

PhD Thesis

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Learning challenges in the interplay between physics and mathematics: the case of the 1-D wave equation

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Learning Challenges in the Interplay between Physics and Mathematics: The Case of the 1-D

Wave Equation

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Resume

Fysiske ligninger er essentielle komponenter til at forklare utallige fysiske fænomener i form af matematiske udtryk. Dog medfører matematikkens tilstedeværelse i fysik mange udfordringer, både ved differentiering af deres roller og ved sammenlægning af begge områder. Mens mange undersøgelser undersøger studerendes perspektiver på fysiske ligninger, foreslår få metoder til at forstå disse ligninger. Denne afhandling sigter mod at udfylde denne mangel ved at undersøge studerendes epistemologiske indramning og læringsvanskeligheder, når de forsøger at forstå en fysikligning, i dette tilfælde 1-D bølgeligningen, fra flere aspekter. To studier i denne afhandling anvender en kvalitativ forskningsmetode, der består af tre trin: udforskning, intervention og evaluering. Det første studie udforsker studerendes forståelse af den grafiske repræsentation af $y(x; t) = f(x - vt)$, hvor studerende præsenteres for en ikke-periodisk bølgeprofil af $y(x)$ og bliver bedt om at finde hastigheden af punkter på profilen. Denne opgave kræver, at de inddrager tid i deres analyse. Resultaterne viser, at denne problemstilling er ekstremt udfordrende for studerende og frembringer flere vanskeligheder. Det andet studie sigter mod at udforske de studerendes epistemologiske indramning af bølgeligningen fra forskellige aspekter. Et spørgeskema bestående af fem spørgsmål stilles til de studerende. Resultaterne viser, at de studerendes svar mangler fysisk betydning, og deres overbevisninger er ikke på linje med eksperter. Interventioner er derefter designet til at hjælpe studerende med bedre at forstå de problemer, der er identificeret i disse studier. Selvom nogle studerende forbedrer sig efter interventionerne, fastholder nogle deres ræsonnement, hvilket indikerer, at de har robuste overbevisninger om emnet. Endelig søger det tredje studie at afsløre kompleksiteten ved bølgeligningen ved at fremhæve konceptuelle finesser, der ofte går ubemærket hen i undervisningen af bølgeligningen.

Abstract

Physics equations are essential components for explaining numerous physical phenomena in the form of mathematical terms. However, the presence of mathematics in physics poses many challenges, both in differentiating their roles and in combining both fields. While numerous studies investigate students' perspectives about physics equations, few propose methods for understanding these equations. This thesis aims to fill this gap by investigating students' epistemological framing and learning difficulties when they attempt to make sense of a physics equation, in this case, the 1-D wave equation (WE), from several aspects. Two studies in this thesis employ a qualitative research method consisting of three steps: exploration, intervention, and evaluation. The first study explores students' understanding of the graphical representation of $y(x; t) = f(x - vt)$, where students are presented with a non-periodical wave profile of $y(x)$ and asked to find the velocity of points on the profile. This task requires them to consider the time dimension in their analysis. The results highlight the significant challenges students face and reveal the various difficulties encountered. The second study aims to explore students' epistemological framing of the WE from various aspects through a questionnaire comprising five questions. The findings indicate that students' responses lack physical meaning and their reasoning is not aligned with those of experts. The interventions are then designed to help students better understand the problems identified in these studies. While some students improve after the interventions, a few persist with their reasoning, indicating they hold robust beliefs about the subject matter. Finally, the third study seeks to uncover the complexity of the WE by highlighting conceptual subtleties that often go unnoticed in teaching the WE.

Preface

Starting as a PhD student three months before the covid pandemic strikes the world and eventually being able to complete it surprises me as a person who lives far from family and close friends. Luckily, on this journey, I received a lot of supports from many people during that difficult time. Therefore, I would like to express my deepest gratitude to those who always believe in me and encourage me in many ways. First, I must send my deepest thanks to my supervisor, Ricardo Karam, for his endless support and motivation. An excitement to discover something about which we had never previously thought and a joy that our efforts have ultimately paid off. This work would not have been like this without your expertise and encouragement. I would also like to thank my co-supervisor, Christian Joas, for his valuable feedback on the research design and the paper.

I never thought I would end up pursuing my PhD in Denmark, a country that was never on my dream list. Although I had received an offer from another place, choosing the Department of Science Education at the University of Copenhagen for my PhD was one of the best decisions I have ever made. Surrounded by very supportive and nice colleagues is a blessing. I want to thank Henriette Holmegaard, who has always been within reach, providing emotional support when the lockdown situation became too much to handle. Also, thank you for the invitation to your lovely house and the enjoyable brunch. Christina Larsen, thank you for always going out of your way to help me. I cannot thank you enough for your kindness. Also, I thank Lene Madsen for her support and assistance. And to Jan Sølberg, thank you for your guidance as the head of the section and for the delightful conversations we have had. I also thank Nadja Normaj, Axel, and all employees at IND for their help and supports.

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My Indonesian friends who lived or live in Copenhagen that always made me feel at home. I want to thank my friends who are currently in Indonesia. Uni Yuvi, Bang Dwi, Kak Huria, Bang Irul, and Bang Ipul, for the beautiful moments we spent together during my first year in Denmark. I am very grateful. Azka and his family have always been within reach, and I am thankful for the enjoyable trip we had in Trømsø. Then, my friends who are currently in Denmark. Randy and his family that have been consistently helpful, and I appreciate their support. I also want to express my gratitude to Bu Kus, who has been a motherly figure, for her advice and for often inviting me to enjoy delicious Indonesian food.

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The thesis is based on these three papers and they are referred to as Study 1-3 throughout the thesis.

Paper I

Rangkuti, M. A., & Karam, R. (2022). Conceptual Challenges with the Graphical Representation of the Propagation of a Pulse in a String. *Physical Review Physics Education Research*, 18(2), 020119.

Paper II

Rangkuti, M. A., & Karam, R. Encouraging Students to Understand the 1-D Wave Equation. The first revision submitted to *Physical Review - Physics Education Research*

Paper III

Karam, R., & Rangkuti, M. A. Conceptual subtleties of the 1-D wave equation. To be submitted in *American Journal of Physics*

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1 Introduction

1.1 The overall purpose of the PhD project

The purpose of this study is to investigate students' main conceptual difficulties in understanding the physical meaning of the WE. The difficulties encountered by students in interpreting the WE were explored in some detail. This study revealed the reasons for these difficulties. Additionally, this study also considered various teaching strategies that may assist students more effectively to develop a conceptual understanding of the WE. Although focusing on the WE as a case study, this research shall also provide ways to help students understand equations more broadly.

1.2 Motivation of this study

Physicists commonly use equations to guide their reasoning, thus an essential part of physics education is to teach students to do so. However, mathematical symbols that are widely used in physics equations present a challenge regarding how to understand their physical meaning. Some studies found that students typically focused on calculating and identifying symbols rather than gaining a conceptual understanding of the principles underlying the equations [1–4]. For example, these studies found that students' performance was better in solving numerical rather than symbolic problems, although the questions shared identical concepts [4, 5], or students failed to transform symbolic into graphical representation of vectors [1].

The epistemological framing regarding physics questions were also explored. Some studies indicated that students believed they understood a physics equation if they could use it to solve problems [6–9]. However, they were confused when asked about the meaning of the equation and tended to treat the physics equation as a calculation recipe, a strategy called plug and chug [6, 10, 11].

It is interesting to see that there is not much literature that attempts to help students understand a physics equation, but exploring students' views about physics equations [6–8, 12]. To fill this gap, this study not only tried to see the epistemological framing regarding a physics equation and the difficulties associated with it, but also designed interventions in order to help students understand deeply a physics equation by exploring different aspects of the equation. Thus, this study not only explored students' views and difficulties in making sense of mathematical representations in physics, but also offered perspectives to connect physics and mathematics by exploring different aspects that are not commonly presented in a teaching situation.

1.3 Research Questions

Mathematical representations in physics come in many forms. The main goal of this study is to investigate students' epistemological framing and difficulties when trying to make sense of

physical phenomena represented mathematically. The WE is a case that can be explored in several aspects.

The mathematical representation of a wave is $y(x; t) = f(x - vt)$, and can be related to the WE. In physics, the wave function is mostly represented in graphs, and since y depends on x and t , the graph of this function can be represented as functions of $y(x)$ and $y(t)$. The dependence of vertical displacement with two variables provides a challenge in understanding it. Therefore, the first study tried to explore the situation where students were asked to find the velocity of the wave profile $y(x)$ which forced them to think about the time dimension in this problem.

After investigating students' difficulties, Study 1 also attempted to design an intervention that addresses the specific difficulties encountered by the students in order to help them better understand the problem presented. Therefore, Study 1 consisted of two research questions, as follows:

1. What are students' reasoning and difficulties related to the conceptual understanding of the graphical representation of waves?
2. What are appropriate interventions that help students understand the mathematical description of waves that always involve x and t ?

Interpreting how students make sense of the WE is the goal of Study 2. The WE is represented by a second-order partial differential equation which can often be deceptively simple at first glance. However, understanding its meaning poses significant challenges. Therefore, Study 2 explored students' epistemological framing when they tried to interpret the WE.

Similarly to Study 1, an intervention that helps students better understand the WE was presented in Study 2. The intervention explored different aspects of the WE as a proposed strategy in this thesis in order to contribute to helping students view a physics equation differently. Therefore, the following two research questions were the core of this study:

3. What are students' reasoning and difficulties related to the physical meaning of the WE?
4. What are possible teaching strategies that help students acquire a deep understanding of the physical meaning behind the WE?

Lastly, this study also explored the conceptual subtleties from the perspective of the historical development of the WE as a potential teaching strategy to deepen students' understanding of the WE. Study 3 answered the following question:

5. What are the conceptual subtleties related to the WE that can help students understand the WE deeply?

1.4 Structure of thesis

This thesis is structured by three studies that address predetermined research questions. Chapters 2 and 3 are based on the three papers included in this study. Chapter 2 focuses on students' difficulties with the propagation of a pulse in a string and how they interpret the WE, as highlighted in the first and second paper. Meanwhile, Chapter 3 highlights some concepts related to the WE that might not be commonly presented in textbooks as a proposed strategy to deepen students' understanding of the WE. Finally, Chapter 4 is the concluding chapter of the thesis.

2 Interplay between physics and mathematics: investigating students' epistemological framing

Physics equations are built from mathematical formalism that explains the nature of the world. However, there is a clear distinction between the treatment of equations in physics and mathematics. For instance, Heck and van Buuren [13] explained that while mathematical formalism can be associated with a physical context, the use of variables in these two subjects depends on how they are treated. In mathematics, variables are commonly used to express unidentified components, while in physics, variables are used for elements that can be measured, quantified, and usually have a unit. Redish [14] provided an example to illustrate this point. He posed a question to both physicists and mathematicians:

$$\text{If } A(x; y) = K(x^2 + y^2); K \text{ is a constant} \quad (1)$$

$$\text{What is } A(r;) =? \quad (2)$$

In physics, the answer is $A(r;) = Kr^2$, where there are some factors that physicists need to consider. The simplest justification is the unit inside the bracket, which must be equivalent. Unlike physicists, mathematicians believed that the answer to that question is $A(r;) = K(r^2 + ^2)$ due to the transformation of two arguments. This example highlights the crucial need to distinguish between the language of physics and mathematics.

Nevertheless, it is also important to emphasize that physics and mathematics cannot be separated, but this situation poses a challenge when one treats them in the same manner. A significant part of this investigation involves exploring the situation where two studies were conducted, resulting in Papers 1 and 2.

2.1 Students' epistemological framing about physics equations

Epistemological framing refers students' perspectives on relevant knowledge in a given situation [15]. In the context of physics education, students' attitudes, beliefs, and expectations play an important role in their success in the subject [16, 17]. Although different studies may use different terms to describe epistemological framing, they generally refer to beliefs about what constitutes knowledge in physics [18]. Redish et al. [16] used the term "cognitive expectation" to describe students' understanding of the process of learning physics and the structure of physics knowledge.

Several studies have investigated students' understanding of physics equations, and some findings suggest that students often hold similar beliefs in comprehending these equations. For example, in a recent study by Airey et al. [6], physics students from three different countries



Figure 1: How students think if they understood a physics equation [6]

were asked to describe how they know when they understand a physics equation. The researchers then classified the students' responses in the first attempt from Swedish university students, which are presented in Figure 1.

From Figure 1, we can see the range of descriptors that students used in their responses. Although the design of this study is limited since it only asked students for a short answer to the question, the diagram provides insight into students' beliefs about understanding physics equations. The authors highlighted some simple descriptors, such as "remember it," "repetition," and "recognize it," as indicating the beliefs that the equation has been memorized. However, the descriptor "can visualize it" needs further exploration since it requires more elaboration.

Comparing the descriptors used by students in the US and Australia, the authors discovered

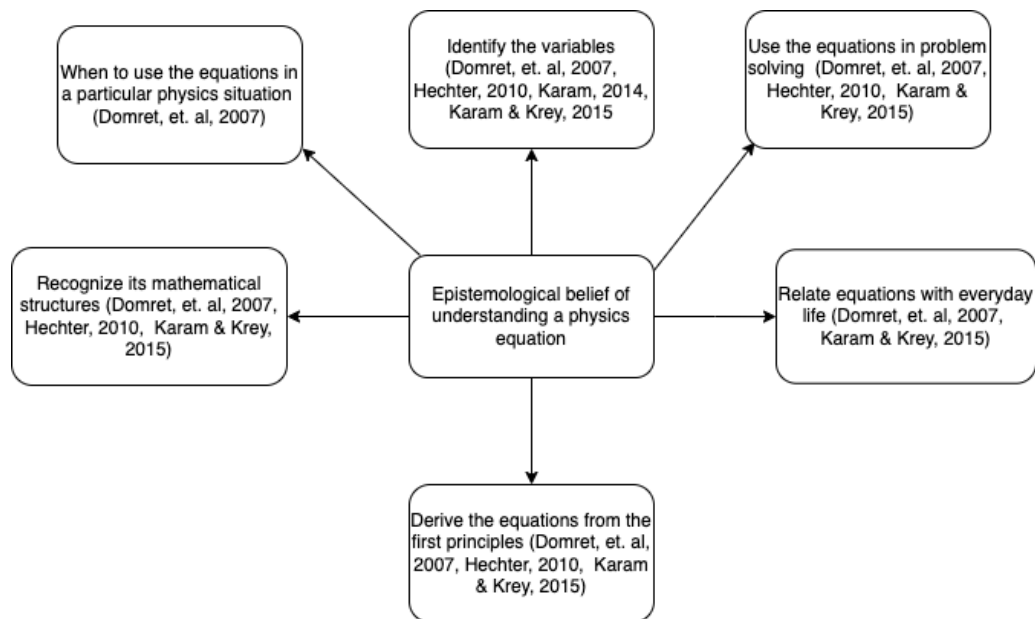


Figure 2: Students’ epistemological framing of understanding physics equations based on literature

that the beliefs were similar and the categorization overlapped. Ultimately, they identified ten categories of students’ beliefs about understanding a physics equation across three countries: significance, origin, description, prediction, parts, relationship, calculation, explanation, repetition, and memorization. These categories, which exclude repetition and memorization, were utilized to formulate questions that can be employed in teaching situations where students encounter physics equations. The aim of these questions is to assist students in directing their attention towards various aspects, thereby enhancing their comprehension of physics equations on a deeper level.

Other studies have also attempted to explore students’ beliefs about understanding physics equations and to examine whether they align with experts’ approaches. These studies have revealed various perspectives commonly observed among students [7–9, 12]. Using the findings of these studies, a diagram illustrating students’ epistemological framing in relation to the understanding of physics equations has been compiled and is presented in Figure 2.

Karam and Krey [9] proposed four epistemological facets to comprehend equations, which are principles, definitions, empirical regularities, and derivation. They also stressed the significance of taking these categories into account when defining equations. The principles describe the facts that can be supported by observations and experiments; definitions describe the justifications to define physical quantities; empirical regularities are scientific phenomena that can be explained with repetitive experiments; and derivations explain how equations can be derived from principles or definitions.

The study investigated changes in the epistemological understanding of equations among pre-service physics teachers between pre-test and post-test, with an intervention conducted in between. The intervention consisted of a series of activities that targeted both epistemological and didactical aspects. These activities involved exploring equations from various perspectives,

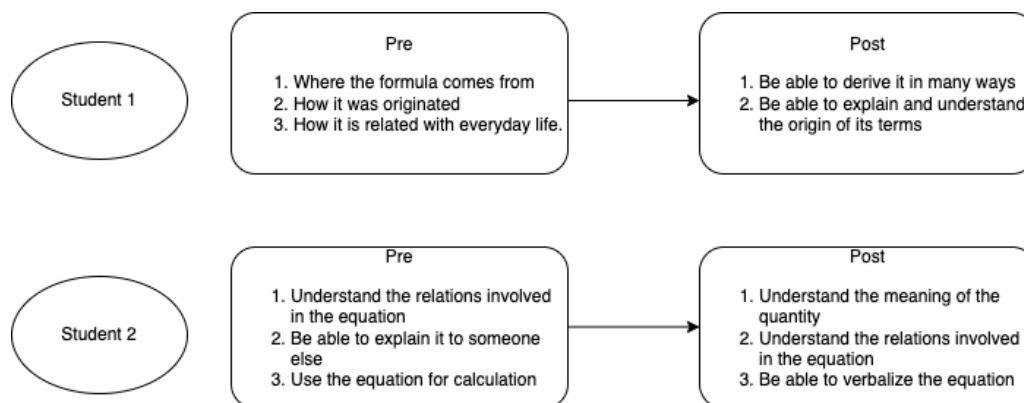


Figure 3: Students' epistemological changes after intervention [9]

including their derivation and historical development. The participants were enrolled in a specialized course aimed at fostering a deeper comprehension of equations through a curated set of tasks and prompts. Figure 3 depicts the epistemological changes observed in two students.

Although some students' responses were still persist, the results shows that the designed intervention led to more nuanced understanding of equations which also led them to change their views of physics equations.

2.2 Epistemological framing and learning difficulties

Some studies have shown that students' views on physics and learning can affect their understanding in certain ways. For example, epistemological framing has been linked to students' conceptual understanding [17,19–21]. Additionally, the way students approach learning physics, including their level of motivation, can also influence their academic achievement [17,22]. However, students' views about physics equations, which may differ from those of experts, can result in learning challenges.

One particular aspect explored in this thesis is students' interpretation of graphical representations of wave profiles. Within the field of PER, students' difficulties in utilizing graphs to solve physics problems have been investigated across various topics [23–27]. In the context of kinematics, McDermott et al. [27] identified certain challenges when students attempted to associate graph information with physical concepts. For instance, students often struggled to use the concept of slope to determine the velocity on a linear graph. Many of them decided to read the height of the line instead of seeing the line with a steeper slope. When presented with a curved graph, these difficulties were found to be even more pronounced.

Planinic, et. al [25] investigated students' interpretation of line graphs by presenting them with parallel questions in both mathematics and physics. The results discovered that students approached problem solving questions involving line graphs differently in physics compared to mathematics. Some students were able to correctly answer questions about slope in mathematics but struggled when the same concepts were applied in physics. These findings were further supported by their subsequent study, where students exhibited greater ease in answering questions related to graphs in the context of mathematics compared to when the same graphs were

presented in a physics context [28].

In the study of wave phenomena, graphical representations are commonly used as important tools to visualize the characteristics and behaviors of waves. However, several issues related to wave graphs have been identified in previous studies [29–31]. Ambrose et al. [29] found that students faced difficulties in interpreting graphical representations of electromagnetic waves. One major mistake observed among students was the belief that magnetic fields exist only within a sine curve. Additionally, some students mistakenly assumed that points located on the x-axis corresponded to zero magnetic fields. Other studies discovered that students held misconceptions regarding the periodic nature of waves. For instance, students often prefer to use the concept of slope to transform a wave graph $y(x)$ into $v(x)$ [30]. Furthermore, students believed that the graphs of $y(x)$ and $y(t)$ represented the waveform [31].

Other studies have explored students' beliefs about wave phenomena, revealing certain conceptual challenges in understanding how waves behave. One such belief is the notion that waves can be treated as objects [31–33]. For example, Wittmann et al. [32] observed that university students tended to perceive sound waves as objects that physically push particles in the direction of wave motion. Other studies have found that students often associate waves solely with periodicity and struggle to grasp fundamental concepts [30–32, 34].

The use of mathematical language in physics is another aspect that was investigated in this thesis. Prior studies have already explored this issue by identifying students' epistemological framing when they believe they understand a physics equation [6–9, 12]. Once again, when students' beliefs contradict those of experts, learning difficulties can arise. For example, a simple equation such as $v = \lambda f$ can not only be treated mathematically without considering its physical relation. This study identified an epistemological stance where some students approached this equation purely from a mathematical perspective, disregarding its physical quantity relationships. They failed to consider that λ can only be manipulated by changing the source of waves or the characteristics of the medium [35]. Another study employed the term "plug and chug," which refers to an instrumental view where students identify an appropriate physics equation for a given problem and simply plug in the given values to obtain a solution [11]. While this strategy may yield correct answers, it does not necessarily foster a deep understanding of the underlying physics concepts involved in the problem.

In the context of problem solving situations, Walsh et al. [36] identified four distinct epistemological stances that students adopt in physics. Only a small number of students employed a scientific approach, aligning their problem-solving strategies with those of experts. The majority of students relied on a plug-and-chug approach, often in an unstructured manner. Some students adopted a memory-based approach, while others lacked a clear approach altogether. These findings indicate that many students tend to rely on memorization and mathematical procedures rather than cultivating a deeper conceptual understanding when tackling physics problems.

2.3 Methodology

The nature of the research in two studies (Study 1 and Study 2) presented in this PhD thesis is similar, as both involve qualitative research methodologies. These studies followed a common framework consisting of three main steps, although the design of the intervention differed between them. Figure 4 illustrates the design of both studies.

Exploration was the first step of data collection, where students' initial reasoning was explored to get a picture of their understanding related to the designed questionnaire. Subsequently, the interventions took place in order to help students overcome their learning difficulties. The last step was the evaluation, where the same questions from the exploration were administered again to the students. This phase aimed to assess the conceptual changes among the students after they received the interventions.

The two papers differ in terms of their contributions to Physics Education Research. Paper 1 investigated students' difficulties in understanding the propagation of a pulse in a string, filling a gap in the research by exploring students' understanding of non-periodic wave profiles. While studies in PER have examined students' difficulties with waves, few have explored this specific area [10, 35, 37–39]. Meanwhile, Paper 2 aimed to explore students' understanding of the 1-D WE, examining their epistemological framing and the difficulties they encountered with the equation. While previous studies have explored students' interpretation of physics equations and their epistemological framing [6–9, 12], Paper 2 aimed to encourage students to develop a deeper conceptual understanding of the equation by exploring various related aspects.

2.3.1 Interviews

All the data in paper 1 and 2 were collected and documented by performing think-aloud interviews with physics university students. The think-aloud interview protocol originated in psychology research, particularly in investigations of problem-solving processes [40, 41]. Nowadays, this approach is widely used across various fields, including PER [42–44]. The nature of open-ended interviews allows researchers to obtain detailed information from students and to

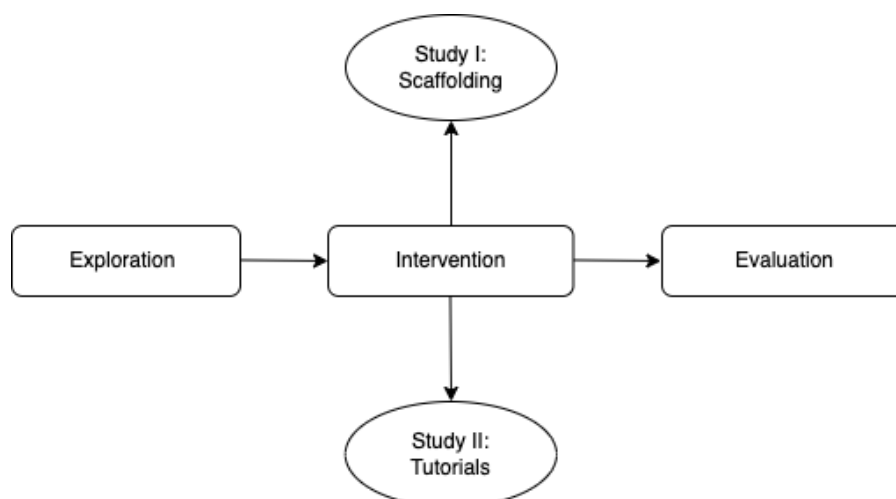


Figure 4: Three main steps of collecting data in Study 1 and 2

ask probing questions and dig deeper into the students' responses [45, 46].

The students were interviewed in pairs due to several reasons. One of the benefits of using paired interviews is the ability to gather a wider range of data, as students engage in discussions and exchange ideas with their peers. According to Houssart and Evans [47], paired interviews, also known as paired depth interviews, involve a researcher interviewing two individuals simultaneously. Previous studies that employed this method have identified advantages, particularly when the pairs have a pre-established relationship [48, 49]. This relationship facilitates openness between the participants, enabling them to complement each other's responses when one person encounters difficulties [47–50].

Furthermore, it is worth emphasizing that the data obtained from the paired interviews are richer when the students encounter the questions for the first time during the interview. This approach avoids fixed or rehearsed answers from the students, as they have not prepared their responses in advance. Houssart and Evans [47] refer to this as "unseen questions," which promotes collaboration among students and encourages them to consider alternative perspectives presented by their peers.

Due to the pandemic, the interviews for Study 1 and Study 2 were conducted remotely using Zoom meetings. Prior to the interviews, all students provided their consent for participation. The interviews were recorded using the embedded recording feature in Zoom, ensuring that the data were captured and stored for subsequent analysis. In instances where students produced drawings or visual representations during interviews, the interviewer requested that the students take a photo of their drawing and share it with the interviewer. The pictures were then shared within the Zoom meeting, allowing peers to provide comments and insights on the drawings.

Taking notes during the data collection process is essential. The interviewer's notes are a valuable addition to the captured data, providing insights and documenting key points raised by the students. By documenting interesting findings and observations during interviews, the interviewer can focus on specific areas for further analysis and exploration. The combination of recorded data and interviewer's notes improves overall data quality and provides a comprehensive understanding of student perspectives and experiences [51].

2.3.2 Data Analysis

The qualitative content analysis was applied to analyze the coded data. Content analysis is a method that enables researchers to analyze data from various sources, such as interviews [52]. While there are three approaches to qualitative content analysis, the conventional content analysis approach was employed in this study due to the nature of the data. According to Hsieh and Shannon [53], conventional content analysis with an open coding technique is more suitable for a study that uses observation as the initial phase, where coding and categorization are conducted during data analysis.

There are several steps involved in employing conventional content analysis [53]. For both Study 1 and Study 2, where data was obtained through interviews, the first step involved

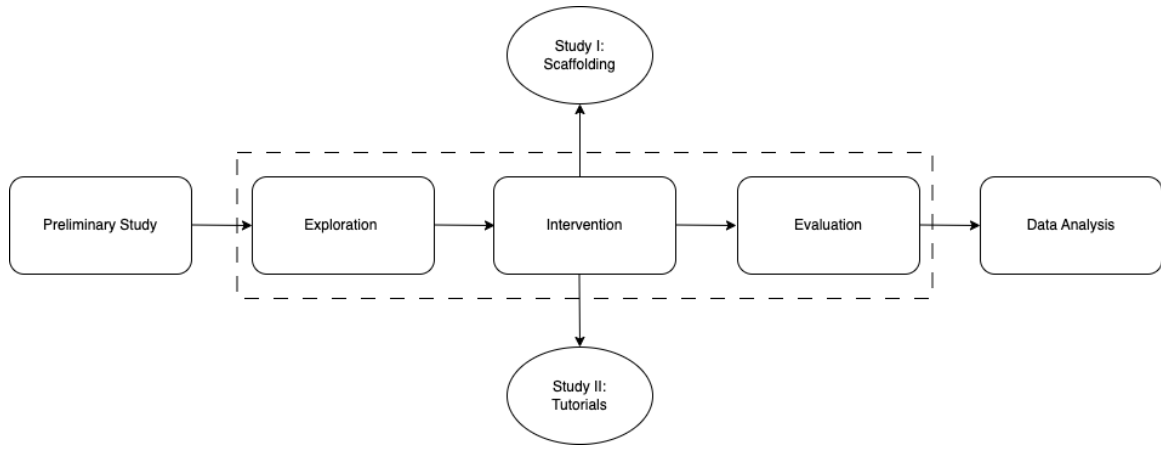


Figure 5: Research design for Study 1 and 2

transcribing the interviews. These transcriptions were then read multiple times to gain an overview of the data. During this reading process, important words and phrases were identified and highlighted to capture the key aspects of the students’ reasoning. Additional notes were made as needed. From this analysis, initial codes were generated to capture the overall ideas expressed in the students’ reasoning. To ensure the reliability and validity of the coding process, the data analysis continues with coding, where the students’ responses to the questions are classified into themes or categories. Experts in the field are recommended to examine and validate the coding process, as suggested in studies by DiCicco-Bloom and Crabtree [54] and Fonteyn et al. [51].

After generating a list of initial codes, the next step is to sort those codes into potential categories/themes. Various techniques can be employed, such as using the tree diagram to aid in clustering [53]. The themes are then reviewed to ensure that the initial codes are placed appropriately in the correct categorization. This is an iterative process of reviewing and refining until a researcher arrives at a satisfactory map.

The themes must then be defined by describing the meaning of each categorization and the trends that these themes captured from the data. This process ensures a well-structured data set that helps researchers in reporting their results. The final step is to write the research report, which was done for both Study 1 and Study 2. Although multiple categorizations may have been identified in these studies, the reported results focused on the most relevant findings that aligned with the existing literature and addressed the research questions.

2.4 Research design

Study 1 and 2 were conducted in a similar research design. The difference was in the type of intervention implemented between the two studies. Figure 5 shows the research design in Study 1 and Study 2.

2.4.1 Preliminary Study

Study 1 A preliminary study was performed to identify possible problems and appropriate methods that will be used during data collection, including refining the research questions. Initially, the instrument was designed to assess students' understanding of the 1-D WE. Several questions were developed and administered to four physics students for a pilot study. During the interviews, it was discovered that students struggled to differentiate between the graphs shown in Figure 6.

In this phase, the results show that the graphical representation of the waves is actually challenging for students to understand because the mathematical description of the waves depends on two variables: x and t . Consequently, it was necessary to investigate this issue prior to assessing students' understanding of the WE, which might present an even greater level of difficulty.

In study 1, the attention was directed towards examining whether students can make sense of the horizontal movement of a pulse and the vertical motion of matter on a non-periodic wave profile. To achieve this, a wave profile $y(x)$ was specifically designed to investigate this matter. During this stage, an intervention was also developed to address the specific learning difficulties identified. The intervention was tested with a group of physics students to ensure its effectiveness before implementing it in the data collection phase. Based on the overall process, the following findings were observed:

1. The instrument designed in Study 1 proved to be extremely challenging for the students, even though they had previously completed a course on waves. As a result, for the purpose of data collection, it is essential that participants have received an advanced course in waves.
2. To effectively address the diverse range of difficulties experienced by students, the intervention should be designed with multiple levels of scaffolding. This approach enables the researcher to observe students' progress and challenges at each level. By doing this, the intervention can meet individual needs.

Study 2 The steps involved in the preliminary study for Study 2 followed a similar framework to Study 1. However, the exploration of materials and the development of instruments in

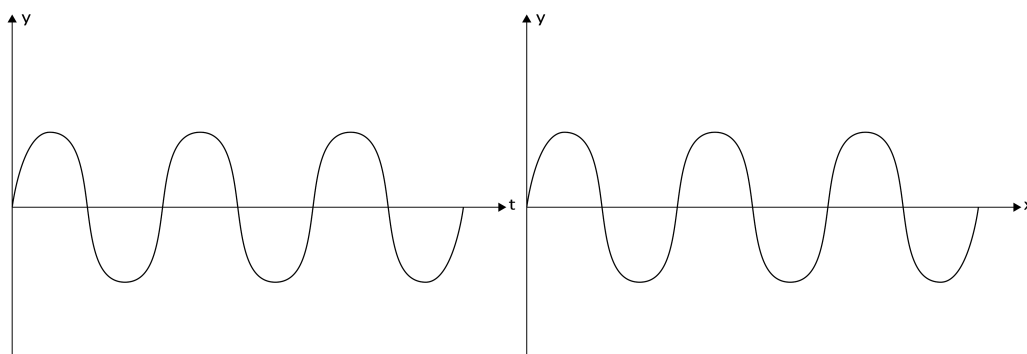


Figure 6: The graphical representation of wave

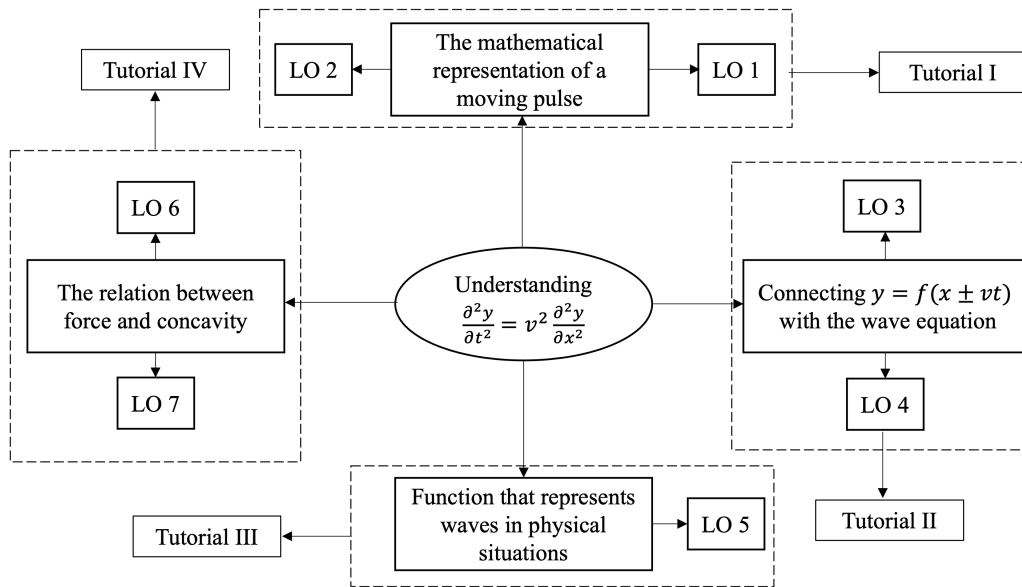


Figure 7: The proposal in study 2 regarding how students understand the 1-D WE

Study 2 posed challenges due to the limited number of studies that had previously explored students' understanding of the WE. In the initial phase, a questionnaire consisting of six questions was developed and administered to a group of physics students. Unlike typical problem-solving physics questions, the questions aimed to explore students' epistemological framing and alignment with expert views. The results obtained from the preliminary study in Study 2 are as follows:

1. The questionnaire were reduced into five, as one question aimed to assess students' understanding of the mathematical structure of the WE was complicated. This difficulty arose because not many students had previously encountered the transport equation.
2. None of the students recognized the mathematical structure of concavity.
3. Students continuously associated their answers with periodic waves, indicating a lack of understanding of non-periodic wave behavior.
4. In general, the students did not have a deep understanding of the WE and its underlying concepts.

Based on these results, interventions were designed to help students explore different aspects of the WE and eventually make them aware that understanding a physics equation is not a simple task. The interventions aimed to address students' difficulties identified in the pilot study, which included understanding the concavity of waves and the tendency to fixate on periodic waves. Therefore, six aspects were proposed to facilitate understanding of the WE. These aspects included the mathematical representation of a moving pulse, the connection between $y = f(x \pm vt)$ and the WE, the distinction between the transport and wave equation, identifying when functions represent a physical wave, the relationship between force and concavity, and differentiating between curvature and concavity. However, two aspects were later removed from the intervention, as they represented conceptual subtleties that would be presented in Study 3. Figure 7 illustrates the framework proposed to understand the WE in this study.

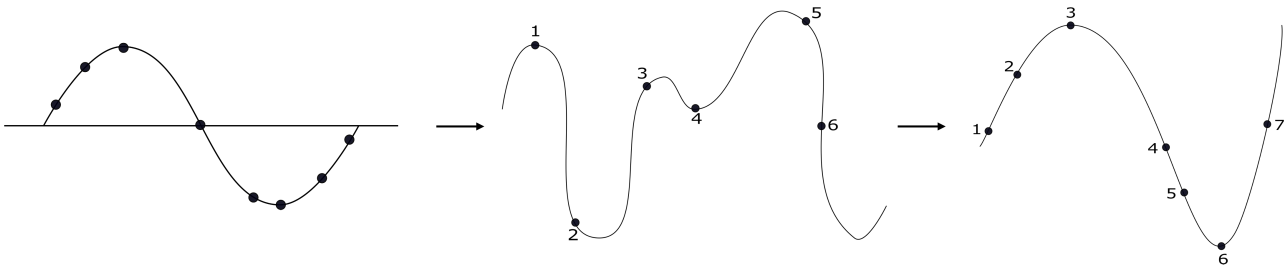


Figure 8: The changes of the wave profile in tutorial IV

Based on the four identified aspects of understanding the WE and the results of the preliminary study, interventions were designed to address different/uncommon aspects of the WE, with the purpose of helping students develop a deep understanding of the WE. Learning objectives (LO) were determined on the basis of these aspects, and tutorials were created accordingly to achieve these objectives. These tutorials were then tested and evaluated with some changes. This modification was designed to ensure that students understood that points located on a more concave profile experience greater force. Figure 8 illustrates the change in the wave profile following the second trial.

2.4.2 Data Collection

Exploration (Pre-test) During this stage, students' prior reasoning was examined by administering the designed questions from studies 1 and 2. In study 1, the question consisting of two items was administered to the students. One part of the question focused on determining the magnitude of the velocity, while the other part involved the velocity and direction of motion of specific points.

Question: A pulse is moving horizontally with constant speed to the right. The profile below represents a given instant, like a picture (Figure 9). As the pulse moves horizontally, the points move vertically (wave does not transfer matter)

- (a) Based on the picture, sort the magnitudes of the (vertical) velocity at each point from the greatest to the smallest. Explain your reasons.

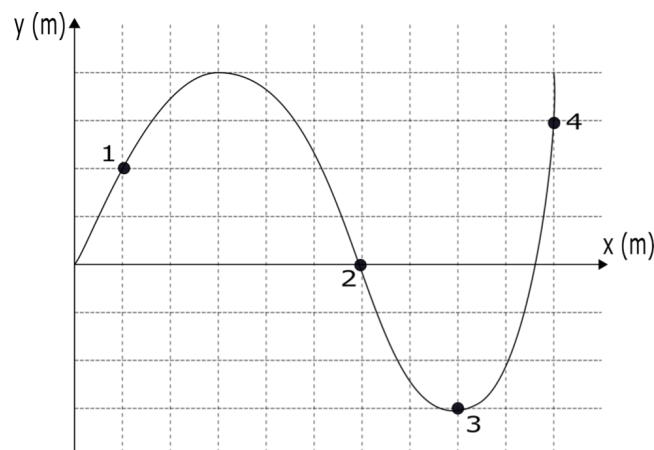


Figure 9: Problem graph

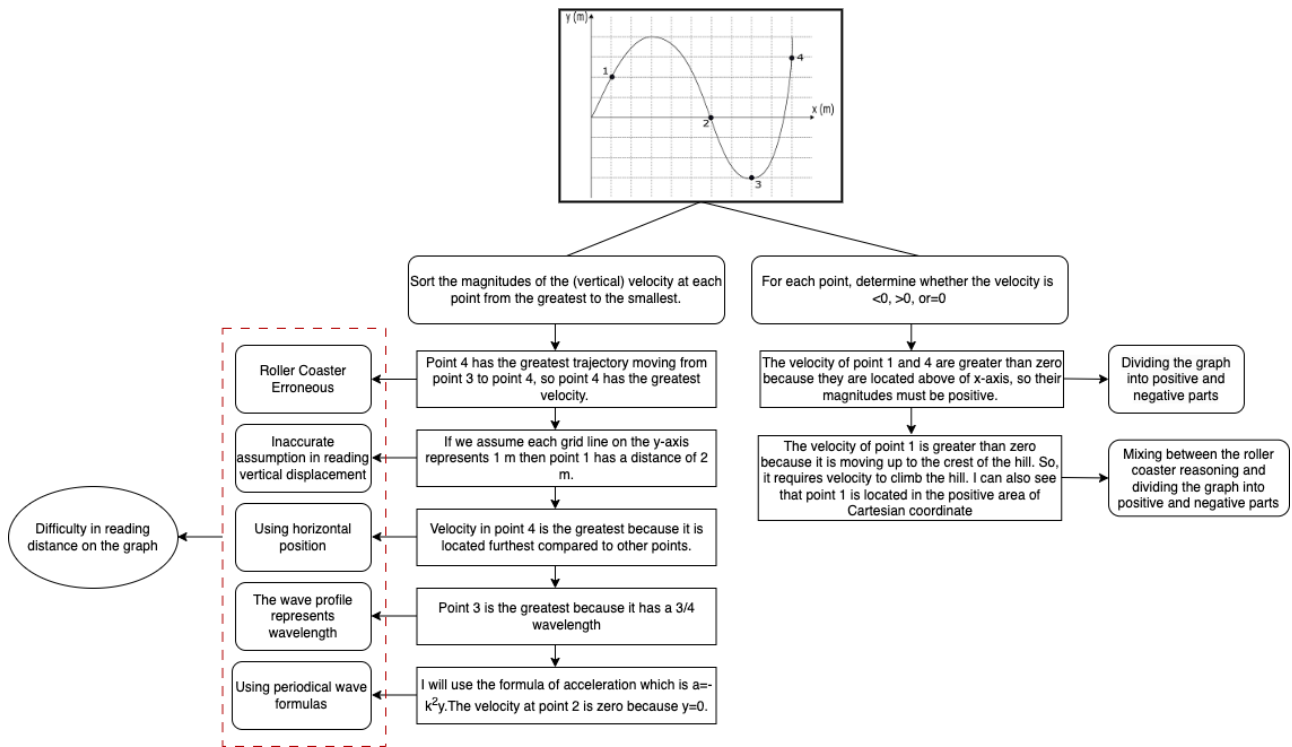


Figure 10: The content analysis with open code technique in study 1

(b) For each point, determine whether the velocity is < 0 ; > 0 ; or $= 0$. Explain your reasons.

Figure 10 illustrates how content analysis with an open-coding technique was applied to analyze students' reasoning in Study 1, providing a visual representation of the analytical process. It visually demonstrates that students often relied on inappropriate information from the graph when attempting to determine distances, highlighting a gap in their understanding. In this thesis, study 1 expands on previous research that has examined graphical representations and identified various challenges encountered by students [23–26]. Furthermore, this study revealed that students tend to fixate on periodicity and assume that waves always exhibit a periodic nature. This finding is consistent with previous studies that have reported similar observations [30, 31, 55].

In study 2, a questionnaire consisting of five question items was administered to the students to assess their understanding of the WE:

Questions: Below you see the 1-D WE consisting of a partial differential equation which describes a physical process.

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

1. If you were to explain the meaning of this equation with your own words (to a non-expert), how would you do that?
2. There are several terms in 1-D WE (x ; y ; t ; v ; $\partial^2 y = \partial t^2$; $\partial^2 y = \partial x^2$). Indicate what those symbols refer to.
3. Based on question 2, indicate the units of those symbols.
4. Describe a physical situation represented by $\partial^2 y = \partial x^2$ / $\partial^2 y = \partial t^2$.

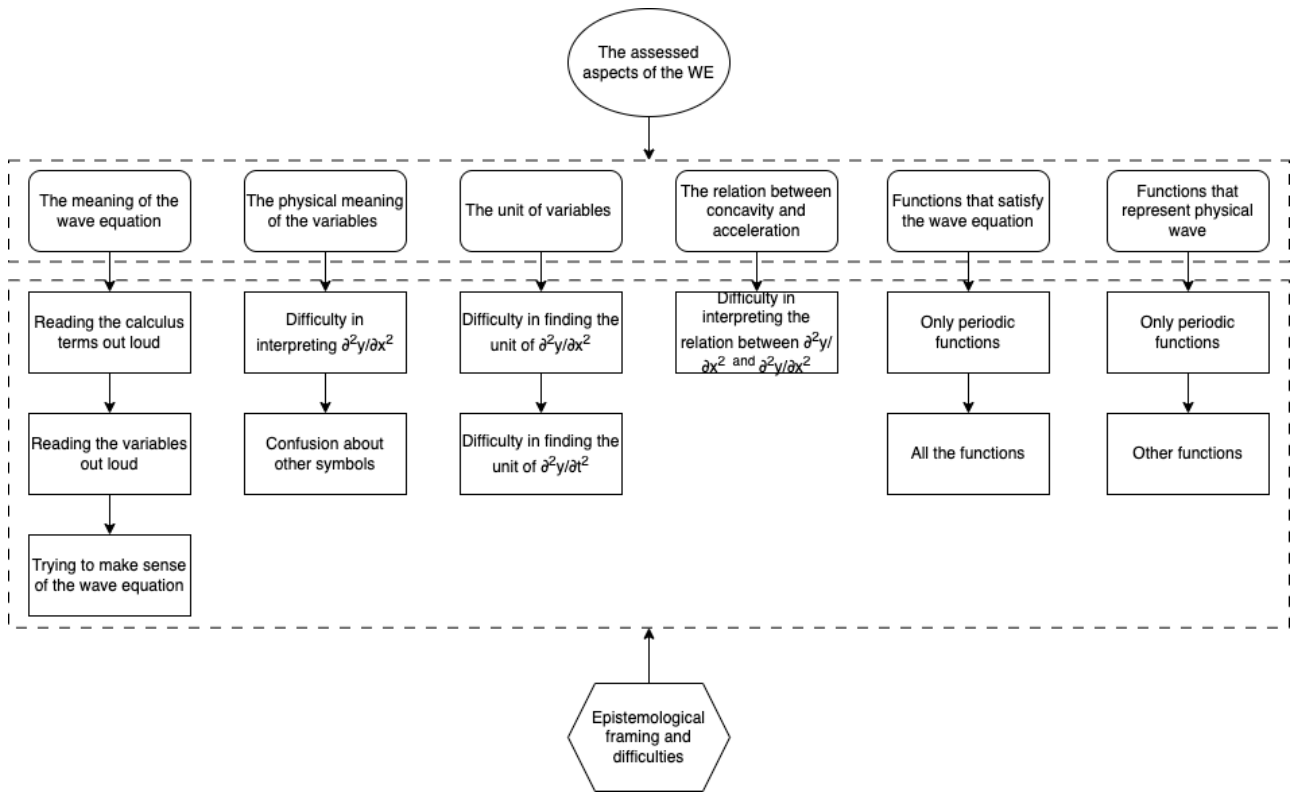


Figure 11: Students' epistemological framing and difficulties regarding the WE

5. Given the functions below, which ones satisfy the WE and which can represent a physical wave? (You can choose more than one). Explain your reason.

- (a) $y(x; t) = f(x + vt)$
- (b) $y(x; t) = A \sin(kx - ! t)$
- (c) $y(x; t) = e^{-(x-vt)^2}$
- (d) $y(x; t) = 2A \sin(kx) \cos(! t)$
- (e) $y(x; t) = (x + vt)^2$

The results indicated that students' interpretations of the WE did not align with expert views, including some conceptual difficulties associated with its fundamental principles. Noticeably, based on one result in study 1 where students quickly used the periodic wave functions as their main reasoning to solve the problem, study 2 further explored these aspects to examine if students also used this approach to make sense of the WE. Figure 11 illustrates students' responses to the questionnaire, providing insights into their epistemological framing and the difficulties they encountered during the study.

The results reveal that students did not have a comprehensive understanding of the WE. Their epistemological framing suggests that they often perceived the WE as mere calculation schemes. For example, some students treated the WE as a combination of single variables, indicating the tendency to plot their magnitudes and obtain results, as the so-called plug-and-chug approach [11,15]. Other students responded to the question without grasping the physical meaning involved, simply reading the calculus terms without understanding the relationships between the variables [10, 37]. Several difficulties identified in Study 2 were also observed in

Study 1, including the tendency to fixate on periodic waves.

Intervention Intervention in the learning process refers to the support and guidance provided by educators or experts to enhance students' understanding and competencies. Effective interventions are guided by specific teaching goals aimed at improving students' ideas and understanding throughout the intervention [56]. This section will provide an overview of the interventions employed in Study 1 and Study 2, as well as the rationale behind their selection for each study.

Scaffolding Scaffolding is a term used to describe a process in which students are supported by teachers to solve problems that are beyond their current competencies [57]. According to Van de Pol et al. [58], scaffolding is a dynamic process that is highly dependent on the nature of the task and can be applied in various situations. Therefore, it is crucial to assess students' prior understanding before designing appropriate scaffolding strategies [59]. However, many teachers face challenges in designing scaffolding because they often directly provide support without first assessing students' understanding [58, 60, 61].

In Study 1, the difficulties faced by students regarding the propagation of a pulse in a string were explored, as discussed in Sec. 2.4.2. Scaffolding was selected as the intervention because it offers temporary solutions for students who are unable to solve problems using expert approaches [57]. Additionally, clear goals were set as important aspects when designing the scaffolding support [62]. The scaffolding was designed to guide the students in developing their own understanding at each level of the learning process. The difficulties encountered at each level were identified and interventions were provided at each level to gradually guide the students to draw the wave profile after some time has elapsed. This approach also allowed the researcher to monitor the students' progress in improving their understanding at each level.

Although there is no consensus, Van de Pol et al. [58] summarized three common characteristics of scaffolding. The first characteristic is contingency, where the supports must be adjusted at the same or slightly higher level than the students' existing understanding. The second characteristic is fading, which involves gradually reducing the amount of support provided. This characteristic is related to the third characteristic named responsibility, where students are expected to take more and more control over their understanding. It is also important to check students' understanding after giving them support in scaffolding [63]. Scaffolding supports are meant not only to help students reduce their learning difficulties, but also to diagnose how students' new understanding has developed in the end. One way to do this is by asking the same questions that were given to see if they changed their prior reasoning.

Tutorials Tutorials were designed based on three stages: elicit, confront, and resolve, as proposed by the physics education research group at the University of Washington [64, 65]. The elicit stage involves exposing students to problems that explore their learning difficulties. These difficulties are then addressed by making students aware that their reasoning is not aligned

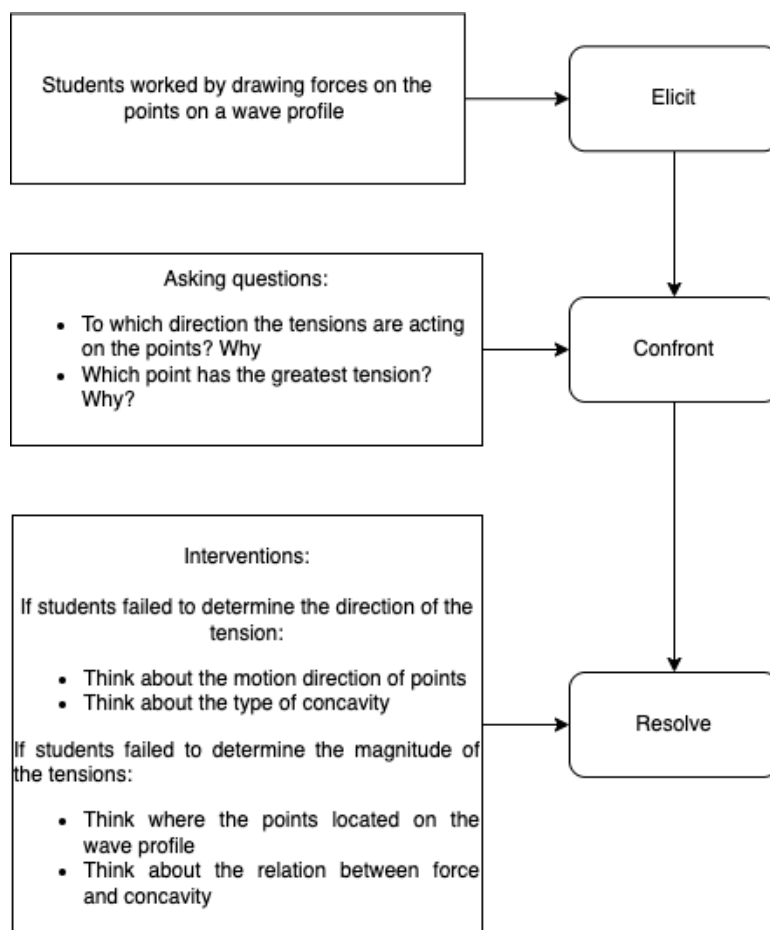


Figure 12: The activities of tutorial 4 was designed based on three stages: elicit, confront, and resolve

with a correct physical situation. Finally, students are guided through necessary interventions to resolve any inconsistencies.

In general, the tutorials in study two were designed based on the stages in which the first activity in each tutorial involved students working and exploring their understanding independently. Subsequently, questions were posed to trigger conceptual conflicts, prompting students to reflect on their initial answers. Finally, additional interventions were provided to students who continued to struggle in order to help them reach the learning goals of each tutorial. Figure 12 provides an example of the activities in Tutorial 4, demonstrating how these three stages were incorporated.

In addition, the interventions implemented in the tutorials were inspired by the conceptual blending framework. This approach was chosen due to the complex nature of the WE, which is presented in second-order PDE and presents challenges in terms of delivering its meaning. The framework employed aimed to assist students in blending the mathematical representation within the equation with meaningful physical situations. Graphical representations were also used extensively throughout the tutorials, as they have been found to facilitate the integration of physics and mathematics, as demonstrated in this study [66].

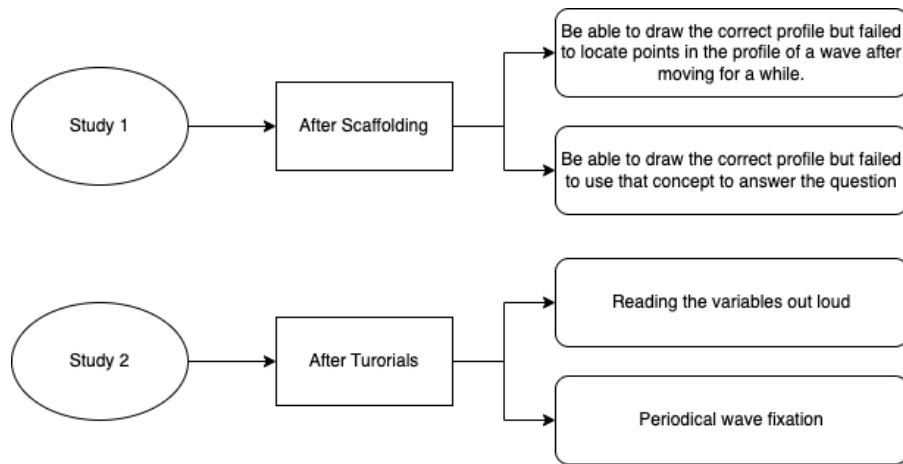


Figure 13: Students' difficulties and epistemological framing after interventions

Evaluation (Post-test) In both Study 1 and Study 2, the same questionnaires used in the pre-test were administered again in the post-test to evaluate students' improvement of the topics and the effectiveness of the interventions. In Study 1, the post-test was given directly after students managed to draw the correct wave profile. In Study 2, the same questions were administered again to the students one month after completing the last tutorials to determine whether the intervention had a lasting impact on their understanding or if they had quickly forgotten the physical intuition behind the interventions.

The results from both studies suggest that the designed interventions contributed to improving students' understanding. While not all students were able to provide comprehensive responses and a few did not demonstrate improvement in their understanding during the evaluation, the overall findings highlight the positive impact of the interventions. Despite these positive results, certain difficulties persisted even after the interventions. Figure 13 illustrates that some students continued to adhere to their prior reasoning or shifted to other incorrect reasoning patterns following the interventions.

Despite the use of different levels of scaffolding, Lin and Singh [62] found that a persistent alternative conception among students was the assumption that static friction is always equal to its maximum value, sF_N . Similarly, in Study 1 and Study 2, although the interventions provided to the students helped them address their learning difficulties, they still struggled to utilize this knowledge effectively in the evaluation. For instance, in Study 1, not all students were able to apply the support provided and change their prior reasoning when faced with the same question again. This observation applies even to the group of students who successfully generated the correct wave profile, which was the main goal of the scaffolding design. These findings demonstrate the inherent challenge of designing scaffolding that is suitable for all students [62, 67]. In Study 2, it was observed that while a few students achieved the learning goals within the tutorials, they still struggled to answer the questions on the questionnaire correctly and some of their responses lacked physical meaning.

One potential factor contributing to students' persistence in incorrect reasoning despite the intervention could be their prior knowledge and skills. Lin and Singh [62] conducted a study

that revealed the effectiveness of a particular intervention in a calculus-based physics course compared to an algebra-based course. However, in the present studies, all participants shared a similar educational background. They had completed introductory physics courses in their first year and advanced courses on waves in subsequent years. Therefore, it can be inferred that the students had similar levels of skill and knowledge in the context of this study.

One recommendation of this study suggested that designing effective interventions is an ongoing process. Lin and Singh [62] proposed three strategies to assist students with strong alternative conceptions. First, interventions should focus on directing students' attention to the learning difficulties in a more detailed manner. One approach could involve activities that explicitly invoke alternative conceptions, creating cognitive conflicts that challenge their existing beliefs. Secondly, it is crucial to provide students with a variety of examples and situations within the intervention that highlight the conflicts associated with their alternative conceptions. This exposure encourages students to actively construct and restructure their understanding of the topic. Lastly, it may be advantageous to prompt students to generate new cases or situations related to the physics concepts covered in the intervention. This approach encourages students to apply their revised understanding to novel contexts, further reinforcing their learning and facilitating a deeper grasp of the subject matter.

3 Exploring the complexity of the wave equation

The 1-D WE may appear simple compared to more complex physics equations such as the Schrödinger equation. However, the exploration of the WE in Study 1 and Study 2 reveals that understanding it is not a trivial matter. This chapter describes some issue related to the complexity of the WE and suggests that exploring different aspects of a physics equation can contribute to a thorough understanding.

3.1 Force and Curvature

In 1715, Brook Taylor published a book called "Methodus Incrementorum Directa et Inversa" and made several assumptions related to the vibrating string. One of these assumptions stated that at each point of the string, the vertical force is proportional to the curvature. This assumption was the only one identified by D'Alembert after deriving the wave equation with the assumption of small vibrations of a string [68,69]. Taylor's work provided an early geometric derivation that gives an idea of the relationship between the tension acting at a point of the string and the curvature at that point.

In derivations of the WE, it is common to analyze only a small segment of the string. Analyzing the small segment of the string enables the use of linear approximation, allowing for the application of Newton's law. Figure 14 illustrates the small segment of the string that needs to be considered to derive the WE.

The WE can also be derived using the assumption of curvature concavity. This assumption arises from approximating that $dy=dx$ is very small. However, it is important to note that the proportionality between force and curvature is the correct relation. The concavity appears to simplify the derivation process. Assuming that force is proportional to concavity gives a reasonable approximation and facilitates the mathematical analysis of wave behavior.

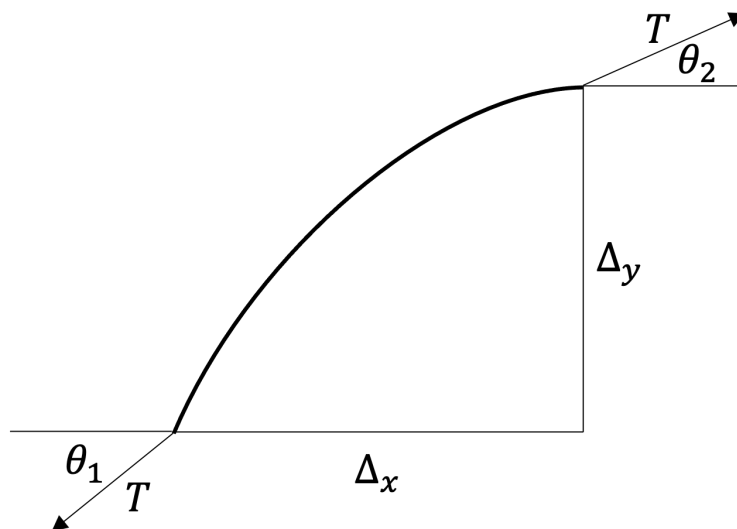


Figure 14: A small segment of the string to derive the WE

3.2 Convolutions of the wave function: The vibrating string controversy

Since D'Alembert proposed the partial differential equation that became the equation of a vibrating string, a controversy arose among four prominent figures during that time. In their paper, Gerald and William [70] depicted the debate over the solution of the WE. They clearly demonstrated that the meaning of $y(x; t)$ was the central issue of the debate.

D'Alembert considered a stretched string fixed at both ends and used Newton's second law until he finally arrived at the WE. He then proposed a general solution of the PDE that he found as:

$$y(x; t) = f(x + t) + g(x - t) \quad (3)$$

and applying the boundary condition $y(0; t) = y(L; t)$, he found this final solution:

$$y(x; t) = f(x + t) + f(x - t) \quad (4)$$

Euler was one of the first to enter the dispute, and his analysis of the vibrating string is not significantly different from that of D'Alembert, except for the interpretation of the function f . Euler proposed that the interpretation of y can be deduced from a plucked string, as shown in Figure 15. Using D'Alembert's solution to the WE, Euler argued that the initial shape of the string would determine its subsequent motion and shape [71]. Wilson [72] referred to this as a "thought experiment" when Euler attempted to impose his physical interpretation on the solution of the WE.

Figure 16 illustrates the physical interpretation of Euler for a plucked string [73]. In his model, after the initial condition, the string splits into two halves and moves in opposite directions, forming the kinks at B and C (Figure 16b). According to his interpretation, OB and C remain stationary after the initial disturbance, while the BC segment, which represents the moving part of the string, moves at a constant speed (Figure 16c). Euler believed that this

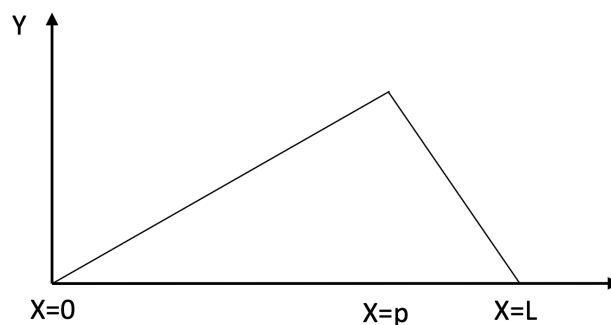


Figure 15: Euler's plucked string.

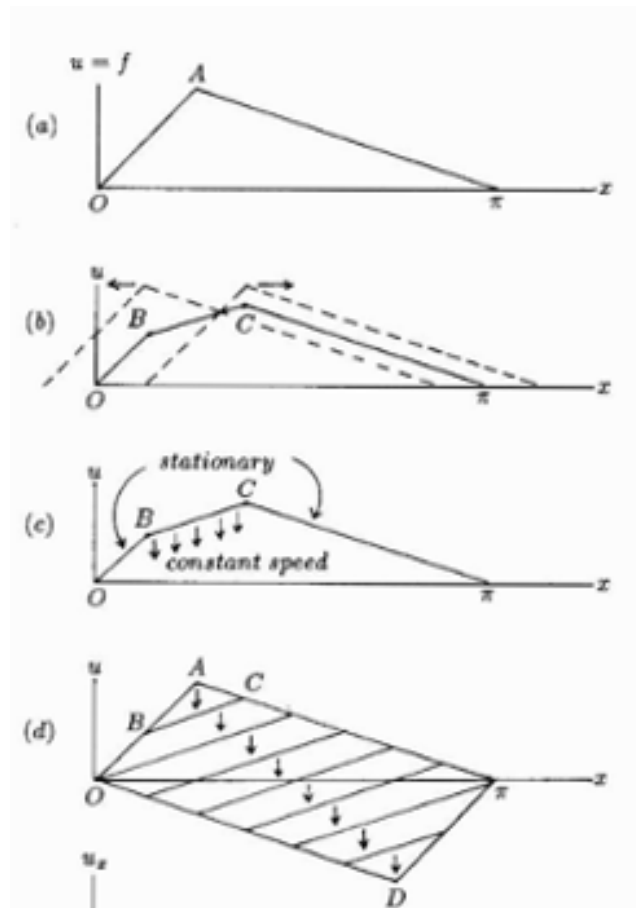


Figure 16: Euler's physical interpretation of a plucked string [73].

segment vibrates like a wave.

A complete downward cycle occurs when B and C steadily move down along the upper edges of the parallelogram OA D (Figure 16d) until it reaches D. After reaching D, the direction changes, and it starts moving back towards point A. This interpretation includes both upward and downward movements associated with a vibrating string.

Later, D'Alembert objected to this argument, saying that if the shape of the string is not smooth, then $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$ will not be equal to $(1/c^2)(\frac{\partial^2 y}{\partial t^2})$ [71]. The force on the plucked string must be concentrated at the corners, and the WE can be applied in this situation [72].

Bernoulli proposed that $y(x; t)$ is the sum of harmonic functions to give his solution a more physical interpretation. However, this argument was disputed by D'Alembert, who claimed that the vibrating string consists only of one frequency. Euler also disagreed with Bernoulli's solution, stating that it was too specific and could not be generalized to other types of waves. Lagrange then entered the debate and proposed a completely new approach to solving the problem of the vibrating string without relying on the WE. He built his interpretation by imagining a string consisting of infinite points connected to each other. His solution of the wave function gives a series of normal modes that consist of sine and cosine functions.

3.3 The wave function

Waves are mathematically expressed by functions of $y(x; t) = f(x - vt)$. Therefore, any function of this form is a wave function. At first glance, this function seems disconnected from the WE. Connecting $y(x; t) = f(x - vt)$ to the WE is quite simple by taking the second derivative of that function with respect to x and t . Letting $x - vt = u$ and performing a double derivation with respect to x results in:

$$\frac{\partial y}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du} \quad (5)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2} \quad (6)$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du} \frac{\partial u}{\partial t} = -v \frac{df}{du} \quad (7)$$

$$\frac{\partial^2 y}{\partial t^2} = (-v)^2 \frac{d^2 f}{du^2} \quad (8)$$

Comparing (2) and (4),

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad (9)$$

which is the 1-D WE.

3.4 Transport Equation

The transport equation, which is a first-order partial differential equation, describes the movement of a wave through a medium with constant velocity. It is depicted as follows:

$$\frac{\partial y}{\partial t} = v \frac{\partial y}{\partial x} \quad (10)$$

In fact, the transport equation can actually explain some limited wave phenomena. Despite its limitations, emphasizing this equation in teaching waves might be essential to provide deep understanding to learners. However, many physics textbooks do not present the transport equation, which could be reasonable due to the general consensus on the mathematical representation of waves.

Presenting the transport equation may answer the question "Why is the WE presented in the second-order partial differential equation (PDE) and not the first order?" This is one unnoticed aspect that could be beneficial to implement when teaching the WE in order to encourage a deeper understanding of it.

3.5 Conceptual subtleties

Numerous research in physics education explored students' difficulties in various physics concepts, including waves [31, 32, 35, 37, 55, 57, 74]. The third paper in this thesis aims to propose an epistemological dimension of teaching by incorporating less commonly encountered concepts related to the WE. These concepts might offer valuable insights and facilitate a deeper understanding of the WE for learners.

Based on the previous sections in this chapter, study 3 highlights three key conceptual subtleties that could be beneficial for teaching the WE. The first concept focuses on the mathematical representation of waves. As also discussed in studies 1 and 2, the mathematical structure of the wave function may appear simple, but understanding its deep meaning poses challenges.

The second aspect focuses on differentiating between the transport equation and the wave equation. This exploration aims to elaborate why the WE is expressed as a second-order partial derivative, while the transport equation falls short in capturing the complexities of wave phenomena. Lastly, the third aspect explores the relationship between $\partial^2 y = \partial x^2$ and $\partial^2 y = \partial t^2$. Understanding this relationship is crucial to understanding the WE and the associated physical properties. Conceptually, one can realize that the force acting on points of the wave is proportional to its curvature and get the WE from that.

In addition, the third paper also touches on the historical episode of the vibrating string. It highlights Taylor's original proposal of the relationship between the curve on a vibrating string and the net force acting on it through geometric analysis. This relation was later used by D'Alembert, who simplified the mathematical formalism by assuming small slope values, leading to the derivation of the WE.

4 Conclusion

This PhD thesis aimed to investigate the interplay between physics and mathematics, using the 1-D WE as a case study. The thesis, along with three accompanying papers, addressed the research questions at hand. The first guiding question focused on examining how students establish connections between graphical representations and the physical interpretation of wave profiles. Additionally, the thesis explored students' epistemological framing and the challenges they encountered in relation to the 1-D WE. Finally, the thesis presented conceptual subtleties with the aim of highlighting uncommon topics associated with the WE.

If we analyze the relationship between the three studies, Figure 17 illustrates their connection in addressing the research questions posed in this thesis.

4.1 Study 1

The use of graphs as a mathematical representation in wave phenomena has been found challenging for students, especially when it comes to applying them to particular physical situa-

tions. Study 1 demonstrated that problems involving graphical representations of waves, which include variables x and t , are extremely difficult for students.

Before the intervention, no student was able to correctly answer the questions. Many students had difficulty extracting relevant information from the graph to solve the problem, and some fell into the trap of focusing merely on periodicity. Students' reasoning did not align with the expert approach, despite being aware that points on the wave only move vertically.

After different levels of scaffolding were implemented, students' performance improved and they were able to solve the problem correctly. However, some students still persisted in using their prior reasoning, even though they were able to draw the correct wave profile, which was the goal of the scaffolding design. This suggests that the scaffolding did not address all the difficulties faced by students.

Nevertheless, Study 1 highlights that students encountered challenges when dealing with the uncommon graphical representation of the wave function. The results show that students' epistemological framing can hinder their ability to see beyond those beliefs.

4.2 Study 2

Study 2 offers a unique perspective on teaching physics equations, specifically focusing on the 1-D WE. While previous studies have primarily investigated students' epistemological framing and difficulties in understanding the WE, this study goes beyond by actively engaging students in exploring different perspectives of the equation.

The pre-test results indicate that students primarily perceive the WE as a mathematical tool, they often read the calculus terms and variables aloud. Consequently, their responses lack a deeper physical meaning.

After the interventions, although it cannot be concluded that all students achieved a deep understanding of the WE in the post-test, as many of them reproduced concepts presented in the tutorials, the intention was to introduce students to alternative ways of thinking about the equation. By equipping them with these new perspectives, the aim was to provide them with valuable tools that could be applied to other physics equations that they encounter in the

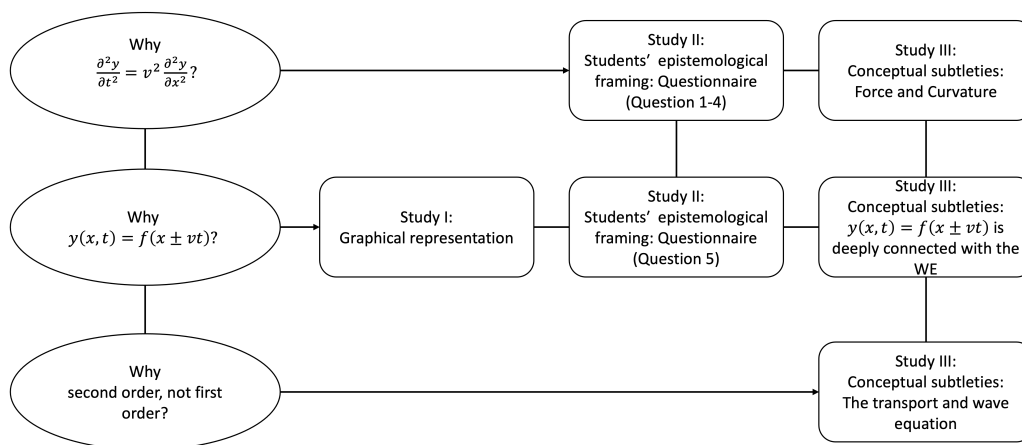


Figure 17: The connection of three studies of the thesis

future. The findings from this study suggest that such an approach is not only feasible but also holds potential for extension to teach other equations and topics in physics education.

4.3 Study 3

In Study 2, it has been shown that understanding the WE poses a significant challenge. Study 3 identified certain concepts associated with the WE that are not commonly addressed in physics textbooks or classrooms to propose ways to understand the WE more deeply.

One of the key concepts explored in this study is the difference between transport and WE. Understanding this distinction enables students to grasp the fundamental idea of a moving pulse and why the WE is represented by a second-order partial derivative. Additionally, students should recognize that WE can be derived from the relationship between force and concavity. However, the derivation of the WE using this relationship fails when one explores deeper the distinction between concavity and curvature. Hence, it is essential to understand that the original relationship proposed by Taylor in 1715 was not between force and concavity but rather between force and curvature.

5 Perspectives for future research

This PhD thesis highlights two cases of interplay between physics and mathematics. The first case involves understanding the graphical representation of waves to determine the velocities of points on a wave profile. It was found that students struggled to use the appropriate information on the graph to find the vertical velocity of points. The second case involves understanding the wave equation (WE), which consists of a second-order partial differential equation. The results also indicated that students lack a proper understanding of the physical meaning associated with the epistemological framing of this equation.

The interventions were designed to help students integrate mathematical representations and physical situations. Although students' performance improved during the study, a few still struggled with learning difficulties after the interventions. Future investigations could consider several aspects. First, the nature of studies 1 and 2 is qualitative. It would be worthwhile to replicate these studies with larger samples so that quantitative data can be obtained. This thesis also laid the foundation for understanding students' difficulties and epistemological framing when interpreting a non-periodic wave profile, as well as their perspectives when making sense of the WE. This is a crucial aspect for developing future interventions related to these topics. As we discovered in studies 1 and 2, the designed interventions did not fully meet the needs of all participants, with a few still failing to provide correct answers and persisting in using their prior reasoning. Thus, in a teaching situation, students' responses to the problems and the designed interventions can serve as valuable tools for researchers and instructors interested in these topics, enabling them to refine and adapt them.

Finally, there are several aspects that could potentially influence students' performances who participated in this study. The curriculum of their universities could be one such fac-

tor. Study 3 offers an approach to embed this epistemological element within teaching. The number of studies that implement conceptual subtleties in physics classes and examine student responses is limited. Hence, more research on this topic could be a necessary component that requires future evaluation.

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**Conceptual challenges with the
graphical representation of the
propagation of a pulse in a string**
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Conceptual challenges with the graphical representation of the propagation of a pulse in a string

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Student difficulties with making sense of graphs in physics have been thoroughly reported. In the study of one-dimensional waves, the issue is even trickier since the amplitude is a function of two variables (position and time). In this work, we investigate students' reasoning and difficulties with interpreting the graphical representation of the propagation of a pulse in a string. A profile $y(x,t)$ of the pulse was provided and students were asked to estimate the velocities of several points at the profile. This forced them to consider the time dimension, by focusing their attention on the motion of these points. This turned out to be extremely challenging to the students, who manifested several conceptual challenges which were categorized and analyzed in the first phase of the study. Based on these findings, three levels of scaffolding support were provided, where each level gradually guided the students to draw the wave profile after some time has elapsed. The scaffolding turned out to be effective, since many students managed to identify the new positions of the points successfully. The study reveals how static representations of intrinsically dynamic phenomena can be challenging for students to grasp.

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I. INTRODUCTION

Graphical representations are widely used as powerful tools to represent concepts and phenomena in physics. In fact, the lack of understanding of graphical representations is often an issue of concern to physics education research (PER). The literature is vast in both the identification of misconceptions and the development of instructional strategies to circumvent them in a variety of topics such as kinematics, thermodynamics, and electrodynamics [1–4].

In wave phenomena, graphical representations are particularly challenging because the mathematical description of waves involves a function of two variables, position x and time t . In the one-dimensional case, this function is generally expressed as $y(x,t) = f(x - vt)$, which is not always treated in mathematics lessons and is quite difficult to grasp. Although one can choose to represent the dependence of the vertical displacement y on each of the variables x and t separately, it is crucial to understand that they are related. The function $y(t)$ describes the movement of a given particle (fixed x) when time is progressing, whereas the function $y(x)$ describes an instantaneous

configuration of a wave, like a screenshot. Investigating how students try to make sense of these conceptual subtleties is the main goal of this study.

The PER literature is also comprehensive in terms of studies investigating student difficulties with wave phenomena. For example, Sadler et al. [5] found that students struggled to distinguish between vertical particle motion and horizontal wave propagation. For the case of transverse waves, students often concluded that matter was transported in the direction of wave propagation. Similar findings also showed that most of the students believed the particles in the air were pushed together towards the direction of motion when a sound wave is traveling [6,7]. These misconceptions occurred because students tend to treat waves as objects and use that reasoning to solve problems [6–9]. Furthermore, some students struggled to distinguish between a mathematical representation and a physical situation, e.g., most students treat the relation between velocity, wavelength, and frequency of periodic waves $v = \lambda f$ mathematically without considering how each variable is related physically [8,10].

In this paper, we explore how university physics students understand graphical representations of waves in a manner which goes beyond other studies in the literature [1–4,11,12]. More specifically, the topic of this study differs from previous ones because most of them investigated students' reasoning in the context of periodic waves [8,10,13–15]. Here, we focus on students' ability to distinguish between the horizontal movement of a pulse and the vertical motion of matter on a nonperiodical wave

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profile. In particular, we provide students with the profile [y(x,t)] of a pulse and analyze whether they can estimate the velocity of some points on the graph, therefore asking them to reason about time evolution. After assessing students' conceptions, three increasingly detailed scaffoldings were provided to see if or how they improved students' performances when solving the task.

II. PHASE I: STUDENTS' REASONING ABOUT THE GRAPHICAL REPRESENTATION OF A PULSE

A. Materials

Our investigation is based on one conceptual question, designed to explore students' understanding of the relationship between the vertical motion of the points on a string, with the (horizontal) propagation of a pulse with constant velocity in this string. Four points were located on a pulse and students were asked to (a) sort out the magnitude of their velocity, and (b) estimate whether the velocity of each point is > 0 , < 0 , or $\frac{1}{4} 0$.

Question: A pulse is moving horizontally with constant speed to the right. The profile below represents a given instant, like a picture (Fig. 1). As the pulse moves horizontally, the points move vertically (wave does not transfer matter)

- (a) Based on the picture, sort the magnitudes of the (vertical) velocity at each point from the greatest to the smallest. Explain your reasons.
- (b) For each point, determine whether the velocity is < 0 ; > 0 , or $\frac{1}{4} 0$. Explain your reasons.

From the expert perspective, one way to solve this question is to draw another profile after some time has elapsed. Figure 2 shows two wave profiles at two different instants.

In Fig. 2, the dotted profile represents the pulse after a short time interval and the red dots show the new vertical positions of the points. It can be seen that point 4 has the greatest speed because it has the greatest displacement

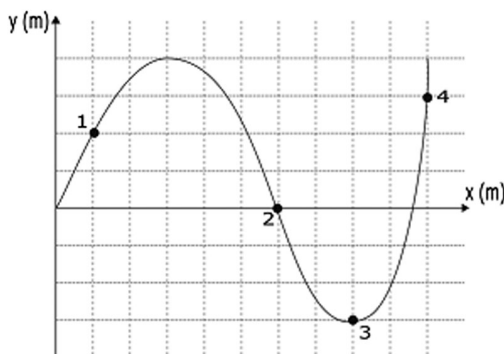


FIG. 1. Problem graph.

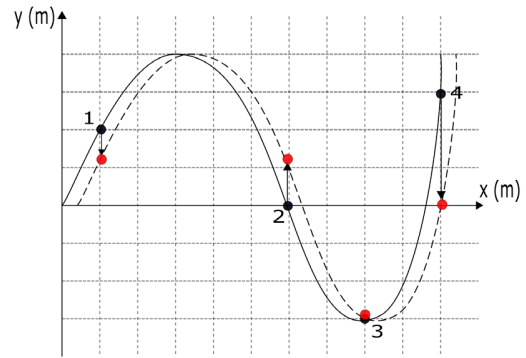


FIG. 2. A graph of two wave profiles at two instants.

compared to other points. Therefore, point 4 requires a greater speed to attain the final position. With this justification, the correct answer for question (a) is 4-2-1-3.

To answer question (b), one needs to see the motion direction of each point. The downward displacement means a negative velocity ($v < 0$) and the upward displacement means a positive velocity ($v > 0$). Using this approach, however, the challenge to answer the velocity for point 3 is inevitable because the point has already moved upward in the new profile. This is because the question asks about the velocity in the initial profile, not when the pulse move after some time has elapsed. In this case, point 3 has zero velocity. Thus, the correct answer for question (b) is that velocities of points 1 and 4 are smaller than zero, the velocity of point 2 is greater than zero, and of point 3 is equal to zero.

In fact, the expert can solve this question (a) by only drawing the slope of each point on the graph to find the speed. However, it is worth saying that the slope of $y(x,t)$ cannot be treated to find the velocity without knowing the relation between the slopes of $y(x,t)$ and $y(t)$. The slope of $y(x,t)$ only addresses the shape of the pulse, but indeed there is a proportionality between the slope of $y(x,t)$ and $y(t)$. Thus, one can infer the velocity based on the slope of each point on the graph of $y(x,t)$ using this relation. Figure 3 shows the slope of each point on the graph.

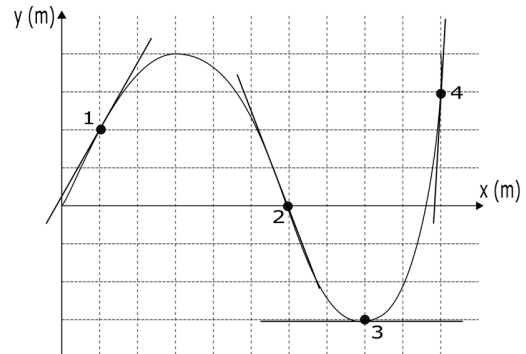


FIG. 3. Solving question (a) by drawing slope.

TABLE I. Students' conceptual challenges in phase I.

Question	Conceptual challenge	Category	Number of students
Part (a)	Difficulty in reading distance on the graph	The roller coaster erroneous reasoning	6
		Inaccurate assumption in reading vertical displacement	6
		Using horizontal position	7
		The wave profile represents wavelength	4
		Using periodical wave formulas	9
	Total		32
Part (b)	Dividing the graph into positive and negative parts Mixing between the roller coaster reasoning and dividing the graph into positive and negative parts		12
			20
	Total		32

Based on Fig. 3, it is clear that point 4 has the greatest slope which results in the greatest speed. Therefore, using this justification, we can also find the correct answer to question (a) is 4–2–1–3.

B. Interviews

In order to explore students' reasoning as fruitfully as possible, we performed paired semistructured interviews with 32 physics students from two Indonesian universities. They were in the second or third year of their studies and all of them had already taken introductory physics courses, including basic notions of wave phenomena. Third year students, especially, had already completed an advanced course on waves.

Arksey and Knight [16] explain that paired interviews have the benefit to bridge the gap between pairs. Consequently, this condition will engage participants in elaborating more on their answers and gaining more interactions during the discussion setting. This type of interview also pushes the participants to work together to answer the questions that they might not be able to respond to individually [17]. Moreover, Houssart and Evens [18] suggest that paired interviews will be more beneficial for unseen questions, meaning that the questions are first encountered at the beginning of the interview. This should provide a collaborative working environment and encourage students to see alternative views from their answers.

We followed the interview procedures based on what has been recommended by the literature. We interviewed students in pairs, and posed the interview question for the first time at the beginning of the interview. They read the questions in a couple of minutes and eventually ask clarification questions to the interviewer. Students first responded individually to the question, and then a discussion in pairs began. In these discussions, students could defend or change their prior reasoning based on each argument from their peers. In the end, we gathered one

agreed final answer from the pairs or individual answers if they could not reach an agreement.

C. Results

We found that no student was able to answer the questions correctly. Their reasons were diverse and their conceptual challenges are categorized based on the common difficulties encountered. Table I shows categories of students' conceptual challenges in phase 1.

In general, we found only one category of students' struggles to answer question (a), which is related to their difficulty to read the appropriate information of distance on the graph but with varieties of conceptual challenges. Meanwhile, the nature of students' reasoning to answer question (b) was based on identifying the position in Cartesian coordinates without considering other physical aspects within it.

1. Question part (a)

Difficulty in reading distance of each point on the graph of $y \times b$.—A few students translated the wave profile as a motion trajectory and used this notion to determine the displacement between two points. We call this the "roller coaster" erroneous reasoning. However, the way this error appeared differed among students. For example Diana¹ divided the profile into four parts and conceived four different motions, the origin (0, 0) moving to point 1, point 1 to point 2, point 2 to point 3, and point 3 to point 4. Point 4 will keep moving upwards. She then related her notion with the proportionality relationship between velocity and displacement. She answered that point 4 has the greatest velocity because it has the greatest trajectory from point 3 to 4. Using that notion, point 2 could also be considered having the greatest trajectory moving from point 1 to 2. When asked about this, she said that she

¹Student names are pseudonyms.

measured the motion trajectory of point 2 from the crest of the wave profile because she assumed that “the point would start its new movement in that position.” She concluded her reasoning by answering 4–2–3–1 for part (a). The following is Diana’s reasoning:

Diana: The greater displacement of a point will result in a greater velocity. Point 4 has the greatest trajectory moving from point 3 to point 4, so point 4 has the greatest velocity. Point 2 is the second order because I calculate its trajectory from the crest of the wave profile. Next is point 3 because it is moving to point 4. Point 1 is the smallest because it has the shortest trajectory moving from point (0, 0).

Another interviewed student, Angela, also assumed that the points on the wave are moving like a roller coaster. However, unlike Diana, she did not measure the motion trajectory of point 2 from the crest to point 3 but from the whole profile from point 2 to point 3. She then decided that point 2 has the greatest speed and sorted the magnitude of speed from the greatest to smallest as 2–4–3–1. Here we see elements of a stronger roller coaster reasoning, as she sometimes relates higher speeds with regions of lower potential energy.

Other students referred to the idea of vertical displacements to determine the velocity, but with inaccurate assumptions. One of the students observed the distance of each point to the x-axis. She said that the greater the distance of each point to the x axis, the greater its velocity and her answer for this question was 4–3–1–2. Julia, also one of our interviewed students, estimated the value of grid lines of 1 m on the y axis and then divided the graph into positive and negative parts.

Julia: If we assume each grid line on the y axis represents 1 m then point 1 has a distance of 2 m, point 2 is 0, point 3 is –3 m, and point 4 is 3 m. So the answer is 4–1–2–3.

Surprisingly, we found some students estimated the horizontal position to determine the distance of each point. They said that the further away a point is from the origin (0, 0) horizontally, the greater its velocity. One of the student’s reasoning is shown below:

Doddy: The answer is 4–3–2–1 because velocity is proportional to the distance based on the velocity formula, which is $v = \frac{1}{4} s = t$. Velocity in point 4 is the greatest because it is located furthest compared to other points.

We do not assume that Doddy considered points on the wave to move horizontally because he did not state any displacement of points to answer the question. Even though this reasoning is simple to understand, using horizontal

displacement seems to be in contradiction to the nature of motion of particles on the string.

Assuming that a wave is always periodic.—Almost half of the students assumed that the wave profile in the question is periodic, even though this was not mentioned in the question. Although their primary goal was to find distances related to each point on the graph, these students associated the distances with wavelengths. We categorized this erroneous reasoning as the “periodicity fixation.”

Ivan, for instance, assumed that the movement of each point starts from the origin (0, 0) and follows the wave profile until it reaches its respective position. This reasoning is also related to the conceptual challenge of the roller coaster. However, it was more plausible to place it into periodicity fixation because he continuously referred to the notion of wavelength in his answer. He conjectured that the distance from each point to the origin (0, 0) determines the magnitude of its wavelength. He then associated it with the proportionality between wavelength and velocity. He said that the greater the wavelength of a point [sic] the greater its velocity. With this notion, he decided that point 3 has the greatest velocity due to its greatest wavelength. This point has a 3=4 wavelength because it consists of one hill and a half valley. Using hills and valleys to determine the wavelength is common when students learn periodic waves in these universities; one wavelength consists of one hill and one valley.

Ivan’s reasoning became more complicated because of his notion of hill and valley. Paradoxically, he did not consider the whole wave profile to determine its wavelength, but asserted a different wavelength to each point of the profile. Moreover, he argued that point 4 is located in a new wavelength, therefore it has the smallest speed. The following is Ivan’s reasoning for question part (a):

Ivan: The order is 3–2–1–4. I calculate the distances of each point to the origin (0, 0) to determine their wavelength. Point 3 is the greatest because it has a 3=4 wavelength, point 2 has a half wavelength, and point 1 has less than a half wavelength. Point 4 is the smallest because it is located in the new wavelength.

This type of conceptual challenge can also be seen from Johan’s reasoning. He actually understood that the points on the wave move vertically, but he believed that the wave profile in the question is a sine wave. Then, he used a sine wave function $y = \frac{1}{4} A \sin \delta kx - t\phi$ to find a formula for velocity, as depicted in Fig. 4.

Johan finally arrived at the velocity formula $v = \frac{1}{4} \sqrt{A^2 - y^2}$. He then determined the magnitude of velocity using those two variables, amplitude (A) and vertical displacement (y). He said that the magnitude of velocity is maximum at $y = \frac{1}{4} 0$ and minimum at $y = \frac{1}{4} A$. With that analysis, he found that point 2 has the greatest velocity because $y = \frac{1}{4} 0$ and point 3 has the smallest velocity because

$$\begin{aligned}
 y &= A \sin(\omega t - kx) \Rightarrow y' = A^2 \sin^2(\omega t - kx) \\
 v &= \frac{dy}{dt} = \omega A \cos(\omega t - kx) \\
 v^2 &= \omega^2 A^2 \cos^2(\omega t - kx) \\
 v^2 &= \omega^2 A^2 (1 - \sin^2(\omega t - kx)) \\
 v^2 &= \omega^2 (A^2 - A^2 \sin^2(\omega t - kx)) \\
 v^2 &= \omega^2 (A^2 - y^2) \\
 v &= \sqrt{\omega^2 (A^2 - y^2)} \\
 v &= \omega \sqrt{A^2 - y^2} \\
 v >> &\sim y = 0 \\
 v << &\sim y = A
 \end{aligned}$$

FIG. 4. Johan’s derivation of a periodical wave to estimate the magnitude of the speed.

$y \frac{1}{4} A$. In the case of a periodical wave, Johan’s reasoning is correct because point 2 is located at the inflection point. However, the wave profile in the question is not periodic. For point 3, in particular, his answer is correct because this point is located precisely in the crest and thus has zero velocity. Finally, he said point 1 has a greater velocity than point 4 because it has a smaller vertical displacement.

Similar to Johan, Adi also operated the concept of periodic waves to solve the problem. However, his method was based on the acceleration formula $a \frac{1}{4} -k^2y$. He assumed that the magnitude of speed can be estimated by using the proportionality relation between acceleration and velocity. The greater the vertical displacement of each point, the greater its acceleration, resulting in greater velocity. He focused on the vertical displacement of each point and measured it based on the displacement of each point to the x axis. With this assumption, he then answered point 4 has the greatest speed. We asked him to clarify his answer since point 3 and point 4 have the same distance to the x axis. He then also considered the horizontal position of a point. Point 4 is located further horizontally than point 3, so that point 4 has the greatest speed. Adi’s reasoning can be seen below:

Adi: I will use the formula of acceleration which is $a \frac{1}{4} -k^2y$ because of the proportionality relation between acceleration and velocity. So, the greater y of a point will result in a greater acceleration, which also produces a greater velocity. The velocity at point 2 is zero because $y \frac{1}{4} 0$.

With slightly different reasoning, Edy immediately noticed that the velocity at point 3 is zero by saying it is

located at the position when a point will move between up and down. This reasoning was undoubtedly correct. He then noticed that point 2 is located at the inflection point and concluded that point 2 has the highest velocity. Again, this answer could be valid if the wave profile in the problem were a periodic wave. Even though Edy never stated any formula regarding a periodical wave, we infer that he also has a periodical wave fixation by his answers to the velocities of points 2 and 3.

2. Question part (b)

Dividing graph into positive and negative parts.—More than half of the students just simply labeled the Cartesian coordinates into negative and positive parts. The velocities of points located above the x axis are positive, and below the x axis are negative. Meanwhile, the velocity of points located exactly at the x axis is zero. The following is one of the student’s reasoning related to this conceptual challenge:

Johan: The velocity of point 1 and 4 are greater than zero because they are located above of x axis, so their magnitudes must be positive. The velocity of point 2 is equal to zero because it is exactly located on the x axis. The velocity of point 3 is smaller than zero because it is located below the x axis, so its magnitude must be negative.

Students in this group merely applied the position of each point based on Cartesian coordinates instead of considering the direction of each point when it is moving. We notice that the majority of students that hold this conceptual challenge also had a false assumption of vertical displacement to answer question (a).

Mixing between the roller coaster reasoning and dividing the graph into positive and negative parts.—Almost half of the students had a conceptual challenge by mixing two different notions to answer this question. First, they claimed that the area above the x axis is positive and the area below the x axis is negative. Then, they combined that notion with their incorrect interpretation of a moving point, the so-called roller coaster reasoning. Here is an example:

Indra: The velocity of point 1 is greater than zero because it is moving up to the crest of the hill. So, it requires velocity to climb the hill. I can also see that point 1 is located in the positive area of Cartesian coordinate. The velocity of point 2 is zero because it is located on the x axis. The velocity of point 3 is smaller than zero because it is located in a valley (moving down). For the same reason as point 1, the velocity of point 4 is bigger than zero.

This group’s reasoning can be associated based on how their method solves question (a). Indra, for example, reasoned that the points on the wave will move along the wave profile. Because of this conceptual challenge, his

approach to answer question (b) was affected by this error. He said that points 1 and 4 are moving up because they are located at the upward profile whereas point 3 is moving down because it is located in the downward profile. Then, he compounded the error by saying that the area above the x axis is positive and the area below the x axis is negative. Indra mixed these two conceptual challenges to solve part (b). Points moving upward (point 1 and 4) will move into the positive area, so their velocities must be greater than zero. A point moving down (point 3) will move into the negative area, so its velocity must be smaller than zero. For point 2, however, he concluded it had a zero velocity because it is located at the x axis. Because of this, Indra's reasoning seems incoherent because he only analyzed the position of point 2 instead of applying his two conceptual challenges like he did when analyzing the other points. However, Diana, who also mixed these two aspects, indicated that velocity of point 2 is smaller than zero because it is located at the downward profile. Thus, her reasoning seems more coherent.

III. PHASE II: SCAFFOLDING SUPPORT

A. Methodology

The result from the questionnaire made it clear how challenging the posed question was to the students, which motivated us to develop instructional strategies to see if they could understand the basic conceptual issues. Three levels of scaffolding were implemented and 22 students who answered the questionnaire were selected randomly to participate in this stage, where, once again, semistructured interviews were conducted in pairs.

Methodologically, our study is similar to the one conducted by Maries et al. [19] who developed scaffoldings to reduce student difficulties with Gauss's law. Our scaffoldings were also designed to incorporate the experts' approach to solve the problems [20,21]. However, our supports were slightly different because we did not provide a complete explanation, but only minor hints to the students. We expected that students could build their understanding and answer the questions based on their own analysis.

The goal of these interventions was to lead the students to draw the wave profile after some time has elapsed to reveal possible changes in students' reasoning to solve the problem. Under these interventions, we expected that students could notice the displacements of each point by comparing two wave profiles at two instants to solve the problem.

Scaffolding is associated with providing suitable support to a learner to overcome something that is difficult to achieve [22]. Originally, the term scaffolding was used to describe a series of steps for a learner to achieve a better performance [23]. Nowadays, scaffolding is used as an intervention to help not only an individual

person, but also pairs and teams in many fields, including physics [24,25].

1. Scaffolding level 1

The purpose of scaffolding level 1 is to provide an illustration to the students of the characteristics of the wave profile after a short time interval. The interviewer demonstrated physically with his hands how to create a single pulse on a string that is moving to the right with constant velocity. Students were then asked to draw the next wave profile after some time has elapsed on the graph in the question. Students were also asked to locate the displacement of points in the new wave profile. Students who failed to draw the correct wave profiles in this stage were given intervention level 2.

2. Scaffolding level 2

In this level, the PhET simulation called "wave on a string" [26] was introduced to the students. This simulation presents the real condition of a vibrating string, and it has a variety of features that are suitable to our wave profile. This simulation can be modified into different situations, for example, showing how a string oscillates with or without reflections. The vibration source can be created manually with the possibility of adjusting damping and tension. Moreover, if the users want to see the movement on the string in detail, a slow-motion feature can be applied to the system.

Students were asked to use the simulation to reproduce the pulse that was given in the question, and they were left to explore the simulation without any help. Students who were able to generate the pulse in the simulation were asked to draw the new wave profile again after a short period of time. Then they were asked once again to answer the question. Even though some students did not create the same wave profile, we asked them to answer the same question because we wanted them to realize that the shape of the wave remains the same when it is progressing. Students who failed to use the appropriate features in the simulation were given intervention stage 3.

3. Scaffolding level 3

In this final support, we showed to the students how to create a pulse moving to the right with a constant velocity. They were instructed to use a manual vibration source, set the damping to zero, choose no-end string, and use the slow-motion feature to see the vibration in detail. After successfully creating the correct simulation, they were asked once again to draw the wave profile after some time has elapsed and then answer the questions once more.

B. Results

Students' performance in the scaffolding environment was diverse at each level with noticeably scaffolding level 1

TABLE II. Students' results in the scaffolding (SCL) environment for question (a) ("1" indicates that students answer the question correctly, "0" indicates that students answer the question incorrectly).

Pair	Student	SCL I	SCL II	SCL III	Drawing
1	1	0	0	0	Failed
	2	0	0	0	Failed
2	3	0	0	1	Succeeded
	4	0	0	1	Succeeded
3	5	0	0	0	Succeeded
	6	0	0	0	Succeeded
4	7	0	0	0	Failed
	8	0	0	0	Failed
5	9	0	0	0	Succeeded
	10	0	0	0	Succeeded
6	11	0	0	0	Succeeded
	12	0	0	0	Succeeded
7	13	0	0	1	Succeeded
	14	0	0	1	Succeeded
8	15	0	1	0	Succeeded
	16	0	1	0	Succeeded
9	17	0	1	0	Succeeded
	18	0	1	0	Succeeded
10	19	0	1	0	Succeeded
	20	0	1	0	Succeeded
11	21	0	0	0	Failed
	22	0	0	0	Failed

TABLE III. Students' results in the scaffolding environment for question (b) (1 indicates that students answer the question correctly, 0 indicates that students answer the question incorrectly).

Pair	Student	SCL I	SCL II	SCL III	Drawing
1	1	0	0	0	Failed
	2	0	0	0	Failed
2	3	0	0	1	Succeeded
	4	0	0	1	Succeeded
3	5	0	0	0	Succeeded
	6	0	0	0	Succeeded
4	7	0	0	0	Failed
	8	0	0	0	Failed
5	9	0	0	1	Succeeded
	10	0	0	1	Succeeded
6	11	0	0	0	Succeeded
	12	0	0	0	Succeeded
7	13	0	0	1	Succeeded
	14	0	0	1	Succeeded
8	15	0	1	0	Succeeded
	16	0	1	0	Succeeded
9	17	0	1	0	Succeeded
	18	0	1	0	Succeeded
10	19	0	1	0	Succeeded
	20	0	1	0	Succeeded
11	21	0	0	0	Failed
	22	0	0	0	Failed

being very challenging for the students. Tables II and III show students' results in the scaffolding environment.

These results show that students' performance has improved by looking at their success in answering the question with correct reasoning after scaffolding level II. Although some students still hold robust erroneous views, the complex understanding of graphs of $y \propto x^2$ and $y \propto t^2$ appeared to be solved by some of our students.

1. Students' results for scaffolding level 1

We found that all students had difficulties imagining the nature of a pulse moving to the right with a constant velocity. Most of the students drew the new wave profile smaller because they said that the wave will lose its energy after moving a bit and its amplitude will diminish slowly. Figure 5 shows Citra's drawing exemplifying this difficulty.

Based on her drawing, Citra's difficulties were not only due to the notion of losing energy, but that she also struggled with locating points in the new profile. She started drawing the new profile from the origin and assumed that the points on the wave would only oscillate in the fixed x-axis position (except point 1). The way she located the new positions of the points also seems inconsistent. When asked for the reason for that choice, she simply said that she located the points randomly in the new wave profile.

Hendri, another of our interviewed students, also thought that the wave will lose its energy. However, unlike Citra, he started drawing the new wave profile after a short time interval to the right from the initial profile. Figure 6 shows Hendri's drawing in scaffolding level 1.

His choice to locate the new positions of the points was based on his conceptual challenge to answer question (a). Based on Fig. 6, he implemented his notion of roller coaster reasoning to locate the displacement of points on the new wave profile. He thought that points on the wave are moving along the wave profile and the numbers with prime symbols indicated this conceptual challenge.

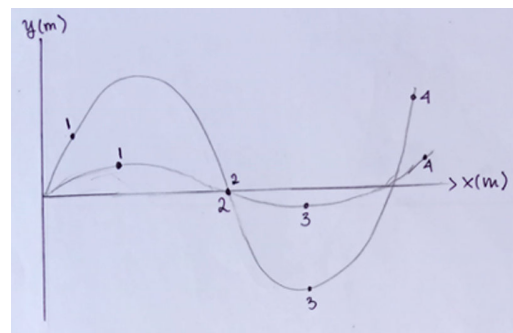


FIG. 5. Student's drawing with the next profile become smaller.

S T 2` A A , 6 B ` b i ` 2 p B b B Q M b m +
S ? v b B + H _ 2 p B 2 r @ S ? v b B + b 1
_ 2 b 2 ` + ?

1 M + Q m ` ; B M ; b i m / 2 M i b i Q m M /
i ? 2 R @ . r p 2 2 [m i B Q M
J m ? K K / b r B M _ M ; F m i B - _ B +

S T2` AAA, hQ #2 bm#KBii2/
K2`B+ M CQm`M H Q7 S?vb

*QM+2Tim H bm#iH2iB2b Q7 i?
2[m iBQM
_B+ `/Q E ` K- Jm? KK / brBM

I. INTRODUCTION

Partial differential equations were key to fostering the fruitful interplay between physics and mathematics. In particular, the wave equation (WE), first introduced by D'Alembert in the late 1700s, has played a significant role in shaping this relationship.

Nowadays, the WE is seen as a fundamental equation in physics, describing a wide range of phenomena. Due to its importance, this equation is broadly taught across physics programs, although with different approaches. Among them, Feynman⁹ derives the WE based on sound propagation and electromagnetic waves, while other authors derive it using Newton's 2nd Law^{10,11}.

Despite the apparent simplicity of $\frac{\partial^2 y}{\partial x^2} = v^2 \frac{\partial^2 y}{\partial t^2}$, understanding this equation deeply is far from being trivial. The purpose of this paper is to present certain conceptual subtleties related to the WE, which are not usually addressed explicitly in teaching, in order to make physics instructors aware of them.

II. WHY SECOND ORDER?

Mathematically, a 1-D wave is represented by a progressive function, i.e., it describes a fixed profile travelling horizontally as time goes by. This can be expressed by functions of the kind

$$y(x; t) = f(x - vt) \quad (1)$$

where f is an arbitrary function¹². Thus, waves are expressed by functions of two variables, space and time, but in a particular way. For example, functions like $y(x; t) = \sin(x - vt)$ or $y(x; t) = xt^2$ do not fulfill this requirement.

In fact, the WE can be derived from the assumption that $y(x; t) = f(x - vt)$. All that is needed is to differentiate it twice with respect to space and time, respectively. For simplicity, let us consider the case where the wave travels to the right, i.e., $y(x; t) = f(x - vt)$. Letting $x - vt = u$ and applying the chain rule:

With respect to x :

$$\frac{\partial y}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du} \quad (2)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2} \quad (3)$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du} \frac{\partial u}{\partial t} = v \frac{df}{du} \quad (4)$$

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{d^2 f}{du^2} \quad (5)$$

By Comparing (3) and (5), we get

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad (6)$$

which is the 1-D WE. Thus, the WE can be seen as the result of simple manipulations of $y(x;t) = f(x - vt)$. But why do we need to differentiate twice? Can we not just stop at the first?

To answer this question, let us differentiate the same function once with respect to x and t. Once again, using the chain rule and letting $x - vt = u$, we have:

With respect to x

$$\frac{\partial y}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} \quad (7)$$

$$= \frac{df}{du} \quad (8)$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du} \frac{\partial u}{\partial t} \quad (9)$$

$$= \frac{df}{du} (-v) \quad (10)$$

Combining the previous results, we arrive at:

$$\frac{\partial y}{\partial t} = -v \frac{\partial y}{\partial x} \quad (11)$$

Eq. (11) is often called the transport equation (TE). If the pulse is moving to the right, then the TE is expressed as:

$$\frac{\partial y}{\partial t} = +v \frac{\partial y}{\partial x} \quad (12)$$

This equation characterizes a wave moving to the left or right with velocity. Thus, the TE is limited in that it can only describe waves propagating in one direction. Therefore, it can not account for wave interactions with boundaries.

Consider a sound wave propagating in a straight line. The transport equation can be used to describe this phenomena. However, when the sound wave collides with an object, the transport equation fails to describe the resulting changes in the wave's behavior, such as alterations in its direction, amplitude, or phase. The transport equation overlooks the influence of boundaries on wave propagation, which stresses that this equation can not capture complex wave phenomena.

We can test whether the TE and the WE satisfy certain wave properties, such as superposition. Consider two pulses traversing the same medium, the first to the right, the second pulse travels to the left, say $y_1(x; t) = f(x - vt)$ and $y_2(x; t) = g(x + vt)$.

At a particular time, the pulses meet, resulting in a wave superposition. The sum of the two wave functions is the total displacement:

$$y = y_1 + y_2 = f(x - vt) + g(x + vt) \quad (13)$$

Let us see if this function satisfies the transport and wave equation. We substitute $x - vt = u$ and $x + vt = v$ and derive it once with respect to x and t .

$$\frac{\partial y}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} + \frac{dg}{dv} \frac{\partial v}{\partial x} \quad (14)$$

$$= \frac{df}{du} 1 + \frac{dg}{dv} 1 \quad (15)$$

$$= \frac{df}{du} + \frac{dg}{dv} \quad (16)$$

$$\frac{\partial y}{\partial t} = \frac{df}{du} \frac{\partial u}{\partial t} + \frac{dg}{dv} \frac{\partial v}{\partial t} \quad (17)$$

$$= \frac{df}{du} (-v) + \frac{dg}{dv} (v) \quad (18)$$

$$(19)$$

Thus, we do not arrive at the transport equation. Now we derive it again with respect to x and t .

$$\frac{\partial y}{\partial x} = \frac{d^2f}{du^2} + \frac{d^2f}{dv^2} \quad (20)$$

$$\frac{\partial y}{\partial t} = \frac{d^2f}{du^2}v^2 + \frac{d^2f}{dv^2}v^2 \quad (21)$$

If we substitute these results, we will arrive at the WE.

$$\frac{\partial y}{\partial t} = v^2 \frac{\partial y}{\partial x} \quad (22)$$

This demonstrates that the transport equation can in fact account for a single pulse travelling to the right and left. However, this equation cannot describe more complex wave phenomena like superposition.

III. SECOND DERIVATIVE OR CURVATURE?

The WE can be interpreted conceptually as a relationship between the resultant force acting on each point, which is proportional to its acceleration $\frac{\partial^2 y}{\partial t^2}$, and the shape of the wave profile, often expressed by the rate of change of the function's spatial derivative $\frac{\partial y}{\partial x}$. Intuitively, the greater the curve deviates from being a straight line at a given point, the greater the force on that point (see Fig. 1).

But is the second derivative the best quantity to represent this aspect related to the profile's shape? Consider, for instance, a parabolic function $y = (x - vt)^2$. Intuitively, the force should be greater at the vertex, but since the second derivative is constant $\frac{\partial^2 y}{\partial x^2}$ gives the same result for all points of the parabola (see Fig. 2). What is going on?

The issue here is a subtle difference between concavity and curvature. Whereas the former describes the slope's rate of change, the latter is inversely proportional to the radius of the tangent circle, at each point. To determine how much a curve deviates from a straight line, i.e., its curvature, we use the following formula:

$$= \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \quad (23)$$

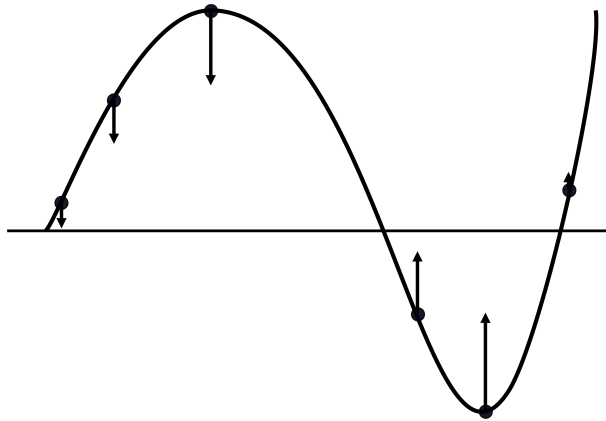


FIG. 1: The tension of each point on a vibrating string is proportional to the second derivative

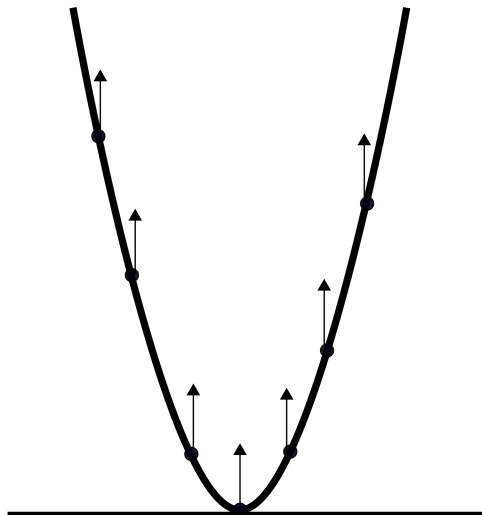


FIG. 2: A parabolic wave profile with constant forces in different points

Let us try to find the curvature of $y = x^2$:

$$dy/dx = 2x; d^2y/dx^2 = 2 \tag{24}$$

The curvature of parabola is defined as:

$$= \frac{2}{[1 + (2x)^2]^{3/2}} \tag{25}$$

$$= \frac{2}{[1 + 4x^2]^{3/2}} \tag{26}$$

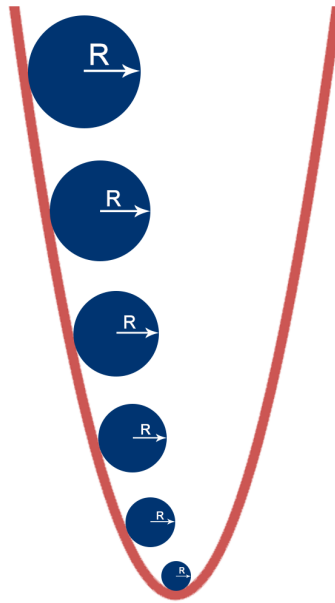


FIG. 3: Curvature of some points on parabola

Thus, we see that the value of (curvature) is dependent on x , as opposed to the second derivative. For example, the curvature of the parabola at the vertex is

$$k(x = 0) = \frac{2}{[1 + 4 \cdot 0^2]^{3/2}} = 2; \quad (27)$$

which is its greater value.

Consequently, if we assume that force is proportional to curvature, the WE should be written as:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \frac{1}{[1 + (\partial y / \partial x)^2]^{3/2}}; \quad (28)$$

which is much more complicated than the usual WE. However, notice that if $\partial y / \partial x = 0$ the usual WE is recovered. We will come back to that.

IV. HISTORICAL INTERLUDE: TAYLOR'S ORIGINAL DERIVATION

Force proportional to curvature was key to one of the first applications of calculus to the study of waves, which was written by the English mathematician Brook Taylor, famous for the "Taylor series", in his "Methodus Incrementorum Directa et Inversa"³. In Lemma IX, Taylor shows that the force acting on a stretched string is proportional to its curvature at

any given point. Due to its historical importance and pedagogical potential, we will try to reconstruct this derivation.

Taylor considered a curve with two adjacent points as shown in Fig. 4. Tangent lines are drawn from each point, represented by t and t' , and normal lines are also drawn from the same points, intersecting in S . The goal is to determine the net force acting on the arc segment Bb .

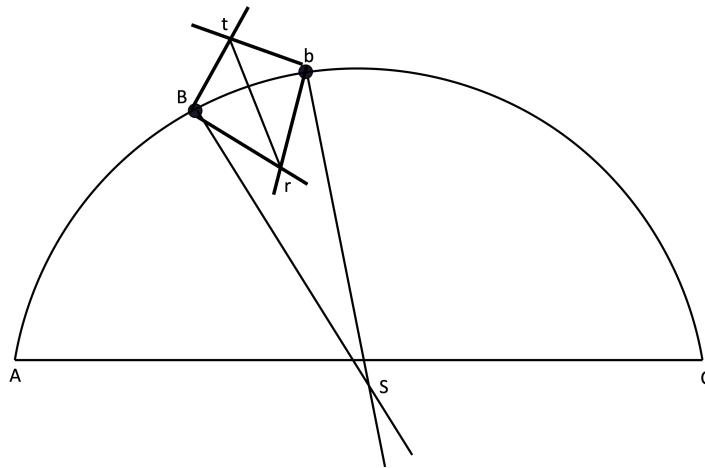


FIG. 4: Taylor's geometrical construction to show that force is proportional to curvature¹³

Fig. 5 highlights the essential features. We can see that the tensions T_B and T_b add up to F_{tr} , which is the net force of the tension due to the curved string. The magnitudes of the forces are proportional to the respective segments, thus

$$\frac{F_{tr}}{T_B} = \frac{tr}{tB} \quad (29)$$

From Fig 5, we can also see that the angles $\angle tBb$ and $\angle t'Br$ are equal, meaning that

$$\frac{tr}{tB} = \frac{Bb}{BS}; \quad (30)$$

yielding

$$\frac{F_{tr}}{T_B} = \frac{Bb}{BS} \quad (31)$$

or

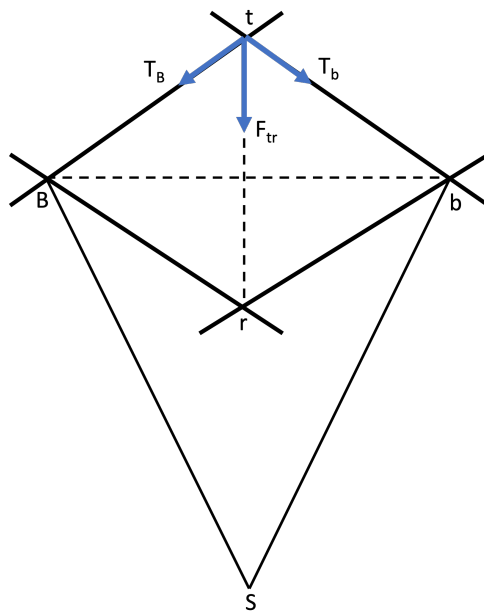


FIG. 5: The parallelogram that shows tensions acting on two points

$$F_{tr} / \frac{T_B Bb}{BS} : \quad (32)$$

As Bb tends to zero, $Bb=BS$ becomes the curvature at point B . Therefore, in the limit, this relation shows that the net force acting on the small line segment Bb is proportional to the tension at point B and the curvature at this point, QED.

V. MAKING SENSE OF THE RESTRICTIVE CONDITIONS OF THE WE

Although it makes sense now to assume that force is proportional to curvature, Eq. 28 becomes very complicated to solve. Therefore, some assumptions are necessary to simplify the mathematical formalism in order to make it easier to solve. From a mathematical argument, let us consider the case where the first derivative of every point is close to zero ($\partial y = \partial x \approx 0$). Under this restrictive condition, we can make an approximation where curvature is approximately equal to concavity.

We can apply this mathematical argument to the case of the vibrating string problem, specifically when the string is fixed at two extremities². This physical situation became the subject of debate between D'Alembert and Euler. D'Alembert considered the condition

where the string has a very small displacement from the straight line and concluded that the derivation of the WE excluded many physical situations. On the other hand, Euler approached the problem by considering a plucked string and argued for the derivation using physical reasoning.

Nevertheless, D'Alembert's derivation of the WE is now commonly accepted to describe classical string vibrations. By using Newton's law, it is mathematically demonstrated why certain assumptions must be made in order to derive the WE. By contemplating only a small portion of the string, this derivation allows us to simplify the problem. Taking into consideration the string's entire curvature would undermine the linear approximation, making it impossible to derive the WE using Newton's laws.

After considering the small segment of the string, the basic assumption that must be made is that the gravity is negligible compared to the tension on the string. This implies that the tension only arises from the vibration of the string and that gravity has little to no effect. We can relate this when playing a guitar and we rotate it in any direction. The sound remains relatively the same, thus illustrating that the gravity has a negligible effect on the string.

If we observe the tension in one edge of the string without making any assumptions, then the force can be decomposed into x and y components, as depicted in Fig. 6. Nevertheless, this condition makes the derivation more complicated.

In order to eliminate the horizontal components on the string, the second assumption made is that the deflections of the string are small. This condition made the string nearly flat. As a result, we can neglect $T \cos \theta_1$ and $T \cos \theta_2$ in the edges of the string because its magnitude in the opposite direction is equal. We can relate this by thinking of a very tight string on a guitar. When we pluck it, the displacement of the string is relatively small

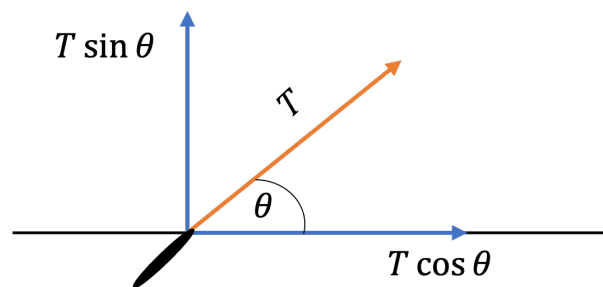


FIG. 6: The condition of one edge of the string with x and y components

compared to its length.

The second assumption enables us to assume that the tension throughout the string is uniform. We can visualize this condition when we increase the tension of a guitar string, that gives the tension at a constant level throughout the entire guitar string. Fig 7 illustrates the condition of the string under three assumptions.

Using the condition of the string in Fig 7, the net force of this small segment of string is given by:

$$T \sin \theta_2 - T \sin \theta_1 = T \Delta m \quad (33)$$

The slope of two edges of the string:

$$m_1 = \frac{\Delta y}{\Delta x} = \tan \theta_1 \quad (34)$$

$$m_2 = \frac{\Delta y}{\Delta x} = \tan \theta_2 \quad (35)$$

Since the string has a small vibration, then $\sin \theta_1 \approx \tan \theta_1$ and $\sin \theta_2 \approx \tan \theta_2$, thus

$$T(m_2 - m_1) = T \Delta m \quad (36)$$

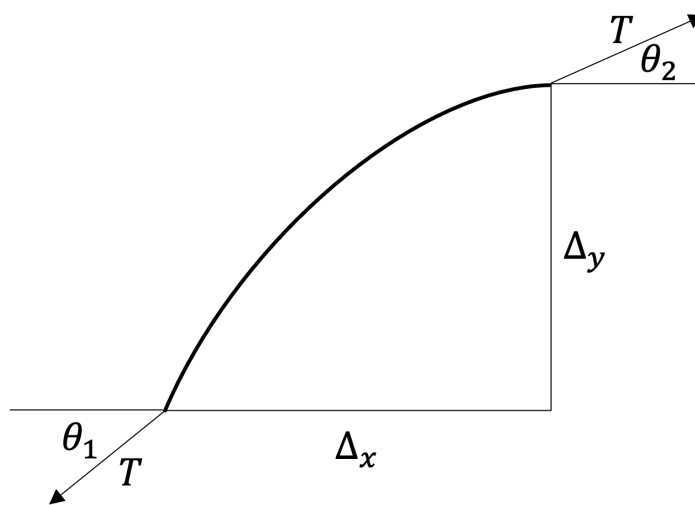


FIG. 7: The condition of the string under two assumptions

We consider the mass on the small segment of the string:

$$m = \mu \Delta x \quad (37)$$

where μ is the mass per unit length.

Applying Newton's second law, we get:

$$T \mu \Delta x = (\Delta x) \left(\frac{\partial^2 y}{\partial t^2} \right) \quad (38)$$

$$T \frac{m}{\Delta x} = \frac{\partial^2 y}{\partial t^2} \quad (39)$$

Then, we consider the limit of $\Delta x \rightarrow 0$

$$\lim_{\Delta x \rightarrow 0} \frac{m}{\Delta x} = \frac{\partial m}{\partial x} = \frac{\partial y}{\partial x} \quad (40)$$

Substituting the results, we will arrive at the WE:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad (41)$$

where $v = \sqrt{\frac{T}{\mu}}$ is the velocity of the propagation.

VI. CONCLUSION

In this work, we have highlighted some conceptual subtleties of the WE that are not commonly presented in textbooks. In Physics Education Research (PER), several studies have found that students often view physics equations solely as mathematical tools, lacking a proper understanding of their physical implications¹⁴⁽¹⁶⁾. We argue that by presenting these unconventional aspects might be one way to help addressing that belief.

Physics equations are typically used in teaching to solve problems that often involve the manipulation of numbers and variables¹⁷. However, this practice may hinder their roles and conceptual status. Some studies have proposed different perspectives to explore physics

equations and concepts that can convey their deep meaning^{18,19}. One more thing that needs to be considered related to this is the advantage of embedding this epistemological dimension in teaching. This includes using the historical aspect^{20,22}. These studies emphasize that using this approach might be more beneficial to pre-service physics teachers in order to improve their educational practices²³, or it offers a new creative perspective of teaching that allows students' curiosity and interest in the topics²⁰.

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