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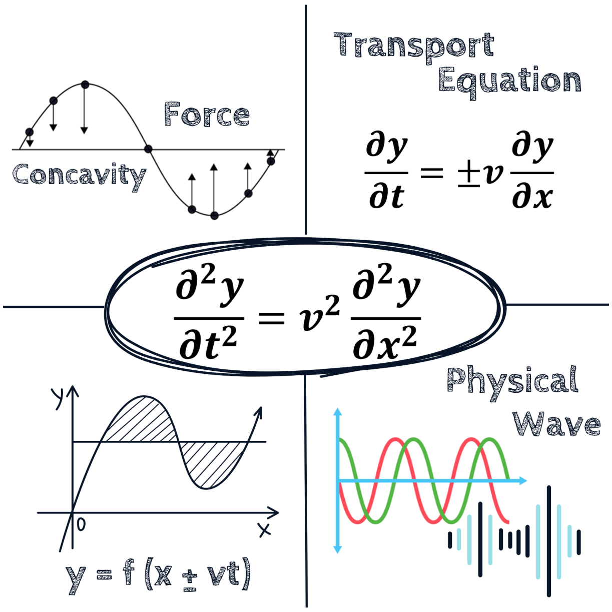

PhD Thesis

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Learning challenges in the interplay between physics and mathematics: the case of the 1-D wave equation

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Learning Challenges in the Interplay between Physics and Mathematics: The Case of the 1-D Wave Equation

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Resume

Fysiske ligninger er essentielle komponenter til at forklare utallige fysiske fænomener i form af matematiske udtryk. Dog medfører matematikkens tilstedeværelse i fysik mange udfordringer, både ved differentiering af deres roller og ved sammenlægning af begge områder. Mens mange undersøgelser undersøger studerendes perspektiver på fysiske ligninger, foreslår få metoder til at forstå disse ligninger. Denne afhandling sigter mod at udfylde denne mangel ved at undersøge studerendes epistemologiske indramning og læringsvanskeligheder, når de forsøger at forstå en fysikligning, i dette tilfælde 1-D bølgeligningen, fra flere aspekter. To studier i denne afhandling anvender en kvalitativ forskningsmetode, der består af tre trin: udforskning, intervention og evaluering. Det første studie udforsker studerendes forståelse af den grafiske repræsentation af $y(x,t) = f(x \pm vt)$, hvor studerende præsenteres for en ikke-periodisk bølgeprofil af $y(x)$ og bliver bedt om at finde hastigheden af punkter på profilen. Denne opgave kræver, at de inddrager tid i deres analyse. Resultaterne viser, at denne problemstilling er ekstremt udfordrende for studerende og frembringer flere vanskeligheder. Det andet studie sigter mod at udforske de studerendes epistemologiske indramning af bølgeligningen fra forskellige aspekter. Et spørgeskema bestående af fem spørgsmål stilles til de studerende. Resultaterne viser, at de studerendes svar mangler fysisk betydning, og deres overbevisninger er ikke på linje med eksperters. Interventioner er derefter designet til at hjælpe studerende med bedre at forstå de problemer, der er identificeret i disse studier. Selvom nogle studerende forbedrer sig efter interventionerne, fastholder nogle deres ræsonnement, hvilket indikerer, at de har robuste overbevisninger om emnet. Endelig søger det tredje studie at afsløre kompleksiteten ved bølgeligningen ved at fremhæve konceptuelle finesser, der ofte går ubemærket hen i undervisningen af bølgeligningen.

Abstract

Physics equations are essential components for explaining numerous physical phenomena in the form of mathematical terms. However, the presence of mathematics in physics poses many challenges, both in differentiating their roles and in combining both fields. While numerous studies investigate students' perspectives about physics equations, few propose methods for understanding these equations. This thesis aims to fill this gap by investigating students' epistemological framing and learning difficulties when they attempt to make sense of a physics equation, in this case, the 1-D wave equation (WE), from several aspects. Two studies in this thesis employ a qualitative research method consisting of three steps: exploration, intervention, and evaluation. The first study explores students' understanding of the graphical representation of $y(x,t) = f(x \pm vt)$, where students are presented with a non-periodical wave profile of $y(x)$ and asked to find the velocity of points on the profile. This task requires them to consider the time dimension in their analysis. The results highlight the significant challenges students face and reveal the various difficulties encountered. The second study aims to explore students' epistemological framing of the WE from various aspects through a questionnaire comprising five questions. The findings indicate that students' responses lack physical meaning and their reasoning is not aligned with those of experts. The interventions are then designed to help students better understand the problems identified in these studies. While some students improve after the interventions, a few persist with their reasoning, indicating they hold robust beliefs about the subject matter. Finally, the third study seeks to uncover the complexity of the WE by highlighting conceptual subtleties that often go unnoticed in teaching the WE.

Preface

Starting as a PhD student three months before the covid pandemic strikes the world and eventually being able to complete it surprises me as a person who lives far from family and close friends. Luckily, on this journey, I received a lot of supports from many people during that difficult time. Therefore, I would like to express my deepest gratitude to those who always believe in me and encourage me in many ways. First, I must send my deepest thanks to my supervisor, Ricardo Karam, for his endless support and motivation. An excitement to discover something about which we had never previously thought and a joy that our efforts have ultimately paid off. This work would not have been like this without your expertise and encouragement. I would also like to thank my co-supervisor, Christian Joas, for his valuable feedback on the research design and the paper.

I never thought I would end up pursuing my PhD in Denmark, a country that was never on my dream list. Although I had received an offer from another place, choosing the Department of Science Education at the University of Copenhagen for my PhD was one of the best decisions I have ever made. Surrounded by very supportive and nice colleagues is a blessing. I want to thank Henriette Holmegaard, who has always been within reach, providing emotional support when the lockdown situation became too much to handle. Also, thank you for the invitation to your lovely house and the enjoyable brunch. Christina Larsen, thank you for always going out of your way to help me. I cannot thank you enough for your kindness. Also, I thank Lene Madsen for her support and assistance. And to Jan Sølberg, thank you for your guidance as the head of the section and for the delightful conversations we have had. I also thank Nadja Normaj, Axel, and all employees at IND for their help and supports.

Thank you so much to Hendra Agustian, with whom I often share many things in Indonesian. Your big support during challenging work situations means a lot to me. I also wish to express my gratitude to Magdalena Kersting, who is always willing to discuss and provide valuable feedback for my manuscripts. Thank you to everyone in KUPER for insightful discussions and feedback.

Thank you so much for all the PhD students at IND. However, I would like to give special thanks to some of them. Pia Tonnesen, who is always very kind and helpful. Thank you for the lovely dinner moments with your family and some of our colleagues. Jonas Jrgensen, for our focused work and lunches, and thanks for teaching me the proper ways to eat rugbrd. Mayu and Rongrong, my office colleagues, who are always nice. Derya Cosan, for sharing some Turkish food and having pleasant conversations. Liv Nøhr, soon to be a PhD student, for making the running club at IND a reality. I also want to thank former PhD students at IND, Andrea Gregersen and Katia Nielsen, who have always been kind.

Thank you to the people who made my stay in the US truly meaningful and enjoyable. First, I express my gratitude to the faculty members of the Physics Education Group at the University of Washington, particularly Paula Heron, for allowing me to learn many valuable things from her. I would like to extend my thanks to Peter Shaffer, Suzanne Brahmia, and

Donna Messina, for our enjoyable discussion and talks. A special appreciation goes to the graduate students who made my visit so fun and memorable: Anne, Taylor, Charlotte, Al, John, Jared, and Kristin. Our time together, whether it was decorating Christmas cookies, doing puzzles, singing at karaoke night, or having dinner in Portland, will forever remain as highlights of the many enjoyable experiences we shared.

I am also grateful to my friends who made living in the US incredibly fun and exciting. Alina, Lars, Annapola, Lucrezia, Francesco, Philip, Jochen, Manato, and Sarah, thank you for our hiking adventures, weekly dinners, and all the other wonderful moments we shared. I want to thank Brian McGinn, a friend, whom I have been talking to since 2013. I still cannot believe that I spent Thanksgiving and late Christmas at your home with your family in Spokane. Thank you for coming all the way to Seattle and introducing me to ”Portage Bay,” which has become my favorite brunch spot with my friends. This PhD achievement was also part of your help. Thank you, Zafran, for willingly coming all the way to Seattle from Pullman to pick me up at the airport and for all your help while I was in the US.

I also want to express my gratitude to the members of some friends of the Overseas Indonesian Students Association Alliance (PPI Dunia), Cendekia Apik Cabinet. These online friends during the pandemic turned out to be real friends when we finally met in person. Mas Faruq, Mba Ika, Mas Chandra, Mba Annisa, and Mas Aas, thank you for our talks and ”friendly bullying” that made my life abroad so much easier.

My Indonesian friends who lived or live in Copenhagen that always made me feel at home. I want to thank my friends who are currently in Indonesia. Uni Yuvi, Bang Dwi, Kak Huria, Bang Irul, and Bang Ipul, for the beautiful moments we spent together during my first year in Denmark. I am very grateful. Azka and his family have always been within reach, and I am thankful for the enjoyable trip we had in Trømso. Then, my friends who are currently in Denmark. Randy and his family that have been consistently helpful, and I appreciate their support. I also want to express my gratitude to Bu Kus, who has been a motherly figure, for her advice and for often inviting me to enjoy delicious Indonesian food.

Lastly, to my family and close friend in Indonesia, I may have given up on this journey without your endless support and care. Although I never anticipated being away from my loved ones during the pandemic, our uncounted video calls and chats have given me strength. Thank you to my very best friend, Wija, for his endless supports. I cannot thank you for everything and for being such an amazing friend. To my brothers, Bang Anda and Bang Oki, thank you for being incredibly supportive siblings. Finally, my parents, the reason why I decided to start this journey. Thank you for raising me to be the person I am today. Thank you for your endless wise advices and unconditional love. Thank you so much.

The thesis is based on these three papers and they are referred to as Study 1-3 throughout the thesis.

Paper I

Rangkuti, M. A., & Karam, R. (2022). Conceptual Challenges with the Graphical Representation of the Propagation of a Pulse in a String. Physical Review Physics Education Research, 18(2), 020119.

Paper II

Rangkuti, M. A., & Karam, R. Encouraging Students to Understand the 1-D Wave Equation.
The first revision submitted to Physical Review - Physics Education Research

Paper III

Karam, R., & Rangkuti, M. A. Conceptual subtleties of the 1-D wave equation.
To be submitted in American Journal of Physics

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1 Introduction

1.1 The overall purpose of the PhD project

The purpose of this study is to investigate students' main conceptual difficulties in understanding the physical meaning of the WE. The difficulties encountered by students in interpreting the WE were explored in some detail. This study revealed the reasons for these difficulties.Additionally, this study also considered various teaching strategies that may assist students more effectively to develop a conceptual understanding of the WE. Although focusing on the WE as a case study, this research shall also provide ways to help students understand equations more broadly.

1.2 Motivation of this study

Physicists commonly use equations to guide their reasoning, thus an essential part of physics education is to teach students to do so. However, mathematical symbols that are widely used in physics equations present a challenge regarding how to understand their physical meaning. Some studies found that students typically focused on calculating and identifying symbols rather than gaining a conceptual understanding of the principles underlying the equations [1–4]. For example, these studies found that students' performance was better in solving numerical rather than symbolic problems, although the questions shared identical concepts [4, 5], or students failed to transform symbolic into graphical representation of vectors [1].

The epistemological framing regarding physics questions were also explored. Some studies indicated that students believed they understood a physics equation if they could use it to solve problems [6–9]. However, they were confused when asked about the meaning of the equation and tended to treat the physics equation as a calculation recipe, a strategy called plug and chug [6, 10, 11].

It is interesting to see that there is not much literature that attempts to help students understand a physics equation, but exploring students' views about physics equations [6–8, 12]. To fill this gap, this study not only tried to see the epistemological framing regarding a physics equation and the difficulties associated with it, but also designed interventions in order to help students understand deeply a physics equation by exploring different aspects of the equation. Thus, this study not only explored students' views and difficulties in making sense of mathematical representations in physics, but also offered perspectives to connect physics and mathematics by exploring different aspects that are not commonly presented in a teaching situation.

1.3 Research Questions

Mathematical representations in physics come in many forms. The main goal of this study is to investigate students' epistemological framing and difficulties when trying to make sense of physical phenomena represented mathematically. The WE is a case that can be explored in several aspects.

The mathematical representation of a wave is $y(x,t) = f(x \pm vt)$, and can be related to the WE. In physics, the wave function is mostly represented in graphs, and since y depends on x and t, the graph of this function can be represented as functions of $y(x)$ and $y(t)$. The dependence of vertical displacement with two variables provides a challenge in understanding it. Therefore, the first study tried to explore the situation where students were asked to find the velocity of the wave profile $y(x)$ which forced them to think about the time dimension in this problem.

After investigating students' difficulties, Study 1 also attempted to design an intervention that addresses the specific difficulties encountered by the students in order to help them better understand the problem presented. Therefore, Study 1 consisted of two research questions, as follows:

1. What are students' reasoning and difficulties related to the conceptual understanding of the graphical representation of waves?

2. What are appropriate interventions that help students understand the mathematical description of waves that always involve x and t?

Interpreting how students make sense of the WE is the goal of Study 2. The WE is represented by a second-order partial differential equation which can often be deceptively simple at first glance. However, understanding its meaning poses significant challenges. Therefore, Study 2 explored students' epistemological framing when they tried to interpret the WE.

Similarly to Study 1, an intervention that helps students better understand the WE was presented in Study 2. The intervention explored different aspects of the WE as a proposed strategy in this thesis in order to contribute to helping students view a physics equation differently. Therefore, the following two research questions were the core of this study:

3. What are students' reasoning and difficulties related to the physical meaning of the WE?

4. What are possible teaching strategies that help students acquire a deep understanding of the physical meaning behind the WE?

Lastly, this study also explored the conceptual subtleties from the perspective of the historical development of the WE as a potential teaching strategy to deepen students' understanding of the WE. Study 3 answered the following question:

5. What are the conceptual subtleties related to the WE that can help students understand the WE deeply?

1.4 Structure of thesis

This thesis is structured by three studies that address predetermined research questions. Chapters 2 and 3 are based on the three papers included in this study. Chapter 2 focuses on students' difficulties with the propagation of a pulse in a string and how they interpret the WE, as highlighted in the first and second paper. Meanwhile, Chapter 3 highlights some concepts related to the WE that might not be commonly presented in textbooks as a proposed strategy to deepen students' understanding of the WE. Finally, Chapter 4 is the concluding chapter of the thesis.

2 Interplay between physics and mathematics: investigating students' epistemological framing

Physics equations are built from mathematical formalism that explains the nature of the world. However, there is a clear distinction between the treatment of equations in physics and mathematics. For instance, Heck and van Buuren [13] explained that while mathematical formalism can be associated with a physical context, the use of variables in these two subjects depends on how they are treated. In mathematics, variables are commonly used to express unidentified components, while in physics, variables are used for elements that can be measured, quantified, and usually have a unit. Redish [14] provided an example to illustrate this point. He posed a question to both physicists and mathematicians:

$$\text{If } A(k, y) = K(x^2 + y^2), \text{K is a constant} \tag{1}$$

$$\text{What is } A(r, \theta) = ? \tag{2}$$

In physics, the answer is $A(r, \theta) = Kr^2$, where there are some factors that physicists need to consider. The simplest justification is the unit inside the bracket, which must be equivalent. Unlike physicists, mathematicians believed that the answer to that question is $A(r, \theta) = K(r^2 + \theta^2)$ due to the transformation of two arguments. This example highlights the crucial need to distinguish between the language of physics and mathematics.

Nevertheless, it is also important to emphasize that physics and mathematics cannot be separated, but this situation poses a challenge when one treats them in the same manner. A significant part of this investigation involves exploring the situation where two studies were conducted, resulting in Papers 1 and 2.

2.1 Students' epistemological framing about physics equations

Epistemological framing refers students' perspectives on relevant knowledge in a given situation [15]. In the context of physics education, students' attitudes, beliefs, and expectations play an important role in their success in the subject [16, 17]. Although different studies may use different terms to describe epistemological framing, they generally refer to beliefs about what constitutes knowledge in physics [18]. Redish et al. [16] used the term "cognitive expectation" to describe students' understanding of the process of learning physics and the structure of physics knowledge.

Several studies have investigated students' understanding of physics equations, and some findings suggest that students often hold similar beliefs in comprehending these equations. For example, in a recent study by Airey et al. [6], physics students from three different countries

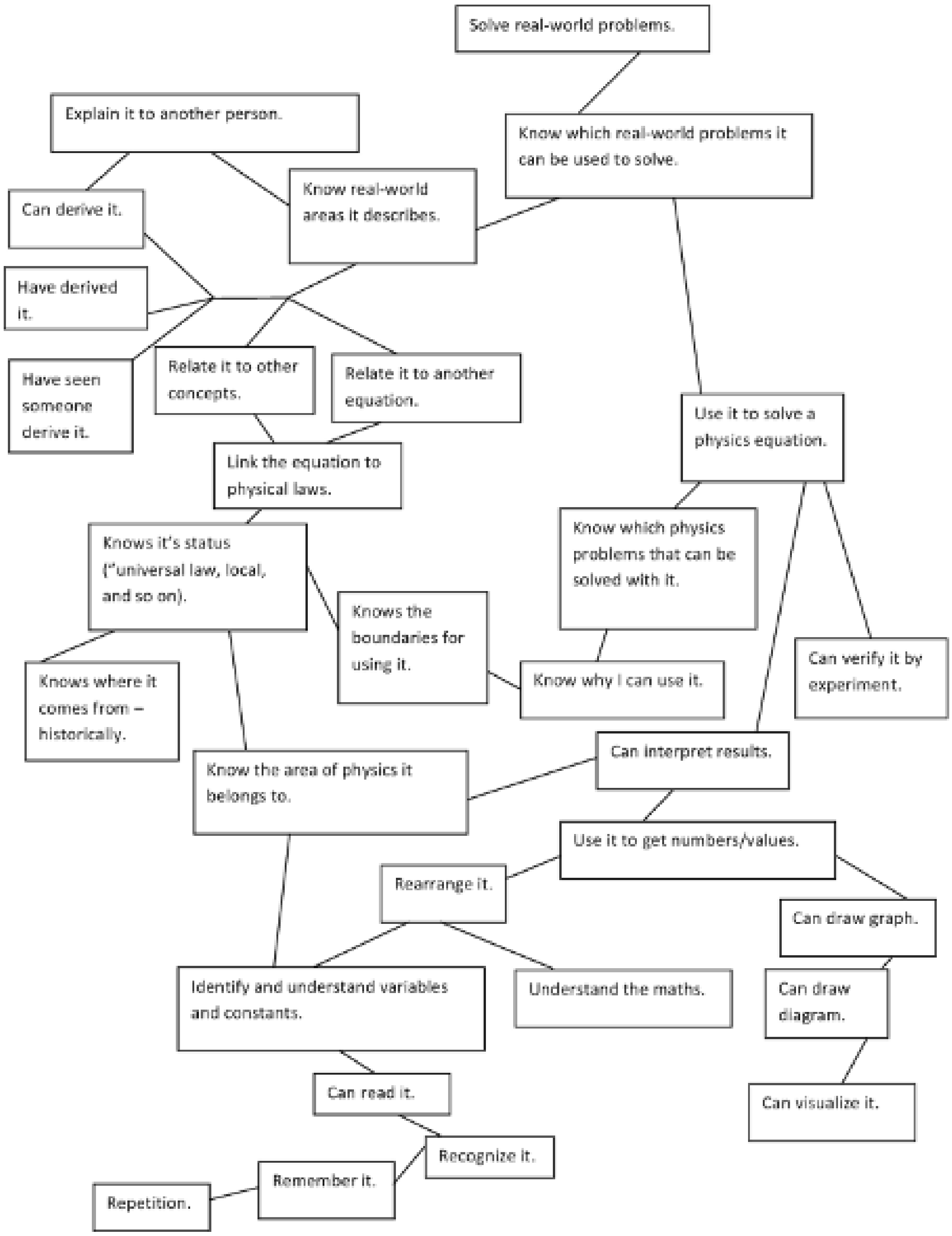

Figure 1: How students think if they understood a physics equation [6]

were asked to describe how they know when they understand a physics equation. The researchers then classified the students' responses in the first attempt from Swedish university students, which are presented in Figure 1.

From Figure 1, we can see the range of descriptors that students used in their responses. Although the design of this study is limited since it only asked students for a short answer to the question, the diagram provides insight into students' beliefs about understanding physics equations. The authors highlighted some simple descriptors, such as "remember it," "repetition," and "recognize it," as indicating the beliefs that the equation has been memorized. However, the descriptor "can visualize it" needs further exploration since it requires more elaboration.

Comparing the descriptors used by students in the US and Australia, the authors discovered

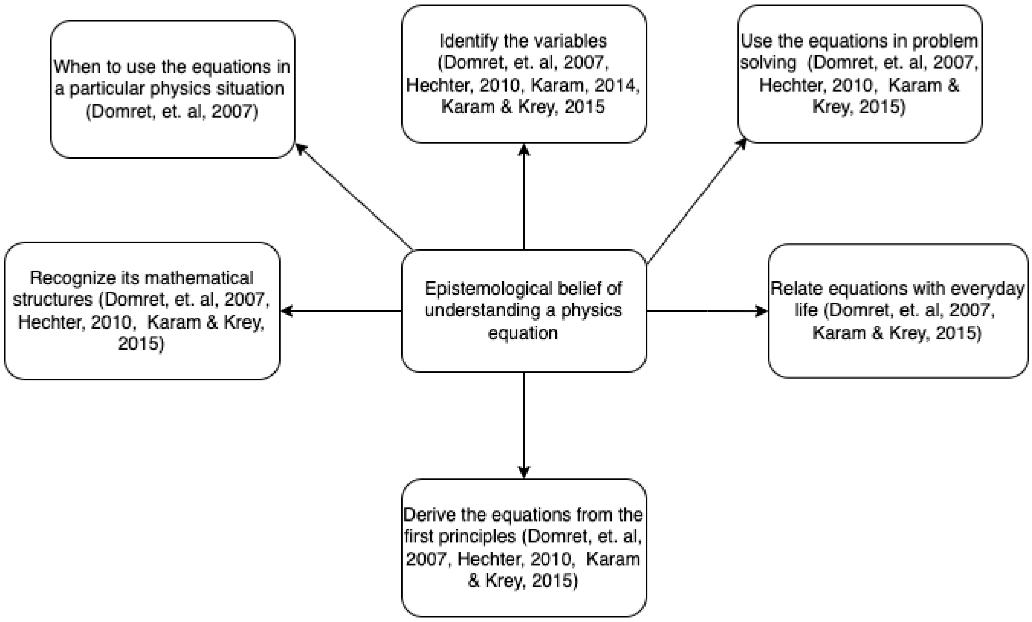

Figure 2: Students' epistemological framing of understanding physics equations based on literature

that the beliefs were similar and the categorization overlapped. Ultimately, they identified ten categories of students' beliefs about understanding a physics equation across three countries: significance, origin, description, prediction, parts, relationship, calculation, explanation, repetition, and memorization. These categories, which exclude repetition and memorization, were utilized to formulate questions that can be employed in teaching situations where students encounter physics equations. The aim of these questions is to assist students in directing their attention towards various aspects, thereby enhancing their comprehension of physics equations on a deeper level.

Other studies have also attempted to explore students' beliefs about understanding physics equations and to examine whether they align with experts' approaches. These studies have revealed various perspectives commonly observed among students [7–9, 12]. Using the findings of these studies, a diagram illustrating students' epistemological framing in relation to the understanding of physics equations has been compiled and is presented in Figure 2.

Karam and Krey [9] proposed four epistemological facets to comprehend equations, which are principles, definitions, empirical regularities, and derivation. They also stressed the significance of taking these categories into account when defining equations. The principles describe the facts that can be supported by observations and experiments; definitions describe the justifications to define physical quantities; empirical regularities are scientific phenomena that can be explained with repetitive experiments; and derivations explain how equations can be derived from principles or definitions.

The study investigated changes in the epistemological understanding of equations among pre-service physics teachers between pre-test and post-test, with an intervention conducted in between. The intervention consisted of a series of activities that targeted both epistemological and didactical aspects. These activities involved exploring equations from various perspectives,

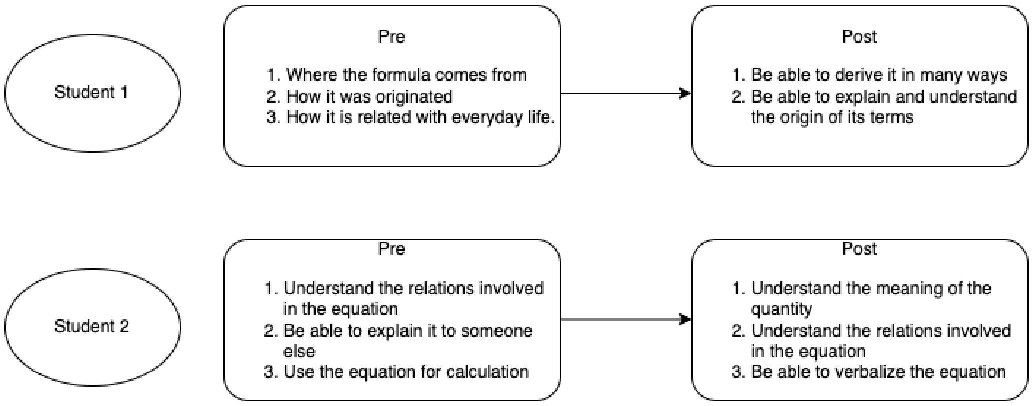

Figure 3: Students' epistemological changes after intervention [9]

including their derivation and historical development. The participants were enrolled in a specialized course aimed at fostering a deeper comprehension of equations through a curated set of tasks and prompts. Figure 3 depicts the epistemological changes observed in two students.

Although some students' responses were still persist, the results shows that the designed intervention led to more nuanced understanding of equations which also led them to change their views of physics equations.

2.2 Epistemological framing and learning difficulties

Some studies have shown that students' views on physics and learning can affect their understanding in certain ways. For example, epistemological framing has been linked to students' conceptual understanding [17,19–21]. Additionally, the way students approach learning physics, including their level of motivation, can also influence their academic achievement [17,22]. However, students' views about physics equations, which may differ from those of experts, can result in learning challenges.

One particular aspect explored in this thesis is students' interpretation of graphical representations of wave profiles. Within the field of PER, students' difficulties in utilizing graphs to solve physics problems have been investigated across various topics [23–27]. In the context of kinematics, McDermott et al. [27] identified certain challenges when students attempted to associate graph information with physical concepts. For instance, students often struggled to use the concept of slope to determine the velocity on a linear graph. Many of them decided to read the height of the line instead of seeing the line with a steeper slope. When presented with a curved graph, these difficulties were found to be even more pronounced.

Planinic, et. al [25] investigated students' interpretation of line graphs by presenting them with parallel questions in both mathematics and physics. The results discovered that students approached problem solving questions involving line graphs differently in physics compared to mathematics. Some students were able to correctly answer questions about slope in mathematics but struggled when the same concepts were applied in physics. These findings were further supported by their subsequent study, where students exhibited greater ease in answering questions related to graphs in the context of mathematics compared to when the same graphs were presented in a physics context [28].

In the study of wave phenomena, graphical representations are commonly used as important tools to visualize the characteristics and behaviors of waves. However, several issues related to wave graphs have been identified in previous studies [29–31]. Ambrose et al. [29] found that students faced difficulties in interpreting graphical representations of electromagnetic waves. One major mistake observed among students was the belief that magnetic fields exist only within a sine curve. Additionally, some students mistakenly assumed that points located on the x-axis corresponded to zero magnetic fields. Other studies discovered that students held misconceptions regarding the periodic nature of waves. For instance, students often prefer to use the concept of slope to transform a wave graph $y(x)$ into $v(x)$ [30]. Furthermore, students believed that the graphs of $y(x)$ and $y(t)$ represented the waveform [31].

Other studies have explored students' beliefs about wave phenomena, revealing certain conceptual challenges in understanding how waves behave. One such belief is the notion that waves can be treated as objects [31–33]. For example, Wittmann et al. [32] observed that university students tended to perceive sound waves as objects that physically push particles in the direction of wave motion. Other studies have found that students often associate waves solely with periodicity and struggle to grasp fundamental concepts [30–32, 34].

The use of mathematical language in physics is another aspect that was investigated in this thesis. Prior studies have already explored this issue by identifying students' epistemological framing when they believe they understand a physics equation [6–9, 12]. Once again, when students' beliefs contradict those of experts, learning difficulties can arise. For example, a simple equation such as $v = \lambda f$ can not only be treated mathematically without considering its physical relation. This study identified an epistemological stance where some students approached this equation purely from a mathematical perspective, disregarding its physical quantity relationships. They failed to consider that λ can only be manipulated by changing the source of waves or the characteristics of the medium [35]. Another study employed the term ”plug and chug,” which refers to an instrumental view where students identify an appropriate physics equation for a given problem and simply plug in the given values to obtain a solution [11]. While this strategy may yield correct answers, it does not necessarily foster a deep understanding of the underlying physics concepts involved in the problem.

In the context of problem solving situations, Walsh et al. [36] identified four distinct epistemological stances that students adopt in physics. Only a small number of students employed a scientific approach, aligning their problem-solving strategies with those of experts. The majority of students relied on a plug-and-chug approach, often in an unstructured manner. Some students adopted a memory-based approach, while others lacked a clear approach altogether. These findings indicate that many students tend to rely on memorization and mathematical procedures rather than cultivating a deeper conceptual understanding when tackling physics problems.

2.3 Methodology

The nature of the research in two studies (Study 1 and Study 2) presented in this PhD thesis is similar, as both involve qualitative research methodologies. These studies followed a common framework consisting of three main steps, although the design of the intervention differed between them. Figure 4 illustrates the design of both studies.

Exploration was the first step of data collection, where students' initial reasoning was explored to get a picture of their understanding related to the designed questionnaire. Subsequently, the interventions took place in order to help students overcome their learning difficulties. The last step was the evaluation, where the same questions from the exploration were administered again to the students. This phase aimed to assess the conceptual changes among the students after they received the interventions.

The two papers differ in terms of their contributions to Physics Education Research. Paper 1 investigated students' difficulties in understanding the propagation of a pulse in a string, filling a gap in the research by exploring students' understanding of non-periodic wave profiles. While studies in PER have examined students' difficulties with waves, few have explored this specific area [10, 35, 37–39]. Meanwhile, Paper 2 aimed to explore students' understanding of the 1-D WE, examining their epistemological framing and the difficulties they encountered with the equation. While previous studies have explored students' interpretation of physics equations and their epistemological framing [6–9, 12], Paper 2 aimed to encourage students to develop a deeper conceptual understanding of the equation by exploring various related aspects.

2.3.1 Interviews

All the data in paper 1 and 2 were collected and documented by performing think-aloud interviews with physics university students. The think-aloud interview protocol originated in psychology research, particularly in investigations of problem-solving processes [40, 41]. Nowadays, this approach is widely used across various fields, including PER [42–44]. The nature of open-ended interviews allows researchers to obtain detailed information from students and to

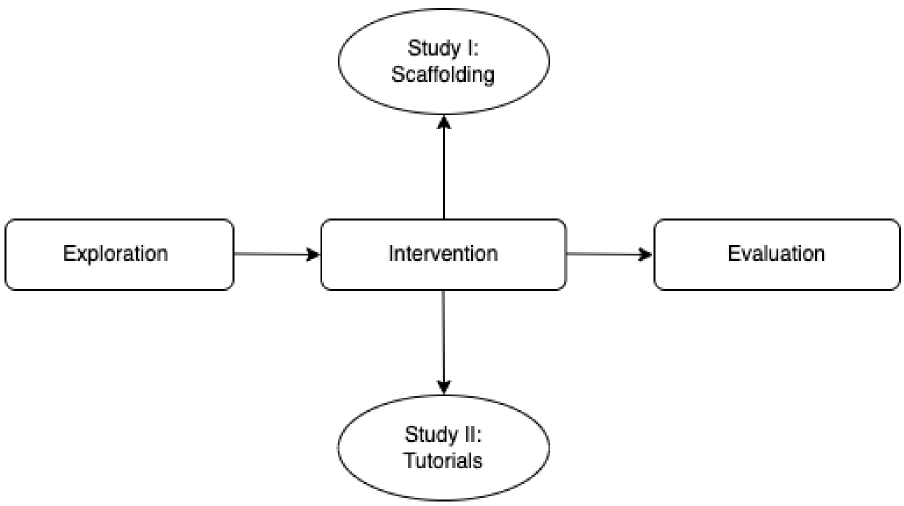

Figure 4: Three main steps of collecting data in Study 1 and 2

ask probing questions and dig deeper into the students' responses [45, 46].

The students were interviewed in pairs due to several reasons. One of the benefits of using paired interviews is the ability to gather a wider range of data, as students engage in discussions and exchange ideas with their peers. According to Houssart and Evans [47], paired interviews, also known as paired depth interviews, involve a researcher interviewing two individuals simultaneously. Previous studies that employed this method have identified advantages, particularly when the pairs have a pre-established relationship [48, 49]. This relationship facilitates openness between the participants, enabling them to complement each other's responses when one person encounters difficulties [47–50].

Furthermore, it is worth emphasizing that the data obtained from the paired interviews are richer when the students encounter the questions for the first time during the interview. This approach avoids fixed or rehearsed answers from the students, as they have not prepared their responses in advance. Houssart and Evans [47] refer to this as "unseen questions," which promotes collaboration among students and encourages them to consider alternative perspectives presented by their peers.

Due to the pandemic, the interviews for Study 1 and Study 2 were conducted remotely using Zoom meetings. Prior to the interviews, all students provided their consent for participation. The interviews were recorded using the embedded recording feature in Zoom, ensuring that the data were captured and stored for subsequent analysis. In instances where students produced drawings or visual representations during interviews, the interviewer requested that the students take a photo of their drawing and share it with the interviewer. The pictures were then shared within the Zoom meeting, allowing peers to provide comments and insights on the drawings.

Taking notes during the data collection process is essential. The interviewer's notes are a valuable addition to the captured data, providing insights and documenting key points raised by the students. By documenting interesting findings and observations during interviews, the interviewer can focus on specific areas for further analysis and exploration. The combination of recorded data and interviewer's notes improves overall data quality and provides a comprehensive understanding of student perspectives and experiences [51].

2.3.2 Data Analysis

The qualitative content analysis was applied to analyze the coded data. Content analysis is a method that enables researchers to analyze data from various sources, such as interviews [52]. While there are three approaches to qualitative content analysis, the conventional content analysis approach was employed in this study due to the nature of the data. According to Hsieh and Shannon [53], conventional content analysis with an open coding technique is more suitable for a study that uses observation as the initial phase, where coding and categorization are conducted during data analysis.

There are several steps involved in employing conventional content analysis [53]. For both Study 1 and Study 2, where data was obtained through interviews, the first step involved

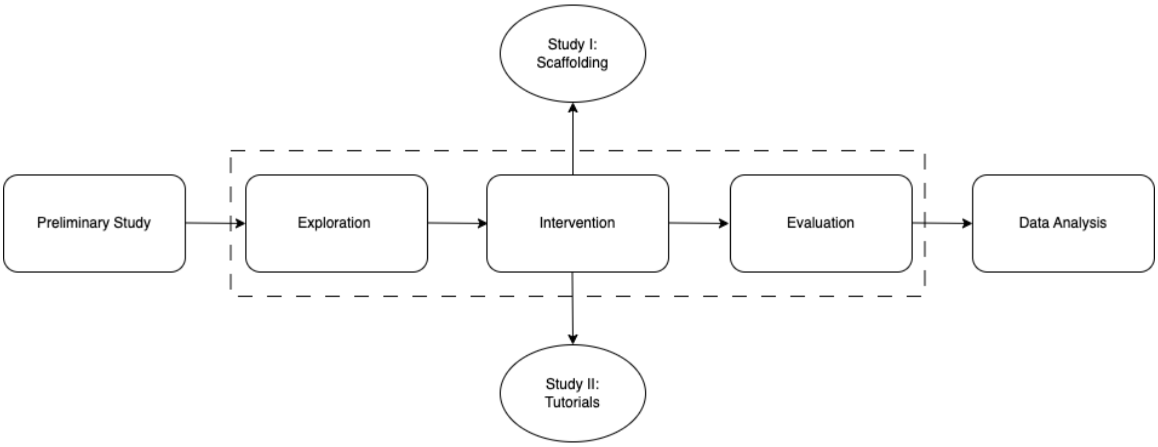

Figure 5: Research design for Study 1 and 2

transcribing the interviews. These transcriptions were then read multiple times to gain an overview of the data. During this reading process, important words and phrases were identified and highlighted to capture the key aspects of the students' reasoning. Additional notes were made as needed. From this analysis, initial codes were generated to capture the overall ideas expressed in the students' reasoning. To ensure the reliability and validity of the coding process, the data analysis continues with coding, where the students' responses to the questions are classified into themes or categories. Experts in the field are recommended to examine and validate the coding process, as suggested in studies by DiCicco-Bloom and Crabtree [54] and Fonteyn et al. [51].

After generating a list of initial codes, the next step is to sort those codes into potential categories/themes. Various techniques can be employed, such as using the tree diagram to aid in clustering [53]. The themes are then reviewed to ensure that the initial codes are placed appropriately in the correct categorization. This is an iterative process of reviewing and refining until a researcher arrives at a satisfactory map.

The themes must then be defined by describing the meaning of each categorization and the trends that these themes captured from the data. This process ensures a well-structured data set that helps researchers in reporting their results. The final step is to write the research report, which was done for both Study 1 and Study 2. Although multiple categorizations may have been identified in these studies, the reported results focused on the most relevant findings that aligned with the existing literature and addressed the research questions.

2.4 Research design

Study 1 and 2 were conducted in a similar research design. The difference was in the type of intervention implemented between the two studies. Figure 5 shows the research design in Study 1 and Study 2.

2.4.1 Preliminary Study

Study 1 A preliminary study was performed to identify possible problems and appropriate methods that will be used during data collection, including refining the research questions. Initially, the instrument was designed to assess students' understanding of the 1-D WE. Several questions were developed and administered to four physics students for a pilot study. During the interviews, it was discovered that students struggled to differentiate between the graphs shown in Figure 6.

In this phase, the results show that the graphical representation of the waves is actually challenging for students to understand because the mathematical description of the waves depends on two variables: x and t. Consequently, it was necessary to investigate this issue prior to assessing students' understanding of the WE, which might present an even greater level of difficulty.

In study 1, the attention was directed towards examining whether students can make sense of the horizontal movement of a pulse and the vertical motion of matter on a non-periodic wave profile. To achieve this, a wave profile $y(x)$ was specifically designed to investigate this matter. During this stage, an intervention was also developed to address the specific learning difficulties identified. The intervention was tested with a group of physics students to ensure its effectiveness before implementing it in the data collection phase. Based on the overall process, the following findings were observed:

1. The instrument designed in Study 1 proved to be extremely challenging for the students, even though they had previously completed a course on waves. As a result, for the purpose of data collection, it is essential that participants have received an advanced course in waves.
2. To effectively address the diverse range of difficulties experienced by students, the intervention should be designed with multiple levels of scaffolding. This approach enables the researcher to observe students' progress and challenges at each level. By doing this, the intervention can meet individual needs.

Study 2 The steps involved in the preliminary study for Study 2 followed a similar framework to Study 1. However, the exploration of materials and the development of instruments in

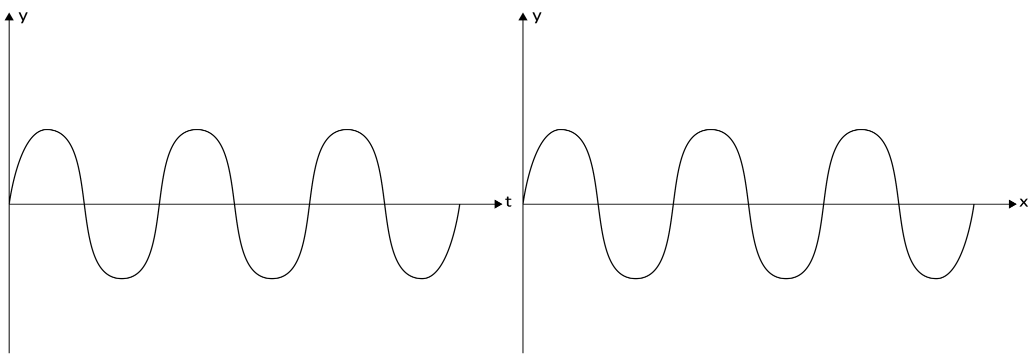

Figure 6: The graphical representation of wave

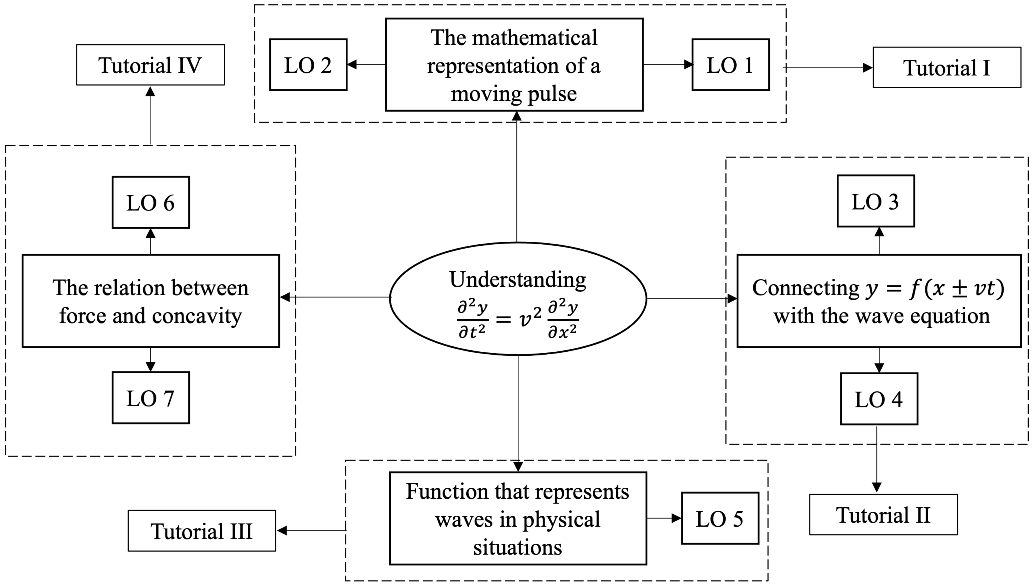

Figure 7: The proposal in study 2 regarding how students understand the 1-D WE

Study 2 posed challenges due to the limited number of studies that had previously explored students' understanding of the WE. In the initial phase, a questionnaire consisting of six questions was developed and administered to a group of physics students. Unlike typical problem-solving physics questions, the questions aimed to explore students' epistemological framing and alignment with expert views. The results obtained from the preliminary study in Study 2 are as follows:

1. The questionnaire were reduced into five, as one question aimed to assess students' understanding of the mathematical structure of the WE was complicated. This difficulty arose because not many students had previously encountered the transport equation.
2. None of the students recognized the mathematical structure of concavity.
3. Students continuously associated their answers with periodic waves, indicating a lack of understanding of non-periodic wave behavior.
4. In general, the students did not have a deep understanding of the WE and its underlying concepts.

Based on these results, interventions were designed to help students explore different aspects of the WE and eventually make them aware that understanding a physics equation is not a simple task. The interventions aimed to address students' difficulties identified in the pilot study, which included understanding the concavity of waves and the tendency to fixate on periodic waves. Therefore, six aspects were proposed to facilitate understanding of the WE. These aspects included the mathematical representation of a moving pulse, the connection between $y = f(x \pm vt)$ and the WE, the distinction between the transport and wave equation, identifying when functions represent a physical wave, the relationship between force and concavity, and differentiating between curvature and concavity. However, two aspects were later removed from the intervention, as they represented conceptual subtleties that would be presented in Study 3. Figure 7 illustrates the framework proposed to understand the WE in this study.

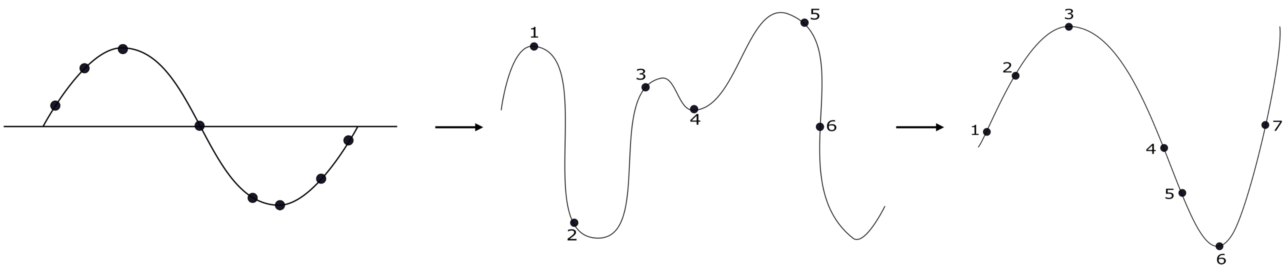

Figure 8: The changes of the wave profile in tutorial IV

Based on the four identified aspects of understanding the WE and the results of the preliminary study, interventions were designed to address different/uncommon aspects of the WE, with the purpose of helping students develop a deep understanding of the WE. Learning objectives (LO) were determined on the basis of these aspects, and tutorials were created accordingly to achieve these objectives. These tutorials were then tested and evaluated with some changes. This modification was designed to ensure that students understood that points located on a more concave profile experience greater force. Figure 8 illustrates the change in the wave profile following the second trial.

2.4.2 Data Collection

Exploration (Pre-test) During this stage, students' prior reasoning was examined by administering the designed questions from studies 1 and 2. In study 1, the question consisting of two items was administered to the students. One part of the question focused on determining the magnitude of the velocity, while the other part involved the velocity and direction of motion of specific points.

Question: A pulse is moving horizontally with constant speed to the right. The profile below represents a given instant, like a picture (Figure 9). As the pulse moves horizontally, the points move vertically (wave does not transfer matter)

(a) Based on the picture, sort the magnitudes of the (vertical) velocity at each point from the greatest to the smallest. Explain your reasons.

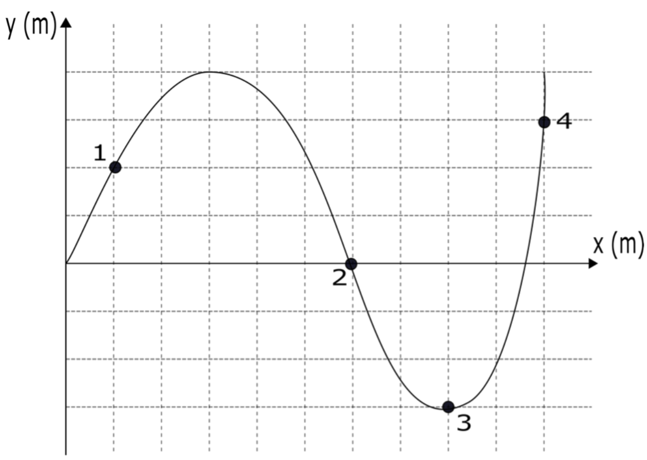

Figure 9: Problem graph

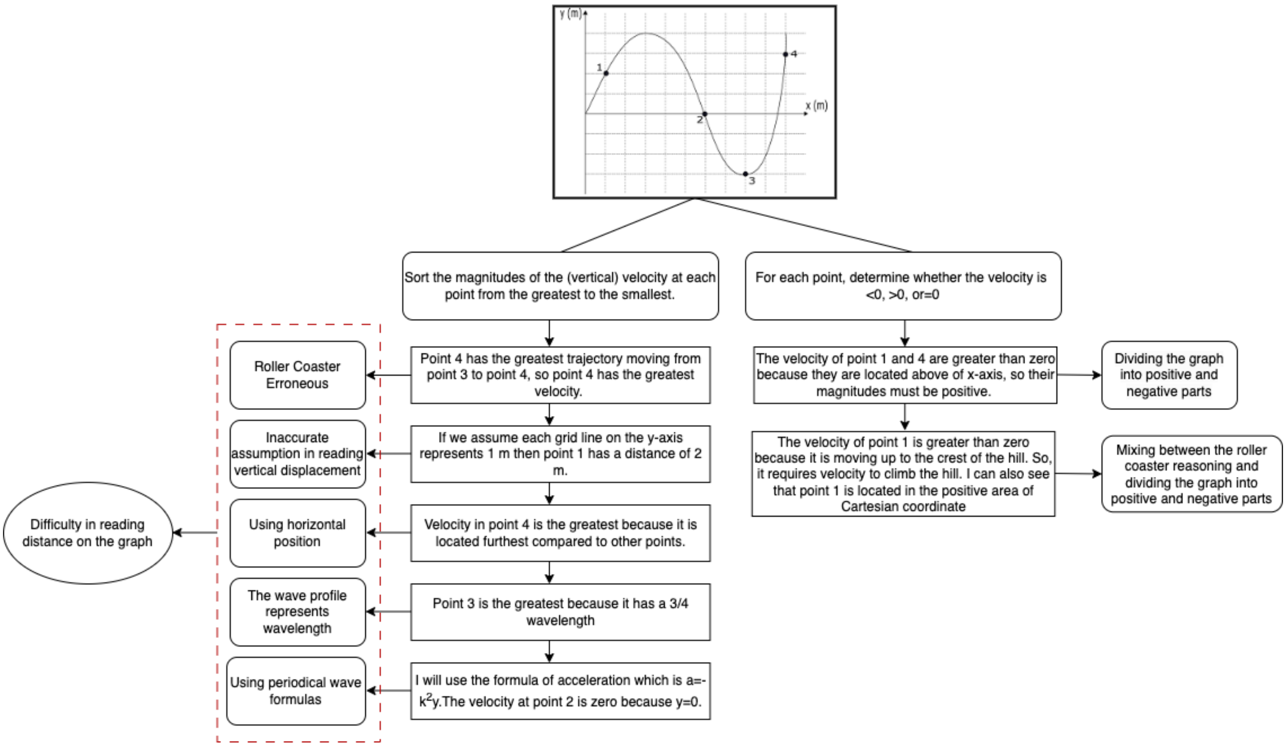

Figure 10: The content analysis with open code technique in study 1

(b) For each point, determine whether the velocity is $< 0, > 0, or = 0$. Explain your reasons.

Figure 10 illustrates how content analysis with an open-coding technique was applied to analyze students' reasoning in Study 1, providing a visual representation of the analytical process. It visually demonstrates that students often relied on inappropriate information from the graph when attempting to determine distances, highlighting a gap in their understanding. In this thesis, study 1 expands on previous research that has examined graphical representations and identified various challenges encountered by students [23–26]. Furthermore, this study revealed that students tend to fixate on periodicity and assume that waves always exhibit a periodic nature. This finding is consistent with previous studies that have reported similar observations [30, 31, 55].

In study 2, a questionnaire consisting of five question items was administered to the students to assess their understanding of the WE:

Questions: Below you see the 1-D WE consisting of a partial differential equation which describes a physical process.

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

1. If you were to explain the meaning of this equation with your own words (to a non-expert), how would you do that?
2. There are several terms in 1-D WE $(x, y, t, v, \partial^2 y/\partial t^2, \partial^2 y/\partial x^2)$. Indicate what those symbols refer to.
3. Based on question 2, indicate the units of those symbols.
4. Describe a physical situation represented by $\partial^2 y/\partial x^2 \propto \partial^2 y/\partial t^2$.

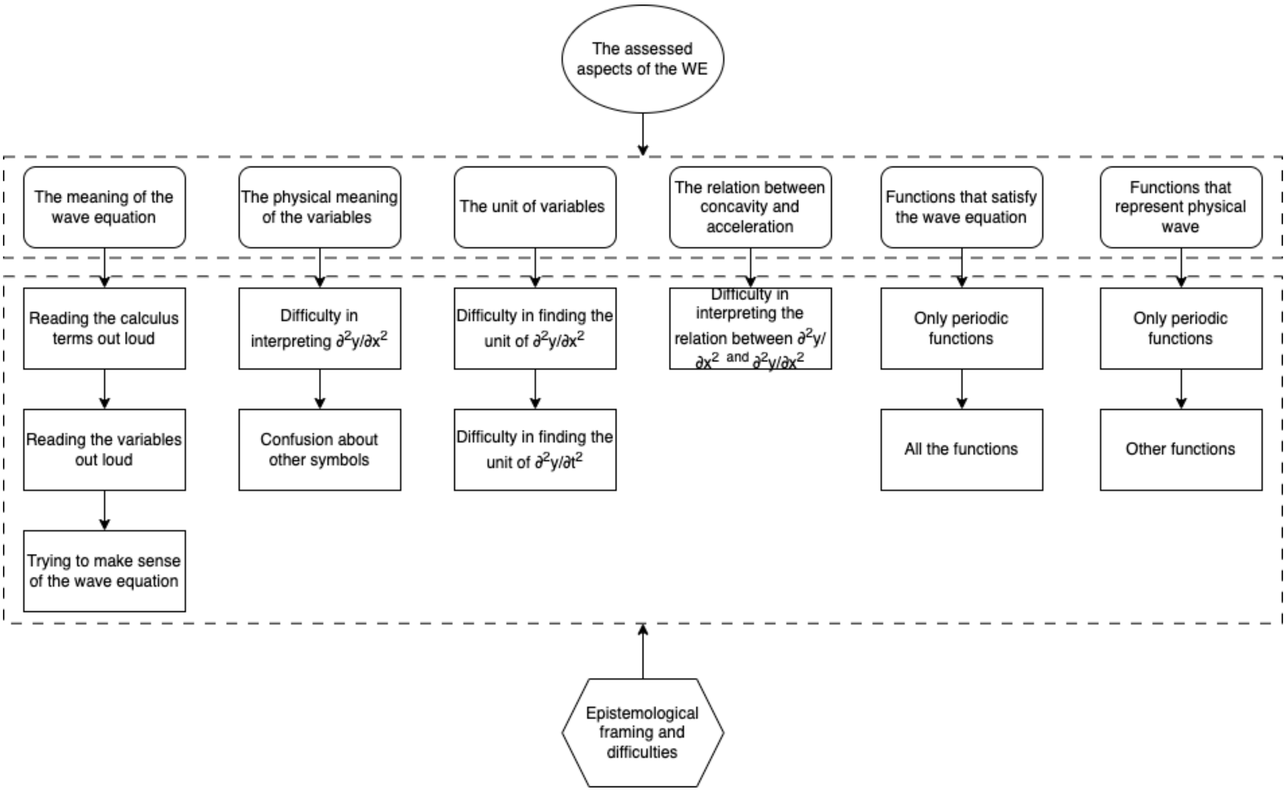

Figure 11: Students' epistemological framing and difficulties regarding the WE

5. Given the functions below, which ones satisfy the WE and which can represent a physical wave? (You can choose more than one). Explain your reason.
   (a) $y(x,t) = f(x + vt)$
   (b) $y(x,t) = A\sin(kx - \omega t)$
   (c) $y(x,t) = e^{-(x-vt)^2}$
   (d) $y(x,t) = 2A\sin(kx)\cos(\omega t)$
   (e) $y(x,t) = (x + vt)^2$

The results indicated that students' interpretations of the WE did not align with expert views, including some conceptual difficulties associated with its fundamental principles. Noticeably, based on one result in study 1 where students quickly used the periodic wave functions as their main reasoning to solve the problem, study 2 further explored these aspects to examine if students also used this approach to make sense of the WE. Figure 11 illustrates students' responses to the questionnaire, providing insights into their epistemological framing and the difficulties they encountered during the study.

The results reveal that students did not have a comprehensive understanding of the WE. Their epistemological framing suggests that they often perceived the WE as mere calculation schemes. For example, some students treated the WE as a combination of single variables, indicating the tendency to plot their magnitudes and obtain results, as the so-called plug-and-chug approach [11,15]. Other students responded to the question without grasping the physical meaning involved, simply reading the calculus terms without understanding the relationships between the variables [10, 37]. Several difficulties identified in Study 2 were also observed in

Study 1, including the tendency to fixate on periodic waves.

Intervention Intervention in the learning process refers to the support and guidance provided by educators or experts to enhance students' understanding and competencies. Effective interventions are guided by specific teaching goals aimed at improving students' ideas and understanding throughout the intervention [56]. This section will provide an overview of the interventions employed in Study 1 and Study 2, as well as the rationale behind their selection for each study.

Scaffolding Scaffolding is a term used to describe a process in which students are supported by teachers to solve problems that are beyond their current competencies [57]. According to Van de Pol et al. [58], scaffolding is a dynamic process that is highly dependent on the nature of the task and can be applied in various situations. Therefore, it is crucial to assess students' prior understanding before designing appropriate scaffolding strategies [59]. However, many teachers face challenges in designing scaffolding because they often directly provide support without first assessing students' understanding [58, 60, 61].

In Study 1, the difficulties faced by students regarding the propagation of a pulse in a string were explored, as discussed in Sec. 2.4.2. Scaffolding was selected as the intervention because it offers temporary solutions for students who are unable to solve problems using expert approaches [57]. Additionally, clear goals were set as important aspects when designing the scaffolding support [62]. The scaffolding was designed to guide the students in developing their own understanding at each level of the learning process. The difficulties encountered at each level were identified and interventions were provided at each level to gradually guide the students to draw the wave profile after some time has elapsed. This approach also allowed the researcher to monitor the students' progress in improving their understanding at each level.

Although there is no consensus, Van de Pol et al. [58] summarized three common characteristics of scaffolding. The first characteristic is contingency, where the supports must be adjusted at the same or slightly higher level than the students' existing understanding. The second characteristic is fading, which involves gradually reducing the amount of support provided. This characteristic is related to the third characteristic named responsibility, where students are expected to take more and more control over their understanding. It is also important to check students' understanding after giving them support in scaffolding [63]. Scaffolding supports are meant not only to help students reduce their learning difficulties, but also to diagnose how students' new understanding has developed in the end. One way to do this is by asking the same questions that were given to see if they changed their prior reasoning.

Tutorials Tutorials were designed based on three stages: elicit, confront, and resolve, as proposed by the physics education research group at the University of Washington [64, 65]. The elicit stage involves exposing students to problems that explore their learning difficulties. These difficulties are then addressed by making students aware that their reasoning is not aligned

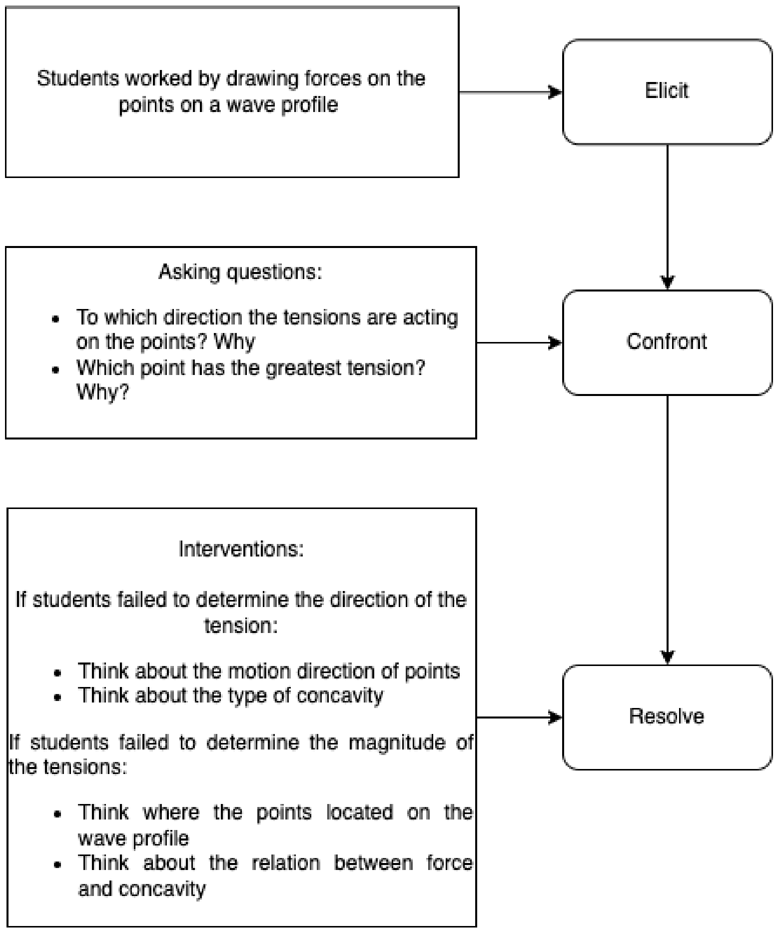

Figure 12: The activities of tutorial 4 was designed based on three stages: elicit, confront, and resolve

with a correct physical situation. Finally, students are guided through necessary interventions to resolve any inconsistencies.

In general, the tutorials in study two were designed based on the stages in which the first activity in each tutorial involved students working and exploring their understanding independently. Subsequently, questions were posed to trigger conceptual conflicts, prompting students to reflect on their initial answers. Finally, additional interventions were provided to students who continued to struggle in order to help them reach the learning goals of each tutorial. Figure 12 provides an example of the activities in Tutorial 4, demonstrating how these three stages were incorporated.

In addition, the interventions implemented in the tutorials were inspired by the conceptual blending framework. This approach was chosen due to the complex nature of the WE, which is presented in second-order PDE and presents challenges in terms of delivering its meaning. The framework employed aimed to assist students in blending the mathematical representation within the equation with meaningful physical situations. Graphical representations were also used extensively throughout the tutorials, as they have been found to facilitate the integration of physics and mathematics, as demonstrated in this study [66].

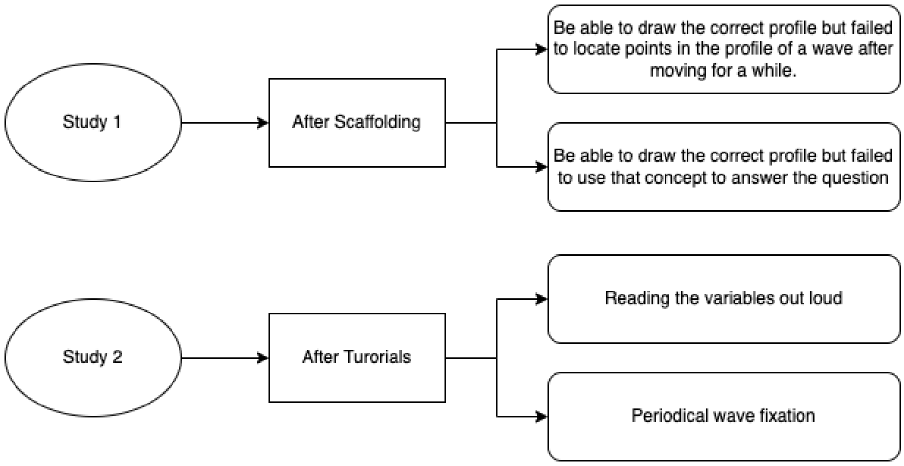

Figure 13: Students' difficulties and epistemological framing after interventions

Evaluation (Post-test) In both Study 1 and Study 2, the same questionnaires used in the pre-test were administered again in the post-test to evaluate students' improvement of the topics and the effectiveness of the interventions. In Study 1, the post-test was given directly after students managed to draw the correct wave profile. In Study 2, the same questions were administered again to the students one month after completing the last tutorials to determine whether the intervention had a lasting impact on their understanding or if they had quickly forgotten the physical intuition behind the interventions.

The results from both studies suggest that the designed interventions contributed to improving students' understanding. While not all students were able to provide comprehensive responses and a few did not demonstrate improvement in their understanding during the evaluation, the overall findings highlight the positive impact of the interventions. Despite these positive results, certain difficulties persisted even after the interventions. Figure 13 illustrates that some students continued to adhere to their prior reasoning or shifted to other incorrect reasoning patterns following the interventions.

Despite the use of different levels of scaffolding, Lin and Singh [62] found that a persistent alternative conception among students was the assumption that static friction is always equal to its maximum value, $\mu_s F_N$. Similarly, in Study 1 and Study 2, although the interventions provided to the students helped them address their learning difficulties, they still struggled to utilize this knowledge effectively in the evaluation. For instance, in Study 1, not all students were able to apply the support provided and change their prior reasoning when faced with the same question again. This observation applies even to the group of students who successfully generated the correct wave profile, which was the main goal of the scaffolding design. These findings demonstrate the inherent challenge of designing scaffolding that is suitable for all students [62, 67]. In Study 2, it was observed that while a few students achieved the learning goals within the tutorials, they still struggled to answer the questions on the questionnaire correctly and some of their responses lacked physical meaning.

One potential factor contributing to students' persistence in incorrect reasoning despite the intervention could be their prior knowledge and skills. Lin and Singh [62] conducted a study that revealed the effectiveness of a particular intervention in a calculus-based physics course compared to an algebra-based course. However, in the present studies, all participants shared a similar educational background. They had completed introductory physics courses in their first year and advanced courses on waves in subsequent years. Therefore, it can be inferred that the students had similar levels of skill and knowledge in the context of this study.

One recommendation of this study suggested that designing effective interventions is an ongoing process. Lin and Singh [62] proposed three strategies to assist students with strong alternative conceptions. First, interventions should focus on directing students' attention to the learning difficulties in a more detailed manner. One approach could involve activities that explicitly invoke alternative conceptions, creating cognitive conflicts that challenge their existing beliefs. Secondly, it is crucial to provide students with a variety of examples and situations within the intervention that highlight the conflicts associated with their alternative conceptions. This exposure encourages students to actively construct and restructure their understanding of the topic. Lastly, it may be advantageous to prompt students to generate new cases or situations related to the physics concepts covered in the intervention. This approach encourages students to apply their revised understanding to novel contexts, further reinforcing their learning and facilitating a deeper grasp of the subject matter.

3 Exploring the complexity of the wave equation

The 1-D WE may appear simple compared to more complex physics equations such as the Schrödinger equation. However, the exploration of the WE in Study 1 and Study 2 reveals that understanding it is not a trivial matter. This chapter describes some issue related to the complexity of the WE and suggests that exploring different aspects of a physics equation can contribute to a thorough understanding.

3.1 Force and Curvature

In 1715, Brook Taylor published a book called "Methodus Incrementorum Directa et Inversa" and made several assumptions related to the vibrating string. One of these assumptions stated that at each point of the string, the vertical force is proportional to the curvature. This assumption was the only one identified by D'Alembert after deriving the wave equation with the assumption of small vibrations of a string [68,69]. Taylor's work provided an early geometric derivation that gives an idea of the relationship between the tension acting at a point of the string and the curvature at that point.

In derivations of the WE, it is common to analyze only a small segment of the string. Analyzing the small segment of the string enables the use of linear approximation, allowing for the application of Newton's law. Figure 14 illustrates the small segment of the string that needs to be considered to derive the WE.

The WE can also be derived using the assumption of curvature $\approx$ concavity. This assumption arises from approximating that dy/dx is very small. However, it is important to note that the proportionality between force and curvature is the correct relation. The concavity appears to simplify the derivation process. Assuming that force is proportional to concavity gives a reasonable approximation and facilitates the mathematical analysis of wave behavior.

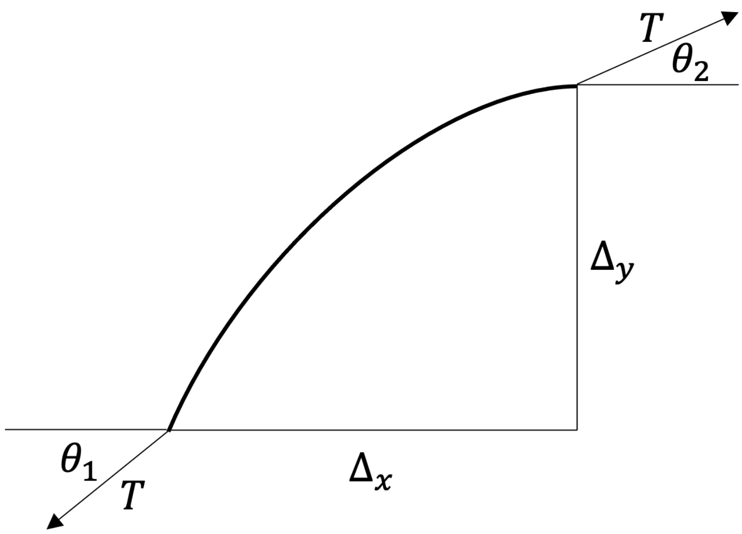

Figure 14: A small segment of the string to derive the WE

3.2 Convolutions of the wave function: The vibrating string controversy

Since D'Alembert proposed the partial differential equation that became the equation of a vibrating string, a controversy arose among four prominent figures during that time. In their paper, Gerald and William [70] depicted the debate over the solution of the WE. They clearly demonstrated that the meaning of $y(x,t)$ was the central issue of the debate.

D'Alembert considered a stretched string fixed at both ends and used Newton's second law until he finally arrived at the WE. He then proposed a general solution of the PDE that he found as:

$$y(x,t) = f(x+t) + g(x-t) \tag{3}$$

and applying the boundary condition $y(0,t) = y(L,t)$, he found this final solution:

$$y(x,t) = f(x+t) + f(x-t) \tag{4}$$

Euler was one of the first to enter the dispute, and his analysis of the vibrating string is not significantly different from that of D'Alembert, except for the interpretation of the function f. Euler proposed that the interpretation of y can be deduced from a plucked string, as shown in Figure 15. Using D'Alembert's solution to the WE, Euler argued that the initial shape of the string would determine its subsequent motion and shape [71]. Wilson [72] referred to this as a "thought experiment" when Euler attempted to impose his physical interpretation on the solution of the WE.

Figure 16 illustrates the physical interpretation of Euler for a plucked string [73]. In his model, after the initial condition, the string splits into two halves and moves in opposite directions, forming the kinks at B and C (Figure 16b). According to his interpretation, OB and Cπ remain stationary after the initial disturbance, while the BC segment, which represents the moving part of the string, moves at a constant speed (Figure 16c). Euler believed that this

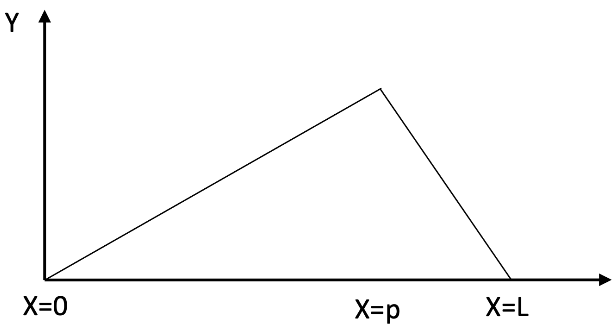

Figure 15: Euler's plucked string.

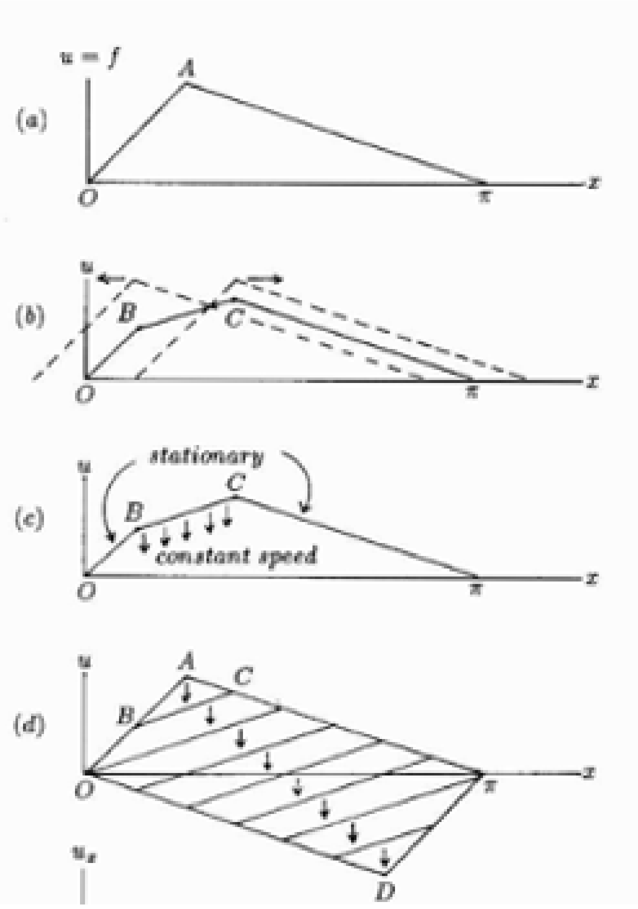

Figure 16: Euler's physical interpretation of a plucked string [73].

segment vibrates like a wave.

A complete downward cycle occurs when B and C steadily move down along the upper edges of the parallelogram OAπ D (Figure 16d) until it reaches D. After reaching D, the direction changes, and it starts moving back towards point A. This interpretation includes both upward and downward movements associated with a vibrating string.

Later, D'Alembert objected to this argument, saying that if the shape of the string is not smooth, then $\partial^2 y/\partial x^2$ will not be equal to $(1/c)(\partial^2 y/\partial t^2)$ [71]. The force on the plucked string must be concentrated at the corners, and the WE can be applied in this situation [72].

Bernoulli proposed that $y(x, t)$ is the sum of harmonic functions to give his solution a more physical interpretation. However, this argument was disputed by D'Alembert, who claimed that the vibrating string consists only of one frequency. Euler also disagreed with Bernoulli's solution, stating that it was too specific and could not be generalized to other types of waves. Lagrange then entered the debate and proposed a completely new approach to solving the problem of the vibrating string without relying on the WE. He built his interpretation by imagining a string consisting of infinite points connected to each other. His solution of the wave function gives a series of normal modes that consist of sine and cosine functions.

3.3 The wave function

Waves are mathematically expressed by functions of $y(x,t) = f(x \pm vt)$. Therefore, any function of this form is a wave function. At first glance, this function seems disconnected from the WE. Connecting $y(x,t) = f(x \pm vt)$ to the WE is quite simple by taking the second derivative of that function with respect to x and t. Letting $x \pm vt = u$ and performing a double derivation with respect to x results in:

$$\frac{\partial y}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du} \tag{5}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2}. \tag{6}$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = \pm v\frac{df}{du} \tag{7}$$

$$\frac{\partial^2 y}{\partial t^2} = (\pm v^2)\frac{d^2 f}{du^2}. \tag{8}$$

Comparing (2) and (4),

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}, \tag{9}$$

which is the 1-D WE.

3.4 Transport Equation

The transport equation, which is a first-order partial differential equation, describes the movement of a wave through a medium with constant velocity. It is depicted as follows:

$$\frac{\partial y}{\partial t} = \pm v\frac{\partial y}{\partial x} \tag{10}$$

In fact, the transport equation can actually explain some limited wave phenomena. Despite its limitations, emphasizing this equation in teaching waves might be essential to provide deep understanding to learners. However, many physics textbooks do not present the transport equation, which could be reasonable due to the general consensus on the mathematical representation of waves.

Presenting the transport equation may answer the question ”Why is the WE presented in the second-order partial differential equation (PDE) and not the first order?” This is one unnoticed aspect that could be beneficial to implement when teaching the WE in order to encourage a deeper understanding of it.

3.5 Conceptual subtleties

Numerous research in physics education explored students' difficulties in various physics concepts, including waves [31, 32, 35, 37, 55, 57, 74]. The third paper in this thesis aims to propose an epistemological dimension of teaching by incorporating less commonly encountered concepts related to the WE. These concepts might offer valuable insights and facilitate a deeper understanding of the WE for learners.

Based on the previous sections in this chapter, study 3 highlights three key conceptual subtleties that could be beneficial for teaching the WE. The first concept focuses on the mathematical representation of waves. As also discussed in studies 1 and 2, the mathematical structure of the wave function may appear simple, but understanding its deep meaning poses challenges.

The second aspect focuses on differentiating between the transport equation and the wave equation. This exploration aims to elaborate why the WE is expressed as a second-order partial derivative, while the transport equation falls short in capturing the complexities of wave phenomena. Lastly, the third aspect explores the relationship between $\partial^2 y/\partial x^2$ and $\partial^2 y/\partial t^2$. Understanding this relationship is crucial to understanding the WE and the associated physical properties. Conceptually, one can realize that the force acting on points of the wave is proportional to its curvature and get the WE from that.

In addition, the third paper also touches on the historical episode of the vibrating string. It highlights Taylor's original proposal of the relationship between the curve on a vibrating string and the net force acting on it through geometric analysis. This relation was later used by D'Alembert, who simplified the mathematical formalism by assuming small slope values, leading to the derivation of the WE.

4 Conclusion

This PhD thesis aimed to investigate the interplay between physics and mathematics, using the 1-D WE as a case study. The thesis, along with three accompanying papers, addressed the research questions at hand. The first guiding question focused on examining how students establish connections between graphical representations and the physical interpretation of wave profiles. Additionally, the thesis explored students' epistemological framing and the challenges they encountered in relation to the 1-D WE. Finally, the thesis presented conceptual subtleties with the aim of highlighting uncommon topics associated with the WE.

If we analyze the relationship between the three studies, Figure 17 illustrates their connection in addressing the research questions posed in this thesis.

4.1 Study 1

The use of graphs as a mathematical representation in wave phenomena has been found challenging for students, especially when it comes to applying them to particular physical situations. Study 1 demonstrated that problems involving graphical representations of waves, which include variables x and t, are extremely difficult for students.

Before the intervention, no student was able to correctly answer the questions. Many students had difficulty extracting relevant information from the graph to solve the problem, and some fell into the trap of focusing merely on periodicity. Students' reasoning did not align with the expert approach, despite being aware that points on the wave only move vertically.

After different levels of scaffolding were implemented, students' performance improved and they were able to solve the problem correctly. However, some students still persisted in using their prior reasoning, even though they were able to draw the correct wave profile, which was the goal of the scaffolding design. This suggests that the scaffolding did not address all the difficulties faced by students.

Nevertheless, Study 1 highlights that students encountered challenges when dealing with the uncommon graphical representation of the wave function. The results show that students' epistemological framing can hinder their ability to see beyond those beliefs.

4.2 Study 2

Study 2 offers a unique perspective on teaching physics equations, specifically focusing on the 1-D WE. While previous studies have primarily investigated students' epistemological framing and difficulties in understanding the WE, this study goes beyond by actively engaging students in exploring different perspectives of the equation.

The pre-test results indicate that students primarily perceive the WE as a mathematical tool, they often read the calculus terms and variables aloud. Consequently, their responses lack a deeper physical meaning.

After the interventions, although it cannot be concluded that all students achieved a deep understanding of the WE in the post-test, as many of them reproduced concepts presented in the tutorials, the intention was to introduce students to alternative ways of thinking about the equation. By equipping them with these new perspectives, the aim was to provide them with valuable tools that could be applied to other physics equations that they encounter in the

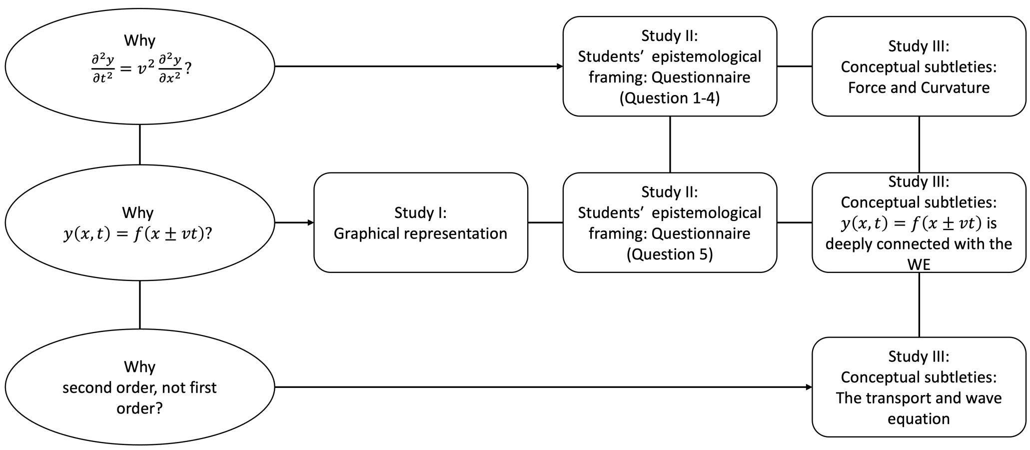

Figure 17: The connection of three studies of the thesis

future. The findings from this study suggest that such an approach is not only feasible but also holds potential for extension to teach other equations and topics in physics education.

4.3 Study 3

In Study 2, it has been shown that understanding the WE poses a significant challenge. Study 3 identified certain concepts associated with the WE that are not commonly addressed in physics textbooks or classrooms to propose ways to understand the WE more deeply.

One of the key concepts explored in this study is the difference between transport and WE. Understanding this distinction enables students to grasp the fundamental idea of a moving pulse and why the WE is represented by a second-order partial derivative. Additionally, students should recognize that WE can be derived from the relationship between force and concavity. However, the derivation of the WE using this relationship fails when one explores deeper the distinction between concavity and curvature. Hence, it is essential to understand that the original relationship proposed by Taylor in 1715 was not between force and concavity but rather between force and curvature.

5 Perspectives for future research

This PhD thesis highlights two cases of interplay between physics and mathematics. The first case involves understanding the graphical representation of waves to determine the velocities of points on a wave profile. It was found that students struggled to use the appropriate information on the graph to find the vertical velocity of points. The second case involves understanding the wave equation (WE), which consists of a second-order partial differential equation. The results also indicated that students lack a proper understanding of the physical meaning associated with the epistemological framing of this equation.

The interventions were designed to help students integrate mathematical representations and physical situations. Although students' performance improved during the study, a few still struggled with learning difficulties after the interventions. Future investigations could consider several aspects. First, the nature of studies 1 and 2 is qualitative. It would be worthwhile to replicate these studies with larger samples so that quantitative data can be obtained. This thesis also laid the foundation for understanding students' difficulties and epistemological framing when interpreting a non-periodic wave profile, as well as their perspectives when making sense of the WE. This is a crucial aspect for developing future interventions related to these topics. As we discovered in studies 1 and 2, the designed interventions did not fully meet the needs of all participants, with a few still failing to provide correct answers and persisting in using their prior reasoning. Thus, in a teaching situation, students' responses to the problems and the designed interventions can serve as valuable tools for researchers and instructors interested in these topics, enabling them to refine and adapt them.

Finally, there are several aspects that could potentially influence students' performances who participated in this study. The curriculum of their universities could be one such factor. Study 3 offers an approach to embed this epistemological element within teaching. The number of studies that implement conceptual subtleties in physics classes and examine student responses is limited. Hence, more research on this topic could be a necessary component that requires future evaluation.

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Conceptual challenges with the graphical representation of the propagation of a pulse in a string

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Conceptual challenges with the graphical representation of the propagation of a pulse in a string

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Student difficulties with making sense of graphs in physics have been thoroughly reported. In the study of one-dimensional waves, the issue is even trickier since the amplitude is a function of two variables (position and time). In this work, we investigate students' reasoning and difficulties with interpreting the graphical representation of the propagation of a pulse in a string. A profile $y(x)$ of the pulse was provided and students were asked to estimate the velocities of several points at the profile. This forced them to consider the time dimension, by focusing their attention on the motion of these points. This turned out to be extremely challenging to the students, who manifested several conceptual challenges which were categorized and analyzed in the first phase of the study. Based on these findings, three levels of scaffolding support were provided, where each level gradually guided the students to draw the wave profile after some time has elapsed. The scaffolding turned out to be effective, since many students managed to identify the new positions of the points successfully. The study reveals how static representations of intrinsically dynamic phenomena can be challenging for students to grasp.

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I. INTRODUCTION

Graphical representations are widely used as powerful tools to represent concepts and phenomena in physics. In fact, the lack of understanding of graphical representations is often an issue of concern to physics education research (PER). The literature is vast in both the identification of misconceptions and the development of instructional strategies to circumvent them in a variety of topics such as kinematics, thermodynamics, and electrodynamics [1–4].

In wave phenomena, graphical representations are particularly challenging because the mathematical description of waves involves a function of two variables, position x and time t. In the one-dimensional case, this function is generally expressed as $y(x,t) = f(x \pm vt)$, which is not always treated in mathematics lessons and is quite difficult to grasp. Although one can choose to represent the dependence of the vertical displacement y on each of the variables x and t separately, it is crucial to understand that they are related. The function $y(t)$ describes the movement of a given particle (fixed x) when time is progressing, whereas the function $y(x)$ describes an instantaneous configuration of a wave, like a screenshot. Investigating how students try to make sense of these conceptual subtleties is the main goal of this study.

The PER literature is also comprehensive in terms of studies investigating student difficulties with wave phenomena. For example, Sadler *et al.* [5] found that students struggled to distinguish between vertical particle motion and horizontal wave propagation. For the case of transverse waves, students often concluded that matter was transported in the direction of wave propagation. Similar findings also showed that most of the students believed the particles in the air were pushed together towards the direction of motion when a sound wave is traveling [6,7]. These misconceptions occurred because students tend to treat waves as objects and use that reasoning to solve problems [6–9]. Furthermore, some students struggled to distinguish between a mathematical representation and a physical situation, e.g., most students treat the relation between velocity, wavelength, and frequency of periodic waves $v = \lambda f$ mathematically without considering how each variable is related physically [8,10].

In this paper, we explore how university physics students understand graphical representations of waves in a manner which goes beyond other studies in the literature [1–4,11,12]. More specifically, the topic of this study differs from previous ones because most of them investigated students' reasoning in the context of periodic waves [8,10,13–15]. Here, we focus on students' ability to distinguish between the horizontal movement of a pulse and the vertical motion of matter on a nonperiodical wave

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profile. In particular, we provide students with the profile [$y(x)$] of a pulse and analyze whether they can estimate the velocity of some points on the graph, therefore asking them to reason about time evolution. After assessing students' conceptions, three increasingly detailed scaffoldings were provided to see if or how they improved students' performances when solving the task.

II. PHASE I: STUDENTS' REASONING ABOUT THE GRAPHICAL REPRESENTATION OF A PULSE

A. Materials

Our investigation is based on one conceptual question, designed to explore students' understanding of the relationship between the vertical motion of the points on a string, with the (horizontal) propagation of a pulse with constant velocity in this string. Four points were located on a pulse and students were asked to (a) sort out the magnitude of their velocity, and (b) estimate whether the velocity of each point is > 0, < 0, or $= 0$.

Question: *A pulse is moving horizontally with constant speed to the right. The profile below represents a given instant, like a picture (Fig. 1). As the pulse moves horizontally, the points move vertically (wave does not transfer matter)*

(a) Based on the picture, sort the magnitudes of the (vertical) velocity at each point from the greatest to the smallest. Explain your reasons.
(b) For each point, determine whether the velocity is $< 0, > 0$, or $= 0$. Explain your reasons.

From the expert perspective, one way to solve this question is to draw another profile after some time has elapsed. Figure 2 shows two wave profiles at two different instants.

In Fig. 2, the dotted profile represents the pulse after a short time interval and the red dots show the new vertical positions of the points. It can be seen that point 4 has the greatest speed because it has the greatest displacement

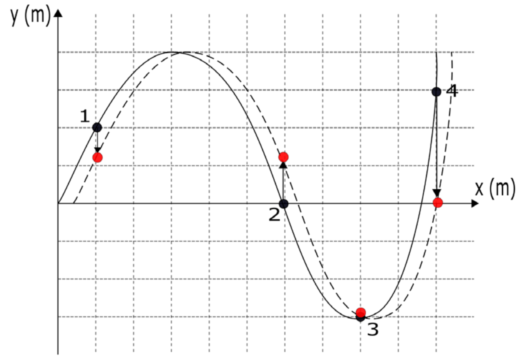

FIG. 2. A graph of two wave profiles at two instants.

compared to other points. Therefore, point 4 requires a greater speed to attain the final position. With this justification, the correct answer for question (a) is 4–2–1–3.

To answer question (b), one needs to see the motion direction of each point. The downward displacement means a negative velocity ($v < 0$) and the upward displacement means a positive velocity ($v > 0$). Using this approach, however, the challenge to answer the velocity for point 3 is inevitable because the point has already moved upward in the new profile. This is because the question asks about the velocity in the initial profile, not when the pulse move after some time has elapsed. In this case, point 3 has zero velocity. Thus, the correct answer for question (b) is that velocities of points 1 and 4 are smaller than zero, the velocity of point 2 is greater than zero, and of point 3 is equal to zero.

In fact, the expert can solve this question (a) by only drawing the slope of each point on the graph to find the speed. However, it is worth saying that the slope of $y(x)$ cannot be treated to find the velocity without knowing the relation between the slopes of $y(x)$ and $y(t)$. The slope of $y(x)$ only addresses the shape of the pulse, but indeed there is a proportionality between the slope of $y(x)$ and $y(t)$. Thus, one can infer the velocity based on the slope of each point on the graph of $y(x)$ using this relation. Figure 3 shows the slope of each point on the graph.

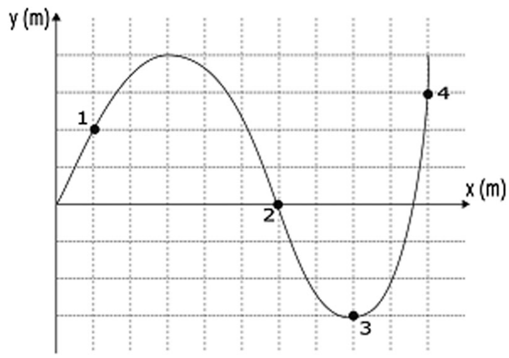

FIG. 1. Problem graph.

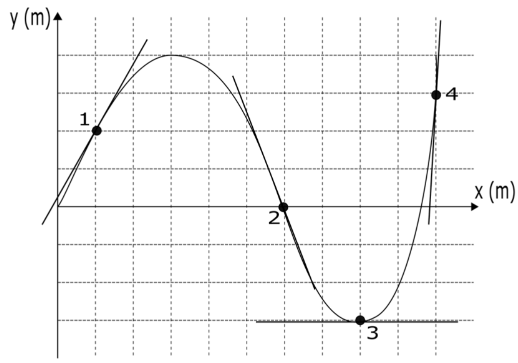

FIG. 3. Solving question (a) by drawing slope.

TABLE I. Students' conceptual challenges in phase I.

Question	Conceptual challenge	Category	Number of students
Part (a)	Difficulty in reading distance on the graph	The roller coaster erroneous reasoning	6
		Inaccurate assumption in reading vertical displacement	6
		Using horizontal position	7
		The wave profile represents wavelength	4
		Using periodical wave formulas	9
		Total	32
Part (b)	Dividing the graph into positive and negative parts		12
	Mixing between the roller coaster reasoning and dividing the graph into positive and negative parts		20
		Total	32

Based on Fig. 3, it is clear that point 4 has the greatest slope which results in the greatest speed. Therefore, using this justification, we can also find the correct answer to question (a) is 4–2–1–3.

B. Interviews

In order to explore students' reasoning as fruitfully as possible, we performed paired semistructured interviews with 32 physics students from two Indonesian universities. They were in the second or third year of their studies and all of them had already taken introductory physics courses, including basic notions of wave phenomena. Third year students, especially, had already completed an advanced course on waves.

Arksey and Knight [16] explain that paired interviews have the benefit to bridge the gap between pairs. Consequently, this condition will engage participants in elaborating more on their answers and gaining more interactions during the discussion setting. This type of interview also pushes the participants to work together to answer the questions that they might not be able to respond to individually [17]. Moreover, Houssart and Evens [18] suggest that paired interviews will be more beneficial for unseen questions, meaning that the questions are first encountered at the beginning of the interview. This should provide a collaborative working environment and encourage students to see alternative views from their answers.

We followed the interview procedures based on what has been recommended by the literature. We interviewed students in pairs, and posed the interview question for the first time at the beginning of the interview. They read the questions in a couple of minutes and eventually ask clarification questions to the interviewer. Students first responded individually to the question, and then a discussion in pairs began. In these discussions, students could defend or change their prior reasoning based on each argument from their peers. In the end, we gathered one agreed final answer from the pairs or individual answers if they could not reach an agreement.

C. Results

We found that *no* student was able to answer the questions correctly. Their reasons were diverse and their conceptual challenges are categorized based on the common difficulties encountered. Table I shows categories of students' conceptual challenges in phase 1.

In general, we found only one category of students' struggles to answer question (a), which is related to their difficulty to read the appropriate information of distance on the graph but with varieties of conceptual challenges. Meanwhile, the nature of students' reasoning to answer question (b) was based on identifying the position in Cartesian coordinates without considering other physical aspects within it.

1. Question part (a)

Difficulty in reading distance of each point on the graph of $y(x)$.—A few students translated the wave profile as a motion trajectory and used this notion to determine the displacement between two points. We call this the "roller coaster" erroneous reasoning. However, the way this error appeared differed among students. For example Diana[1] divided the profile into four parts and conceived four different motions, the origin (0, 0) moving to point 1, point 1 to point 2, point 2 to point 3, and point 3 to point 4. Point 4 will keep moving upwards. She then related her notion with the proportionality relationship between velocity and displacement. She answered that point 4 has the greatest velocity because it has the greatest trajectory from point 3 to 4. Using that notion, point 2 could also be considered having the greatest trajectory moving from point 1 to 2. When asked about this, she said that she

[1]Student names are pseudonyms.

measured the motion trajectory of point 2 from the crest of the wave profile because she assumed that "the point would start its new movement in that position." She concluded her reasoning by answering 4–2–3–1 for part (a). The following is Diana's reasoning:

> *Diana: The greater displacement of a point will result in a greater velocity. Point 4 has the greatest trajectory moving from point 3 to point 4, so point 4 has the greatest velocity. Point 2 is the second order because I calculate its trajectory from the crest of the wave profile. Next is point 3 because it is moving to point 4. Point 1 is the smallest because it has the shortest trajectory moving from point (0, 0).*

Another interviewed student, Angela, also assumed that the points on the wave are moving like a roller coaster. However, unlike Diana, she did not measure the motion trajectory of point 2 from the crest to point 3 but from the whole profile from point 2 to point 3. She then decided that point 2 has the greatest speed and sorted the magnitude of speed from the greatest to smallest as 2–4–3–1. Here we see elements of a stronger roller coaster reasoning, as she sometimes relates higher speeds with regions of lower potential energy.

Other students referred to the idea of vertical displacements to determine the velocity, but with inaccurate assumptions. One of the students observed the distance of each point to the *x-axis*. She said that the greater the distance of each point to the *x axis*, the greater its velocity and her answer for this question was 4–3–1–2. Julia, also one of our interviewed students, estimated the value of grid lines of 1 m on the *y axis* and then divided the graph into positive and negative parts.

> *Julia: If we assume each grid line on the y axis represents 1 m then point 1 has a distance of 2 m, point 2 is 0, point 3 is* −3 m*, and point 4 is 3 m. So the answer is 4–1–2–3.*

Surprisingly, we found some students estimated the horizontal position to determine the distance of each point. They said that the further away a point is from the origin (0, 0) horizontally, the greater its velocity. One of the student's reasoning is shown below:

> *Doddy: The answer is 4–3–2–1 because velocity is proportional to the distance based on the velocity formula, which is $v = s/t$. Velocity in point 4 is the greatest because it is located furthest compared to other points.*

We do not assume that Doddy considered points on the wave to move horizontally because he did not state any displacement of points to answer the question. Even though this reasoning is simple to understand, using horizontal displacement seems to be in contradiction to the nature of motion of particles on the string.

Assuming that a wave is always periodic.—Almost half of the students assumed that the wave profile in the question is periodic, even though this was not mentioned in the question. Although their primary goal was to find distances related to each point on the graph, these students associated the distances with wavelengths. We categorized this erroneous reasoning as the "periodicity fixation."

Ivan, for instance, assumed that the movement of each point starts from the origin (0, 0) and follows the wave profile until it reaches its respective position. This reasoning is also related to the conceptual challenge of the roller coaster. However, it was more plausible to place it into periodicity fixation because he continuously referred to the notion of wavelength in his answer. He conjectured that the distance from each point to the origin (0, 0) determines the magnitude of its wavelength. He then associated it with the proportionality between wavelength and velocity. He said that the greater the wavelength of a point [sic] the greater its velocity. With this notion, he decided that point 3 has the greatest velocity due to its greatest wavelength. This point has a $3/4$ wavelength because it consists of one hill and a half valley. Using hills and valleys to determine the wavelength is common when students learn periodic waves in these universities; one wavelength consists of one hill and one valley.

Ivan's reasoning became more complicated because of his notion of hill and valley. Paradoxically, he did not consider the whole wave profile to determine its wavelength, but asserted a different wavelength to each point of the profile. Moreover, he argued that point 4 is located in a new wavelength, therefore it has the smallest speed. The following is Ivan's reasoning for question part (a):

> *Ivan: The order is 3–2–1–4. I calculate the distances of each point to the origin (0, 0) to determine their wavelength. Point 3 is the greatest because it has a $3/4$ wavelength, point 2 has a half wavelength, and point 1 has less than a half wavelength. Point 4 is the smallest because it is located in the new wavelength.*

This type of conceptual challenge can also be seen from Johan's reasoning. He actually understood that the points on the wave move vertically, but he believed that the wave profile in the question is a sine wave. Then, he used a sine wave function $y = A\sin(kx - \omega t)$ to find a formula for velocity, as depicted in Fig. 4.

Johan finally arrived at the velocity formula $v = \omega\sqrt{A^2 - y^2}$. He then determined the magnitude of velocity using those two variables, amplitude (A) and vertical displacement (y). He said that the magnitude of velocity is maximum at $y = 0$ and minimum at $y = A$. With that analysis, he found that point 2 has the greatest velocity because $y = 0$ and point 3 has the smallest velocity because

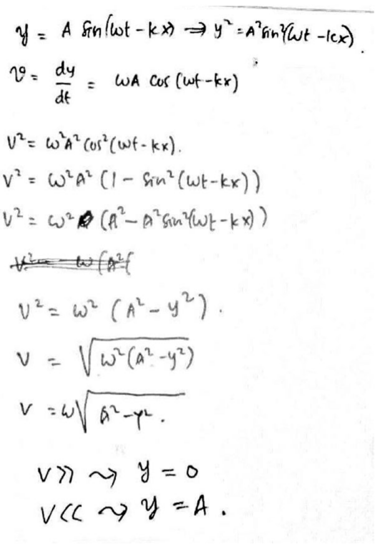

FIG. 4. Johan's derivation of a periodical wave to estimate the magnitude of the speed.

$y = A$. In the case of a periodical wave, Johan's reasoning is correct because point 2 is located at the inflection point. However, the wave profile in the question is not periodic. For point 3, in particular, his answer is correct because this point is located precisely in the crest and thus has zero velocity. Finally, he said point 1 has a greater velocity than point 4 because it has a smaller vertical displacement.

Similar to Johan, Adi also operated the concept of periodic waves to solve the problem. However, his method was based on the acceleration formula $a = -k^2 y$. He assumed that the magnitude of speed can be estimated by using the proportionality relation between acceleration and velocity. The greater the vertical displacement of each point, the greater its acceleration, resulting in greater velocity. He focused on the vertical displacement of each point and measured it based on the displacement of each point to the *x axis*. With this assumption, he then answered point 4 has the greatest speed. We asked him to clarify his answer since point 3 and point 4 have the same distance to the *x axis*. He then also considered the horizontal position of a point. Point 4 is located further horizontally than point 3, so that point 4 has the greatest speed. Adi's reasoning can be seen below:

> *Adi: I will use the formula of acceleration which is $a = -k^2 y$ because of the proportionality relation between acceleration and velocity. So, the greater y of a point will result in a greater acceleration, which also produces a greater velocity. The velocity at point 2 is zero because $y = 0$.*

With slightly different reasoning, Edy immediately noticed that the velocity at point 3 is zero by saying it is located at the position when a point will move between up and down. This reasoning was undoubtedly correct. He then noticed that point 2 is located at the inflection point and concluded that point 2 has the highest velocity. Again, this answer could be valid if the wave profile in the problem were a periodic wave. Even though Edy never stated any formula regarding a periodical wave, we infer that he also has a periodical wave fixation by his answers to the velocities of points 2 and 3.

2. Question part (b)

Dividing graph into positive and negative parts.—More than half of the students just simply labeled the Cartesian coordinates into negative and positive parts. The velocities of points located above the *x axis* are positive, and below the *x axis* are negative. Meanwhile, the velocity of points located exactly at the *x axis* is zero. The following is one of the student's reasoning related to this conceptual challenge:

> *Johan: The velocity of point 1 and 4 are greater than zero because they are located above of x axis, so their magnitudes must be positive. The velocity of point 2 is equal to zero because it is exactly located on the x axis. The velocity of point 3 is smaller than zero because it is located below the x axis, so its magnitude must be negative.*

Students in this group merely applied the position of each point based on Cartesian coordinates instead of considering the direction of each point when it is moving. We notice that the majority of students that hold this conceptual challenge also had a false assumption of vertical displacement to answer question (a).

Mixing between the roller coaster reasoning and dividing the graph into positive and negative parts.—Almost half of the students had a conceptual challenge by mixing two different notions to answer this question. First, they claimed that the area above the *x axis* is positive and the area below the *x axis* is negative. Then, they combined that notion with their incorrect interpretation of a moving point, the so-called roller coaster reasoning. Here is an example:

> *Indra: The velocity of point 1 is greater than zero because it is moving up to the crest of the hill. So, it requires velocity to climb the hill. I can also see that point 1 is located in the positive area of Cartesian coordinate. The velocity of point 2 is zero because it is located on the x axis. The velocity of point 3 is smaller than zero because it is located in a valley (moving down). For the same reason as point 1, the velocity of point 4 is bigger than zero.*

This group's reasoning can be associated based on how their method solves question (a). Indra, for example, reasoned that the points on the wave will move along the wave profile. Because of this conceptual challenge, his approach to answer question (b) was affected by this error. He said that points 1 and 4 are moving up because they are located at the upward profile whereas point 3 is moving down because it is located in the downward profile. Then, he compounded the error by saying that the area above the *x axis* is positive and the area below the *x axis* is negative. Indra mixed these two conceptual challenges to solve part (b). Points moving upward (point 1 and 4) will move into the positive area, so their velocities must be greater than zero. A point moving down (point 3) will move into the negative area, so its velocity must be smaller than zero. For point 2, however, he concluded it had a zero velocity because it is located at the *x axis*. Because of this, Indra's reasoning seems incoherent because he only analyzed the position of point 2 instead of applying his two conceptual challenges like he did when analyzing the other points. However, Diana, who also mixed these two aspects, indicated that velocity of point 2 is smaller than zero because it is located at the downward profile. Thus, her reasoning seems more coherent.

III. PHASE II: SCAFFOLDING SUPPORT

A. Methodology

The result from the questionnaire made it clear how challenging the posed question was to the students, which motivated us to develop instructional strategies to see if they could understand the basic conceptual issues. Three levels of scaffolding were implemented and 22 students who answered the questionnaire were selected randomly to participate in this stage, where, once again, semistructured interviews were conducted in pairs.

Methodologically, our study is similar to the one conducted by Maries *et al.* [19] who developed scaffoldings to reduce student difficulties with Gauss's law. Our scaffoldings were also designed to incorporate the experts' approach to solve the problems [20,21]. However, our supports were slightly different because we did not provide a complete explanation, but only minor hints to the students. We expected that students could build their understanding and answer the questions based on their own analysis.

The goal of these interventions was to lead the students to draw the wave profile after some time has elapsed to reveal possible changes in students' reasoning to solve the problem. Under these interventions, we expected that students could notice the displacements of each point by comparing two wave profiles at two instants to solve the problem.

Scaffolding is associated with providing suitable support to a learner to overcome something that is difficult to achieve [22]. Originally, the term scaffolding was used to describe a series of steps for a learner to achieve a better performance [23]. Nowadays, scaffolding is used as an intervention to help not only an individual person, but also pairs and teams in many fields, including physics [24,25].

1. Scaffolding level 1

The purpose of scaffolding level 1 is to provide an illustration to the students of the characteristics of the wave profile after a short time interval. The interviewer demonstrated physically with his hands how to create a single pulse on a string that is moving to the right with constant velocity. Students were then asked to draw the next wave profile after some time has elapsed on the graph in the question. Students were also asked to locate the displacement of points in the new wave profile. Students who failed to draw the correct wave profiles in this stage were given intervention level 2.

2. Scaffolding level 2

In this level, the PhET simulation called "wave on a string" [26] was introduced to the students. This simulation presents the real condition of a vibrating string, and it has a variety of features that are suitable to our wave profile. This simulation can be modified into different situations, for example, showing how a string oscillates with or without reflections. The vibration source can be created manually with the possibility of adjusting damping and tension. Moreover, if the users want to see the movement on the string in detail, a slow-motion feature can be applied to the system.

Students were asked to use the simulation to reproduce the pulse that was given in the question, and they were left to explore the simulation without any help. Students who were able to generate the pulse in the simulation were asked to draw the new wave profile again after a short period of time. Then they were asked once again to answer the question. Even though some students did not create the same wave profile, we asked them to answer the same question because we wanted them to realize that the shape of the wave remains the same when it is progressing. Students who failed to use the appropriate features in the simulation were given intervention stage 3.

3. Scaffolding level 3

In this final support, we showed to the students how to create a pulse moving to the right with a constant velocity. They were instructed to use a manual vibration source, set the damping to zero, choose no-end string, and use the slow-motion feature to see the vibration in detail. After successfully creating the correct simulation, they were asked once again to draw the wave profile after some time has elapsed and then answer the questions once more.

B. Results

Students' performance in the scaffolding environment was diverse at each level with noticeably scaffolding level 1

TABLE II. Students' results in the scaffolding (SCL) environment for question (a) ("1" indicates that students answer the question correctly, "0" indicates that students answer the question incorrectly).

Pair	Student	SCL I	SCL II	SCL III	Drawing
1	1	0	0	0	Failed
	2	0	0	0	Failed
2	3	0	0	1	Succeeded
	4	0	0	1	Succeeded
3	5	0	0	0	Succeeded
	6	0	0	0	Succeeded
4	7	0	0	0	Failed
	8	0	0	0	Failed
5	9	0	0	0	Succeeded
	10	0	0	0	Succeeded
6	11	0	0	0	Succeeded
	12	0	0	0	Succeeded
7	13	0	0	1	Succeeded
	14	0	0	1	Succeeded
8	15	0	1	0	Succeeded
	16	0	1	0	Succeeded
9	17	0	1	0	Succeeded
	18	0	1	0	Succeeded
10	19	0	1	0	Succeeded
	20	0	1	0	Succeeded
11	21	0	0	0	Failed
	22	0	0	0	Failed

TABLE III. Students' results in the scaffolding environment for question (b) (1 indicates that students answer the question correctly, 0 indicates that students answer the question incorrectly).

Pair	Student	SCL I	SCL II	SCL III	Drawing
1	1	0	0	0	Failed
	2	0	0	0	Failed
2	3	0	0	1	Succeeded
	4	0	0	1	Succeeded
3	5	0	0	0	Succeeded
	6	0	0	0	Succeeded
4	7	0	0	0	Failed
	8	0	0	0	Failed
5	9	0	0	1	Succeeded
	10	0	0	1	Succeeded
6	11	0	0	0	Succeeded
	12	0	0	0	Succeeded
7	13	0	0	1	Succeeded
	14	0	0	1	Succeeded
8	15	0	1	0	Succeeded
	16	0	1	0	Succeeded
9	17	0	1	0	Succeeded
	18	0	1	0	Succeeded
10	19	0	1	0	Succeeded
	20	0	1	0	Succeeded
11	21	0	0	0	Failed
	22	0	0	0	Failed

being very challenging for the students. Tables II and III show students' results in the scaffolding environment.

These results show that students' performance has improved by looking at their success in answering the question with correct reasoning after scaffolding level II. Although some students still hold robust erroneous views, the complex understanding of graphs of $y(x)$ and $y(t)$ appeared to be solved by some of our students.

1. Students' results for scaffolding level 1

We found that all students had difficulties imagining the nature of a pulse moving to the right with a constant velocity. Most of the students drew the new wave profile smaller because they said that the wave will lose its energy after moving a bit and its amplitude will diminish slowly. Figure 5 shows Citra's drawing exemplifying this difficulty.

Based on her drawing, Citra's difficulties were not only due to the notion of losing energy, but that she also struggled with locating points in the new profile. She started drawing the new profile from the origin and assumed that the points on the wave would only oscillate in the fixed *x-axis* position (except point 1). The way she located the new positions of the points also seems inconsistent. When asked for the reason for that choice, she simply said that she located the points randomly in the new wave profile.

Hendri, another of our interviewed students, also thought that the wave will lose its energy. However, unlike Citra, he started drawing the new wave profile after a short time interval to the right from the initial profile. Figure 6 shows Hendri's drawing in scaffolding level 1.

His choice to locate the new positions of the points was based on his conceptual challenge to answer question (a). Based on Fig. 6, he implemented his notion of roller coaster reasoning to locate the displacement of points on the new wave profile. He thought that points on the wave are moving along the wave profile and the numbers with prime symbols indicated this conceptual challenge.

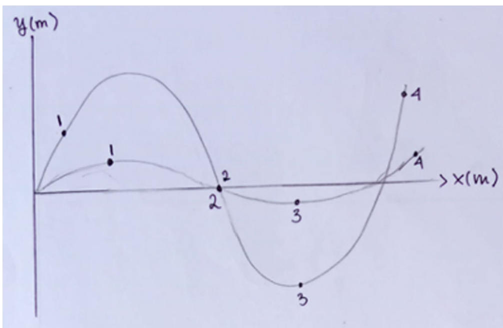

FIG. 5. Student's drawing with the next profile become smaller.

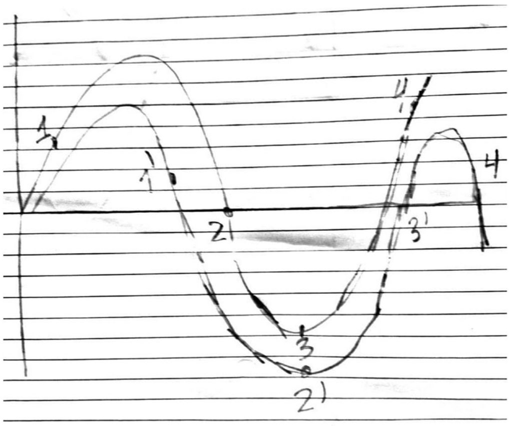

FIG. 6. Hendri's drawing in scaffolding level 1.

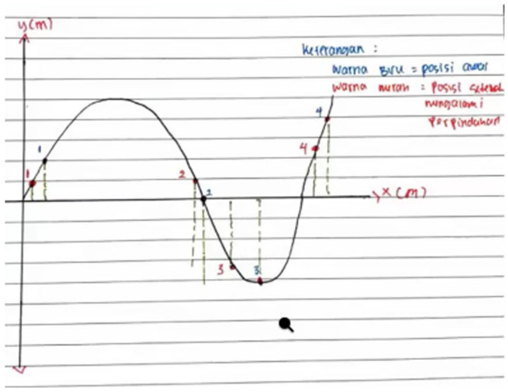

FIG. 8. Yuda's difficulty drawing two wave profiles in one graph.

A few students could not draw two wave profiles separated by a short time interval. Ivan, for example, could only visualize the next wave profile after it moves a complete wavelength. He drew the next wave profile from the end of the original profile, which can be seen in the red profile from Fig. 7.

Based on Ivan's drawing, it is impossible for him to imagine a wave profile after a short time interval. The way he located points on the red profile also indicated he had a roller coaster erroneous view. For question (a), we categorized Ivan's reasoning in the periodical wave fixation because he analyzed the problem by considering the wavelength to solve it. However, in this stage, we also found that Ivan holds a roller coaster reasoning.

One of our interviewed students, Yuda, recognized that the wave profile will always be identical when it is progressing. Only the points will be moving up and down vertically. However, he did not have a picture how to draw the two wave profiles in one graph. Yuda's drawing can be seen in Fig. 8.

Figure 8 shows that Yuda did not manage to draw the new profile after a short time interval. He only drew one wave profile and located the points at two different conditions. The blue dots represent points at the original profile and the red dots represent the displacement of points after a short time interval. In the beginning, we thought that he had a roller coaster reasoning because the points appeared moving along the wave profile, but this was not his reasoning. He said that point 1 will be moving down, point 2 will be moving up, point 3 will be moving up, and point 4 will be moving down after some time has elapsed. Points on the wave only move vertically and his reasoning regarding this was correct. When we asked him to draw once again the two wave profiles in one Cartesian coordinate, he was still puzzled about how to do that.

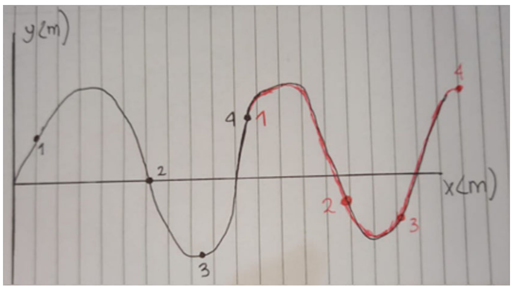

FIG. 7. Ivan drew the next wave profile after a complete wavelength.

Some students understood that the shape of the wave profile will be identical when it is progressing. The difficulty arrived when they located the displacement of points on the new wave profile. Figure 9 shows that Edy could draw two wave profiles at two instant times and the new wave profile is represented with the dotted line. The displacement of points was still inaccurate except point 3 which was located correctly, showing that point 3 is moving upward vertically. However, his reasoning regarding the motion direction of the points was correct. We also noticed that the dotted profile he drew looked like a repeating continuous pattern which is the characteristic of periodical waves.

On the other hand, Indra drew his new wave profile as if it was traveling as shown in Fig. 10. This result came

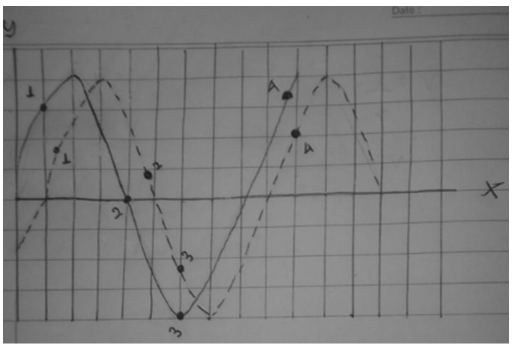

FIG. 9. Student's difficulty to place the displacement of points on the new wave profile.

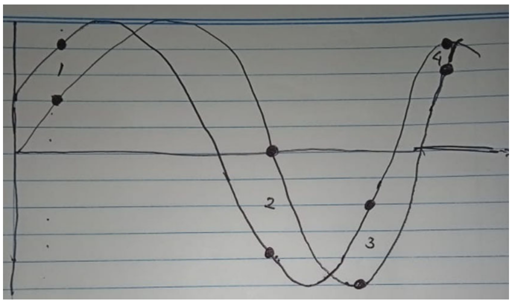

FIG. 10. Indra’s drawing as if the wave moves to the left.

because of his notion about the motion direction of the points. He said that point 1 will move upward after a short time interval. His reasoning would be correct if the wave were moving to the left. From the positions of the points on the new wave profile, it shows that Indra understood that each point moves vertically.

2. Students’ results for scaffolding level 2

Seven pairs of students were not able to use the simulation correctly to imitate the wave profile. All pairs eventually tried to use different settings in the simulation several times. However, students tended to use periodic wave vibration in the simulation instead of using manual vibration. We also noticed most of them hold a periodical wave fixation in phase I. One pair of students used the manual setting, but the damping was not set to zero. The simulation output was the wave that loses its energy when it is progressing, and its amplitude diminishes after some time has elapsed. We noted that this pair had difficulty drawing the new wave profile smaller in intervention level 1.

Regarding the reflection, two pairs applied the fixed end and no end setting, respectively, and the rest of the pairs applied the loose end setting. They paused the simulation and showed a periodical wave profile at an instant time to the interviewer. However, most of them realized that the way they used the simulation was incorrect because it was not a single pulse moving to the right with a constant velocity.

Four pairs of students could set the features properly and imitate the similar wave profile in the simulation. However, only three pairs applied the precise settings so that they could create the same profile as in the question. Meanwhile, one pair of students failed to move the manual vibration precisely, so the result of their wave profile was not similar to the question. However, at this point, they noticed that the wave profile will always be identical, and the points are moving vertically in a straight line when it is progressing.

Students who managed to notice this were asked once again to draw the wave profile after some time has elapsed and locate the positions of the points. Figure 11 shows one of student’s drawings after scaffolding level 2.

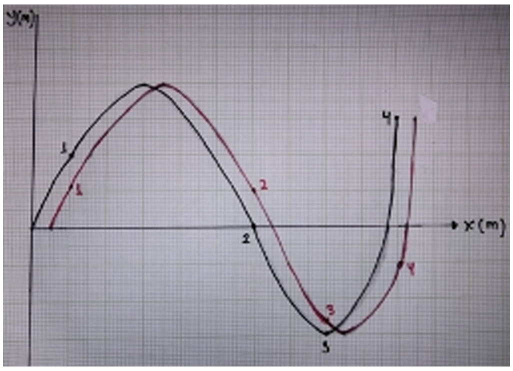

FIG. 11. Student’s drawing after scaffolding level 2.

With their drawings, most of them immediately noticed that point 4 has the greatest displacement compared to the other points. They then relate this notion to distinguish the difference of displacement of each point to solve question part (a). One interviewed student, Edy, who was categorized as having a periodical wave fixation in phase I, changed his reasoning to answer this question. He now focused on the displacement of each point and said that point 4 has the greatest velocity because of its displacement.

> *Edy: I think the greater the displacement of a point, the greater its velocity. We know that the velocity is proportional to the displacement. So, the velocity in point 4 is the greatest because it has the greatest displacement. Also, we can see from the simulation that point 4 has the greatest speed moving downward compared to other points.*

In contrast, one pair of students who also managed to grasp the conceptual understanding of a traveling pulse using the simulation did not use this support to change their prior reasoning to solve the problem. We note that this pair failed to create identical wave profiles in the simulation consistent with the question, but they understood the concept behind it. Figure 12 shows one of their drawings after scaffolding level 2.

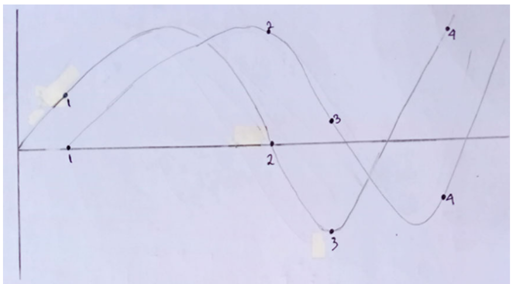

FIG. 12. A correct drawing from a student, but it did not help him to change his prior reasoning.

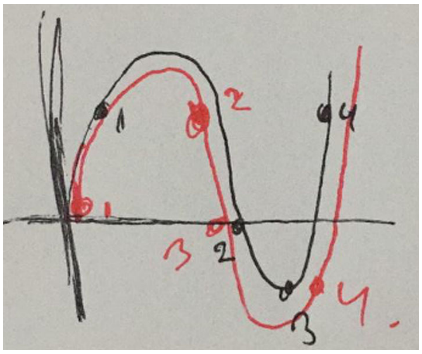

FIG. 13. A student still could not imagine how to draw two identical wave profiles in scaffolding level 3.

Their drawing was still inaccurate because of their mistake drawing the initial wave profile, affecting their drawing on the new profile. We intended to ask them to draw the wave again with the correct initial profile, but we decided to ask them first whether they wanted to change their answer. They said that they still hold to their prior reasoning, and they did not know how to answer the question based on their drawing. Even though the difference of displacement of each point it is clearly seen in their drawing, they still assumed that the greatest speed was point 4 because it is located the furthest horizontally.

For question (b), the group of students who conceived the concept of traveling pulse in this level managed to recognize the motion direction of each point correctly. Three pairs changed their answers by observing the motion direction of each point in the simulation. They said that points moving downwards will have a negative velocity, and points moving upwards will have a positive velocity. They said that points 1 and 4 are moving downward, points 2 and 3 are moving upward.

It is worth noting that our scaffolding is a bit tricky for point 3. When we emphasized that the question asked the velocity at the initial profile, then at that point, they realized that velocity in point 3 is equal to zero. The following is one student conversation in scaffolding level 2 to solve question part (b):

Irvin: Points 1 and 4 are moving downward so their velocities are negative. Point 2 is moving upwards so its velocity is positive. Velocity in point 3 is equal to zero.

Ruth: Yes, I agree. Point 3 is located in the greatest vertical displacement. It is the position where a point in the wave could move farthest. In that condition, the velocity of a point is equal to zero.

3. Students' results for scaffolding level 3

At this stage, the remaining pairs were able to visualize the wave profile with the simulation and they had a better understanding of the problem. However, there were still three pairs who were unsuccessful in drawing the wave profiles. From their pictures, we noticed that they could not imagine how to draw two wave profiles at two instant times in one Cartesian coordinate. Figure 13 shows students' difficulties drawing the new profile after some time has elapsed.

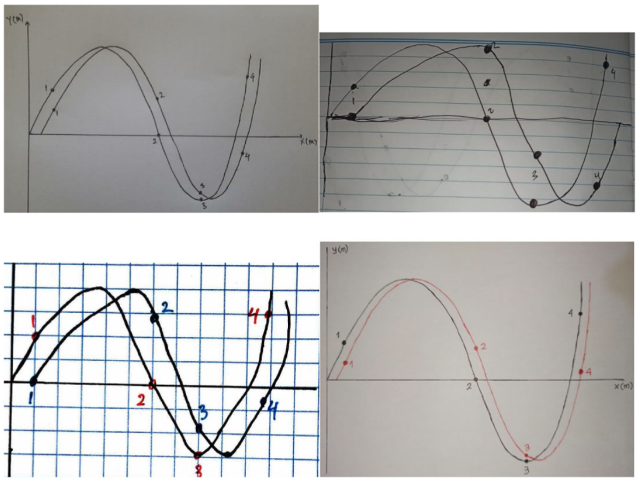

FIG. 14. Students' drawings after scaffolding level 3.

Diana managed to improve her conceptual understanding from the simulation but failed to draw the correct wave profile. She could explain the motion direction of points correctly, but when we asked her to draw the new wave profile, she was unable to do that. She tried several times to draw several wave profiles, but none were correct. From Fig. 13, it seems like Diana could not imagine how to draw two identical wave profiles crossing each other in one coordinate Cartesian.

Nevertheless, four pairs of students were finally able to draw the wave profile and locate the points correctly in scaffolding level three. Two pairs could answer questions (a) and (b), one pair only could answer question (b), and one pair failed to answer questions (a) and (b).

Figure 14 shows that all the remaining pairs were able to draw two wave profiles correctly but only two pairs could notice the different displacement of each point to answer question (a). Meanwhile, we still found one pair holding a strong conceptual challenge and they did not change their prior reasoning even though their drawing was correct. They still insisted on dividing the graph into positive and negative parts and determining the magnitude of the speed based on the incorrect assumption of vertical displacement. However, for question part (b), three pairs could analyze the motion direction of points and provide the correct reasoning to answer the question.

IV. DISCUSSION

A. Students' difficulties before scaffolding support

Before the scaffolding support, students held strong conceptual challenges regarding these wave phenomena.

None of the students' reasoning is scientifically acceptable and some are quite difficult to understand. The dominant difficulty for question part (a) is students' incorrect interpretation of distance on the graph of $y(x)$ and how they relate that with the proportionality relation between velocity and distance ($v = s/t$). Noticeably, most of them did not mention time explicitly in their reasoning and a few of them assumed that time is fixed. Overall, most students tried to find distances randomly to make assertions about the velocities of each point.

The first conceptual challenge arose when a few students misinterpreted the wave profile as the motion trajectory, which we called roller coaster erroneous reasoning. Some studies show that students assumed that the wave pushes the particle in front of it forward when it is traveling and they often treat waves like objects [7,27]. Wittmann [7] found that students related points in the wave to the movement of an object in kinematic and failed to distinguish between propagating object and propagating wave. He described this conceptual challenge like a surfer riding on an ocean wave because it moves everything in front of it. In our study, students also mentioned that the points move along the wave profile because of the disturbance on the string, which can be assumed that they treat the disturbance like a kick to a ball. However, this reasoning seems to be contradictory because students were aware that the wave only transfers energy which was already emphasized in our question. Therefore, the strong assumption that a wave should be treated as an object prevented them from solving the problem correctly.

Another reason why students failed to distinguish the concept between pulse and particle motion on the wave is the confusion between the graphs of $y(x)$ and $y(t)$. This can be even more complicated when the students combine it with the assumption that a wave is periodic. In our study, almost half of the students assumed that a wave is always periodic, which is in agreement with several findings [6,7,28]. The confusion between $y(x)$ and $y(t)$ is identified by one study showing that students failed to sketch different graphical functions using the graph of $y(x)$. For instance, students misunderstood that $y(x)$ can be directly transformed to sketch the graph $v(x)$ by looking at the slope of the graph [28]. Another difficulty came when students were asked to interpret the wave properties within the graphs of $y(x)$ and $y(t)$. The result shows that most of the students were confident that two wave profiles represent the waveform [6]. In periodic waves, one can use $y(x)$ to determine the wavelength and $y(t)$ to determine the wave's period. Our wave profile was meant to represent just a pulse, thus it should not be used to determine wavelength.

It is clear that the periodical wave fixation is robust among the students. Many referred to periodical properties and formulas like $y = A\sin(kx - \omega t)$ to solve the problem. The profile provided in the question even had a lack of symmetry, which did not prevent students from thinking in terms of sines or cosines. This periodical wave fixation was detected in one study that found more than two-thirds of the students had a very strong belief that the motion of a particle will form a sine wave pattern when they were presented with three different $y(t)$ graphs [6]. Another study also found that students drew the sine wave curve when they were asked to transform the graph of $y(x)$ which is represented in sawtooth shape into different graph functions [28]. During the interview, we asked students about their decision to use periodical waves to solve the problem and the majority of them said that they are only familiar with sine or cosine waves, which was also found in similar studies [6,7,28,29].

In a curved graph of $y(t)$, one can see the slopes at the points to find whether the velocity is positive, negative, or equal to zero but not with the graph of $y(x)$.[2] Again, mixing the reasoning between the graph of $y(x)$ and $y(t)$ and treating a wave like an object greatly complicated the students attempts to answer question (b). This phenomenon can be found when students were presented with a position-time graph located above of *x axis*. Many of them could not imagine that the points have negative velocity due to the position of the graph in the Cartesian coordinate [30]. Although the function of the graph is different since we plot the graph of $y(x)$, we can relate that finding by how students respond to answer question (b). Most of our students just simply observed where the points are located in Cartesian coordinates, whether a point is in the $+y$ *axis*, $-y$ *axis*, or exactly in the *x axis* without considering the motion direction of the points on the wave. In the curved graph of $y(t)$, students simply observed the position of points on the graph instead of analyzing the slope of each point to find the velocity [11]. Moreover, many students in our study mentioned that a point located in the *x axis* treated having a zero or lowest velocity due to its zero position, which was also one of the highlighted findings in Eshach [30] and Mcdermott [11].

B. Students' performances after scaffolding support

In our scaffolding, we tried to address the conceptual challenges that were found in phase I by creating two approaches. The complex relationship between the motion of points and the pulse on the wave was addressed with a simulation. Here, students who thought that the points on the wave move horizontally would finally see that the points only move up and down. This also tackles the conceptual challenge of using the position of points in Cartesian coordinates to define the sign of velocity since students can notice the motion direction of points. Particularly, for point 2, students realized that the velocity

[2]In this situation (pulse moving with constant horizontal speed) one can indeed use the slope at $y(x)$ to infer velocity (dy/dt). This is related to the fact that the pulse also satisfies the transport equation, which states that dy/dx is proportional to dy/dt.

in that position is not zero but that this is the case of the points located in the crest or trough of the graph. The problem of determining distance to obtain the velocity on the graph of $y(x)$ was addressed by asking students to draw the second wave profile after it moves a bit. This created a cognitive conflict among students which used inappropriate ways to define the distance or vertical displacement of each point on the graph.

Scaffolding level 1 was less effective because we found that all the students failed to draw the new wave profile. They believed that the wave would lose its energy and the amplitude would slowly shrink. Our result is in line with a study from Wittmann [9], who presented a pulse at $x = 0$ and $t = 0$ on the graph of $y(x)$ which propagates in the x direction to the students. He then asked them to draw the condition of the pulse after moving at $x = x_0$. Most of the students sketched the amplitude lower after moving at $x = x_0$ due to energy losses. This reasoning could be accurate if dissipative effects were taken into account. However, in the idealized situation, students should recognize that the shape of the pulse will remain the same when it is moving.

Scaffolding level 2 and 3 were more helpful to the students. Some of them have improved their performances when the PhET simulation was used, and it helped students distinguish the (horizontal) wave motion from the (vertical) particle motion. One study reported significant improvement in performance of the students using this simulation, showing that 71% of them correctly recognized the nature of velocity of the points of the string of a violin compared to only 21% of students in traditional teaching [31]. The simulation brings a dynamic component to our static wave profile because the wave is traveling. The students can now focus on the motion of some specific points when the wave is progressing.

In essence, to create the wave profile precisely like in the problem, students need to use the manual feature to move the string, set the damping to zero, and use no end string. How to move the source of vibration is also essential. There are four important steps to create the wave profile exactly like in the question: (i) the string should be placed from above first; (ii) pull the string downward at a specific position; (iii) pull the string back upward at the initial position; (iv) pull the string downward again and place it in the middle position between the whole movement upward and downward. To make a more precise wave profile, students need to pull the string a little bit faster downward than the upward movement. To observe the motion in detail, students need to apply the slow-motion feature.

Four pairs of students were able to create the correct profile without any help on the simulation and changed their prior reasoning. We note that students had different performances when they precisely imitated the wave profile to match the question compared to the students who could only create a similar pulse. Three pairs who managed to simulate the identical pulse immediately spotted that the shape of the profile will remain the same and realized that there was a significant difference between point 4 and the other three points. The motion in point 4, especially, is more noticeable to observe because it moves downward faster. Thus, it was easy for them to determine that point 4 has the greatest velocity even though they had not yet considered the concept of displacement in their reasoning.

One pair who only managed to create a similar pulse did not change their prior reasoning despite successfully drawing the correct wave profile. Although there is a mistake in failing to draw the wave profile consistent with the question, they understood that the shape of the wave will always be identical when it is progressing. One of their drawings can be seen in Fig. 12, showing two identical wave profiles in a graph. The difference of displacement and direction can be clearly seen from those two wave profiles, but they kept insisting on using their prior reasoning. As a side note, we did not provide any further hints after students finished their drawings.

In scaffolding level 3, all the remaining pairs were able to use the simulation to improve their performances. However, three pairs still failed to draw the correct profile and solve the question. Lin and Singh [32] suggested that the scaffolding's effectiveness depends on students' initial knowledge and skills. Based on our results in scaffolding level 2 and 3, it is not guaranteed that our designed scaffolding was effective for all the students even though we have set a clear goal in our interventions which is strongly suggested when designing the scaffolding environment [32]. The careful design of our intervention was not enough to help students change their prior reasoning despite their successful drawing of the correct wave profile.

Students' performances were also diverse in every level of scaffolding. We found that a few students were able to grasp the consistent shape of the wave when it progressed, and our intended goal of this support was accomplished only by implementing scaffolding level 2. In the end, all the students were able to draw the correct wave profile and the purpose of our scaffolding supports were fulfilled. However, a few students still hold their robust alternative conception and did not use that support to change their prior reasoning. Our findings are similar to some studies that implemented scaffolding support as a tool to help students overcome their conceptual challenges. They found outcomes varied when the students were engaged with various scaffolding support levels. These studies suggested that the level of competence is probably the reason why student performances are diverse in each scaffolding environment [24,33].

V. CONCLUSION

Our study highlights some of the main difficulties regarding the propagation of a pulse in a string. All our students initially tried to extract inappropriate information to find the distance on the graph of $y(x)$. This becomes even more complex because many students are mostly exposed to periodic waves and always think that the nature of waves is periodic. In consequence, they hold a robust erroneous reasoning if they are presented with an unusual wave profile. Because of their simplicity, it is understandable to use regular and periodic waves in teaching. However, this practice may convey to capture a thorough understanding of wave phenomena.

The second issue is the difficulty of extracting information from physics graphs. We found many students quickly fixated on unsuitable features on the curved graph that led them into a broader area of erroneous reasoning. Many phenomena on waves are presented in the graph, and this condition requires a comprehensive understanding regarding how to relate correct information on the graph into physical concepts.

In our study, designing scaffolding support becomes a challenge since students' response to these interventions is diverse and not all of them in the end successfully answered the question correctly. Our purpose of scaffolding support to lead students to draw the new wave profile after some time has elapsed was literally achieved. However, not all students used that help to change their prior reasoning. This happened because of the prior knowledge and procedural competences of the students. Further studies are needed to answer why these phenomena happened among university physics students.

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Encouraging students to understand the 1-D wave equation

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Encouraging students to understand the 1-D wave equation

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Despite its crucial importance in physics, there aren't many studies focusing on student difficulties and teaching strategies related to the (1-D) wave equation in the PER literature. In order to contribute to fill this gap, we conducted a study with university students which focused on specific aspects that are crucial for understanding this equation. Our results include not only key learning difficulties and potential teaching strategies to circumvent them, but they also suggest that students can search for a deeper understanding of physics equations when prompted to do so.

I. Introduction

It is plausible to assume that all physics instructors wish their students understood physics equations, but there exists no consensus in the PER literature about what this actually means. Some of the aspects addressed in previous studies include being able to identify variables [1–5], derive equations from first principles [1–3, 5], recognize mathematical structures [1–3, 5], relate equations with everyday life [1, 2, 5], and use equations in problem solving [1–3, 5].

Some studies pointed out that students' epistemological framing[1] sometimes resulted in the tendency to treat physics equations merely as calculation schemes [7–9]. For instance, students were satisfied when they could perform mathematical calculations with physics equations but failed to understand how the variables within the equations are related physically [1, 10, 11].

Our study focused on encouraging students to make sense of the 1-D wave equation (WE), chosen due to its convoluted mathematical structure. At first glance, this equation may look simple since it only consists of two second-order partial derivatives and a constant, but explaining its meaning is far from being trivial. We argue that by trying to make sense of this particular equation, students may overcome conceptual difficulties that some studies already discovered related to wave phenomena [10–16].

In order to answer the question "what does it mean to understand the WE?", we proposed a scheme that consists of three essential aspects: the mathematical representation of a moving pulse, connecting $y = f(x \pm vt)$ with the WE, and the relationship between force and concavity. These aspects guide us to assess students' interpretation of the main physical concepts behind the WE. Based on these guidelines, seven learning objectives (LOs) were defined and integrated in the design of tutorials to help students understand the WE.

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[1]We use the term of epistemological framing to describe students' perspectives on relevant knowledge in a given situation [6]

II. Fundamental aspects related to understanding the WE

There are many things that contribute to developing a physical understanding of the WE. In the following, three crucial aspects are described and justified.

A. The mathematical representation of waves

Broadly speaking, waves are moving profiles. Mathematically, this is expressed by functions of the kind $y(x,t) = f(x \pm vt)$. Thus, in order to be able to understand the WE, students should first be able to understand how to represent waves mathematically.

Furthermore, many studies have shown that students tend to refer to periodic elements (e.g. frequency, wavelength) when asked about essential characteristics of waves, but in principle a wave does not need to be periodic. In this sense, we wish to move away from this "periodic fixation" and stress the more fundamental and general description of a traveling wave by $y(x,t) = f(x \pm vt)$.

The way we describe a traveling wave is by performing horizontal graph transformations, as shown in Fig. 1.

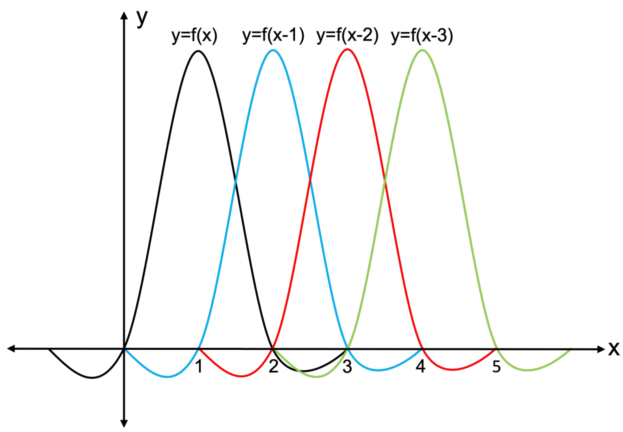

FIG. 1: Horizontal graph transformations of $f(x)$

In order to move the graph to the right, we substitute

x by $x-a$, where a is a positive number. If we let $a = vt$, then $y(x,t) = f(x - vt)$ represents a pulse propagating to the right with a fixed shape and constant speed v. A pulse moving to the left can be expressed by the function of $y(x,t) = f(x + vt)$.

B. Connecting $y(x,t) = f(x \pm vt)$ with the WE

Although apparently disconnected, it is possible to show that by assuming $y(x,t) = f(x \pm vt)$, we can obtain the WE. Thus, $y(x,t) = f(x \pm vt)$ and $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$ are (almost) mathematically equivalent, and being aware of this connection is a crucial part of understanding the WE deeply.

We obtain the WE by deriving $y(x,t) = f(x \pm vt)$ twice with respect to the independent variables x and t and comparing the results. Considering $x \pm vt = u$, double derivation with respect to x yields.

$$\frac{\partial y}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du} \tag{1}$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2}. \tag{2}$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = \pm v\frac{df}{du} \tag{3}$$
$$\frac{\partial^2 y}{\partial t^2} = (\pm v^2)\frac{d^2 f}{du^2}. \tag{4}$$

Comparing (2) and (4),

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}, \tag{5}$$

which is the 1-D WE.

Although any function of the form $y(x,t) = f(x \pm vt)$ satisfies the WE, not all of them represent a physical wave. For instance, the function of $y = (x \pm vt)^2$ is an infinite parabolic graph and therefore cannot express waves in the physical world. However, other functions, such as $y = A\sin(kx - \omega t)$ or $y = 2A\sin(kx)\cos(\omega t)$, are examples of wave functions that describe physical waves[2].

C. Concavity and Force

It is also crucial to encourage students to interpret the WE physically. Seeing this equation as a relation between force and concavity[34] is one promising way to do that. This should allow students to tell a "physical story" when describing the meaning of this equation, instead of seeing it merely as a calculation scheme and/or an abstract mathematical relation.

Fig. 2 shows a bent string and the resultant internal forces acting on several points. These forces differ in magnitude; the greater the concavity, the greater the force. We can express this relation mathematically:

$$\frac{\partial^2 y}{\partial x^2} \propto F \tag{6}$$
$$\frac{\partial^2 y}{\partial x^2} \propto ma \tag{7}$$

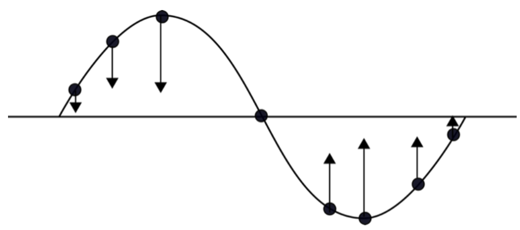

FIG. 2: The proportional relation between force and concavity.

If we assume the mass on the string is uniform:

$$\frac{\partial^2 y}{\partial x^2} \propto a \tag{8}$$
$$\frac{\partial^2 y}{\partial x^2} \propto \frac{\partial^2 y}{\partial t^2} \tag{9}$$

Then, adding a proportionality constant k and using dimensional analysis, we arrive at:

$$\frac{\partial^2 y}{\partial x^2} = k\frac{\partial^2 y}{\partial t^2} \tag{10}$$
$$\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial t^2}, \tag{11}$$

which is the 1-D WE.

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[2] A physical wave is a wave that exists in physical reality.

[3] The term concavity is used to here describe the second derivative of y with respect to x.

[4] A more precise formulation of this relationship would be to state that force is proportional to curvature. But with some approximations, curvature and concavity become equivalent. This subtlety was not discussed with the students, the important thing for us was to make them relate the shape of the profile with the resultant force at each point.

III. Methodology

A. Participants and Data Collection

Semi-structured interviews were performed in pairs, and twenty physics students from four Indonesian universities participated in our study, capturing a broader and more representative population. The universities included in the study consisted of two top-ranked institutions and two with mid-range rankings, as assessed by the Ministry of Education, Research, and Culture of Indonesia.

All students were in the third year of their studies and all of them had already completed an advanced course on waves. This condition allowed us to mitigate the different approaches and instructions that might exist among these institutions, as all students had already been exposed to the WE. In these universities, the basic concept of waves is introduced in the introductory physics course. However, the 1-D WE is typically covered in more advanced wave courses. The specific names of these courses may vary across universities, such as Waves, Physical Waves, or Waves I and II.

Using semi-structured interviews in qualitative research allowed us to thoroughly explore students' reasoning when answering the questionnaire. The questions involved were not typical problem-solving questions. Instead, students needed to elaborate on their answers, and the interviewer followed up with them until clarity was achieved.

Two days before the interviews, we told the students that the topic of discussion would be the 1-D WE. The pair interview was chosen because we wanted the students to work together during the entire data collection process. In this way, students could help each other, speak their "own language". [17, 18]. The students discussed their answers with their friends with whom they already had a relationship, so they felt more comfortable sharing their thoughts. This made it more likely for them to provide honest and authentic answers [19, 20].

The interviews were conducted in Indonesian as all physics courses in those four universities were taught in the same language. The interviewer, who is the first author of the paper, conducted the interviews. Due to the pandemic, the entire data collection was conducted online via Zoom. The interviews were recorded using the embedded recording feature in Zoom, and the interviewer also took notes to capture important reasoning and discussions during the interviews. When students needed to draw or perform calculations, students were asked to send the pictures of their works to the interviewer and then the interviewer shared and discussed in Zoom. The same method was used when students worked in GeoGebra, with the interviewer asking them to share their screen in Zoom.

The data was collected in three stages: pre-test, tutorials and post-test, the students remained in the same pairs throughout the study. The questions were first encountered by the students at the beginning of the interview and the students were encouraged to "think aloud" during the whole process of data collection. The first tutorial began afterwards, and approximately almost all the students completed the designed tutorials within two weeks, depending on their availability. After one month of finishing the last tutorial, the post-test with the same questionnaire was given again to the students. Due to the pandemic, the entire data collection was conducted online.

B. Questionnaire

The questionnaire consisted of five open-ended questions to assess students' understanding of the WE.

Questions

Below you see the 1-D WE consisting of a partial differential equation which describes a physical process.

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

1. *If you were to explain the meaning of this equation with your own words (to a non-expert), how would you do that?*
2. *There are several terms in 1-D WE* $(x, y, t, v, \partial^2 y/\partial t^2, \partial^2 y/\partial x^2)$. *Indicate what those symbols refer to.*
3. *Based on the question 2, indicate the units of those symbols.*
4. *Describe a physical situation represented by* $\partial^2 y/\partial x^2 \propto \partial^2 y/\partial t^2$.
5. *Given the functions below, which ones satisfy the WE and which can represent a physical wave? (You can choose more than one). Explain your reason.*
   - *(a)* $y(x,t) = f(x + vt)$
   - *(b)* $y(x,t) = A\sin(kx - \omega t)$
   - *(c)* $y(x,t) = e^{-(x-vt)^2}$
   - *(d)* $y(x,t) = 2A\sin(kx)\cos(\omega t)$
   - *(e)* $y(x,t) = (x + vt)^2$

The goal of the first question is to explore students' intuition about the WE. In the second and third questions, we asked them to explain the meaning and unit of each symbol in the WE $(x, y, t, v, \partial^2 y/\partial t^2, \partial^2 y/\partial x^2)$. This question aimed at revealing students' views about the symbols condensed in the WE, where one should be able to relate the meaning of physical symbols to the appropriate situations.

The most noticeable mathematical structure in the WE is the proportionality relationship between $\partial^2 y/\partial t^2$ and $\partial^2 y/\partial x^2$. Thus, we asked the students if they could physically interpret the relationship in question 4. In fact, this relation is important because it allows students to see it as the relationship between force and concavity. In the last question, different functions were given and the students asked which ones i) satisfy the WE and ii) represent physical waves.

C. Tutorials

The tutorials were designed to provide a platform for students to work collaboratively and discuss their learning difficulties [21]. When designing the tutorials, we were inspired by the conceptual blending framework, which has been recently used to investigate how students connect mathematics and physics [22–25]. This framework was initially proposed by Fauconnier & Turner [26] and explains cognitive processes by combining thinking elements such as symbols and images to create meaning.

Using a conceptual blending framework, Witmman [27] proposed two schemes of students' reasoning about wave propagation that combined three mental spaces, namely, gestural, perceptual, and verbal to make a new meaning. Eynde et. al [25] used dynamic conceptual blending (DBD) to investigate students' performance in integrating physics and mathematics when dealing with the heat equation, which has many similarities to the WE. Students' difficulties were identified when they missed individual elements of or connections between mental spaces. For example, some students failed to describe the meaning of heat flow and related that to entropy. Therefore, they missed the physical element in their reasoning. As a consequence, they made a mistake by relating entropy with $\partial/\partial t$ in a blended space. To follow up on these results, the authors suggested that graphical representations impacted positively on blending mathematical and physical meaning. This encouraged students to make sense of the partial derivatives of the heat equation [28].

Our study did not focus on modeling students' reasoning with conceptual blending. Still, we perceived the potential of using graphical representations to help students connect the two mental spaces. In essence, our tutorials consisted of algebraic and graphical representations, used in combination to make sense of the WE. For instance, GeoGebra is one tool used in our intervention that transforms static mathematical representations into observable dynamic physical situations where students could plot the mathematical functions and make physical meaning of them.

Fig. 3 is one example of how we use the conceptual blending framework to design tutorial IV, where the solid circles represent the mental spaces and the dotted box represents the intervention that plays a role of integrating those mental spaces.

The activities in the tutorials were designed based on the framework: elicit, confront, and resolve [21]. In the elicit phase, students were asked to work on tasks, allowing us to explore students' understanding and identify any difficulties on each tutorials. Next, students were asked questions that confronted their difficulties and

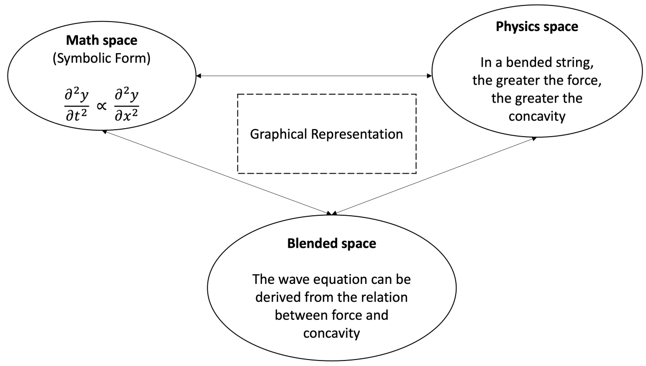


FIG. 3: The design of tutorial IV using the conceptual blending framework.

prompted them to reflect on their prior reasoning. Finally, if students continue to struggle, their difficulties were addressed by other interventions in order to assist them in achieving the goals set for each tutorial.

We designed 4 tutorials with a total of 7 learning objectives (LOs) to understand the WE, as shown in Fig. 4. These domains were used to classify and design the tutorials. The LOs were determined as follows:

1. Apply horizontal graph transformation of $y = f(x)$ when x is substituted by $x + a$ or $x - a$.
2. Relate the intuitive meaning between the graph transformation and the general solution of the WE.
3. Derive the WE from the assumption of the mathematical description of waves.
4. Reflect about why the WE must be a second-order partial differential equation.
5. Realize that not every function that satisfies the WE represents a physical wave.
6. Interpret the WE based on the proportionality relationship between force and concavity.
7. Use the previous relationship and dimensional analysis to derive the WE.

We addressed these learning objects across four tutorials that were designed to address four key aspects: the mathematical representation of a moving pulse, connecting $y = f(x \pm vt)$ with the WE, determining when functions represent a physical wave, and the relationship between force and concavity.

1. *Tutorial I*

Students were introduced to the concept of progressive functions in the first tutorial. They were given three functions: $y = x$, $y = x^2$ and $y = e^{2x}$ and asked to plot these functions in GeoGebra. We then instructed them to modify the functions by changing the argument of $y = x$. At this point, we expected that the students would have recognized that progressive functions describe graphs that move to the left or right with a fixed shape.

If the students were able to identify the concept of pro-

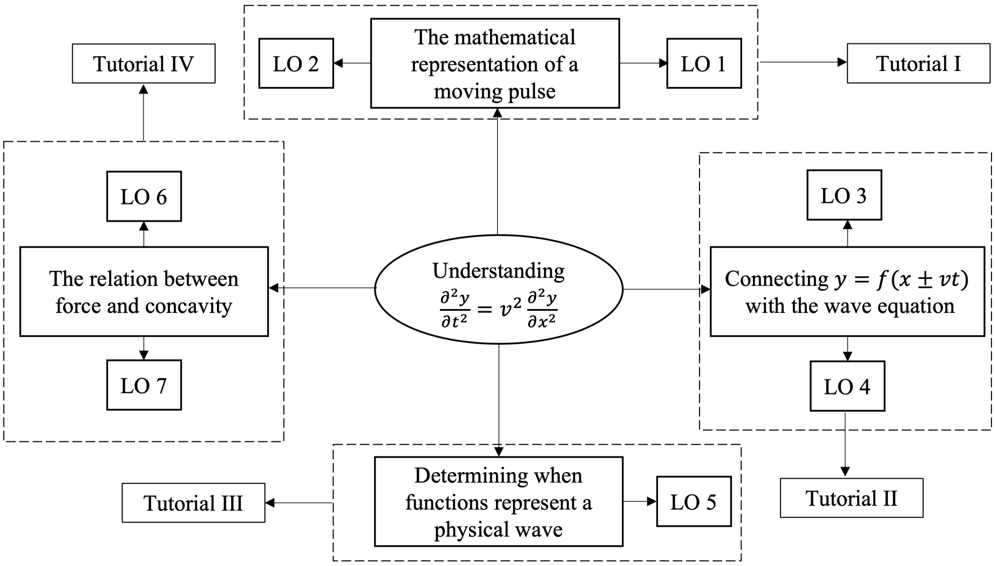


FIG. 4: How students understand the 1-D WE

gressive function, we brought them to the second phase of the tutorial. We asked a question about a general function that illustrates an arbitrary graph with a continuous motion. Students were expected to answer that $y = f(x \pm vt)$ represents a graph that moves with a constant speed to the left or right without changing its shape. In the end, we asked them to transform the three functions provided at the beginning of this tutorial in the form of $y = f(x \pm vt)$.

2. *Tutorial II*

From tutorial I, students had three functions in the form of $y = f(x \pm vt)$. Here, they were asked to differentiate the functions twice with respect to time t and position x. If they managed to differentiate the functions correctly, we would make them aware that functions that appear in the form of $y = f(x \pm vt)$ satisfy the WE, or one could find the WE by taking the derivative of these kinds of functions twice with respect to t and x. The process of taking the derivative presented as the primary goal of this tutorial. At the end, students were asked to provide one or two examples of functions that satisfy the WE.

Although the functions provided to the students satisfy the WE, one could differentiate these functions once with respect to t and x to obtain an equation, so-called the transport equation as follows:

$$\frac{\partial y}{\partial t} = \pm v \frac{\partial y}{\partial x} \tag{12}$$

The transport equation was introduced briefly after students managed to reach the primary goals of tutorial II. We tried to induce a cognitive conflict by asking students what if they only differentiate the functions once with respect to x and t. At this point, students should have realized that the transport equation also satisfies those functions and wondered why the WE is represented in the second-order partial derivative, not the first-order. Students were told that this was just a question that they needed to think deeply in order to encourage them to view a physics equation differently. We ended the tutorial session with that problem and did not ask for a follow-up answer in the next tutorials.

3. *Tutorial III*

In tutorial III, students were asked to plot the functions from tutorial II with GeoGebra. Up to this point, students only encountered wave functions that do not represent physical waves when the shape of these functions are graphs that go to infinity. After analyzing the three functions, students were given one more function of $y = e^{-(x-vt)^2}$, which satisfies the WE and represents a travelling wave. The shape of this function is a single pulse moving to the right which can be found in many wave phenomena, like a wave in a string. When students realized that not all functions satisfying the WE represent physical waves, our goal was achieved in this tutorial.

4. *Tutorial IV*

This tutorial is the most important session in our study since the goals were meant to reveal some important physical intuitions behind the WE. It is a crucial concept

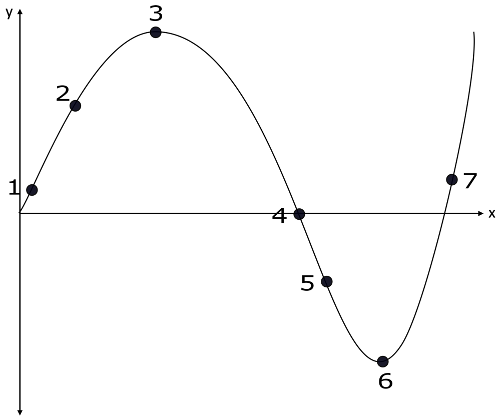


FIG. 5: Wave profile posed in the first phase of tutorial IV

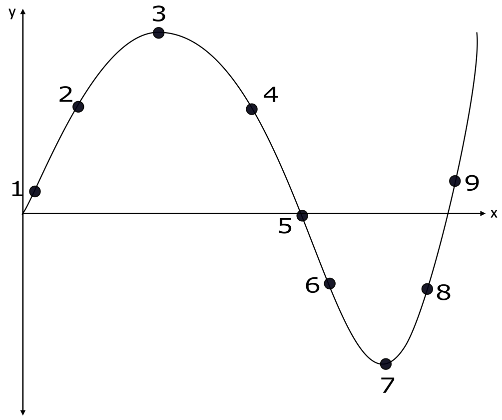


FIG. 6: Wave profile posed in the second phase of tutorial IV

because it allows students to see the WE as a relation between force and concavity.

The interview began when students were asked to analyze a wave profile in the first phase of tutorial IV. Seven points were located on the wave profile (Fig. 5) and students were required to draw the forces acting on these points. If students could grasp the idea of the proportionality relation between force and concavity, they should be able to recognize that the magnitude of forces acting on each point is different due to the different tension acting on them. The direction of the force can be found based on the type of concavity where the points are located. The direction is downward for the points located in the concave down and upward for the points located in the concave up.

Concavity was a tricky concept, and part of this tutorial was designed to address the mathematical formulation of this idea. Therefore, we added two more points (nine in total, Fig. 6) on the same wave profile. Students were then asked to draw the slope on each point, as shown in Fig. 7, and tell us how fast/intense the slope changes from one point to another.

At this point, students should have recognized that $\partial^2 y/\partial x^2$ describes the rate of change of the slope, which is related to the concavity at each point.

In the last phase of this tutorial, students were asked to find the WE using the concept of the proportional relation between force and concavity. We provided two physical conditions before they started to derive it. First, the mass of all the points on the wave is fixed and they have a uniform density. This condition would allow them to neglect mass in their derivation. Second, we told them to include k as a constant that describes the proportionality between force and acceleration.

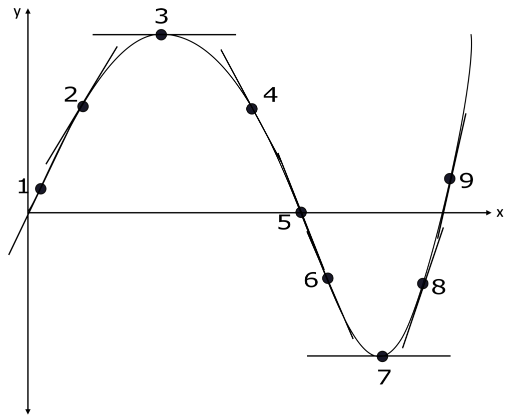


FIG. 7: Slopes on the points in the wave profile

D. Data analysis

Since the nature of this study is exploratory, we discussed students' performance from three stages of data collection: pre-test, tutorials, and post-test. The changes in students' reasoning and difficulties to answer our questionnaire were explored between the pre-test and post-test. We also highlighted the discussion of students' answers to question 1. This particular question is essential in our study because it reveals the changes in students' epistemological framing when they try to make sense of the WE after the interventions.

The interviews conducted during the three stages of the data collection were transcribed and translated in English. The transcription included students' drawings, calculations, and some pictures of their works on GeoGebra. Qualitative content analysis was then employed to interpret the findings from the transcriptions, where the open coding approach was used to synthesize the data [29]. The coding and categorization process involved grouping the data based on students' epistemological framing and difficulties encountered in each questionnaire item. The same technique was also applied to categorize students' difficulties when implementing the tutorials.

The first author of this paper performed the initial analysis. Later, the results were reviewed and discussed with two physicists, who examined the transcriptions and initial coding. One of them has extensive experience with teaching an advanced course on waves at university level. If there were differences in coding and interpretations, these were deliberated until a consensus was reached. The final coding and categorization were then verified by the second author. The coding process also combined the transcriptions across all four universities, ensuring that the final coding of the students' responses to the questionnaire and tutorials represents the perspectives of all participants.

IV. Results

A. Students' performance from three stages of data collection: pre-test, tutorials, and post-test.

1. *Pre-test*

We found that students struggled to answer the questions in our questionnaire. Looking more into specific content of the questions, we grouped students' answers in the pre-test as shown in Table I.

Students' answers were divided into three different categories for question 1. A few students described the WE as a mathematical description by reading the mathematical terms out loud. Although they mentioned some physical quantities, their interpretations showed that they did not have a deep understanding of the WE. These are two examples of students' answers describing this situation:

Example I:

> Eric[5]: *v is a wave velocity. So, when y is differentiated twice with respect to time will be equal to the second derivative of y with respect to x multiplied by squared wave velocity.*

Example II:

> Ivan: *This is a second order partial differential equation. It explains the relation of function y which was differentiated twice with respect to time will be equal to the second derivative of y with respect to x.*

Some students read the variables out loud to make sense of the WE. Reading variables in our context is being able to mention the definition of individual or combination variables, like reading the definition of variables in the glossary of a physics textbook. For example, it is correct that students recognize y as a vertical displacement, but it would be ideal if they could elaborate y is a vertical displacement at a given x and t.

Example I:

> Ryan: *This equation describes the time dimension of a wave, because we can see time in the left side of that equation. In the right side, there is x which describes the position. Based on this equation, we can determine the wave velocity, time, and position.*

Example II:

> Sara: *This equation shows the wave acceleration. This equation also has y which describes the vertical displacement, t is time, v is wave velocity, and x is position.*

Ryan and Sara's answers show a strong tendency in reading variables out loud to make sense of the WE. These students described individual components of the equation, but they did not explain the equation as a whole. Furthermore, we also noticed the difference between their answers, where Ryan tended to say out loud the definition of individual variables but Sara immediately mentioned the wave acceleration at the beginning of her answer. Sara's answer indicated that she identified $\partial^2 y/\partial t^2$ but avoided answering the definition of $\partial^2 y/\partial x^2$.

We found similar answers from a group of students who focused their attention on velocity and acceleration. They were able to define the meaning of this symbol, but when they tried to relate them with $\partial^2 y/\partial x^2$, they were confused because they were not familiar with the mathematical representation of concavity.

Some students tried to be more descriptive to make sense of the WE. These are the examples that describe this situation:

Example I:

> Ari: *Wave is travelling in dimensions x and t. Wave function (y) is differentiated partially with respect to x and t because the wave propagates depends on space and time. We use the second-order partial derivative in order to get the solution of the WE which has a nature of harmonic. The nature of a wave is always harmonic or periodic.*

Example II:

IAN: *The left side of this equation describes the nature of waves because of time and the right side of this equation describes the nature of waves because of the space. In one dimensional context, the space is only in the dimension of x which means the wave propagates only in the x direction. v is a wave velocity.*

INTERVIEWER: *Could you elaborate more about the nature of wave because of the time and space?*

[5] Students' names are pseudonyms.

TABLE I: Students' response to the questionnaire during the pre-test.

Question	Aspects of understanding the WE	Students' response	Percentage of students
1	The meaning of the WE	Trying to make sense of the equation	45 %
		Reading the variables out loud	35 %
		Reading the calculus terms out loud	20 %
2	The physical meaning of variables within the WE	Difficulty in interpreting physically $\partial^2 y/\partial x^2$	100 %
		Confusion about other symbols	50 %
3	The unit of variables	Difficulty in finding the unit of $\partial^2 y/\partial x^2$	100 %
		Difficulty in finding the unit of $\partial^2 y/\partial t^2$	10 %
4	The relation between concavity and acceleration	Difficulty in interpreting the relation between $\partial^2 y/\partial x^2$ and $\partial^2 y/\partial t^2$	100 %
5	Functions that satisfy the WE	Only periodic functions	65 %
		All the functions	35 %
	Functions that represent physical wave	Only periodic functions	70 %
		Function (b)	10 %
		Function (b),(c), and (d)	10 %
		Function (a) and (b)	10 %

IAN: *In the textbook, y also can be written as u. y describes the vibration of a particle, moving up and down repeatedly. So, we can see the behavior of the point when time is progressing. The shape will look like a sine wave.*

Even though their reasoning does not make a lot sense physically, seen from an epistemological framing, Ari and Ian tried to answer this question differently. Their answers focused on describing y, x and t, indicating that they understood how the vertical displacement varies with position and time. We also noticed that they clearly assumed that the nature of a wave is periodic. Looking at other answers, it is obvious that the periodical wave is robust among them to make sense of the WE. Here is another example of a student who tried to explain the meaning of the WE using a periodical wave function.

NORA: *This equation is always influenced by x as a position and t as time, and we can write it as $y(x,t)$. For example, we can write a function as $y = A\sin(kx - \omega t)$. If there is a phase change, it can be written as $y = A\sin(kx - \omega t - \theta)$. The sign also can be negative or positive, depending on its direction. This function is a solution to the WE. Let me differentiate this function twice with respect to x and t.*

INTERVIEWER: *Why did you come up with that function?*

NORA: *I am not sure, but it is often found in waves. Based on my derivation, if we substitute the result to the WE, $y = A\sin(kx - \omega t)$ is the solution of the WE.*

Students' answers in this group showed that they did not relate the specific parts of the WE. Although their answers pictured a few concepts about waves, we can see that there was an attempt from them to tell the meaning of the WE at a different level. From these results, our goal was students' attention on conceptual understanding.

It was easy for students to mention the definition of x, y, t, and v to answer question 2. However, when we asked them to relate the definition of those variables to the wave phenomena, some of them struggled. For example, they recognized x as a physical quantity that describes "distance", but they did not explore what x actually is. The variable of x can be referred as the spatial direction, which describes the direction of the wave when it travels, or it tells the fixed position of particles on the wave in the x-axis.

No student described the meaning of $\partial^2 y/\partial x^2$, or even just provide a definition. Most of them simply said that they had no idea about it. A few said it describes the displacement of a string (y) based on the position (x). Some of them just read the mathematical term out loud, saying that it is the second-order partial derivative of y with respect to x. One student even said that $\partial^2 y/\partial x^2$ is an acceleration due to its similar structure to $\partial^2 y/\partial t^2$.

When analyzing the units, many students correctly answered the units of x, y, t, and v. However, none of them managed to answer the unit of $\partial^2 y/\partial x^2$. Some said the unit is meter because the variables in the derivative only consist of x and y. The rest said that $\partial^2 y/\partial x^2$ has no unit due to its derivative structure and assumed that the numerator and denominator have the same unit of m^2.

We also noticed that a few students were confused with the different labels of physical quantities in textbooks and finally made a mistake with the units. For instance, one student said that v is frequency because he remembered that ν (nu) is a symbol of frequency in a textbook which looks similar to v. This student probably tried to relate his answer with periodic waves that always are characterized by a frequency. One student also thought that y is a wave function (ψ). This is an example of conversations from a pair describing this struggle:

DEVIN: *What is the unit of y? Is it ψ?*
ANNA: *If y is ψ, then it has no unit.*
ANNA: *I forgot what actually y is.*
DEVIN: *What is actually ψ?*
DEVIN: *I remember deriving the WE using force.*
ANNA: *I am not sure. let's say for now that y has no unit.*

The difficulties continued when students tried to make sense of the proportionality relationship between concavity and acceleration. A few of them tried to relate the individual symbols of the derivatives which can be seen in this answer:

> Ira: *$\partial^2 y/\partial x^2$ is distance, and $\partial^2 y/\partial t^2$ is acceleration. The greater the distance, the greater the acceleration. However, distance is inversely proportional to time.*

From her answer, Ira only recognized the mathematical expression of acceleration but not the concavity. She then related the proportional relation between concavity and acceleration by looking at a single variable in those derivatives. Here, we found again that students tend to break combined variables into a single-known variable if they had no idea what the description of a derivative was. Another example can be seen in the following conversation, where these students only analyzed the variable of y and t to make sense of this relation.

ARI: *For example, wave on the string. The further the wave travels, the greater time is needed. It can be seen that $\partial^2 y/\partial x^2$ is proportional to $\partial^2 y/\partial t^2$. Suppose $y = A\sin(kx - \omega t)$ and we differentiate it twice with respect to x and t. If we substitute the results into the WE, we will get the result $1 = 1$.*
INTERVIEWER: *Could you elaborate more on what you mean by the further the wave travels, the greater time is needed?*
ARI: *It is like a wave travelling. For example, the wave travels from 0 to $x = L$. So, the greater the time needed when the wave travels from 0 to L.*
NOVA: *I agree with Zulmi. In mechanics, the further the displacement, the greater the time needed as long as the velocity is constant.*

Based on this conversation, it implied that this pair only describes the relation between x and t which is commonly found in linear motions. The decision not to discuss y to make sense of this relation was probably due to the similar form of numerator between $\partial^2 y/\partial x^2$ and $\partial^2 y/\partial t^2$.

For question 5, we found that 65 % of students assumed that only periodic functions (functions (b) and (d)) satisfy the WE. They reasoned that the shape of those functions generated the sine or cosine alike if they were plotted into graphs. They also mentioned that the wave must be continuous. Meanwhile, six students tried to differentiate all the functions and found that all the functions satisfy the WE. However, when we asked them about the similarity between those functions, they could not explain it. A few students recognized that the wave function must consist of x, y and t but failed to conclude the general form of the wave function. This difficulty prompted them to choose functions that represent a physical wave, with 70 % of them saying only functions (b) and (d) are physical waves.

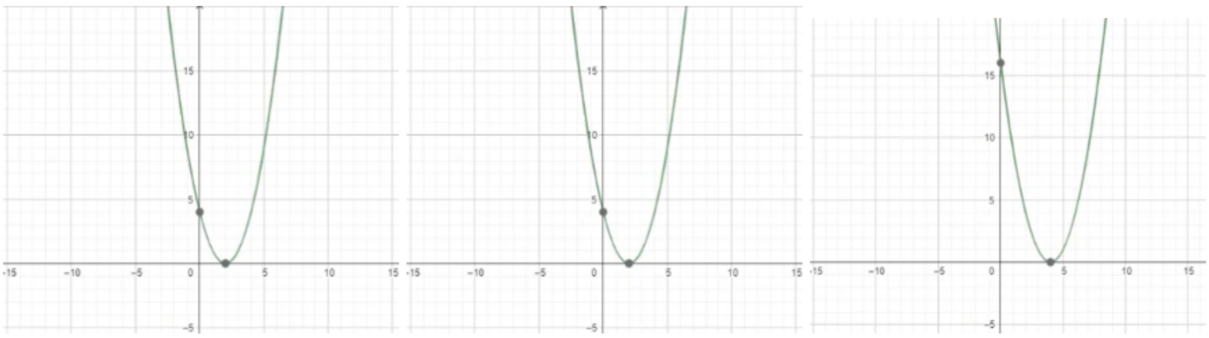
2. Tutorials

a. Tutorial I All the students recognized that the graph would move to the right by subtracting the magnitude of x and would to the left by adding the magnitude of x with a fixed shape using GeoGebra. Fig. 8 shows one example of how a pair plotted $y = x^2$ and changed the argument of x in GeoGebra.

The difficulty emerged when students tried to transform the progressive functions into a general form of the wave function. They could displace the functions from one position to the other but struggled to express how they should keep moving. We helped students by focusing their attention to translating a physically, and then they realized that time must be involved in order to move the functions continuously. With this justification, students now came up with $y = f(x \pm t)$. Some of them managed to find the general form of the wave function by using dimensional analysis which led them to include velocity. We told the rest of the students about how to make the functions move faster until they finally were able to transform the functions in the form of $y = f(x \pm vt)$. By the end of the tutorial, students now had functions of $y = x \pm vt$, $y = (x \pm vt)^2$, and $y = e^{2(x \pm vt)}$ where these functions would be used again in tutorial II and III.

b. Tutorial II In tutorial II, all the students were able to differentiate all the functions given to them and showed that they satisfied the wave function. A few of them had difficulties with the chain rule, but their pairs helped them resolve this issue. The goals of these tutorials were achieved because students understood that the wave function must appear as $y = f(x \pm vt)$. They also recognized that the WE could be obtained by taking the derivative of $y = f(x \pm vt)$ twice with respect to x and t.

We then asked the students to give one example of a function that satisfies the WE, and most of them presented examples of arbitrary functions with the form of $y = f(x \pm vt)$. However, we still found a few of them reasoned the periodical function to answer this question, such as $y = A\sin(kx - \omega t)$ or $y = A\cos(kx - \omega t)$. It was on purpose that our tutorials never mentioned any periodical wave functions to circumvent students' assumption of periodical waves in the pre-test. However, for some of them, the wave periodicity seemed robust and led them to always relate their answers with periodical wave conceptions. This collaborated the results found in



(a) $y = x^2$ (b) $y = (x-2)^2$ (c) $y = (x-4)^2$

FIG. 8: A pair plotted $y = x^2$ in GeoGebra and changed the argument from x to $x - 2$ and $x - 4$.

our previous study [13].

At the end of this tutorial, students were surprised to find that all the functions were also satisfied by only the first derivative. We then told them to discuss it with their friends after the tutorial ended.

c. Tutorial III In this part, students now plotted the three wave functions to GeoGebra and found that none of those graphs represents physical waves as shown in Fig. 9. For instance, they said that the function $y = (x - vt)$ represented a straight line, and the shape of a wave must be "wavy", indicating a wave must be presented in a curved graph. Some said that the amplitude must be involved, and none of these graphs has amplitude. Furthermore, students could not see the oscillation when these graphs were moving in GeoGebra. Some said that the graphs indicated the transfer of matter due to the line on those graphs shifting to the left or right. A few students also assumed that a wave must be harmonic, which consists of "hills" and "valleys". We noted that the latter reasoning was answered by the students who always related their answers with periodical wave conceptions in tutorial II. Some students also noticed that all the functions represent infinite graphs, and they said that there is no wave in physics representing infinite value.

When we posed the function of $y = e^{-(x-vt)^2}$ and told the students to plot this in GeoGebra (Fig. 10, almost all of them said that the shape of that function represented a physical wave. Students recognized that this was a single pulse travelling to the right. When we asked students how one can find this pulse in everyday life, many of them reflected on the shape of a single disturbance up and down from the end of a string. However, we still found two pairs of students who said that this profile does not represent waves because they thought it is not repeated and the wave must be periodic. At this point, we expected students to have a broader view regarding the nature of waves, but a few students did not change their prior reasoning and kept relating their answers to periodical waves.

d. Tutorial IV In this tutorial, we found that students failed to draw the forces acting on the points of the wave profile. For instance, some of them were distracted by the shape of the wave profile to determine the direction of the force, as shown in Fig. 11

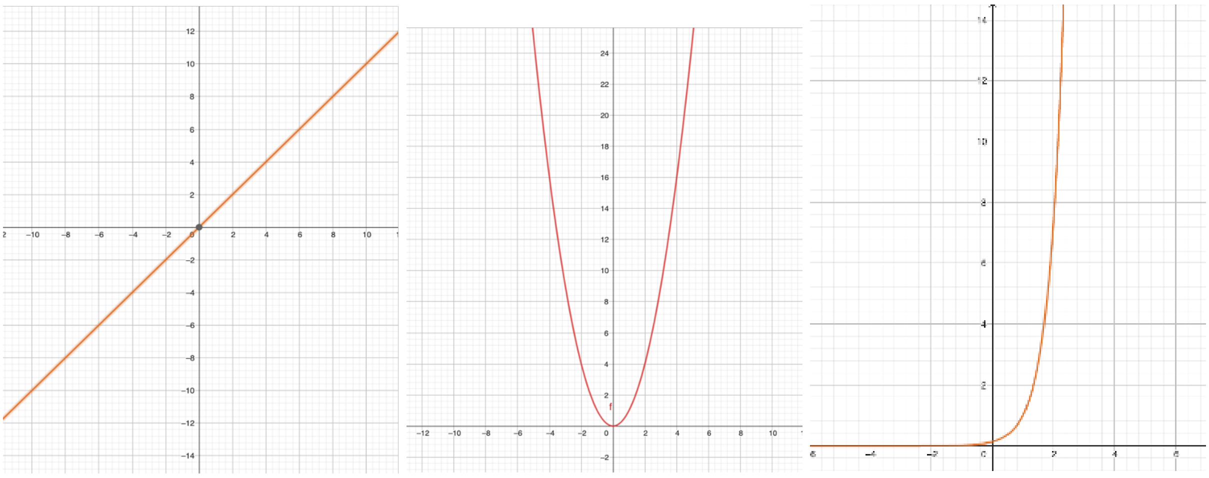
We thought this student assumed that the points were moving along the wave profile. However, his choice to draw the forces was based on the wave motion to the right, which we never mentioned in our tutorial. When we confronted him about this, he just assumed that the wave was moving to the right. The following is one reasoning from the student:

> Eric: *The direction of forces is the same as the motion direction. For example, when the wave is traveling from the left to the right, the direction of forces will be pointing to the right. However, it will be aligned with the wave profile.*

A few students thought that the force on the points could be decomposed into x and y components, as depicted in Fig 12. This reasoning is correct in a non-ideal string but not in an ideal string where the string is assumed strictly flexible. When we asked for their reasons, they said they were distracted by the force components of parabolic motion and eventually confused with their answer.

We circumvented students' difficulties by encouraging them to think about the motion of particles in an ideal string. For a while, students were still struggle to determine the direction of force with this hint, but some noticed that they only had two options to answer this problem. If the motion direction of points is to be a consideration, thus the direction of force is only up or down. We provided another hint to the students by suggesting that they think about the type of concavity. With this intervention, they were then able to find the correct answer, as expected. In the end, they said that the direction of forces on the points in the concave down is downward, and the direction of forces on the points in the concave up is upward.

It was easier for students to notice the difference between the magnitudes of points. They said that points



(a) $y = (x - vt)$ (b) $y = (x - vt)^2$ (c) $y = e^{2(x-vt)}$

FIG. 9: Three wave functions plotted in GeoGebra.

FIG. 10: The graph of $y = e^{-(x-vt)^2}$.

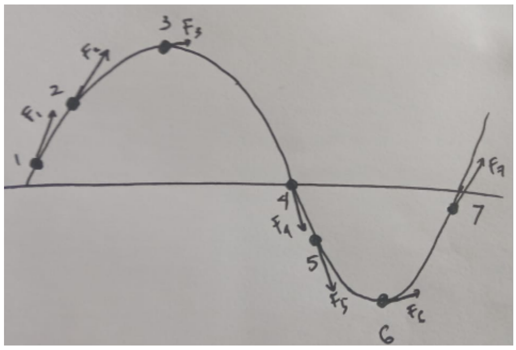


FIG. 11: Students' drawing based on the assumption of wave motion.

located in a more concave profile resulting a greater force. They also said that points in the crest and trough have the greatest force because they are in the most concave part of the profile. To confirm their understanding, we asked them to compare the magnitude of forces in points 3 and 6 since the concave down in our wave profile is narrower. They then said that the magnitude of the force in point 6 is greater than in point 3 because it is located in the more concave profile.

A few students had difficulty finding the magnitude of the force because their justification was based on the source of vibration. They said that point 1 has the greatest magnitude due to its position close to the source of vibration. This reasoning implied that these students thought the source of vibration was close to point 1 and the energy would be transferred to the neighborhood points; thus, point 7 has the smallest force. After several failed attempts, we then told students to think about the relationship between force and concavity until they were able to draw the correct forces on the points of the wave profile. Fig. 13 shows some correct drawings from the students.

After the students grasped the idea of the proportional relation between force and concavity, we plotted more points in the same wave profile and asked them to draw slopes on those points. We found that almost all students were able to draw the slope on the point correctly, but we still found a few of them struggled to do it, as shown in Fig. 14. The discussion with their pair finally helped them to solve this difficulty.

When students had correctly drawn the slope, we then asked them to formulate the slope mathematically. At this point, many students said that slope is $y = ax + b$ or $\Delta y/\Delta x$. We followed up this by encouraging them to think about the slope at a particular point, not between two points. Some students then realized that the mathematical expression of the slope is $\partial y/\partial x$. However, we still found that a few students had no idea how to an-

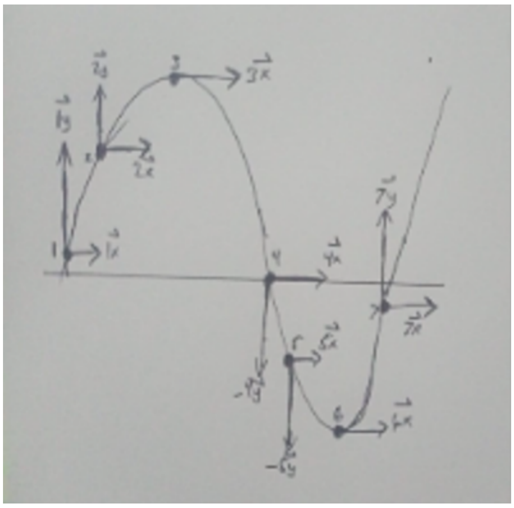


FIG. 12: A students drew the forces that can be decomposed into x and y components.

swer that question, and in this situation, we just provided them what the slope formula.

After that, we asked students to compare the slopes on the points between concave up and concave down of the wave profile. Many noticed the difference of slopes between two sections of the graph and said that the slopes in concave down are steeper and they change abruptly. We then asked students to describe the change of slope mathematically, and most of the students managed to tell us that $\partial^2 y/\partial x^2$ expressed the concavity. We noticed that a few students were surprised to find this because they were more familiar with the expression of $\partial^2 y/\partial t^2$ in physics courses.

At this point, students now tried to find the WE using the relation between force and concavity. However, the difficulty came when they arrived at this step:

$$\frac{\partial^2 y}{\partial t^2} = k\frac{\partial^2 y}{\partial x^2} \tag{13}$$

It is easy for the students to substitute k as v^2 because they have already recognized the form of the WE. However, we then told them to imagine if they had never encountered the WE before; how do they define k? Many students employed dimensional analysis, but the problem arose when they realized they did not know the unit of concavity. Then, they attempted to find the unit of concavity using the dimensional analysis of the WE and found that the unit is $1/m$. This was indeed correct, but this was because they already defined k as v^2 in their analysis.

From the expert perspective, the easiest way to find the unit of concavity is by comparing it to the acceleration because they are in the same form of second-order partial derivative.

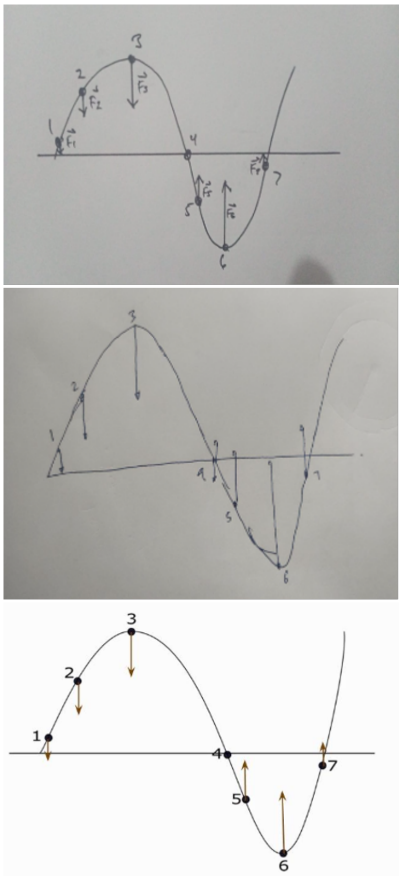


FIG. 13: Students drew the forces acting on the points based on the analysis of concavity on the graph.

$$acceleration = \frac{\partial^2 y}{\partial t^2} = \frac{m}{s^2} \tag{14}$$

$$concavity = \frac{\partial^2 y}{\partial x^2} = \frac{m}{m^2} = \frac{1}{m} \tag{15}$$

A few students managed to find the unit of concavity using this method, but many of them said that they had no idea to answer it if they were not allowed to define k as v^2 in their analysis. After a while, we encouraged students to compare the unit of $\partial^2 y/\partial t^2$ to find the unit of $\partial^2 y/\partial x^2$. With this intervention, students then managed to find the unit of concavity and continued working on their derivation to find the WE.

3. Post-test

Approximately one month after finishing the tutorials, we noticed that students' performance improved when they were encountered with the same question posed in

TABLE II: Students' response to the assessment during the post-test.

Question	Aspects of understanding the WE	Students' response	Percentage of students
1	The meaning of the WE	Trying to make sense of the WE at different level	70 %
		Reading the variables out loud	30 %
2	The physical meaning of variables within the WE	Elaborating the physical sense of some symbols, including the derivatives	100 %
3	The unit of variables	Telling the unit of each symbol correctly, including the derivatives	100 %
4	The relation between concavity and acceleration	Trying to make sense this relation	100 %
5	Functions that satisfy the WE	All the functions in the question.	70 %
		All the functions except (d)	30 %
	Functions that represent a physical wave	Function (b), (c), and (d)	50 %
		Periodic function (b) and (d)	20 %
		Function (b)	20 %
		Function (b) and (c)	10 %

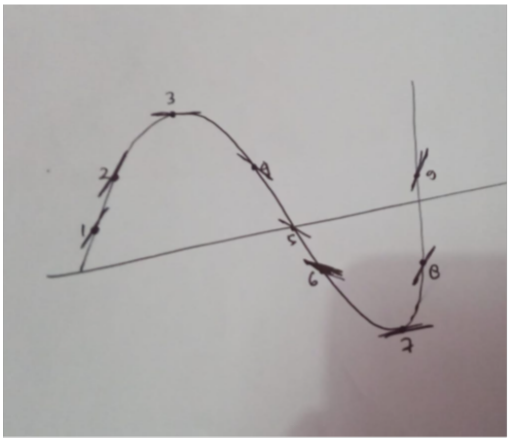


FIG. 14: A student failed to draw the slope on the points correctly.

the pre-test. Table II shows students' performance in the post-test.

When we asked about the meaning of the 1-D WE in the post-test, we noticed an attempt from the students to make sense of the WE at a different level. In the pre-test, we described three categories of students' views to make sense of the WE, and one of those was seeing it by only a mathematical description. Here are their answers in the post-test:

Example I:

> Eric: $\partial^2 y/\partial t^2$ *describes the wave acceleration, v describes the velocity, and* $\partial^2 y/\partial x^2$ *describes the change of slope or describes the concavity of a wave. So, the WE tells the acceleration is proportional to the concavity in any points on the wave.*

Example II:

> Ivan: *The acceleration* ($\partial^2 y/\partial t^2$) *will be proportional to the velocity squared and* $\partial^2 y/\partial x^2$ *which is the change of slope or concavity.*

We also noted the changes in students' views who tended to read the individual variables of the WE out loud into a deeper understanding level. These are the answers from Ryan and Sara in the post-test:

Example I:

> Ryan: $\partial^2 y/\partial t^2$ *is wave acceleration which is equal to concavity* $\partial^2 y/\partial x^2$*.*

Example II:

> Sara: *The WE describes the wave acceleration depending on the squared velocity and concavity. We also can see that concavity is proportional to the force.*

It is clear now that Ryan and Sara analyzed the WE as a whole instead of distinguishing each variable of the WE as they did in the pre-test. Sara, in particular, now mentioned concavity, where she avoided this mathematical term in her prior answer.

The nature of students' answer who already attempted to make sense the WE in the pre-test was also different. In the pre-test, they focused on describing the vertical displacement that depends on the distance and time. However, in the post-test, they elaborated their prior reasoning by emphasizing the relationship between force and concavity. We also noted that these students now did not mention any periodical wave conception to justify their answers.

Example I:

> Ari: *This equation explains that wave propagates in the dimension of x and t. That is why this equation is differentiated with t and x. The second derivation of* $\partial^2 y/\partial x^2$ *explains the gradients of points in the graph.*

Example II:

Ian: *We can see this equation based on the relation between force and concavity. On the left side, we can related it with $F = ma$, and we can neglect m in the system because it is uniform. We know that $a = \partial^2 y/\partial t^2$ will be equal to $\partial^2 y/\partial x^2$ shows the relation between concavity and acceleration.*

Example III:

NORA: *This equation tells the acceleration of a point which is moving at a particular position because of the disturbance on a string.*
IRA: *$\partial^2 y/\partial x^2$ is concavity*
NORA: *We can also see that this is a second-order partial derivative equation. So, this equation also tells the nature of waves that we already proved before (she referred to a wave function that represents a physical wave)*

In our tutorials, we provided different answers to question 1. For instance, one can answer that the WE is about the relationship between force and concavity. The other approach is that the WE describes the mathematical functions that move at a given velocity. However, it was interesting to see that the majority of students' answers preferred to relate force and concavity. This suggests that this relation is more easily understandable, which could be used in teaching.

More thorough answers from the students can be seen in question 2, where some of them not only mentioned the meaning of each variable but also attempted to elaborate them in a particular physical situation. For example, in the pre-test, many students merely described the definition of x and y as the length dimension. However, some students related these variables to the wave phenomena in the post-test. These are examples of students' answers to explain x and y.

Layla: *y is a vertical displacement and x is a position. These variables describe the position of the string at any particular time or the condition of points on the string in terms of its position (x) and vertical displacement (y).*

Mika: *y is a vertical position. When we observe a particular point in a wave, y shows the position of that point on the y-axis or vertically. At the same time, x is the position of a point in the x-axis when the wave is travelling.*

Noticeably, all students were able to mention the definition of concavity in the post-test, whereas all of them failed in the pre-test. The discussion within the pairs helped those who did not remember the meaning of $\partial^2 y/\partial x^2$. Furthermore, all the students managed to define the unit of each variable, including concavity which was also the dominant problem in the pre-test. A few students forgot about the unit of concavity, but they finally managed to find it using dimensional analysis. Here is one example of a student describing concavity:

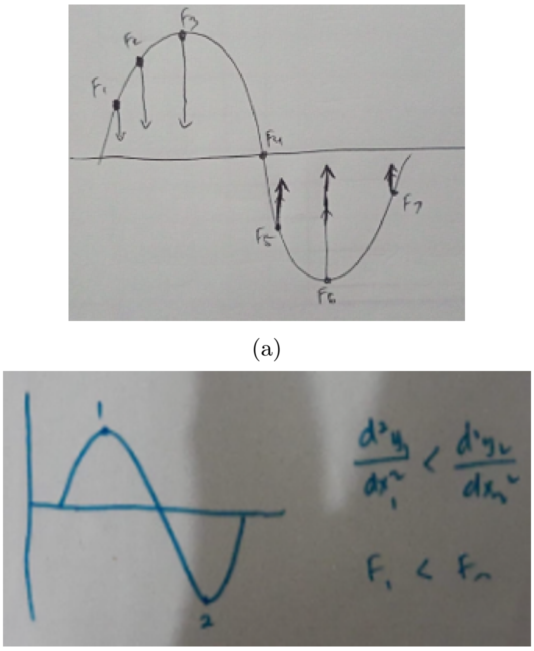
Eric: *$\partial^2 y/\partial x^2$ explains the slopes on the points with different gradients. It shows that the slopes change when the points are moving.*

We can see that there was confusion and understanding regarding the relationship between the slope changes and motion from Eric's answer. In fact, the $y(x)$ graph is a static quantity that represents in a geometrical shape where time is not involved in this situation. However, here, we noticed the manifestation of an idea from his answer that shows he tried to make sense of the concavity with his own words. We also found another student with a similar situation when she tried to elaborate more about the concavity. In question 1, Nora did not mention any periodical reasoning in her answer, but this conception emerged again when she tried to describe the significant of concavity. Here is how Noras' reasoning that describes this situation:

NORA: *$\partial^2 y/\partial x^2$ is concavity*
INTERVIEWER: *Could you elaborate more about it?*
NORA: *A wave usually is represented in the form of sine/cosine. That is why we can always relate the concavity in wave phenomena. So, each point in the wave will move, and all of them will form a concavity. The concavity itself is also related to the acceleration of each point on the wave.*

When we asked again about the relation between acceleration and concavity in the post-test, many students referred to the forces acting on points of a wave. They said that the force is proportional to the concavity. Thus, the acceleration is also proportional to the concavity. We then asked students to elaborate on their answers and draw a graph to justify their reasoning. Here is one example of a pair conversation to answer question 4:

DAVID: *The concavity is proportional to the force. The greater the concavity, the greater the force. When the force is greater, the acceleration is also greater because the force is related to acceleration.*
MIKA: *Based on my drawing (Fig. 15a, we can see that point 1-4 are located in a concave down and points 5-7 are located in a concave up. The longer arrow indicates a greater force because they are located in a more concave profile. Furthermore, if we compare the forces in points 1 - 3, point 3 has the greatest force because it is located in the most concave part of the profile. The greater the slope changes also indicates the greater concavity.*
DAVID: *My drawing is not that detail (Fig. 15b. I just want to explain that point 1 is located in*



(a)

(b)

FIG. 15: The drawing of Mika (a) and David (b) describes the proportionality relation between concavity and acceleration.

the less concave profile compared to point 2. Point 2 is located in the slopes that change abruptly.

MIKA: *The direction of forces will toward to the zero point (x-axis). If it is concave down, the forces will be going downward, and vice versa.*

For question 5, 70 % of students believed that all the functions in the question satisfy the WE, whereas 30 % of them said that function $y = 2A\sin(kx)\cos(\omega t)$ (d) is not a wave function. They genuinely understood that the general form of the wave function is $y(x,t) = f(x \pm t)$, but some of them were distracted by the form of function (d) which is not presented in the general form of wave function. However, some students believed that function (d) is just a different form of the wave function, and it is only the manipulation of trigonometry identity. One pair of students even recognized that function (d) describes the wave superposition. They said that the initial form of the function is just the addition or subtraction of two wave functions with the form of $y(x,t) = f(x \pm t)$.

Half of the students said that functions $y = A\sin(kx - \omega t)$ (b), $y = e^{-(x-vt)^2}$ (c), and $y = 2A\sin(kx)\cos(\omega t)$ (d) represent a physical wave. Some students said that the shape of a wave could be anything, including a single traveling pulse. A few of them doubted the representation of function (c) and tried to plot it again in GeoGebra. However, 40% of the students still thought that the wave must be presented in the form of sine/cosine. A few students even said that only function (b) represents wave because it is commonly found in waves. Meanwhile, one pair of students said that function (d) does not represent a wave because it is not in the form of $y(x,t) = f(x \pm t)$. They said that if a function does not satisfy the WE, it does not represent a physical wave.

V. Discussion

In physics education, interventions play a key role to improve students' reasoning towards expert-like approaches [5, 30–34]. In this section, we discuss the fundamental changes when students tried to make sense of the WE between pre- and post-test. We do not claim that our tutorials helped all the students to a full understanding of the WE. However, data from the qualitative content analysis suggest that students looked at this physics equation differently after the interventions.

Students' answers to question 1 in the post-test showed that they now attempted to make more sense of the WE in certain ways. Although some were still confused about some physical concepts, their answers manifested a pursuit of a deeper understanding, demonstrating a positive impact of our tutorials. The "think aloud" interviews allowed us to identify three kinds of epistemological framing when they were asked about the meaning of the WE: reading the mathematical operations out loud, reading the variables out loud, and trying to make sense of the equation. A group of students who generally viewed the WE only as describing mathematical calculations, now considered the physical quantities involved and how they are related. The same changes happened to the students who read the variables out loud in the pre-test. Although a focus on mathematical procedures was still present in their reasoning, some students now at least tried to see the WE as a whole without separating its variables or components to make sense of it.

Comparing the epistemological framing on understanding physics equations, Karam and Krey [5] identified different approaches from two physics students. In the evaluation, one student said that understanding a physics equation is being able to explain the equation in meaningful ways. Meanwhile, another student believed that one needs to recognize the meaning of symbols of a physics equation and perform calculation in order to understand it. We found a similar case in our study where almost one-third of students did not change their prior reasoning to answering question 1, where some of them still read the variables out loud. We did not state that this reasoning is not acceptable since this is an open question, and one could perceive a physics equation differently. However, at this point, we hoped that students would have provided a deeper analysis of the WE after the interventions.

Our results indicate that the relationship between force and concavity was tempting to the students, as they often referred to it throughout their post-test. However, by only stating it, we cannot conclude that they fully understood this relationship. It is worth investigating whether relating force and concavity makes sense for them in some kind of physical situation. For instance, students could produce a pulse in a real string and then realize that one needs more force to make a more concave wave profile.

Regarding concavity, we did not find many studies in PER investigating second-order partial derivatives. Instead, some explored students' performance of first-order partial derivative in thermodynamics [35–37]. In the mathematics education literature, however, a few studies explored this topic [38–42]. Jones [41], in particular, distinguished the meaning of concavity into four components: the shape of a graph, the sign of the second derivative of a function, the increasing or decreasing of the first derivative of a function, and the change of rates. From the mathematical context, many students mentioned that concavity describes the shape of the graph and the sign of the derivative. They then struggled to relate concavity with slope, with just a few including the rates of change in their answers.

Although Jones [41] stated that there are four criteria for understanding the concavity, we only expected that students could describe it as the shape of the graph and the rate of change of the first derivative, since these two are the most relevant for understanding the WE. Where the slope changes abruptly, the graph is more concave. In the pre-test, no student succeeded in interpreting the meaning and the unit of $\partial^2 y/\partial x^2$. However, again, we found that students manifested the concept of concavity differently in the post-test, mentioning at least two meanings of concavity. Understanding the concavity was an eye-opener for some of them, many were surprised to finally make sense of $\partial^2 y/\partial x^2$. We did not ask how they felt after finishing each tutorial, but some said they never thought they could explore so much physical meaning from only one physics equation.

Without a doubt, some students performed well in mathematics because they managed to derive all the functions in question 5 and found that those functions satisfied the WE in the pre-test. However, as some studies emphasized, proficiency in mathematics does not always link to success in physics [43–46]. For example, discussing graphical representations, one study found that students succeeded in describing the concept of slope in mathematics but failed in translating it into physics, although the graphs reflect the same idea [45].

Another noticeable aspect of students' epistemological framing was about the nature of wave. For some of them, $y = A\sin(kx \pm \omega t)$ or $y = A\cos(kx \pm \omega t)$ were like a "template", assuming that these functions are the only mathematical representation of wave. Although many physical phenomena in our everyday are presented in terms of periodicity, the students neglected the basic concepts of waves. Other studies also found similar situations, where students thought that the graphical representations of waves always represent waveform [12], a sound wave triggered a sine wave pattern of a particle [12], drawing a sine wave was proper when when a non-sinusoidal waveform was posed to them [47]. One reason to explain the cause of this issue is a strong exposure to the periodical waves in physics courses that lead students to think that the nature of a wave is always periodic [12, 13, 15, 47, 48].

Nevertheless, although our tutorials had an effect on the students to some extent, not all of them managed to change their prior reasoning and improve their performance in the post-test. As some PER literature suggests, developing the sequence in tutorials should be an ongoing process [49, 50]. Our tutorials were designed for an online format, but we argue by providing hands-on experiences, such as physical activities, may lead to better improvements in students' performance.

VI. Conclusion

In our study, we investigated how students tried to interpret a physics equation and some conceptual challenges that they had associated with it. Our tutorials attempted to address this issue by asking specific questions, where we drew students' attention to make sense of the meaning, relation, and interpretation of the symbols related to this physics equation. After the tutorials, students tried to read and interpret the equation physically and conceptually with a different level of epistemological framing.

We also argued that the way of asking questions about a physics equation in our study could trigger students' curiosity about the nature of physics equations. In our case, although not all the students manifested a deep detail about the meaning of the WE, we hope that they look at physics equations differently in the future. Students can perceive the Schrödinger, Laplace, heat equation, etc., in a more meaningful way. In our study, we focused on some specific aspects related to understanding the WE which are not usually common in the classroom. Our study does show that this is possible and it is probably worth emphasizing this epistemological dimension in teaching.

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A. Appendix I: Tutorial 1

The mathematical representation of a moving pulse

Learning Objectives:

1. Translate the idea of graph transformation of $y = f(x)$ when x is substituted by $x + a$ or $x - a$.
2. Relate the intuitive meaning between the graph transformation and the general solution of the WE.

(a) What would the graph look like if we plot $y = x$ to GeoGebra?
(b) If we substitute x by $x - 2$, what does happen to the graph?
(c) If we substitute x by $x - 4$, what does happen to the graph?
(d) If we substitute x by $x - 6$, what does happen to the graph?
(e) If we substitute x by $x + 2$, what does happen to the graph?
(f) If we substitute x by $x + 4$, what does happen to the graph?
(g) If we substitute x by $x + 6$, what does happen to the graph?
(h) Now, do the same process for the functions of $y = x^2$ and $y = e^{2x}$. You can change the magnitude of addition and subtraction as you wish.
(i) Based on this, could you provide a function that illustrates an arbitrary graph with a continuous motion?

After the students find $y = f(x \pm vt)$

(j) Can you transform three functions in the beginning in the form of $y = f(x \pm vt)$?
(k) What is your conclusion from this tutorial?

B. Appendix II: Tutorial II

Connecting $y = f(x \pm vt)$ with the WE

Learning Objectives:

1. Derive the WE from the assumption of the mathematical description of waves.
2. Reflect about why the WE must be a second-order partial differential equation.

You know that a function that moves to the left and right is expressed as:

$$y = f(x \pm vt)$$

From tutorial I, you have functions of $y = x \pm vt$, $y = (x \pm vt)^2$, and $y = e^{2(x \pm vt)}$.

(a) Differentiate those functions twice with respect to x and t. What can you obtain from your derivation?

After the students find that any function in the form of $y = f(x \pm vt)$ satisfies the WE,

(b) You know that when we derive twice the function with the form of $y = f(x \pm vt)$ with respect to x and t, we will get the WE. Why don't we just stop at the first derivative? What do you think about that?
(c) What is your conclusion from this tutorial?

C. Appendix III: Tutorial III

Determining when functions represent a physical wave

Learning Objectives:

1. Realize that not every function that satisfies the WE represents a physical wave.

From tutorial II, we have proofed that $y = x \pm vt$, $y = (x \pm vt)^2$, and $y = e^{2(x \pm vt)}$ satisfy the WE

(a) Plot those functions in GeoGebra with a constant v and $t = 0 - 10s$.
(b) What can you see from the graphs when you move the slider of time (t)?
(c) Do those functions represent physical waves? Why?
(d) Now, plot $y = e^{-(x-vt)^2}$ in GeoGebra. Do this function represent a physical wave? Why?
(e) What is your conclusion from this tutorial?

D. Appendix IV: Tutorial IV

Force and concavity

Learning Objectives:

1. Interpret the WE based on the proportionality relationship between force and concavity.
2. Use the previous relationship and dimensional analysis to derive the WE.

Suppose you shake a tight string up and down with your hand, and then it will form a pulse which is shown in the picture below:

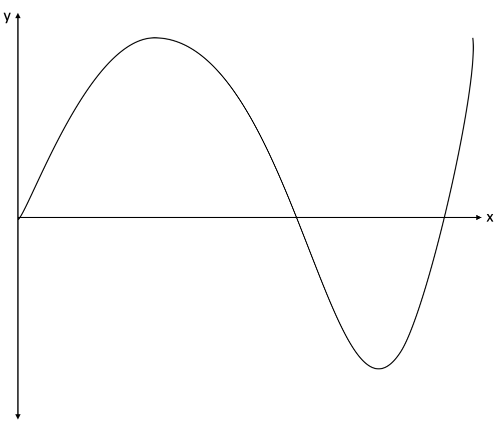

FIG. 16: The pulse profile.

Because we give disturbance on the string, now a tension force is acting on each point of the string. Let's locate several points on the string like in the picture below:

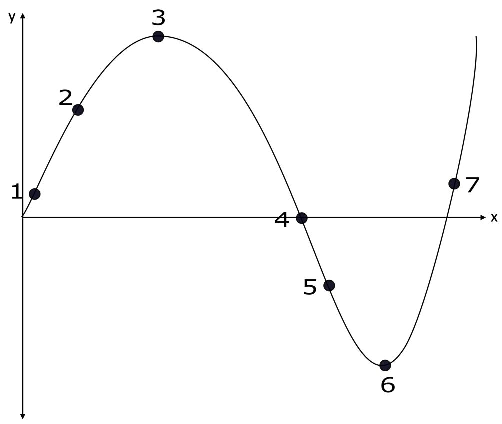

FIG. 17: The pulse profile has seven points placed on it.

(a) Draw the tension forces acting on that profile.

From the concave down part of the wave profile

(b) To which direction are the tensions acting on the points? Why?
(c) Which point has the greatest tension? Why?

From the concave up part of the wave profile

(d) To which direction are the tensions acting on the points? Why?
(e) Which point that has the greatest tension? Why?

(f) Between points 3 and 6, which point has the greater tension? Why?

(g) How do you write this relation mathematically?

Let's add two more points on the same wave profile shown below:

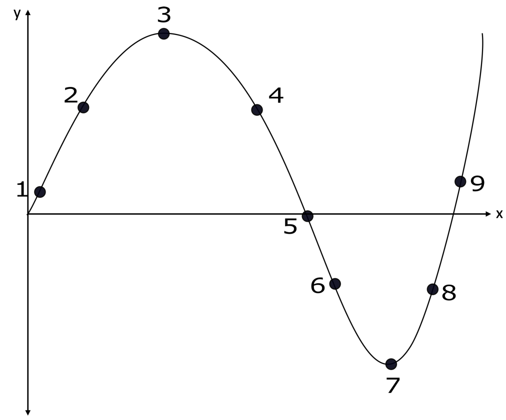

FIG. 18: The pulse profile has nine points placed on it.

(h) Draw the slope on the points of the graph.
(i) What is the slope? How do you express the slope of a point mathematically?
(j) Could you notice the difference in slopes between concave up and concave down? Could you describe the difference?
(k) How do you express the change of slope mathematically?

Now, you know that force is proportional to concavity.

(l) Find the WE based on the relation between force and concavity. (Mass of particles on the string is fixed and has a uniform density, k is a variable that describes the proportionality between force and concavity).

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Paper III: To be submitted in American Journal of Physics

Conceptual subtleties of the 1-D wave equation

Ricardo Karam, Muhammad Aswin Rangkuti

Conceptual subtleties of the 1-D wave equation

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Abstract

The 1-D wave equation is one of the most important equations in physics describing how waves propagate in space and time. Despite being a type of partial differential equation, understanding this equation is not trivial. There are several hidden concepts founded while exploring this equation deeply. In this work, we uncover the complexity of the wave equation by highlighting conceptual subtleties that often go unnoticed in teaching this equation.

I. INTRODUCTION

Partial differential equations were key to fostering the fruitful interplay between physics and mathematics. In particular, the wave equation (WE), first introduced by D'Alembert in the late 1700s, has been played a significant role in shaping this relationship[1–7].

Nowadays, the WE is seen as a fundamental equation in physics, describing a wide range of phenomena. Due to its importance, this equation is broadly taught across physics programs, although with different approaches. Among them, Feynman[8,9] derives the WE based on sound propagation and electromagnetic waves, while other authors derive it using Newton's 2nd Law[10,11].

Despite the apparent simplicity of $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$, understanding this equation deeply is far from being trivial. The purpose of this paper is to present certain conceptual subtleties related to the WE, which are not usually addressed explicitly in teaching, in order to make physics instructors aware of them.

II. WHY SECOND ORDER?

Mathematically, a 1-D wave is represented by a progressive function, i.e., it describes a fixed profile travelling horizontally as time goes by. This can be expressed by functions of the kind

$$y(x,t) = f(x \pm vt) \tag{1}$$

where f is an arbitrary function[12]. Thus, waves are expressed by functions of *two* variables, space and time, but in a particular way. For example, functions like $y(x,t) = \sin x \cdot vt$ or $y(x,t) = xt^2$ do not fulfill this requirement.

In fact, the WE can be *derived* from the assumption that $y(x,t) = f(x \pm vt)$. All that is needed is to differentiate it twice with respect to space and time, respectively. For simplicity, let us consider the case where the wave travels to the right, i.e., $y(x,t) = f(x - vt)$. Letting $x - vt = u$ and applying the chain rule:

With respect to x:

$$\frac{\partial y}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du} \tag{2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2}. \tag{3}$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = -v\frac{df}{du} \tag{4}$$

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{d^2 f}{du^2}. \tag{5}$$

By Comparing (3) and (5), we get

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}, \tag{6}$$

which is the 1-D WE. Thus, the WE can be seen as the result of simple manipulations of $y(x,t) = f(x \pm vt)$. But why do we need to differentiate twice? Can we not just stop at the first?

To answer this question, let us differentiate the same function once with respect to x and t. Once again, using the chain rule and letting $x - vt = u$, we have:

With respect to x

$$\frac{\partial y}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} \tag{7}$$

$$= \frac{df}{du} \tag{8}$$

With respect to t

$$\frac{\partial y}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} \tag{9}$$

$$= \frac{df}{du}(-v) \tag{10}$$

Combining the previous results, we arrive at:

$$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x} \tag{11}$$

Eq. (11) is often called the *transport equation* (TE). If the pulse is moving to the right, then the TE is expressed as:

$$\frac{\partial y}{\partial t} = +v\frac{\partial y}{\partial x} \tag{12}$$

This equation characterizes a wave moving to the left or right with velocity v. Thus, the TE is limited in that it can only describe waves propagating in one direction. Therefore, it can not account for wave interactions with boundaries.

Consider a sound wave propagating in a straight line. The transport equation can be used to describe this phenomena. However, when the sound wave collides with an object, the transport equation fails to describe the resulting changes in the wave's behavior, such as alterations in its direction, amplitude, or phase. The transport equation overlooks the influence of boundaries on wave propagation, which stresses that this equation can not capture complex wave phenomena.

We can test whether the TE and the WE satisfy certain wave properties, such as superposition. Consider two pulses traversing the same medium, the first to the right, the second pulse travels to the left, say $y_1(x,t) = f(x - vt)$ and $y_2(x,t) = g(x + vt)$.

At a particular time, the pulses meet, resulting in a wave superposition. The sum of the two wave functions is the total displacement:

$$y = y_1 + y_2 = f(x - vt) + g(x + vt) \tag{13}$$

Let us see if this function satisfies the transport and wave equation. We substitute $x - vt = u$ and $x + vt = v$ and derive it once with respect to x and t.

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{df}{du}\frac{\partial u}{\partial x} + \frac{df}{dv}\frac{\partial v}{\partial x} && (14)\\
&= \frac{df}{du}1 + \frac{df}{dv}1 && (15)\\
&= \frac{df}{du} + \frac{df}{dv} && (16)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{df}{du}\frac{\partial u}{\partial t} + \frac{df}{dv}\frac{\partial v}{\partial t} && (17)\\
&= \frac{df}{du}(-v) + \frac{df}{dv}(v) && (18)\\
& && (19)
\end{aligned}$$

Thus, we do not arrive at the transport equation. Now we derive it again with respect to x and t.

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2} + \frac{d^2 f}{dv^2} \tag{20}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{d^2 f}{du^2}v^2 + \frac{d^2 f}{dv^2}v^2 \tag{21}$$

If we substitute these results, we will arrive at the WE.

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \tag{22}$$

This demonstrates that the transport equation can in fact account for a single pulse travelling to the right and left. However, this equation cannot describe more complex wave phenomena like superposition.

III. SECOND DERIVATIVE OR CURVATURE?

The WE can be interpreted conceptually as a relationship between the resultant force acting on each point, which is proportional to its acceleration ($\frac{\partial^2 y}{\partial t^2}$), and the shape of the wave profile, often expressed by the rate of change of the function's spatial derivative, ($\frac{\partial^2 y}{\partial x^2}$). Intuitively, the greater the curve deviates from being a straight line at a given point, the greater the force on that point (see Fig. 1).

But is the second derivative the best quantity to represent this aspect related to the profile's shape? Consider, for instance, a parabolic function $y = (x - vt)^2$. Intuitively, the force should be greater at the vertex, but since the second derivative is constant, ($\frac{\partial^2 y}{\partial x^2}$) gives the same result for all points of the parabola (see Fig. 2). What is going on?

The issue here is a subtle difference between concavity and curvature. Whereas the former describes the slope's rate of change, the latter is inversely proportional to the radius of the tangent circle, at each point. To determine how much a curve deviates from a straight line, i.e., its curvature, we use the following formula:

$$\kappa = \frac{d^2 y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{23}$$

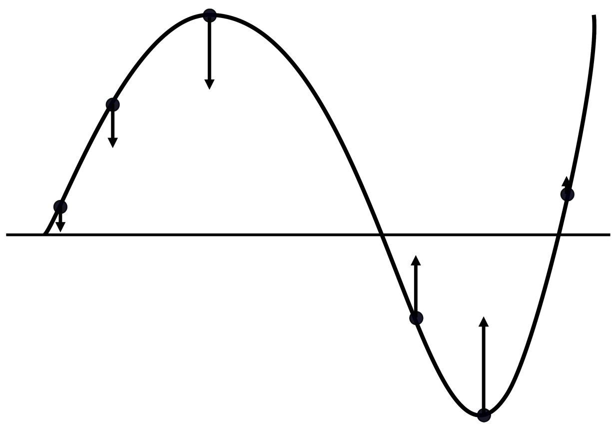

FIG. 1: The tension of each point on a vibrating string is proportional to the second derivative

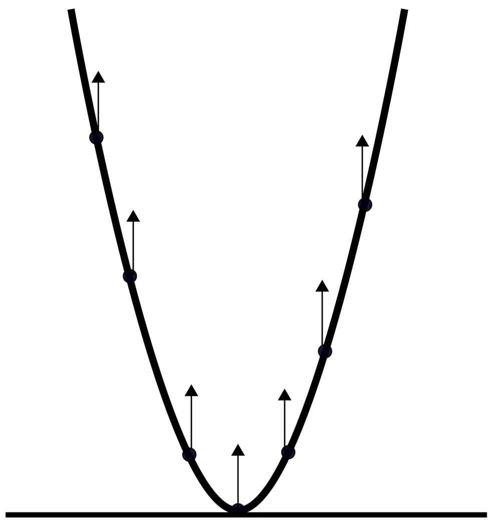

FIG. 2: A parabolic wave profile with constant forces in different points

Let us try to find the curvature of $y = x^2$:

$$dy/dx = 2x; d^2y/dx^2 = 2 \tag{24}$$

The curvature of parabola is defined as:

$$\kappa = \frac{2}{[1 + (2x)^2]^{3/2}} \tag{25}$$

$$\kappa = \frac{2}{[1 + 4x^2]^{3/2}} \tag{26}$$

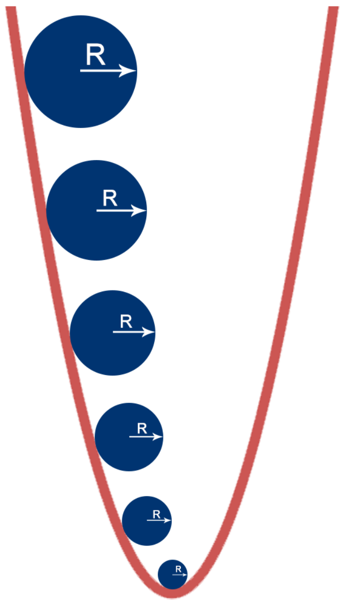

FIG. 3: Curvature of some points on parabola

Thus, we see that the value of κ (curvature) *is* dependent on x, as opposed to the second derivative. For example, the curvature of the parabola at the vertex is

$$\kappa(x=0) = \frac{2}{[1+4.0^2]^{3/2}} = 2, \tag{27}$$

which is its greater value.

Consequently, if we assume that force is proportional to curvature, the WE should be written as:

$$\frac{\partial^2 y}{\partial t^2} \propto \frac{\partial^2 y/\partial x^2}{[1+(\partial y/\partial x)^2]^{3/2}}, \tag{28}$$

which is much more complicated than the usual WE. However, notice that if $\partial y/\partial x \to 0$ the usual WE is recovered. We will come back to that.

IV. HISTORICAL INTERLUDE: TAYLOR'S ORIGINAL DERIVATION

Force proportional to curvature was key to one of the first applications of calculus to the study of waves, which was written by the English mathematician Brook Taylor, famous for the "Taylor series", in his "Methodus Incrementorum Directa et Inversa"[13]. In Lemma IX, Taylor shows that the force acting on a stretched string is proportional to its curvature at any given point. Due to its historical importance and pedagogical potential, we will try to reconstruct this derivation.

Taylor considered a curve with two adjacent points as shown in Fig. 4. Tangent lines are drawn from each point, represented by Bt and bt, and normal lines are also drawn from the same points, intersecting in S. The goal is to determine the net force acting on the arc segment $\overset{\frown}{Bb}$.

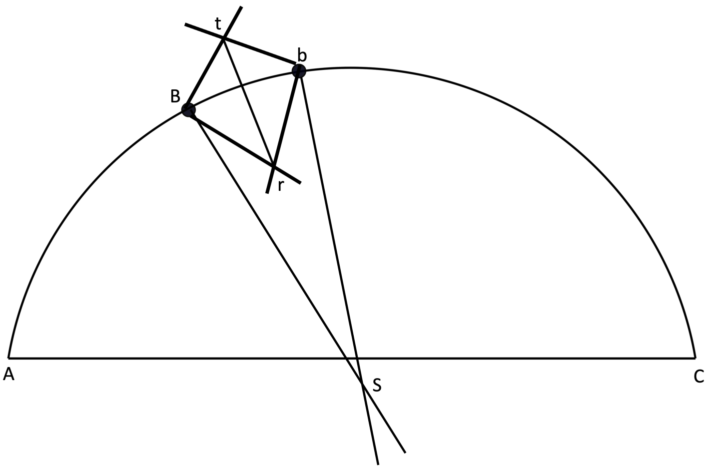

FIG. 4: Taylor's geometrical construction to show that force is proportional to curvature[13]

Fig. 5 highlights the essential features. We can see that the tensions T_B and T_b add up to F_{tr}, which is the net force of the tension due to the curved string. The magnitudes of the forces are proportional to the respective segments, thus

$$\frac{F_{tr}}{T_B} = \frac{tr}{tB} \tag{29}$$

From Fig 5, we can also see that the angles $\widehat{Bsb}$ and $\widehat{tBr}$ are equal, meaning that

$$\frac{tr}{tB} = \frac{Bb}{BS}, \tag{30}$$

yielding

$$\frac{F_{tr}}{T_B} = \frac{Bb}{BS} \tag{31}$$

or

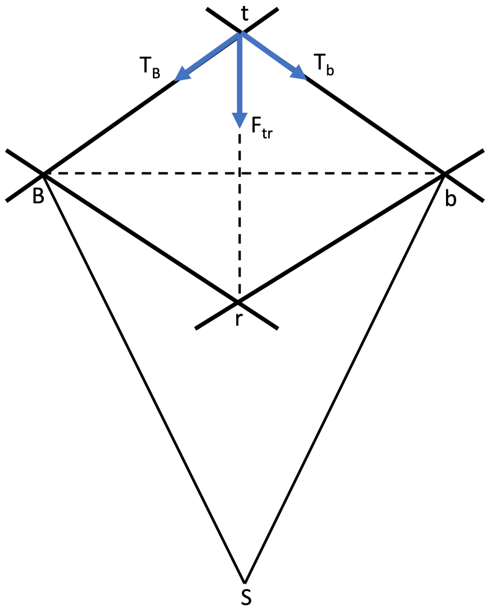

FIG. 5: The parallelogram that shows tensions acting on two points

$$F_{tr} \propto \frac{T_B Bb}{BS}. \tag{32}$$

As Bb tends to zero, Bb/BS becomes the curvature at point B. Therefore, in the limit, this relation shows that the net force acting on the small line segment Bb is proportional to the tension at point B and the curvature at this point, QED.

V. MAKING SENSE OF THE RESTRICTIVE CONDITIONS OF THE WE

Although it makes sense now to assume that force is proportional to curvature, Eq. 28 becomes very complicated to solve. Therefore, some assumptions are necessary to simplify the mathematical formalism in order to make it easier to solve. From a mathematical argument, let us consider the case where the first derivative of every point is close to zero ($\partial y/\partial x \approx 0$). Under this restrictive condition, we can make an approximation where curvature is approximately equal to concavity.

We can apply this mathematical argument to the case of the vibrating string problem, specifically when the string is fixed at two extremities[2]. This physical situation became the subject of debate between D'Alembert and Euler. D'Alembert considered the condition where the string has a very small displacement from the straight line and concluded that the derivation of the WE excluded many physical situations. On the other hand, Euler approached the problem by considering a plucked string and argued for the derivation using physical reasoning.

Nevertheless, D'Alembert's derivation of the WE is now commonly accepted to describe classical string vibrations. By using Newton's law, it is mathematically demonstrated why certain assumptions must be made in order to derive the WE. By contemplating only a small portion of the string, this derivation allows us to simplify the problem. Taking into consideration the string's entire curvature would undermine the linear approximation, making it impossible to derive the WE using Newton's laws.

After considering the small segment of the string, the basic assumption that must to be made is that the gravity is negligible compared to the tension on the string. This implies that the tension only arises from the vibration of the string and that gravity has little to no effect. We can related this when playing a guitar and we rotate it in any direction. The sound remains relatively the same, thus illustrating that the gravity has a negligible effect on the string.

If we observe the tension in one edge of the string in without making any assumptions, then the force can be decomposed into x and y components, as depicted in Fig. 6 Nevertheless, this condition makes the derivation more complicated.

In order to eliminate the horizontal components on the string, the second assumption made is that the deflections of the string are small. This condition made the the string is nearly flat. As a result, we can neglect $T\cos\theta_1$ and $T\cos\theta_2$ in the edges of the string because its magnitude in the opposite direction is equal. We can relate this by thinking a very tight string on a guitar. When we pluck it, the displacement of string is relatively small

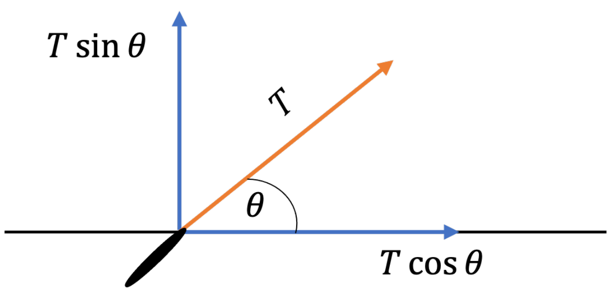

FIG. 6: The condition of one edge of the string with x and y components

compared to its length.

The second assumption enables us to assume that the tension throughout the string is uniform. We can visualize this condition when we increase the tension of a guitar string, that gives the tension at a constant level throughout the entire guitar string. Fig 7 illustrates the condition of the string under three assumptions.

Using the condition of the string in Fig 7, the net force of this small segment of string is given by:

$$T \sin\theta_2 - T \sin\theta_1 = T\Delta m \tag{33}$$

The slope of two edges of the string:

$$m_1 = \frac{\partial y_1}{\partial x} = \tan\theta_1 \tag{34}$$

$$m_2 = \frac{\partial y_2}{\partial x} = \tan\theta_2 \tag{35}$$

Since the string has a small vibration, then $\sin\theta_1 \approx \tan\theta_1$ and $\sin\theta_2 \approx \tan\theta_2$, thus

$$T(m_2 - m_1) = T\Delta m \tag{36}$$

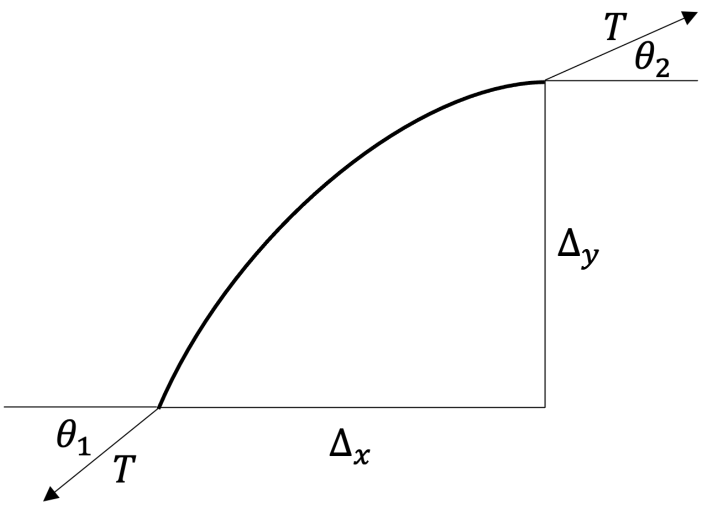

FIG. 7: The condition of the string under two assumptions

We consider the mass on the small segment of the string:

$$m = \mu \Delta x \tag{37}$$

where μ is the mass per unit length.

Applying Newtons' second law, we get:

$$T\Delta m = (\mu \Delta x)(\frac{\partial^2 y}{\partial t^2}) \tag{38}$$

$$T\frac{\Delta m}{\Delta x} = \frac{\partial^2 \mu y}{\partial t^2} \tag{39}$$

Then, we consider the limit of $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{\partial m}{\partial x} = \frac{\partial^2 y}{\partial x^2} \tag{40}$$

Substituting the results, we will arrive at the WE:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \tag{41}$$

where $v = \sqrt{T/\mu}$ is the velocity of the propagation.

VI. CONCLUSION

In this work, we have highlighted some conceptual subtleties of the WE that are not commonly presented in textbooks. In Physics Education Research (PER), several studies have found that students often view physics equations solely as mathematical tools, lacking a proper understanding of their physical implications[14–16]. We argue that by presenting these unconventional aspects might be one way to help addressing that belief.

Physics equations are typically used in teaching to solve problems that often involve the manipulation of numbers and variables[17]. However, this practice may hinder their roles and conceptual status. Some studies have proposed different perspectives to explore physics equations and concepts that can convey their deep meaning[18,19]. One more thing that needs to be considered related to this is the advantage of embedding this epistemological dimension in teaching. This includes using the historical aspect[20–22]. These studies emphasize that using this approach might be more beneficial to pre-service physics teachers in order to improve their educational practices[23], or it offers a new creative perspective of teaching that allows students' curiosity and interest in the topics[20].

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