UNIVERSITY OF COPENHAGEN
DEPARTMENT OF SCIENCE EDUCATION

Teachers' Practice and Knowledge on School Algebra with CAS

PhD Thesis

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This thesis has been submitted to the PhD School of the Faculty of Science, University of Copenhagen July 2020.

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ACKNOWLEDGEMENTS

I want to thank the following people for participating in my research and supporting me and my work for the completion of this thesis.

- *Carl Winsløw* for many hours of advice and patience. I could not have asked for anything more.
- *Toke Meier Carlsen*, for all the help and support.
- *Marianna Bosch* for advice, communication of the ATD, and great stays in Barcelona.
- *The Faroese Research Council* and the *University of the Faroe Islands* for financial support.
- The preservice teachers at the *University of the Faroe Islands* who bravely ventured into CAS-based teaching.
- The pupils at *Skúlin a Fløtum*, *Lindegårdsskolen*, and *Engelsborgskolen* for being good sports.
- The teachers *Jens* and *Mads* for adopting my lesson plans and turning them into lessons.
- *Irene* and *Leo* for all their love.

POPULAR ABSTRACT

Computer algebra systems (CAS) are an integrated part of teaching and learning mathematics in upper secondary school, which has resulted in more and more teachers in lower secondary school considering implementing CAS. However, in lower secondary school pupils are introduced to elementary algebra, and CAS-use therefor does not seem as obvious. Indeed, some questions remain, including the following: 1) How is CAS compatible with the current approach to teaching school algebra, when the dominant tasks are solving equations? 2) How can teacher education support preservice teachers in implementing CAS when little well-established knowledge exists about CAS-based teaching?

This thesis comprises of a literature review and three papers. The first paper considers the epistemic value of the current paper-and-pencil approach to teaching algebra and how this is influenced by a naïve implementation of CAS. The second paper, on the other hand, studies how CAS can lever the pupils' knowledge and practice for manipulating and solving equations. The third study examines formats for supporting preservice teachers in developing and formulating knowledge and practice for implementing CAS in the teaching and learning of school algebra.

SUMMARY IN DANISH

Denne afhandling består af en gennemgang af litteraturen og tre artikler. En af artiklerne er allerede publiceret, mens de to andre er indsendt til publikation. Dette afsnit beskriver de tre forskningsaspekter som afhandlingen adresserer, et resumé af de tre artikler samt strukturen for afhandlingen.

Denne afhandling behandler det større spørgsmål: ”(hvordan) kan computer-algebra-systemer (CAS) anvendes til at styrke den tidlige undervisning i algebra?” For at tackle dette store forskningsspørgsmål, adresserer denne afhandling tre aspekter. Først, kompatibiliteten af den nuværende tilgang til undervisning af algebra i skolen med implementeringen af CAS. For det andet, ”lever” potentialet for CAS samt designværktøjer til udformning af CAS-baserede aktiviteter. Til sidst, formater for læreruddannelse til støtte for lærerstuderendes implementering af CAS, og hvad det er muligt for lærerstuderende at lære om undervisning og læring af tidlig algebra med CAS. Desuden undersøger afhandlingen også, hvordan forskningsprogrammet den Antropologiske Teori om Didaktik (ATD) kan bruges til at studere didaktiske situationer, der involverer et digitalt værktøj så som CAS.

Den først artikel, *What algebraic knowledge may not be learned with CAS -a praxeological analysis of Faroese exam exercises*, undersøger foreneligheden af den nuværende tilgang til undervisning i algebra i folkeskolen med implementeringen af CAS. Artiklen identificerer og situerer ved brug af begrebet praxeologi fra ATD de fundamentale algebraiske (papir-og-blyant) teknikker i mellemskolen og deres tilhørende læringsmål, samt beskriver de anvendte teknikker der er til stede, når CAS anvendes konsekvent. Analysen sammenligner de to tilgange med hensyn til mangfoldighed af teknikker, ”effektiviteten” af teknikker og antallet af nødvendige teknikker. Derudover undersøger artiklen også om og hvordan de fundamentale algebraiske (blyant-og-papir) teknikker er til stedet i et CAS-miljø.

Den anden artikel, *Designing activities for CAS-based student work realising the lever potential*, undersøger hvordan CAS kan kapitaliseres til at udvikle og formulere studerendes matematiske diskurs af teori, og hvordan begrebet praxeologi kan bruges til at designe CAS-baserede aktiviteter, der realisere ”lever” potentialet for CAS. Artiklen præsenterer to design af aktiviteter, der fokuserer på at udvikler og eksplicit formulere elevernes koncept af ligninger, og teknikkerne og diskursen for at

manipulere ligninger med teorien om at løsningen forbliver den samme. I analysen bruges begrebet praxeologi til at analysere teknikker og viden der er udviklet, formuleret og sat spørgsmålstegn ved af de studerende. Analysen viser et rigt didaktisk miljø, hvor CAS udnyttes til at inkludere eksempler på ligninger der ellers ikke er til stede i mellemskolen, og hvorved elevernes forståelse af ligninger udvikles. Derudover viser analysen af den anden aktivitet at CAS kan bruges til at udvikle og formulere eksplicitte teknikker og diskurs til manipulation af ligninger, og relaterer dette med teorien om at løsningen forbliver den samme.

Den tredje artikel, *A study of a preservice teacher course on the use of CAS in school algebra*, undersøger hvordan formater i læreruddannelsen kan understøtte lærerstuderendes udvikling og formulering af praksis og viden relateret til implementering af CAS i undervisningen af skolealgebra. Artiklen undersøger et lektionsstudieinspireret kursus, hvor grupper af lærerstuderende deltager i planlægning, forskningstimer i 7. og 8. klasse, reflektionsmøder og omskrivning af lektionsplaner. Analysen koncentrerer sig om to grupper af lærerstuderende og deres sidste cyklus med planlægning - undervisning/observationer - reflektionsmøder - revidering af lektions plan. Til at identificere den didaktiske praksis og viden der er udviklet og formuleret af de lærerstuderende, kombineres begreberne praxeologi og didaktiske momenter fra ATD med instrumental genesis og orkestrering. Analysen viser et kursusformat, hvor observationer i klasseværelset fungerer som en katalysator for refleksioner, hvorved der udvikles og formuleres didaktisk viden og praksis.

For at situere forskningen udført i de tre artikler med den eksisterende litteratur, begynder afhandlingen med en baggrundsgennemgang. Gennemgangen begynder med didaktikken for skolealgebra, dvs. betydningen af skolealgebra og problemet med skolealgebra. Derefter i afsnittet *the ATD perspective*, præsenteres de forskningsværktøjer, der er anvendt i afhandlingen, samt ATD-perspektivet på skolealgebra. Det næste afsnit af afhandlingen, *Literature review*, består af tre litterære gennemgange. Den første gennemgang opridser hvordan ATD har udviklet sig og været anvendt til at formulere forskningsspørgsmål og analysere didaktiske situationer inkluderende digitale værktøjer. Den anden gennemgang overvejer mulighederne og hindringerne for implementering af CAS, derudover er gennemgangen struktureret ved hjælp af niveauerne for didaktisk medbestemmelse (levels of didactical co-determination) fra ATD. Den sidste gennemgang studerer formater for læreruddannelsen, der har til hensigt at understøtte lærerstuderendes udvikling af viden og praksis til implementering af CAS i undervisningen af skolealgebra. I det efterfølgende afsnit, *Presentation of research questions*, præsenteres forskningsspørgsmålene til de tre artikler i relation til litteraturgennemgangen. Dernæst gives et kort resumé af de tre artikler. Det sidste afsnit før selve artiklerne, *Conclusion and reflections*, opsummerer konklusionerne fra artiklerne for der ud over at reflektere over fremtidig forskning.

SUMMARY IN ENGLISH

The present thesis is comprised of a literature background and three papers. One of the papers has been published and the other two have been submitted for publication. This section will present the research aspects, a summary of the three papers and the structure of the thesis.

The present thesis considers the broad research question of whether (and how) computer algebra systems (CAS) can be used to strengthen the early teaching of algebra. To address this larger research question, the thesis addresses three aspects. First, the compatibility of the current approach to teaching school algebra with the implementation of CAS; second, the lever potential of CAS and design tools for crafting CAS-based activities that realise the lever potential of CAS; and third, formats for teacher education to support preservice teachers' implementation of CAS and what is possible for preservice teachers to learn about teaching and learning early algebra with CAS. In addition, the thesis explores how the research programme of the Anthropological Theory of Didactic (ATD) can be used to study didactical situations which involve a digital tool such as CAS.

The first paper, *What algebraic knowledge may not be learned with CAS -a praxeological analysis of Faroese exam exercises*, studies the compatibility of the current approach to teaching school algebra with the implementation of CAS. The paper identifies and situates, with the use of the notion of praxeology from the ATD, the fundamental algebraic (paper-and-pencil) techniques of lower secondary school and their related epistemic value, as well as describing the techniques used when consistently employing CAS. The analysis compares the two approaches in terms of the diversity of techniques, the "effectiveness" of techniques, and the number of techniques necessary. In addition, the analysis also examines how the fundamental algebraic (paper-and-pencil) techniques are present in a CAS environment.

The second paper, *Designing activities for CAS-based student work realising the lever potential*, studies how CAS can be capitalised upon to develop and formulate students' mathematical discourse and theory, and how the notion of praxeology from the ATD can be used to design CAS-based activities that realise the lever potential of CAS. The paper presents two designs of activities focusing on developing and explicitly formulating the students' concept of equations and the techniques and discourse for

manipulating equations with the theory of the solution staying the same. For the analysis, the notion of praxeology is utilised to analyse the techniques and knowledge developed, formulated, and questioned by the students. The analysis shows a rich didactical environment where CAS is capitalised upon to include examples of equations otherwise not present in school algebra, which furthers the students' concept of equations. In addition, the analysis of the second activity shows that CAS can be used to develop and make explicit techniques and discourse for manipulating equations, as well as relating the techniques and discourse for manipulating equations with the theory of the solution staying the same.

The third paper, *A study of a preservice teacher course on the use of CAS in school algebra*, studies how formats in teacher education can support preservice teachers' developing and formulating practice and knowledge related to implementing CAS in the teaching of school algebra. The paper examines a lesson-study-inspired course where groups of preservice teachers participate in planning and presenting research lessons in grade 7 and 8, followed by reflection meetings and rewriting the lesson plan. The analysis concentrates on two groups of preservice teachers and their last cycle of planning-teaching/observing-reflecting-replanning. To identify the didactical practice and knowledge developed and explicitly formulated, the paper combines the notion of praxeology and didactical moments from the ATD with instrumental genesis and orchestrations. The analysis shows a course format where the observations in the classroom act as a catalyst for reflections on developing and formulating didactical knowledge.

To situate the research done in the three papers, the thesis begins with a background review. The background review commences with the didactics of school algebra, that is, the importance of school algebra as well as the problem of school algebra. Then, in the section "The ATD perspective," the research tools employed in the present thesis from the ATD are presented in addition to the perspective of the ATD on school algebra. The next section of the present thesis, "Literature review," consists of three reviews. The first review considers how the ATD has evolved and been utilised to formulate research questions and analyse didactical situations. The second review considers the potential and obstacles for the implementation of CAS; in addition, the section has been structured using the levels of didactical co-determination from the ATD. The last review considers formats in teacher education which are intended to support preservice teachers’ development of knowledge and practice for implementing CAS in the teaching of school algebra. Following, the section “Presentation of research questions,” presents the research questions of the three papers based upon the literature reviews. Before short summaries of the papers are given, the last section, “Conclusion and reflections,” summarises the conclusions of the papers in addition to reflections for future research.

LIST OF PUBLICATIONS

In this section is listed all articles which has been published as part of the studies related to the present thesis.

2015 Carlsen, L. M. (2015). Combining CAS-based teaching with lesson study: A Theoretical paper. In M. Achiam, & C. Winsløw (Eds.), *Mathematics and Science: The relationships and disconnections between research and education: Papers from a doctoral course at the University of Copenhagen* (pp. 51-60). Department of Science Education, University of Copenhagen, Denmark. IND's Skriftserie, Bind. 39 http://www.ind.ku.dk/publikationer/inds_skriftserie/2015-39/Hoveddokument.pdf

Carlsen, L. M. (2015). Analysing student teachers' lesson plans: Mathematical and didactical organisations. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2949-2950). https://hal.archives-ouvertes.fr/hal-01289663

2016 Carlsen, L. M. (2016). A lesson study approach to develop instrumental orchestration for student teachers. In M. Achiam, & C. Winsløw (Eds.), *Educational design in math and science: The collective aspect: Peer-reviewed papers from a doctoral course at the University of Copenhagen* (pp. 25-32). Department of Science Education, University of Copenhagen, Denmark. IND's Skriftserie, Bind. 46 http://www.ind.ku.dk/publikationer/inds_skriftserie/educational-design/INDs_skriftserie_nr46.pdf

2017 Carlsen, L. M. (2017). *A study of the development of mathematics student teachers' knowledge on the implementation of CAS in the teaching of mathematics -a praxeological analysis*. Poster presented at Congress of the European Society for Research in Mathematics Education, Dublin, Ireland. https://hal.archives-ouvertes.fr/hal-01942114

2019 Carlsen, L. M. (2019). What algebraic knowledge may not be learned with CAS -a praxeological analysis of Faroese exam exercises. *Educaçao Matemática Pesquissa*, 85-99. https://doi.org/10.23925/1983-3156.2019v21i4p085-099

In addition, the paper *A study of a preservice teacher course on the use of CAS in school algebra* is under revision, and the paper *Designing activities for CAS-based student work realising the lever potential* is still in initial review.

MOTIVATION

The embedding of a computer algebra system (CAS) in teaching school algebra in upper secondary schools is rapidly growing around the world, and becoming an integrated necessity in textbooks, as well as in exams. In Denmark, CAS has been an important part of high school mathematics for decades, and since 2005, the use of CAS has been allowed during the main part of the "baccalaureate" written exams in mathematics. However, problematic effects on teaching have increasingly been reported, and in 2017, a general reform of the Danish high school system included a modest reduction of CAS use in most high-stakes exams in mathematics.

The extensive use of CAS in upper secondary schools in Denmark has led some teachers in lower secondary schools to also consider implementing CAS in their teaching (and even doing so). Beyond local and individual initiatives in this direction, there has been a coordinated national effort. In 2016, the Danish mathematics teacher association, together with other organisations, launched a national project for the integration of CAS in lower secondary schools; the project included the publication and distribution of a book on the implementation of CAS, national conferences for all mathematics teachers, workshops around the country and so forth.

Because my research is about the implementation of CAS, I was invited to the kick-off conference in 2016 and the final conference in 2017. On both occasions, I had the opportunity to observe and talk with many of the teachers and other interested parties. I met a teacher presenting an activity where the students animated dance moves with functions in GeoGebra, which involved relatively advanced algebra. I observed a discussion about the dialectic between techniques and thinking and how to develop and make the thinking explicit when using the app Dragon Box (a dynamic algebra programme). On the other hand, I also came across a teacher who presented uncritical use of a triangle-solver on traditional (paper-and-pencil) problems, leaving me with the following question: "Where is the mathematics in that?" I also met (the largest group of) teachers, who were merely considering how to implement CAS in their teaching; they were looking for inspiration and guidance on how to do so.

This is where the current project comes in: to develop research-based knowledge that can eventually be used to support and monitor the implementation of CAS in the teaching and learning of school algebra. The current study analyses what happens to mathematics if a CAS is used consistently in the traditional approach towards school algebra. We will develop and examine “good” examples of CAS-use that develop and explicit formulate elements of theoretical school algebra. Finally, we will consider teacher education, exploring what is possible for preservice teachers to learn about the implementation of CAS in school algebra from teaching and observing a class.

INTRODUCTION

From the perspective of the teacher, the present thesis explores the use of computer algebra systems (CAS) in the teaching of algebra in lower secondary school (grades 7–9). The thesis is based on three papers that each present valuable results for the successful implementation of CAS for researchers, teacher educators, task designers, teachers and the research programme of the Anthropological Theory of the Didactic (ATD). The papers are not presented chronologically (as they were developed) but in an order that hopefully will enlighten the reader about the complexity of implementing CAS, the didactical potential of CAS to further the learning of school algebra, along with showing how teacher education can support preservice teachers in CAS-based teaching.

To elucidate the complexity of the implementation of CAS in lower secondary school, in addition to situating the current research, we consider the importance and the problem of school algebra, the development and application of the ATD for studying the implementation of CAS, the potential and obstacles for integrating CAS into school systems and efforts in teacher education to support preservice teachers in implementing CAS. For this purpose, we introduce a Venn diagram that consists of the sets CAS, the ATD and teacher education (TE). As shown in Figure 1, we primarily consider the three areas within (i.e., restricted to) didactical research on algebra.

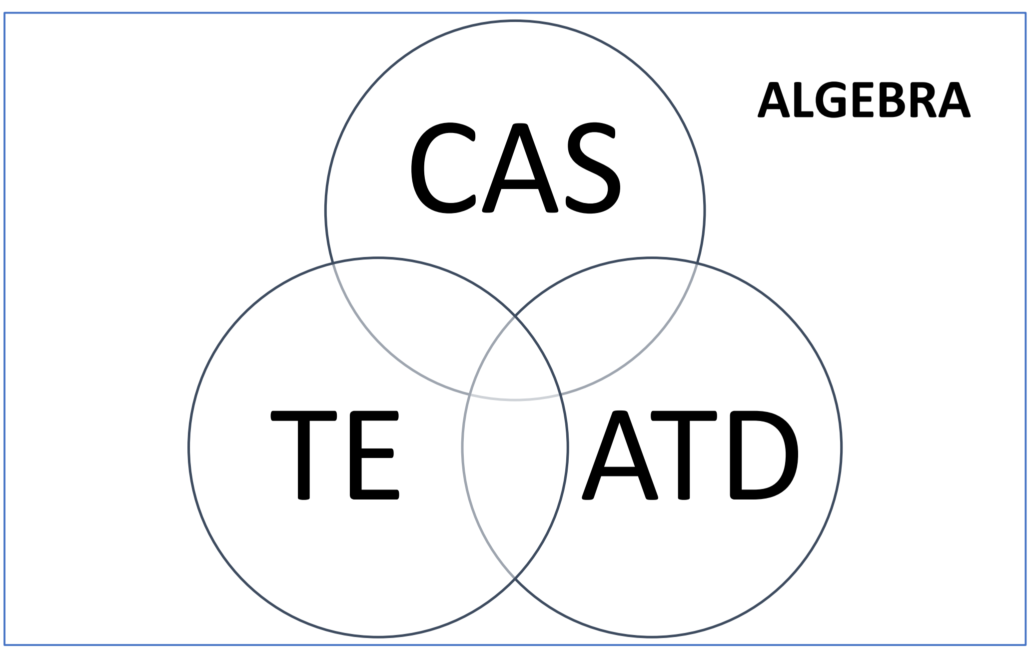


Figure 1. Venn diagram with the sets CAS, teacher education and ATD in the didactics of algebra.

The current thesis will begin by considering the general subject of school algebra, in particular the importance of algebra in an educational context, as well as the classical struggles of students with algebraic work. In the next section, “the ATD perspective”, a tenant of the ATD on posing research questions, the notions from the ATD that will be employed throughout the current thesis and the proposal of the ATD regarding how to consider school algebra, will be presented. The following section presents three literature studies. The first concentrates on the intersection of the sets ATD and CAS, specifically the development of the ATD to include studies involving digital tools, such as CAS, -as well as the research employing the ATD- to examine CAS-based teaching and learning. The second study considers the whole of the set CAS, more specifically studies on the potentials and obstacles related to pupils’ learning with CAS. The review considers the last 20 years of research and is structured based on the levels of didactical co-determination (see later for explanation). The last section, the intersection of CAS and TE, examines the literature on teacher education, which focuses on the development of preservice teachers’ didactical knowledge and practice on teaching school algebra with CAS.

What is CAS?

To start, we present some examples and descriptions of what a CAS is. According to Wikipedia (2020b) “A computer algebra system (CAS) is any mathematical software with the ability to manipulate mathematical expressions in a way similar to the traditional manual computations of mathematicians and scientists”. However, the description on Wikipedia does not include all types of CAS, and at the same time includes apps that are not a CAS. A CAS, such as Maple, which was developed for professionals, can perform mathematical work that cannot be imitated manually, such as solving a large system of equations. In addition, it is not the similarity to manual computation that is the strength of a CAS; rather, it is the feature of outsourcing laborious mathematical work to CAS, and the ability to efficiently obtain an output using CAS (see Figure 2).

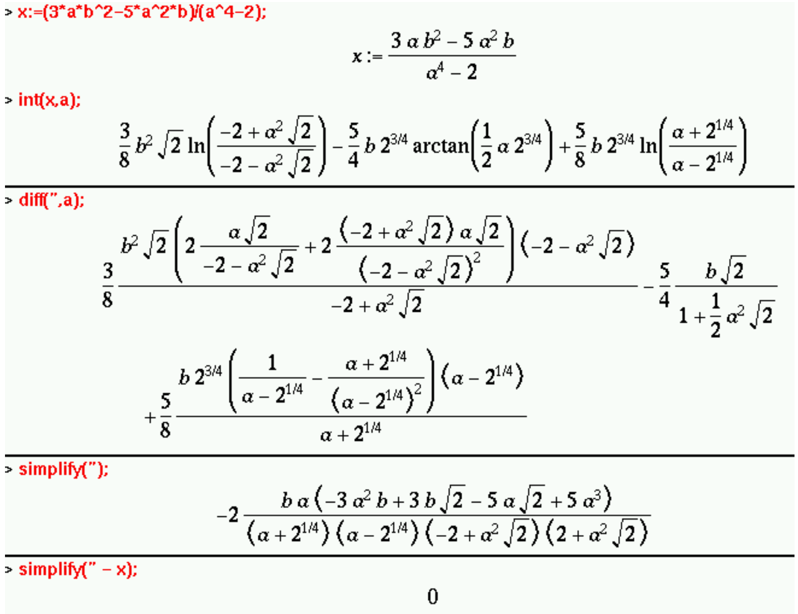


Figure 2. Illustration of Maple

In, what is now considered to be, one of the key contributions to studying the implementation of a CAS in school algebra (Drijvers, 2003b), the following definition is given: “Computer algebra is software that performs algebraic calculations and formula manipulations” (p. 82). This definition of CAS is far closer to our own perception because it includes the word “perform”, indicating that a CAS can be used as an agent to which mathematical work is outsourced. This feature is, from an educational perspective, volatile for the students’ learning and, thus, must be included in the description.

An example of an app that we consider an example of software that can simulate algebraic manipulations is Dragon Box (see Figure 3). It is not a CAS; rather, it is a type of dynamic algebra software.

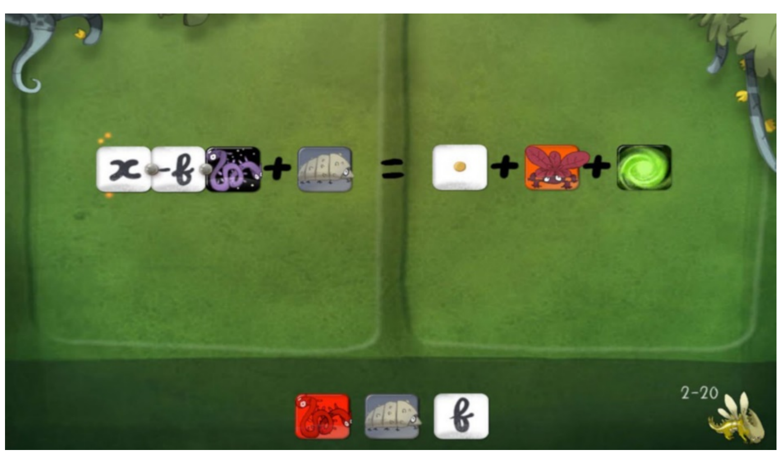


Figure 3. Screenshot of the app Dragon Box.

In this specific app, the user can insert tiles that represent numbers or parameters. A bubble around a group of tiles represents parenthesis; the tiles, which have the same motive, but different colour schemes, represent additive inverses; and the vortex is the unknown. The aim is to isolate the vortex in as few steps as possible. Although the app, in particular during the final levels, simulates algebraic manipulations and the user solves equations, such as $\frac{a}{x} + d + \frac{-b}{x} = 0$, the app does not perform algebraic manipulations; it only simulates.

Another type of algebraic software that can be integrated in teaching and learning of school algebra but is not a CAS, is algebraic microworlds. According to J. F. Nicaud, Pavard, and Bouhineau (2001), this type of software can be characterised by the feature of ”direct manipulation”, that is, with the microworld, the user can perform elementary manipulations, such as adding a number to both sides of an equation (see Figure 4).

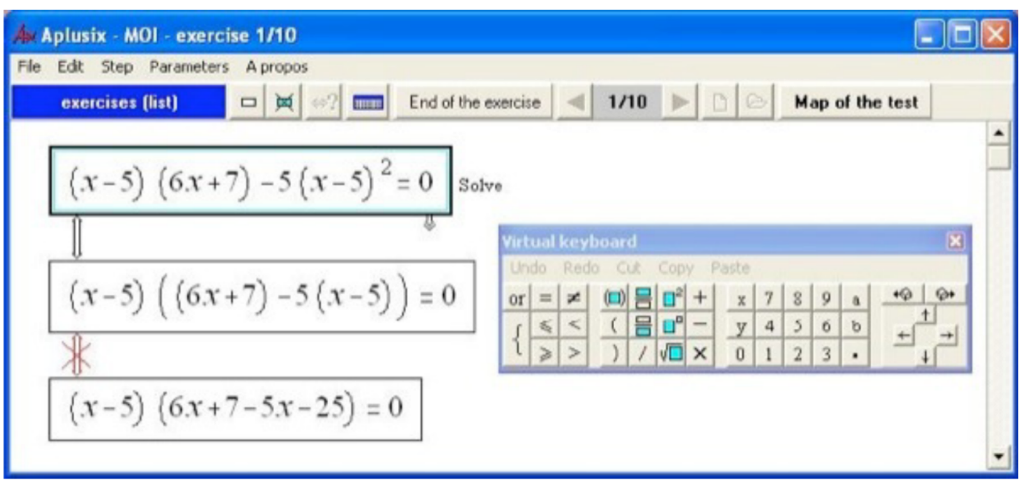


Figure 4. Screenshot of Aplusix (Chaachoua, 2010).

However, with this type of software, the algebraic manipulations are not outsourced to the programme. The user performs the step-by-step algebraic manipulations, and the programme validates the equivalence between expressions.

The last characteristic of a CAS to consider is the variety of algebraic work that is a necessary feature for software to be a CAS. For example, the CAS part of GeoGebra can, at least, factorise or expand an expression, solve an equation or a system of equations, differentiate a function and perform substitutions. However, based on programmes such as GeoGebra, it is possible to create applets, which only allow the user specific, often dynamic, manipulations or input. For instance, there is an applet where sliders are employed to determine the value of the roots in a second-degree polynomial and the value of the coefficient of x^2. Then, the applet displays the graph of the polynomial, the factorisation of the polynomial and the expansion of the polynomial in the form $ax^2 + bx + c$ (see Figure 5).

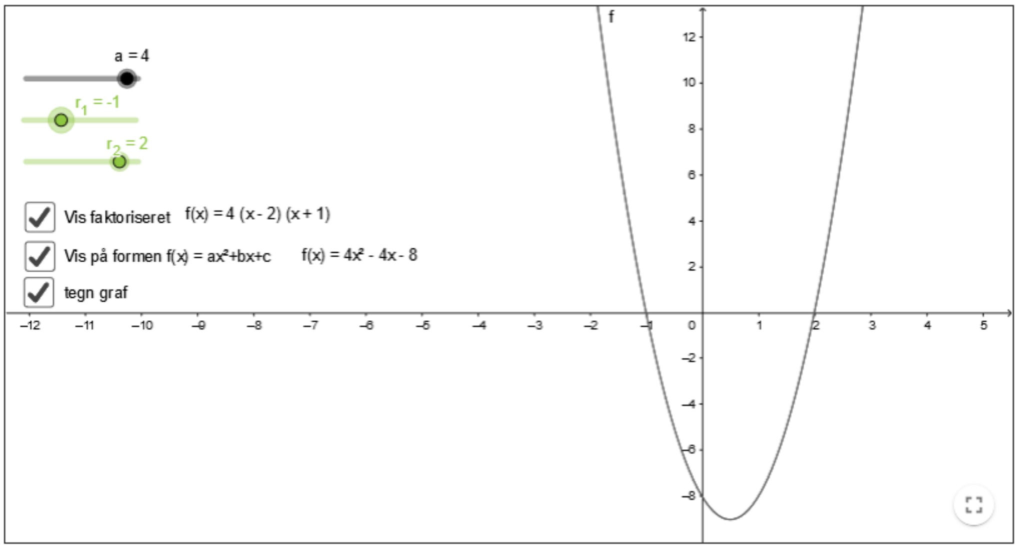


Figure 5. Screenshot of an applet.

However, the applet does not allow the user to substitute, or work with third degree polynomials, or change the roots of the polynomial to rational number or numbers higher than 5.

Based on our three examples, we consider a CAS as having three main features. Concurring with the first two descriptions, a CAS can perform algebraic work such as manipulations. Second, the user can outsource algebraic work, such as factorisation, to the software. And last, the software offers a variety of algebraic work.

Didactics of algebra

Before we go into detail about the importance of school algebra and the related struggles of students, we shortly clarify what school algebra is in secondary schools as there exists different approaches to the teaching and learning of school algebra. Kieran (2007) state that there are three main approaches and contents of courses on school algebra. The traditional algebra courses, which have a strong focus on symbols and the manipulation of symbols, involve tasks such as reducing expressions and solving equations. The reformist algebra courses, where functions are at the heart of the curriculum involving "real-world" problems. And then, there are the types of algebra courses that are in between the two prior ones, emphasising equally equations/algebraic expressions and functions. In this section, where we consider the importance of school algebra and the problem of school algebra, we include all three of the approaches.

The importance of school algebra

To begin describing the importance of school algebra, I would like to cite from the first page of a textbook on abstract algebra (Pinter, 2010) that sums up the importance of school algebra in five lines:

> In elementary algebra we learned the basic symbolism and methodology of algebra; we came to see how problems of the real world can be reduced to sets of equations and how these equations can be solved to yield numerical answers. This technique for translating complicated problems into symbols is the basis for all further work in mathematics and the exact sciences, and is one of the triumphs of the human mind. (p. 1)

In the following, we elaborate and clarify in more detail on the imperative role that certain elements of school algebra play in students' educational advances within mathematics, as well as other disciplines.

The domain of algebra is characterised by the use of symbols, in particular the use of letters, which enables studies involving variables, parameters, equations, functions, inequalities and so forth. The utilisation of letters and symbols allows for the formulation of general formulas across mathematical domains, such as geometry, where the general formula for the area of a circle can be written as $A = r^2 \cdot \pi$. In addition, the use of letters to generalise arithmetic calculations and relations allow students to describe, consider and study an entire family of problems, such as the sum of angles in an n-polygon. In addition, many problems can be formalised using letters and symbols to express variables and parameters, allowing students to study problems, such as the relation of cost and demand, to find the price of a product that yields the highest profit. Another type of object that is fundamental in mathematics is algebraic expressions; these can be utilised to describe and study patterns or generalities. A classic example is the problem where students are tasked to find the expression that describes the number of people who can be seated around a row of *n* tables.

Another characteristic associated with algebra is equations. Equations are pivotal in mathematics because they can be utilised to model a problem with an unknown that can then be solved with algebraic manipulations or CAS. For example, consider the following problem: If your parents start a savings account for you and they want the account to hold 50,000 DKKR by your eighteenth birthday, what should the initial deposit be? Equations can also be used to describe equal relations. For example, students can analyse the number of squares used to build a pyramid, where the base consists of an odd number of squares, where for each row above, two squares are subtracted ($(2n-1)+(2n-3)+\cdots+3+1$), and the number of squares used to build a n x n-square (n^2). Generally, most domains of mathematics in secondary school, whether statistics or geometry, employ algebra in some form or other. In mathematics education at higher education levels, letters, symbols, equations and so forth are used, from the first course to the last across all fields of mathematics. The English Wikipedia page for algebra states, “It is a unifying thread of almost all of mathematics” (Wikipedia, 2020a).

In secondary school, disciplines apart from mathematics also make frequent use of more or less advanced algebra, including physics, biology, chemistry, geography, astronomy and others. The fields sometimes use mathematics as a way of describing an equal relation such as $E = m \cdot c^2$, and at other times to model, such as modelling of the race of a formula 1 car. Now and again, the dialectic between algebra and other disciplines becomes pivotal in developing knowledge. For example, for recommending the amount and frequency of taking a specific medicine, algebra can be used to model the absorption and decomposition of the medicine, which enables students to estimate and make recommendations. At higher education levels, any science contains courses on mathematics and statistics, which make use of algebra extensively whether it is a course on differential equations, linear algebra, or an introduction to statistics. In recent years, a "de-mathematisation" of calculus courses in science education has appeared, and many formal mathematical proofs have had to give way for less formal argumentation. However, algebraic knowledge -such as which grows faster when n increases, 99^n or n^{99}- is still necessary knowledge. Furthermore, equations are fundamental building blocks in many courses, such as linear algebra and stochastic processes, where equations are used to model mappings and time series, such as digital animations or the price of a stock. With the increase in science-related jobs to solve the many problems of our society and the growth of jobs requiring a science education, school algebra becomes even more important.

The problem of school algebra

Although algebra is pivotal as a domain in mathematics and as a tool in many science disciplines, learning algebra is often associated with many difficulties. Research has examined the elements of students' difficulties with algebraic work, such as the minus sign in equations and algebraic expressions (Vlassis, 2004). In this section, we will allude to some of the struggles of learning school algebra.

When working with equations, one of the greatest difficulties for students is the shift in meaning of the equal sign. Prior to the introduction of algebraic work, the equal sign is most often used as a "do-something" sign. However, when working with equations, students have to extend the meaning of the equal sign to include equivalence between two algebraic expressions and as a symbol that signals a study for which the conditions of two algebraic expression are equal (Kieran, 1981). In an abstract task such as $8 + 4 = [\ ] + 5$, students who have not yet developed the multiple meanings of the equal sign will most likely answer 12, 17 or 17 and 12 (Falkner, Levi, & Carpenter, 1999).

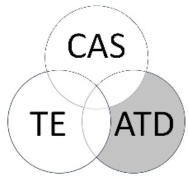
Algebraic work most often includes the use of letters to symbolise unknowns, variables and parameters. In prior work, letters are used as labels or place holders in formulas; thus, when doing algebraic work, students have to develop the use and meaning of letters to include unknowns, variables and later parameters. This process of transition of the students' concept of the use of letters is also connected with struggles (Bills, 1997, 2001; Clement, 1982; Fujii, 1993; Stacey & MacGregor, 1997; Trigueros & Ursini, 1999; Ursini & Trigueros, 1997; Wagner, 1981) as cited in (Kieran, 2006).

In addition, work with equations often includes the students solving them with paper and pencil. This type of work is also associated with great difficulties, such as "ignoring the minus sign preceding a pair of numbers to be combined (Herscovics & Linchevski, 1994); reduction errors (Carry & Bernard, 1979); and erroneous checking behaviour (Pawley, 1999; Perrenet & Wolters, 1994)" (Kieran, 2006. p 14). Solving equations also requires knowledge of algebraic structure and syntax, which is associated with many struggles. Kieran (2006) begins a section on students' difficulties with the theme of algebraic structures and the following three examples:

> Wagner, Rachlin, and Jensen (1984) found that algebra students have difficulty dealing with multiterm expressions as a single unit and do not perceive that the structure of, for examples, $4(2r + 1)7 = 35$ is the same as $4x + 7 + 35$. According to Kieran (1984), students also find it demanding to judge, without actually solving, whether equations such as $x + 37 = 150$ and $x + 37 - 10 = 150 + 10$ are equivalent, that is, whether they have the same solution. More recently, Linchevski and Livneh (1999) found that 12-year-old students' difficulties with interpreting equations containing several numerical terms and an unknown were a reflection of the same difficulties that they experienced in purely numerical contexts. (p. 16)

Although we only presented some of the difficulties for students' algebraic work, these problems are varied and a serious hindrance for algebraic work. The literature on the subject documents the extensive "problem of algebra". Therefore, the allure to implement CAS is attractive to many researchers as well as teachers and students. With the use of CAS, students can skip the troublesome and time-consuming algebraic manipulations of expressions and equations and are freed to focus on more theoretical and conceptual mathematical knowledge.

The ATD perspective



The current thesis will employ and develop the programme of research known as the Anthropological Theory of the Didactic (ATD) when formulating and answering our research questions. The ATD is an extensive framework, and in this section, we present the general perspectives of the ATD that help us question how we can implement CAS in the teaching and learning of school algebra. We first describe the positioning of an ATD researcher relative to the school institution and the tools from the ATD that we employ in our studies before presenting the subject of school algebra through the lens of the ATD.

Posing research questions within the programme of ATD

A main tenet in ATD-based research is the *detachment principle*, posing that researchers must distance themselves from the institutions whose practices they are studying, in the sense of explicitly model those practices, rather than simply adopting implicit assumptions which can reside, for instance, in the institutional terminology. If we consider questioning school algebra in grades 7 to 9, we must model the school's dominant point of view on algebra to describe, analyse and suggest alternatives for the current practice (Bosch, 2015).

Using praxeology, the ATD suggests a tool for researchers to detach themselves from the dominant points of view in institutions, particularly in the institution that is being studied. The tool can be used to identify a main outline of the algebra taught and learned in lower secondary school by analysing textbooks, exam questions, syllabi and other relevant documents. In school algebra, for example, rewriting an expression of the type $(nx + m)l$ into $nlx + ml$ can be related to terminology mainly used in other institutions working with algebra, such as the distributive axiom $(a + b)c = ac + bc$, reflecting the fundamental connection between multiplication and addition represented by the distributive axiom.

To analyse human activity, the ATD suggests considering it as an amalgam of practice and theory. When we do something such as cooking soup in a more or less automatic fashion, there is a practice that can, at will, be made explicit: the practice can be explained and validated, for example, in relation to alternative techniques. In turn, this explanatory discourse forms new knowledge that can be used to develop new practices.

For denoting the dialectic of practice and theory, the ATD proposes the notion of *praxeology*, which is a composition of the two words *praxis* and *logos*. Logos comes from Greek meaning discourse and thinking. The praxis of the praxeology can be described by two components. The first component is *the type of task* that the praxis is intended to solve. The type of task, in some sense, characterises the entire praxeology, such as “cooking vegetable soup”. The second component of the praxis is the *techniques* employed to solve the type of task. The techniques are the gestures performed, such as dicing and frying some vegetables, transferring them into a pot, adding stock and cream and so forth. In a teaching situation, the types of task are often clearly stated, and the techniques can be described by the teacher or by the researcher who is observing the students, not knowing the students’ justification of the techniques. The logos of the praxeology consists of two components. The first component is denoted as *technology* and is defined as the description and discourse of the techniques. The technology used in the technique of rewriting an expression of the type $(nx + m)l$ into $nlx + ml$ could be that the sum $nx + m$ multiplied with the number l is the same as multiplying each term of the sum with l. The second component of the logos is denoted *theory*. The theory of the praxeology exists to justify and unify a collection of technologies. For the technology just described, the theory could be that there is a hidden multiplication sign between the parentheses and the number “outside” of the parentheses, and that the parentheses modify the order of operations because the operations inside the parenthesis must be carried out first.

The notion of praxeology can be used to analyse human activity by explicitly and objectively describing the abovementioned four elements. In a classroom, we can consider the students’ mathematical praxeologies along with the teacher’s praxeology, which is aimed at guiding the students’ mathematical praxeology. This is called the didactical praxeology.

The notion of praxeology can also be used to study lessons involving the use of digital mathematics tools. In such cases, we can see the appearance of instrumented techniques within mathematical praxeologies. For example, when solving an equation with CAS, the techniques that are employed involve entering an equation, employing a “solve” type command and interpreting the output. These techniques are referred to as instrumented techniques because they involve the use of an “instrument” (Guin & Trouche, 1998). The notion *instrument* is sometimes used to indicate the amalgam of a tool and the technical knowledge involved in using it.

To identify and analyse the conditions and constrains that influence the didactical organisation of activities -which is a series of related didactical praxeologies- in teaching and learning situations, the ATD suggests considering the levels of didactical co-determination (Chevallard, 2002). We consider six levels. The first level is the level of pedagogy, which is determined by the teaching principles. The second level is the school disciplines, such as mathematics or physics in their entirety. The third level is the domain, such as school algebra. Then, there is the sector level, such as equations, theme level, such as first-degree equations, and the subject level, such as solving a “simple” first-degree equation (see Figure 6).

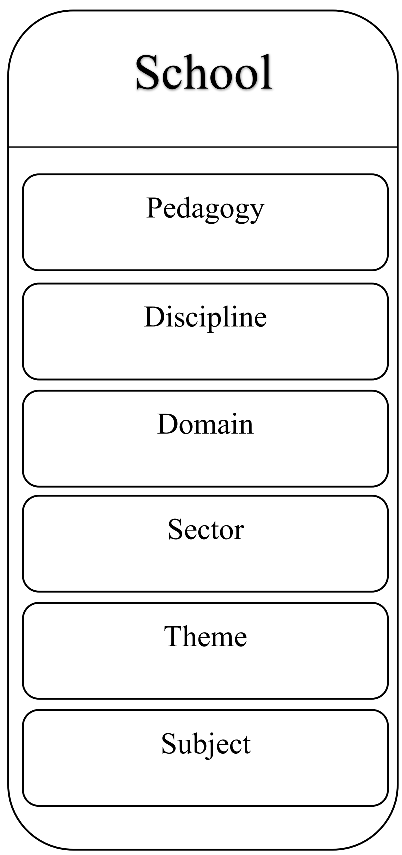


Figure 6. Levels of didactical co-determination (figure adapted from Artigue and Winsløw (2010)).

In our review of the background literature, we employ the levels of didactical co-determination to structure and differentiate between different types of potential and hindrances for the use of CAS.

School algebra

Algebra has been at the heart of many studies within the ATD, including some early and foundational ones (Bolea, Bosch, & Gascón, 1998, 2001, 2010; Bosch, 1994; Chevallard, 1989; Gascon, 1994; Munzón, 2010). However, in this section, we do not give an exhaustive account of the research done on algebra utilising tools from the ATD instead merely explaining how school algebra is viewed by the research programme ATD. A more throughout account can be read in the paper by Bosch (2015), based on her regular lecture at the 12th International Congress on Mathematics Education. At the same time, we also relate the societal context in which our study takes place to the basic assumptions of the ATD.

Studying school algebra within the research programme of ATD, we should detach ourselves from the dominant points of view of algebra in different institutions, such as the school institution in which it lives, but we also need to think of the scholarly institutions where algebraic knowledge is or was produced to avoid the impression that there exists only one form of school algebra. Time should also be considered because the views of school algebra have changed over time and will continue to do so.

The tools and principles of the ATD, especially the notion of praxeology, let us state and analyse the following series of aspects:

> "What is being taught" and show also its undefined features. What is this thing called "school algebra"? What kind of praxeologies is it made of? What could it be made of under other institutional constraints? How does it vary from one school institution to another, both in time (from one historical period to another) and in the institutional space (from one country or educational system to another)? Where does it come from? What legitimates its learning? (Bosch, 2015, p. 55).

Studying the curriculum of elementary algebra in France and Spain, with the new mathematics reform, the traditional triad of algebra, geometry and arithmetic, which takes on a formal approach towards algebra, has changed into other content categories, including the strand “change and relations” (Catalan, 2007). In Denmark and the Faroe Islands (2020), the traditional triad still stands. However, in Denmark, a large section of the curriculum now is formulated in terms of “competencies” that include ideas such as “communication about mathematics”, “posing and solving problems” and “symbols and formalism competence”, here alluding to a shift in the approach towards teaching school algebra, at least at the political level. In studying the types of task present in secondary schools, “Elementary algebra is largely identified with solving equation, mainly first and second degree equations, with some subsequent “applications” to a set of “word problems” coming out of nowhere” (Bosch, 2012, p. 56).

In our studies of school algebra in the Faroe Islands (Carlsen, 2019), where school algebra is present starting from grade 7, the above description of school algebra primarily consisting of solving equations and “applications” holds true, yet here with the addition of solving systems of two linear equations with two unknowns, inequalities and algebraic manipulations such as reducing an expression or factoring an expression. In addition, we can affirm what Bosch (2015) states about the current school algebra: “ [It] is unable to recreate the big variety of manipulations that are needed to use algebra in a functional way, which will be required when students arrive at higher secondary education and find “completely algebraized” mathematics” (p. 56).

In questioning what legitimates its learning, school algebra was prior to the new mathematics reform considered as a mere generalisation of arithmetic, allowing to systematically perform arithmetic calculations, and preparing for further education including formulas and functions (Bosch, 2015). Following the new mathematics period, reformed school algebra has been reduced to a tool for modelling problems of a different type.

In considering society and culture, we can further identify the reasons for this reduction of school algebra. In Western society, orality seems to be privileged, while algebra depends crucially on writing and cannot be “spoken” in its essence. At the level of culture, Bosch (2015) exhibits and illustrates the disdain of algebra by presenting several examples of how formulas and other algebraic work is frowned upon by scholars and in textbooks of other school disciplines. As a future direction for school algebra, Bosch (2015) writes the following:

> Our proposal is to interpret it as a *process of algebraization* of already existing mathematical praxeologies, considering it as a *tool* to carry out modelling activities that ends up affecting all sectors of mathematics. Therefore, algebra does not appear as ”one more content” of compulsory mathematics, at the same level as the other mathematical praxeologies learned at school (like arithmetic, statistic or geometry) but as a general modelling tool of *any* school mathematical praxeology, that is, as a tool to model previously mathematised systems (Bolea et al., 2001; Bolea, Bosch, & Gascón, 2004; Gascón, Bosch, & Bolea, 2001; Munzón, 2010; Ruiz-Munzón, Bosch, & Gascón, 2011; Ruiz, Bosch, & Gascón, 2007). In this interpretation, algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also of questioning, explaining and rearranging already existing bodies of knowledge. (p. 61)

An example of implementing algebra work to generalise other mathematical themes is the generalisation of geometric patterns, such as how many blocks are required to build a (2D) pyramid where the foundation is built from an odd number of squares and for each new layer, the outer squares (two) are subtracted (Måsøval, 2011). A different type of problem for realising algebraic work is the problems that question the world and generate a series of studies where algebra can be developed and used as a tool, such as how to finance a student’s trip by selling t-shirts (Munzón, 2010).

When considering culture, our studies were carried out in the Faroe Islands (an autonomous part of Denmark) and in Denmark. Although the countries have different curricula and exams, the triad of geometry, arithmetic and algebra continue to be explicit domains in both the curricula for lower secondary school, coming with the more recent addition of stochastics. Furthermore, both on the Faroe Islands and in Denmark, school algebra is comparable to school algebra in Spain and France before the new mathematics reform, where algebra appeared as an abstraction of arithmetic calculations and as a tool for describing and solving simple “word problems”. Differences can also be detected, though, such as a larger part of the written exam being devoted to evaluating the students’ proficiency in school algebra on the Faroe

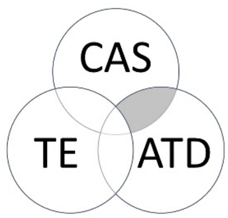
Islands than in Denmark. However, compared with other European countries, Denmark and the Faroe Islands still examine school algebra rather extensively and broadly. Because our studies took place on the Faroe Islands and in Denmark, they involved schools that considered algebra as a domain to be studied in and of itself.

Literature review

In this section, we give a systematic review of the literature that our research builds upon. Going back to the Venn diagram from earlier, it illustrates the themes of the thesis (see Figure 1). Because our research falls into the smaller sets of the Venn diagram, our review will concentrate on these three sets. The first set is the intersection of the ATD and CAS, where we focus on the development of the ATD to include research about the implementation of CAS and the tools employed from the ATD; this includes which types of research questions can be brought forward to examine the topic. The second review considers didactical use of CAS in school algebra. To mediate teachers' weighing of the pros and cons for the implementation of CAS in the teaching of school algebra, we focus on studies that describe either the potentials of CAS to develop algebraic knowledge and practice or the obstacles for CAS-based teaching. In the third and last section of the review, we concentrate on the intersection of CAS and teacher education. We describe studies that focus on and identify explicitly the development of preservice teachers' knowledge and practice for implementing CAS in the teaching and learning of school algebra.

To encapsulate the main tendencies of research about the themes mentioned above, we selected research devoted specifically to studies involving digital tools. Because the ATD originated in France, we have also included one French journal and the conference proceedings of CITAD available. The list of reference works consists of the following: the journals *International Journal of Mathematics Education in Science and Technology, International Journal of Computers for Mathematics Learning, International Journal for Technology in Mathematics Education, The International Journal of Computer Algebra,* and *Recherches en Didactique des Mathématiques*, the book series *Mathematics Education in the Digital Era* and the conference proceedings of CITAD (International conferences on the ATD). We also searched the reference lists of the papers we found. Because of language issues, we primarily concentrated our search to articles in English. For the journals, the search word "algebra" was used for a first sorting of the articles. Then, the list of articles was manually sorted by title and abstract, excluding any paper not concerned about the teaching and learning of algebra at the secondary level using CAS. Therefore, we excluded papers only focusing on teachers' views or any wider issues. The last sorting, which was based on skimming the remaining articles, excluded articles that did not study either the obstacles or positive influences of CAS in explicit detail, employed the ATD or focused on the development of preservice teachers' practice and knowledge.

The ATD on CAS



The initial development and foundation of the ATD was when CAS was not yet employed in the teaching and learning of school algebra, and its description of theories, tools for analyses and reflections did not include digital tools. Researchers have since adopted and developed selected notions from the ATD to question and study the implementation of CAS. In particular, the term "technique" has gained acceptance and is used outside of the ATD circles to examine CAS-based teaching. The current thesis aims at furthering the employment of tools from the ATD to study the implementation of CAS. Our review of the intersection of the ATD and CAS will hence focus on the development of notions from the ATD to analyse and reflect on the use of CAS in the learning and teaching of school algebra. In addition, we will describe how tools from the ATD have been employed and how research questions have been asked.

We will begin by reviewing an article by John Monaghan in which he describes the development and contributions of the ATD to do research on the implementation of CAS. We then consider some of the early key developments and reflections by Michèle Artigue, Jean-Baptiste Lagrange and Carolyn Kieran on the didactic use of CAS. We end the section by outlining points from articles employing ideas from the ATD to study specific questions concerning the implementation and use of CAS.

Monaghan (2007) outlines the historical development of the ATD approaches for studying didactic CAS-use. Monaghan considers the position of France in the growing development of innovative activities in secondary school and college mathematics classrooms involving CAS. He relays that a research team led by Michèle Artigue, which was supported by the French Ministry, questioned the actual use of CAS, its constraints and obstacles and its emergent principles. To establish a basis for their initial research, the team connected to the work of Verillon and Rabardel (1995) on instrumentation and to the notion of praxeology from the ATD. This led to the idea that "tool use does not exist in a vacuum, tools are used in a context/ in practice/ in activity" (p. 4) and view that new elements of practice or activity could be captured through praxeology. To distinguish different values or functions of a technique (for users), Artigue and Lagrange talk about pragmatic and epistemic values (more on this later). Monaghan considers the most controversial part of their work to be "their claim that the relationship between techniques and conceptual understanding is a highly complex one (or to put it bluntly, that using technology to bypass techniques is not a quick way to conceptual thinking)" (p. 5). One of the controversies is whether a theoretical understanding of a topic may be approached through sequential procedures, speaking about and addressing techniques and exploring the limits and possibilities of these techniques. In a closing remark, the author comments how "the anthropological approach, especially its focus on the technical-conceptual cut and epistemic/ pragmatic values, has, I believe, taken us forward in theorizing the complexity of supporting learners in technological environments" (p. 10).

The next series of articles, that we now turn to, focuses upon how to use and develop the tools offered by the ATD to include the implementation of CAS. These papers include, in particular, a number of works by Artigue, Lagrange and Kieran.

Lagrange (2005a) deploys the notion of didactical transposition from the ATD to reflect on the transposition of computer tools, such as CAS, from being efficient tools used in mathematical research to being implemented in the teaching of mathematics. These two uses of computer tools seem to substantially differ. The use of computer tools in the mathematical sciences is motivated by their effectiveness in carrying out technical work, while teaching "is not primarily interested in improving mathematical productivity by way of new tools but rather in the transmission of a mathematical culture" (p. 70), and the kernel of the mathematical culture was built only when traditional tools existed. Lagrange considers two examples of transposition from the mathematical sciences to the teaching of mathematics, which have been influenced and made possible by the implementation of digital tools. One of them is the transposition of the Euclidean algorithm into the school curriculum. He notes how the presentation in the textbooks offers neither practical nor theoretical dimensions, and the algorithm appears as an isolated object without visible interest or in relation to other mathematical results. As another example, he considers the transposition of the experimental dimension of mathematical culture. Lagrange points out several obstacles for this transposition and concludes with the following:

> Experimental computer-aided approaches to teaching and learning cannot be thought of as simply a matter of using a machine to ease problem solving or to enhance inductive activity, but requires reflection on what prior knowledge students need both in algebra and about the machine, on what questions can serve to provoke inductive thinking, and on what form students' representation of concepts and of the machine operation takes. (p. 79)

Artigue (2002a) has what is now considered to be one of the key articles in research on the implementation of CAS in the teaching and learning of mathematics, specifically a proposal of theoretical innovation. Building on the dialectic of practice and knowledge from the ATD, to study the constraints that institutions impose and the potentials for the implementation of CAS, Artigue defines the aforementioned values of a technique: a "pragmatic value, that is to say, by focusing on their productive potential (efficiency, cost, field of validity)" and "an epistemic value, as they contribute to the understanding of the objects they involve" (p. 248). The traditional paper-and-pencil techniques for manipulating algebraic expression for teachers has a well-established epistemic value in school algebra. However, the use of instrumented techniques struggles to gain mathematical status, and the epistemic value appears weak to several agents in the school institution and beyond.

Further advances for studying the implementation of CAS using the ATD have been accomplished in several articles (Hitt & Kieran, 2009; Kieran, 2008; Kieran & Drijvers, 2006); these studies build on the works of Lagrange and Artigue, with additional developments of the idea of the epistemic value of techniques. From the ATD, the articles adopt the notions of task, techniques and theory, giving theory a wider meaning by including the associated technology. In the first article, the T triad (task, techniques and theory) is used as a design tool for developing the activities involving CAS. The activities intertwine both paper-and-pencil techniques with instrumented techniques, (instrumented and non-instrumented) techniques with theory and vice versa. In the reflections on these intertwinings, Kieran and Drijvers (2006, p. 256) note the following:

> This interaction proved to be very productive in cases of confrontation, or even that of conflict, between the techniques—particular the CAS techniques—and

the students' theoretical thinking… the epistemic value of CAS techniques by themselves may depend both on the nature of the task and on the limits of students' existing learning. When students cannot explain, in terms of their current theoretical and technical knowledge, that which a CAS technique produces, reliance on additional CAS techniques may not suffice. In such cases, the epistemic value of paper-and-pencil techniques would seem to play a complementary, but essential role.

The quoted article reinforces the conclusion of the first article (Hitt & Kieran, 2009) of using the T triad to design and analyse CAS-based activities that build on developing students' theory by crisscrossing CAS and paper-and-pencil activities. In addition, the T triad can be utilised to study the development described in the quote above. In particular, the structure of the tasks that the students work on, the techniques they develop and employ and the theory they articulate can be mapped out for concrete observations.

The focus of the next series of articles is primarily on what the studies bring to the area of research on implementing CAS. We will describe which ATD tools have been employed and what results they contribute with.

Lagrange (2005b) examines the possible impact of CAS on the study of mathematics by using task and technique, analysing the pragmatic and epistemic value of instrumented and non-instrumented techniques. He considers a task initially studied by Chevallard (1999) to reduce an expression $\frac{a+b\sqrt{2}}{c+d\sqrt{2}}$, where a, b, c and d are integers into an expression of the form $\alpha + \beta\sqrt{2}$, where α and β are rational numbers. Lagrange argues that the non-routine solving of such a task using a standard technique to obtain an integer denominator includes a series of elementary actions requiring an algebraic analysis of the expression. The epistemic role of the technique is "developing knowledge of algebraic properties of quotients and radicals, developing a procedure for obtaining canonical forms, and provides some familiarity with the structure of the field $\mathbb{Q}[\sqrt{2}]$"; with the use of CAS "it is possible to do more examples and orient the activity towards *pattern discovery*…and *generalization* building a praxeology for $\mathbb{Q}[\sqrt{k}]$ or $\mathbb{Q}[\sqrt[3]{2}]$" (p. 5). As he continues to present examples of tasks and analysing paper-and-pencil and instrumented techniques, he comments, "We cannot envisage students doing mathematics only by using CAS. Rather, we envisage a “CAS assisted” practice intertwining technology and paper-and-pencil. Thus, we should think of the use of CAS as calling for an interrelation between new techniques and paper-and-pencil techniques” (p. 11). In considering the role of the teacher and the introduction of new praxeologies, Lagrange concludes that “s/he has to integrate these techniques into his/her own understanding of the domain, into his/her own personality and to crate relevant situations, certainly not an easy task” (p. 17). He furthermore recounts an example of two teachers who wanted to integrate a CAS in the study of logarithmic functions, requiring the teachers to build an entirely new approach to the domain.

The article (Hitt & Kieran, 2009) studies the (epistemic) value of techniques to develop a conceptual understanding and the vision of Lagrange to intertwine technology and paper-and-pencil knowledge and practice to further the development of school algebra. The authors adapt the model of praxeology, reducing it to the T triad (task, techniques, and theory). The T triad is used to design a series of eight activities that examine, conjecture, verify and explain factorisation patterns of polynomials of the form $x^n - 1$. In the account and analysis of those activities, the paper-and-pencil techniques play just as important a role as CAS techniques in developing theoretical knowledge. In a closing remark, the authors conjecture ”The T-T-T design of tasks, where an equilibrium exists between paper-and-pencil and technology activities, as we followed in our experimentation, can fill the gap between the practices of the teachers who rejects the use of technology in the classroom and those of the enthusiastic teacher who uses technology in a somewhat naïve way” (p. 151).

In a study of modelling tasks, where the students use CAS, Zehavi and Mann (1999) use the notion of praxeology to identify “a new praxeology of teaching and learning mathematics with CAS” (p. 15). The activities are based on modelling tasks such as the following: “How long did Diophantus live given that Diophantus spent 1/6 of his life as a child, 1/12 as a young man, and then 1/7 of his life as a bachelor. Five years after he got married, he left his hometown. He returned to his hometown after 4 years before his death. Diophantus stayed away from his hometown ½ of his life” (p. 254). In the authors’ analysis, they describe the traditional didactical praxeology and identify a new didactical praxeology offered by the implementation of CAS. At the task level, the traditional activities of translating a word problem into equations can be supplemented by the new type of task: inventing problems for a given type of model. The traditional didactical techniques of demonstration by the teacher were replaced with “teachers transferring the control over the process of modelling to the pupils who debug their work using CAS commands” (p. 264).

Braukmüller (2018) studies a discussion between textbook authors on the implementation of a digital system (Algebra Tiles) developed specifically for learning school algebra. The notion of praxeology is employed in a preliminary analysis to identify elements of the didactical praxeologies. The analyses allow the researcher in “finding obstacles and constraints that are in the authors’ minds and gives space to a process of approaching the integration of Algebra Tiles” (p. 579). One obstacle identified in the didactical technology is “the balance model is easier” (p. 579), referring to the idea of representing an equation as a scale where the equilibrium must be kept when performing manipulations.

Martínez, Kieran, & Guzmán (2012) use the T triad to “analyze and discuss students’ performance in a CAS environment related to the simplification of rational expression” (p. 1089), such as $\frac{2(a+b)}{2}$. The use of the terms types of task, techniques and theory allowed the authors to identify the students’ development of mathematical discourse and the role of previously obtained paper-and-pen techniques in developing of new practice and knowledge. At first, the students “expanded the expression, and after that, they cancelled out the repeated elements in both the numerator and denominator” (p. 1092). In a second series of activities, CAS was used to explore other cases of rational expressions, along with a discourse and description of the techniques evolved. With further exploration of examples and the use of the command simplify, the students developed the idea of factoring to simplify a rational expression.

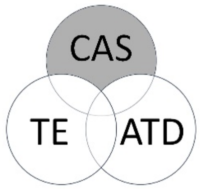
In his habilitation thesis, Chaachoua (2010) uses the notion of praxeology to examine school algebra in extensive details, describing the intricate structure of solving algebraic problems. For example, for the type of task, such as solve an equation of the form $ax^4 + bx^2 + c = 0$, the related techniques consist of completing three subtypes of task: substitute $u = x^2$, find the solution for an equation of type $au^2 + bu + c = 0$, and solve the equation $x^2 = s$. He further structures the types of task into families of locally related types of task, which again are associated regionally with other families of types of task. The structure and web of algebraic praxeologies is used to construct and manage the use of the digital tool Aplusix.

Reflections/ our contribution

The review shows the development of the ATD to include research on the implementation of digital tools. It also describes what the ATD can bring to this research area in terms of research questions and tools for a more precise examination of the mathematics actually taught and learned. With our research, we hope to develop these early endeavours, carried out with the ATD, to study the implementation of CAS.

In our use of the ATD to question CAS-based teaching and learning, we use the affordances of the full notion of praxeology to articulate in explicit detail the epistemic value of the paper-and-pencil techniques in school algebra and their relation to a CAS environment (Paper I) and what values might be lost with the careless use of CAS. In Paper II, the full four T structure of praxeology -in particular the distinction between technology and theory- is used to design and analyse activities for the exploration of the didactic potentials of CAS. The structure of praxeology is used to describe how the relation between paper-and-pencil techniques and elements of theory can be articulated in CAS-based activities. Additionally, the notion of theory is used to design activities that focus on the abstract concept of equations. In Paper III, a course for preservice teachers about the implementation of CAS in school algebra is described and studied. To analyse the activities utilising CAS in the classroom, including the teachers' didactical organisation and the students' activities, several frameworks, including the ATD, are combined. From the instrumental approach, instrumental orchestration and instrumental genesis are used. In favour of explicitly describing and relating the instrumental orchestration, the instrumental genesis of the pupils, and the development of didactical knowledge of the preservice teachers, moments of didactic processes and praxeology are employed. This allows us to describe and identify the preservice teachers' development of didactical practice and knowledge based on their experiences and observations in the classroom.

CAS and school algebra



The implementation of CAS in teaching and learning school algebra introduces a series of new techniques, providing opportunities for new types of mathematical and didactical praxeologies, and changing the scenery of school algebra as new themes and approaches become available. However, the utilisation of CAS also introduces new obstacles for students' learning.

In this section, we review the literature focusing on the potential of CAS and the obstacles associated with their use. The objective is not the frameworks used but rather the results of the studies we explore. To structure the review, we employ the levels of didactical co-determination from the ATD (see Figure 2). In addition, the levels are used to categorise and differentiate between the different types of potentials and obstacles. With this categorisation, based on the levels of didactical co-determination, we can identify further research potential and situate our own studies in relation to research beyond what has been done within the ATD. The review is limited to studies published after 2000 and that explicitly describe students' practices and development of knowledge.

The lever potential

The most commonly cited *raison d'être* for implementing CAS in the teaching of school algebra is what is sometimes called *the lever potential* (Dreyfus, 1994; Winsløw, 2003) of CAS. For example, in a teaching situation, the focus can be more on activities concerned with the study of theoretical elements or about explicitly developing and formulating theoretical elements, i.e. technology or parts of the theory. In addition, the implementation of CAS makes it possible to include themes from other domains of mathematics, even themes that traditionally belong to university studies. In one of the first articles systematically addressing the lever potential and obstacles for the implementation of CAS in school algebra, the following is stated:

> Computer tools have the potential to contribute to the learning process not only as amplifiers (saving time on computations and making graphing easy in the above example) but also, and more importantly, as reorganizers: Mathematics itself becomes different for the learner; new tools change cognition.

> Representations can be linked. Diagrammatic and qualitative approaches can be taken. (Dreyfus, 1994, p. 210).

Here, Dreyfus envisions how CAS can influence the teaching of school algebra. At the lower levels of didactical co-determination, the didactical use of CAS can shift the focus of the teaching of algebra from techniques (the level of subject) to technology and theory (the levels of theme and sector). At the upper level of pedagogy, a more qualitative and experimental approach can be realised.

For most of the studies before 2000, concerned about teaching and learning using CAS, the main interest was, in fact, on what we now call the lever potential. Going through the more recent research on the implementation of CAS, a larger number of examples still demonstrates the lever potential of didactical CAS-use. These studies indicate commonly held hypotheses and motivations for investigating CAS in the learning of school algebra and explain why the majority of studies on CAS and school algebra are still, after more than 30 years of research on the subject, concerned with the lever potential (in some form). However, detailed and explicit accounts and analyses are pivotal in apprehending a more nuanced understanding of the specifics of the potentials that CAS can offer.

We structure our review of the potentials of didactical CAS-use by using the levels of didactical co-determination, beginning with the pedagogical level and ending with the level of subject. One potential does not exclude another potential, so a given study may be included more than once.

The level of pedagogy

At the level of pedagogy, which can be described as teaching principles and the relation between teacher and pupils, we have identified three potentials. In general, it is observed how the use of CAS facilitates diversified work on a problem, and in that way allowing for different "types" of pupils. The potentials at this level generally are not explicitly described but are an implicit condition for the disclosure of the lever potential.

One type of potential is the possibility of what could be called mathematical experimentation, that is, generating a series of examples to study, discover patterns, formulate conjectures and validate conjectures. Although the most prominent changes occur at the level of pedagogy, it might not change all of the other levels, so the students would still be working with factorisations of polynomials. However, the specific praxeologies are new to the pupils and have the potential to strengthen the pupils' knowledge up to the level of theme. Lagrange (2005a) envisions how, with the implementation of CAS, it is possible to organise explorative activities that allow the inclusion of participation of pupils who have relatively weak algebraic knowledge; he describes several examples and starts off with the following:

> In these authors' view, experimenting with new tools will be a remedy. For instance, students, even with weak abilities in arithmetic or algebraic procedures, might be able to use symbolic calculators to explore advanced mathematical domains or to try several approaches to problems that they could not do by hand. Thus, with new tools, mathematical teaching would become more interesting and accessible to more students. (Lagrange, 2005a, p. 74)

Some explicit examples of this can be found in the episodes studied by Hitt and Kieran (2009), where students are examining the factorisation of polynomials of the type $x^n - 1$, or in Heid and Edwards (2001), where expressions of the type $(x+n)^3$ for $n = 1, 2, 3, ...$, are examined. Another series of examples is from Ball (2001), where she reflects on solving equations and how students can approach solving equations more generally with the use of CAS. For example, if students consider the equations $a = \frac{b}{f(x)} + d$ for simple functions $f(x)$, they could study what the solution would be for a series of different functions and values of a, b c and d. This type of activity attempts to develop more general knowledge about the meaning of equivalent forms of equations. In another article, Mok, Johnson, Cheung, and Lee (2000) reflect on how the implementation of CAS in teaching school algebra in Hong Kong can influence teaching and learning; they envision how using CAS has the potential to transition the students' activities from what they call a procedurally oriented paradigm to offer opportunities for inquiry. One of the examples mentioned is an activity about the relation between the value of A in the expression $(x-1)(x+A)$ and the graph of the parabola $f(x) = (x-1)(x+A)$. Another series of examples of mathematical experimentation with CAS-based teaching is presented by Aidoo, Manthey, & Ward, (2010); they investigate how students, with the aid of CAS, can discover simple theorems concerning the roots of quadratic and cubic polynomial equations, such as the problem: "Consider the quadratic equation $x^2 + bx + c = 0$. For what values of

b and c do the roots lie in the interval $0 < x < 1$?". This can lead to the general result of exactly one root in the interval $(0,1)$ if and only if $\min(0, -b-1) < c < \max(0, -b-1)$.

One type of example of inquiry-based teaching that several papers pursue is the students using CAS as a type of black box to discover mathematical notation, axioms and definitions, such as $a \cdot a \cdot a \cdot a \cdots a = a^n$. Thomas (2001) describes how students can discover the distributive law by first asking them to store the values 2.5 in A and 0.1 in B and then trying to predict the value of the expression $2(A + B)$.

Another potential of CAS at the level of pedagogy is the possibility of students getting immediate feedback. In an example studied in Zehavi and Mann (1999), the students first used algebraic technology for developing a model, and then, a CAS was used to rapidly obtain various results. This enabled the students to validate the quality of the model by relating a series of inputs and outputs. In another study (Heid & Edwards, 2001), the immediate feedback is mentioned in relation to the ability of CAS to always perform correct algebraic manipulations. The authors note how a student wanted to subtract two on both sides of the equation, $2x + 7 = 3$, to get $x + 7$ on the left-hand side of the equation. Because the CAS, TI-89, reduced the equation correctly to $2x + 5 = 1$, the student got immediate feedback on her technique. The potential of the immediate feedback of CAS has also served as a feature in designing specific types of CAS (Bokhove & Drijvers, 2010; Bouhineau, Bronner, Chaachoua, Mezerette, & Nicaud, 2005; J.-F. Nicaud, Bouhineau, & Chaachoua, 2004; Jean Francois Nicaud, Bittar, Chaachoua, Inamdar, & Maffei, 2006). Bouhineau et al., (2005) present how one CAS, Aplusix, can provide feedback to students, for example, by validating the students' algebraic manipulations, such as verifying the equivalence of two algebraic expression. This feature can be used to identify in which step of solving an equation the student has made an error.

An additional potential of implementing CAS is situated at the level of pedagogy and is the prospect of creating a richer activity that involves real-world applications. The focus on real-world applications is rarely the object of study in these papers but an implicit principle of the activity design, such as activities based on the realistic mathematics education framework. Heid, Blume, Hollebrands, and Piez (2002) mention this potential in an article directed at teachers. Schneider (2000) describes more specifically how the use of CAS can increase the application of the exponential and the logarithmic function to model disease spreading, population growth, radioactive decay and financial applications such as investment funds. These studies may also relate mathematics to other school disciplines, and we now turn to studies pursuing that aim.

The level of discipline

At the level of discipline, such as biology or history, CAS can be used to facilitate the intertwining of school algebra with other disciplines. In general, certain themes, such as linear regressions, are taught not for their importance in mathematics but because of their use as a tool in other disciplines. Also, more classical themes could presumably be more easily implemented in other disciplines with CAS-based techniques. For example, in one paper (Gerny & Alpers, 2004), a group of grade 12 pupils set up a toy model of a formula 1 track. Intertwining physics and school algebra, the pupils modelled a run of the racetrack using concepts such as velocity, acceleration, motion, continuity, piecewise function and so forth. Maple was used by the students to define and handle the variables, piecewise functions, plotting functions, differentiating functions and integrating functions.

The level of domain

At the level of domain, such as statistics or geometry, CAS and algebra can be integrated as tools. In Schumann and Green (2000), the following task appears: "Given a ladder of length 5 m is to be leaned against a wall. This is hindered by a plinth of height 1 m and depth 1 m placed next to the wall over which the ladder must pass… At what distance from the wall should the base of the ladder be placed to reach as high as possible up the wall?" This can be solved both geometrically and algebraically, and the authors exemplify how the use of algebra -in particular the use of CAS- can be efficiently used to solve a system of equations. The article (Mann, Dana-Picard, & Zehavi, 2007) study a series of traditionally geometric tasks such as "Find the locus of intersection points of perpendicular tangents to the hyperbola whose equation is $\frac{x^2}{9} - \frac{y^2}{4} = 1$". To solve the problem, the students set up a general model for the tangent. The model is manipulated and conditioned by the hyperbola using CAS, and the solution $x^2 + y^2 = 5$ can be obtained.

The level of sector

At the level of sector, such as equations or variables, we have identified one potential. Indeed, this level is where we find most of the studies considered for our review. Traditionally, pupils would not work at the level of sector because it is a general level and requires a certain level of abstraction and generalisation. However, by including activities that articulate and develop more general knowledge across different themes, increased theoretical connectedness of students' knowledge is the aim.

In a way, this level of didactical co-determination also concerns the theory of students' praxeologies. For example, when solving an equation, an element of the theory would be the students' perceptions of variables -what we call the students' personal definitions of the mathematical entities involved. Although the plus sign for most students at secondary school is a well-established entity based on their experience with addition, the equal sign can still be associated purely with having to perform calculations, and the students' definition of the equal sign is restricted to a sign that prompts calculations to obtain a result. In such a case, the students' definition of the equal sign can be developed further to include an indication of equality between two expressions. In this subsection, we describe the affordance CAS can offer in including activities that develop and articulate students' definitions of mathematical entities.

A key study for the implementation of CAS comes from Drijvers (2003b); in the thesis, the potential of CAS to develop students' "definition" of parameters is studied. The use of CAS enabled the students to work with a series of activities involving different parameter uses. The students' "definition" of parameters started as a placeholder ("the students substitute numerical values for the parameter and are reluctant to accept that the parameter value is unknown" (p. 193)) but transitioned to the view of changing quantity ("concerns systematically changing the parameter value, so that the parameter acquires the character of a really changing constant" (p. 199)). To some extent, the students also developed the view of parameters as generalisers ("unified a class of situations, formulas and solutions" (p. 206)) and to a limited range of parameters as unknowns ("the parameter as unknown selects cases that fulfil a specific condition of this set" (p. 213)). Another paper (Heid et al., 2002) accentuates the affordances of CAS to make variables, parameters and their difference an object of study. As an example, they propose studying different interest rates and initial deposits over time, that is, the function $A(time) = deposit(1 + rate)^{time}$.

Another sector identified as an entity for study is algebraic syntax. Artigue (2002a) recounts how in other papers (Artigue, 2002b; Guin & Delgoulet, 1997), CAS can "promote a work on the syntax of algebraic expression", for example, the use and meaning of parenthesis, which often must be entered additionally when using CAS. A fourth sector that is not traditionally an object of study or otherwise explicitly stated in secondary school is algebraic equivalence. In a study, that considers the CAS *Casio Class-Pad* (www.classpad.org), Gjone (2009) relays how examples, such as the two functions $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$, where the command $judge(f(x) = g(x))$ with the output $true$, may be used as a catalyst for the study of equivalence of algebraic expression. Another set of CAS features that can be used to develop students' concepts of algebraic equivalence are the commands *Expand*, *CommonDenominator* and *Factor* because all three commands manipulate algebraic expressions (Lagrange, 2005b).

The level of theme

At the level of theme, such as first-degree equations or right triangles, we have identified several potential. In a way, the level of theme connects to the students' technology. When students solve a first-degree equation, part of the technology is that the first-degree equation probably has a unique solution. The lever potential at the level of theme is how activities aided by CAS-use can develop and explicitly formulate technology, such as first-degree equations having no, exactly one or infinitely many solutions.

One of the potentials of implementing CAS is how CAS can support the relation between the different representations of a theme. In turn, this would conjure a broadening of the students' concept of the entity. A prominent intertwining of representations is between algebraic expression in two variables and a two-dimensional graphic representation. The study (Gjone, 2009) presents such examples. For the task of solving the system of equations $x^2 + y^2 = p$ and $x + y = q$ for different values of the parameters p and q. By first considering the graphical representation of the system, the students were able to structure their (algebraic) arguments based on the three cases, no intersection, one intersection and two intersections. A paper by Kieran & Yerushalmy (2004) also study the opportunity for multi representation with CAS-based teaching, including tables, graphical representation and algebraic representation. One of the examples they study is a task in which the students are required to describe the equation $x + y = 2x - y$. The students developed three descriptions and strategies for evaluating the equations: a comparison between the two functions $f(x, y) = x + y$ and $g(x, y) = 2x - y$ in a three dimensional space; choosing one of the variables as a parameter and obtaining a family of functions; or first rewriting the equation into $y = \frac{x}{2}$ and then finding the intersection of the line with the x-axis. In all three cases, CAS was used to support the description and evaluate the equivalence and, afterwards, to shift efficiently between representations.

Another potential of CAS that is well described and documented is the potential of using already established algebraic paper-and-pencil technology for justifying a pattern generated using CAS. In a series of articles with Kieran and colleagues (Hitt & Kieran, 2009; Kieran, 2008; Kieran & Drijvers, 2006; Martínez et al., 2012), one of the central themes of the activities is the factorisation of polynomials. The students worked on finding and explaining patterns for the factorisation of polynomials of the type $x^n - 1$. The telescoping technique, which traditionally is a paper-and-pencil technique, was used for justifying and validating the hypothesis that the term $(x - 1)$ will always be a part of the factorisation and that when *n* is a prime number, the polynomial only has two factors. Here, CAS was used to factorise cases of the polynomial $x^n - 1$, rewriting some of factors, and testing cases for a large *n*, such as $x^{2003} - 1$. Another example of intertwining traditional paper-and-pencil technology with CAS-use is the study by Zeller and Barzel (2010). A class of 13-year-old students worked on comparing different offers for planning an event. The students used already known algebraic paper-and-pencil technology for developing and formulating general expressions for calculating the expenses dependent on the number of people attending the event. As comparing several equations was (still) outside the students' paper-and-pencil praxeologies, CAS was used to perform work with the more advanced algebraic expression, resulting in the students being able to efficiently compare the equations to find the best solution.

The use of new instrumented techniques changes the technology for the praxeology. However, the above studies allude to the view of looking at CAS-use not as an opposition to the traditional paper-and-pencil technology, but instead as a potential for developing and formulating traditional paper-and-pencil technology. As Lagrange (2005b) writes, "think of the use of CAS as calling for an interrelation between [instrumented] techniques and paper-and-pencil techniques" (p. 11).

The implementation of CAS and the ability of CAS to work with more advanced algebraic expressions provide the possibility of approaching themes that would not otherwise be considered to be accessible to students in secondary school. In a study (Abramovich & Brouwer, 2003), integer partitions are brought forward as a possible theme from discrete mathematics to be included in secondary school. An integer partition of a positive number n is a rewriting of n into an ordered sum of positive integers, such as $4 = 2 + 1 + 1$. The number of integer partitions of a positive number can be found by considering the generating function. For example, to find the number of partitions consisting of one odd and two even parts of n, one would consider the product of the generating functions $(x + x^3 + x^5 + \cdots)$ and $(1 + x^2 + x^4 + \cdots)^2$. The answer to the question would be the coefficient of the n^{th} term. The use of CAS would free the students from time-consuming calculations, permitting them to look for patterns of the coefficients (up to higher powers), include more complex partitions and establish relations between the number of different types of partitions. A series of other advanced mathematical themes that could be accessible in similar ways are described by Cuoco and Goldenberg (2003), such as Lagrange interpolation, polynomials of the form $S_2(n) = 0^2 + 1^2 + \cdots + (n-1)^2$, generating functions and structural similarities between integers and polynomials. Another theme closer to school algebra is presented by Abramovich (2014), who lists a series of tasks involving more advanced use of inequalities, such as inequalities and systems of inequalities with more than one parameter, for example, $x - a > -1$ and $x^2 - 3x < a - 1$. The efficiency of CAS enabled the students to obtain a series of results when running through the values of the parameter a. The students could then work with a family of inequalities or systems of inequalities. In the paper (Roanes-Lozano, 2011), a series of examples of how CAS can be used to work with students on themes traditionally not included in secondary school is presented. Roanes-Lozano reflects on how many problems that can be modelled as an algebraic system can be approached and -with the use of CAS- solved by students in secondary school, for example, themes such as graph three-colouring, logical puzzles and decision making in railway interlocking systems.

Level of subject

At the level of subject, we are concerned about the techniques and types of task, such as solving simple first-degree equations and the techniques used to solve a simple equation. A potential identified in several papers is how the mastering of an instrumented technique may develop mathematical technology (and theory) in students. In the paper (Drijvers & Gravemeijer, 2005), they describe how the command *Solve*, when the variable must explicitly be stated, may develop the technology that an equation is solved with respect to a variable. Drijvers (2004) recounts how the command $substitute$, where a variable is substituted with an expression, can lead students to develop a discourse reflecting their view of an expression as an object, and can develop new explicit (mathematical) knowledge about substitution. The article (Artigue, 2002a), recounts how, when choosing appropriate plotting options, plotting a function, such as $f(x) = x(x + 7) + \frac{9}{x}$, requires or can develop technology for analysing the algebraic structure of the function.

Reflections and other potentials

The list of different types of potentials for implementing CAS in the teaching and learning of school algebra includes, at every level, many detailed examples. One level of potentials seems to draw more attention and is seen as having more potential to influence the students' algebraic powers: the level of sector, such as parameters. As mentioned above, at this level, CAS-based activities can foster a more general study of algebraic objects. The students' development and explicit formulation of algebraic theory (the level of sector) will influence entire families of praxeologies. Although the above review of possible potentials for the implementation of CAS in school algebra is extensive, we can with the notion of praxeologies and levels of didactical co-determination envision more types of potentials than what was identified. For example, at the level of sector, other sectors of school algebra can be studied, such as equations. It is also possible to study how CAS can be used to develop relations between different levels. Working to fill this gap, in Paper II, we study two CAS-based activities. The first activity is the study of the theme of equations. With a series of examples of equations, the students developed and explicitly formulated their definition of equations. In the second activity, the students developed and made explicit the relation between traditional paper-and-pencil techniques for solving equations and the mathematical theory of the solution staying the same. For the main activity, the students were given an equation and had to work on further complicating the equation without changing the solution.

The obstacles

Although the majority of articles portray a positive influence of CAS in the teaching and learning of school algebra, increasing scepticism can be detected. Regarding the studies that portray the potential of CAS, many are conducted by a researcher and under "artificial" conditions not comparable with regular school settings. When it comes to the gap between the exemplary research studies and regular classes implementing CAS, Artigue (2009) writes in her article inspired by her closing lecture of the ICMI study Conference in Hanoi about research on the use of CAS:

> The contrast between the idealistic discourse of the experts of the group, totally coherent with the literature on the educational use of CAS at that time and what was revealed by observations made in their classrooms, turned for us quickly into a research question: how to understand such a gap? (p.466).

In this section, we describe some of the types of obstacles that can occur when implementing CAS, hence hindering the development of mathematical practice and knowledge. As in the previous section, we have tried to differentiate between different types of obstacles. In our review, we were not able to locate studies that described obstacles for the implementation of CAS at the level of domain, discipline and pedagogy, and at the same time also related them to students' activities with CAS; there is research concerned about teachers' beliefs towards CAS-based teaching, such as (Ball, 2014; Pierce, Ball, & Stacey, 2009). However, we hypothesise that a perceived hindrance by the teacher at the upper levels of didactical co-determination would result in the teachers not or limiting the implementing CAS; thus, these studies cannot relate the hindrances at the upper levels with students' CAS-based activities. From our knowledge of Denmark, where extensive CAS-use is part of the curriculum and examination, we know that CAS-based teaching, together with standardised problems for the final exam, lead to a trivialisation of mathematical knowledge and more focus on templates that are produced in class or acquired from outside. Because this is undocumented in the research literature, our review begins at the level of sector.

The level of sector

Just as CAS-based teaching can lever the sectors of school algebra with activities that are focused on developing and explicitly formulating the elements of mathematical theory, such as studying the entity of parameters, it can also hinder the development of mathematical theory. In an article, Jankvist and Misfeldt (2015) describe how CAS-use might blur the differences between equations and differential equations, and between variables and functions, for the students. They analyse a Danish high school teacher's examples, which include the commands *Solve* and *deSolve* from TI-Nspire. The commands can be used to solve an equation and an initial value problem, such as $\frac{dy}{dx} = -16y + 32$ and $y(10) = 1$, respectively. The use of the commands *Solve* and *deSolve* require the input (problem, name of the variable) and (problem, name of the function), respectively. The problems belong to different sectors i.e. equations and differential equations, so the type of task is different: solving an equation or solving an initial value problem. When the teaching takes a pragmatic approach, the similarity of the commands was hypothesised to be causing the students to have problems distinguishing between equations and differential equations and between variables and functions.

The level of theme

One of the more obvious obstacles that can hinder the use of CAS is insufficient mathematical technology. For some instrumented techniques, the mathematical technology required is significantly different from what is used in the context of the corresponding non-instrumented technique. For example, when employing a command such as *Solve* or *Substitute*, the students are required to consider an expression as an object in and of itself in order to substitute a variable with an expression or to know that when you solve an equation, you solve it with respect to a variable that you may have to indicate explicitly. Further, students may be required to choose (among several variables) the one variable for which they want to solve the equation. In a list of obstacles, Drijvers (2000) mentions that when a solution to an equation is an expression containing parameters, it can be a hindrance because the students must be able to differentiate between variables and parameters; he also observed how students' insufficient algebraic strategies can be an obstacle during CAS work. He recounts how the students were unable to "help" the CAS overcome its limitations by performing a relevant rewriting of the expression when the CAS was unable to perform a command due to the input containing square roots. Drijvers & Gravemeijer (2005) also present an example of how students use TI-89 to solve a system of linear equations, such as $x^2 + y^2 = 25^2$ and $y = 31 - x$. Several times when (trying) to use the nested command $Solve(x^2 + y^2 = 25^2 | y = 31 - x, x)$, the students failed. The first entry of the command requires an equation to be manipulated, i.e. $x^2 + y^2 = 25^2$. The vertical line symbolises a substitution, the equation $y = 31 - x$ tells the CAS to substitute y with $31 - x$, and the last entry of the command tells the CAS for which variable the equation should be solved. Using the nested command, the students doubted which variable to enter in the last part of the input and several times chose the variable y instead of x. However, if the students had had a more developed mathematical technology for the instrumented technique, they would not have doubted which variable to enter as the last entry of the input.

Obstacles at the level of theme are extensively documented and can be differentiated further based on their specific mathematical theme, such as systems of equations (as seen with the last example above). We have also identified insufficient knowledge of the theme equivalence of algebraic expressions as a hindrance for implementing CAS. In his research (Drijvers, 2000, 2004), Drijvers comments on how students' inadequate technology for algebraic equivalence is a hindrance for their work with CAS, as CAS might represent an algebraic expression differently than what the students would expect. He relays how the students became confused about the difference between the output $-\frac{(12-x)}{\sqrt{5^2+(12-x)^2}}$ and the expected numerator $x - 12$. A series of other studies (Ball, Pierce, & Stacey, 2003; Heid et al., 2002; Pierce, 2001) also report on the same type of hindrance, concluding how the technology is crucial in situations where the students have to interpret the output of CAS, particularly in handling the automatic simplification feature of some CAS. Heid et al. (2002) write, "CAS can produce results in forms that students may not expect, so students must interpret new forms of expression and develop an enhanced ability to identify equivalent forms of symbolic expressions" (p. 588). Under the same mathematical theme falls the technology of identifying and explaining when algebraic expressions are not equivalent. Kadijevich (2009) studies how a CAS, the TI-Nspire tool (www.ti-nspire.com), rewrites algebraic expressions and equations without informing the user of the limiting equivalence between the algebraic forms. For example, the CAS rewrites $\sqrt{x^2}$ into x without warning users of the limited validity of this equivalence. Another example is how when the denominator of the form $x - n$ in a fraction that is part of an inequality, the TI-Nspire does not bring attention to the fact that the value $x = n$ is invalid when solving the inequality.

Apart from acquiring new techniques, the use of CAS to some extent necessitates knowledge about the general features of the programme, such as how to delete the last entry, the execution of commands and how to define variables and so forth. Thus, it requires an inclusion of new themes related to CAS in the teaching of school algebra, such as viewing options. Drijvers (2000) mentions how when not knowing the syntax sufficiently for the TI-92, the students entered the command $y1(solve(d(y1(x), x) = 0, x))$, asking the CAS to differentiate the function y1, find the solution for $y1' = 0$ and asking for the value of y1 the vertex all in one command line. The CAS was unable to compute the desired request due to the order in which the commands were executed. In an article where DERIVE is the CAS, Artigue and Lagrange (1997) describe how the students, when factoring polynomials, were required to select a ”level” of factorisation. This type of requirement, which is a class of possible outputs, calls for extra teaching of technology related to the types of factorisation in order for the students to master the command.

Another hindrance for the implementation of CAS in school algebra can be described as the discrepancy between conventional (paper-and-pencil) notation and that used by CAS in outputs, which, in some cases, are specific to the CAS. For the students to successfully be able to interpret the output of the CAS, technology about different standards for notation is required. In analysing six different CAS and their ability to handle inequalities, Sangwin (2015) compares the outputs of the CAS for solving the inequalities $x^2 > -1$ and $x^2 < -1$ to that of conventional notation. In all six CAS, there are points of concern. One CAS has the output x and an empty output, respectively, while another CAS has the outputs $\{-\infty < x\}$ and { }. He concludes that the difference between traditionally written mathematics and the CAS representation can be a hindrance for the users.

The level of subject

The implementation of CAS also influences the differentiation between different types of task. For example, the command used for solving a simple equation, such as

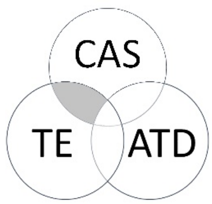
$3(4-x)+7x=(2+x)2+12$, is the same command used for solving advanced equations, such as $\frac{4-x}{2}+(x-3)7=\frac{3x}{5-x}+4x$. So, unless entering the equations or reading the output requires different techniques, the type of task is the same when using a CAS. In a sense, a technique can become very powerful due to its efficiency. This can lead to the inclusion of further types of task into the technique. Lagrange (2005b) describes an example from Monaghan, Sun, and Tall (1994) where a student linked the concept of limits of functions too closely with the corresponding command, *Limit*, in a CAS so that even simple tasks such as finding $\lim_{x\to\infty}\frac{1}{x+1}$ were solved with the CAS rather than reasoning either algebraically or graphically. Another example of this is described by Drijvers (2003a), where a student said that "I don't know how to do it without calculator" when asked to make one equation from the following two equations where y does not appear: $y=a-z$ and $x^2+y^2=10$.

Reflections and other obstacles

To round off the list of obstacles, let us emphasise that many of the researchers express that obstacles can be turned into opportunities to further the students' development of mathematical knowledge (Drijvers, 2004; Drijvers & Gravemeijer, 2005; Heid et al., 2002; Tonisson, 2015). A study (Clark-Wilson & Noss, 2015) investigates the spontaneous obstacles occurring in the classroom, denoted as "hiccups". One of the activities examined involves the study of a car driving. A coordinate system with the axis time and position illustrates how far the car travels. A hiccup for this activity is that part of the line segment, i.e. the length the car travels, can "disappear" outside the window. In their closing section, they reflect on the spontaneous obstacles that occur in the class room and "conjecture that it is both possible and desirable to prepare teachers for such occurrences … make sense of it [hiccups] from an epistemological perspective" (p. 106). Thus, for the "hiccups" not to be a hindrance in the implementation of CAS, a change at the level of pedagogy is required. They further envision that the potential of "hiccups" should be part of any teacher development course.

Finally, we would like to draw attention to another type of hindrance for students' learning when implementing CAS, which has been alluded to previously (Ball, 2014). It is a type of pedagogy, where CAS is enthusiastically applied to almost any type of task. This type of pedagogical approach towards the teaching of school algebra, when the activities have been designed for the use of paper and pencil, can undoubtedly decreases their epistemic value. In Paper I, we study exactly what happens to the traditional algebraic praxeologies if all problems are solved using CAS. What traditional paper-and-pencil techniques are present? What are their epistemic value? What happens to the techniques and their epistemic value when a CAS is used?

Teacher education and CAS



Implementing CAS in the teaching and learning of school algebra requires not only handling the added challenges for students using the CAS mentioned above. In addition, it necessitates a different approach towards teaching school algebra and a restructuring of many of the themes, such as variables and exponential functions. To successfully manage and prevail the implementation of CAS, teachers cannot rely on their own education on school algebra to guide their didactical organisation as most likely, their education did not include CAS-use. To support the transition and overcoming the challenges of implementing CAS in the teaching of school algebra, teacher education is seen as the key (Artigue, 1998). She writes the following:

> We are convinced that these strategies do not enable students to overcome such resistant obstacles as those mentioned above and do not necessarily provide teachers with the didactical tools they need. Efficient teacher training cannot only rely on imitative strategies which cannot correctly take into account the differences between experts and novices, homology techniques obscure the fact that one's mathematical knowledge strongly shapes the use of computer technologies, and if the teachers are not provided with didactical tools for analysis, if observations and experiments are not carefully prepared, they only serve to reinforce initial representations or perceptions. (p. 127)

In this section, we present research on efforts in teacher education on supporting the implementation of CAS. We focus on studies concerned with and presenting developments of preservice teachers' didactical knowledge and practice. This means that papers concerned about the structure or approaches to teaching and learning about school algebra with CAS for preservice teachers (Grugeon et al., 2009; Man, 2006) or papers only examining possible mathematical or didactical problems appropriate for teacher education (Abramovich & Brouwer, 2003; Gierdien, 2011; Grassl & Mingus, 2002; Hitt, 2011) were excluded. Out of the more than 20 papers on teacher education and the implementation of CAS in school algebra, only three papers examined and identified the development of preservice teachers' didactical praxeologies.

The paper (Davis, 2015) studies a 12-week element concerned about preservice teachers evaluation and hypothetical adaptation of textbook lessons that incorporates CAS to varying degrees. An example is an investigation of the predicted income, I, for a given price, p, at a bungee jump carnival ride: $I = p(50 - p)$. The preservice teachers worked with 28 lessons, of which 11 included the use of CAS. In the hypothetical lesson plans, the CAS-based activities were typically retained. In addition, the preservice teachers added supplementary questions to the students about explaining their thinking, make predictions about the solutions they would get from CAS, and reflect on the strengths and weaknesses of CAS. In addition, they also added activities for the students so that they could learn commands such as finding the maximum of a function. However, the preservice teacher did not relate different representations, such as graph and algebraic expressions. The paper ends by commenting that the next important step to investigate is how the preservice teachers actually implement the CAS-based activities.

Gorev & Gurevich-Leibman (2015) studied the effect of integrating digital tools in both (pure) mathematics and didactics courses with a group of 17 preservice teacher having no prior experience of learning mathematics with digital tools. The study presents examples of the development of the preservice teachers' mathematical practice and knowledge, as well as their approach to using CAS for courses concerned about (pure) mathematics. For example, a preservice teacher reflected how the use of a digital tool could enabled the students to understand the relation between the parameter and the graph of the hyperbola. In addition, the preservice teachers shifted proficiently between what the study calls graphical, algebraic and written representations for the task of finding functions $f(x)$, such that the functions $f(x)$ and $g(x) = \sqrt{f(x)}$ have $0, 1$ or 2 points of intersection. In a course concerned about the didactic of mathematics, the preservice teachers were tasked with writing (hypothetical) lesson plans and reasoning about their choice of digital tool and its value. In the reasoning, one of the preservice teachers argued that the digital tool would allow the students to consider different values of parameters and, in this way, study more general cases. In addition, the use of digital tools permits shifting between representations and a trial-and-error approach towards solving problems. Unfortunately, the majority of the preservice teachers, despite having lesson plans and appropriate digital tools, conceded that they were hesitant to implement a digital tool in their first year of teaching because they foresaw a variety of difficulties in the classroom.

Niess (2005) studied how mathematics and science teachers' education can guide preservice teachers' development of didactical knowledge on implementing digital tools. In an element of the programme that was focused on practical experiences in the classroom, the preservice teachers planned, taught, and reflected on teaching with digital tools. One of the five cases studied is a mathematics preservice teacher with a minor in computer science. Her sequence of activities focused on relating graphs modelling the rates of change, such as distance or time, with the slope of the linear function. Some of her reflections on implementing digital tools is relayed in the paper, alluding to the possible development of didactical knowledge. Her overarching view was "that technology was integral to mathematics and thus to learning mathematics". In addition, the digital tool "allows students to visualize and experience with in previous impossible ways". Based on an experience in the classroom where students did not perceive the computer lab as doing mathematics, she proposed that she must include explicit questions so that the students could reflect, and develop more general mathematical knowledge. As pivotal elements of the course for the development of didactical knowledge for implementing digital tools, the study points to the requirement to teach a sequence of technology-based lessons because doing so forced the preservice teachers to consider the potential for implementing digital tools. In addition, the teacher educators challenged the preservice teachers to reflect more generally on the effect on the curriculum when implementing digital tools.

Conclusion and reflections

Based on our review of the literature, the research on how teacher education can support preservice teachers' development of didactical organisation for the implementation of CAS in school algebra is only just beginning. In the search, we located a little over 20 papers, of which only three studies relay some of the specific details of the preservice teachers' knowledge and practice of implementing CAS.

After having asked the question of "how can" and only alluding to the preservice teachers' development of knowledge and practice, the next question -and the focus of Paper III- is "how, can and what". In the paper, using praxeology, we analyse preservice teachers' development of didactical praxeologies for CAS-based teaching, and in addition, we systematically relate the development of the preservice teachers' knowledge and practice to the elements of the course.

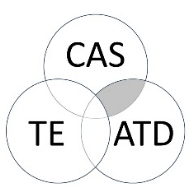
PRESENTATION OF RESEARCH QUESTIONS

To present and situate our research questions for Paper I, Paper II and Paper III, we will present the overarching research goal of the present thesis and the three aspects of the research goal which we consider. For each research aspect, we will relay the main points from the review that the research questions are a furtherance of before presenting our research questions and giving a short description of our methods for pursuing the research questions.

The thesis has the overarching goal of contributing to answering: whether and (how) can CAS be used to strengthen the early teaching of algebra? To shed light upon this broad research goal, we consider three aspects:

- The compatibility of the prevailing approach to school algebra with the implementation of CAS.
- The potential of CAS to introduce pupils to more advanced mathematics topics and working modes, as well as design tools for crafting such CAS-based activities.
- Formats for teacher education to support preservice teachers in developing didactical practice and knowledge about CAS-based teaching of school algebra.

The compatibility of the prevailing approach to school algebra with the implementation of CAS



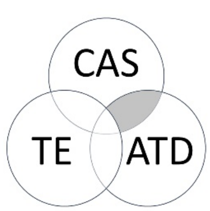
To consider the aspects of the suitability of implementing CAS in the teaching of school algebra, we first reflect upon the literature background. We know from the section "Didactics of mathematics" that teachers who are considering implementing CAS in the teaching of school algebra are submerged in a curriculum with a traditional approach to teaching school algebra and surrounded by materials that support a traditional approach. The most prominent tasks are solving equations and manipulating algebraic expressions (with paper and pencil). Introducing CAS in the traditional approach will at least change the techniques possible. But according to Artigue (see the section "ATD on CAS"), techniques in an educational context have, what she calls, epistemic value. The solutions for the specific equations are not of importance, and the students will likely never be asked to solve the specific equations again. It is not the tasks or the solution that are in focus, but the techniques that are utilised and the knowledge developed and employed for performing the techniques. If CAS is introduced into the traditional approach to school algebra, the tasks will still stand, but the techniques utilised will have changed, and so will, to a large extent, the knowledge that said techniques would develop or employ. We therefore examine what the exact epistemic values are of the procedures for solving equations or reducing the algebraic expressions of present school algebra, as well as what will happen to those values if CAS is used to solve the same set of traditional problems.

The first paper, Paper I, poses the research questions:

- What are the algebraic non-instrumented techniques of lower secondary school?
- What will happen to the algebraic non-instrumented techniques in a CAS environment?
- How are the algebraic non-instrumented techniques related to the instrumented techniques?

As a method for answering the three research questions, the notion of techniques is given a slightly different definition. The paper categorizes elementary techniques, such as the distributive axiom from right to left i.e. rewriting $3x + 12$ into $3(x + 4)$, to enable the identification of fundamental algebraic practice and knowledge. As a representation of the current school algebra, the article examines the last ten years of the final written exam of lower secondary school (grade 9) on the Faroe Islands.

The potential of CAS to introduce pupils to more advanced mathematics topics and working modes, as well as design tools for crafting such CAS-based activities



In recognising that the traditional algebraic tasks are not a fit for the combination of CAS-use and learning of the fundamental structure of school algebra, such as the distributive axiom, we then consider what the potential of CAS is. From the section "The lever potential" we know that with the right tasks the implementation of CAS can support a range of potentials that results in the development of rich algebraic knowledge and practice. An activity that focuses on the sector, developing and explicitly formulating elements of mathematical theory, affects several technologies, which again relates to families of series of techniques (see Figure 7). Thus, an activity that studies mathematical objects at the level of sector can perhaps realise the greatest lever potential of CAS.

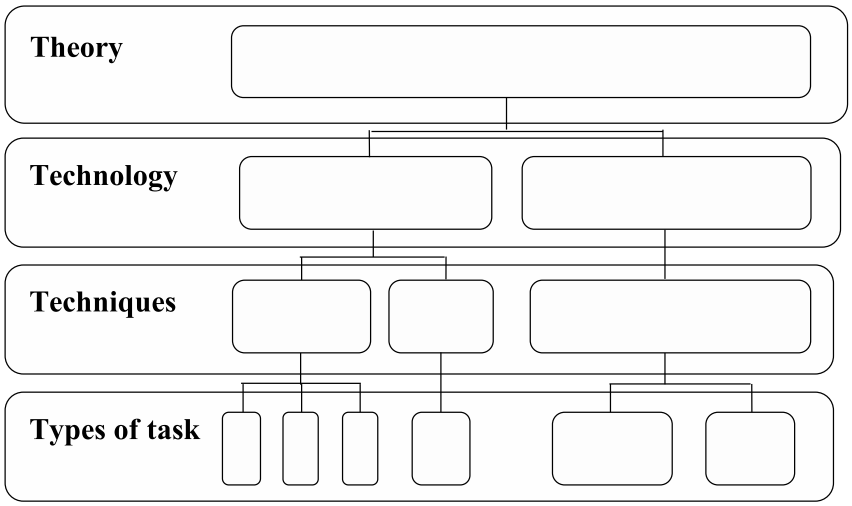


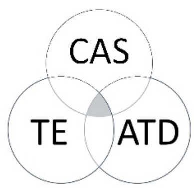
Figure 7. Illustration of possible structure of praxeologies from Paper II.

Solving equations is one of the pillars of school algebra (see sections “Didactics of algebra” and “Importance of school algebra”) but is also related to great difficulties for the students (see section “The problem of school algebra”). However, the sector of equations is not a target of any of the studies examined in section “The lever potential”. Paper II therefore considers the mathematical sector of equations and at the same time poses more general research questions for the lever potential of CAS. In addition, the paper considers the notion of praxeology as a design tool for crafting CAS-based activities which realise the lever potential:

- How can CAS be used to engage students to work with elements of the theory block for praxeologies in school algebra?
- How can CAS be used to strengthen students’ technology related to standard techniques (such as rewriting equations) in school algebra?
- How can one design tasks that realise the use of CAS described in the previous research questions?

The paper considers two designed activities. For the first design, the main activity is for the pupils to develop and formulate their description of what an equation is based on a series of equations that are solved with CAS. In the second design, for the main activity, the pupils are given a list of equations (all have solution 2) that they have to complicate further but keep the solution 2. Concerning the analysis of the data, the notion of praxeology is used to identify the pupils' techniques, technology and theory developed and formulated.

New formats for teacher education to support preservice teachers in developing didactical practice and knowledge about CAS-based teaching of school algebra.



The lever potential of CAS has been studied and recognised by many; however, most of the examples examined are "artificial," that is, either the teaching is conducted by a researcher, the students are only a smaller group and not an entire class, or the CAS-based activities do not correlate with regular classroom teaching. However, even with conditions as close to regular mathematics classes as possible, that is, regular classes of students with their regular mathematics teachers and CAS-based activities that correlate with the school curriculum, the activities have been carefully crafted by a researcher. This leaves the question of how we can achieve and support implementation of CAS that realises the lever potential of CAS in everyday teaching and learning situations, unanswered.

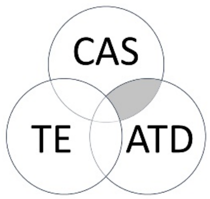
For this, teacher education is seen as one of the keys (see section "Teacher education and CAS"). However, our search of the literature resulted in only a bit more than twenty articles of which only three studies alluded to the development of the preservice teachers' knowledge and practice in explicit detail. From the first of these three papers (Davis, 2015), we have an example of preservice teachers working with a textbook which has integrated CAS-use. The study relays how, in hypothetical teaching plans based on the textbooks, the preservice teachers realise the lever potential of CAS to develop and make explicit elements of the technology and theory by adding more theoretical questions. In the second paper (Gorev & Gurevich-Leibman, 2015), we have an example of teacher education which integrates digital tools intensively across all courses. Based on the preservice teachers' reflections, the study describes how the preservice teachers realise the lever potential of CAS on several levels. At the level of pedagogy, the preservice teachers recognise the lever potential of CAS for teaching experimental-based mathematics. At the level of sector and theme, the preservice teachers voice the potential of CAS to include activities that have a more general focus by using CAS to work with functions and equations containing parameters. The last article (Niess, 2005) presents reflections of a preservice teacher based on her experiences teaching mathematics with digital tools. As part of her reflections she points to the lever potential of CAS for multiple representations and recommends making more general studies as work with parameters using CAS becomes more efficient.

The three articles show that teacher education is far from redundant in supporting preservice teachers' development of didactical theory for implementing CAS in school algebra. However, the didactical theory developed by the preservice teachers described in the first two papers has yet to be tried out in the real classroom, and therefore rests on uncertain grounds. The third paper, where the preservice teacher's reflections are based on experiences in the classroom, the developed didactical theory has more credibility. However, the didactical theory formulated by the preservice teacher is very general and meagre. The three papers leave a need for systematic and detailed studies of what didactical practice and knowledge it is possible to develop in the context of teacher education and what conditions support the development and formulation of sustainable didactical practice and knowledge. In the last paper of the present thesis, Paper III, we therefore examine a course for preservice teachers on the implementation of CAS inspired by lesson study and the protocol for planning-teaching-reflections. We study based on planning-teaching-reflections-replanning what didactical practice and knowledge preservice teachers can develop and formulate about CAS-based teaching of school algebra. Secondly, we explore how tools from the ATD can be utilized to systematically describe and analyse CAS use in the classroom and the development of didactical knowledge and practice related to the implementation of CAS in school algebra.

- What didactical praxeologies can be developed by preservice teachers when teaching elementary algebra using CAS?
- What potential does the implementation of lesson-study-like practice hold for preservice teacher education related to the successful development of instrumental orchestrations?

To examine the research questions, we consider a lesson-study-inspired course which is part of the teacher education of the Faroe Islands. Two groups of preservice teachers are followed in their last cycle of planning - research lesson - reflection meeting - rewriting the lesson plan. To identify and describe in explicit detail the variety of didactical practice and knowledge developed by the preservice teachers, the notion of didactical moments and praxeology from the ATD is combined with instrumental genesis and orchestration.

Paper I: What algebraic knowledge may not be learned with CAS -a praxeological analysis of Faroese exam exercises



This paper investigates the possible consequences of a naive implementation of CAS into the current approach to teaching school algebra. We study the epistemic value of the present paper-and-pencil practice and how this is influenced by consistent CAS use. The paper has been published in Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática in 2019.

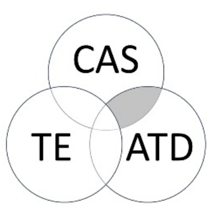
To examine the present school algebra and the influence of CAS, the last ten years of final written exams were used as data. All problems requiring any form of algebraic work were included. All algebra and algebra-related problems were solved with paper and pencil, and the step-by-step procedure for solving each problem was written down.

For the analysis, the notion of types of task and elementary techniques were employed. The problems were categorised into 11 different types of task, such as solving a first-degree equation or evaluating an algebraic expression given the value of the unknown. The two most dominant types of task, solving first-degree equations and reducing algebraic expressions, were further split into sub-types of task. For the analysis of the epistemic value of the techniques, techniques were identified on an elementary level, such as rewriting the expression $-4a + 2b + 4a$ into $2b$. To clarify the epistemic value of the elementary techniques, they were described based on their scholarly "origin", such as the additive inverse field axiom, i.e. $a + (-a) = 0$. In addition, some of the composite series of elementary techniques were presented to illustrate another aspect of algebraic work, the ability to select and combine elementary techniques to reduce or expand an algebraic expression.

Following, the same set of problems were solved using CAS consistently. For each problem, the CAS-related techniques (from now on called the "instrumented techniques") employed were identified.

To compare and differentiate between paper-and-pencil and CAS-based work, the distributions of the elementary techniques and the instrumented techniques are presented. In addition, the total number of both elementary and instrumented techniques were counted, as well as the number of techniques employed for each type of task. The analysis ends with relating the elementary techniques to their instrumented counterpart, concluding that the elementary techniques, such as the additive inverse axiom, definition of powers, and the distributive axiom from left to right can all be represented by the same instrumented technique of evaluating an algebraic expression. In addition, compositions of elementary techniques are replaced by one instrumented technique, diminishing the algebraic work of having to select and combine elementary techniques. Further, it is noted that one instrumented technique can solve several different types of task, effectively re-categorizing types of task to include a wider range of problems. However, one type of task could be said to be neutral to the use of CAS: problems which require modelling a non-abstract problem utilising the elementary properties of school algebra.

Paper II: Designing activities for CAS-based student work realising the lever potential



This paper investigates two aspects of the lever potential of CAS. The possibility to develop and explicitly formulate mathematical theory, and the relation between techniques and technology with theory. In addition, the paper examines the notion of praxeology from the ATD as a design tool for CAS-based activities that realise the lever potential of CAS. The paper has been submitted for publication to the International Journal of Mathematics Education in Science and Technology.

The paper presents two activities designed based on the notion of praxeology. Both CAS-based activities are concerned about one of the pillars of school algebra: equations.

The first activity is designed to develop and formulate the students' notion of equations. As part of the activity the students have to explicitly describe what an equation is, alongside solving types of equations that are not traditionally part of lower secondary school. The CAS is used to solve types of equations uncharacteristic for lower secondary school, such as the second-degree equation $(x-1)(x-2)$ or equations with no solution.

The second activity is designed to develop, formulate, and relate the techniques and technology for manipulating equations with the theory that the solution stays the same. For the main activity, the students are given equations that have solution 2, the students then have to advance and further complicate the equations, but the solution must remain 2. The CAS is used to experiment and develop the equations as well as check the solution of the equations produced.

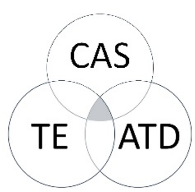
The design of the two lessons is tested in three grade eight classes (13-14-year olds), with the classes' regular teachers. Two of the classes are used to using GeoGebra, while the third class have not used GeoGebra or digital tools prior to the lessons. Preceding the lessons, the design of activities has been presented to the teachers in the form of two lesson plans. To collect data, Dictaphones recorded the students' discussions and the general structure of the lesson. To make the students' work utilising CAS obtainable for study, the students' recorded their CAS-work with screencast. In addition, all of the students' written work was collected at the end of each lesson.

To identify the students' development of knowledge and practice, the notion of praxeology in particular types of task, techniques, technology, and theory was utilised.

The analysis of the data from both activities shows prosperous discussions among the pupils, where elements of theory are developed, formulated, and clarified. From the first activity, for example, the students discuss questions like the following. How do you substitute the unknown in an equation when the equation has two solutions? Does an equation require an unknown or an equal sign? Can a solution be a parenthesis? The second activity generates discussions with questions such as the following. What are the strategies for making equations with solution 2? How do you compensate when a (simple) equation has solution 4 and you want the solution to be 2? In addition, the students explored developing and explicitly formulating techniques and technology for manipulating an equation but keeping the solution, such as adding or subtracting the same number on both sides, dividing part of the equation with an expression that equals zero, and so forth.

The paper concluded that the use of CAS allowed the students to work with examples of equations that are not traditionally part of school algebra, which prompted the development of the students' concept (and description) of equations. In the second activity, the use of CAS allowed the students to experiment and quickly obtain solutions for the equations, establishing and developing the relation between manipulating equations and having the solution to the equation stay the same. In addition, the activities led to further questioning of the theory and technology than what the a priori analysis had foreseen. Furthermore, the paper concludes that the use of the notion of praxeology proved pivotal in the design of the CAS-based activities, as it allowed for the identification of potentials for the implementation of CAS to support and enhance the development of knowledge and practice concerned with equations.

Paper III: A study of a preservice teacher course on the use of CAS in school algebra



The paper studies what knowledge and practice about implementing CAS in school algebra is developed by preservice teachers and how elements of the course, such as sharing observation of classroom practice, support this development. Furthermore, it explores how the notions of didactical moments and praxeology from the ATD in addition to instrumental genesis and orchestration can be utilised to explicit, identify, and analyse instrumented didactical organisation and knowledge. The paper has been accepted with revision in the journal Recherches en Didactique des Mathématiques.

The paper examines a course for preservice teachers with the objective of studying the implementation of CAS in school algebra. The course is inspired by the Japanese lesson study and consists of bi-weekly research lessons implementing CAS in the teaching of school algebra for grade seven and eight. As data for the study, all reflection meetings are recorded, the lesson plans before and after the research lesson are collected, and field notes during research lessons and reflection meetings are made. In addition, the last research lessons, of each group of preservice teachers, are recorded with a Dictaphone.

In the paper, the evolution of two groups of preservice teachers is presented, based on one research lesson each.

The first group of preservice teachers have designed a set of problems for the pupils to work on with CAS which rests upon their own CAS-based endeavours. The main pedagogical approach is to capitalise upon the implicit assumptions of the traditional approach to school algebra and the contradictory examples of equations possible with CAS. The lesson consists of first having the pupils work on traditional school algebra problems where the pupils solve simple equations with paper and pencil. The second activity consists of the students solving equations with CAS. In the series of equations, one of the equations has a fraction as the solution and another "equation" has no equal sign. This spurs some discussion among the pupils.

The second group of preservice teachers have adopted an activity that capitalises upon the possible multiple representation of CAS and the possibility of approaching experimental mathematics. The objective of the lesson is for the students to discover the distributive axiom by realising the equivalence between algebraic expressions of the types $n(x + m)$ and a$x + b$ based on their graphical representation. In the lesson, the pupils are handed a picture-booklet as a guide for the technical demonstration. The pupils set up, with great difficulties, two sliders a and b in GeoGebra and draw the generic line $y = ax + b$. Then the pupils generate and study a series of examples of lines $y = n(x + m)$, where n and m are given. After this, the pupils slide the generic line on top of the generated lines and look for a pattern that relates a and b with n and m. Last, the pupils try to explain their findings. As much more time than anticipated was used for the set-up of sliders, the preservice teachers cut the end of the lesson short and the planned presentations by the pupils are conducted by the preservice teacher.

The analysis shows that elements of the course structure, such as sharing detailed observations of the pupils' practice and knowledge, in addition to having to improve the lesson design, were pivotal in the preservice teachers' development of practice and knowledge related to the implementation of CAS. In the two cases presented, the shared observations of the pupils' activities either worked as a catalyst for reflections about didactical organisation and thus formation of didactical knowledge or contributed to the refining of didactical theory.

Based on the structure of planning-teaching-reflecting-replanning, the preservice teachers realised the lever potential of CAS to conduct experimental mathematics, to shift between multiple representations, and to direct “higher” levels of study, such as developing the concept of equations. In addition, the preservice teachers learned that the pupils were able to autonomously develop instrumented techniques if the instrumented techniques required mathematical knowledge, such as extra parentheses or multiplication symbols. But if the instrumented techniques required general knowledge about the programme or commands, such as syntax, the pupils failed at employing the instrument technique.

CONCLUSION AND REFLECTIONS

In this section we summarize some main conclusions from our work and provide a few final reflections to identify possible future research goals.

Based on Paper I, the implementation of CAS in the traditional approach to school algebra, which primarily includes problems of solving equations and manipulating algebraic expressions, amounts to replacing a large number of pen-and-paper techniques with a small number of instrumented techniques, which, in the setting of the traditional tasks, has very limited epistemic value. Most elementary and fundamental principles of school algebra, such as the distributive axiom, would no longer be required to solve the tasks, and as long as solving tasks of these types remains a main criterion for success in school mathematics, it is therefore very likely that students would not learn about those principles, even at the basic practical level.

The problem with integrating CAS into the traditional approach to school algebra is that "on paper", such as in curricula, one could avoid changing the official learning objectives, like solving first degree equations, reducing algebraic expressions, and so forth. The problems posed and the solutions found could still remain the same. However, it is the techniques utilised to solve such tasks, along with the associated technology and theory that are the pillars of school algebra and all of its subsequent functions in upper secondary and tertiary level mathematics. However, it is typically only the types of task (not the techniques) which appear explicitly in the curricula for secondary school, in some cases even in rather vague terms. We believe that a shift in the view of school algebra, from yet another domain of mathematics to teach, to specific methods and types of works that explicitly value what theoretical and practical knowledge school algebra encompasses, will be needed to meet the challenges for implementing CAS in school algebra. Here, one should not underestimate the institutional interests that may be served by dissimulating a drastic reduction in students' learning through unchanged curricula accompanied by massive CAS use.

On the positive side, the review of research literature suggests that the lever potential associated with CAS is vast and many-sided. Structuring and relating the studies based on the levels of didactical co-determination allow us to identify further possibilities for CAS to develop and explicitly formulate students' practice and knowledge of school algebra. A relatively ambitious attempt in this direction is proposed in Paper II, where CAS-based activities are used to help students develop and formulate theory *about* equations and to relate techniques and discourse for solving equations to the theory; an example of this would be working with equivalent equations.

In order to disseminate ideas from the studies exploring the potential of CAS, for instance, to teachers, this literature needs to be synthesised in a structured manner. In particular, this could contribute to advances related to new design tools for crafting CAS-based activities that enhance students' algebraic work in specific and controlled ways.

The examination in Paper II of the notion of praxeology as a design tool is a contribution in that direction. The notion of praxeology explicitly describes and helps to identify elements -and relations between elements- of the students' activity and knowledge, which can be capitalised upon to craft CAS-based activities that realise some of the potentials identified in the literature.

Structured and detailed reviews of the literature are also needed to better comprehend different aspects of the implementation of CAS and establish conclusive knowledge on the results of CAS-based activities.

Paper III addresses the area where the need for such reviews is imminent: what is indispensable knowledge for future teachers' use of CAS, given that they can typically not draw on personal experiences from their own school time (a source of teaching which is questionable in general, but may be simply impossible here). How can teacher education support the "successful" and sustainable implementation of CAS?

Teacher education has its own set of challenges, such as the gap between what is taught in the courses at university and the professional life of a teacher. When CAS-based teaching is added to the curriculum of teacher education, then an additional challenge is added: identifying established knowledge about implementing CAS, but also acting in a situation where such knowledge may still be scarce and continuously in need of update, if not for other reasons, then because of the continuous development of CAS tools and computer technology. In fact, research to date offers close to nothing about how to support preservice teachers in developing practice and knowledge about implementing CAS. This calls for studies that document innovative ways to frame future teachers' work with CAS through both personal and classroom-based experience and through being introduced to tools that can help teachers in their first experiences: preparing and orchestrating students' CAS use, observing it, and turning observations into didactical theory.

Paper III starts from the assumption that teacher education is a key to improving future implementation of CAS in teaching and learning of school algebra. It proposes to employ, in this context, the structure of lesson study (planning and the formulation of a lesson plan, research lesson, detailed observations of pupils' practice and knowledge, and the protocol for the reflection meeting) to organize the preservice teachers' first experience with CAS. The preservice teachers in the study realised, based on a number of lesson-study cycles, several aspects of the lever potential of CAS. In addition, the notions of praxeology and didactical moments from the ATD, along with the notions of instrumental genesis and orchestration, were used as tools to identify and describe in explicit detail the didactical practice and knowledge developed in the lesson-study activity.

"The devil is in the details" could in retrospect have been the title of this thesis. It is intuitively evident to many teachers that when CAS is used in the traditional approach to school algebra, "something" essential is lost. But what exactly is this "something" and how can it be recovered? Could CAS help students to learn more and better algebra, rather than just less algebra? We are convinced that with the right activities, CAS can lever the students' learning of school algebra. We also accept that implementing CAS in the classroom comes with its own set of challenges and consider that teacher education is an important key to meeting them, at least in the long term. However, we do not yet know enough about how to configure or activate this "key."

To further comprehend the different aspects of the implementation of CAS in school algebra, we are convinced, based on our own studies, that an attentive care about details is pivotal in gaining useable knowledge and developing well-established results. Based on our own work so far, and on that of other researchers employing tools from the ATD, we are convinced that its methods and ideas, such as praxeology, didactical moments, and levels of didactical co-determination, really do enable us to provide that needed care.

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PAPERS

ISSN 1983-3156
A2
http://dx.doi.org/10.23925/1983-3156.2019v21i4p085-099

What algebraic knowledge may not be learned with CAS -a praxeological analysis of Faroese exam exercises

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Abstract. We are interested in the potentials and pitfalls of introducing computer algebra systems in lower secondary school, investigating the case of the Faroese Islands. In order to identify what algebraic knowledge is tested in the final written exam in mathematics after the ninth grade, and how this would change if computer algebra systems were allowed at that exam, we analyse all exam exercises from the past 10 years in terms of the techniques required to solve the exercises both with and without symbolic tools. The comparison suggests that fundamental algebraic structures may not be learned if students consistently use computer algebra systems for the tasks given in the exam.

Résumé. Nous sommes intéressés par les potentiels et les risques liés à l'introduction de logiciels symboliques au niveau du collège, dans le cas de l'école publique des îles Féroé. Afin d'identifier quelles sont les connaissances algébriques testées à l'examen écrit en mathématiques à la fin de la neuvième année, et comment cela pourrait changer si les logiciels symboliques étaient autorisés à cet examen, nous avons analysé tous les exercices de l'examen des 10 dernières années en termes de techniques nécessaires pour résoudre les exercices avec et sans logiciels symbolique. La comparaison suggère que certaines structures spécifiques et fondamentales de l'algèbre ne seraient peut-être pas apprises si les étudiants utilisent de façon consistante des logiciels symboliques pour les tâches rencontrées à l'examen.

Introduction

The students of lower secondary school (grade 7 - 9) are introduced to the formalism of algebra by syntactically-guided manipulation, such as factorization, or simplification of simple algebraic expressions, or solving a first order equation (Kaput & Blanton, 2001; Måsøval, 2011). These techniques play a crucial role in the students learning of mathematics; through these techniques, the students learn the fundament of algebraic structures, work with and manipulations of these. The techniques are later used to further study mathematics including formalistic algebra and algebra as a tool for generalization, modelling and problem solving. How will the implementation of CAS in lower secondary school influence these fundamental techniques?

To study the potential influence of CAS on traditional algebra exercises we have chosen examine how the use of CAS applies to standard exam exercises. In the literature, two studies consider this problem: Flynn and McCrae (2001); Kokol-Voljc (1999). The studies conclude that for traditional exercises mathematics is devaluated to some extent. However, the studies do not give an explicit and exact answer to what mathematical knowledge is no longer present. Such answers are sought, in the present studies, through praxeological analysis.

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1. Notes on praxeology

We assume that the reader is familiar with the concept of praxeologies, a model suggested by the Anthropological Theory of the Didactic to study human activity (Bosch & Gascón, 2014; Chevallard, 1999).

We will adopt the notation T for types of tasks and τ for techniques. Furthermore we will distinguish between techniques in a paper-and-pencil environment and in a CAS environment and will refer to them as non-instrumented techniques and instrumented techniques respectively (Trouche, 2005).

Techniques change over time as students' activities become more routinized. When introduced to the formalism of manipulation of an equation, the technique of solving $3x - 1 = 2$ would be to first add 1 on both sides of the equation: $3x - 1 = 2 \rightarrow 3x - 1 + 1 = 2 + 1$. Later on, when students are acquainted with solving this type of tasks, the technique of adding the same constant to both sides of the equation will change into a technology. Instead the technique regrouping the constants on one side will emerge: $3x - 1 = 2 \rightarrow 3x = 2 + 1$ and even later on directly merging the constants on one side: $3x - 1 = 2 \rightarrow 3x = 3$.

For our praxeological model we will consider the techniques on elementary level such as adding a constant to both sides of the equation. We will define these as techniques that are described by and based on definitions and axioms. For example, a technique could be to apply the distributive field axiom rewriting the expression $3x^2 + 21$ into $3(x^2 + 7)$.

We can now, with the notions of praxeology formulate our research goals and questions:

- What are the algebraic non-instrumented techniques of lower secondary school?
- What will happen to the algebraic non-instrumented techniques in a CAS environment?
- How are the algebraic non-instrumented techniques related to the instrumented techniques?

2. Context and rational

Our data material is the set of exercises from the last ten years of the final written exam of lower secondary school on the Faroe Islands. We see the exam exercises as a representation of the minimal requirements of lower secondary school students.

From the set of exams we consider only a subset of exercises. We study the exercises in which variables or unknowns are used, either in manipulation of algebraic expressions, solving of equations or inequalities, in modelling or problem based exercises. This means that several exercises pose a geometric problem but are solved with algebraic techniques.

First all selected exercises were solved by the author using paper-and-pencil, and all solutions have been documented. The solutions for the exercises were made with techniques supposedly known by students of lower secondary school, and thus the technique chosen can be considered as a minimum level of actions required to solve the exercises. In the cases where several different solutions were possible, a ninth grade teacher was consulted or the solution requiring the least number of techniques chosen, and if still undecided a minimum set of techniques were chosen.

Following, the same set of exercises were solved using GeoGebra and the input, the command and the output documented. GeoGebra was chosen as the CAS, since it is the most frequently used CAS program on the Faroe Islands, based on a questionnaire 2015 (unpublished). A few of the exercises were more easily solved using the geometric environment of GeoGebra. They are therefore not a part of the exercises forming the basis for the development of our praxeological model involving the instrumented techniques.

3. Praxeological reference model

The praxeological model we developed is not only a tool for our study, but also one of the main results for our study in order to answer our research questions. Our praxeological model includes both types of tasks and instrumented and non-instrumented techniques.

3.1. Types of tasks

The first part of the practice block of a praxeology, and what is observable to us, is the types of tasks. The type of tasks is constituted by the form of the tasks.

Though the students of lower secondary school are supposed to operate in the field of real numbers, in our set of selected exercises only the field of the rational numbers was in play.

The types of tasks and following the elementary techniques identified are not exhaustive for 9'TH grade, but what are present in the last ten years of written exams.

Let $T_{\text{solve.eqn}}$ denote the type of tasks of **solving a first order equation**.

Example: A simple example of $T_{\text{solve.eqn}}$, is exercise 18 from 2014: $x + 3 = 24$, a more advanced example of such type of tasks is exercise 6a) from 2013: Solve the equation: $6x - 30 = 3(x - 4)$.

Let $T_{\text{solve.stm}}$ denote the type of tasks of **solving a system of two linear first order equations.**

Example: A standard example of $T_{\text{solve.stm}}$ is exercise 6d) from 2013: Solve the system of equations: $y = -3x - 4$ *and* $y = 2x + 6$.

Let $T_{\text{solve.scnd}}$ denote the type of tasks of **solving a second-degree equation** of the form $ax^2 + bx + c = d$, where a, b, c and d are in $\mathbb{N}$.

Example: An example of $T_{\text{solve.scnd}}$ is exercise 7b) from 2012: Solve the equation: $4x^2 - 28x = 0$.

Let $T_{\text{eval.ineql}}$ denote the type of tasks of, given a finite set of given values, **evaluating an inequality** of the form $ax + b \le c$ where a, b and c are in $\mathbb{N}$.

Example: An example of $T_{\text{eval.ineql}}$ is exercise 45 from 2010: Which of the numbers 2,3,4,5 and 6 are solutions of the inequality: $3x - 2 \le 10$.

Let $T_{\text{solve.ineql}}$ denote the type of tasks of **solving an inequality** with one variable and constant and coefficients in $\mathbb{N}$.

Example: An example of $T_{\text{solve.ineql}}$ is exercise 6c) from 2011: Solve the inequality $8 + 3x > 2(x - 2)$.

Let $T_{\text{eval.expr}}$ denote the type of tasks of **evaluating an algebraic expression** for given values of the variables.

Example: An example of $T_{\text{eval.expr}}$ is exercise 26 from 2011: $a = -2$ and $b = 4, 3a + 3b =$ _____ .

Let $T_{\text{reduce.expr}}$ denote the type of tasks of **reducing an algebraic expression**.

An advanced type of tasks $T_{\text{reduce.expr}}$ is exercise 5a) from 2011: Reduce the expression: $(a + 3b)^2 - (a - 2b)^2$. A simpler example of such is exercise 23 from 2010: $5a - 2b - 4a + 3b =$ __________.

Let $T_{\text{factor.expr}}$ denote the type of tasks of **factoring an algebraic expression**.

Example: An example of $T_{\text{factor.expr}}$ is exercise 6c) from 2008: Put as much as possible outside of brackets: $28x^2 - 14x + 21x^2$.

Let T_{text} denote the type of tasks that begins with a text description of a real world situation. The students are then asked a question in which they should define a variable and relations to information given in the text.

Let T_{geom} denote the type of tasks containing **geometric problem**.

Example: Exercise 4d) from 2013: Are the triangles ABC and DEF similar?

Let T_{other} denote all other of the selected exercises, which do not fall into other types of tasks.

Example: Exercise is 17 from 2014: Mark which of the following expressions have the greatest value for $p = 3$: $p \cdot 4, p^2 + 5, 5p - 4$.

3.2. $T_{\text{reduce.expr}}$ and $T_{\text{solve.eqn}}$

The two most frequent occurring types of tasks are $T_{\text{solve.eqn}}$ and $T_{\text{reduce.expr}}$. We therefore further divide these types of tasks into more fine grained types of tasks. We define the following four types of tasks based on $T_{\text{solve.eqn}}$, due to notational reasons we have introduced the notation $T_{1.1}$, $T_{1.2}$, $T_{1.3}$ and $T_{1.4}$:

Type of tasks	Description
$T_{1.1}$	Solve first order equation of the form $x + a = b$, where a and b are non-zero numbers in $\mathbb{N}$.
$T_{1.2}$	Solve first order equation of the form $cx + a = b$, where a, b and c are non-zero numbers in $\mathbb{N}$.
$T_{1.3}$	Solve first order equation of the form $d(cx + a) = b$, where a, b, c and d are non-zero numbers in $\mathbb{N}$, or d of the form $\frac{1}{e}$ where e is a non-zero number in $\mathbb{Z}$.
$T_{1.4}$	Solve first order equation of different form with constants in $\mathbb{Q}$.

Table 1. Types of tasks within $T_{\text{solve.eqn}}$

Example: An example of a task of type $T_{1.4}$ is exercise 5b) from 2011: Solve the equation $\frac{x}{2} + 3x = 7$.

For the type of tasks $T_{\text{reduce.expr}}$ we get the following five types of tasks, for notational reasons we have introduced the notation $T_{7.1}$, $T_{7.1}$, $T_{7.3}$,..., $T_{7.6}$:

Type of tasks	Description
$T_{7.1}$	Reduce an algebraic expression of the form $ax + by + c + dx + ey + f$, where a, b, c, d, e and f are numbers in $\mathbb{N}$.
$T_{7.2}$	Reduce an algebraic expression of the form $a(bx + cy) + dy$, where a, b, c and d are numbers in $\mathbb{N}$.
$T_{7.3}$	Reduce an algebraic expression of the form $ax + y + b(cy + s) + d(ey + t)$, where a, b, c, d and e are numbers in $\mathbb{N}$ and s and t are numbers in $\mathbb{Q}$.
$T_{7.4}$	Reduce an algebraic expression containing a squared variable with constants in $\mathbb{N}$.
$T_{7.5}$	Reduce an algebraic expression of the form $x^n x^m x^l y^s y^t$, where n, m, l and t are numbers in $\mathbb{N}$ and n and m are different from zero.

$T_{7.6}$	Reduce an algebraic expression of other form.

Table 2. Type of tasks within $T_{\text{reduce.expr}}$

Example: An example of an exercise of the type $T_{7.6}$ is exercise 6a) from 2008: Reduce the expression $\frac{4+3a}{3a} - \frac{2a+a^2}{3a} + \frac{a}{3}$.

3.3. Non-instrumented techniques

To reduce the expression $3a + 4b + a - 2b$ we group terms by applying the additive commutative axiom and the distributive axiom from right to left:

$$3a + 4b + a - 2b \rightarrow (3 + 1)a + (4 - 2)b \rightarrow 2a + 2b.$$

We are only interested in the techniques including letters, thus we do not consider the arithmetic techniques such as rewriting $4 - 2$ into 2 using the ring axioms to rewrite $(((1 + 1) + 1) + 1) - (1 + 1)$ into $(1 + 1)$.

3.4. Non-instrumented techniques based on the field axioms

A field is a fundamental algebraic structure consisting of a set of elements, including a neutral and zero-element, together with two compatible operations satisfying the field axioms. In our study we will be referencing the following axioms:

- The distributive axiom: $a(b + c) = ab + ac$,
- The additive inverse axiom: $a + (-a) = 0$,
- The multiplicative inverse axiom: $aa^{-1} = 1$, whenever $a \neq 0$,

for all a, b and c in the field. The students of lower secondary school operate on the field of real polynomials in two variables $\mathbb{R}[x, y]$.

For the distributive axiom we will not distinguish between the right and the left distributive axiom, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ respectively. Nevertheless we will distinguish between applying the axiom from the left to the right or from the right to the left, $a(b + c) \rightarrow ab + ac$ and $ab + ac \rightarrow a(b + c)$ respectively.

> Let $\tau_{\text{right.left}}$ denote the technique of applying the distributive field axiom from the right to the left. That is, an expression of the form $ab + ac$ is rewritten into the form $a(b + c)$.

Example: The technique, $\tau_{\text{right.left}}$, is used such as in exercise 30 from 2013: Reduce the expression: $3a - 2b - 6a + 5b$. As part of the solution the students will have to apply $\tau_{\text{right.left}}$ in order to arrive at $(3 - 6)a + (-2 + 5)b$. $\tau_{\text{right.left}}$ is also used such as in exercise 6c) from 2013: put outside of brackets: $6x^2 + 21$. Here the students will have to apply the technique $\tau_{\text{right.left}}$ to arrive at $3(2x^2 + 7)$.

> Let $\tau_{\text{left.right}}$ denote the technique of applying the distributive axiom from left to the right. That is, an expression of the form $a(b + c)$ is rewritten into the form $ab + ac$.

Example: The technique, $\tau_{\text{left.right}}$, is used in exercises such as exercise 31 from 2013: Reduce the expression: $2(-2a + b) + 7a$. Here the technique $\tau_{\text{left.right}}$ is applied in order to arrive at the expression $2(-2)a + 2b + 7a$. In other types of task the technique, $\tau_{\text{left.right}}$, is used nine times such as in 5a) from 2011: Reduce the expression: $(a + 3b)^2 - (a - 2b)^2$ to arrive at the expression $a^2 + 3ab + 3ab + 9b^2 - a^2 + 2ab + 2ab - 4b^2$.

Let $\tau_{add.inv}$ denote the technique of applying the additive inverse field axiom from left to right. That is, an expression of the form $a + (-a)$ is rewritten into 0.

Example: This technique is used to solve an exercise such as exercise 24 from 2010: $2(-2a + b) + 4a$. The technique computes the following step: $-4a + 2b + 4a \rightarrow 2b$. Thus, $\tau_{add.inv}$ substitutes the technique $\tau_{right.left}$ in cases where the coefficients are additive inverses of each other.

3.5. Non-instrumented techniques based on the axiom of substitution

To substitute a variable by a number in any relation is often referred to by the substitution property in introductory courses at universities. Further, the following was found at a scholarly discussion forum (theage, 2015):

> If $\phi(x)$ is a statement and if $\phi(a)$ is true and $a = b$ is true, then $\phi(b)$ is true. An example of this axiom is if we have the statement $\phi(x)$: $x\ is\ red$ and if for an object a the statement a is red is true and another object b is identical to a then we can conclude that the object b is red.

By applying the axiom of substitution and introducing functions, we get that if $\phi(x)$ is the statement and $f(x) = f(a)$ then $\phi(a)$ is true. Since $a = b$ it follows from the axiom of substitution that $\phi(b)$ is true and thus $f(a) = f(b)$.

Thus, the non-instrumented techniques of this section can be deduced from the axiom of substitution:

Let $\tau_{add.eqn}$ denote the technique of adding a real number or a variable on both sides of an equation. That is, an expression $a = b$ is rewritten into $a + c = b + c$.

Example: The technique, τ_{add}, is applied in exercises, where the object of the exercise is to find a solution for a first order equation or inequality, such as in exercise 18 from 2014: $x + 3 = 24$. The technique τ_{add} is applied in the following computation $x + 3 - 3 = 24 - 3$.

Let $\tau_{multi.eqn}$ denote the technique of multiplying on both sides of a first order equation with a real number.

Example: The technique, $\tau_{multi.eqn}$, computes the following step $5x = 30 \rightarrow 5x \cdot \frac{1}{5} = 30 \cdot \frac{1}{5}$ in exercise 29 from 2006: $5x = 30$.

Let $\tau_{sub.num}$ denote the technique of substituting a variable with a number in a first order equation or inequality. That is, given an algebraic expression $ax + by$, where a and b are in R, and values s and t for x and y, respectably, then we have the rewriting into $as + bt$.

Example: The technique, $\tau_{sub.num}$, is applied in exercises such as 35 from 2007: $a = -2$, $b = 4$, $-5a - 2b =$ ______ and computes the following step $-5a - 2b \rightarrow (-5)(-2) - 2 \cdot 4$.

Let $\tau_{sub.expr}$ denote the technique of substituting a variable with an algebraic expression. That is, given a system equations $y = ax + b$ and $y = cx + d$ then we have the computation $ax + b = cx + d$.

Example: The technique, $\tau_{sub.expr}$, is applied in exercises of the type where students are asked to find the solution of a system of two linear equations such as exercise 6b) from 2008: Solve the system of equations: $y = 2x - 1$ and $y = -\frac{2}{3}x + 7$ and computes the following expression $2x - 1 = -\frac{2}{3}x + 7$.

> Let $\tau_{add.ineq}$ denote the technique of adding a real number or a variable on both sides of an inequality. That is, an expression of the form $a \leq b$ is rewritten into $a + c \leq b + c$.

Example: The technique, $\tau_{add.ineq}$, to compute the following step $8 + 3x > 2(2 - x) \rightarrow 8 + 3x - 8 > 2(2 - x) - 8$ in order to solve the exercise 6c) from 2013: Solve the inequality: $8 + 3x > 2(2 - x)$.

Note that the technique, $\tau_{add.ineq}$, does not extend to include multiplying of variables.

3.6. Non-instrumented techniques based on the definition of exponents

Exponentiation of a natural number b to the n'th power is defined by $b^n = b \cdot b \cdots b$ (n times multiplication of b by itself). The following technique is justified based on this definition.

> Let τ_{power} denote the technique of multiplying one variable raised to a power with another variable raised to a power, where both variables are denoted with the same letter. That is an expression of the form $a^n a^m$ is rewritten into a^{n+m}.

Example: The technique, τ_{power}, computes the following step $a^2 \cdot a^3 = a^{2 \cdot 3}$ in exercise 36 from 2014: $a^2 \cdot a^3 \cdot a^{-1} =$ _______. τ_{power} is also applied when doing the rewriting of $b \cdot b$ into b^2 such as in exercise 6 b) from 2005: Reduce the expression: $3(b - 1) - (b + 1)(b - 2) + b^2$.

3.7. Example

To exemplify the non-instrumented techniques defined earlier we consider again exercise 7a) from 2005: Solve the system of equations: $y = x + 4$ and $y = -\frac{1}{2}x + 1$:

$x + 4$	$=$	$-\frac{1}{2}x + 1$	($\tau_{sub.expr}$)
$x + 4 + \frac{1}{2}x$	$=$	$-\frac{1}{2}x + 1 + \frac{1}{2}x$	(τ_{add})
$\left(1 + \frac{1}{2}\right)x + 4$	$=$	1	($\tau_{right.left}$, $\tau_{add.inv}$)
$\frac{3}{2}x + 4$	$=$	1	
$\frac{3}{2}x + 4 - 4$	$=$	$1 - 4$	(τ_{add})
$\frac{3}{2}x$	$=$	-3	
$\frac{2}{3}\frac{3}{2}x$	$=$	$\frac{2}{3}(-3)$	(τ_{multi})
x	$=$	-2	
x	$=$	$-2 + 4$	($\tau_{sub.num}$)
x	$=$	2	

Table 3. Exercise 7a) from 2005

Note that the techniques are disjoint and that they do not describe every elementary step in order to solve an exercise. Instead, they aim at describing every elementary step involving a letter.

3.8. Instrumented techniques

We categorize the instrumented techniques based on the command used, the type of input and the type of output. These criteria are based on GeoGebra, thus if one was to use e.g. Maple instead, one might use the criteria only of the command used or even a class of commands. GeoGebra is a piece of software designed for teaching and learning mathematics and science from the level of primary school to university. In the GeoGebra window for conducting CAS work there are twelve commands. Relevant for our level of mathematics and the exercises are the four commands: Evaluate, Factor, Expand and Solve. We note that we did not need to use the command Substitute due to the effectiveness of other commands and that substitution of a variable with a number is done, not by a command, but when entering the expression, equation or inequality such as in exercise 35 from 2007: $a = -2$, $b = 4$, $-5a - 2b =$ _____.

> Let $\tau_{solve.\ eqn}$ denote the technique of using the command **Solve** on a first order equation.

Example: The technique, $\tau_{solve.\ eqn}$, is used in exercises such as 18 from 2014: $x + 3 = 24$. The input is $x + 3 = 24$, the command Solve giving the output Solve: $\{x = 21\}$.

> Let $\tau_{solve.\ ineqn}$ denote the technique of the command **Solve** on a first order inequality.

Example: The technique, $\tau_{solve.\ ineqn}$, is used in exercises such as exercise 6c) from 2013: Solve the inequality: $8 + 3x > 2(2 - x)$. The input is $8 + 3x > 2(2 - x)$, the command Solve giving the output $\left\{x > \frac{(-4)}{5}\right\}$.

> Let $\tau_{solve.system}$ denote the technique of using the command **Solve** with an input of a system of two linear first order equations.

Example: The technique, $\tau_{solve.system}$, is used in exercises such as 7a) from 2005: Solve the system of equations: $y = x + 4$ and $y = -\frac{1}{2}x + 1$. The exercise is solved by entering each linear equation followed by pressing enter, such that GeoGebra stores each linear equation as an equation. Then both equations need to be highlighted before pressing the button "Solve" resulting in the output: Solve: $\{x = -2, y = 2\}$.

> Let $\tau_{eval.num}$ denote the technique of using the command **Evaluate** with an input of only a numerical expression.

Example: This technique, $\tau_{eval.num}$, is used in exercises such as 35 from 2007: $a = -2$, $b = 4$, $-5a - 2b =$ _____, where the substitution of the variables with numbers are completed while entering the expression $-5 * (-2) - 2 * 4$. Note that the technique is not used in exercises such as 38 from 2008: Which of the numbers $-2, -1, 0, 1, 2, 3$ *and* 4 are solutions for the inequality: $4x - 2 < 2$ because it would require the technique seven times, and the command Solve produces the solution with less effort.

> Let $\tau_{eval.expr}$ denote the technique of employing the command **Simplify** with an input of an algebraic expression.

Example: The technique, $\tau_{eval.expr}$, is employed in exercises such as 31 from 2013: Reduce the expression: $2(-2a + b) + 7a$. The exercise is solved by entering the expression followed by the command "Symbolic Evaluation" which results in the output $\rightarrow 3a + 2b$.

> Let τ_{factor} denote the technique of employing the command **Factor** with an input of an algebraic expression.

Example: The technique, τ_{factor}, is used in exercises such as 6b) from 2013: Put outside of brackets: $6x^2 + 21$. The exercise is solved by entering the expression followed by the command "Factor", which results in the output: " Factor: $3(2x^2 + 7)$".

Let τ_{brackets} denote the technique of **inserting brackets** into an expression in order for CAS to correctly read the expression.

Example: The technique, τ_{brackets},is used in exercises such as 7a) from 2014: Solve the equation: $\frac{2x-4}{5} = 6$. In order for GeoGebra to correctly read and distinguish between the numerator and denominator brackets must be inserted: $(2x - 4)/5 \; = 6$.

Let $\tau_{\text{interpret}}$ denote the technique of **interpreting the output**.

Example: The technique, $\tau_{\text{interpret}}$, is used in exercises such as 45 from 2012: Which of the numbers -2, 0, 2, 6 and 7 are solutions for the inequality: $5x - 2 \leq 10$, where GeoGebra returns the output " Solve: $\left\{\frac{12}{5} \geq x\right\}$". The student must then further interpret the output from GeoGebra in order to reach a solution for the exercise.

In all of the exercises, a solution can be reached with only one technique as it is necessary to employ only one command in order to solve an exercise.

4. Analysis and results

In this section we will give a short overview of the quantitative result of our praxeological reference model on the selected exercises, followed by establishing relations between non-instrumented and instrumented techniques.

4.1. Types of tasks

For the selected exercises in our study, we get the following distribution of types of tasks:

Type of tasks	Frequency
$T_{\text{solve.eqn}}$	25
$T_{\text{solve.stm}}$	7
$T_{\text{solve.scnd}}$	5
$T_{\text{eval.ineql}}$	9
$T_{\text{solve.ineql}}$	1
$T_{\text{eval.expr}}$	8
$T_{\text{reduce.expr}}$	30
$T_{\text{factor.expr}}$	2
T_{text}	18
T_{geom}	4
T_{other}	1

Table 4. Frequency of types of tasks

We see that the most frequent occurring types of tasks are $T_{\text{solve.eqn}}$, $T_{\text{reduce.expr}}$ and T_{text} constituting more than 66% percent of the exercises.

By considering types of task within the $T_{\text{solve.eqn}}$ we get the following distribution:

Type of tasks	Frequency
$T_{1.1}$	5
$T_{1.2}$	12
$T_{1.3}$	5
$T_{1.4}$	3

Table 5. Frequency of types of tasks $T_{1.1}$, $T_{1.2}$, $T_{1.3}$ and $T_{1.4}$

By considering types of task within $T_{reduce.expr}$ we get the following distribution:

Type of tasks	Number of occurrences
$T_{7.1}$	12
$T_{7.2}$	10
$T_{7.3}$	2
$T_{7.4}$	3
$T_{7.5}$	2
$T_{7.6}$	2

Table 6. Frequency of the types of tasks $T_{7.1}$, $T_{7.2}$, $T_{7.3}$, $T_{7.4}$, $T_{7.5}$ and $T_{7.6}$

4.2. Structure of types of tasks

When solving the tasks using paper and pencil several of the types of tasks are relational. For example, the task $T_{solve.stm}$ includes the task $T_{solve.eqn}$ and $T_{solve.eqn}$ can include the task $T_{reduce.expr}$, thus we can draw the follwoing diagram of relations between types of tasks when solving using paper and pencil, see Table 7.

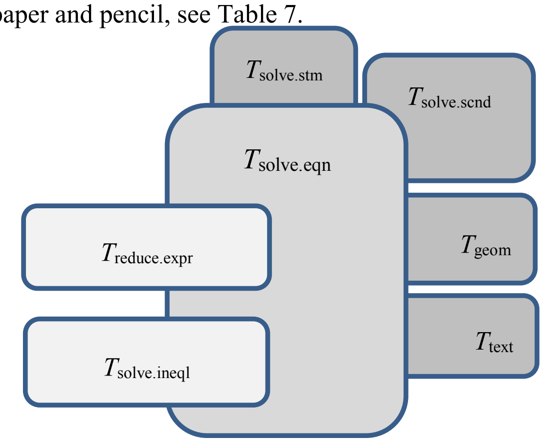

Table 7. Relation between tasks in a non-instrumented environment.

However, when we solve the same set of exercises using GeoGebra only two types of tasks are relational, the T_{text} and the $T_{solve.eqn}$. Thus the relation of traditional algebraic exercises is considerable weakened when solved using GeoGebra.

4.3. Techniques

Applying our praxeological model for non-instrumented techniques we get the following distribution of non-instrumented techniques:

Non-instrumented technique	Number of uses in solutions
$\tau_{right.left}$	59
$\tau_{left.right}$	49
$\tau_{add.inv}$	23
τ_{add}	47
τ_{multi}	46
$\tau_{sub.num}$	98
$\tau_{sub.expr}$	7
τ_{power}	15
τ_{text}	11

Table 8. Frequency of non-instrumented techniques

Furthermore, we get the following distribution of number of non-instrumented techniques used per exercise:

Number of non-instrumented techniques per exercise	Frequency
1	17
2	44
3	20
4	7
5	5
6	8
7	4
8	1
9	3
10	3
12	1

Table 9. Frequency of number of non-instrumented techniques per exercise

It follows from the table that most exercises require a composition of non-instrumented techniques. If we consider the praxeology, then the technology is the explanation for and justification of techniques. Thus in exercises where a composition of two or more elementary atomic techniques are required to reach a solution, then a richer technology is present in order to successfully choose the non-instrumented techniques.

Applying our model for the instrumented techniques, we get the following distribution of instrumented techniques:

$\tau_{solve.\ eqn}$	$\tau_{solve.\ ineqn}$	$\tau_{solve.system}$	$\tau_{eval.num}$	$\tau_{eval.expr}$	τ_{factor}
48	10	7	8	30	2

Table 10. Frequency of instrumented techniques

Furthermore, in 105 out of 110 exercises only one of the instrumented techniques was necessary to obtain the solution. In 4 of the remaining 5 exercises the geometric environment of GeoGebra was preferable to obtain the solution for the exercises and has therefor been left out. The last exercise we consider an exception, and we are uncertain of what instrumented technique that would most effortlessly solve the exercise.

4.4. Relations between non-instrumented and instrumented techniques

In this section we will present our study of the relations between the non-instrumented techniques and the instrumented techniques. We have selected two different approaches to investigate this relation. The first investigation is a direct correspondence between the non-instrumented techniques and the instrumented techniques. The second investigation considers the relations between the non-instrumented techniques and the instrumented techniques via types of tasks to get a more explicit relation that relies on exercises.

4.5. Relations between non-instrumented and instrumented techniques through definitions

In our first analysis we begin with the non-instrumented techniques and determine what instrumented technique(s) are capable of accomplishing the same action as the non-instrumented technique. Thus, if considering applying the distributive field axiom, what instrumented techniques could return the same result?

Consider the non-instrumented technique $\tau_{\text{right.left}}$, equivalent to the action of applying the distributive field axiom from the right to the left: $ab + ac = a(b + c)$. The same result can be achieved by applying the instrumented technique τ_{factor}. However none of the other instrumented techniques yields the output $a(b + c)$. For the non-instrumented technique $\tau_{\text{left.right}}$, we establish a relation to the instrumented technique $\tau_{\text{eval.expr}}$, with similar method.

For the non-instrumented techniques $\tau_{\text{add.inv}}$ and τ_{power} corresponding respectively to the technique of applying the additive inverse axiom from the left to the right and applying the definition of exponentiation, we reach the same results with applying the instrumented technique $\tau_{\text{eval.expr}}$.

For the non-instrumented techniques τ_{add}, τ_{multi}, $\tau_{\text{sub.num}}$, $\tau_{\text{sub.expr}}$ τ_{text} we are not able to obtain identical outcome with any of our instrumented techniques from our praxeological reference model. However GeoGebra still accommodates methods and commands to carry out these non-instrumented techniques. Furthermore, other methods and commands not included in our praxeological reference model will be able to execute the same actions as the previous mentioned non-instrumented techniques. This means that though GeoGebra affords instrumented techniques to accomplish non-instrumented techniques, because of the types of tasks and the presence of other instrumented techniques, they are not used.

We get the following visualization based on a direct relation between non-instrumented fundamental techniques and instrumented techniques:

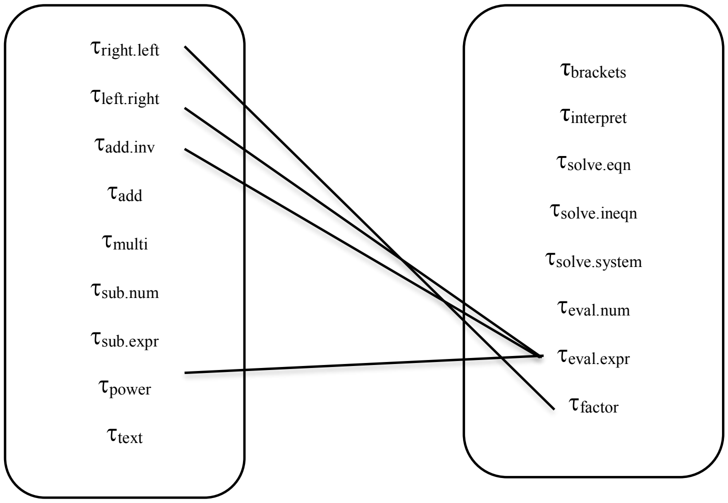

Table 11. Relations by definition.

4.6. Relation of non-instrumented and instrumented techniques through types of task

Due to the relations (or lack of) between instrumented and non-instrumented techniques our second analysis relates the non-instrumented and the instrumented techniques through types of task. By looking at exercises within $T_{solve.eqn}$ we determine the relation between the instrumented and non-instrumented techniques. Since only one instrumented technique is applied per exercise, one could also see the relation as the relation between a composition of non-instrumented techniques to an instrumented technique.

Consider the type of tasks $T_{solve.eqn}$. All exercises within $T_{solve.eqn}$ can be solved applying the instrumented technique $\tau_{solve.\ eqn}$. Regarding the non-instrumented techniques, we get the following relations between types of task and series of non-instrumented techniques for $T_{1.1}$, $T_{1.2}$ and $T_{1.3}$:

$T_{1.1}$	⟺	(τ_{add})
$T_{1.2}$	⟺	$(\tau_{add}, \tau_{multi})$
$T_{1.3}$	⟺	$(\tau_{left.right}, \tau_{add}, \tau_{multi})$ $(\tau_{multi}, \tau_{add}, \tau_{multi})$

Table 12. Relation between types of task and non-instrumented techniques

For the $T_{1.3}$ we have two cases of series of non-instrumented techniques. The series is dependent on whether the number d is written as a fraction or a whole number, in the expression $d(cx + a) = b$, where a, b and c are non-zero numbers in $\mathbb{N}$.

For $T_{1.4}$ we have less uniformity of the order and types of the non-instrumented techniques applied, nonetheless all exercises in $T_{1.4}$ can be solved by applying a composition of the non-instrumented techniques: $\tau_{right.left}$, $\tau_{left.right}$, τ_{add}, and τ_{multi}.

Thus within $T_{solve.eqn}$ we get that a composition of the non-instrumented fundamental techniques $\tau_{right.left}$, $\tau_{left.right}$, τ_{add}, and τ_{multi} is replaceable with the instrumented technique $\tau_{solve.eqn}$. We also get that the instrumented technique $T_{solve.eqn}$ can replace several different compositions of non-instrumented techniques.

By similar analysis of non-instrumented and instrumented techniques via types of tasks, except for T_{text}, we get that any composition of non-instrumented techniques is replaceable with an instrumented technique. But also that one instrumented technique can replace several different compositions of non-instrumented techniques. Furthermore we see that one instrumented technique can solve several different types of task, which is not the case with non-instrumented techniques.

5. Conclusion and reflection

In section 5, we observe that direct relations between non-instrumented and instrumented techniques via definitions can, for some cases of non-instrumented techniques, not be established. Furthermore, we observe that the instrumented technique $\tau_{\text{eval.expr}}$ can replace all of the non-instrumented techniques $\tau_{\text{left.right}}$, $\tau_{\text{add.inv}}$ and τ_{power}. This means, that with the current exercises within the domain of algebra, it is not possible to distinguish between applying the distributive field axiom, the additive inverse field axiom or applying the definition of exponents when using GeoGebra.

Furthermore when considering relations between non-instrumented and instrumented techniques through types of task, we saw that the four series of non-instrumented techniques: (τ_{add}), $(\tau_{\text{add}}, \tau_{\text{multi}})$, $(\tau_{\text{left.right}}, \tau_{\text{add}}, \tau_{\text{multi}})$ and $(\tau_{\text{multi}}, \tau_{\text{add}}, \tau_{\text{multi}})$ can all be replaced by the instrumented technique $\tau_{\text{solve.eqn}}$. Therefore, it is not possible to explicitly distinguish what series of non-instrumented techniques the instrumented technique $\tau_{\text{solve.eqn}}$ is substituting.

The conclusion of section 5 being that it is not possible, when using GeoGebra on traditional algebra exercises, to distinguish between individual non-instrumented techniques or distinguishing between different series of non-instrumented techniques.

In addition, we consider the relation among the types of tasks. The relation between the types of tasks are considerable weaker when solving using GeoGebra, compared to paper and pencil.

But what occurs? One type of exercise, when solved in the CAS environment, causes a new technique to emerge: having to determine the intersection of two sets of numbers. For example exercise 48 from 2006: Which of the numbers -3, -2, -1, 0, 1, 2 and 3 are solutions for the inequality: $2x - 3 > -2$, where students applying the instrumented technique τ_{10} to the given inequality and get the output: Solve: $\left\{x > \frac{1}{2}\right\}$. The students then have to find the intersection of the set {-3, -2, -1, 0, 1, 2, 3} and the set $[\frac{1}{2}; \infty[$.

Using CAS does not exclude the presence of non-instrumented techniques as seen in several results from the literature (Hitt & Kieran, 2009; Lagrange, 2005; Pierce, 2001). The non-instrumented techniques might not be part of the praxis, but they can be part of the logos for solving an exercise. Thus it becomes a question of task design.

We suggest that more work on the transition to and interplay between non-instrumented and instrumented environments are necessary such as (Chaachoua, 2010).

With the current algebraic praxeology one non-instrumented technique was unaffected by the instrumented techniques: τ_{text} present in the task type of T_{text}. Thus the future of algebra in lower secondary schools might lie as a tool in modelling activities that goes across the sectors of mathematics and as a process of algebraization (Bosch, 2012).

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Paper II: Designing activities for CAS-based student work realising the lever potential

Abstract: This paper explores two types of lever potentials of CAS. In the first activity, CAS is used for the students to study the concept of equations. In the second activity, CAS is used to strengthen the relation of traditional algebraic paper-and-pencil manipulations of equations with the theory that the solution of the equation must stay the same. To design CAS-based activities that develops and explicitly state algebraic knowledge the notion of praxeology from the Anthropological Theory of Didactical is employed.

Keywords: Computer algebra system; school algebra; task design; lever potential

Introduction

In the early days of research on implementing CAS in mathematics education, the entrance of this digital tool was met by applause by the community of researchers, and the focus was on the potential that CAS could offer mathematics education (Dreyfus, 1994). The idea is, that, at the level of upper secondary school, 'low level' work, such as solving an equation, can be outsourced (Bang, Grønbæk, & Larsen, 2017) to the CAS. This would leave more time to focus on more complex issues such as the mathematical discourse. This potential is called the lever potential (Winsløw, 2003).

The lever potential has been documented in several articles since the nineteen-nineties (which we will further elaborate on in the background section). However, using a CAS on any problem in the subject of algebra does not automatically lift the students' learning to a higher level, quite the contrary. When solving a traditional algebraic task, such as finding the solution for an equation with paper and pencil, the mathematical discourse of the students includes the use of fundamental algebraic structures, such as the distributive axiom. When solving the equation with a CAS, the mathematical discourse becomes very different, and most fundamental algebraic structures are hidden (Carlsen, 2019).

In our study, we will explore how CAS can be used to make explicit the mathematical discourse in the teaching of elementary algebra. Further, we will study how the four categories of task, technique, (technical) discourse and theoretical discourse can be used to design appropriate activities in elementary algebra for the implementation of CAS. Paragraph: use this for the first paragraph in a section, or to continue after an extract.

Background

In this section, we will give an insight into the potential that the use of CAS in the teaching of algebra offers from a selection of researchers and their respective results. The articles selected have been chosen based on their practical approach to study the potential of CAS. Further, we have selected articles that focus on outsourcing the time-consuming algebraic manipulations to CAS, i.e. the use of, for instance, commands such as solve or reduce expression.

One type of potential is that CAS can introduce other, perhaps more interesting, and more advanced, topics of mathematics into the curriculum. The article by Cuoco and Goldenberg (2003) lists a series of example of exercises within the topics Lagrange interpolation, polynomials of the form $S_2\ (n) = 0^2 + 1^2 + \cdots + (n-1)^2$, generating functions, and structural similarities between integers and polynomials “to illustrate some possibilities of CAS use that supplements rather than supplant, traditional algebra curricula”.

Another opportunity is an increased focus on the application to the real-world context. The article by Schneider (2000) documented that the use of CAS, due to the outsourcing of the algebraic manipulation, increasingly emphasised the application of the exponential and logarithmic functions on models of population growth, disease spreading, radioactive decay etc.

A type of potential that is studied and documented explicitly and extensively, is the potential of CAS to free the students from time-consuming manipulations and allow them to instead reflect upon the mathematical objects that are used, and thereby develop deeper conceptual knowledge. The doctoral thesis by Drijvers (2003) is an example of this viewpoint. In his thesis, Drijvers studies how the concept of parameters can be taught and shows that a greater understanding can be reached by outsourcing time-consuming algebraic manipulations to CAS.

The fourth type of potential we will consider, is how students through carefully designed activities, despite outsourcing of algebraic manipulations to CAS, can investigate a series of examples, conjecture, validate and justify employing algebraic discourse (Kieran & Drijvers, 2006; Martínez, Kieran, & Guzmán, 2012). In particular the article by Hitt and Kieran (2009) where the factorisation of the polynomial $x^n - 1$ is studied. In this study, the students examine a series of examples, exploring the number of factors by using the telescoping technique.

The last article mentioned, utilizes the dialectic between practice and reasoning (Artigue, 2002). This dialectic is structured by a model using the three notions of task (the task to be solved), technique (that is applied to solve the task), and theory (the reasoning that explains and validates the technique). The idea is taken from the Anthropological Theory of Didactic (ATD) that suggests considering the theory part of the dialectic as consisting of two parts. One part is technology, the discourse that explains and justifies the technique. The second part is theory, that is the discourse that justifies and validates the technology. A structural division that we think can further benefit the design of CAS-based activities.

In our study, we want to show how the notion of praxeology can be used to design activities that realise the potentials of CAS. In particularly the fourth type of potential where traditional algebraic technical discourses such as the telescoping technique are in focus; and the third type where theoretical discourses such as the concept of parameters are the focus.

Framework

In general, the ATD suggests to consider human activity, for example cooking a soup, as consisting of an amalgam of practice and reasoning (Bosch & Gascón, 2014; Chevallard, 1998). To cook a soup, one could start with frying onions in a pot (the practice), the reasoning being that onions give a nice flavour and frying the onions sweetens and enhances their flavour, or you might be frying onions because it is generally perceived as a good way to start cooking a soup. Our knowledge is based on practice, and our practice is shaped by our knowledge.

The ATD suggest denoting the dialectic of practice and knowledge by *praxeology*. Further to structure the practice as a twofold. One part *the type of task*, which is the type of task that is aimed at being solved. For example, cook a soup or solve an equation of the type $s(x-t) = n \cdot s - s \cdot m \cdot x$, where s, t, m and n are integers. The other part, *the techniques*, which are the gestures that are utilized in order to solve the type of task such as frying onions, or in the case of solving the equation above: rewriting the equation into $x - t = n - m \cdot x$, and then another rewriting into the equation $x = \frac{n-t}{-m}$. The knowledge related to the practice, which is called *logos* from Greek meaning theory, can be split into two components. *The technology*, which is the discourse that justifies and explains the techniques, such as we multiply with $\frac{1}{s}$ on both sides of the equation. *The theory*, which justifies and explains the technology, such as we must keep the balance between the right side and the left side of the equation. Notice here that theory, or even technology, does not need to be "formal mathematics" but can, to individuals and even large institutions, also include mere convictions and even falsities, such as «we must maintain the balance between the right-hand side and the left-hand side of the equation».

The four components, task, techniques, technology, and theory enable us to consider structures and relations between praxeologies across themes such as solving equations (of different types) or manipulation of algebraic expressions. A praxeology is determined by its type of task (and the institution in which it lives). One technique can be applied in several different praxeologies or several times in the same praxeology, i.e. we can reduce an expression also when the task is not to solve an equation. One technology can unify several techniques, for instance, multiplying both sides of the equation by a number does not necessarily reduce the equation. The theory, such as we must maintain the balance between the right-hand side and the left-hand side of the equation, can join several technologies, since it is also the theory for adding on both sides of the equation. A possible structure of related praxeologies can be seen in Figure 1.

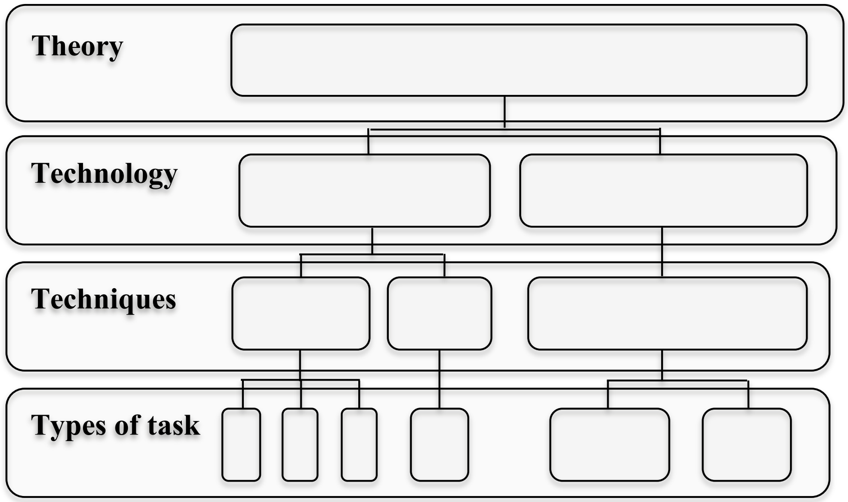

Figure 1. A possible structure of related praxeologies.

Studying the structure of students' praxeologies can help us identify some of their difficulties in learning algebra. If we consider the two tasks of expanding the following expressions: $3(x-2)$ and $(x-2)3$, often students find the second expression a lot harder to expand than the first expression (the case of Danish upper secondary students (Poulsen, 2015)). For the students of lower secondary school, the two praxeologies generated by the tasks are not related, not even by theory, because the students of lower secondary school are not familiar with the distributive axiom that links the two praxeologies. This example emphasizes that if the theory block is not made explicit in the teaching such as putting it into words, or being the object of study in a lesson, then many of the praxeologies are not linked, and thus techniques are not related to each other. This makes the more than 90 elementary algebraic techniques of lower secondary school (Poulsen, 2015) a wilderness of rules.

If we consider the praxeologies of the students in lower secondary school for solving equations, the use of CAS will change part of the praxeology quite dramatically (Carlsen, 2019). Without CAS, it would make sense to distinguish between solving the equations $2(x-3) = 5 \cdot 2 - 2x$, and $12 - 3x + 2 = -2$, because they require different techniques to solve. However, when solving with a CAS, the two equations do not require different techniques in order to solve, so the praxeologies would be the same. Further, since the technique has changed into entering the equation, entering the command solve, and interpret the out-put, the technology has changed accordingly. However, a part of the theory block is unchanged, i.e. the solution to the equation is a number, so that when substituted with x, both sides of the equation yield the same number. By assigning technology and theory the same category and notion, as in (Hitt & Kieran, 2009; Kieran & Drijvers, 2006; Kieran & Saldanha, 2008) we might miss possible information or design opportunities for CAS-based activities.

If we consider the potential of CAS through the lens of praxeology and ATD, the fourth type of potential can be described as carefully designed tasks that can only (reasonably) be approached with CAS-based techniques. The logos for making the conjectures and the validations of the conjectures, are technology known from traditional algebraic paper and pencil tasks such as reducing or expanding an expression. The third type of potential for the use of CAS, reflecting on mathematical objects such as parameters, can be considered as making explicit and formulating elements of the algebraic theory. As this will strengthen the theory, it will benefit a range of algebraic praxeologies that includes the concept of equations as part of the theory. In terms of ATD, an object that plays the role of a work tool without traditionally being the object of study such as equations, is called a paramathematical concept (Chevallard & Johsua, 1985).

Research questions

- How can CAS be used to engage students to work with elements of the theory block for praxeologies in school algebra?
- How can CAS be used to strengthen students' technology related to standard techniques (such as rewriting equations) in school algebra?
- How can one design tasks that realise the use of CAS described in the previous research questions?

Framework

To explore how the notion of praxeology can guide the design of tasks for the teaching of school algebra realising the potential of CAS, two lessons were designed. A series of tasks were designed, and two lesson plans were created to describe the different didactical situations throughout the lessons. The two lessons were carried out in three grade 8 (14 - 15-year-old) classes conducted by their regular mathematics teachers. Prior to the lesson plans being carried out, a meeting was held with each teacher to familiarize the teachers with GeoGebra, Screencast-O-Matic, and the learning objectives of the lessons. The first two classes participating were already familiar with using GeoGebra, though they had never used the CAS application, while the third class had never used GeoGebra, CAS tools, or other dynamic geometric environments.

To collect data for the study, seven recording devices were placed in the classroom to record the students' and the teacher's voices. Further, the students' work in GeoGebra was recorded with Screencast-O-Matic. Everything written on the blackboard was photographed, and field notes were written by the researcher during each lesson.

For each of the three lessons, one recording was chosen for a full transcript. For the remaining recordings, only episodes including not yet transcribed work or discussions were transcribed. All the transcriptions were analysed with the notion of praxeology. The students written materials were also analysed using the notion of praxeology. The screencasts and written materials were used to support the transcription and the analyses of students praxeologies.

The lesson plans and the rational of the designs

In this section, we will give a description of the design of the lessons based on the lesson plans written for the teachers, and part of the a priori analysis of the lesson. The description will be supplemented by the rational of the design.

Lesson A: Describe what is an equation

The first lesson, the learning objective is for the students to further develop their praxeology for solving equations, in particularly making explicit and formulating what makes an equation such as the possible solutions and whether or not the equal sign or a variable can be left out. Thus, the object of the lesson can be categorised as paramathematical.

The first task that the students are given in the lesson, is to write down an equation of their own, and following try to describe, what an equation is. A very important point of this activity is that the teacher does not give the students an example of an equation earlier in the lesson or interfere with the students' work. The use of CAS is not intended for this activity. The expectation is for the students to give a somewhat vague definition, mentioning an unknown, and that the unknown needs to be found.

The students are then given a series of equations to solve in the CAS window of GeoGebra, while recording their screenwork with Screencast-O-Matic. For the series of equations containing different types of equations, see Figure 2. For each equation solved with CAS, the students are tasked with reconsidering their description of an equation.

	Equation	Addition to the description of what an equation is
1)	$14\,a + 2 = 72$	
2)	$14\,y + 4 = 7 + (y - 11)34$	
3)	$x - 2 = 0$	
4)	$(x - 1)(x - 2) = 0$	
5)	$(x - 1)(x - 2)(x - 3) = 0$	
6)	$2x + 4 = (x - 2)2 + 8$	
7)	$r - 23 = (3r - 4)2 - 5r$	
8)	$4(x - 436) - 326 = t - 6434$	
9)	$2(3 - 4) + 7 = 5 - 3(4 - 2) + 18$	
10)	$3(23 - 11) = \frac{1}{5} + \frac{22(3 - 5)}{2}$	
11)	$3t + 12 = 3(t + 4)$ $= 4(t - 2) + 20 - t$	
12)	$12(3 - 7) + 4 + y$	

Figure 2. A minimized version of the students' worksheet for solving equations with CAS.

The first equation served the purpose of getting the students familiar with entering and solving an equation using CAS in GeoGebra. The second equation yields the output $\{y = \frac{371}{20}\}$. This equation was included as we expected most of the students to implicit define an equation to have integer solutions. Further, it is thought of as a warm-up task where the students meet an "easy" new type of equation. The fourth, and the fifth equations have the solution 1 and 2, and 1, 2, and 3 respectively, the output from CAS is $\{x = 1, x = 2\}$ and $\{x = 1, x = 2, x = 3\}$ respectively. The equations have a form so that the students easily can test or find the solutions by trial and error by hand in order to relate the output of CAS into the solutions for the equations. It is expected that the students will add that it is possible for an equation to have several solutions to their description of an equation. Equation six is true for all values of x which CAS writes $\{x = x\}$, while equation seven does not have a solution which CAS writes { }. It is expected that the students will spend a considerable time on interpreting the output of CAS but will end with adding that equations can have an infinite number of solutions or no solution at all. Equation eight has the solution $\frac{1}{4}\, t - 1091$, which is to show the students a type of equation where the solution contains an unknown. It is expected that the students will add that an equation can have an unknown as part of the solution. Equation nine and ten are equations without an unknown, and CAS gives the output { } and $\{x = x\}$ respectively. We expect that all students in their first description of an equation will mention something about an unknown, and thus equation nine and ten are included to prompt a discussion of whether an unknown is a requirement for an equation. Expression eleven and twelve contain two or no equal sign, respectively, and CAS gives the outputs $\{t = t\}$ and $\{y = 44\}$, respectively. The expressions have been included in the hope that the students will discuss the role of the equal sign in an equation. It is expected, that due to CAS being able to give a solution for both expressions, the students will question the role of the equal sign.

For the next section of the lesson, the teacher conducts the students' sharing of their discoveries of additional description and clarifications, followed by the teacher's reformulation and clarifications of the description. For the last section of the lesson the teacher furthers the students' reflections by challenging the students' description, i.e. asking if an equation can have four and a half solution, if it is fair that you can manipulate an equation (with an unknown) into something that is not an equation (equation without an unknown).

Lesson B: Make an equation where $x = 2$

For the second lesson, the learning objective is for the students to strengthen their praxeology for solving equations, in particular the relation between the techniques for manipulating an equation, such as adding a number on both sides of the equation, and the theory that the solution must stay the same.

The lesson starts with the teacher telling the students that today they will have a competition about making ugly equations. He then proceeds to write his two contributions on the board $\frac{(2(x+4))}{3} - \frac{7x+2}{4} = \frac{3x+2}{2} - \frac{33x-18}{12}$ and $3x - 4 = \frac{4x-2}{3}$. He further shows how to solve the two equations in CAS, and that both equations have the solution 2.

Then the students are given five minutes to make their own ugly equation, but the solution must be 2. The students are asked to use CAS to experiment and to check that the solution is 2. After the five minutes, one student for each group then presents the group's equation to the class, and the ugliest equation is crowned.

The next section of the lesson is introduced by the teacher. The students will keep on working to uglify equations; however, the focus is now, given an equation with solution two, to find, develop, and describe techniques that complicates the equation further, but the solution is still two. The students are given a series of equations that all have solution two but are free to set off with any equation that has solution two, see example in Figure 3. The students are asked to use CAS to experiment and check that the equations have solution two.

Old equation with solution x=2	New equation with solution x=2
$3x - 4 = \frac{(4x-2)}{3}$	
Method:	

Figure 3. An extract of the worksheet for developing methods for complicating equations.

The students get 15 minutes to develop their methods. It is expected that the students will develop the techniques of adding a number on both sides of the equation, multiplying with a number except from 0 on each side of the equation, adding an expression equal to zero on one side of the equation, multiplying with an expression equal to one on one side of the equation etc. Following, the teacher orchestrates the students' sharing of different methods while taking notes on the board.

For the last section of the lesson, a new competition of making the ugliest equation with solution 2 is started (the students get five minutes), where the students get to try out the collection of shared methods. The students are asked to use CAS to experiment and verify that the equations have solution two. Following, a student from each group presents the group's ugliest equation, and the ugliest equation of the class is chosen.

The teacher ends the lesson with repeating the learning objective of the lesson, in particularly relating the traditional algebraic technology for manipulating an equation, with the theory that the solution stays the same.

Empirical findings

In this section we will present the students' praxeologies from the data collected. In lesson A, we are interested in how and what theory is developed and clarified through the lesson. The discussions cited and the replay of the lesson are based on the transcripts from the voice recordings, and the students' work presented are cuttings from their worksheets. In lesson B, we are interested in the techniques and the technology developed related to making equations that has solution two. We use mainly the transcriptions to give a replay of the lesson, and the students' production noted on their worksheets to illustrate the students' technology related to the praxeology of retaining the solution for the equations.

Lesson A: Describe what is an equation

Though the lesson was carried out in three different classes, the students' praxeologies related to the learning objective of the lesson were similar. The most noticeable difference between the classes was the difference in time spend on introducing the use of CAS and Screencast-O-Matic. In our analysis we will focus on the theory that the students developed, related to the praxeologies of solving equations.

Of interest for our study of the lesson A are the episodes around solving new types of equations in CAS and revising the description of an equation. We will go through those parts of the lesson chronologically, and relay and present episodes from our data.

In this section of the lesson, the students work in groups. We will now consider the students' work and discussions. Equation two, where the solution is $\frac{371}{20}$, for the first two classes, who regularly solve equations with a non-integer solution doing the calculations with a calculator, the output $\{ x = \frac{371}{20} \}$ does not generate any discussion related to whether or not it is a viable solution. Those groups who do reflect on the output come to the conclusion that the fraction is a division that is not yet preformed and should be written as a decimal number instead. Many of them go on to perform the calculation and writing the first three to four decimals of the number. In the third class, the solution $\frac{371}{20}$ was cause for the first discussion in the groups. The students at first were not accepting a fraction as a solution for the equation. After either confronting the teacher or solving the equation by hand, the groups accept a fraction as a possible output for CAS. In the discussions some groups expressed that CAS had isolated x on the left-hand side of the equation but had yet to finish the calculations on the right-hand side of the equation that would make the fraction a decimal number. One group disappointedly remarked that CAS cannot do everything for them. Subsequently each group added a statement like "The solution to an equation can be a fraction", see Figure 4.

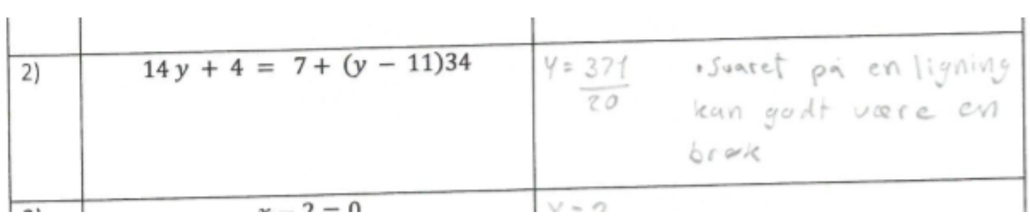

Figure 4. Addition of a fraction being a possible solution to an equation to the description of an equation, cut out from worksheet.

The fourth equation with two solutions is the task the groups discuss the most. It is generally accepted, after rechecking the input in CAS, that CAS is able to solve the equation, and that the (one) solution to the equation can be read from the output $\{x = 1, x = 2\}$. The disconnect between the students' view of equations having exactly one solution, and the solution offered by CAS sparks varied discussions. One group tries the hypothesis that you can choose between one or two, another group that one and two must be added, and the solution is three. One group has the following discussion about substitution and the role of the unknown:

> Aza: But, it is x equal to one, and x equal to two!
>
> Bab: Well, it [x] is the same [referring to the equation].
>
> Aza: What do you mean? It is two different x's [referring to the output]. [Small pause]
>
> Cala: Are there no multiplication sign in between [the expression (x-1) and (x-2)]?
>
> Aza: There is at least a multiplication sign there.
>
> Cala: I think it makes sense. x minus one is one thing[equation], and x minus two is another thing[equation], and then they both must be solved.
>
> Bab: But, both x's [in the equation] must be the same.
>
> Cala: Well, yes, it is like that.

The larger part of the groups ends up reaching the conclusion that an equation can have two solutions by substituting (by hand) x with a value, and then discovering that either the term $(x - 1)$ or the term $(x - 2)$ must be zero, realising thus that both 1 and 2 are solutions. The groups all wrote additions to their description of an equation, see Figure 5 for example.

Figure 5. Addition of «can have several solutions» to the description of an equation, cut out from worksheet.

We have cited the above part of a discussion as it shows how the students are beginning to develop and articulate elements of theory for solving equations, in particular what is the syntax for writing equations, and what is the role of the unknown if it appears more than one time in the equation and if an equation has more than one solution how do you substitute.

In some groups unforeseen inquiry into the theory continues, in one group a student asks how one knows if an equation has two solutions or one solution. Another group discusses if the number of solutions for an equation can be two and a half.

The sixth equation with an infinite number of solutions is not an example that all groups reach. The groups that reach the equation is met by the output $\{x = x\}$. In general, the students are not confused about the CAS output, but instead on what the expression $x = x$ means.

Xia: x is equal to x. That makes sense [ironic].

Yao: Then the solution is x -that is just super [ironic]. [small pause] I do not understand x equal to x, how can that [x=x] be?

Xia: I don't think we entered everything correctly.

Zhou: We entered correctly.

Yao: What does it mean: x is equal to x [directed towards the teacher]?

Teacher: That is a very good question.

Yao: We entered correctly.

Teacher: Yes, you did. But there is something very interesting here.

…

Xia: It [CAS] wrote that the x, that the unknown is equal to the unknown.

Teacher: So, what numbers can you substitute with x in order to solve the equation?

Xia: All.

Teacher: That is an interesting observation.

Yao: So, no matter if we substitute four or seven, then it would be correct.

Teacher: So, how many solutions does the equation have?

Zhou: Infinite.

Teacher: Okay Zhou, that is an interesting observation, which you might want to note on your paper.

Yao: And x is equal to x [writes aloud while she writes]

Zhou: Does it also apply for negative numbers?

Xia: I don't think so. We can ask our teacher.

Zhou: Teacher, does it also apply for the negative numbers?

Teacher: Try. What happens if you substitute with a negative number?

…

We have chosen this extract from one transcript of the groups, as it illustrates not only the formulation of a new type of equation (containing an unknown) with infinite many solutions, but also makes the students reflect upon and formulate further elements of the theory block.

Equation seven is an equation with no solution, which GeoGebra prints as the output { }. The groups of students who have reached the equation conclude that a solution does not exist since the curly parenthesis that beforehand have encompassed the solution(s) are now empty. One group, before making the conclusion of the equation not having any solution, first discuss, if the unknown, in this case r, can be substituted by a curly parenthesis:

Sekai: What does it say?

Tabia: It just says the wrong [type of] parenthesis.

Sekai: Okay, we interpret it as if there should be a parenthesis on both.

Tabia: On both?

Sekai: Yes, because we must keep the balance [of the equation].

Tabia: No, that must be wrong, it [the solution] cannot be parentheses.

Sekai: If this is an equation, then there was a parenthesis here, and it is really heavy, then you also need a parenthesis here [on the other side of the equation], so it becomes equally heavy. Do you get it?

Ronja: No!

It is finally dismissed with the argument that a parenthesis is not a solution for an equation.

This bit of discussion between the students shows that the questioning of part of the theory leads to further exposure of the theory, i.e. an equation is a seesaw where the equilibrium must be kept, and a clarification of what is possible and not possible to substitute with the unknown. Further, the students implicitly ask the question what do you substitute the unknown with, if there is no solution to the equation?

The eighth equation is an equation with two different variables, and the output of CAS is $\{ x = \frac{1}{4} t - 1091\}$. The groups that reach this equation do not seem perplexed with now having a solution for an equation that contains an unknown and relates the solution to previous obtained solution type with infinitely many solutions. In one group a student remarks: "x is equal to one fourth t [stops mid-sentence]. It is that thing where x can have one or several unknown [as a solution]. Next!" later she reflects "I had not considered that one could put an equation in relation to, err, an unknown in relation to another unknown".

The ninth equation is an equation with no unknowns, and it is true. If each side of the equation is reduced by calculations, then it will yield $5 = 5$. The output of CAS is $\{x = x\}$. The groups of students who reach the equation immediately dismiss the equation as an equation, since it does not contain an unknown: “There is no x -that is not an equation!”.

The next episode of interest is the last episode of the lesson where the students share their findings, and the teacher reformulates the students’ findings and write key phrases on the board. Following, the teacher outsets further discussions and clarifications. In one of the lessons, the output $\{x = x\}$ is further discussed. During the students’ presentation of the implications of this type of output, i.e. an equation can have an infinite number of solutions, it was implicitly understood that the solution type was considered a number. The teacher relates this type of equation with that of equation eight, where the solution contains an expression with a second unknown, and asks the students if perhaps when $x = x$, then the solution can also be an expression containing an unknown.

The curly parentheses are also one of the objects that the teachers consider. After the students have had their try at guessing the meaning of the curly parentheses, the teacher explains that the curly parentheses are usually used to denote a set. The students have not only encountered equations with more than one solution but are also taught how to write such solutions.

Equation nine and ten are also brought to the attention of the students since they do not include an unknown. The students all agree that they do not qualify as equations even though CAS can solve them either with an output of $\{x = x\}$ or { }. In one of the lessons, the teacher then suggests that they can be called statements, then the statements can either be true or false, just as the so-called equations can either be a true statement or a false statement.

In the last episode of the lesson, the students formulate elements of their theory. The teacher then further questions the theory, and together the students and the teacher develop and formulate theory for equations but also on other related subjects. New paragraph: use this style when you need to begin a new paragraph

Lesson B: Make an equation where x=2

We will now study the empirical findings for lesson B. In our analysis we will focus on the technology that the students developed related to the tasks of making and further develop equations with solution two. As the tasks are open, there are many possible ways of attacking the task. Though the lesson was carried out three times the students' praxeologies developed in the lesson were similar. We will present our findings in chronological order i.e. first the task of making an equation with solution two, and then the task of further developing an equation with solution two maintaining the same solution.

The first section of interest is the students' first try to create an equation with solution equal to two, one group of students suggest three main technologies. The first main technology:

> Irene: Can't we just make a random equation and then compensate?
>
> Leo: What! Oh, you mean like just add minus [a number] at the end.
>
> …
>
> Leo: We just copy the one up there [from the board] but we make it longer.
>
> …
>
> Irene: I have got the greatest idea ever! Can we just do like last lesson, where x is equal to x. Infinitely many answers.
>
> Joules: How do you get x equal to x?
>
> Leo: Then you have to make both sides [of the equation] such that when reduced they are the same.

The group of students has formulated the three main technologies for this section of the lesson. We will refer to them as θ_1, for the main technology of for making a "random" equation, checking the solution with CAS, and then compensating until the solution is two. θ_2, for starting with an equation with solution two, complicating the equation further, and checking the solution with CAS. θ_3, for making an equation where the right-hand side is equal to the left-hand side, thus obtaining an equation with infinitely many solutions.

The group cited above ends up trying out θ_1. After entering a "random" equation, they get the output from CAS $\{x = 4\}$. This prompts the discussion of how to compensate such that the solution is two.

Leo: What if we divide it [the equation] with two?

Irene: Then it would be one-half x equal to two, that would still give the result x equal to four. If we want two less on the right side perhaps, we should either add or subtract two up here.

The group's work and discussion show the variety of technologies the students hold, their formulation of their technology, and the choice of technology. Even more specific technology of how to manipulate a simple equation with solution four such that the solution becomes two.

All the groups work with similar or variations of the above mentioned three main technologies.

Another group, who works with θ_3, uses the techniques of making two expressions that are equal to five, and then put the expressions on different sides of the equal sign, such that the equation has infinitely many solutions. The result is $\frac{4x+x-\frac{1}{2}x}{x+\frac{1}{2}x}+x=3+x$.

Another group employing θ_3 uses the technique of adding and subtracting the same on both sides of the equation.

Gerð: We use a lot of difficult numbers, and then we subtract them in the end.

Fróði: Minus one thousand-seven-hundred-and-sixty-one minus something plus x minus seven-hundred-and-eleven. Then x is equal to x!

Gerð: I think we should use pi.

…

The students' discussion and the results of their work show that the students have the theory that equations with infinitely many solutions exist and know how such equations can be made with the technology of making an equation where the right-hand side of the equation is equivalent with the left-hand side.

Though θ_2, further developing an equation with solution two, is discussed by some groups, none of the groups choose to work with the strategy.

In the next section of the lesson of interest, the classes are tasked with θ2. They are given a list of equations, all having solution two, to make more complex. Our main interest is not the ugly equations produced, but the techniques and technology that the students develop and employ. We will first recount the main techniques and technologies developed and formulated by the students, and then recount some of the theory that was discussed and clarified.

We will present two of the techniques developed and formulated by the students. The first technique, which we will denote by τ_1, is to divide part of the equation with an expression that is equal to one when x is equal to two. In the following transcript the technology and the technique are developed and the expression $3x - 4$ is rewritten into $\frac{3x-4}{(x^3 - 2x)-3}$.

> Margrethe: Perhaps we can make a fraction here, and then make some weird expression [as the denominator] that is just equal to one … Then we say four minus two, all we need is for it to equal one. x and we have two, we need [pause]. Here we have two x, thus we need four, minus x squared. Here we have got. No. x to the power of three, then we have minus four, then we need to subtract with three on the other side somehow [$(x^3 - 2x) - 3$]…That is a very overcomplicated way of writing one.

As a method the group writes "overcomplicating by dividing with one".

The other technique, which we will denote as τ_2, developed, and clarified is to add the same number on both sides of the equation. A group has the following discussion:

> Emmy: We must make this one ugly.
>
> Elliot: You can change this to three and then minus thee here [referring to the other side of the equation sign in the equation].
>
> Emmy: Plus six, plus two.
>
> Elliot: [interrupts] You are doing it wrong. You cannot put plus on each side [of the equal sign] because you have to put a minus. [Directed to the teacher] Isn't it true that if you put a plus on one side, then on the other side is has to be minus?

> Teacher: Hmm, you have to keep the balance of the equation. If I have an equation and I make one of the sides heavier, then in order of keeping the balance, the other sides must also get heavier. I.e. if I add something on one side then I need to add the same thing on the other side.
>
> Elliot: Does this make sense? [shows the teacher her equation]
>
> Teacher: Test it [with CAS]
>
> …
>
> Elliot: Okay, we add plus three and twenty on both sides. And minus nineteen on both sides. And plus four on both sides.
>
> ...

The extracts show the two techniques, τ_1 and τ_2, developed, formulated, and employed by students. τ_1, dividing a term of the equation with an expression that is equivalent to one, and τ_2, adding the same number on both sides of the equation. Though the focus of the activity was to develop techniques and technology, discussions, and clarification of elements of the theory emerged. The main theory present was that the solution must stay the same, as it was the condition for rewriting of the equations. The students often referred to another main theory during the lesson, that an equation can be seen as a seesaw and that the seesaw must be kept in balance i.e. if you add a number on one side of the equation you must add a number or an expression that is equally heavy on the other side of the equation.

We will now present some more of the technology and theory that emerged in this section of the lesson. In a group of students that were employing τ_1, a student wanted to divide by zero, in another group, also employing τ_1, a student asked what happens if you divided by x. As time was of the essence the respective groups did not engage in theoretical discussion but refocused on the task at hand and developing techniques for which the technology and theory were already part of the students' inventory.

The activity of handling equations all with solution two, seemed to generate, in several groups, a questioning about whether the equations were not the same. After seeing the teachers' two examples of equations with solution equal to two, a student asks

Uliuk: How can they all give the same result? Such a long equation can give the same result as that one.

Vaaltimaat: That is the question!

Teacher: I think that is a very good question!

Uliuk: Then you can just write that [the first equation] instead of all that [the second equation]. That [the second] one takes a lot longer to reduce.

In the second section of the lesson, other students commented that it did not feel like they had made new equations as it was a rewriting of the first equation. This happened in particularly if the students had done simpler manipulations such as rewriting $4x$ into $3x + x$ or $2x$ into $x + x$.

The extracts show that elements of theory are discussed and clarified but also inspire further questioning of the theory, that could lead to entire new types of praxeologies as to when are two equations the same? The students are becoming aware of equations as an object of study.

In the next part of the lesson the students share their methods while the teacher reformulates and writes them on the board. The techniques shared are: multiplying a term of the equation with an expression that are equal to one when x is equal to two; dividing a part of the equation with an expression that are equal to one when x is equal to two; adding an expression that is equal to zero when x is equal to two, to one side of the equation; adding the same number or equivalent expressions on both sides of the equation; and multiplying with the same number or expression on both sides of the equation.

The students' sharing of their methods developed, shows the variety of the different technologies present in the lesson, and the similarity to the technology for solving equations. Thus, embedding the theory element that the solution is maintained into the praxeology for solving equations.

In the last attempt at making an ugly equation with solution two, another piece of theory concerning the equal sign emerges. In a group the necessity of the equal sign in an equation is questioned. However, after wondering how one would solve such an equation, or how one would check if the solution is two, an agreement is reached that the equal sign is a requirement for an expression to be an equation.

The recount of the groups discussion shows that again the students are considering equations as an object of study, and though the question of whether or not the equal sign is a necessity for an equation was raised in the first lesson, the discussion in this account includes technology.

Conclusion

In this section we will try an answer our research questions into the potential of CAS and the notion of praxeology as a tool for designing tasks.

The use of CAS in the activities allowed for the students to approach new tasks that would otherwise have been buried in time-consuming algebraic work such as solving equations. The new tasks involved themes within school algebra that strengthened, developed, and clarified both technology and theory. In lesson A, the students worked with describing what is an equation. The students developed, formulated, and clarified a great range of technology and theory involving equations and related topics. Therefore, strengthening an entire series of related praxeologies. In lesson B, the students worked with establishing a greater connection between the theory of maintaining the solution for an equation with the technology and techniques used for manipulating equations. Thus, sustaining a grander logos for school algebra. However, lesson B did not solve all problems related to the praxeologies of solving equations. During the lesson both students and teachers mentioned the metaphor for an equation as a seesaw, where the balance must be kept, and used it as a valid argument for justifying techniques. There were instances where this magic trick could have been put into question such as when a student wanted to divide by zero, or one student wanted to take the square root on both sides of the equation. But it did not happen.

By outsourcing time-consuming algebraic work, CAS can be used to introduce new types of equation i.e. equations with more than one solution, making it possible to study equations as an object and not just using equations as a tool. This in turn developed both elements of the students' theory and technology such as in lesson A. In lesson B, the use of CAS provided the students with a technique for verifying the solution of their equations, allowing the students to experiment with creating and developing the equations. This led the students to develop and formulate not only an abundance of technology, but also formulate and clarify elements of theory.

The work of the students with the logos further prompted the students to questioning, formulated and discuss elements of theory, such as rules and the role of substitution (from lesson A) or what happens if you divide by the unknown, thus further enhancing the activity.

As both lesson A and lesson B contained many questions inquiring further into elements of technology and theory, we conjecture that the activities could be further enriched by not being bounded by a lesson plan. This would allow the students to control the path of inquiry and development of technology and theory. In addition, the object of lesson B, to further develop and explicitly state the relation between the techniques and technology (of manipulating equations) with the theory (of the solution staying the same), the institutionalisation must be further emphasised.

The use of the notion praxeology, in particular the four notions of task, technique, technology, and theory, allowed us to analyse the prevailing paper-and-pencil praxeology for solving an equation, and thus pick the themes “the students' definition of an equation”, and “the relation between algebraic manipulations of an equation, and maintaining the same solution”. Both themes within school algebra, thus the designed activities would strengthen, by further developing, formulating, and clarifying the students' logos.

The notion of praxeology lets us study the lever potential of CAS further, and thus enabling the design of future activities more easily. In lesson A, the lever potential can be described as developing and formulating the students' concept of equation, which is an element of the theory block. This characterisation can also be used to describe the potential of CAS studied in the thesis by Drijvers (2003). In lesson B, the lever potential is the strengthening of the relation between the theory of maintaining the same solution, with the techniques and technology for traditional algebraic manipulations). In the article by Hitt and Kieran (2009) the lever potential can be described as the (paper-and-pencil) telescoping technique being used not only as a technique, but also as technology for developing, justifying, and validating conjectures. The use of the notion praxeology enables us to describe the students' learning explicit and justifies the lever potential of CAS.

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A STUDY OF A PRESERVICE TEACHER COURSE ON THE USE OF CAS IN SCHOOL ALGEBRA

Louise Meier Carlsen

A STUDY OF A PRESERVICE TEACHER COURSE ON THE USE OF CAS IN SCHOOL ALGEBRA

Abstract – In this article, a lesson-study-inspired course in mathematics teacher education on the Faroe Islands is studied. The preservice teachers have to conduct four lessons in grade 8, teaching early algebra using CAS. We analyse the preservice teachers' practice and knowledge gained during two lessons, combining the instrumental approach and the notions of praxeology and moments from the Anthropological Theory of the Didactic.

Keywords: teacher education, CAS, the instrumental approach, praxeology, lesson study.

ETUDE D'UN COURS D'ENSEIGNANT DE PRESERVICE SUR L'UTILISATION DU CAS EN ALGÉBRE SCOLAIRE

Resumen – Dans cet article, un cours inspiré de la leçon-étude de la formation des enseignants de mathématiques aux îles Féroé est étudié. Les enseignants préposés à l'entretien doivent dispenser quatre leçons en 8e année et enseigner l'algèbre précoce en utilisant la CAS. Nous analysons la pratique et les connaissances acquises par les enseignants en formation initiale au cours de deux leçons, combinant l'approche instrumentale et les notions de praxéologie et les moments de la théorie anthropologique de la didactique.

Palabras-claves: formación docente, CAS, el enfoque instrumental, praxeología, étude de cours.

UN ESTUDIO DE UN CURSO DE PROFESORA DE CONSERVACIÓN SOBRE EL USO DE CAS EN LA ESCUELA ÁLGEBRA

En este artículo, se estudia un curso inspirado en el estudio de lecciones en la formación del profesorado de matemáticas en las Islas Feroe. Los maestros en servicio tienen que llevar a cabo cuatro lecciones en el octavo grado, enseñando álgebra temprana usando CAS. Analizamos la práctica y el conocimiento adquirido por los maestros en formación durante dos lecciones, combinando el enfoque instrumental y las nociones de praxeología y momentos de la Teoría Antropológica de la Didáctica.

Mots-Clés : formation des enseignants, CAS, approche instrumentale, praxéologie, Estudio de la lección.

INTRODUCTION

Computer algebra systems (CAS), such as Maple, were developed foremost with a view to efficiency in carrying out cumbersome routines, such as symbolic computations for engineers and research mathematicians. In an educational context, when applying a program such as Maple or GeoGebra for pupils to use in the learning of mathematics, the time-saving aspect must take a background position, while the epistemic value for supporting and promoting mathematical learning and understanding must stand in the forefront (Artigue, 2005).

The use of CAS has been an integrated part of upper secondary school in Denmark since 2005 and in many other countries for a considerable time. The intensive use of CAS in upper secondary school has inspired some teachers in lower secondary school to adopt CAS in their teaching as well. Talking with and observing teachers in lower secondary school implementing CAS reveals a variety of different approaches to the use of CAS. In some cases, CAS is used merely to check the results of tasks solved with pen and paper. In a few cases, CAS is used for mathematical explorations, and in other cases, CAS is used as a substitute for pen-and-paper manipulations. With a traditional algebra exercise in lower secondary school, such as solving an equation like $\frac{x}{2} + 3x = 7$, the techniques used to solve the exercise convey fundamental algebraic structures and principles when being solved with pen and paper. With the replacement of pen and paper with CAS, the epistemic value of learning about fundamental algebraic structures, such as the distributive axiom, is meagre (Carlsen, 2019).

For a successful implementation of CAS in the teaching of school algebra, teacher education is seen as a key issue (Artigue, 1998). However, it is unclear how courses can be structured and what is possible and necessary for preservice teachers to learn when they have no prior experiences with CAS.

In our study, we will explore what is possible for preservice teachers to learn about teaching with CAS tools in the following context: planning and teaching a grade 8 lesson using CAS and subsequently reflecting on observations (as in Japanese lesson study). In order to describe and analyse our findings, we also present a combination of the Instrumental Approach and of the Anthropological Theory of the Didactic (ATD), in part drawing on earlier work by Artigue (2002, 2007) and Lagrange (2005). The use of praxeology and moments from ATD contribute to a systematic and detailed description of instrumental geneses and instrumental orchestrations.

BACKGROUND

In this section, we will first take a look at the literature regarding preservice teachers' education that includes the teaching of algebra using CAS in the curriculum. We will then connect the pivotal elements of the preservice teacher education course to the format of lesson study. Next, we will consider selected aspects of using CAS in the teaching of school algebra, and, finally, we will present an introduction to the complementation of the instrumental approach from the perspective of ATD.

We will consider the few research articles on preservice teacher education where the use of CAS in the teaching of algebra has been part of the curriculum. Generally, we know that teaching algebra using CAS is part of the curriculum in several preservice teacher educations around the world (Grugeon et al., 2009) and that there are several methods and perspectives for developing preservice teachers' knowledge on teaching algebra using CAS (Grugeon et al., 2009), such as "classroom acting" where the preservice teachers act as the pupils and the teacher educator acts as the teacher of the lesson. The works of Özgün-Koca (2010) and Özgün-Koca, Meagher, & Edwards (2010) describe and analyse a method course in preservice teacher education, the latter using the framework of Technological, Pedagogical and Content Knowledge (TPACK) (Niess, 2005), where the preservice teachers designed and implemented technology-rich activities during their practicum. The articles contain extracts of the preservice teachers' reflections, giving us insight into the type of knowledge developed by the preservice teachers on teaching algebra using CAS, such as how technology can be used to test conjectures much more easily than previously with pen and paper. The articles stress that the development of TPACK and beliefs related to the use of CAS in the teaching of algebra are related to the preservice teachers' experiences as acting teachers in the classroom as well as their reflections (in the form of journal writing or interviews).

The pivotal elements (for developing knowledge on teaching algebra using CAS) in the methods course are also to be found in the format of lesson study. Lesson study includes both teaching and a reflection session based on the teaching experience and is also to be found in preservice teacher education (Chen & Zhang, 2019; Elipane, 2012; Rasmussen, 2015). Lesson study in preservice teacher education has shown great potential. From a broad perspective, lesson study breaches the gap between theory and practice and lessens the gap between preservice teacher education and the teaching profession. From an educational perspective, it can contribute to the development of practice and knowledge about problem-based teaching or more specific practice and knowledge about fostering and anticipating pupils' responses to specific tasks (Chen & Zhang, 2019; Elipane, 2012; Rasmussen, 2015). In particular, the reflection session is seen as a rich environment for the development of didactical knowledge (Miyakawa & Winsløw, 2013). Thus, the combination of implementing lesson study in preservice teacher education with the aspect of developing non-trivial knowledge and practice about teaching algebra using CAS seems promising.

The implementation of CAS in the teaching of school algebra provides both opportunities and obstacles for learning mathematics. Among some of the opportunities can be mentioned the prospect of the study of mathematical entities such as algebraic equivalence (Gjone, 2009; Lagrange, 2005). Gjone exemplifies how two functions, such as $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$, can be compared for equality with CAS (which gives an output of "true") and thus can give rise to discussion of algebraic equivalence. Although the objects are part of lower secondary school education, they are traditionally used as tools and not as an object of study. However, including activities that explore mathematical entities such as algebraic equivalence has the possibility, not just to explicitly formulate algebraic equivalence, but also to strengthen the students' abilities to solve other types of algebraic tasks. Other potential uses for the implementation of CAS are algebraic explorations such as studying the factorization of polynomials of the type $x^n - 1$ (Hitt & Kieran, 2009). The CAS is used to efficiently obtain a series of results from which the pupils can detect a pattern. The CAS is used again to test their hypothesis and support their arguments for the hypothesis. A third potential is how graphical representation of the algebraic expression or equation can support students' development of knowledge (Gjone, 2009; Kieran & Yerushalmy, 2004). In Gjone, the use of the graphical representation of the two equations $x^2 + y^2 = p$ and $x + y = q$ supports the pupils in structuring an algebraic argument for the solution of the system.

The obstacles for the implementation of CAS are many and varied. We would like to mention the process of teaching and familiarizing the pupils with the general use of the program and commands (e.g., how to undo the last entry, how to change viewing settings, what data type is required for a command etc.) and how the importance of this process is often diminished or the process itself entirely neglected. (Artigue, 2009). Clark-Wilson and Noss (2015) talk about "smaller" types of obstacles, denoted "hiccups", occurring during activities with CAS. For example, pupils' difficulties in entering x^2 or inserting extra parentheses. Clark-Wilson and Noss stress how such hiccups can be capitalized upon to further the pupils' mathematical knowledge and that the potential of hiccups should be an integrated part of teacher development courses.

Against the background of complementing ATD with the instrumental approach, Artigue (2002) adopted the dialectic between practice and knowledge to analyse the position of instrumented techniques in the development of mathematical knowledge. The complementation of the institutional perspective from ATD to the instrumental approach has also been studied (Artigue, 2007; Trouche, 2005). Artigue studied the possible tensions and complements of the two theories, while Trouche used ATD to develop a typology that describes the didactical configuration of the classroom when a digital tool is included. Our vision is to further the combination of the instrumental approach and ATD already accomplished in order to give a description and analysis of an entire lesson that includes the use of CAS, to include the teacher's didactical configuration of the lesson in explicit detail and to describe and analyse the didactical knowledge developed based on reflections about the use of CAS in the lesson.

THEORETICAL FRAMEWORKS AND THEIR COMPLEMENTATION

In this section, we first describe the key elements of lesson study, then two well-established theoretical frameworks for studying learning and teaching situations involving the use of CAS. The first of the frameworks, Instrumental Genesis, brings into focus the role a digital tool plays in the development of an individual pupil's knowledge. The second, closely related framework, Instrumental Orchestration, describes the teachers' guidance of the pupils' instrumental genesis. We then introduce the notions of praxeology and moments from the Anthropological Theory of the Didactic (ATD) to complement Instrumental Genesis and Instrumental Orchestration as they envision a more collective vision of knowledge (Artigue, 2007) and provide a structure for describing in explicit detail. Finally, we will state our research questions.

Lesson study

Lesson study is a format for professional development that originated in Asia but has now spread to a wide range of countries from its origin in Japan and China to the United States and from South Africa to Norway (Hart, Alston, & Murata, 2011; Huang, 2019). The structure of lesson study can be characterised by three phases that are cyclic and repeated: the first phase is the preparation, which includes some studying of resources by the teacher, planning of a lesson, and writing a lesson plan. In the second phase, which we will refer to as the research lesson, the lesson plan from the first phase is carried out in a class. A group of interested parties is invited to observe the lesson.

The third phase, which we will refer to as the reflection meeting, is a meeting of the teacher, invited observers and other parties. During the reflection meeting, aspects of the lesson are discussed, such as specific aspects of the task worked on by the pupils or, more generally, the meaning of the notion "clarity through drawing" (Miyakawa & Winsløw, 2013).

Instrumental Genesis

Instrumental Genesis (Artigue, 2002; Guin, Ruthven, & Trouche, 2006) is a generic theory on human/machine interaction (Verillon & Rabardel, 1995) and has its origins in cognitive ergonomics. The theory suggests that when a person picks up, appropriates and applies a digital tool, the person goes through an evolution, the instrumental genesis, that involves the creation of schemes related to the uses of the tool and the tool itself. The instrument is defined as a combination of the tool, viewed as an object or artefact, and those schemes.

The theory considers the process of instrumental genesis as having two directions. One direction is towards the artefact. This is called instrumentalisation, a process during which schemes for use of the tool are created, such as creating keyboard shortcuts or specific techniques for uses of a command. This process is influenced by the person's knowledge related to the purposes of the tool (e.g., solving equations or creating a route between two addresses). For instance, one cannot learn about commands available in a CAS for solving an equation without prior insight into the subjects of variables, equations, and solution sets. In the process of instrumentalization, the focus is on the tool and one's prior knowledge about the tasks; therefore, while it can help us to solve problems, it is merely supporting this acquisition of schemes about how to use the tool.

The other direction of the process, which is directed towards the user's knowledge about what the tools may help the subject to do (such as equations and solving them), is called instrumentation, and we talk in general about instrumented knowledge (techniques, and so on). For instance, a specific command has been used to plot a function to examine its graph, or an algebraic expression has been factorized in order to determine the number of roots of a given polynomial; this may provide knowledge about the mathematical object both specifically (regarding the function in question) and more generally (what may hold true for functions of a given class, etc.). Consequently, the digital tool affects the user's knowledge about the subject matter through its commands, display and, not least, outputs (the latter are the tool's "reaction" to the user's input and make the tool function as a kind of milieu for the user).

We will use the notions of instrumentalization and instrumentation to distinguish, relate and describe actions and knowledge both mathematically and didactically.

Instrumental Orchestration

Our second theoretical framework is the theory of Instrumental Orchestration, (Guin & Trouche, 2002; Trouche, 2004); this framework is specific to the study and design of situations where a teacher actively supports a user's instrumental genesis. It has mainly been used in the case of teaching mathematics with digital tools, such as a dynamic geometry system or CAS. Instrumental orchestration complements instrumental genesis by focusing on the teachers' work to frame and direct pupils' instrumental genesis. It involves descriptive models of teachers' systematic and planned configurations of the pupils' work with the tool, uses of digital tools available and the choices related to the different stages in the treatment of the mathematical subject. Two dimensions of the instrumental orchestration are emphasized, namely, the didactical configurations and their exploitation modes.

The didactical configurations can be described as the layout of the classroom with regard to the objects already available in the classroom and the objects brought into the classroom, distinguishing between different didactical configurations found in each stage of the didactical situation (Brousseau, 2006). Having the pupils in turn cast the screen of their iPad onto the smartboard is part of a didactical configuration. We distinguish between the didactical configurations for different stages of the didactical situation, such as the situation of introducing a problem to be worked on or the situation of organising the pupils' sharing of findings.

The exploitation mode of the didactical configuration can be described as the way the teacher and the pupils are utilizing the didactical configurations. Continuing the example of the situation of the pupils sharing their findings, the way in which the teacher conducts and organises the sharing and what should be shared as well as the emphasis on certain aspects of the pupils' presentation and not on others is part of the exploitation mode.

The notion of instrumental orchestration will be employed to describe and differentiate between the different situations of the activities.

Praxeology

The Anthropological Theory of the Didactic considers knowledge (such as mathematical or didactical knowledge) as a product of human activity (Bosch & Gascón, 2014; Chevallard, 1999). For instance, the didactical knowledge of a teacher is shaped by her activities and experiences as a teacher and usually also by her past activity as a pupil and preservice student. The theory further posits that human activity takes place within an institution. Here an institution is determined by culture and social context. For instance, a school, a family, and a class of pupils may all be viewed as institutions. Including the institutional view of ATD enables us to consider the instrumental genesis of the institution made of the class of pupils and how the instrumental orchestration effects the instrumental genesis of the class.

In order to study elements of knowledge of an institution, we must have precise models of the practices (activities) that produce the knowledge and, at the same time, of the knowledge that produces the praxis. In order to name this amalgam of knowledge and practice, ATD proposes the notion of praxeology. A praxeology consists of two main components: the praxis and the logos. The praxis component is, furthermore, considered as having two closely related parts: the constituting types of task that the institution aims to complete and the techniques that the institution employs in order to solve the task. We distinguish between an instrumented and a non-instrumented technique based on whether a digital tool instrument is employed.

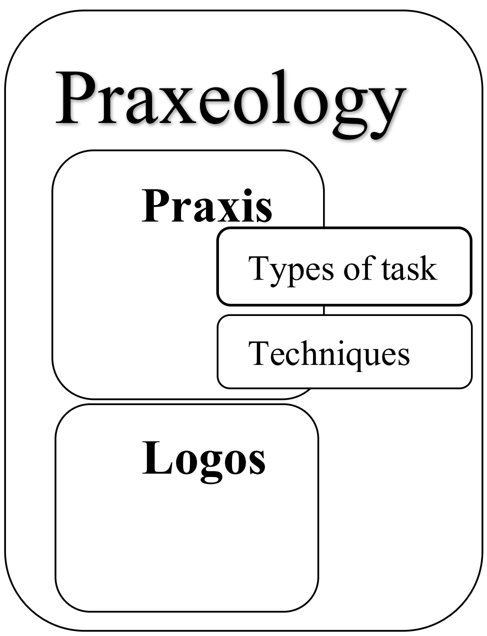

Figure 1 - Praxeology structure

In our study, we will describe and analyse interrelated families of praxeologies: the pupils' and teacher's mathematical praxeologies and the teacher's didactical praxeologies. The teacher's didactical praxeologies aim at developing the pupils' mathematical praxeologies to include an intended, new element (e.g. a technique, a theoretical notion); this new element is called the didactical stake.

The notion of praxeology will be used to complement instrumental genesis and instrumental orchestration by systematically and explicitly providing description of both their mathematical and didactical character.

In order to distinguish between different stages of the development of pupils' and teacher's praxeologies, we adopt the notion of didactic moments from ATD. As explained in more detail by Chevallard (1999), the theory distinguishes between six different moments which may occur in a different order than that given below and which (in a given lesson) may not all be present. The six moments are as follows (Bosch & Gascón, 2014):

- the *first encounter* with the question or tasks related to a new mathematical praxeology to be learned,
- the *exploration* of the corresponding type of tasks and the emergence of a technique,
- the *work on the technique* (refining and improving it) and the study of its scope,
- the elaboration of a *theoretical environment*,
- the *institutionalization* and
- the *evaluation* of the work done.

The notion of moment from ATD will complement instrumental orchestration by providing a systematic description of the teachers' intention of the activity.

Research questions

- What didactical praxeologies can be developed by preservice teachers when teaching elementary algebra using CAS?
- What potential does the implementation of lesson-study-like practice hold for preservice teacher education related to the successful development of instrumental orchestrations?

CONTEXT AND METHODOLOGY

In this section we will describe the context and the methodology for our study.

In order to explore what didactical praxeologies can be developed by preservice teachers about teaching elementary algebra using CAS, we designed a lesson-study-inspired course. The course was incorporated as part of a larger course with the title "Numbers, Arithmetic and Algebra" (NAA). The course NAA was the first course in a series of courses lasting for a year in order to reach the status of teaching competence in mathematics for grades 1 to 10. The course was attended by third- and fourth-year preservice teachers at the teacher education programme in the Faroe Islands. The course which the students had previous taken concerned pedagogy, psychology, and general didactics such as constructivism and cooperative learning. Generally, this teacher education programme does not include classroom practice until the end of the third and fourth years, and thus a larger part of the participants of the course had no prior experience in teaching, and none of the participants were familiar with the concept of lesson study. The preservice teachers also had no prior experiences with the use of CAS in their primary, secondary or teacher education. Furthermore, explorative approaches to learning and teaching were also unfamiliar.

Our interest in the NAA course was only the algebra part that included the use of CAS. It started with three weeks of classroom teaching to prepare the preservice teachers to conduct lesson study during the following eight weeks. The three weeks included getting familiar with the format of lesson study from Japan and with the CAS tools in GeoGebra. For the following eight weeks, the course consisted of bi-weekly lesson studies and a half-day of regular class per week to plan, discuss and analyse with the teacher educator, if necessary. For the preservice teachers to practice lesson study, they were divided into groups of 3-4 preservice teachers. Each group alternated between observing a lesson and being responsible for the enactment of a lesson. Prior to each research lesson, a lesson plan had to be written. The lesson plan followed a template that first included a description of the didactical stake. In the next section, the preservice teachers gave a content analysis of the pupils' planned activities in the lesson. In the following section of the lesson plan, the preservice teachers gave a description of the different orchestration planned for the lesson. The preservice teachers were asked to describe the different orchestrations planned for the lesson to realise the many possibilities and reflect on their choices. The last section of the lesson plan contained the script for the lesson. The script was structured along a vertical timeline and had three horizontal columns: the first column was the type of the didactical moment and a time estimate, the second column included the teacher's planned actions and reactions to the pupils' performance and the third column contained a description of the pupils' expected performance. All the lesson plans written prior to the research lessons and related worksheets were collected as data.

During the research lessons, field notes were written to note the instrumental orchestrations, specific didactical techniques, and pupils' praxeologies. The last research lesson of each group was recorded.

After the research lesson, a reflection meeting was held the same day, following a protocol. The reflection meeting started with the teacher's recollection of the lesson. Then the observers of the lesson were asked to recall their observations of the pupils' performance. The teacher educator also shared observations of the pupils' praxeologies. At the first reflection meetings, the teacher educators' observations were pivotal to exemplify for the preservice teachers the level of detail in observations that is beneficial for reflections. In the latter reflection meetings, the preservice teachers were responsible for most of the detailed observations. Reflections on the observations of the pupils' performance were welcomed. The reflection meetings ended with suggestions for improving the lesson. For the oral exam, the preservice teachers had to rewrite the lesson plans and present one of the lesson plans.

For all the reflection meetings, we recorded what was said by the participants. We consider the reflection meetings our main source of data on the development of didactical praxeologies, based on the article by Miyakawa and Winsløw (2013). In general, all the groups of preservice teachers conducted at least one successful lesson implementing CAS, that is, a lesson which included mathematical explorations using CAS. Some groups started out carefully with using CAS to check results of paper-and-pencil tasks, wanting to familiarize the pupils with CAS and see how the pupils were able to handle CAS and, more generally, an iPad in the classroom. Other and more brave groups started out with mathematical explorations using CAS from the beginning.

For our analysis, the choices of lessons to analyse and present in this study was based on criteria following to our research questions, which emphasise realised possibilities. The first criterion was to choose lessons that where mathematically interesting in the sense that they gave rise to non-trivial development of praxeologies for the teaching of elementary algebra using CAS. The second criterion was to choose lessons where the reflection meeting focused on different perspectives of teaching using CAS, and to present and analyse two different praxeologies. Finally, we chose to analyse some of the last research lessons carried out by the preservice students, as their designs of these lessons were braver and more inquisitive in terms of didactical stake, instrumental orchestration etc., than what we saw in their first lesson studies. It is also clear that at later stages, the preservice teachers were more comfortable with the format of lesson study, and more skilled in observing the pupils' performances.

Two reflection meetings were selected for transcription. Then detailed practice reports of the research lessons were written, and the types of task were identified. Following, the reflection meetings and lesson plans were analysed using the full model of praxeology following the categorization of types of tasks from the accounts of the research lessons.

Our detailed findings related to the two chosen lesson studies are presented in the next two sections. Each section begins with an account of the research lesson around which the preservice teachers' reflections and development of didactical praxeologies were centred. Then we will describe the didactical development of the preservice teachers based on the lesson plans made prior to the research lesson, the reflection meeting, and the revised lesson plan. This gives rise to many observed potentials related to instrumental orchestration. We summarize the findings from both cases in the conclusion.

LESSON A: WHAT IS AN EQUATION?

Description of the research lesson

The didactical stake of the lesson was for the pupils to develop and formulate a tentative definition of an equation, centred around the following five types of tasks:

T_1: Describe what an equation is,
T_2: Solve simple first degree equations with pen and paper,
T_3: Solve non-simple first degree equations with CAS,
T_4: Add together two first degree equations and
T_5: Determine if a symbol string is an equation or not.

The lesson started with a didactical moment of first encounter. The didactical task of the teacher was to guide the pupils to a tentative definition of an equation. The teacher posed the initial question, "What is an equation?" When the pupils seemed lost, she used the didactical techniques of reformulating the task: "When do we see that it is an equation? What is necessary for it to be an equation? What symbols do we need? What does an equation look like?"

The first encounter was then followed by a moment of exploration where the pupils formulated a description of an equation as a tentative definition. The teacher orchestrated that the pupils work in pairs and then used the didactical technique of having the pupils note their work on paper while she walked among the pupils to observe their work.

After a moment of exploration, the lesson changed into a moment of institutionalisation where the teacher orchestrated that the pupils in turn share an element of their work (i.e., "equal sign", "parenthesis", "variable", etc.). A main didactical technique used by the teacher was to write key words on the whiteboard.

The pupils were then given a sheet of paper where a number of tasks of type T_2 were stated, and we entered a moment of work on the technique. The mathematical techniques employed related to traditional algebraic manipulations established in previous lessons. The didactical type of task was intended to reinforce and extend the pupils' praxeology constituted by T_2. The didactical techniques included a list of equations and the stipulation that they must be solved with pen and paper.

The next moment of the lesson was work on techniques where T_3 and T_5 were posed. For examples of equations and symbol strings, see Figure 1. For T_3 the instrumented technique involved entering the expression in CAS then reading the solution of the equation from the output. The T_5 type of task was not stated in the lesson by the teacher but appeared when the pupils were encountering an equation with no solution. The pupils then checked what they had entered and what they had been asked to enter and realised that the equation did not contain any variable.

$$2x - 12 - x = -3x + 8$$
$$\frac{2}{5}x + 5 + 5x + 1 = 2(-x - 3) + 8 + 13$$
$$3 + \frac{1}{2} + 14 - 8 = 3(2 + 1) + 2$$
$$8x + 7 - 27 + 4x = 6 + 5x - x(-3 + 2) + 3$$
$$2\frac{2}{3}x + 21 + 2x = 42 - 4\frac{1}{3}x + 15 + x$$
$$4x + 8 - 14 + 3x = 17 + 5x - 2 - 2x$$
$$2(x - 2) + 4 + 3x - 5$$
$$= 5 + 2(2 - x) + 3 - 4$$
$$3(x - 3) + 5x = 2(-x - 4) + 9$$
$$2x + \frac{1}{2} + 4(x + 2) - \frac{1}{2} + 4 - 2$$
$$4x + 12 + x = 2x + 6 + 12$$

Figure 1. Examples of equations for T_3

The didactical type of task was designed to advance the pupils' definition of an equation. As a didactical technique, the preservice teachers used the affordances of the CAS to introduce a new set of equations. The new equations differed from the types of equations included in previous algebraic praxeologies, which not only reinforced elements of the pupils' definition of an equation, but also extended the definition. Another didactical technique was to have the pupils write down the solutions for the equations on a sheet of paper to push the pupils to read the solution to the equation from the output.

The instrumental orchestration was to have the pupils resume their work in groups, sharing an iPad. Since the pupils were re-evaluating their description of an equation and formulating additions to the description, the moment included a theoretical environment.

The next section of the lesson shifted between moments of institutionalisation and moments of evaluation. The instrumental orchestration was to project the screen of an iPad onto the smartboard, alternating between pupils, while the pupils described what they had done, the output of the CAS and how it related to their (previous) concept of an equation. For example, the equation $\frac{2}{5}x + 5 + 5x + 1 = 2(-x - 3) + 8 + 13$ was considered with the output $\{x = \frac{45}{37}\}$. The pupils recounted that the output made them recheck what they had entered because they did not consider a fraction to be a solution to an equation. After rechecking, they realised that, in fact, a fraction can be a solution to an equation. The teacher then reformulated the pupils' conclusion about fractions as solutions of a first degree equation:

> [A solution to an equation can be] not so nice… but still a [correct] value for x… A solution can also be [a] very small [number] or something with a comma, that does not end [an infinite number of decimals] like three point eight two seven and so on [3.827…], but sometimes they are easier to write in that way [fraction], it is just a way of writing a number.

The equation $3 + 1/2 + 14 - 8 = 3(2 + 1) + 2$ was also considered in similar fashion, and it was concluded first by the pupils and then by the teacher that an equation must contain at least one variable. The symbol string $2x + 1/2 + 4(x + 2) - 1/2 + 4 - 2$ gave rise to a reflection about whether or not an equation must have an equal sign because the CAS was able to find a solution for the equation. Apart from concluding that a main requirement for an equation is an equal sign, it was also concluded that the pupils' and the teacher's instrumentalisation process was unfinished since neither could figure out why CAS could possibly solve the "equation".

The lesson changed to address T_4, but we will go no further with our description since the remainder of the lesson is not related to the didactical stake of making a tentative definition of an equation.

Development of didactical praxeology

Although solving T_1 did not directly require the use of CAS per se, it was because of CAS that such a type of task was being formulated, and with the implementation of CAS it gave rise to further mathematical investigations.

In the lesson plan written prior to the research lesson, we could identify T_1 and some elements of the related didactical praxeology. For the moment of exploration, the preservice teachers described the didactical technique of having the pupils discuss together in pairs. For the moment of work on the technique, the preservice teachers specified the didactical technique of having the pupils write the result of their discussion on a piece of paper. As didactical logos for T_1, the preservice teachers asserted the intention to strengthen the pupils' ability to formulate their mathematical knowledge and to reflect on equations as an object of study.

In the research lesson, two further didactical techniques could be identified that were not included in the lesson plan and were pivotal for T_1. The first was to reformulate the initial question during the first encounter, and the second was to write key words on the board during the moment of institutionalization.

In the reflection meeting, the task T_1 was reformulated more explicitly by a preservice teacher as was the didactical technique of reformulating the task:

> [T]he pupils have worked a lot [with equations] and I see it as if you wish for the pupils to make a definition of an equation–close to, anyway.

Then a preservice teacher shared her observations of the pupils' responses in relation to the didactical technique of reformulating the task:

> The teacher asks, "What is an equation?" Then the pupils are stunned, perplexed. But then the teacher asks [the] supplementary [questions]: "When do we see that it is an equation? What is necessary?" This is actually a great addition; they can answer [the questions]. Then the teacher also asks, "What symbols do we need?" And then the ball starts rolling for the pupils.

In the revised lesson plan, we could explicitly find the expression "definition of an equation". It was used as a guide for the reader, but not as an expression to be specifically stated during the lesson. In the column of the teacher's activities in the script of the lesson, we could identify the didactical technique of reformulating the task and even explicit suggestions for the reformulations. In the description of the orchestration, where before there was not even a description of an orchestration, we could now find the orchestration related to T_1 from the research lesson and could even identify the didactical technique of writing key words on the whiteboard and that the key words must remain visible through the lesson. In contrast to the previous lesson plan, we could now identify more of the pupils' praxeology for T_1 in the column of the expected pupil activities such as the following:

> [I]t will be difficult to formulate for the pupils, no pupils will come up with a definition of an equation, and they will say equal sign and the like.

Although the preservice teachers still did not give the task T_1 its own content analysis, they were still giving more importance to the task. The didactical techniques related to T_1 were now explicitly identifiable in the lesson plan, and the descriptions contained more professional terms. In addition, the pupils' performance, which before was a wild card, was now predicted.

The preservice teachers, with their posing of T_1, realised the potential of CAS to study mathematical entities such as the study of an equation. To plan a journey on such a study with a class of pupils, the preservice teachers realised the importance of the initial question, of handing over the question to the pupils and of the pupils being unfamiliar with such types of tasks and needing reformulations.

The task T_2, as it was familiar to the preservice teachers and pupils from previous lessons, did not warrant a change in the lesson plan nor was it subject of further reflections.

For the task T_3 in the lesson plan, we could identify the instrumented techniques such as "entering equations", "placing [extra] parenthesis" and logos such as "knowing syntax on how CAS reads what has been entered", "the addition of extra parenthesis and their placement" and more general logos such as "definition of equations, [definition of] parenthesis, … syntax of CAS". The content analysis did not give the complete praxeology of the pupils; nevertheless, it showed us that the preservice teachers had some insight into the necessary instrumentalisation process for entering and solving equations with CAS.

In the reflection meeting, the first observation shared by one of the preservice teachers in relation to T_3 concerned the pupils' responses to the didactical technique of including a type of equation of the form $\frac{n_1}{n_2}x + n_3 + n_4x + n_5 = n_6(-x + n_7) + n_8 + n_9$, where the n_i's are non-zero integers and n_1 and n_2 are mutually prime. The sharing of observations of the pupils' activities started with the recalling of a question asked of fellow pupils about how to enter the expression $\frac{2}{5}x$ in CAS. Then an observation followed of the pupils' reactions to including a type of equation that had a non-integer solution:

> It is really good with all the big ugly fractions [as solution for the equation] because then the pupils think it is incorrect. But then they check [what they have entered] and everybody else have forty-five thirty-seventh [$\frac{45}{37}$]. They have never had such a solution before.

The two shared observations then sparked a reflection by one of the preservice teachers:

> This [introducing new types of equations] is among other what we can do with CAS … They [the pupils when encountering new types of equations] run with it. Usually they would have had a raised a hand to ask the teacher for help. But they are experimenting and investigating, I think this is great.

Another preservice teacher shared her reflection about the pupils' willingness to establish new instrumented techniques:

> I think that when CAS is involved then it is as if they [the pupils] dare better to experiment than if they were using pen and paper. They feel safer with CAS even though they do not know [for sure].

Another observation of the pupils' responses propelled further development of didactical logos. A preservice teacher shared an observation:

> One [an equation] was x parenthesis [$x($]. A pupil tried [entering the expression] but it did not work. But then he writes x times the parenthesis and then it works. He discovered that if x is next to the parenthesis then you have to enter an additional multiplication sign.

Another preservice teacher commented:

> It [the new type of equations] can also enhance the mathematical understanding, the theory.

Then the first preservice teacher replied:

> Yes, you need to understand that x is multiplied into the parentheses. We do it implicitly, but now the pupil is forced to write it. That is great.

The preservice teachers' observations and reflections about the instrumented technique of entering expressions of the form $x($ and $\frac{n}{m}x$ exemplified how CAS could be capitalized upon to explicitly write the otherwise implicit multiplication sign. Such smaller autonomous instrumentalisation processes were possible because of the pupils' previous algebraic praxeologies. This also illustrated that the use of instrumented techniques requires mathematical logos.

Another line of observations was related to the didactical technique of including symbolic strings that did not include an equal sign or variable and the emergence of T_5. It was pointed out that the pupils, who noticed that there was no variable or no equal sign, immediately concluded that the given expression was not an equation, due to the key words on the whiteboard from the beginning of the class. The teacher confirmed that he intended the key words to remain on the whiteboard to guide the pupils during the moment of the work on the technique.

In the lesson plan written after the research lesson, we could explicitly identify several of the didactical techniques that occurred in the research lesson, such as including new types of equations and symbol strings that did not contain an equal sign or a variable. Further, a new type of orchestration was added and another one was modified. Inspired by the article on instrumental orchestration by Drijvers, Doorman, Boon, Reed, and Gravemeijer (2010), the preservice teachers called one orchestration "technical-demo". However, the orchestration did not include any collective element or interference by the teacher. Instead, it was expected that the pupils could develop the appropriate instrumented techniques through autonomous experimentation. The other and new orchestration was called Sherpa-at-work, which is where each pupil in turn will "present and describe her work process". The mathematical techniques, both instrumented and non-instrumented, such as placement of extra parentheses and other mathematical symbols, were also more elaborately described in the timeline.

Now that the preservice teachers were able to create a series of tasks, T_3, they realised the potential of CAS to study mathematical entities (mentioned in the background section).

A moment of institutionalisation that did not occur in the research lesson was added to the timeline:

> The teacher orchestrates a recollection of what they have learned today, in particular, the conditions and the characteristics of an equation. The pupils will answer considerably better ... presenting how the definition of an equation has gradually developed.

Further, the preservice teachers added moments of institutionalisation of the theoretical environment conforming to the structure of some forms of inquiry-based teaching.

LESSON B: THE DISTRIBUTIVE LAW $a(x+b) = ax + ab$

Description of the research lesson

The full lesson was divided into two activities. In the first activity the pupils worked in groups and solved simple equations with pen and paper, sharing their techniques and technology with the group and later with the class. In the second half of the lesson the preservice teachers had planned an activity with the didactical stake of having the pupils formulate the abstract distributive law $a(x+b) = ax + ab$ for real numbers a, b and x. As the activities are not closely related, we will concentrate on the second activity that involved the use of CAS. The activity revolved around the following mathematical tasks:

> T_1: Set up two integer sliders named a and b in the interval from -10 to 10, (see Figure 3),
> T_2: Set up the generic line $y = ax + b$, (see Figure 3),
> T_3: Generate an example of the form $y = n(x + m)$, where n and m are integers,
> T_4: Slide the line $y = ax + b$ on top of the generated example and note the values for a and b (see Figure 4), and
> T_5: Answer the question, "What is the relation between the equivalent expressions?"

The second half of the research lesson started with a moment of first encounter of T_1-T_5, with the main didactical technique of giving a short oral presentation of the tasks T_1-T_5. Following, the lesson changed to a moment of work on the techniques, where the main didactical technique was to hand the pupils a booklet. The booklet contained illustrations for the techniques required for T_1 and T_2, such as setting up sliders and lines in GeoGebra (see Figure 3).

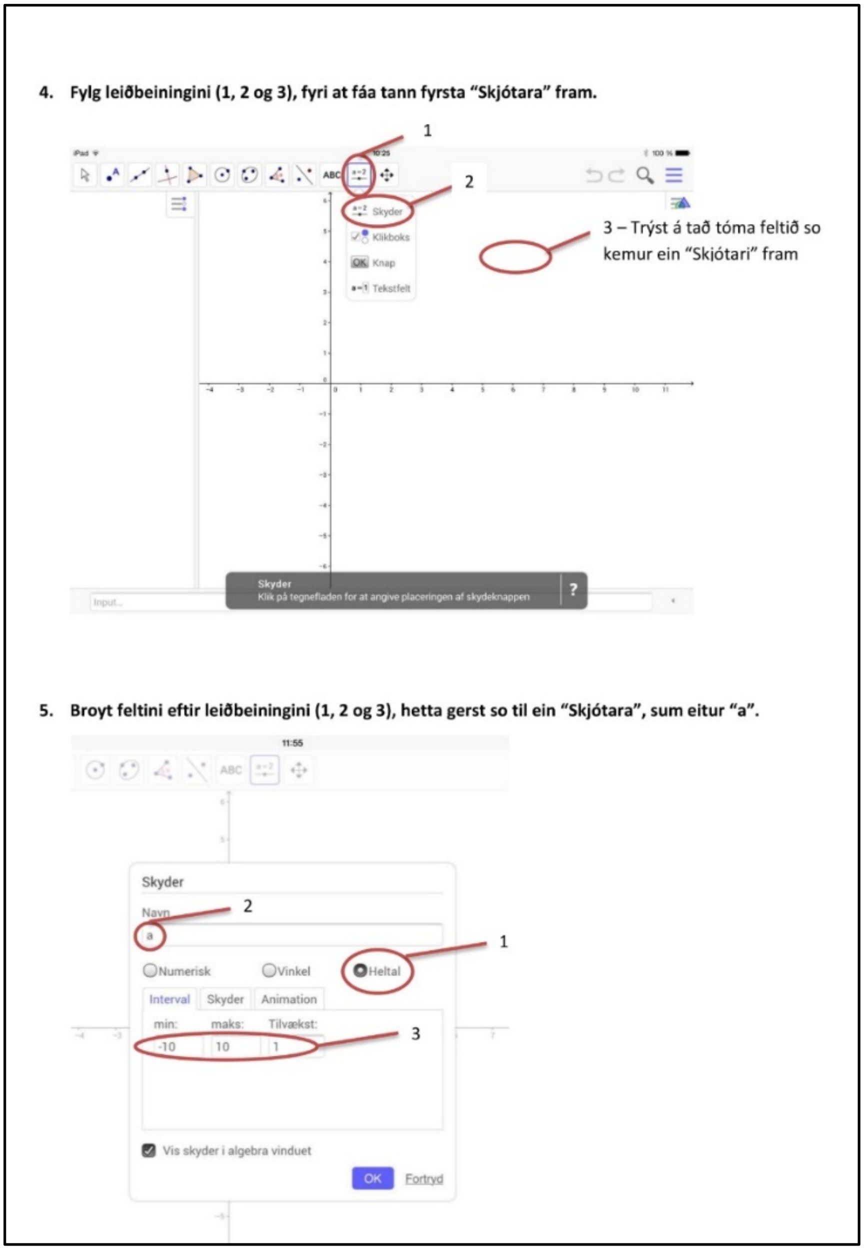

Figure 3. Extract of booklet, Illustration 4 and 5

Whenever the pupils encountered an instance that was not described in the booklet, such as slider *b* being above slider *a* instead of below slider *a* (see Figure 2), they requested assistance from the teacher. It also became apparent in the moment that the order in which the instrumented techniques were performed was pivotal (i.e., if in Figure 3, Illustration 5, the instrumented technique numbered one was performed last, then the changed values for the name and the interval (the instrumented techniques numbered by two and three) changed to a pre-set standard). The moment did not contain any explicitly stated logos for any of the instrumented techniques present. The completion of the task T_2 resulted in the blue line in figure 6.

The next moment of the lesson was again a work on techniques.

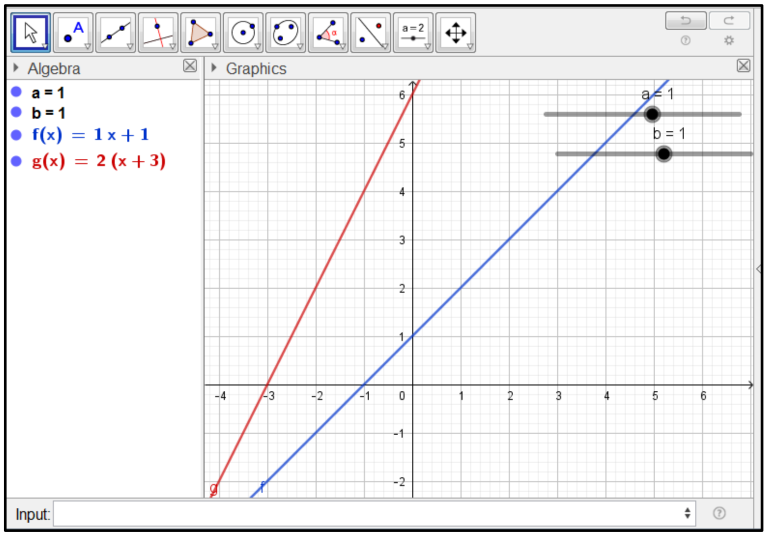

Figure 4. Screen shot of GeoGebra

The moment involved the tasks T_3 and T_4. The pupils first generated an expression of the form $n(x+m)$, where n and m were integers, then they noted the expression in their work sheet, and then the expression was entered into the input bar in GeoGebra, which generated the coloured line $y = n(x+m)$ in the coordinate system, the red line in Figure 4. The pupils then moved on to the task T_4. The instrumented technique was to press down on a slider and slide it back and forth. When the slider changed value, the parameter in the generic line changed value accordingly. By alternating between slider a and b, the pupils moved the generic red line on top of the other blue line (see Figure 6). The pupils then noted the expression for the generic line on their worksheet next to the expression from T_3. The pupils repeated T_3 and T_4 with different examples.

For the pupils who managed to complete T_3 and T_4 with four different examples, the lesson changed to a moment of theoretical environment. The task of the moment was T_5 that was stated on the worksheet handed out in the beginning of the activity.

Shortly afterward, the lesson changed to a moment of institutionalisation. A didactical technique of the moment was for the pupils to use an online version of GeoGebra on the smartboard and institutionalise every instrumented technique of setting up the sliders and the generic line. The group of pupils also shared an example of a line where they described how they entered the line in GeoGebra and how they moved the generic line on top of their example. The group of pupils then explained the relation between the two equivalent expressions. In the moment, both the didactical stake and the instrumented techniques were being institutionalised.

The moment then changed to evaluation of the work done, and the teacher reformulated and expanded the pupils' conclusion, emphasising the geometric illustration of the expressions and the parentheses. (The two lines are matching, so the expressions must be the same. Since the expressions are the same, then a is equal to n and b is equal to n times m, hence when we go from the expression $n(x + m)$ to the expression $ax + b$, the number n (outside the parentheses) is multiplied with the terms x and m inside the parentheses. This is the property of the parentheses.) She then went on to illustrate "the property of the parentheses" with another example.

Development of didactical praxeologies

The techniques related to answering T_1 could be identified prior to the research lesson as the preservice teachers had compiled a booklet. We could determine techniques such as pressing the slider icon, then choosing the slider from the fold-out menu, and then pressing an empty space on the coordinate system (see Figure 3, Illustration 4). Apart from the techniques required to set up sliders, nothing was written about the didactical praxeologies.

In the reflection meeting, the first shared observations and reflections were about the orchestration of the instrumentalisation process, in particular, the didactical technique of using a booklet to guide the pupils' instrumented techniques. The first observation shared by a preservice teacher was the pupils' reaction when they were handed the picture booklet. The joy that the pupils had expressed earlier when being asked to take out their iPads was being replaced with expressions conveying their demotivation at being handed a booklet, and even before opening the booklet a pupil claimed not to be able to read it. There was general agreement among the preservice teachers that the pupils found it difficult to follow the instructions in the booklet. However, the following shared observations of the pupils' interaction with GeoGebra give a more nuanced impression. A preservice teacher recalled:

> There were some pupils that did not get the same screen [as depicted in the booklet], the slider a was below slider b. So, they were, "Oh, no! We did it wrong, we have to re-do it".

The observation exemplified that, even though the instrumented techniques in the booklet were very carefully and meticulously depicted, the booklet did not describe for what and how (part of the logos) the sliders were intended to be used, enabling the pupils to draw the conclusion that the positioning of the sliders in relation to each other was irrelevant.

Then followed a series of observations of instrumented techniques that were not planned. A preservice teacher shared two of her observations:

> They [the pupils] pressed somewhere on the screen, and then a message or other pops up. Then one of the pupils says, "What have you done!" Another group of pupils had entered a great deal, but then suddenly they delete everything, and started over!

The observations shared showed the preservice teachers that a lengthy series of instrumented techniques, such as setting up sliders, required more instrumentalisation in order to edit what they had entered and not to be so easily shaken by unforeseen actions of GeoGebra.

Different didactical techniques were suggested, either to avoid the lengthy set up of the sliders completely or to further the pupils' instrumented techniques. A didactical technique suggested was to send the pupils a GeoGebra file, where the sliders had already been set up. However, the didactical technique was dismissed since it would create other technical issues, such as the pupils all accessing the internet at the same time, logging into a shared folder and downloading a file, and it still would not resolve the pupils' insufficient knowledge of instrumented techniques related to the general use of GeoGebra. Another instrumental orchestration was suggested to integrate the booklet and further the pupils' instrumentalisation process:

> It [the setup of sliders] is a good idea to have on paper [referring to the booklet], but perhaps for the next time …, we give the booklet anyway in case they do not remember all the techniques, but we go through the techniques on the smartboard so that everyone gets the understanding.

The preservice teachers suggested and agreed to a new instrumental orchestration, where the teacher would go through the setup of the sliders in order to be able to provide some logos along the description of the instrumented techniques.

The preservice teachers experienced that during a lengthy moment of work on techniques, when it involves instrumented techniques commended upon the inclusion of logos of the instrumented techniques, the logos must contain knowledge in relation to the mathematical object and more general knowledge about the use of GeoGebra, such as how to edit or go back one step of what has just been entered.

In the revised lesson plan, we could identify the orchestration, including the booklet, complemented by the teachers going through the setup of the sliders on the smart board. The preservice teachers gave more importance to the instrumentalisation process.

For the task T_4, we could not detect any mathematical or didactical praxeologies from the lesson plan written prior to the research lesson. In the reflection meeting, the preservice teachers shared observations and reflections related to the task T_4. A preservice teacher recalled a group of pupils that were sliding the generic line ($y = ax + b$) on top of their own example of a line ($y = n(x + m)$), where n and m are fixed integers. She recalled that the pupils slid one line on top of the other, but then the visualisation of the line $y = n(x + m)$ disappeared, and only the generic line was visible. This confused the pupils, and the preservice teacher recalled that one of the pupils exclaimed that she didn't see it, but that the other pupils in the group helped her by clarifying that the "disappeared" line was behind the visible line. Another preservice teacher then added her observation of a group of pupils using the sliders. She observed that even though the pupils could adjust on two sliders, one of the pupils insistently only adjusted one of the sliders:

> She starts with changing [the slider] a, but she doesn't get the line on top of the other, because she is only changing the slope of the line. She tries for a very long time, back and forth, back and forth [with the slider], all of four times! At the end, another [pupil from the group] asks her if she can try. Then she [the other pupil] moves slider b, and then slider a again. [I think] This is great!

This exemplified for the preservice teachers several more instrumented techniques for T_4 other than pressing down on the slider and moving the slider back and forth. There was the instrumented technique of interpreting when the lines are on top of each other and the instrumented technique of changing between sliders with the logos of relating one slider to the slope of the line and the other to the vertical move of the line (the intersection with the y-axis).

The final observation shared related to T_4 concerned the didactical technique of limiting the sliders to the interval between -10 and 10. A preservice teacher recalled that several groups of pupils encountered that they were not able to slide the generic line ($y = ax + b$) on top of all lines and that one group dealt with this phenomenon by deleting the line that was outside of reach, while another group started to reflect on why the generic line could not reach the other line. A preservice teacher suggested including supportive questions from the teacher to the groups who had encountered this so that all the pupils could benefit from this information.

The preservice teachers observed a use of the slider that for some pupils was a constraint but that others were able to capitalize upon.

The preservice teachers suggested that the lesson could benefit from the phenomenon to enrich the pupils' learning.

In the revised lesson plan, the preservice teachers described the case where the multiple of the variables *n* and *m* was above 10 or below -1 and prepared the teacher to ask the pupils why they could not slide the generic line on top of their example, how the slider *a* moved the line and how the slider *b* moved the line.

By adding to the lesson plan or turning an obstacle into an opportunity, the preservice teachers realised the potential of turning a hiccup into an opportunity for further learning (as described in the background section).

CONCLUSIONS

Based on the detailed findings which were described in the preceding sections, we now present our main conclusions related to each research question.

The preservice teachers' praxeologies developed related to the use of CAS

For the didactical moment of work on the technique, the preservice teachers experienced that the use of CAS can enrich the exploration of non-instrumented techniques, such as looking for the presence of an equal sign in order to decide whether a given symbol string is an equation, as new types of expressions can be examined. In order for the moment to be enriched, the deliberate construction of new types of examples plays a crucial role, as seen in Lesson A.

The preservice teachers experienced that the pupils were able to autonomously develop new instrumented techniques based on their previously established mathematical praxeologies, as in Lesson A, but that they failed if the logos for the instrumented technique required generic knowledge of CAS, such as how to edit what had been entered (experienced both in Lesson A and Lesson B).

As a general didactical hypothesis for the use of CAS in the teaching of elementary mathematics, the preservice teachers experienced that relative theoretical mathematics could be introduced, such as in Lesson A where the didactical stake was "What characterises an equation?" or as in Lesson B, where the pupils discovered the distributive law. Further, they experienced that hiccups could be capitalized upon to enrich the activity (as in Lesson B). The preservice teachers experienced the potential of CAS for graphical representation (as in Lesson B) and that the instrumentalisation process should not be diminished.

The preservice teachers' observations and reflections supported them in developing a sharpened sense for details regarding instrumental orchestrations including the details of descriptions, the repertoire of different instrumental orchestrations and the distinction between different types, as is evident in the revised lesson plans of the two lessons.

The potentials for implementing lesson-study-like practice in preservice teacher education related to the successful development of instrumental orchestrations

The lesson-study-inspired course with lesson plans, research lessons and reflection meetings allowed us get insights into the preservice teachers' development of didactical praxeologies related to the use of CAS when teaching elementary algebra. We hypothesise that the use of lesson-study-type activities as an explorative device for professional development has led the preservice teachers to attempt more courageous lessons with untraditional tasks, such as Lessons A and B, and thus to engage in genuine experimental inquiry about teaching elementary algebra with the use of CAS. We have the impression that this explorative nature has also enhanced the development of didactical praxeologies related to using CAS for elementary algebra far beyond what is found in the preservice teachers' teacher education.

The development of the preservice teachers' didactical praxeologies had its origin in the shared observations of the pupils' praxeologies. In Lesson A, there was a clear pattern of how shared observations contribute to general reflections about teaching elementary algebra using CAS, while in Lesson B, it was clear that the first impression about pupils' praxeologies, such as that the booklet was difficult to read, was being modified and described in explicit detail by shared observations.

The preservice teachers developed instrumental orchestrations realising the potentials of CAS described in the literature, such as mathematical explorations (both Lessons A and B), studying mathematical entities (Lesson A), capitalizing on hiccups to further enrich the mathematical activity (Lesson B) and using graphical representation to develop algebraic knowledge.

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