequences. For instance, one PST wrote as follows:

When the pupils see numbers between $\frac{2}{5}$ and $\frac{4}{5}$, I assume that they see the number, $\frac{3}{5}$, as part of the sequence of numbers with nominators 2, 3, and 4. Between 0.4 and 0.8, they similarly see 3 number namely 0.5, 0.6, and 0.7 (DS$_{4c}$).

Some Danish PSTs claimed that a lack of knowledge about numbers in general could be a possible reason why they assumed principles for natural numbers would also hold “in general”. The pupils should notice that decimals and fractions are representations of rational numbers. Indeed, using a mathematical technique to convert fractions to decimals, they argued that pupils could see that $\frac{2}{5}$ equals to 0.4, and $\frac{4}{5}$ equals to 0.8.

A few Danish PSTs provided inappropriate examples to expose their mathematical knowledge. For instance, one PST gave an explanation

…but this requires pupils to know about conversion to decimal places and have an understanding that 0.5 is the same as $\frac{1}{5}$ (DS$_{11a}$).

DS$_{11a}$ did not notice the different value or size of these numbers, and she just turned the digit behind the comma into the denominator.

The Indonesian PSTs’ written answers, on the other hand, are more explicit on their judgment than on giving technological discourses for the pupils’ given answers to the task of type $T_a$. Twenty PSTs’ answers agreed that there was only one fraction between $\frac{2}{5}$ and $\frac{4}{5}$, and three decimals between 0.4 and 0.8. Most of their answers were followed by presenting a mathematical technique. For instance, one PST wrote as follows:

I think the pupils’ answers are correct, because we know [that] the only fraction between $\frac{2}{5}$ and $\frac{4}{5}$ is $\frac{3}{5}$, meanwhile decimals between 0.4 and 0.8 are 0.5, 0.6, and 0.7 (IS$_{7a}$).
IS$_{13a}$ just applied the mathematical technique of ordering fractions and decimals because she just showed that both fractions and decimals in the tasks were already in the format of fractions with a common denominator and a one-digit decimal. Meanwhile, only three PSTs’ written answers indicate that there are many numbers between two fractions but not between two decimals, and other four PSTs’ written answers indicate that the there are many numbers between two decimals but not between two fractions. This is illustrated by IS$_{13a}$ on his written answer

I think if the pupil answers that there is a number between $\frac{2}{5}$ and $\frac{4}{5}$, namely $\frac{3}{5}$, it means that his understanding of fractions is correct. Meanwhile, if he thinks that there are three numbers between 0.4 and 0.8, it means his understanding of decimals is still not sufficient because decimals are different from fractions. If he thinks after 0.4 there are 0.5, 0.6, 0.7, and then 0.8, it is not correct because there are also 0.51, 0.52, etc. (IS$_{13a}$, in Putra, 2017).

IS$_{13a}$ considered that fractions and decimals were two different kinds of number. He did not realise (or use) that a fraction such as $\frac{2}{5}$ has the same value as 0.4.

Two Indonesian PSTs provided a didactical technology for pupils’ misconceptions: the pupils just apply their common and simple mathematical technique of ordering natural numbers to ordering fractions and decimals. For example, IS$_{13b}$ wrote on her worksheet as

From the pupils’ answers, I think their answers are based on their simple understanding because between $\frac{2}{5}$ and $\frac{4}{5}$, there is $\frac{3}{5}$ since they have a common denominator, and there are 3 numbers between 0.4 and 0.8 because they are in the same form that is a one-digit decimal (IS$_{13b}$).

Even though IS$_{13b}$ gave the interpretation behind pupils’ misconception, she did not provide any more explanation which indicates that she knows there are infinitely many numbers between them.

Only one Indonesian PST, IS$_{1a}$, explicitly noted that there are infinitely many numbers between the two fractions and also between the two decimals. She supported her answers by prolonging fractions and decimals and showing that the tasks are the same, since $\frac{2}{5} = 0.4$ and $\frac{4}{5} = 0.8$. 

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The praxeological analysis of Danish and Indonesian PSTs’ collective work

The Danish and Indonesian PSTs did the collective work after they completed the written answer to the type of task $T_a$. The discussion focused on the type of didactical task $T_b$, but this could not be separated from their individual work. We analyse their collaborative praxeologies on the three following categories.

Discrete praxeology: no or limited knowledge to support the praxeological change

From the *a posteriori* analysis of Danish PSTs’ written answers and their collaborative work, none of them seems to be in this category. This is because all the Danish pairs were actually able to figure out the pupils’ misconception on both tasks: that the pupils just considered fractions with a common denominator and one digit decimals, and order them as natural numbers. In fact, the pupils did not see that the given numbers could be converted from fractions to decimals or vice versa, and that they have the same value, e.g. $\frac{2}{5}$ equals to 0.4.

In contrast, ten Indonesian pairs agreed that there were finitely many numbers between two fractions and also between two decimals. The mathematical technique underlying their answers was to consider the order of fractions or decimals as a simple extension of the ordering of integers. Even though three pairs applied a mathematical technique of changing fractions into decimals or vice versa, they were still trapped by the ordering problem. For instance, $S_{11b}$ changed $\frac{2}{5}$ to 0.4 and $\frac{4}{5}$ to 0.8, and then realised that both tasks were the same but with different representations. She proposed that there were also three numbers between two fractions because a fraction, such as 0.5 lying between the two decimals, equals $\frac{1}{2}$, and it is also between the two fractions. In addition, her partner, $S_{11a}$, did not provide any suggestion or question that could lead them to consider other rational numbers between the two rational numbers, but she emphasized a generic didactical technique, organising a classroom discussion without giving any explanation on how pupils’ misconceptions might be resolved in such a discussion.

Besides directly instructing pupils to order fractions with a common denominator and order decimals with the same digit, five pairs of Indonesian PSTs in this category suggested a didactical technique of teaching pupils through representing fractions and decimals on number lines. Since they represented fractions and decimals in two different number lines, they could not realise that a fraction, such as $\frac{2}{5}$, would be in the same position as 0.4 if they located both numbers in the same number line. For instance, the following excerpt presents a discussion between IS$_{15a}$ and IS$_{15b}$:
IS\textsubscript{15b}: How can we explain it with number lines?

Then, IS\textsubscript{15a} draws a number line in the paper and writes 0 in the middle.

IS\textsubscript{15b}: I don’t understand it. (points her written answer presented in figure 1a)

IS\textsubscript{15a}: Hah! Why is that?

IS\textsubscript{15b}: That’s why I don’t understand. What if it is represented to a number line? Or \( \frac{1}{5} \) is written under the line. Or like this, \( \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \). (write those fractions under the number line, presented in figure 1b).

IS\textsubscript{15a}: Here is 0. (points the left edge of the number line, presented in figure 1b)

IS\textsubscript{15b}: Oh yes, then the answer is 1. [Between] \( \frac{2}{5} \) [and] \( \frac{4}{5} \) is 1, [and] this is only \( \frac{3}{5} \).

IS\textsubscript{15a}: Yes, that is correct. Let we agree on that (Then, both of them laugh). How about 0.4 and 0.8?

IS\textsubscript{15a}: If it is represented in a number line.

IS\textsubscript{15b}: Correct. There are also 3 numbers.

(Then, IS\textsubscript{15b} writes another number line under the previous one, presented in figure 1b, and shows that there are also 3 decimals between 0.4 and 0.8).

![Figure 1a. IS\textsubscript{15b}'s written answer](image1a.png)

![Figure 1b. Separated number line representations for fraction and decimal tasks](image1b.png)

Even though they succeed to represent fractions and decimals into two separate number lines, they do not figure out that between two consecutive fractions or decimals, one can always find other fractions or decimals (a technology that can lead to the theoretical result of the density of rational numbers). The representations of fractions with a common denominator and decimals with one-digit lead them to consider the mathematical task as more on the order of integers than on the density of rational numbers.

Another interesting finding is from the discussion of IS\textsubscript{4a} and IS\textsubscript{4b}. They agreed that they would teach pupils based on how the textbooks explain the techniques of ordering fractions and
decimals, but they might not have been aware that the tasks in the textbooks just focus on ordering some given fractions and decimals, not on the set of numbers between given numbers. For this pair, direct instruction based on the textbooks is a general strategy to teach mathematics.

**Discrete-denseness praxeology: insufficient knowledge to support praxeological change**

Two Danish pairs are placed in this category. The reason to put them here is their difficulties to synthesise their thoughts about how many numbers there are between two rational numbers. For instance, in one moment in the beginning of the discussion DS$_{1b}$ stated that there were more numbers between $\frac{2}{5}$ and $\frac{4}{5}$, but in the end her partner and her agreed that there was only one number between the two fractions. It seems like they do not have consistent answers to the same task.

Even though the two pairs were not really consistent to the density property of rational numbers, they still suggested some potential didactical techniques to show that both tasks are the same. The first didactical technique is based on a contextual problem, called a pizza experiment discussed by DS$_{1a}$ and DS$_{1b}$. The idea is to link decimals, fractions, and percentages.

We could change the decimals to percentages and get 40% and 80%. Then, we could ask them to remove 40% of the pizza and $\frac{2}{5}$ of the pizza. Then, we could help them understand that they have the same value even though they have different notations (DS$_{1a}$).

They know well that the tasks are the same, and there must be the same numbers in between, but they do not provide any mathematical and didactical justification that there are infinitely many numbers between the two numbers.

Another example is from the discussion of DS$_{9a}$ and DS$_{9b}$. Their collaborative work also focused on showing the two tasks are the same. DS$_{9b}$ reflected what they experienced during a practicum at the teacher training.

Pupils have only learned about fractions and decimals separately. There has not been such a link between the two, which can be quite important for the essential understanding of what fractions are. I remember that in practicum we used a specific application for IPad that is called virtual manipulatives which show with the exact same shapes/figures that one third corresponds to zero point three three three…. And then the bars are equal in size so you can get them as decimals, fractions and percentages (DS$_{9b}$).

DS$_{9b}$ gave a technological discourse on the necessity to recognise both representations, fractions and decimals as pertaining to rational numbers. This could be supported by using an appropriate
media, such as IPad or computer technology, to show that \( \frac{1}{3} \) equals to 0.\( \bar{3} \), but she did not try to link this knowledge to justify that there are infinitely many numbers between them.

Meanwhile, five Indonesian pairs gave inconsistent answers when asked how many numbers there are between two rational numbers. Four of those pairs seem to agree that there is only a fraction between \( \frac{2}{5} \) and \( \frac{4}{5} \), but many decimals between 0.4 and 0.8. For instance, In the discussion IS\(_{13a}\) said that there was only \( \frac{3}{5} \) between the two fractions, because after the numerator 2, it had to be an integer, and that was 3. It seems that his prior praxeologies of natural numbers lead to the misconception of fractions, and he did not use, for instance, that a mathematical technique of prolonging a fraction by multiplying each numerator and denominator by an integer. Meanwhile, for the decimal task, IS\(_{13a}\) argued

Pupils understanding is not yet stable. [They] still lack of understanding about decimals because after 0.4 there are still 0.41, 0.42, etc. Aren’t there? It depends on the numbers of digits behind the comma, but more often there are 3 digits behind the comma…(IS\(_{13a}\)).

IS\(_{13a}\) seems to believe that decimals were more flexible than fractions (a possible technology to justify his answer), as he could easily identify many numbers between two decimals through a mathematical technique of putting some 0s digits behind the comma. He was more dominant in sharing his mathematical and didactical ideas, so that his partner, IS\(_{13b}\), did not give many contributions to construct other possible techniques and technological discourse. IS\(_{13b}\) just relayed what she had written for the question a that the pupils just view the mathematical task based on their simple knowledge of ordering fractions with a common denominator and decimals according to their digits.

One pair in this category, group 16, changed their knowledge from considering many to few numbers between two fractions and two decimals. In the beginning of the discussion, IS\(_{16a}\) asked IS\(_{16b}\) about a fraction, \( \frac{2.5}{5} \), whether it could be between \( \frac{2}{5} \) and \( \frac{4}{5} \), so there were many fractions could be written by the numerators from 2.1 until 3 and then 3.1 and so on. IS\(_{16b}\) found these weird, because a number was written as a fraction and also as a decimal. She realised that 2.5 equals to \( 2 + \frac{1}{2} = \frac{5}{2} \), but she then ignored the denominator. She could not find a correct fraction representation for \( \frac{2.5}{5} \) and supposed \( \frac{5}{2} \) be the result, and this fraction was not in between \( \frac{2}{5} \) and \( \frac{4}{5} \).

IS\(_{16a}\) also agreed with IS\(_{16b}\)’s explanation, without verifying that mathematical technique. Finally, they agreed to instruct the pupils through the mathematical technique of the order of fractions. Then, it was also observed in the discussion on the decimal task. IS\(_{16a}\) considered many decimals
between 0.4 and 0.8, but he was not so sure with his idea and lacked support from his partner. In this case, IS$_{16a}$, a PsT who has potential praxeological change, was unable to maintain his idea because of less technological means. He could not explain why such a fraction, $\frac{2.5}{5}$, can represent a number between the two fractions.

**Denseness praxeology: Sufficient knowledge to support praxeological change**

Thirteen Danish pairs agreed that there are more than just one number between the two fractions and three numbers between the two decimals. They stated clearly during the discussion that there are many numbers between them, and seven of them said that there are infinitely many numbers. For instance, DS$_{2b}$ first said

> There are a lot! There are a lot of numbers between them. There are, for example, complex numbers and much more between them (DS$_{2b}$).

Actually, the example of complex numbers is not appropriate because they do not belong to the numbers pupils know and cannot be ordered consistently. It seems that he just provided an example that he did not really know well. For a further explanation, he argued

> …the numbers are the same. They are only written in different notations. This way we see that there are many other numbers in between. There are also decimal numbers. There are infinitely many numbers (DS$_{2b}$).

In fact, knowing both tasks are the same becomes one technological discourse to grasp the density property of rational numbers.

The two most common didactical techniques suggested by the Danish pairs, are to teach pupils through converting fractions to decimals and sometimes vice versa (12 pairs), and to represent fractions and decimals in a number line (9 pairs). One technological discourse underlying the idea of conversion is to let pupils work more flexibly with rational numbers. For example, DS$_{4b}$ said

> They have to understand the relation between decimals and fractions such that they can transform the numbers. If they struggle with one of the notations, then they can transform it to the other notation (DS$_{4b}$).

Most of the Danish pairs actually tended to convert fractions to decimals more often than vice versa. It seems that decimal representations are more comfortable to see the density of rational numbers. In addition, most of them connected the didactical technique of a number line
representation to conversion, or vice versa. One example comes from the discussion of DS₆ as follows:

**DS₆a:** I (also) think it is fine to start by. It does not seem like they have fully understood some of the things yet, to do one thing at a time. [We] just do fractions to begin with, put those on the number line in relation to each other or what? What do you think?

**DS₆b:** I do not think it clearly appears if they understand fractions or for... or well to a certain degree.

**DS₆a:** Since they say there are only three numbers in between here (decimals) and one number (fractions)… I think there is something we have to work with in both cases. The task is fine, but perhaps in two parts, where you first place fractions and then you place the other, and then you place the third and then possibly a joint all in one.

**DS₆b:** Yes, and then start with a low level of complexity and then mix it together and have different representations.

**DS₆a:** And then we could us it in the future lessons as a tool. The number line they have made, make a joint one or something that could be hung on a larger piece of paper in the class.

**DS₆b:** I just came to think that we could also work with conversion from decimal numbers to fractions and vice versa.

**DS₆a:** Yes.

**DS₆b:** So they really get to understand numbers can be represented in different ways but still have the same value.

From the excerpt, it seems that in the beginning they would like to present both tasks, fractions and decimals, in two different number lines. The idea is to avoid the complexity of the tasks, but it also challenges them to realise that both tasks are the same when presented on a number line. In addition, providing a classroom activity to let the pupils use different representations, decimals, fractions, and also possible percentages, will lead them to the conversion of representations of rational numbers because they need to put a same value of numbers but different representations in the same exact position. This kind of mathematical activities can support pupils’ praxeological change from seeing numbers as part of a discrete set (integers), to seeing them as dense sets.

Two Danish pairs proposed a didactical technique that might be based on the arithmetic mean, but they did not present detailed explanation on how they found it. For instance, DS₂b explained
We could also write \( \frac{2.5}{5} \) between \( \frac{2}{5} \) and \( \frac{3}{5} \). They correspond to the same values. Those numbers that they thought were the numbers in between the decimals (DS\(_{2b}\)).

It seems that DS\(_{2b}\) tried to confirm that \( \frac{2.5}{5} \) is a number that should be in between two fractions as well as decimals. For a further discussion, he provided a technological discourse behind the common principle on choosing rational number representations because some numbers were prettier to write in fractions than decimals, or vice versa. The reason was that some numbers could only be written precisely using fraction notations such as \( \frac{1}{3} \).

Another common didactical technique is to embed the mathematical tasks into contextual situations, discussed by three pairs in this category. The contextual situations presented by PSTs sometimes seem not to be precise didactical praxeologies to teach the density of rational numbers. They tend to focus on a pedagogical aim that the pupils engage in the activity. For instance, DS\(_{5b}\) suggested to doing an experiment of pouring one jug of water into five cups. She proposed to use a cup with 200 ml and a jug of 1 litre of water. The question given to the pupils was how many times could you pour water into the cups? Her idea is to let the pupils consider that pouring water into 2 and 4 cups is similar to 400 ml and 800 ml respectively. This activity supports the pupils to know about equivalent values of two different rational number representations, decimals and fractions, but does not suffice to bring out the concept of denseness.

In contrast, only one Indonesian pair, IS\(_{1a}\) and IS\(_{1b}\), can be placed in this category. In the beginning, only IS\(_{1a}\) considered that there are infinitely many numbers between the two decimals and the two fractions. While, IS\(_{1b}\) stated in her worksheet that she agreed with the pupils’ answers, and explains it based on ordering fractions and decimals by “counting”.

During the collaborative work, IS\(_{1a}\) dominated the discussion, and she confidently stated that there are many numbers between two fractions and also two decimals, as infinitely many numbers. One of her technological discourse is based on equivalent value like 0.4 is equal to 0.400000… and so on, so between 0.4 and 08, there are many numbers. To teach the pupils, she suggested two didactical techniques. First, she provided a contextual situation of dividing a cake, that if one divides a cake into two equal parts, each of part is a half, then one can do it again by dividing a half into more pieces and so on. The second one is to represent fractions and decimals on number lines and then showing that between two fractions or decimals, one can find another number. It seems that she indirectly tries to employ the technological idea behind the arithmetic mean (Brousseau 1997; Vamvakoussi, & Vosniadou, 2004).
Discussion

The overall aim of this study is to present the idea of praxeological change which describes the need for changing practical and theoretical knowledge at some critical points. This idea is applied to a comparative case study of the Danish and Indonesian PSTs. We now summarise the major findings of this study as explicit partial answers to the three research questions, and point out implications for pre-service teacher training related to the rational numbers.

The differences between Danish and Indonesian PSTs’ mathematical and didactical knowledge (RQ₁) are clearly indicated by their individual and collaborative work. There is a wide gap between them, both at the practical and theoretical level, and both mathematically and didactically. The basic difference is that the Danish pairs are more successful with applying the mathematical technique of converting fractions to decimals and vice versa. It leads them to perceive that the pairs of numbers, involving fractions and decimals, have the same value, so the theory of infinitely many numbers between two fractions as well as between two decimals could be achieved. Meanwhile, the Indonesian pairs assume that fractions and decimals are two different kinds of number. They tend to use their pre-established knowledge about whole numbers on both kinds, and apply the mathematical technique of ordering by counting, which amounts to the misconceptions on rational numbers that have also been observed in the study of Widjaya et al., (2008), in the case of decimals. This explains the inability of the Indonesian PSTs to propose sufficient mathematical praxeologies for the density of rational numbers (Vamvakoussi & Vosniadou, 2004). Therefore, acquiring the concept of density is not only a slow process for the Indonesian PSTs, but it can be impossible to achieve for most of them by the end of the teacher training programme. The root of the misconception likely lies with the insufficient instruction in most Indonesian schools (e.g., Revina & Leung, 2018). The school experience is likely limited to working with procedural techniques. This knowledge is then used by the Indonesian PSTs without realise the need for the praxeological change both on the practices and theory related to order of numbers (figure 1); it is not rectified by the instruction during their studies in the teaching training institution.

Through categorising PSTs’ collaborative work into three categories; discrete, discrete-denseness, and denseness praxeologies, it can be seen to what extent PSTs themselves have realised the praxeological change (RQ₂). PSTs in the first category, as most of the Indonesian pairs, did not realise the need for students’ praxeological change although they master some potentially relevant mathematical techniques, like converting fractions to decimals or vice versa. One main reason is their limited mathematical knowledge on subsets of rational numbers (Vamvakoussi &
Vosniadou, 2004), and their tendency to apply the counting order from natural numbers directly to rational numbers. PSTs with discrete-denseness praxeologies mostly realize that fractions and decimals are different representations of the same numbers. They tend to change their praxeologies when they work with decimals and fractions at the same time. This is because finding “many” numbers between two given numbers is much easier for them if they work with decimals than they work with fractions. They just need to add 0s digits after comma. As in other studies (Vamvakoussi & Vosniadou, 2010; Vamvakoussi et al., 2011) we have seen here that the type of the interval endpoints affects their judgment and the mathematical techniques they propose, and also the corresponding technology. In the last category, in which we can say that PSTs have realised the need for conceptual change in the fictitious pupils, we find most of the Danish pairs. One reason for that is that most of them individually recognise that the two mathematical tasks are essentially the same, and they bring this knowledge to the collaborative work. Using the mathematical technique of converting fractions to decimals or vice versa, they show the equivalent value of fractions and decimals such as $\frac{1}{2} = 0.5$. Indeed, one can say that to achieve the denseness properties of rational numbers, one need to have a firm practical knowledge of equivalence class of rational numbers (Wu, 2014). The one Indonesian pair whose collaborative praxeology can be put in the denseness category, only one of the PSTs (SI$_{1a}$) had sufficient individual mathematical knowledge but she was able to convince her partner to accept the infinity of numbers between the two fractions and decimals.

Answering the third research question related to the PSTs’ collective work supporting pupils’ praxeological change (RQ$_3$), it is obvious that the gap between Danish and Indonesian PSTs’ mathematical praxeologies affects the didactical praxeologies they are able to develop, even in pairs. The Danish pairs propose more adequate didactical praxeologies than Indonesian PSTs. Most of Danish pairs employ two specific didactical techniques: emphasize conversion between decimals and fractions, and use the representations of both kinds of number on a number line. They attempt to combine the two tasks into a situation that can support the pupils to conceive several properties of rational numbers, not only density but also others such as multiple representations and equivalent classes of rational numbers. Meanwhile, the Indonesian pairs try to solve the problem by principles used in standard tasks on ordering given fractions and decimals. Some of them also try to use number line representations, while they locate fractions and decimals in separate number lines; they also do not really make use of the fact that any line segment contains infinitely many points (Vamvakoussi, 2017). In addition, a lack of technological discourse or questioning a proposed didactical technique leads to limitations in further elaborations of
didactical techniques. Number line representations seem to be ineffective to support Indonesian PSTs’ praxeological change because they still basically rely on locating natural numbers when working on rational numbers. In fact, the linear model representations of number lines employed by the two groups do not show a similar result to support pupils to develop the density of rational numbers, and they really depend on mathematical knowledge the PSTs have. While, proposing an activity involving contextual situations may support pupils to learn about equivalent representations of rational numbers, it is not enough to develop pupils’ knowledge about the density (McMullen et al., 2015).

It certainly appears from these results that there is a need for the teacher training institutions to focus on the important challenges related to praxeological change from natural to rational numbers, not only in terms of theory (notions, “concepts” and so on) but also at the level of basic practice (figure 1).

**Concluding remarks**

Praxeological change is needed at critical points of learning mathematics, when it does not suffice to smoothly extend “old” mathematical praxeologies. If teachers are struggling with an incomplete praxeological change themselves, they will find it difficult to support pupils’ praxeological change. Through the Danish and Indonesian case, we present the extent the need for such praxeological change is observed both in and by PSTs during their individual and collective work on a task related to the density of rational numbers.

This change, indeed, involves both practice and theory (figure 1) while many previous studies address the changes of practical and theoretical “concepts” separately through the lens of conceptual change, but the point of a praxeological analysis is to consider these levels as closely related. Without putting aside the existing theory of conceptual change, praxeological change can provide a more solid framework for learning the change of human knowledge.

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