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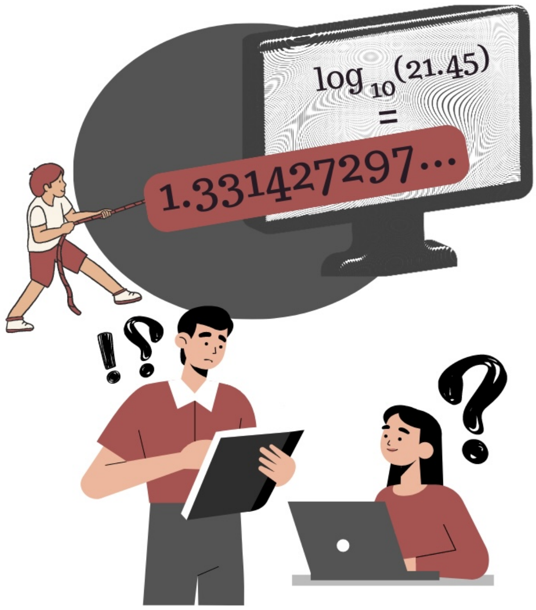

PhD Thesis

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Future teachers' knowledge of real numbers and functions on computers

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A PhD thesis by

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Resume

Mange gymnasielærere i matematik finder det vanskeligt at forbinde den teoretiske viden de tilegnede sig på universitetet med de praktiske opgaver som er knyttet til gymnasial matematikundervisning. Med udgangspunkt i Klein's anden diskontuinitet søger dette PhD projekt forsøger at undersøge denne vanskelighed ved at analysere den matematiske viden hos universitetsstuderende som har afsluttet deres matematiske studier og planlægger at blive gymnasielærere, idet vi fokuserer på et specifikt matematisk emne – reelle tal. Eftersom computer software har indtaget en stadig mere dominerende rolle i dansk matematikundervisning på sekundært niveau, ikke mindst hvad angår beregninger med og af reelle tal, omfatter denne afhandling tre forskningsgenstande: reelle tal i universitær matematik, reelle tal i gymnasial matematik, og reelle tal som de repræsenteres i computere. Afhandlingen ser på modellen "uendelige decimalbrøker" som en mulig bro mellem de modeller af de reelle tal som findes i gymnasiet og på universitetet. Herudover introduceres begrebet beregnelighed for at analysere betydningen af computergenererede decimalrepræsentationer af reelle tal. At forstå reelle tal som uendelige decimalbrøker er afgørende for at linke de tre nævnte forskningsgenstande. Den antropologiske teori om det didaktiske (ATD) er grundlæggende for afhandlingen. ATD-begreberne institutionel relation og praxeologi udgør de centrale elementer i vores model af Klein's anden diskontinuitet og i analysen af muligheder for at overkomme denne. Udforskningen af de studerendes arbejde med uendelige decimalbrøk-modeller af de reelle tal, og af beregnelighed, gennemføres i konteksten af "capstone" kurset UvMat ved Københavns Universitet. Studiet involverede design af to ugeopgaver som adresserer disse to begreber, analyse af de studerendes besvarelser af ugeopgaverne, og interviews gennemført i relation til besvarelserne. Analysen af de studerendes besvarelser viser, at tilgangen til uendelige decimalbrøker sammen med beregnelighedssynspunktet til en vis grad kan understøtte de kommende læreres anvendelse af universitetsmatematisk viden på praktiske opgaver fra gymnasial matematik, og hjælpe dem med at åbne "black boxes" i matematikværktøjer til brug ifm. reelle tal. Men der er også mange studerende som ikke kan etablere en forbindelse mellem gymnasiets værktøjsbaserede matematikundervisning og de to ugeopgaver. Dette forhold fordrer yderligere udforskning, særlig ift at undersøge om explicit inddragelse af beregnelighedsbegrebet i kurset kan hjælpe de studerende til at etablere denne forbindelse.

Abstract

Many high school mathematics teachers struggle to relate the theoretical knowledge they gained in university to the practical mathematics teaching in high school. This PhD project, inspired by Klein's second discontinuity, seeks to bridge this gap by investigating the knowledge of university students who have completed their university mathematics study and plan to become high school mathematics teachers, focusing on a specific mathematical domain - real numbers. The primary goal of the project is to aid students in developing an advanced understanding of real numbers. As computer software has become ever more prevalent in Danish secondary mathematics education, particularly when dealing with real number calculations, the thesis considers three main research objects: real numbers in university, real numbers in high school, and real numbers on computers. The thesis explores the teaching of infinite decimal models of real numbers to bridge the models of real numbers in high school and university. Additionally, the concept of computability is introduced to analyze the meaning of computer-generated decimal representations of real numbers. Understanding real numbers as infinite decimal representations through the lens of computability is crucial for linking these three research objects. The theoretical framework of the Anthropological Theory of the Didactics (ATD) is foundational to this thesis. The notions of institutional relations and praxeology from ATD are presented as key elements for modeling Klein's second discontinuity and establishing a connection to address this gap. The exploration of students' work with infinite decimal models of real numbers and computability is carried out within a "capstone" course called UvMat at the University of Copenhagen. The present study involved the design of two weekly assignments that incorporate these concepts, the analysis of students' responses to the two assignments, and interviews conducted in relation to these tasks. The analysis of students' work reveals that engaging with infinite decimals from the standpoint of computability can, to a certain extent, assist future mathematics teachers in applying their university-acquired knowledge to address practical tasks in high school and aids them in opening the "black box" of computers when handling real numbers. However, many students fail to establish a link between mathematics teaching with digital tools in high school and the two assignments. This aspect needs further investigation, particularly in exploring whether explicitly teaching the concept of computability as a mathematical object within the course can help students perceive the connection.

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Being a member of the Department of Science Education at the University of Copenhagen fills me with pride. I express my thanks to everyone in the department for their kindness. A shout-out to my old office friends, Jonas and Aswin, who provided assistance with any challenges I encountered in the department.

None of this would have been possible without the unwavering support of my parents in China. My gratitude also extends to my family and friends in Denmark and China for their continuous support and encouragement.

In the end, I extend special thanks to Henriette Tolstrup Holmegaard, whose recommendation led me to the research group I am proud to be a part of today.

The thesis is based on the following four papers.

Paper I

Winsløw, C., Bosch, M., González-Martín, A. S., & Huo, R. (2023). Technology in University Mathematics Education. In B. Pepin, G. Gueudet, & J. Choppin (Eds.), *Handbook of Digital Resources in Mathematics Education*. Springer International Handbooks of Education. Springer, Cham. https://doi.org/10.1007/978-3-030-95060-6_34-1

Paper II

Huo, R. (2023). Drawing on a computer algorithm to advance future teachers' knowledge of real numbers: A case study of task design. *European Journal of Science and Mathematics Education, 11*(2), 283-296. https://doi.org/10.30935/scimath/12640

Paper III

Winsløw, C., & Huo, R. From global to local aspects of Klein's second discontinuity.
Accepted by the special FUMME issue of the journal *Recherches en Didactique des Mathématiques.*

Paper IV

Huo, R. Secondary teacher knowledge of real numbers and functions as handled by computers: the critical notion of computability.
Submitted to the journal *For the learning of mathematics* and currently **under revision**.

Other contributions during the PhD period.

Huo, R. (to appear). Future teachers' knowledge of real numbers. *The 7th International Conference on the Anthropological Theory of the Didactic (CITAD7).* Springer Nature Swizerland AG.

Winsløw, C., & Huo, R. (2022). Task design for Klein's double discontinuity. In M. Trigueros, B. Barquero, R. Hochmuth, & J. Peters (Eds.), *Proceedings of the INDRUM2022* (pp. 558-567). University of Hannover and INDRUM.

Huo, R. (to appear). Computability as a key aspect of future teachers' knowledge of real numbers and functions. *The 13th Congress of the European Society for Research in Mathematics Education (CERME13).* ERME

Table of content

1 Introduction

During my time as a pure mathematics student in China, I encountered a significant loss of motivation. It felt like the content I was learning in university had little connection to what I had studied in high school. As someone who excelled in high school mathematics, I found myself struggling when transitioning to university-level mathematics. I often questioned the practical applications and real-life significance of the theoretical mathematics knowledge I was acquiring. It was not until I began studying programming that I started to appreciate the importance of theoretical mathematics knowledge in understanding and creating algorithms. Later, when I had to use Matlab for mathematical modeling, I began to see the value of mathematical software. However, my involvement in computer-related mathematics came to a halt when I moved to my research on ergodic optimization for asymptotically additive potentials in connection with the final thesis for my Master's degree (cf. Huo et al., 2023).

Upon completing my Master's program, I felt confident about carrying my seven years of university-level mathematics knowledge with me as I aspired to become a high school teacher. However, when I revisited my former high school assignments, I realized that I had lost many problem-solving "tricks" for high school tasks over the years. I was no longer as skilled as a high school graduate in mathematics. My university mathematics knowledge seemed to have no compensating effect in this situation, as I contemplated the outlook of becoming a high school teacher. With one exception, however: as I looked again at high school mathematics questions, I noticed that many of them could be easily solved using computer software, at least for most real-value calculations. However, high school mathematics education in China typically does not involve the use of digital tools that I used in university like Matlab (except for calculators, which however are not allowed during exams).

With these doubts and questions, I began my PhD project at the University of Copenhagen. One significant reason for choosing Denmark was its integration of digital tools into high school mathematics education. I was also introduced to the issue that had been brewing within me, known as *Klein's double discontinuity*, which I will elaborate on later. Due to language barriers, I was unable to carry out empirical studies at the Danish high schools, so I directed my focus towards university mathematics education, with a particular emphasis on mathematics students who aspire to become secondary school teachers, called *future teachers* in this thesis.

1.1 The overall purpose of the PhD project

The challenges I have faced are rooted in an intricate lack of clarity regarding the interrelationships between three fundamental components: university mathematics education, high school mathematics education, and digital technology. The most conspicuous discrepancy lies in the variance between university-level mathematical knowledge and the sort of mathematical understanding expected in high school, which is also known as *Klein's second discontinuity*.

One could visualize each piece of mathematical knowledge in high school akin to individual beads that need stringing together in a particular sequence. Upon completing high school mathematics, students should possess an ordered string of these "beads" instead of merely an assortment of disparate elements. This calls for teachers to provide students with a figurative "string" and a systematic "method of arranging the beads" before commencing teaching. To facilitate this, teachers need a robust mathematical foundation, particularly in theoretical mathematics, to connect individual pieces of knowledge. This means high school teachers should approach teaching materials from a comprehensive perspective, comprehending and mastering the knowledge rather than merely focusing on sharing techniques for solving practical tasks. Even if teachers' university-level mathematical knowledge might not be visibly applied in high school classrooms, it could still significantly shape teaching strategies and their implementation. Hence, a pivotal question arises: What specific university mathematics knowledge can effectively bolster high school mathematics teachers in comprehending and mastering the goals of high school teaching, and how can this be achieved?

In this thesis, the relationship between mathematics education in universities or high schools and the use of digital technology shares a similar goal of facilitating more accessible and efficient mathematics learning and teaching. Digital technology, within the context of this thesis, refers primarily to computer software and calculators, with the latter encompassed under computers due to their similar functional aspects. For high school mathematics teachers, effective mathematics teaching is not just about understanding the content but also about adeptly using computers. However, most teachers' grasp of computer use tends to focus primarily on using specific functions within the software. For instance, they might know how to use a command in some mathematics software to compute values of certain special functions but might lack an understanding of the software's underlying principles or the rationale behind employing a particular command within the mathematics software. In this sense, the gap between learning university-level mathematics and teaching high school mathematics is closely related to the disparity in computer utilization. Reflecting on personal experience, the computer knowledge acquired during high school teachers' university mathematics studies does not readily translate to the teaching of mathematics in high schools. Thus, the question could be reformulated to inquire: What specific university mathematics knowledge, combined with advanced computer proficiency, can effectively support high school mathematics teachers in understanding and mastering the computer-related mathematical objectives in high school teaching, and how can this be achieved?

Addressing this pivotal question necessitates more than simply attending standard high school or university mathematics classes. It requires the design and experimentation of interventions that seamlessly intertwine these fundamental components. Thus, focusing on future teachers preparing to engage in high school mathematics teaching in a context that involves extensive use of advanced mathematical software, becomes crucial. The core objective of my PhD project is to present results from such experimental work on ways to enrich the mathematical knowledge of these future teachers, specifically in the context of computers. The entire project centers around a crucial yet intricate mathematical domain — real numbers. It critically examines three closely related research objects that are all concerned with prospective teachers' understanding of real numbers in a variety of contexts. These research objects encompass real numbers in university mathematics learning, real numbers in high school mathematics teaching, and real numbers on computers (as depicted in Figure 1). Ultimately, the project endeavors to comprehensively connect these diverse research objects, aiming to bridge the gaps and provide vital insights into the central question:

> **Q**: How can future teachers bridge the gap between the model of real numbers acquired at university and the model they are expected to teach in high school within the context of computers?

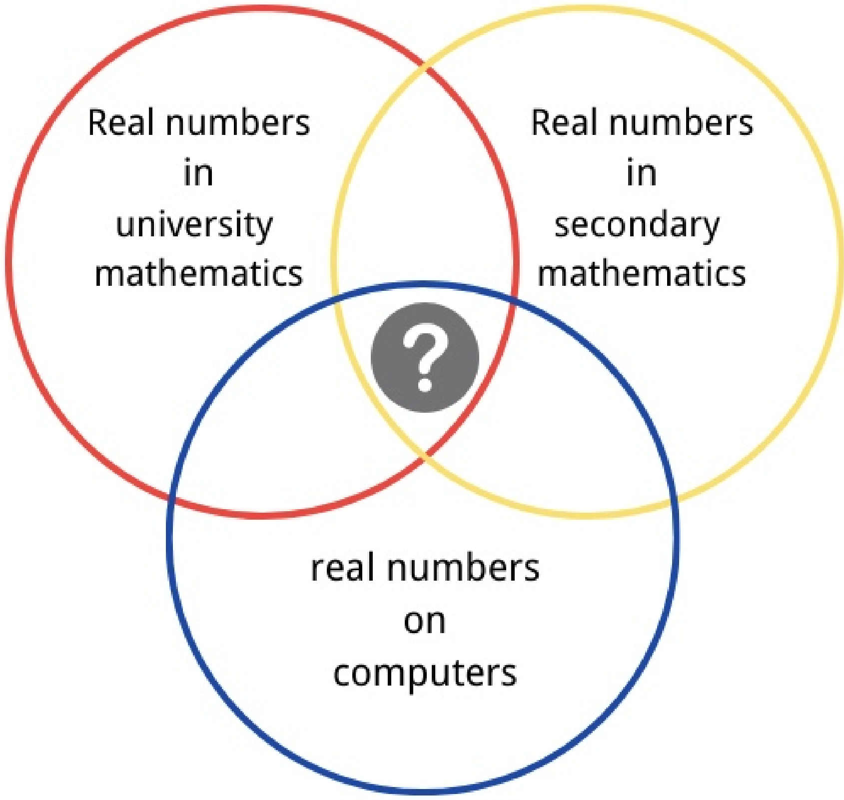

Figure 1: The research objects of this PhD project.

1.2 Structure of the thesis

This thesis will provide a comprehensive account of my entire research project conducted throughout my PhD studies. The primary focus of this thesis revolves around the knowledge of prospective mathematics teachers regarding real numbers as infinite decimal numbers on computers.

Section 2 will briefly recall the main points of the literature review contained in Paper I, which focuses on research concerning the digital learning environment for university-level mathematics students. Then we proceed to elucidate the distinct models of real numbers that appear in high school and university. Following this, I will enter the core issue that motivated my research project, which is known as Klein's double discontinuity. I will provide a brief review of previous research on this problem. Following this, I will introduce some mathematical prerequisites for the specific case of real numbers - the *infinite decimal model of real numbers* and the concept of *computability*, which will be essential for addressing the second discontinuity in this particular mathematical context.

To guide the exploration of the second discontinuity, I have employed the Anthropological Theory of the Didactics (ATD) as a foundational framework. This framework is reviewed in Section 3, for the parts actually used in the thesis. It will play a pivotal role in modeling and analyzing the second discontinuity. Based on these theoretical preliminaries, I can then formulate the overall research questions for the thesis.

The research questions have been developed and investigated in relation to a specific institutional context, namely a "capstone" course for future mathematics teachers at the University of Copenhagen. In section 4 we provide an outline of pertinent features of this context at a level of detail which goes slightly beyond what is feasible within the frame of a journal paper.

As we shall see, the research questions contain both theoretical and empirical dimensions. Identifying necessary extensions and elaborations of students' mathematical knowledge in a given context of undergraduate study naturally begins with theoretical analyses, as does the construction of tasks which support the students' learning corresponding to these goals. Then, the implementation must be observed in the given context with appropriate methods to gather data. Both the preliminary analysis, the task design and the data collection must be done with due regard to the specific conditions in the context, besides naturally being guided by the research questions and the underlying theoretical framework. Section 5 is devoted to explain the methodological choices made to meet these needs.

In section 6, we provide an overview of the results from the three main papers, including both task design and the results of analysing empirical material like students' answers to the tasks.

In section 7, we discuss in more detail how the results are related to each other and to previous research, as well as the limitations of the results which are implied, for instance, by the context and small-scale nature of our theoretical analysis, designs, and experiments.

In the concluding section 8 of the thesis, we will summarize the key findings and implications of the research presented and also outline potential avenues for future research in this domain.

2 Definitions and background research on main components of the thesis subject

As Figure 1 illustrates, the subject of the thesis can be seen as situated in the intersection of larger research problematiques in which more or less extensive research has been done. In this section, we review some of that research and also provide some of the basic notions which structure these areas.

2.1 Digital resources in university mathematics education

At the outset of my research project, I investigated the utilization of digital resources within university mathematics education. Several factors guided my choice to focus on digital tools at the university level rather than in high school mathematics education. Drawing from the example of Denmark, although various digital tools, particularly *Computer Algebra Systems* (CAS) such as Maple and (a part of) GeoGebra, are widely used in schools, their primary function revolves around calculations and graphing. They are not extensively employed as instruments for students to explore deeper theoretical mathematics knowledge beyond the scope of textbooks. However, the personal knowledge and utilization of digital resources by secondary mathematics teachers could be expected to have significant implications for their mathematics teaching, especially concerning the integration of technological tools. Unfortunately, there are not any university courses in Denmark solely dedicated to enhancing future teachers' understanding of mathematical digital tools and establishing a connection with theoretical mathematics.

Paper I provides a categorization of four classes of digital resources based on previous research on their use in university mathematics education (refer to Figure 2 for more details, as discussed in Paper I). This thesis, however, will exclusively focus on digital tools. In essence, digital tools refer to computer software and calculators utilized for solving mathematics tasks. Based on the use of these tools, they can be categorized into two groups: *ready-made tools* and self-made or *programming tools*. It is worth emphasizing that a single piece of software can supply tools in both categories depending on the user's specific purpose. For instance, Maple can be considered a ready-made tool when used for calculations, and it can also be viewed as a programming tool when its coding functionality is utilized.

	Receptive use	Productive use
Tools	*Ready-made tools*	*Programming*
Media	*Library media*	*Interactive media*

Figure 2: Four categories of digital resource (see Table 1 in Paper I).

The most frequently mentioned digital tools in the literature are the ready-made ones (e.g.

Lavicza, 2010; Teodoro & Neves, 2011; Oates, 2009; Winsløw, 2003a; Troup, 2019; Buteau et al., 2010; Buteau et al., 2014; Jarvis et al., 2014; Kilicman et al., 2010; Turgut & Uygan, 2014; Sümmermann et al., 2021). Several of these tools are extensively used in high schools. While their usage becomes more advanced at the university level compared to high schools, their primary purpose is to simplify complex or inaccessible steps in graphing or numerical computation. Therefore, it is important to note that the complexity lies in the mathematical content these tools support, not in understanding how these tools operate. In other words, whether at the high school or university level, these ready-made tools are essentially "black boxes", and hardly any discussion of their operational principles, neither in teaching nor in the mathematics education literature.

Conversely, to engage in mathematical use of programming tools requires students to develop new, structured strategies for solving mathematical problems, thus establishing a bridge corresponding to an essential link between between computer science and mathematics (e.g. Buteau et al., 2020; Broley et al., 2018; Sangwin & O'Toole, 2017; Cline et al., 2020). Currently, according to the literature reviewed in Paper I, many university students pursuing pure mathematics have encountered at least one programming language during university study. Using such productive digital tools requires (and might therefore lead to acquiring) a solid grasp of mathematical theory to construct algorithms and translate them into programming code. In recent years, programming has gained traction in both university and high school mathematics education. Although programming can provide some counterpart to the "black box" nature of ready-made tools, the convergence of these two aspects does not seem to be addressed in the research literature on the mathematical education of prospective teachers.

One of the key questions driving this project is how programming can be employed to gain insight into the inner workings of CAS software and how this knowledge can be effectively conveyed to future mathematics teachers. I will explore this question in conjunction with the disparity in the understanding of real numbers between high school and university mathematics education, aiming to evaluate what knowledge future high school mathematics teachers possess about real numbers in a computer-based context.

2.2 Two models of real numbers

2.2.1 Real numbers in high school

Knowledge about real numbers has been integrated into our mathematical study from primary school. However, the term "real numbers" appears after students have worked with various concepts related to real numbers, such as the number line, decimals, rational and irrational numbers. Durand-Guerrier (2016, p341) briefly described how numbers are treated in the French curriculum before the term "real numbers" is introduced to students:

> In the primary schooling, the elaboration of the concept of natural number is expected from the syllabus, relying on finite discrete collections and on one-to-one

> correspondences between finite discrete collections and initial sequences of counting numbers names. Rational numbers (fractions) and finite decimal expansions (that from now on we will call decimal numbers) are introduced in the context of measurement of continuous magnitudes, along with an arithmetic treatment. The number line plays an important role. At middle school, students go on developing competences about natural numbers, decimal numbers and fractions. They meet irrational numbers through the square root of natural numbers that are not perfect square, such as $\sqrt{2}$. The letter π is introduced in the formula for the circumference of a circle and the area of a disk, but students use mostly its decimal approximation 3.14.

González-Martín et al. (2013) discovered that in Brazilian textbooks, which are used in state secondary schools and approved by the Ministry of Education, the definition of real numbers is always introduced after rational and irrational numbers and real numbers are usually defined as the union of rational and irrational numbers.

Branchetti (2016) in the background reading of the thesis summarized known difficulties of high school students and teachers, related to real numbers in terms of five main themes, including irrational numbers and the number line, drawing from previous research, noting that this research is actually quite limited.

Voskoglou and Kosyvas (2012) emphasized the common challenge faced by both school and university students in understanding the definitions of rational and irrational numbers. González-Martín et al. (2013) provided examples from Brazilian textbooks where rational numbers are represented by fractions or decimals, and irrational numbers are those that cannot be expressed as fractions or are infinite non-periodic decimals. None of the textbooks in their study offered a mathematical explanation to clarify the equivalence between fraction and decimal representations. Furthermore, the definition for irrational numbers in this case assumes the existence of a broader category, later termed "real numbers" in the textbooks, which encompasses both rational and irrational numbers. However, what exactly constitutes these "real numbers" and why they are invented are often left unexplained in textbooks. In other words, a formal definition for real numbers is rarely provided. Students and teachers tend to accept these ambiguous objects intuitively. As noted by Voskoglou and Kosyvas (2012), university level knowledge for these concepts necessitates students to first have a formal approach to rational numbers. Additionally, it is observed that "the concept of rational numbers in general remains isolated from the wider class of real numbers (Moseley, 2005; Toepliz, 2007)" (Voskoglou & Kosyvas, 2012, p30). Fischbein et al. (1995) found that high school students and some preservice teachers encounter challenges in providing correct definitions for rational, irrational, and real numbers due to a lack of emphasis on teaching the hierarchical structure of the various classes of numbers in high school mathematics education. Interestingly, they noted that "the concept of irrational numbers does not encounter a particular intuitive difficulty in the students' mind" (Fischbein et al., 1995, p43). Branchetti (2016, p30) identified these challenges as stemming from "the incomplete understanding of rational numbers, the incommensurability and nondenumerability of irrational numbers".

Branchetti (2016, p31) identified another factor contributing to the challenges of knowledge related to real numbers as "the difficulties in managing and making sense of different representations". Some studies (e.g. Sirotic & Zazkis 2007a, 2007b, 2004) have proposed that some of these difficulties arise from teachers' knowledge of real numbers. For instance, Sirotic & Zazkis (2007b) found prospective secondary mathematics teachers encounter challenges with different representations, particularly when dealing with irrational numbers, even if they are familiar with the definitions. This also includes the difficulties for prospective teachers in recognizing infinite decimal representations as irrational numbers (Sirotic & Zazkis, 2007b) and in establishing connections between points on the number line and the decimal approximation of irrational numbers (Sirotic & Zazkis, 2007a).

In high school, decimal representations are commonly used in calculations, primarily handled by calculators or computers (Durand-Guerrier, 2016). Sirotic & Zazkis (2004) found that the tendency to rely heavily on calculators has led teachers and students to prefer decimal representations over fractional representations, contributing to their confusion about irrationality and infinite decimal representations. Therefore, as students in France, "students acceding to university have in general no idea of the differences and interplay between finite decimal numbers, rational numbers and non-terminating decimal expansions, and thus are not prepared for what they will be taught at university." (Durand-Guerrier, 2016, p341). Working with these different representations of real numbers without a clear connection between them, the relationship between students (and even teachers) and real numbers is like "blind men and an elephant".

2.2.2 Real numbers at university

Bergé (2010, p217) introduced the role of the set of real numbers $\mathbb{R}$ in university mathematics studies as follows:

> In most universities, the set of real numbers is approached in a progressive manner throughout several courses. In the first Calculus courses the set $\mathbb{R}$ is not explicitly defined; instead, what is used is the naive idea of considering $\mathbb{R}$ as all possible numbers, influenced by the image of the number line. This idea is compatible with the kind of work usually carried out in these courses, and it allows instructors to advance through the curriculum fairly quickly. Further on, when studies in Analysis start and the formally defined set $\mathbb{R}$ becomes the natural domain of functions, other properties become relevant. $\mathbb{R}$ is then usually introduced by means of the axioms of a complete ordered field. In advanced courses, $\mathbb{R}$ is presented as the set of rational cuts or rational Cauchy sequences.

Hence, real numbers at university no longer function as distinct and separated representations, as was the case in high school, but rather as a unified set to be characterized and utilized.

At the beginning of Analysis, students are often introduced to a new property of real numbers, frequently through *the supremum property* (e.g., in many French universities, as noted by Durand-Guerrier, 2016). When I studied Analysis, the textbook tried to connect the high school curriculum by asserting that all real numbers can be represented by infinite decimal representations, without an unambiguous construction. Occasionally, discussions with students regarding infinite decimal representations might also involve some properties, such as $0.\overline{9} = 1$. (e.g. Njomgang-Ngansop and Durand-Guerrier, 2013).

Dedekind cuts, as proposed by Dedekind (1963/1872), stand out as one of the most renowned method for constructing real numbers. Another method commonly employed for real number construction is through Cauchy sequences (Durand-Guerrier, 2016), which is more frequently used in university, particularly in connection with the property of completeness. However, in university mathematics studies, the process of constructing real numbers via Dedekind cuts or Cauchy sequences is commonly presented as supplementary material or assigned as part of reading tasks, with no extensive elaboration.

In university mathematics, real numbers take on a much more theoretical nature. Njomgang-Ngansop and Durand-Guerrier (Njomgang-Ngansop and Durand-Guerrier, 2013) pointed out that university students need to not only grasp how to validate related statements about real numbers but also understand the "truth" behind them (Durand-Guerrier et al., 2012). They supported this point based on a study involving nine fresh undergraduate students in France, focusing on the case of $0.\overline{9} = 1$.

In reviewing the literature on real numbers in university mathematics (e.g., Bergé, 2010 Njomgang-Ngansop and Durand-Guerrier, 2013 Durand-Guerrier et al., 2012 Durand-Guerrier, 2016), one can observe that certain representations, such as infinite decimals that persist from high school, are rarely subjected to explicit discussions about specific decimals in university mathematics education. Therefore, techniques based on such as using computers for decimal computations seldomly arise in the theoretically oriented learning process of university mathematics.

2.3 Klein's second discontinuity

Klein (2016/1908, p. 1) introduced the concept of double discontinuity to describe two gaps between high school and university mathematics. The first discontinuity occurs when a high school graduate enters university to study mathematics, often finding it challenging to bridge the contents between university and high school mathematics. This disconnection can lead some students to believe that high school mathematics has limited relevance to university-level mathematics. It is essential to understand that in this transition from high school to university, the role of the person as a student (learner) remains unchanged; what undergoes a transformation is the *mathematics objects*. These mathematics objects are taught differently at various institutions and are received by students with distinct expectations, posing a risk that students, like I once experienced, may struggle to connect the two. For example, in high school, a student is expected to solve practical mathematics tasks, while in university, the emphasis shifts to proving theorems and engaging in more abstract mathematical thinking. Klein's second discontinuity pertains to university students who aspire to become high school teachers, the future teachers discussed in this thesis. These students then encounter a scenario where the mathematics they studied in college does not appear useful to the task of teaching high school mathematics. In this discontinuity, both the position (from students to teachers) and the mathematics objects shift, due to the change of institutional context. These changes could result in university students, upon becoming high school mathematics teachers, adopting teaching methods "in the old pedantic way" (Klein, 1932/1924, p.1) which might be similar to those used by their own high school mathematics teachers (Lindgren, 1996), and their university mathematics with time will be as Klein described "remain only a more or less pleasant memory" (Klein, 1932/1924, p.1).

Klein's first discontinuity is evident in the transition from the high school model of real numbers to the university model of real numbers. This gap emerges as students move from the fragmented elements of the high school real numbers model, which includes concepts like irrational numbers, decimal approximations, and the number line, to the more formal real numbers model in university, which involves concepts such as Dedekind cuts and completeness. Numerous studies have highlighted the challenges associated with teaching real numbers in both high schools and universities, particularly considering the first discontinuity (e.g. Durand-Guerrier, 2016).

In my research project, the focus is on the second discontinuity, which pertains to the disconnection between the real numbers model future teachers acquire at university and the real numbers model they will teach in high school. This disconnection can be attributed to three possible factors.

The first factor involves the disparity in the use of digital technology. At university, though mathematics students have many experiences with digital tools (not only in mathematics learning), computers or calculators are rarely employed in the study of real numbers, as occurs at the beginning of first courses on real analysis. In contrast, high school teachers heavily rely on digital tools, especially CAS and calculators, for teaching subjects related to real numbers. Digital technology plays a crucial role in students' work with concrete real functions, like the computation of function values, zeros, etc. Therefore, future mathematics teachers face the challenge of comprehending real numbers within a computerized context and using computers effectively in teaching real numbers.

The second factor is the difficulty future teachers encounter in bridging the gap between their theoretical knowledge of real numbers acquired at university and the practical real numbers tasks they are expected to teach in high school. The transition from theoretical understanding to practical application in the high school classroom can be challenging without specific guidance or instructions from university education.

The third factor relates to the transition from a student role to a teacher role. Future teachers may experience confusion regarding the type of real numbers knowledge required for instructing students in solving high school-level real numbers tasks. This shift in roles and responsibilities can be accompanied by uncertainties about effectively teaching real numbers in a high school context.

Future teachers' education plays a crucial role in addressing Klein's second discontinuity, particularly in bolstering their knowledge of the mathematics subject they will teach in the future. As described by Adams (1998, p35):

> A major goal of teacher education programs is to provide opportunities for prospective teachers to acquire subject matter knowledge. The individual who proposes to teach subject matter to children should demonstrate knowledge and understanding of the subject matter to facilitate children's understanding of that subject matter (Ball, 1990; Even, 1990; Shulman, 1986; Simon, 1993). Mathematics subject matter knowledge includes understanding of concepts (conceptual knowledge), understanding of skills, symbolism, rules and procedures (procedural knowledge), and the relationships which may exist between conceptual knowledge and procedural knowledge (Ball, 1991; Even & Lappan, 1994; Leinhardt & Smith, 1985; Van de Walle, 1994). Assessment of prospective teachers to effectively develop and implement mathematics curriculum, instruction, and assessment relies heavily upon the teachers' demonstration of mathematics subject matter knowledge.

It is not the purpose of the present thesis to discuss the extent and effects of the (often rather superficial) contents that students are exposed to at university, in relation to the theoretical foundations of the real number system. Rather, its objective is to address and propose bridging measures for Klein's second discontinuity by enhancing future teachers' knowledge of real numbers based on the above three factors. Essentially, the aim is to construct a bridge between the two educational institutions, connecting the university model of real numbers with the high school model of real numbers. To achieve this objective, we need some foundational "materials" for the bridge's construction.

1. The infinite decimal model of real numbers - a third model of real numbers, that bridges the gap between the high school and university models, and is suitable for describing real numbers as they are handled by computers.

2. Computability - a paramathematical notion which is needed to understand the representation of real numbers on computers, and whose study we argue could help future teachers to gain insight into the "black box" of CAS and calculators, incorporating programming tools.

A detailed explanation of these two "materials" of the bridge will be given in the next subsections.

2.4 Infinite decimal model of real numbers

Decimal representations of real numbers are relevant to both high school and university mathematics education. The infinite decimal model of real numbers serves as a valuable bridge, connecting the distinct models used in high school and university settings. It is important to emphasize that the specific presentation and treatment of the infinite decimal model is not a novel creation introduced in this thesis; rather, it is drawn from established content, for example, from the book *The mathematics that every secondary school math teacher needs to know* (Sultan & Artzt, 2018).

Now, I will provide a brief overview of how real numbers can be formally modeled using infinite decimals, drawing from the second part of Chapter 8 of the book *The mathematics that every secondary school math teacher needs to know* (Sultan & Artzt, 2018, p331-358). The first question addressed in this model is whether all real numbers as defined as at least outlined in undergraduate mathematics can be represented as infinite decimals. In this representation, an infinite decimal $x = 0.c_1c_2c_3.... \in [0,1)$ can be expressed as a series $x = \sum_{i=1}^{\infty} c_i * 10^{-i}$, where $c_i \in \{0,1,2,...,9\}$ and $i = 1,2,....$ The *Archimedean property* of $\mathbb{R}$ ensures that all real numbers can indeed be represented by infinite decimals, which is formalized in the following two theorems (Sultan & Artzt, 2018, Theorem 8.52, p336; Theorem 8.52, p337):

Theorem 1 *Any decimal number $.d_1d_2d_3...$ represents a series that has a finite sum.*

Theorem 2 *Every real number N, where $0 \leq N < 1$ can be written as a decimal.*

The second question pertains to whether a real number possesses only one infinite decimal representation. Addressing this, Sultan and Artzt (2018, p348, Theorem 8.65) proved another crucial theorem in Chapter 8:

Theorem 3 *Every nonnegative real number x between 0 and 1 is represented by a unique infinite decimal, except those numbers whose decimal representations terminate in an infinite number of zeros or an infinite number of 9's. These and only these decimals can also be represented in two ways.*

Noting that the result in these theorems can be extended from the interval $[0,1)$ to all real numbers by adding integers.

In this thesis, the unique infinite decimal representation of a real number that does not terminate in an infinite number of 9's is called the *canonical decimal representation* of the real number.

These three fundamental theorems elucidate the rationale behind representing real numbers using decimals. The proofs of these theorems, as fully presented by Sultan and Artzt (2018), incorporate concepts from university mathematics, such as series and convergence. Leveraging university-level concepts to explain decimal representations, which are intuitively used as real numbers in high school mathematics education, makes this model a valuable bridge between high school and college models of real numbers.

2.5 Computability

It is evident that a general infinite decimal cannot be fully contained in a computer. Therefore, the question arises: how does a computer handle infinite decimals when performing calculations involving real numbers, particularly irrational numbers? In essence, how is the infinite decimal model of real numbers adapted to a computer? This prompts the introduction of a new concept - computability.

In the 1930s, pioneering researchers such as Alan Turing and Alonzo Church laid the foundation for computability theory, often referred to as "recursion theory" (Soare, 1999). The term "computability" finds its place in various scientific domains, including mathematical logic and computer science. However, there is a noticeable absence of research in the field of mathematics education addressing this concept.

Ménissier-Morain (2003) provided three equivalent definitions of computable real numbers from the area of computer science along with all relevant properties and corresponding proofs. Lucier (2022) presented an informal translation of one of the definitions, named "B-approximable real numbers"(Ménissier-Morain, 2005), into a more accessible version for those not studying computer science:

> we say that a real number x is computable if and only if there can exist a computer program that computes for each positive integer k (roughly, the number of digits we want to the right of the decimal point) an integer N such that
>
> $$|N - 10^k x| < 1, \text{or equivalently } |\frac{N}{10^k} - x| < \frac{1}{10^k}. \qquad (*)$$

This characterization as "informal" primarily stems from Lucier's omission of a clear definition for the term "computer program". In an effort to maintain simplicity, Paper IV (p3) provides an informal yet improved version (compared to Lucier's) of the definition for "computer program":

> A branch of coding is called a *computer program*, when provided with an input and a natural number n, which could produce a decimal with n decimal digits. A computer program has to have an end condition and be restricted to basic arithmetic operations — addition, subtraction, multiplication, and division - of integers. In other words, a computer program does not function like a "magic button"; its algorithm should facilitate a process that can also in principle be performed manually with paper and pen, which we refer to as a "manual operation".

However, Lucier's explanation of the computability of real numbers lacks rigor. To illustrate, let us consider $\sqrt{2}$ as an example. Assume we look at a concrete computer program that computes $\sqrt{2}$, in the sense of $(*)$. With the notion in $(*)$, when $k = 2$, the value of N could be either 142 or 143. This implies that when approximating to the second decimal place, $\sqrt{2}$ could be represented as either 1.42 or 1.43. So while the infinite deicmal representation of $\sqrt{2}$ is unique, the finite approximations according to (∗) are not. Therefore, building upon Lucier's definition of computability, I will introduce a refined version named *absolute computability.* Paper IV defines absolute computability as follows:

Definition 2.1 *A real number x is called* absolutely computable *if there exists a computer program that, for any given $n \in \mathbb{N}$, computes the first n decimals of the canonical decimal representation of x, namely $\hat{x}(n) = \frac{\lfloor 10^n \cdot x \rfloor}{10^n}$ for all $n \in \mathbb{N}$.*

Figure 3 illustrates a potential computer program providing evidence for the absolute computability of $\sqrt{2}$. A concise explanation of the coding in Figure 3 is provided here, with a detailed discussion available in Paper I.

```
K := 1 :;
for i from 0 to 10 do
      for j from 0 to 9 do
           if (K + j * 10^(-i))^2-2 <= 0 then
                p := K + j*10^(-i);
           end if ;
end do;
K := p :;
print(x(i) = evalf(p, i + 1));
end do :;
```

Figure 3: A computer program demonstrating the absolute computability of $\sqrt{2}$ (see Figure 2 in Paper IV).

Executing the code in Figure 3 on Maple (a CAS software) allows for obtaining the canonical decimal representation of $\sqrt{2}$ with precision ranging from the first decimal place to the 10th decimal place. The code adheres to the fundamental principles of a computer program: it possesses an end condition and comprises only basic arithmetic operations. Thus, it can be classified as a computer program. In this specific computer program, $\sqrt{2}$ is treated as a zero of the function $f(x) = x^2 - 2$. The approach to determining the unique digit in each decimal place relies on the intermediate value theorem, a concept typically covered in university mathematics education. By adjusting the range of i, one can obtain the canonical representation of $\sqrt{2}$ with the desired level of accuracy.

In addition to determining the canonical decimal representation of a real number, particularly an irrational one, students and high school teachers heavily rely on computers when computing values of transcendental functions, such as $\sin x$, $\cos x$, and $\log_{10} x$. While manually calculating special values like $\sin \pi$, $\cos \pi$, or $\log_{10} 100$ may be straightforward, dealing with a decimal input, like 1.41, for functions such as $\sin 1.41$, $\cos 1.41$, and $\log_{10} 1.41$ requires the use of computers. Thus, it becomes crucial to understand how computers generate values for transcendental functions with decimal inputs from a computability perspective. Paper IV introduces the definition of absolute computability for a function as follows:

Definition 2.2 *Let $\mathbb{D}_n = \{10^{-n}y : y \in \mathbb{Z}\}$ and $\mathbb{D} = \bigcup_{n=\mathbb{N}} \mathbb{D}_n$. A function $f : \mathbb{R} \to \mathbb{R}$ is called* absolutely computable *if there exists a computer program that, for any given $x \in \mathbb{D}$ and $n \in \mathbb{N}$, computes $\widehat{f(x)}(n) \in \mathbb{D}_n$ for all $n \in \mathbb{N}$.*

Figure 4 presents an illustrative approach to calculate the first 4 decimal digits of the canonical decimal representation of the value of the function $f(x) = \log_{10} x$ with a given $M \in \mathbb{D}$. Similar to the code in Figure 3, the code in Figure 4 adheres to the two principles, establishing it as a computer program. By adjusting the parameter n in the code, one can obtain an increasing number of decimal digits for the canonical representation of $\log_{10} M$. It is crucial to note that the precision of the final result is independent of the number of decimal digits in the initial input M. A detailed explanation of the computer program is presented in Paper III.

```
compLog := proc (M, d := 10, n := 4);
    local a, i, res, Mnew;
    Mnew := M;
    res := 0;
    for i from 0 to n do
         for a from 0 by 1 while d^(a+1) <= Mnew do
         end do;
         Mnew := evalf((Mnew/d^a)^d);
         res := res+evalf(a*d^(-i));
    end do;
    return res;
end proc:;
```

Figure 4: A computer program demonstrating the absolute computability of the function $f(M) = \log_{10} M$ (see Figure 3 in Paper IV).

The distinction between Definition 2.1 and Definition 2.2 lies in the target of the computer program. For instance, when considering $\log_{10} 1.41$ and $\log_{10} 1.42$ as two distinct real numbers, the computer programs used to generate their canonical decimal representations may be the same or different. However, when treating them as two values of the function $f(x) = \log_{10} x$ with given $x = 1.41$ and $x = 1.42$, the computer programs employed to derive their canonical decimal representations must be the same.

3 Theoretical framework-ATD and research questions

3.1 Anthropological theory of the didactics

ATD, the theoretical framework of this thesis, was formulated by Chevallard in the 1980s and serves as a theory for modeling mathematical activity carried out within a given institutional context, as initially outlined in (Chevallard, 1991, 1999).

In this thesis, ATD will be employed in particular to model Klein's second discontinuity. Before entering this part, it is essential to introduce two crucial notions from ATD that will serve as key elements in modeling Klein's second discontinuity: *institutional relations*, and *praxeology*.

3.1.1 Institutional relations

ATD evolved from the *didactic transposition theory*, which is currently acknowledged as a subtheory within the broader framework of ATD (see Figure 5). Instead of an exhaustive introduction of the entire didactic transposition framework along with its constituent elements, I will first concentrate on a central aspect — namely, mathematical knowledge. A comprehensive overview and elaboration of each element involved in this transposition can be found in Chevallard (2019).

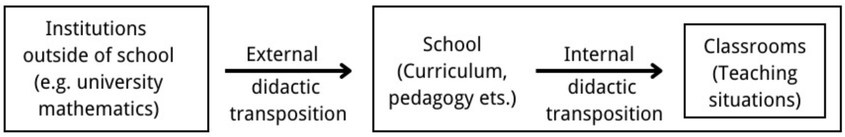

Figure 5: The didactic transposition: how scholarly knowledge outside of school is transposed into classroom.

Each piece of mathematical knowledge undergoes varied presentations contingent on its context and audience. To enhance clarity, the ensuing discussion specifically centers on a particular facet of mathematical knowledge within the domain of real numbers denoted as h, with a specific focus on the university and high school.

The knowledge of real numbers acquired through university mathematics study by students is denoted as the *scholarly* version of h, labeled as h_u, encompassing concepts like Dedekind cuts and completeness. The real number knowledge involved in high school mathematics study is termed the high school version of h, denoted as h_s, and includes concepts such as irrational numbers. h_u focuses on the theoretical aspects of real numbers, enabling university students to engage in more profound theoretical explorations in mathematics, for example, the completeness of real numbers plays an indispensable role in proving some theorems in Analysis. "In order to become teachable and (learnable)", h_s "has to be "transposed" from some hypothetical scholarly world to adapt to conditions specific to the schools" (Chevallard, 2019, p76). The theoretical aspects of real numbers are usually taken for granted by most students as well as teachers, which makes h_s places emphasis on the practical application of real numbers, aiding high school students to address practical problems such as calculating the values of functions. Thus, the gap between h_u and h_s appears.

This thesis does not aim to investigate hypothetical strategies for the reduction of the gap between h_s and h_u such as altering existing curricula or modifying the content related to real numbers for learning or teaching at the university or high school level. While acknowledging that such a gap is an inherent aspect of educational institutions, this thesis operates under the assumption they are what they are. Therefore, the focus of this thesis shifts away from the study on h_u and h_s themselves towards the exploration of what could be done to connect them for future teachers.

A piece of mathematical knowledge encompasses at least one mathematical object. For instance, within h_s, not only the uncountably many real numbers but also classes of numbers like the irrational numbers, constitute such mathematical objects. Moreover, a mathematical object cannot be studied didactically without considering an institution and its different types of agents, referred to as *positions* by Chevallard. For example, we say "High school students use computers to calculate decimal approximations of $\sqrt{2}$". Here, the mathematical object is "decimal approximations of $\sqrt{2}$", the position is students, and the institution is high school. The relationship of the position x and a mathematical object o within an institution I is called institutional relation, denoted as $R_I(x, o)$ (Chevallard, 2019, p78).

In the high school setting (HS), multiple relationships can revolve around two distinct positions: mathematics teachers (t) and students (s). The fundamental distinction lies in the dynamics of teaching and learning the object o. Mathematics teachers shape their approach to teaching the object o based on their experience or beliefs regarding how students can effectively grasp and learn it. Concurrently, students acquire the object o as a result of their activity in the context of mathematics teaching. Thus, $R_{HS}(s, o)$ and $R_{HS}(t, o)$ represent two distinct yet interrelated relationships, each influencing the other.

For mathematics teachers, o is not a novel concept; hence, the relationship $R_{HS}(t, o)$ encompasses a crucial aspect — the teachers' own knowledge of o. Their relationship to o significantly impacts their teaching approach and, consequently, affects students' learning. While it develops through their teaching experiences in high school, a substantial portion is derived from their prior relationships, dating back to their time as students in university or even high school. Klein's second discontinuity considers teachers' knowledge within $R_{HS}(t, o)$ and explores the relationship between $R_{HS}(t, o)$ and their previous personal and institutional relations before assuming the role of mathematics teachers in high school.

3.1.2 Praxeology

How does one interpret $R_I(x, o)$? To provide a more tangible example, consider the activity "A high school mathematics teacher teaches real numbers-related knowledge". What specific aspects should be considered when understanding this activity? In a more general context, what contents should be encompassed in the process of teaching the mathematics objects involving real numbers? Addressing this question, Chevallard introduced a central subtheory of ATD, known as *the theory of praxeologies* (cf. Chevallard, 1991, 1999, 2019), designed to model the mathematics object o. Praxeological analysis can guide us in exploring and comprehending the motivations and outcomes of human activities, as explained by Chevallard (2019):

> What I had to do, what I do — handle something, think about it, love it, discard it, etc.—determines my (personal) relation to the objects that make up my cognitive universe. The notion of praxeology was introduced as an essential means of analyzing human activity—be it mathematical or otherwise. This is where the reason for labeling "anthropological" the theory developed is most obvious.

Within each praxeology, there are two integral components: the *praxis block* and the *logos block* (see Figure 6), denoted as Π and Λ respectively. A prexeology can be represented as a pair (Π, Λ). The praxis block delineates a practical activity, such as making a pizza, while the corresponding logos block provides an explanation for all the steps involved in the process of making a pizza, rendering the activity meaningful. Each praxis block comprises a *type of tasks* and a technique utilized to solve these tasks. This technique is explicated through a discourse known as *technology*, and the technology can be generated and justified by a higher-level discourse termed *theory*. Together, a technique and a theory constitute the logos block. It is crucial to clarify that the term "technology" in the context of praxeology differs from the technology discussed in our introduction to digital tools. In praxeology, technology is an abstract conception that encompasses mathematical theorems, whereas, in this thesis, digital technology is simply understood as computers.

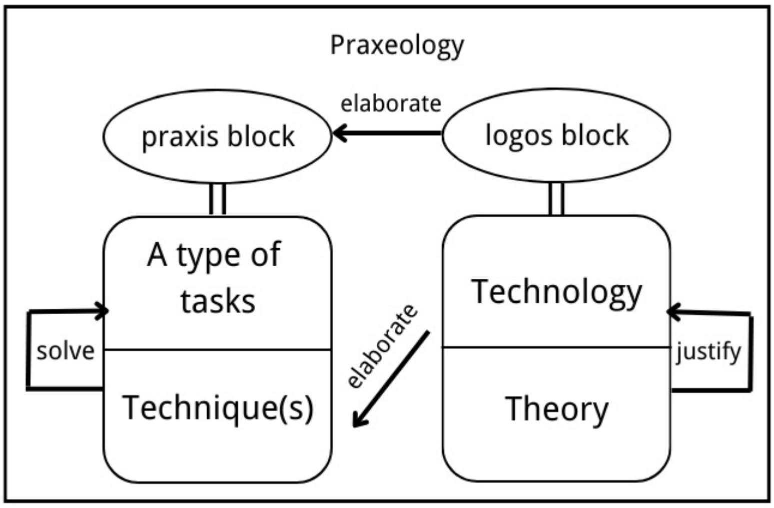

Figure 6: The components of a praxeology and their relations.

Now, let us return to the mathematical objects we are primarily focusing on in this thesis — real numbers, especially in the context of computers. In high school, students can efficiently tackle almost all calculations involving real numbers through computers, including tasks like computing square roots and function values. In this context, we can conceptualize the computer as a set of "buttons" that, when pressed, provide answers to a host of different questions. Most types of task they meet and which relate to real numbers can be resolved by pushing the "buttons" and how the "buttons" work can be theoretically explained by the corresponding logos blocks.

Every praxeology must incorporate both a praxis block and a corresponding logos block. However, not all praxis blocks and logos blocks are always well-presented through an institutional relation. In general, high school mathematics emphasizes praxis blocks related to real numbers, such as calculation activities among real numbers. However, logos blocks related to real numbers are sometimes inadequately constructed or are directly and confidently accepted without precise explanations through teaching or learning. For instance, Durand-Guerrier (2016) found that in the context of French high school, students in grade 12 "learn the *mean value theorem* without proof and without a discussion on the fact that this theorem holds in the set of real numbers." Thus, the influence of logos blocks related to real numbers seems somewhat limited in the relationship $R_{HS}(s, o)$, when a high school student s is learning about a mathematical object o involving real number knowledge. A challenge in the relationship $R_{HS}(t, o)$ for a high school mathematics teacher t is to address the gaps in logos blocks that may not be adequately defined or explained in textbooks and to establish connections with the corresponding praxis blocks.

Conversely, in university mathematics, there is a greater emphasis on logos blocks, such as characterizing real numbers via the supremum property. However, new praxis blocks related to this property of real numbers are not extensively taught or presented in the courses or textbooks at the university. Therefore, a university student σ might not establish a strong connection with the praxis blocks that could be explained by the logos blocks so that the students through $R_U(\sigma, \omega)$ become mainly a relationship with the logos block of ω. This variation in mathematics study based on different institutions is a potential reason that students who have completed high school and are beginning university may find it challenging to engage in university mathematics study.

The challenges related to praxis blocks and logos blocks within different institutional relations and the difficulties arising during transitions between these relations can be viewed as potential factors that contribute to Klein's double discontinuity.

3.2 Theoretical model for Klein's second discountinuity

Klein's double discontinuity comprises three crucial transitions: the institutional transition, involving shifts between university and high school; the mathematical objects transition, entailing shifts between theory-centric university mathematical objects and practice-centric high school mathematical objects; and the position transition, encompassing the transformations from high school students to university students to high school teachers. Therefore, personal and institutional relations effectively encapsulate the interplay among these three transitions.

Winsløw and Grønbæk (2014) formulated a model for Klein's double discontinuity as follows:

$$R_{HS}(s, o) \to R_U(\sigma, \omega) \to R_{HS}(t, o) \tag{1}$$

where o represents a mathematical praxeology taught by teachers (t) to students (s) in high school (HS), and ω represents a mathematical praxeology received by students (σ) at the university (U).

This thesis explores the second arrow in transition (1), representing Klein's second discontinuity. Focusing specifically on the knowledge of real numbers, I utilize $\omega_{\mathbb{R}}$ to denote the praxeologies related to real numbers and to be acquired by university students, and $o_{\mathbb{R}}$ to denote the praxeologies related to real numbers that are intended to be taught in high schools. Consequently, the second discontinuity, concerning specific knowledge of real numbers, can be modeled as:

$$R_U(\sigma, \omega_{\mathbb{R}}) \to R_{HS}(t, o_{\mathbb{R}}) \tag{2}$$

Klein proposed two approaches to address the second discontinuity, as summarized by Kilpatrick (2019, p218):

> (a) offering university courses that would show connections between problems in various fields of mathematics (e.g., algebra and number theory), and (b) developing university courses in elementary mathematics from a higher standpoint.

This thesis considers the inclusion of a "capstone" course to address the gap in transition (2) based on the two proposed approaches. Kilpatrick (2019) describes a "capstone" course as one that "came near the end of our program and was designed to demonstrate our mastery of mathematics". In the context of this thesis, participants in the "capstone" course would be university students who have completed the rest of their mathematics studies and plan to become high school mathematics teachers upon graduation (i.e. future teachers), denoted as σ_{FT}. This course introduces future teachers to various mathematics domains relevant to high school and assists them in establishing connections between university mathematics and high school mathematics, specifically between $\omega_{\mathbb{R}}$ and $o_{\mathbb{R}}$ in the case of this thesis. The mathematics praxeologies related to real numbers in this course are denoted as $\omega^*_{\mathbb{R}}$. The new relationship to be achieved through this course is denoted as $R_U(\sigma_{FT}, \omega^*_{\mathbb{R}})$, updating the transition (2):

$$R_U(\sigma, \omega_{\mathbb{R}}) \to R_U(\sigma_{FT}, \omega^*_{\mathbb{R}}) \to R_{HS}(t, o_{\mathbb{R}}) \tag{3}$$

The introduction of the new relationship $R_U(\sigma_{FT}, \omega^*_{\mathbb{R}})$ functions as a bridge connecting $R_U(\sigma, \omega_{\mathbb{R}})$ and $R_{HS}(t, o_{\mathbb{R}})$, thereby creating two possible or potential sub-discontinuities represented by the two arrows ($\to$). So the capstone course success to address Klein's second discontinuity depends in particular, on these new, potential sub-discontinuities. More specifically we can address these here in terms of the following three questions:

1. In the construction of $\omega^*_{\mathbb{R}}$, which specific elements pertaining to real numbers are necessary to include in this praxeology, in order to establish a link between $\omega_{\mathbb{R}}$ and $o_{\mathbb{R}}$?

2. With regard to the first arrow, how might future teachers effectively engage with the newly devised praxeology, leveraging the undergraduate relationship to academic knowledge in view of autonomously examining mathematical and didactical questions at the high school level?

3. Concerning the second arrow, what novel insights into high school real number teaching could future teachers gain through their involvement with the new praxeology?

3.3 Research questions

Formalizing and sharpening the aforementioned three questions, this thesis endeavors to address and resolve the following three research questions:

RQ1. How could the praxeology $\omega^*_{\mathbb{R}}$ be built so as to include the theoretical perspective of computability to connect the high school model of real numbers and the university model of real numbers? In particular, how could task design contribute to developing and assessing $R_U(\sigma_{FT}, \omega^*_{\mathbb{R}})$?

RQ2. How could future teachers work with computer algorithms through the designed tasks from RQ1 to build the connection with $R_U(\sigma_{FT}, \omega^*_{\mathbb{R}})$?

RQ3. What new relationships to real numbers and functions as they appear in high school could the work with the designed tasks lead future teachers to acquire?

Let $o_{\mathbb{R}}$ be represented by the pair (Π_{HS}, Λ_{HS}), and $\omega_{\mathbb{R}}$ be denoted by the pair (Π_U, Λ_U). The purpose of $R_U(\sigma_{FT}, \omega^*_{\mathbb{R}})$ is to establish a connection between Π_{HS} and Λ_U for future teachers. Therefore, constructing $\omega^*_{\mathbb{R}}$ requires the inclusion of Π_{HS} and Λ_U. However, directly incorporating these two components into one praxeology is insufficient. Thus, it becomes crucial to adjust specific elements in Π_{HS} to align it with the structure of a university-level praxis block. In this thesis, computer algorithms were employed to perform certain instrumented techniques from Π_{HS}. This adapted praxis block, derived from Π_{HS}, is denoted as Π^*_{HS}. The logos block in $\omega^*_{\mathbb{R}}$ used to elucidate Π^*_{HS} is denoted as Λ^*_U. The elements included in Λ^*_U need to provide a theoretical explanation for the computer algorithms. These elements used to explain the computer algorithms also need to have a connection with some elements in Λ_U. This requires the computer algorithms to involve university-level real numbers knowledge. Therefore, how to construct the computer algorithms is crucial for building the logos block Λ^*_U in $\omega^*_{\mathbb{R}}$.

The exploration of the three research questions is distinguished across three main papers in my PhD project (see the paper list). Each paper contributes distinct perspectives to the overarching investigation centered around the UvMat capstone course.

Paper II: This empirical paper delves into the design and implementation of assignments featuring computer algorithms to construct $\omega^*_{\mathbb{R}}$ for future teachers. The data for Paper II was gathered during the UvMat course in 2021.

Paper III: Positioned as a theoretical paper, Paper III addresses Klein's second discontinuity, transitioning from a global to a local perspective. It utilizes a newly designed assignment from the UvMat course in 2022 as a concrete example to discuss the second discontinuity locally.

Paper IV: Another theoretical contribution, Paper IV centers on how the concept of computability can aid future teachers in comprehending real numbers as infinite decimal representations on computers. This paper draws from the examples of both assignments in Paper II and Paper III, showcasing part of future teachers' engagement with the two computer algorithms.

In summary, Paper II and Paper III collectively tackle RQ1, Paper II and Paper IV are intertwined in addressing RQ2, and Paper III and Paper IV synergize to address RQ3 (see Figure 7).

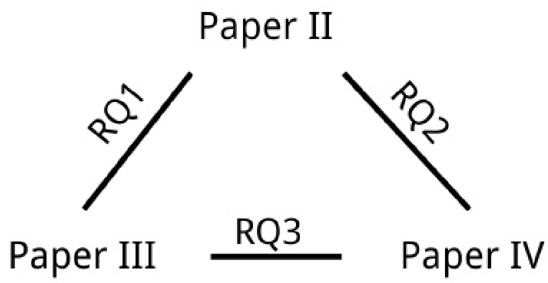

Figure 7: The relations between my work in papers and research questions.

Paper II and Paper III provide relevant knowledge on infinite decimal representation, which is used in constructing $\omega^*_{\mathbb{R}}$ in this thesis. Both papers respectively introduced and explained each task of a designed assignment, from a theoretical standpoint. Hence, the content of these papers contributes to addressing RQ1.

Paper II and Paper IV encompass the two computer programs featured in Section 2.5 (refer to Figure 3 and Figure 4). These papers present students' engagement with the provided computer programs through the designed assignments from RQ1, accompanied by analysis. The insights drawn from this exploration contribute to addressing RQ2 in this thesis.

Paper III and Paper IV cover students' responses to the interplay between future teaching in real numbers and functions and the knowledge they acquired through their engagement with the two designed assignments (from RQ2). These papers explore two distinct aspects of the connections. Paper III centers on the theoretical knowledge of real numbers and algorithmic functions, contributing to the advancement of teachers' understanding of mathematical teaching objects. Paper IV concentrates on digital technology, offering insights from the perspective of compatibility to enhance future teachers' understanding in dealing with decimal representations on computers.

4 Context - “Capstone” course UvMat

The context and date treated in Paper II, Paper III, and Paper IV come from the bachelor course UvMat at the University of Copenhagen. This annual course typically accommodates 20-30 participants. UvMat is categorized as a “capstone” course due to its primary objective being to offer a higher-level perspective on secondary school mathematics. It should provide future teachers with a profound understanding of the content they will teach in secondary school. While the course is tailored for future mathematics teachers, it does not directly teach pedagogical or didactical knowledge. The students in this course are identified as future teachers in this thesis.

UvMat is an optional course offered at the end of a so-called minor in mathematics. In Denmark, teachers are generally required to teach two subjects. Therefore, students in this course are mainly university students majoring in subjects such as Danish, history, sports, etc., with a minor in mathematics. Given the shortage of mathematics teachers in Danish upper secondary schools, it is common for university students to enroll in this course to enhance their possibilities of employment by enabling them to teach mathematics alongside their main subjects. The students who enroll in this course usually do not possess a strong mathematical background akin to mathematics major students, but all of them have completed a minimum of two years of university mathematics courses. According to the requirements of the University of Copenhagen, students are mandated to have completed the following essential courses in mathematics before attending UvMat:

- Introduction to the Mathematical Sciences (MatIntroMat)
- Linear Algebra in the Mathematical Sciences (LinAlgMat)
- Analysis 0 (An0)
- Analysis 1 (An1)
- Discrete Mathematical Methods (DisMat)
- Probability Theory and Statistics (SS)
- Probability Theory and Statistics (SS)
- Algebra 1 (Alg1)
- History of Mathematics (Hist1)
- Geometry 1 (Geom1)

These requirements for university students minoring in mathematics at the University of Copenhagen, as well as course descriptions, can be accessed on the website (University of Copenhagen, n.d.).

The three main papers pertain to the UvMat course in 2021 and 2022. In both years, the courses had seven weeks of teaching, each focusing on different mathematical topics, with the fourth week of the course focusing on real numbers and in particular their infinite decimal representations. Due to the impact of COVID-19, over half of the course in 2021 was conducted online through Zoom, including the entirety of the fourth week. The entire course in 2022 was held onsite as usual. Students were offered 9 hours of teaching each week in the course. In 2021, the course schedule included a 3-hour lecture on Monday afternoon, a 3-hour exercise class on Wednesday morning, and 3 hours allocated for working on a weekly assignment with access to supervision on Wednesday afternoon. In 2022, there was a slight adjustment to the structure of lectures and exercise classes. Students attended a 2-hour lecture followed by a one-hour exercise class on Monday afternoon and a 2-hour exercise class followed by a one-hour lecture on Wednesday morning.

The professor giving the lectures was responsible for determining the weekly mathematics exercises and lecture content. In both years, the textbook "The Mathematics That Every Secondary School Math Teacher Needs to Know" by Sultan and Artzt (2018) served as the primary textbook for this course. The mathematical objects covered in the fourth week of both years are based on the second part of chapter 8 in the textbook. Students were introduced to the representation of real numbers as infinite decimals, developing the content outlined in section 2.2, as well as the proofs of some main theorems stated in the textbook and further developed in the lectures. However, the concept of computability, which served as the basic idea for designing the two computer programs, was not explicitly part of the standard curriculum in this course.

In lectures, the professor introduced and explained some specific content from the textbook and assigned exercises for students, often selecting them from the textbook, to be addressed during exercise classes. These exercise classes, led by a teaching assistant (TA), aimed to ensure that students not only completed the exercises but also comprehended the solutions. All lectures and exercise classes were in Danish. Students' attendance in both exercise classes and lectures was not counted towards the final course assessment.

The professor is also responsible for drafting the weekly assignments, which are new every year to prevent copying answers by students. As the rest of the course, the weekly assignments were in Danish. These weekly assignments went beyond the scope of the textbook, requiring a broader understanding of the week's content and presenting a challenge to students. Students had the option to tackle these assignments individually or in groups of 2-3 members. Group compositions were fixed at the course's outset and could only be altered with valid reasons. After students submitted their assignments, the TA reviewed students' answers and provided feedback based on the guidelines from the professor. Submissions that were not accepted were returned to students. Groups or individuals receiving a return could revise their assignment answers based on the feedback and then resubmit. However, revised versions might still face rejection. Assignments not accepted in a second attempt were considered a failure. Each student must submit at least 4 weekly assignments, and these submissions must be finally accepted; otherwise, the student would directly fail the course, and could not attend the final exam.

In 2021, the professor and I collaboratively designed an assignment for the fourth week of UvMat. Additionally, we created the computer program presented in Figure 3, which played a central role in the assignment. The initial draft was in English, as shown in Appendix A, and was translated into Danish by the professor for student use. This assignment is covered in Paper II and Paper IV. In 2022, the professor and I, with assistance from the Department of Computer Science at the University of Copenhagen for creating the computer program shown in Figure 4, formulated another assignment for the fourth week of UvMat. This assignment was initially drafted in Danish, actually used for students, and then translated into English by the professor. The English version is presented in Appendix B. This assignment is covered in both Paper III and Paper IV.

The 7th week of UvMat completed the statistics part of the course and also functioned as a preparation period for the final exam, providing students with the opportunity to engage with exam tasks from previous years. During this week, students also had supervision with the professor to address any questions from the course. The final exam comprised 5 tasks. Students were required to complete these tasks individually in 5 hours. All aids are allowed. A satisfactory answer corresponding to 2 tasks is awarded 02 (the line to pass the exam) and a satisfactory answer to all tasks is awarded 12. Students were encouraged to refer to the results from the course reading materials. The professor and an internal censor reviewed the exam answers from students and determined their exam grades. The responsibility for drafting the final exam tasks also rested with the professor, with the principles of task design discussed in (Winsløw & Huo, 2022).

5 Methodology

This thesis primarily emphasizes the theoretical dimension, with the empirical dimension serving to support our theoretical analysis. Therefore, the methodology employed in this thesis is relatively straightforward.

5.1 Methodology for RQ1

Infinite decimal representations of real numbers are encountered through practical tasks in high school and theoretical constructions of real numbers at university, as discussed in Section 2.2. The same section notes that while infinite decimal representations are occasionally discussed in university studies, there is a lack of theoretical explanations regarding how real numbers can be modeled by infinite decimals. Furthermore, the study of how computers generate decimals is not covered in either high school or university curricula. Therefore, a comprehensive and theoretical exploration of the interconnections among infinite decimal representations, real numbers, and computers is crucial for constructing $\omega^*_{\mathbb{R}}$. The content presented in Sections 2.4 and 2.5 can be seen as foundational elements in the construction of $\omega^*_{\mathbb{R}}$, addressing the first part of RQ1.

Students in the UvMat course had the opportunity to acquire related content regarding infinite decimal representations of real numbers from the textbook and lectures. However, the concept of computability was not covered in either the textbook or lectures. To assist students in enhancing their understanding of real numbers as decimal representations on computers, we incorporated computability into two weekly assignments of UvMat in both 2021 and 2022. The design of these two assignments adhered to the following three general principles:

1. The assignment must include one or more high school tasks.
2. The assignment must incorporate a computer program as a tool to address the high school task.
3. The assignment must encompass specific university-level knowledge related to real numbers, which is applied to solve questions associated with how the computer program works.

The actual design of the two assignments based on the three principles will be detailed in Section 6.1.

The assignment, crafted in accordance with the three principles, evaluates students' proficiency in applying university theoretical knowledge to elucidate high school tasks, aligning with the objective of constructing $\omega^*_{\mathbb{R}}$. Moreover, this assignment prompts students to explore computability by comprehending the provided computer program, facilitating the development of their understanding of computer operations. Hence, the task design, grounded in the three principles, addresses the second part of Research Question 1.

5.2 Methodology for RQ2 and RQ3

Before the fourth week of UvMat, students were provided with two requests for informed consent. One consent pertained to permission to researchers' use of their assignment answers for the 4th weekly assignment, including both initial and revised submissions. The other consent was related to permission to use information obtained from the interviews. The purpose of data collection, the type of data (assignment answers), and the assurance of anonymization of personal information before storage and analysis were clearly outlined in the consent for assignment answers. All students from UvMat in 2021 and 2022 signed this consent. Similarly, the consent for interviews outlined the purpose of the interviews. In 2021, due to the pandemic, interviews were conducted online via Zoom. The consent specified that the interviews would be recorded as videos and that each interview would not exceed 15 minutes. It assured students that video recordings would be destroyed after transcription. The latter is in an anonymized form. No parts of the interviews would be shared with anyone, including course-related individuals like the professor (responsible for final grades). In 2022, there were two differences in the interview consent. First, the interviews were conducted onsite, and the interview records were in audio format. Second, students participated in two interviews — a pre-interview before the fourth week and an post-interview after revising their assignment answers. The handling of the interview records remained the same as in 2021. All interviews were voluntary, and students had the right to withdraw from the study and revoke consent at any time without consequences. A total of 5 students from 5 different groups participated in the interviews in 2021, and 8 students from 5 different groups participated in the interviews in 2022. All signed consents were securely stored at the Department of Science Education, the University of Copenhagen. The results from students' answers and a portion of the interviews are used to address RQ2, while other segments of the interviews are employed to address RQ3.

5.2.1 How to process students' assignment answers in both years

In 2021, nine group assignment answers were received, and in 2022, eight group assignment answers were received. Eight groups in 2021 and 6 groups in 2022 were asked to revise their answers, and all revisions were eventually accepted. Notably, in 2022, one group initially decided not to submit an answer to the fourth assignment. The analysis of these assignments involved two versions of answers: the initial answers from all students with the TA's comments, and the revised versions submitted by groups that were requested to revise. All answers were submitted online as PDF files in both 2021 and 2022.

Two types of formats were included in the online submissions, with some possible mixture. The most common format used by students was the digital version, where they solved the questions on Maple (the primary CAS software used in UvMat) or typed the answers on other online platforms such as Word. Screenshots of separate answers were then compiled into a single answer sheet. The second format was a semi-physical version, where students wrote down answers on paper and took pictures of them. All answers and the TA's comments were in Danish. Due to the language barrier, the textual content of these answers, including TA comments, had to be translated into English. Answers in the digital version were easily translated using Google Translator through copy-paste, while answers from pictures of physical papers, which could not be copied, including TA comments provided in digital handwriting, were translated by the professor of the course.

Before students began working on the 2021 assignment, a survey about students' experience with Maple was conducted via Zoom during the first lecture of the course week. The survey consisted of 6 questions, and all students had to answer all the questions. The survey was in Danish and translated from English by the professor of the course. The results of students' answers to the survey, translated back into English, are presented in Appendix C. From the survey, only 30% of students reported having used Maple in high school. TI Npire and Geogebra were the most commonly used mathematical tools among them as high school students. Additionally, 15% of students did not have any experience with mathematical tools in high school, but all of them had some experience with Maple before the course. The survey results supported the hypothesis that some questions such as question c) in the 2021 assignment could be handled and explored by students using Maple.

The evaluation of students' work on the 2021 assignment began with a review of students' answers to each assignment question, along with the TA's comments on the initial submissions. The assignment question that elicited the most revision requests was scrutinized, and the revised answers to the same question were analyzed. Attention then was directed to the two open questions c) and f) (see Appendix A). For question c), which required students to provide a visual explanation of the given computer program, all visualizations were in the form of figures presented through screenshots from Maple. The analysis focused on two aspects: the type of figure and the content that students aimed to express through the figures. Question f) asked students to discuss the addition of infinite decimals. Students' answers were initially categorized into two groups: those who gave misleading answers and those who highlighted issues with the addition. These satisfactory answers were then analyzed based on how they presented the problem, either through a visualized approach with textual explanations or purely through textual explanations.

The analysis of students' work on the 2022 assignment concentrated on the mathematical content that students provided through their answers because most questions involved providing mathematical proofs. The evaluation explores whether students presented comprehensive mathematical proofs without relying on any mathematical phenomena as evidence, based on the TA's comments. Similar to 2021, the assignment question that garnered the most revision requests was typically scrutinized alongside the revised answers. Students' answers to the last question (see question f) in Appendix B) which asked students to explain the given computer program were also included in the analysis.

Analysis merely based on students' assignment answers does not provide direct insight into the impact of working with the two assignments on students' knowledge of real numbers on computers (RQ2). Therefore, it was considered to be useful to elicit students' oral explanations of their assignment answers.

5.2.2 Interviews in 2021

All interviews adhered to a standardized set of guidelines and were conducted in English. The interview guidelines consisted of two parts. In the first part of each interview, students were instructed to elaborate on their answers to questions c) and f), responding to the following main questions:

1. Could you take me through how your figures visualize the routine in question c)?
2. Are you satisfied with your figures and why?
3. What challenges did you encounter while creating visualizations or utilizing Maple?
4. Can you explain about your answers to question f)?
5. What is the meaning of question f)?

Apart from the initial five questions, additional relevant questions were posed based on students' responses, particularly concerning question f). For instance, if a student provides only a conclusion, follow-up questions like "Can you provide an example?" were asked, though students had the option to decline answering. The aim of these interviews was not only to gather information but also to assist students in comprehending and addressing the assignment questions by using guiding questions. The responses from students to this part served as supplementary explanations for students' answers to the assignment questions and were also used to answer RQ2.

The second part of each interview involves two questions from a broad perspective:

1. What do you think the most important mathematical point of this assignment is?
2. Do you think the knowledge from this assignment is relevant to a high school mathematics teacher and why?

This part was used to answer RQ3.

The video recording of each interview was securely saved in a safe drive accessible only to me. Upon completion of all interviews, the students' responses were transcribed, and the saved video recordings were then deleted.

5.2.3 Interviews in 2022

In 2022, a pre-interview was added before the fourth week. In this interview, students were asked to answer the following 3 questions:

1. What is your major and do you have any teaching experience in mathematics?

2. Could you find out how many digits are in 242^{17}?

3. How do you define logarithms?

Some follow-up questions were also posed through the conversions with the students. This part serves as a reference base for the after-interviews, aiming to better capture students' learning of infinite decimals throughout the week and assess the impact of completing the assignment on their understanding of logarithms. The pre-interview is also intended to provide students with a clearer direction for their study through the week.

The after-interviews consisted of two parts. The first part aimed to contrast with the pre-interview, incorporating the last two questions from the pre-interview. To relate to the assignment, interviewees were also prompted to utilize the approach from the assignment to explain how to calculate $\log_{10} 25$. The second part of the post-interviews mirrored the second part of the interviews conducted in 2021. The responses of students in the first part, including those from the pre-interviews, were used to address RQ2, while their answers to the second part were employed to answer RQ3.

In 2022, all interviews were conducted in English, taking place in a quiet one-to-one setting. These were onsite interviews, and the recording of students' responses was in the form of audio recordings. Students had the option to halt the interview at any time, and they could choose not to participate in the post-interview even if they completed the pre-interview. All recordings were securely stored in a safe drive, and after the conclusion of the interviews, student responses were transcribed in anonymized form and then all recordings were deleted.

6 Main results

6.1 Main results for RQ1

Paper II and Paper III provide detailed explanations and analysis of the assignments used for the fourth week of the UvMat course in 2021 and 2022, respectively. In particular, the mathematical content of both assignments, along with the answers to each question and their analysis, is thoroughly explicated. In this subsection, I will explain the design idea of the two assignments, based on the three principles outlined in the previous section, and then provide a brief comprehensive analysis.

The weekly assignment in 2021 (see Appendix A) commences with a typical high school task: determining the decimal approximation of $\sqrt{2}$ up to the nth decimal place. The assignment includes the presentation of the computer program (called routine in the assignment) shown in Figure 3, which serves as a tool for solving the task. Through this program, students can find the first 10 decimal digits of $\sqrt{2}$ by executing it in Maple. A key task in this assignment requires students to provide a theoretical explanation of the provided computer program. To accomplish this, students are expected to apply their real numbers-related knowledge acquired from university mathematics studies, such as limits and the intermediate value theorem. Another significant task involves students modifying and applying the given computer program to discuss the addition of two infinite decimal numbers.

The weekly assignment in 2022 (see Appendix B) introduces another familiar high school task: calculating the value of function $f(M) = \log_{10} M$ with a given $M > 0$. The assignment provides the computer program illustrated in Figure 4. This computer program can produce the first 4 decimal digits of $\log_{10} M$ for a given $M > 0$ and can be executed using Maple. The assignment challenges students to prove certain mathematical claims and employ them to elaborate the given computer program. To prove these mathematical claims, students are required to use their university-level knowledge and proof skills, such as series and induction which are commonly employed for series proofs. The assignment incorporates numerous symbols, guiding students to address the questions and comprehend the computer program theoretically rather than relying on some concrete numbers. This design follows the typical style in university mathematics studies.

The most significant difference between the designs of the two assignments lies in the placement of the two computer programs. In 2021, the computer program (for $\sqrt{2}$) was presented at the beginning, requiring students to demonstrate their understanding by trying it out in Maple. Then, the computer program served as a tool to facilitate the exploration of additional questions, such as determining the first 10 decimal digits of $\sqrt{3}$. While in 2022, the computer program (for $\log_{10}$) was provided in the final task. Students were asked to solve five related problems first, using the obtained results to explain how the computer program procedures. This necessitated students to find the corresponding code segments in the program that matched their earlier results.

Neither of the assignments was intended to require students to create a computer program for solving high school-related tasks. The primary focus of the two assignments is not on the process of creating such computer programs but rather on providing students with insights into how computers work when handling infinite decimals. Although the two assignments share a common goal, their different question arrangements are based on some additional purposes. The 2021 assignment aimed to provide students with insights into the limitations in yielding precise answers from computers, such as the example in Figure 8. The assignment in 2022 aimed to help students open the "black box" of transcendental functions, offering them an opportunity to understand the inner workings of these functions, which they completely rely on computers to compute.

$$\textit{evalf}(2/3) - \textit{evalf}(1/3) - \textit{evalf}(1/3)$$

$$1. \times 10^{-10}$$

Figure 8: An example of "mistake" from Maple.

Both arrangements have their advantages and disadvantages. In the 2021 assignment, the inclusion of open-ended questions, such as visualizing the computer program and discussing the addition of two infinite decimals could be seen as attempts to broaden the scope of knowledge covered and encourage independent thinking. However, this approach may lack a standardized criterion for evaluation, leading to informal answers. Students might also put too much effort into additional knowledge such as how to make visualization and fail to think about the meaning of the assignment as a whole, even though all tasks are connected. The 2022 assignment addressed this issue by presenting strongly connected questions in a formal mathematical way. The entire assignment was dedicated to exploring the theoretical explanation of how the function $f(M) = \log_{10} M$ works without computers. However, students may face challenges in completing all questions, and there was no direct connection with instrumented techniques, except for the computer program presented at the end.

6.2 Main results for RQ2

The analysis of students' performance on each task in the weekly assignments for 2021 and 2022 is detailed in Papers II and III, respectively. Paper IV also incorporates some students' work on the two provided computer programs. In this subsection, the main results from students' efforts on the two assignments will be presented.

The primary findings for RQ2 center on two key aspects: the form of answers employed by students in responding to assignment questions, particularly those requiring mathematical reasoning, and their comprehension of the two provided computer programs. The following will showcase some representative examples from students' submissions.

It was surprising to observe that when university students were presented with tasks related to high school, they tended to adopt the perspective of high school students, as highlighted in Paper II and Paper III. They seemed to habitually apply the knowledge and methods they had acquired in high school to solve these questions. All submissions from both years were eventually accepted. A common issue observed in the initial versions of students' answers in both years was the informal arguments by which some groups responded to certain questions. In the 2021 assignment, the task most frequently requested for revision was question b), which required students to explain why $x(n) \to \sqrt{2}$. Only one group provided a satisfactory answer by employing formal reasoning, as in formal mathematical proof at the university level. Other groups offered informal explanations on descriptions akin to oral discourse, such as "$x(n)$ gets closer and closer to $\sqrt{2}$ as n becomes larger". Similarly, in the 2022 assignment, most students did not provide a clear proof to show the uniqueness of $C(M) \in \mathbb{N} \cup \{0\}$ in question b), where they took it as evident that $C(M)$ is unique when $C(M) \leq 10^x < C(M) + 1$, where $x > 0$. Additionally, some students used the pseudo decimal form, like $x_0.x_1x_2....$, to represent an arbitrary infinite decimal, despite having learned to model infinite decimals with series in lectures and the textbook. Students modified their answers in a more formal way through their revisions, guided by hints from the TA.

This phenomenon could be attributed to two possible reasons. First, there might be a flaw in the design of the assignment questions. The questions might not have provided sufficiently clear guidance to direct students to use formal university-level proof. Second, students may not have placed themselves in the position of university students but reverted to a high school student position, in view of the task being about what they perceive as high school objects. As a consequence of the first discontinuity, these could remain, for them, strictly separate from university objects.

The most interesting aspect of students' answers from 2021 was their responses to the two open questions c) and f). The visualizations created by students for question c) could be categorized into two main groups: images of the function $f(x) = x^2 - 2$ and dot plots displaying the results produced by the computer program. Some students combined both approaches by superposition. Even though some of the figures created by students were similar, the intended messages were not always the same. For instance, some students aimed to explain the limits $x(n) \to \sqrt{2}$ using dot plots, while others simply intended to showcase the results obtained by running the computer program. Therefore, these visualizations gained meaning when students attached specific descriptions and explanations. All students who participated in the interviews expressed satisfaction with the visualizations created by their respective groups. Some mentioned that it required significant effort to produce these visualizations, as they lacked sufficient knowledge on how to use Maple for advanced visualizations, like animations, as they initially thought were expected. Students' answers to question f) did not come as a surprise, as they emphasized that one could not determine the nth decimal of the sum of two infinite decimals without knowing all the infinitely many decimals of each summand. However, how the similar conclusions were presented by students, varied. Some students included screenshots from the computer to illustrate examples, while others provided textual explanations. Thus, it was not immediately clear from their answers whether the conclusions were obtained using the provided computer program if they were considered evident, and if they were acquired prior to the course. In the interview, a student from a group that did not include any examples explained that they did use the computer programs but did not include the output in their and-in. Therefore, it is difficult to determine the extent to which working with the computer program aided students in discussing and exploring the addition of two infinite decimals. Two groups that did not provide the expected conclusions did not attend the interviews, leaving the reasons for their misleading answers unknown.

In the 2022 assignment, the task directly connected to the given computer program was question f). In fact, students were required in question e) to use the results from previous questions to experiment with a concrete example - to find the first 4 decimals of $\log_{10}(57.64)$. This meant that, in question e), students had already gained familiarity with the algorithm underpinning the given computer program before answering question f). Through this process, students learned to approximately compute the function $f(M) = \log_{10} M$ with $M > 0$ without relying entirely on computing tools (although they might use computers for basic arithmetic). Therefore, explaining code segments using the results from earlier questions was not a difficult task for students, and all answers to f) from their initial submissions were considered satisfactory. In the pre-interviews, none of the students could confidently define logarithms. However, while most students could demonstrate the process of calculating logarithms by hand in post-interviews, none of them could provide a formal definition of logarithms after working on the assignment. This indicates that the assignment played a crucial role in improving students' practical understanding of how computers process logarithms but did not significantly enhance their theoretical understanding of logarithms.

Summarizing students' work with the two given computer programs in both years, it is evident that students have gained insights into how real numbers can be handled as infinite decimal representations by computers. This understanding was developed through the completion of the assignment tasks, indicating a theoretical grasp of the two computer programs.

6.3 Main Results for RQ3

While the RQ2 study revealed that students developed a theoretical understanding of the two given computer programs, the RQ3 study indicated that when students were required to evaluate the entire assignment from a higher perspective, particularly from the viewpoint of a future teacher, their focus remained on specific tasks. They struggled to provide a theoretical summary. Consequently, the connections to future high school teaching derived from the assignments were often more technical, such as understanding how to calculate logarithms, rather than reaching a profound understanding of a specific mathematical concept. The following presents some typical and varied responses from students in the 2021 and 2020 interviews.

Students' responses on the most important mathematical aspect of the assignment during the interviews in 2021 went in two directions. Two students focused on the given computer program, asserting that the primary objective of the assignment was to understand how a computer deals with infinite decimals. One student expressed:

> All the lessons to learn from this is to be aware of how differently computer pro-

> grams approximate things depending on what you put into it, whether it is a sum or a square root. That was the most important thing we got from this. (Question) b) was the one that was the heaviest in terms of the math for us and explaining what the routine does and how to use it and think about what maple actually produces for us. That was hard for us to figure out what it is doing and how to do it. We tested it in different ways as well just to see what happens when we add different decimals. That was really interesting.

The other three students directed their attention to real numbers themselves, particularly as infinite decimal representations. One student stated:

> I think that the most important thing about this is, it gives a very clear illustration about how to approximate real numbers with what would be equated to irrational numbers. In the construction of the real numbers, we used these "Cauchy sequences" to show that when we complete the fields of rational numbers, we get the real numbers. I think this is a really good illustration of how we always have a rational sequence converging to a real number. For me that was an interesting takeaway that goes beyond the scope of this assignment, knowing something about real numbers.

To the same question, students were very consistent in their responses in the interviews in 2022, which were about how to calculate logarithms. However, students were not so confident to show their opinions on this question. Most of the students start with uncertain words such as "I do not know" and "maybe". The student from the group who gave up submitting the assignment said:

> I figure it is to develop a method to actually calculate the log sequentially one decimal at a time...We kind of got the idea of what we were to work with, like a way to calculate the log, but the way the questions were actually put, we somehow did not manage to figure out exactly what we were going to do in different points. So having spent some time reading and discussing and getting confused we just decided we simply do not have the time for it this week so we skipped it.

During interviews, most of the students' responses regarding the connection between the 2021 assignment and the tasks of a high school mathematics teacher, were somewhat trivial. Some students provided answers like: "No matter what you teach, it is always a good thing to know a lot more than you actually need in what you teach". Besides agreeing that the assignment could help to advance teachers' knowledge, two students estimated that the topic covered in the assignment has no relation to what high school mathematics teachers should teach. One student denied any connection between this assignment and high school mathematics teaching as follows:

> I don't think they are going to teach rational and irrational numbers and a high level of decimals. Not even on a slightly less abstract level. The curriculum of high school students is very far from this. I think these kinds of numbers and adding these numbers are too far from the high school curriculum. The students care more about the knowledge they need to pass an exam instead of having to get more insight into these kinds of numbers.

Still, one student expressed an opinion on the relationship between teachers and computers:

> I think you (teachers) should be able to figure out when maple is good to use and when it is not. We saw during that week how maple can make mistakes that differ from the theories we know from mathematics. You should understand the limits Maple can have when you are teaching.

The responses from interviewees in 2022 regarding the relationship between the assignment and the teaching of logarithms in high school varied based on students' mathematics teaching experience. Those students who had some experience with teaching mathematics at the secondary level believed the main point of the assignment was not crucial for teaching, as it might make teaching logarithms in high school too challenging. On the other hand, those without any experience in mathematics teaching found the assignment quite helpful for a mathematics teacher teaching logarithms in high school. One student shared an opinion drawn from her/his own experience:

> I always thought that the logarithm is a bit of a mysterious box when you use the computer ... They (teachers) are going away from all these CAS machine works and all that in high school or Denmark I don't know about other countries, but if they are doing that then maybe the logarithm can be used to calculate with big numbers by hand and maybe that can make students understand what is happening better because the computer does not help you understand it but just helps you to calculate.

7 Discussion

One of the challenges related to RQ1 was to integrate the knowledge on infinite decimals and approximate computation, which is taught in high school mathematics, with a university level approach to real numbers. It turns out that students do not spontaneously mobilize theoretical knowledge aspects of infinite decimals when faced with a comptational algorithm such as the one presented in the assignment on the logarithm, which poses a significant challenge for task design in this context. Thus, students might not have advanced their theoretical understanding of real numbers and functions when focusing on how decimal representations could be produced as approximate values of logarithms, even though infinite decimal representations were the theme throughout that course week. We sometimes overestimate the tendency of students to seek connections between the lectures and the meaning of the assignments that are proposed in direct connection to them. Students may find it challenging to take a higher perspective on seemingly elementary mathematical tasks, potentially leading them to believe that the assignment's purpose is to just learn a method to calculate logarithms. Therefore, they may overlook more theoretical points concerning the meaning of computation of a function by a digital device, and about the logarithm in particular. This challenge is evident in students' interviews after working on the 2022 assignment, as they found it easy to explain how to calculate logarithms by hand but struggled with even basic theoretical questions on logarithms, such as providing a formal definition.

In high school, the difficulties often encountered with logarithms by students can be generated by "specific mistakes in manipulating logarithmic expressions, and more general problems in understanding the meaning of the logarithmic concept" (Weber, 2016, p.S81). Weber (2016, p.S78) introduced a method to describe logarithms as "repeated divisions", forming the foundational idea for the author's interpretation of the "manual calculation of logarithms". More specifically, a logarithm $\log_a b$ (where a is the base of the logarithm) can be conceptualized as finding how many times a is a factor in b (naturally, this is informal as the number of times may not be an integer): $b \div a \div a .... \div a = 1$. The computer program (see Figure 4) provided in the 2022 assignment was designed based on Weber's approach to the "manual calculation of logarithms" specifically with a base 10. Furthermore, this approach was also used by Weber (2016, p.S89) in the heuristic description of logarithms as "counting the number of digits", aimed at helping secondary school students understand logarithms by connecting them to familiar mathematical concepts like decimals. Based on this model, one could manually determine the number of digits in the integer part of a large number, such as 242^{17} (an example used in interviews in 2022), in its decimal representation. However, students' work in 2022 with the assignment demonstrated proficiency in the technical aspects of this "manual calculation" but fell short in reaching the theoretical dimension of the approach, particularly in interpreting "what are logarithms". Especially for those students who participated in the interviews in 2022, even though they were asked to determine the number of digits of 242^{17} and the definition of logarithms in the pre-interviews, they did not develop the intended theoretical knowledge of logarithms through the two questions, from the course and the assignment. Therefore, when asked the same questions in their post-interviews, they were still unable to systematically and theoretically relate these concepts to explain the corresponding meaning or definition of logarithms, except being able to calculate the number of digits of 242^{17} by hand.

Another challenge in RQ1 is to integrate mathematical and didactical knowledge. In secondary school, "with multimedia capabilities, students are able to visualize mathematical concepts that are difficult to imagine using traditional methods of teaching" (Bakara et al., 2021, p. 4650). Therefore, how mathematics teachers make didactical use of visualizations produced with digital tools could come to play a crucial role in students' understanding of abstract mathematical notions. A visualized approach to explaining a mathematical procedure, such as an algorithm for computing the decimal representations of square roots, was integrated into the assignment in 2021 (question b)). The purpose of this task is to enhance future teachers' knowledge of how the procedure works mathematically, and, simultaneously, to develop their ability to communicate and share this knowledge through visualizations produced with Maple, thus to use the latter didactically.

However, this combination of mathematical and didactic purposes did not function effectively in students' work. Many students agreed in the interviews that visualizations can be helpful in mathematics teaching, but they did not perceive a strong connection to the concrete question of the assignment, out of the common assumption that the knowledge of real numbers would not be that complex. Furthermore, according to students in the interviews, many of them expressed that they spent more time creating the figures than they had used to answer the mathematics questions, which was not the intention of the designer or of the course. Therefore, although this task to some extent trained students in techniques for creating visualizations, it did not significantly contribute to advancing their knowledge of real numbers. While we cannot entirely dismiss the potential value of this approach for combining mathematical and didactical techniques, it failed to establish meaningful connections for students with the mathematical objects they would encounter in future teaching. Hence, we decided not to include this type of task from the 2022 assignment.

CAS is one of the most widely used types of in mathematics teaching and learning (Lagrange, 2019; Drijvers and Trouche, 2007; Winsløw, 2003b). Lagrange (2019, p. 129) asserts that:

> The intriguing fact is that even when the introduction of a technology has been well prepared by an epistemological analysis and situations have been proposed, implementation by teachers still looks like a struggle to give birth to a more personal creation. [...] When a teacher wants to introduce technology, s/he has to integrate these techniques into his/her own understanding of the domain, into his/her own personality and to create relevant situations, certainly not an easy task.

Therefore, another combining mathematical and didactic knowledge in the assignment for future teachers was how CAS deals with infinite decimals before they teach related knowledge in the future using CAS. The traditional paper-and-pencil techniques, to some extent, helped students with reasoning and conceptualizing. However, the "push button" techniques from CAS cannot directly replace the role of paper-and-pencil although they evidently facilitate calculations (Lagrange, 2019). For example, Schneider (1999) found that students learned theoretically important properties of logarithmic functions by solving exponential equations when asked to use paper-and-pencil techniques, so that even if the teachers were aware that TI-92 could easily solve the equations they realized the epistemic and didactic value of the non-instrumented techniques. The two computer programs in Figure 3 and Figure 4 could also be conceived of as techniques that would allow students to calculate decimal representations of square roots and logarithms by hand. In other words, the two computer programs transformed paper-and-pencil techniques into codes while at the same time, avoided using time on the actual calculations. As students engaged with understanding and interpreting these codes, this was intended to have similar epistemic and didactic value and at the same time, make the students reflect on how computers may determine real numbers like these function values. Moreover, it was hoped that these two assignments enhanced the teacher's knowledge of digital technology, which could be helpful for their future teaching with CAS.

In addressing RQ2 and RQ3, our data do not allow us to directly determine whether students' work on these two assignments will be beneficial to their actual teaching in the future. What is certain, however, is that students acquired or improved some aspects of their knowledge on decimal representation and their meaning in relation to computing function values approximately. A notable outcome is the transition from confusion about logarithms to the ability to calculate logarithmic values by hand. Understanding the true impact on students' future teaching of the knowledge gained through assignments is challenging, given that they are not yet mathematics teachers. The gains, including their comprehension of the assignments and the connection between the theoretical perspectives of the course material and future teaching, remain subjective and could ultimately only be assessed if we were able to follow these students into their roles as high school mathematics teachers years later. Therefore, evaluating our study's contribution to smoothen Klein's second discontinuity must be done in less direct ways.

It is evident that our experiment has several limitations. Firstly, the sample size was relatively small, consisting of 20-30 students each year. This makes it challenging to conclusively establish the impact of the content and assignments we designed. Moreover, not all students worked on every assignment question as these were done in groups who sometimes shared the questions among members in somewhat exclusive ways. For instance, in the 2021 interview, one student mentioned not being involved in creating visualizations but participating in the discussion of adding infinite decimals. Hence, the final presentation of the assignment answers could not attribute each student's work within the group accurately. Secondly, as the interviews were voluntary, those who chose to participate were likely individuals confident in their work or willing to actively engage in the course. This could potentially introduce bias, as their responses could potentially be more confident and positive.

8 Conclusion and perspectives for future research

This thesis addressed Klein's second discontinuity by exploring strategies to enhance future teachers' knowledge of real numbers. The approach involves bridging the gap in future teachers' practical and theoretical knowledge about real numbers as taught in high school and university, by constructing an intermediary model which is also useful to analyze more deeply the functioning and limitation of computing devices in relation to real numbers. In high school, challenges arise from unclear concepts and relationships among different representations of real numbers, impacting both didactic and mathematical praxeologies found in this institution, according to the background literature reviewed earlier. Our focus is specifically on the decimal representations commonly used in high school tasks involving calculations. Given that decimal presentations are also treated from a theoretical perspective in the book by Sultan and Artzt (2018), more specifically in the chapter concerning real numbers, this thesis utilizes the infinite decimal model as a bridge to connect the high school and university models of real numbers. With computers playing a crucial role as computational tools in high school, we also explored the relationship between decimal representations and computers. The concept of computability was introduced to elucidate how computers handle real numbers in the form of decimal presentations, especially when dealing with irrational numbers and transcendental functions. Two typical examples, square roots and logarithmic functions, were employed to illustrate how computers generate decimal representations digit by digit. The theoretical construction of the infinite decimal model and the concept of computability enabled us to establish new teaching-oriented connections among the three main research objects: real numbers in high school, real numbers in university, and real numbers on computers. Together, they contributed to the development of an intermediate praxeology related to real numbers, bridging the gap between high school and university praxeologies concerning real numbers.

This intermediate praxeology encompasses a high school praxis block related to real numbers and a university logos block related to real numbers, the latter being disseminated in a capstone course named UvMat. We have focused on the fourth week of the course, in which completed assignments whose content can be understood in terms of the theoretical notion of computability, high school tasks related to real numbers as a starting point. In the students' work on the assignments, it was observed that when faced with a problem, students often struggled to adopt the perspective of a future teacher, tending to automatically place themselves in the positions of either university students or high school students, thus entering into the relationship $R_{HS}(s, o)$ or $R_U(\sigma, \omega)$. Similar findings were reported by Barquero and Winsløw (2022) in their study of the UvMat course. Moreover, when not specifically instructed to use a certain type of reasoning, students turned by default to their high school knowledge and answered questions in a manner similar to that of high school students. In other words, the assignment made them face but not necessarily succeed a number of challenges in autonomously analyzing common high school problems from a theoretical standpoint and responding using a rigorous university mathematics approach.

It is difficult to evaluate the overall impact of students' work with the intermediate praxeology for real numbers. There is no doubt that the work with the assignments made students gain technical understanding from working with the two computer programs we designed. Some of the students also consider that, in more generic terms, it is important to "know the basics", understood here as having a more connected and coherent knowledge about the number system that underlies most secondary mathematics. The knowledge about computation could be particularly important in relation to a situation, like the Danish high school, where the use of technology is quite dominant in mathematics. Our results suggest that students have to a large extent succeeded in opening the "black box" of procedures for which the assignment proposes possible codes, and symbols like $\sqrt{}$ and log are in this sense, and to these students, no longer mere "buttons" on a computer or calculator. While it is challenging to prove the significant impact of such results on students' future practical teaching, they demonstrate the didactic relevance of the notion of computability both as a design tool and as a potential theoretical element in capstone courses for future teachers who prepared to teach.

This study is devoted to addressing the central question (refer to Section 1.1):

> **Q**: How can future teachers bridge the gap between the model of real numbers acquired at university and the model they are expected to teach in high school within the context of computers?

The primary contributions of this study to this question come from three main aspects. First, from a theoretical perspective, we constructed design tools and concrete assignments aiming to connect two different models of real numbers, integrating practical problems from high school with theoretical knowledge from university. This theoretically addresses the challenge of Klein's second discontinuity. The content planned for the capstone course on infinite decimal representations of real numbers and the corresponding assignments we designed offers a promising and feasible example for addressing this discontinuity, providing valuable insights into enhancing future teachers' knowledge of the mathematical concepts they will teach in the future. However, there is still room for improvement in the design of the assignments.

Second, this study confirms the necessity and difficulty for future teachers to overcome the limitations and stereotypes imposed by institutions when connecting high school and university real number models such as high school tasks corresponding to high school knowledge, and advanced theoretical knowledge for theorem proving. However, the content and knowledge designed in this study have the potential to help future teachers break through these fixed mindsets, which would allow them to consider high school tasks from a higher standpoint. It enables them to flexibly apply university theoretical knowledge to explain and analyze high school tasks. To achieve the desired goal, future teachers might need much more extensive training of this kind than what is provided in the isolated seven-week course considered as context in this study.

Third, this study establishes a connection between real computations on computers and computer algorithms. This integration, within the contemporary digital context, offers a promising and future-oriented way to link the pivotal role of computers in high school in dealing with real numbers and the otherwise isolated courses on programming in undergraduate mathematics education. Through the students' work, it was observed that such an approach could assist future teachers in enhancing their theoretical knowledge of mathematical concepts while simultaneously improving their understanding of the operational principles of digital technology in producing real numbers. This represents a substantial contribution for future teachers. It further equips them with further invites theoretical, didactical reflections on the utilization of digital technology at the secondary level, for the teachers concerned and for university teachers as well.

In our study, the concept of computability served as the foundation for a part of our design and data analysis and provided theoretical support for the above-mentioned intermediate praxeology of decimal representations. However, the concept of computability was not directly included in assignments or in the lectures of the course as we designed and observed them. While certainly gain practical (mute) knowledge about of computability through completing assignments, very few students considered infinite decimals from this perspective (which was in fact also not our primary goal). Given that students' engagement with computer algorithms significantly contributes to their technical knowledge of real numbers, we believe that incorporating computability as a theoretical component of the intermediate praxeology on real numbers is a prominent hypothesis for future, continuing study of the topics of this thesis.

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Appendix A: Week assignment in 2021

Week Assignment 4

For $n \in \mathbb{N}$ we define $\mathbb{D}_n = \{10^{-n}x : x \in \mathbb{Z}\}$, and we define $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$. Also define $\mathbb{D}_\infty$ to be the set of formal expressions $\pm N.c_1c_2...$ where $N \in \mathbb{N} \cup \{0\}$ and $c_k \in \{0, 1, ..., 9\}$ for all $k \in \mathbb{N}$, and finally let $\mathbb{D}_0$ be the set of formal expressions $\pm N.c_1c_2...c_k000...$ where $N \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ and $c_1, ..., c_k \in \{0, 1, ..., 9\}$.

a) Prove that there exists bijections $\varphi : \mathbb{D}_0 \to \mathbb{D}$ and $\psi : \mathbb{D}_\infty \setminus \mathbb{D}_0 \to \mathbb{R}$, but that no bijection exists between $\mathbb{R}$ and $\mathbb{D}$.

b) Consider the following routine in Maple (try it out!):

```
K := 1 :;
for i from 1 to 10 do
      for j from 0 to 9 do
           if (K + j * 10^(-i))^2-2 <= 0 then
                p := K + j*10^(-i);
           end if ;
end do;
K := p :;
print(x(i) = evalf(p, i + 1));
end do :;
```

Explain what the routine does, why $x(n) \in \mathbb{D}_n$, and why $x(n) \to \sqrt{2}$.

c) Use Maple to produce a visual explanation of how the routine from b) works.

d) Explain how a similar routine can be made for any continuous function f, to find a zero between $a \in \mathbb{Z}$ and $a + 1$, when $f(a)f(a + 1) < 0$. How does the intermediate value theorem come into play? How can you use this idea to approximate $\sqrt{3}$ by numbers from $\mathbb{D}$?

e) Find a polynomial p such that $p(\sqrt{2} + \sqrt{3}) = 0$, and use the idea from d) to approximate $\sqrt{2} + \sqrt{3}$ by numbers from $\mathbb{D}$.

f) Investigate what the results from b), d) and e) tell you about addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$.

Appendix B: Week assignment in 2022

Week Assignment 4

Recall from the textbook (and the lectures) that for any $x > 0$, we can determine a sequence (c_k) with $c_k \in \{0, 1, ..., 9\}$ for all k, such that not all c_k equal 9, and with

$$(*) \qquad x = \lfloor x \rfloor + \sum_{k=1}^{\infty} c_k 10^{-k}.$$

where $\lfloor x \rfloor =$ the integer part of x. In this assignment, we provide a way to compute $\log_{10}(M)$ for any $M > 0$, using only simple arithmetical operations. The idea is that to calculate (or define) $x = \log_{10}(M)$, we must find a number x that satisfies $10^x = M$. With the notation in (*), the method successively gives $\lfloor x \rfloor$, c_1, c_2 and so on.

a) Based on properties of $x \mapsto 10^x$, show that for any $M > 0$, there is a unique $x \in \mathbb{R}$ such that $10^x = M$. If we know how to calculate this x when $M \geq 1$, how can we do it for $0 < M < 1$?

Because of a), we assume $M \geq 1$ from now on.

b) Show that there is a unique $m \in \mathbb{N} \cup \{0\}$ satisfying

$$10^m \leq M < 10^{m+1}.$$

In the sequel we use the notation $C(M)$ for the number m, determined as above.

c) The number $C(M)$ from b) gives a certain property of $\lfloor M \rfloor$. Which, and why?

d) We now assume that $10^x = M$ (cf. a)). Show that $\lfloor x \rfloor = C(M)$, and that if x is written as in (*), then $c_k = C(M_k)$ $(k \in \mathbb{N})$ where the sequence $(M_k) \subseteq [1, 10^{10})$ is defined by

$$M_1 = \left(\frac{M}{10^{C(M)}} \right)^{10}$$

$$M_{k+1} = \left(\frac{M_k}{10^{C(M_k)}} \right)^{10}, \quad k = 1, 2, ...$$

e) Find $\log_{10}(57.64)$ with 4 decimals by using the method from d) - explain your calculations!

f) Explain briefly how the Maple procedure below (due to Mikkel Abrahamsen, DIKU) works, and verify your result from e) by using the procedure.

```
compLog := proc (M, d := 10, n := 4);
    local a, i, res, Mnew;
    Mnew := M;
    res := 0;
    for i from 0 to n do
        for a from 0 by 1 while d^(a+1) <= Mnew do
        end do;
        Mnew := evalf((Mnew/d^a)^d);
        res := res+evalf(a*d^(-i));
    end do;
    return res;
end proc:;
```

Appendix C: Results of the survey in 2021

1. Which mathematics tool(s) did you use in high school? (Multiple choice)

Answer 1: Maple	(6)30%
Answer 2: TI Nspire	(8)40%
Answer 3: Geogebra	(8)40%
Answer 4: Other	(5)25%
Answer 5: Nothing	(3)15%

2. How much experience did you have with Maple before this course? (Single choice)

Answer 1: No experience	(0)0%
Answer 2: Same experience	(17)85%
Answer 3: a lot of experience	(3)15%

3. Please indicate how much you agree with the following statement: “A high school mathematics teacher should know more about CAS than the students” (Single choice)

Answer 1: Strongly agree	(11)55%
Answer 2: Agree	(7)35%
Answer 3: Disagree	(1)5%
Answer 4: Strongly disagree	(1)5%

4. Please indicate how much you agree with the following statement: “CAS is an important part of high school mathematics education” (Single choice)

Answer 1: Strongly agree	(6)30%
Answer 2: Agree	(11)55%
Answer 3: Disagree	(1)5%
Answer 4: Strongly disagree	(2)10%

5. Please indicate how much you agree with the following statement: “CAS is an important part of high school mathematics education” (Single choice)

Answer 1: Strongly agree	(5)25%
Answer 2: Agree	(10)50%
Answer 3: Disagree	(3)15%
Answer 4: Strongly disagree	(2)10%

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Technology in university mathematics education

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Contents

Abstract

We present a new classification of technologies and their didactic functions, based on the anthropological theory of the didactic and, in particular, on the media-milieu dialectic. Drawing on this framework, we examine how research on university mathematics education (UME) has focused on different aspects of technology use, while other aspects remain to be considered more closely. We

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observe, in particular, that a large part of the existing research literature has focused on small-scale experiments with ready-made tools (such as computer algebra systems) and more recently also on the use of online platforms for teaching and disseminating mathematics. Adopting largely the viewpoint of university mathematics teachers, such research explores new possibilities for undergraduate mathematics teaching, taking into account its great heterogeneity of audiences. Research has so far been much less attentive to the institutional forces that drive or inhibit the use of technology in universities, as well as to the changes in mathematical contents which such use often implies, locally (e.g., certain mathematical methods cease to be taught) and more globally (e.g., certain mathematical domains become more or less privileged). Some parts of the research object are intensively investigated, most notably the use of CAS in basic university mathematics teaching, while others – such as programming as a tool for learning and doing mathematics – are only emerging as phenomena to be studied.

Keywords

University mathematics education · Technology · CAS · Programming · Anthropological theory of the didactic (ATD)

Introduction

This chapter focuses on the use of digital resources in university mathematics education. A specific and classically emphasized trait of university institutions is the cohabitation of scientific research and teaching. One could then expect new developments in mathematical research – like increased use of technological tools or important interactions with computer science – to be quickly and seamlessly reflected in teaching. However, research suggests that the impact of research is less direct on teaching (Madsen and Winsløw 2009) and that, in particular, undergraduate curricula in mathematics are surprisingly static, with the impact of technological developments typically limited to some specific topics (such as numerical analysis or statistics) or to a few first-year courses (Bosch et al. 2021). There are relatively many studies of small-scale experiments with innovation of university mathematics teaching through the use of ready-made technology and programming, to some extent based on the researchers' own use or discovery of those tools in research contexts (see, for instance, the works presented in TWG14 in the CERME12 conference, González-Martín et al. 2022), which in some cases lead to permanent, local changes (examples are given in this chapter, sections "Digital Media in UME" and "Digital Tools in UME"). The extent and mechanisms of such impacts of mathematics education research need further investigation, as we discuss in this chapter.

Mathematics education at university level is, however, not only influenced by technology through occasional adjustments to the developments in the mathematical sciences and their neighborhood, such as mathematics education research. Other influences come from trends in society and education, such as the more widespread use of online and hybrid modes of teaching or students' general habits concerning computer use, for instance, to take notes or search for information. These influences, although coming from general and well-known phenomena, still have particular repercussions on mathematics in higher education, for instance, due to the use of symbolic writing in mathematics and mathematics-dependent fields. Other impacts are related to the vast variety of media for higher mathematics that has become available on the Internet, including discussion forums and videotaped lectures on almost any mathematical topic. Teachers may use such online resources (in the sense of Gueudet 2017) actively in their teaching, for instance, by curating them as links that are shared with students through a course management system. Even when these are not known to the teachers, they may still be used by students and thus, in a way, influence the educational processes. Research on teachers' uses of online sources has emerged over the past decade or so, while much less research has been devoted to students' spontaneous use of these resources and its impact on teaching and learning (for an exception, see Hausberger 2016).

This chapter aims to further outline and synthesize international research into technology use in university mathematics education (UME), including university teachers' choices and beliefs related to technology, as well as to point out some open and less explored questions. A main contribution of the chapter is to present and demonstrate a new way to categorize research on digital technologies in mathematics education. The framework (described in the next section) is based on the anthropological theory of the didactic (ATD hereinafter – Chevallard 2019; Chevallard et al. 2022). It serves, in this chapter, to explain and justify:

- The categories used to classify and describe individual research studies
- The methods for selecting and analyzing particular references

We then present outlines of the selected references in four broad categories and conclude with a summarizing discussion and perspectives for future research.

Methodological Framework

By *technology,* we refer here broadly to software on electronic devices, from a video conferencing app to a programming environment. This obviously covers a large variety of software, all of which could in principle be used in university mathematics teaching. To organize our categorization of studies of such uses, we begin with basic assumptions and terminology from ATD (e.g., Chevallard 2019; Lagrange 2005) that we use throughout the chapter. It was a deliberate a priori choice from the beginning to develop our categorization based on this framework, to explore its potential in the special case of university mathematics education, and to test its operationalization while classifying different studies.

Technology Use in Mathematics Teaching

ATD studies human activity as it unfolds in (and is specific to) social institutions. Human activity is modeled as consisting in *praxis* (actions) and corresponding *logos* (discourse). Praxis consists in solving problems or *tasks* by applying one or more *techniques*. Logos falls in two parts: *technical discourse,* which is directly concerned with describing and justifying techniques, and *theoretical discourse*, which is more general and takes technical discourse as an object, so that we can, for instance, link different technical discourses and discuss their terminology, assumptions, validity, and scope. We use the term *praxeology* to refer to collections of related praxis and logos (so the logos serves to describe, develop, justify, and link the praxis). Note that in ATD, the term "technology" usually has a specific meaning, different from the widespread use of the term; to avoid confusion, in this chapter we use "technical discourse" to refer to this term.

The distinction between praxis and logos leads us to a first classification of technology use (by a user, x).

- The technology is used as a *tool* by x, when it allows x to carry out a *technique*.
- The technology is used as *media* by x, when it allows x to access and/or produce praxeologies and, in particular, develop *logos*.

We emphasize that this distinction is not a classification of digital technologies but of *usages*: abstractly speaking, the technology can directly affect the praxeology activated by a user or can mediate certain relations between the praxeology and the user, within some institutional context. To provide a simple example, we can think of a university student working with a mathematical exercise (task), using the computer algebra system called *Maple* (see www.maplesoft.com). This same piece of technology may be used as a tool (offering techniques in the form of ready-made commands or for writing mathematical text) and as media (to look up mathematical definitions or theorems).

We define *teaching* as the efforts made by someone (called a teacher) to enable someone else (students) to engage with some praxeology (e.g., mathematical). In particular, the praxis could involve the use of technology as a tool or as media or both. Moreover, students' intended use of tools and media could be more or less *guided* by the teacher, and the students could be intended to engage not only in using ready-made tools or media but also in producing them. Teaching is in itself a praxis with a more or less explicit logos, and studies of teaching often lead to more shared technical and theoretical discourses about this kind of praxis – as when teachers or researchers give names to particular techniques or principles of teaching. Therefore, studies of technology use in teaching will be more or less explicit about the choices and capacities of the teachers carrying out teaching practices. We also emphasize that involving technology in teaching may change not only the teaching techniques but also the mathematical praxeologies that are being taught. This especially holds for tools, since they provide new mathematical techniques, many times requiring a modified *logos,* with new technical, and sometimes even theoretical, discourses.

Categories of Studies

We have organized our search, analysis, and presentation of the literature in four categories, according to the above theoretical framework, as we now explain (see also Table 1).

Considering research studies emphasizing the use of technology in mathematics teaching at university level, we found that a significant part is related to students' use of ready-made (technology based) techniques, like the use of a computer algebra system (CAS) to perform certain symbolic calculations (thus, carrying out one or more techniques). In these cases, the tool is typically a black box to students, in the sense that they do not always know how it works, for instance, how it graphs functions or inverts matrices. In contrast to this common situation, there is a newer focus (with surprisingly old roots, as we discuss later in this chapter) on teaching that emphasizes students' *production* of tools, mainly by programming (sometimes also referred to as coding). The intentions behind such an emphasis can both be to integrate praxeologies from computer science and software development into a mathematics course and to pursue the aim that students get a less black box about the mathematical techniques carried out with the software. Thus, we end up with two ways of involving technology-based techniques in students' mathematical work: engaging students in the use of *ready-made tools* that can solve certain mathematical types of tasks and asking students to *produce tools* (by programming) that can solve such tasks.

When it comes to media, we can make a similar distinction between *productive* and *receptive* usage modes (as in linguistics, when distinguishing production and reception of language or text; notice that there is no value judgement in any of the terms). The most commonly studied mode is the productive one, that is, where students may *produce* praxeologies, as in an online teaching platform or while editing an assignment using a word processor. Most of the commonly used technology (like video conferencing apps, content management systems, and web browsers) is not specific to mathematics, although university students will often meet and use mathematics-specific technology for editing text (such as *TeX* or even *Maple*). A less commonly studied phenomenon concerns students accessing mathematical logos online, much as in a classical library, but now with the vastly enhanced volume and search engines available on the Internet. We thus distinguish *interactive media* from *library media*. Interactive media are mainly used to exchange praxeologies and to

Table 1 Summary of categories of technology use

	Receptive use	Productive use
Tools	*Ready-made tools* (Section "Ready-Made Tools: Use of Dedicated Software")	*Programming* (Section "Programming and Links with Computer Science")
Media	*Library media* (Section "Library Media: Use of Online Mathematics Resources")	*Interactive media* (Section "Interactive Media: Online UME")

work collaboratively. Library media are mainly used to search and access information and knowledge.

Selecting and Analyzing Studies

The identification of studies for the survey began with:

(a) An initial screening of all issues of the *International Journal of Research in Undergraduate Mathematics Education*, for papers focusing on technology use in teaching; appearing since 2015, it is the main international research journal on university level mathematics education.
(b) A search on Google Scholar using the keywords *university*, *mathematics education, computer* and *undergraduate*, *mathematics*, and *computer*, restricting to the period from 2010 to 2021. We did find a large amount of literature in this way and then filtered out articles and books that fit the theme of the chapter.

To get a rough impression of the landscape and fine-tune our use of the model in Table 1, we classified the studies identified in these two searches. In fact, all studies were clearly classifiable in terms of the technology use that was the focus of the study. Table 2 shows how many appeared in each category, with *n/m* meaning that *n* studies were found in search (*a*) above and *m* in search (*b*).

Certainly, this coarse search was only a beginning. We also considered papers identified from the references in papers appearing in the search, as well as papers known by the authors to be relevant to the theme (including some not written in English), which led us to consider some references published before 2010.

We then categorized the papers identified according to their focus on ready-made tools, the production of tools by coding, interactive media, and library media. We focused on the findings as expressed in the abstract and the conclusion, and then we identified the institutional context and aims of the usage under study and more generally the mathematical praxeologies involved (to the extent they are given). We have in the end selected references to present in each of the four categories, in view of our assessment of the importance and originality of the findings, the specificity of findings to university institutions, the level of explicit and detailed study of actual uses of technology, and a reasonable representativity in terms of mathematical domains and institutional contexts studied in these references. Throughout, the above framework served to identify the research objects, the questions, and findings of the studies, and, in particular, the classification proposed in Table 1 led us to grouping connected studies.

Table 2 Number of studies found in initial screening

	Receptive use	Productive use
Tools	2/14	1/7
Media	1/2	6/10

Digital Media in UME

This section considers studies about the use of digital technology in undergraduate mathematics education as *media*: a way to access, disseminate, and interact about mathematical works. The Internet is nowadays the *library media* per excellence, which enables access to many resources presenting information, like written texts, hypertexts, open online courses, animations, video presentations, etc. Its role as interactive media allows the organization of the exchanges between lecturers and students and among students themselves. The technological resources associated with these uses (from a simple web browser to communication, edition, and collaboration platforms) are extremely diverse but are also becoming more and more familiar, especially after their dissemination during the forced switch to fully online teaching during the pandemic lockdown that began in 2020. In the exploration of the possibilities, role, and impact of these digital media in undergraduate mathematics education, we distinguish the case where the entire teaching process is carried out almost entirely online from the case where online resources complement an instructional process that is mainly conceived and implemented face-to-face.

A general and striking observation from our reading of studies in these two areas is that research generally focuses on the *effects* of using digital media and the *kind of interaction and uses* that are made by students and by teachers. Barring a few exceptions, the focus is not on the kind of mathematics that is taught, the choices made to structure the content, or the main pedagogical strategy adopted (if any). This means that most of the instructional processes remain undescribed in the papers, as if there was nothing special to notice about them. At the risk of oversimplification, we can describe the mainstream assumptions by saying that, in a typical elementary university mathematics course like calculus, algebra, linear algebra, or differential equations, the content is well established and organized in topics defined by some notions and theoretical developments, associated to a set of exercises and more or less complex problems. Finding, structuring, and presenting the main content are the responsibility of the lecturers, while students are mainly required to learn how to solve the problems by practicing with exercises and paradigmatic examples. Many studies analyzing the role of technology as media consider this kind of instructional organization. These assumptions could be related to the fact that research is often carried out by (or in close proximity with) university teachers and tends to give prominence to the teacher's perspective, for whom contents are determined by syllabi and the main question is that of their delivery.

Interactive Media: Online UME

We can distinguish two types of fully online mathematics courses depending on whether the interaction teacher-students is synchronous or asynchronous. The first case appears mainly in recent studies related to the Covid19 pandemic (Radmehr and Goodchild 2022) or in specific courses like developmental, remedial, or bridging courses offered prior to or during the first months of tertiary education (Biehler et al.

2011; Webel et al. 2017). The second case corresponds to the typical "distance learning" modality, supported by an institutional *learning management system* (Moodle, Blackboard, Canvas, etc.) that stores the course content and provides an infrastructure for interactivity. In both cases, teachers use the system to provide learning resources to students, usually in sequential order and with some milestones to pass at certain points, for instance, in the form of tests to take, or work to deliver, at some specific dates. Then, the teachers provide feedback to students' work – possibly, on top of automated feedback provided by the system – and attend to students' questions and doubts. In the asynchronous modality, this will be done in chats, forums, mail exchanges, or similar; in the synchronous mode, face-to-face interactions are also considered. Similar devices can be used by students to collaborate, in a private or public way.

We can consider the US "Math Emporium" as an example of synchronous online mathematics course, among others (see, for instance, Biehler et al. 2011). This type is entirely organized and "delivered" online and requires additional teacher work. For instance, students go to computer labs in groups under the guidance of an instructor who provides on-demand assistance to students who are struggling with the material. Unlike asynchronous courses, mandatory group meetings ensure students spend sufficient time on learning activities and also help build a feeling of community among students and with instructors. According to Webel et al. (2017, p. 356), "rather than listening to in-person lectures, students' progress through the course topics at their own pace, moving to more advanced topics only when they are ready." Technology is here essential to provide worked-out examples, video lectures, digitized textbook pages, opportunities to complete the exercises, and feedback on solutions with guidelines about materials to review or new practice problems. Moreover, technology is also used to provide immediate feedback to students through automatic grading and to sequence their work by requiring the completion of specific activities related to a topic before granting access to the next one.

The design, structure, and content of any online course respond to specific instructional principles that involve not only what it means to study a given content but also what this given content is. The research of Webel et al. (2017) is very illustrative in this respect. To contrast reports on the impact of Math Emporium courses, they start by noticing the "little information that exists about the nature of mathematics learning" (p. 359) in such courses. They then question "whether the efficiencies gained by the use of computer assistance and the diffusion of instructional roles come at the expense of meaningful learning experiences for the students" (p. 359). The close relationship between the format of the course and the kind of mathematics that is consequently taught is examined for the case of algebraic activities. Their results point at many deficiencies in the students' capacity to create models to represent situations and interpret equations in terms of their meaning in the considered situations. In other words, technology-assisted online courses tend to foster mathematical activities that may be tested and graded automatically and to which it is possible to provide automatic feedback because students' errors can be foreseen. However, it seems to exclude work with open-ended questions about complex or extramathematical situations, since such problems tend to admit a great variety of approaches and answers. It is also difficult to imagine automated feedback to students' written or oral presentation and defense of proposed solutions and procedures related to such problems. The authors conclude that there is need for research that takes into account not only passing rates, exam scores, and cost savings but also explores the nature of the mathematical reasoning that online courses promote, as well as how students interact and perceive their experiences.

Similar considerations are described by Boyce and O'Halloran (2020) in their shift from an emporium-style format of a college algebra course to a blended format where active learning can find a more prominent place. The course evolution is not only supported by the kind of digital devices proposed but also by changes in the rhythm imposed – weekly modules – to facilitate cooperative learning activities and modifications in the course content, like adjusting "the topics and sequencing of topics [. . .] to make room for problem-solving sessions focused on mathematical concepts" (p. 459). Students' discussions of mathematics, written reports, and presentations of their group work through poster sessions were also integrated. The authors analyze the kind of mathematics that is to be taught according to the digitally supported structure of the course. They conclude that "knowing how to use a mathematical tool is arguably less important than knowing how to use the right tool to solve a problem, knowing how tools are related to one another, or knowing how to communicate mathematics with others" (p. 472). Finally, they also point at challenges in implementing computer-based instruction like "supporting students' understanding of the concepts of mathematics, promoting student discourse, and engaging students in challenging problem-solving" (p. 472).

In their paper about fully online (FO) teaching of undergraduate mathematics, Trenholm and Peschke (2020) address a broader set of courses, focusing on the asynchronous case, and compare them with current face-to-face (F2F) practices. They identify key differences between the two paradigms, like the need for a deeper engagement of FO students; the prevalence of one-on-one experiences with little use of collaboration; and the difficulties in communicating mathematics via notations and diagrams or other semiotic resources like gestures and multimodal communication. Concerning the use of digital resources, there are reasons to think that technological developments can continue to reduce these differences, by providing new tools to communicate through simultaneous multiple channels and modalities, as well as procedures to ensure online reliable proctoring. Technology is bringing the two modes of education ever closer together, and Trenholm and Peschke (2020) notice "a realignment in pedagogical approaches between the two communities of F2F and FO teaching" (p. 27).

In all synchronic and asynchronic modalities, online mathematics teaching requires specific technological resources to ensure the material dimension of the mathematical activity: writing, speaking, gesturing, diagramming, and sketching. These dimensions involve both individual and collaborative activities and communicating the outcome to others. In this respect, technological resources can foster certain practices and hinder others, in the same way as a given classroom organization, schedule, or blackboard size do so. However, the problem of what kind of mathematics to teach, and how to teach it, will remain.

The work of Rosa and Lerman (2011) provides an interesting case in this respect. They describe an online 40-h course that included synchronous and asynchronous activities such as chats, forums, email exchanges, and the elaboration of a final project. An aspect that stands out in their proposal is the innovative character of the type of mathematical activities that are proposed in the online environment, even if the technological tools were not as developed at the time as they are today. Instead of being introduced to some previously established topic – definite integrals – along with some sets of exercises and problems to solve, students were required to engage in a role-playing game to help a farm widow manage her property (which included calculating the area of irregular fields). The function of the digital resource and the online organization of this type of instruction is, of course, radically different from those of the Math Emporium perspective. At the same time, the questions raised might be very similar to those of analogous F2F proposals, proving that the considerations about digital technologies must take into account other kinds of teaching and learning resources, modalities, and organizations, as well as the mathematical activities that constitute the goal of the course. In their review about FO tertiary mathematics, Trenholm et al. (2016) confirm this as they note that the very nature of mathematics is not being addressed by current professional development efforts and propose the use of technology-enabled peer assessment processes to help students engage in higher-level mathematical activities – praxeologies – and student-student interactivity. They call for progress in FO *mathematics* pedagogy and suggest that such a focus would also help improve F2F practice.

Library Media: Use of Online Mathematics Resources

Today's face-to-face teaching involves a large number of online mathematics resources that are used for different purposes. The next section considers resources like CAS that are integrated within the mathematical activity and can be thought of as part of the mathematical praxeologies (more specifically, while supplying certain techniques). This section focuses on digital media that are mainly used to get or provide access to mathematical knowledge: texts, videos, or animations presenting topics, a corpus of exercises and problems, examples of problem resolutions, activities to test the mastery of a given content, etc. These media can incorporate interaction in the form of feedback or options for cooperative use. Several studies address different kinds of exploitation of such tools: solving tasks (Kanwal 2020; Rønning 2017), receiving lecturers' feedback (Robinson et al. 2015), passing tests (Kinnear et al. 2021), learning collaboratively (Heinrich et al. 2020), doing homework (Dorko 2020), or, more generally, the interaction of students with different digital resources (Anastasakis et al. 2017; Fleischmann et al. 2020; Oates et al. 2014). As with the previous case, we first discuss the importance of the link between the choice of digital media and the type of mathematical activities these resources support or hinder. We finish by pointing at some uses of digital media that do not seem to have been considered in research and seem also mostly unmanaged in university education.

Fleischmann et al. (2020) focus on the benefits and appreciation of the integration of digital learning materials from an online course for students in an attendance-based learning environment. Their design corresponds to a blended learning scenario based on an already existing course concept. One interesting, expected result concerns students providing positive overall feedback about the support provided by the digital learning material, especially "passive elements" like dynamic applets and videos as part of the lecturers' presentation, which did not change in a major way their previous in-class learning strategy. We can interpret this in terms of the prevailing *didactic contract* (Brousseau 1997): students appreciate the incorporation of digital tools as long as they do not increase students' level of responsibility for the content, in the learning situation. A related point is found by Anastasakis et al. (2017), who studied the kind of resources students use when studying mathematics and the goals and reasons for their choices. They find that students use a variety of resources (like online videos, WolframAlpha, and online encyclopaedias) but mostly those provided by the university and their written notes. Students' aim is mainly related to passing the exams and getting a high mark, so they choose what they think is helpful to that end, according to their reading of the didactic contract. Oates et al. (2014) also note the crucial role of the lecturers' practices as an example, privileging the use of technology (videos, websites, simulators, and CAS), as a key factor to explain students' engagement with digital tools. As noticed in the previous section, we also observe here that, in none of the investigations considered, the researchers comment about the kind of instructional organization that is implemented nor about the content structure associated with it.

Other authors put a stronger emphasis on the connection between both. For instance, Rønning (2017) directly studies the influence of computer-aided assessment (CAA – here *Maple* T.A.) on the way students work with mathematics. He finds that students develop strategies to obtain quick, direct answers to problems ("hunting for the answer") and pay less attention to writing out the solution process carefully. At the same time, students report learning more from the lecturers' feedback than from the automated one, because teachers also address the solution process and their way of reasoning. We can see here two different modalities not only of learning but also of *doing* mathematics. The author states that "problems handled with CAA must have clear, objective answers, whereas human markers can handle both objective and subjective problems, and also human markers can act flexibly when faced with ill-posed or unanticipated student responses" (p. 98).

Similar considerations are reported by Kanwal (2020). In her study about students' interactions with an online environment, the author also found the use of a variety of resources on top of those provided by the course, conducted through Pearson's MyMathLab (MML) and based on a concrete textbook of *Mathematics for Engineers*. In this case, the author provides some details about some mathematical activities performed by the students. She finds deviations from what is initially expected – like "hunting for the answer" – due to the way the automated system conditions the kinds of activities that are proposed and the strategies that are fostered: division of problems into sequences of operations, choice of tasks that can be divided into single steps, etc. The term "black-boxed mathematics" is used to explain that the change produced also affects the theoretical developments that are needed to justify the procedures and results obtained. The author says that "the implementation of [the online] environment does not ensure that the students engage with the mathematical tasks in the expected manner" (p. 62). We could go further and discuss specific transformations of the mathematical content that is taught and learnt in both its technical and theoretical dimensions.

The work of Dorko (2020) addresses this transformation when searching to characterize the nature of Calculus II students' activity as they complete an online homework assignment – here, about sequences. Instead of talking about "hunting for the answer," the author characterizes students' multiple attempts and formative feedback as a cyclic activity and compares it to mathematicians' problem-solving activity. However, she also notices a critical difference: mathematicians validate their results by themselves, while it is the online platform that does the verification for students. The author does not analyze the kind of mathematical problems that constitute the online homework nor whether or how the choice of the online modality affects what kinds of problems can be worked on. Again, the mathematical content is taken as a parameter instead of a variable to question.

Two other important aspects remain unexplored in the investigations considered above. One corresponds to the way digital tools can help develop – or to the contrary, hinder – the collective dimension of the mathematical activity. To this respect, in their exploration of the perceived experiences of mathematics lecturers and students in Norway as they transitioned to fully online education in 2020, Radmehr and Goodchild (2022) point at the difficulties for interaction and collaboration as one of the major challenges to address. We can mention to this respect that university mathematics education does not always foster the implementation of cooperative activities and tends to be dominated by individual mathematical practices. Moreover, teamwork in mathematics is more present in research than it is in teaching. Therefore, teachers could find ways to exploit more and better the spread of digital platforms to share and capitalize on teaching strategies, thus developing a more collective vision of the profession.

The last aspect that appears unexamined – and remains unmanaged in many university teaching settings – concerns the opportunities provided by digital media to access information and how this modifies what doing mathematics means, not only for students but also for mathematicians. Several studies (Anastasakis et al. 2017; Dorko 2020; Kanwal 2020; Oates et al. 2014) indicate that students spontaneously use a variety of digital tools (forums, videos, websites, etc.) to access information outside the media provided by the institution (course material, lecture notes, and the like). Being able to read, confront, test, and validate the mathematical information that appears in such a variety of online sources is a critical competence nowadays to do mathematics – not only to learn it. We still know little about how students deal with those alternative media and, more importantly, how to manage it within the instructional process (Hausberger 2016).

Digital Tools in UME

We now consider research on university teachers' and students' use and production of digital *tools* that can solve certain mathematical types of tasks. We note, initially, that there seems to be some delay – if not a more permanent discrepancy – between the overall roles of digital tools in mathematical research and practice (e.g., Lockwood et al. 2019) and its presence in undergraduate and graduate curricula, particularly within pure mathematics.

Programming has emerged in mathematical practices over more than 70 years ago and has gained some place in undergraduate education as well. From the 1990s, ready-made tools (CAS, spreadsheets, etc.) have become gradually more available and efficient, as has their use in secondary and tertiary education. We examine research on these developments in this section.

Ready-Made Tools: Use of Dedicated Software

Different ready-made tools have been used in UME, such as spreadsheets, for the teaching of probability and statistics (e.g., Lagrange and Kiet 2016), or in some engineering courses (e.g., Castela and Romo Vázquez 2022). Other environments that allow for modeling and visualization have also been used (e.g., Hogstad et al. 2016). Among the different tools available, CAS have received considerable attention in the UME literature. For this reason, we focus a part of this section on CAS use by university teachers and then zoom in on studies focused on students' learning. We finish this section identifying some potential risks related to the integration of ready-made tools at the tertiary level.

Regarding the use of CAS by university teachers, Lavicza (2008a) noted that "little attention has been paid to why and how CAS is being integrated into the university curriculum, what factors influence CAS integration, or the extent to which CAS remains permanently used in a university environment" (p. 121). We note that two views are present in the literature: (1) university mathematics teachers use technology at least as much as school teachers, although a large part of this use is not reported in the literature (Buteau et al. 2010a), and (2) the impression that undergraduate mathematics education seems blind to technological advances, which are, however, much more present nowadays in many areas of mathematical research (Artigue 2016).

To produce a clearer view of trends in the existing literature on the use of CAS at the postsecondary level, Buteau et al. (2010a) analyzed a corpus of 204 papers. This first corpus considered the journals *International Journal for Computers in Mathematical Learning* (since its beginning in 1996) and *Educational Studies in Mathematics* (since 1990). They also selected proceedings from the conferences *Computer Algebra in Mathematics Education* (since 1999) and the *International Conference on Technology in Collegiate Mathematics* (since 1994). A first remark is that whereas 88% of these papers are practice reports by practitioners (presentation of examples, examples with practitioner reflections, classroom study or survey, examinations of a specific issue, or only abstracts), only 10% were education research papers. In the sub-corpus of practitioner reports, the most integrated tools that are mentioned at that time are graphing calculators (40%), *Maple* (25%), and *Mathematica* (20%) (see also Buteau et al. 2010b). The factors which influence teachers' decisions about CAS integration, as reported in these papers, were grouped in three main categories: (1) technical issues (lab availability, reliability of technical support, system requirements, troubleshooting), (2) cost-related issues, and (3) pedagogical issues. The last category is the most present in the analyzed corpus, and among the 11 issues identified, the more frequently mentioned are (a) the difficulty to design appropriate assessment integrating technology (related to the time it takes to properly address this issue); (b) the difficulty for students to learn the new syntax; (c) the fact that technology may provide answers in an unexpected format that does not match the paper-and-pencil expected solution; and (d) the need for time for faculty to design courses and meaningful activities with technology. These issues are identified as obstacles to an extensive integration of CAS in teaching. Moreover, this integration is mostly studied in individual courses (67% of the corpus), with few reports about integration in a group of courses or program-wide. We note that some of these issues point at difficulties related to the didactic transposition work, since integrating CAS requires the creation of praxeologies to be taught that integrate new types of tasks adapted to the instrumented techniques and new theoretical developments. We return to this point below.

The results above thus identify major reasons that influence faculty's decision to integrate or not technology in their teaching. Buteau et al. (2010b) also offer some interesting information about specific uses of CAS by university teachers, with calculus (including precalculus and multivariable calculus) being the courses most present in their corpus. Regarding the programs where CAS are integrated, Buteau et al. (2014) collected data from 302 Canadian instructors, indicating that 92% use CAS for mathematics and computer science majors, 87–89% use it for science or engineering majors, and 70% use it for mathematics education majors. The limited geographical scope of the instructor survey calls for further research to have a better view of the programs where CAS (and other dedicated tools) is being used. Regarding the actual uses of CAS, the most reported use was to provide an experimental laboratory where students could explore mathematical objects (Buteau et al. 2010b, p. 59), followed by visualization and exploring real world or complex problems; another use consists on using the CAS to assign projects and homework (Buteau et al. 2014). Buteau et al. (2010b) also note that an important number of papers emphasize the potential benefits for students of using CAS: promoting a greater understanding of mathematics (for instance, allowing the use of several representations), supporting students' development to achieve and learn independently (for instance, through exploration), increasing student motivation to learn, facilitating access to harder and more realistic mathematics, and being responsive to twenty-first-century workplace needs. The idea that the use of CAS can also help focus more on conceptual understanding, leaving tedious calculations aside, is also mentioned (we return to this point later in this section). Among the examples of innovative uses of CAS, Buteau et al. (2010b) mention examples like explore integration by approximating area with finite Riemann sums, visualizing different terms of a Taylor series, and visual experimentation with the formal epsilon-delta definition of a limit; visualization of complicated three-dimensional surfaces or solution of systems of linear equations is also cited. It seems, however, that the use of CAS is rather local, affecting a topic or a type of tasks, with little studies of a systematic integration of CAS in the praxeologies that are taught, with their consequent transformations.

The previous results led to an interest to go deeper into the factors that encourage teachers to use CAS in their teaching. This led to the abovementioned survey with 302 Canadian postsecondary instructors (Buteau et al. 2014; Jarvis et al. 2014). As many as 81% participants used CAS outside teaching (e.g., research), while 69% used CAS in their teaching. An important result of this study is that the analyses suggest that the strongest predictor for the use of CAS in one's teaching is the use of CAS in one's research; this result agrees with a similar finding from an international study considering university mathematics teachers in Hungary, the UK, and the USA (Lavicza 2008b). This could be seen as a possible explanation for a spontaneous integration of technology in tertiary teaching (Buteau et al. 2014), since it is already a tool in many instructors' research practices. This result is reinforced by the data concerning the inverse tendency: the participants who reported never using CAS in their research all mentioned either never (67%) or rarely (33%) using it in their teaching. It is also worth noting that respondents who used CAS in their teaching exhibited two dominant ideas about the role of CAS: CAS is a tool (not a purpose in itself) that helps learn mathematics, and CAS is used when it is believed it will help students understand better (this is similar to the common distinction between pragmatic and epistemic values of instrumented techniques, defined by Artigue 2002, p. 248 – we return to this distinction below). The first case involves a transformation of praxeologies to include CAS, at least in the praxis component (techniques and types of tasks), while the second may overlook the changes of mathematical praxeologies that come with CAS techniques. Some participants also mentioned that CAS is not the most relevant technology for their courses, in particular for statistics courses. More information from this survey concerning programming is discussed in the next section. Finally, among the conditions identified by the participants as important for a successful integration of technology, beyond individual initiatives, there are a key proponent in a decision-making position in the department, a strong and shared incentive for change, strategic hiring practices, an administration that supports creative pedagogical work, and a continuous and determined revisiting of the original program vision.

The previous paragraphs provide a broad view of the uses of CAS in postsecondary education as presented in recent literature, in particular, the practices of Canadian instructors. We now examine some of the reported uses of CAS and other ready-made tools, as well as some of the reported effects of this use. Calculus courses have received special attention since very early. Some examples include studies related to the visualization of the local linearity property related to the derivability of a function at a point (Tall 1996), the qualitative study of differential equations (Artigue 1987), the visualization of epsilon-delta strips (Roh and Lee 2017), etc. These different initiatives over the years have led, for instance, to the revision of whole calculus courses with the integration of technology, such as DIRACC Calculus (Thompson 2019; Thompson et al. 2013). It is undeniable that the use of different dedicated technology has been connected to the evolution of research about the teaching and learning of calculus. Among others, the role of technology as a support for visualization and coordination between registers has been highlighted (e.g., Artigue et al. 2007). However, we note that for studies that focus on punctual activities, it is typically not clear how the integration of these tools (to produce some techniques, such as graphing) impacts on the whole course or lead to the development of stable new praxeologies for the students.

The use of dedicated technology in studies related to topics other than calculus has also grown in the last years. For instance, Troup (2019) describes the use of *Geometer's Sketchpad* (GSP) to develop a conceptualization of the derivative of a complex-valued function. In this study, speech, gesture, technology, and reasoning about complex numbers are interconnected, helping students "discover that for a complex-valued function, its derivative describes how a small disk around a point is rotated and dilated by the function" (p. 4). This is seen as an important leap for students, moving from the derivative of a real-valued function to the derivative of a complex-valued one. The use of GSP allowed for an inversion in the tasks: instead of starting with formulae and interpreting them geometrically later on, Troup (2019) constructed an activity that starts with an embodied, geometric reasoning to then move toward symbolic algebraic reasoning. This allowed to reason about disks which need to be small enough to "stay away from bad points" (p. 22). Other studies are not mainly concerned with visualization and aim at constructing activities that foster students' relating of theoretical and practical aspects in real analysis. Gyöngyösi et al. (2011) follow the distinction (e.g., Artigue 2002) between the pragmatic value (the efficiency for solving tasks) of instrumented techniques and their epistemic value (the insight they provide into the mathematical objects and theories to be studied). Drawing on the notion of praxeology, they propose an organization of tasks concerning sequences and series, to be solved using *Maple*, which allows students to produce some examples that can help them grasp and use theoretical results. This way, instrumented techniques can support theoretical reasoning. Their experimentation allows for identifying two main groups of students: *the proud purists* (p. 2011), who are successful in analysis and are not very eager to use *Maple* in a real analysis course, and *the challenged but helped*, who are students struggling with some parts of the course and who benefited from the tasks where instrumented techniques can be used. Studies about the use of ready-made tools in advanced, traditionally theoretical courses definitely challenge our own (and students'!) epistemological views concerning mathematical activity.

Many writings in this area – and not only in journals accepting practitioners' reports from practice – take a less critical stance. They report on students' improved learning or easier access to grasp some notions, as a consequence of some activities designed for this purpose, and involving some dedicated technology. However, there are studies that highlight how the use of tools for algorithmic tasks also has the potential to raise new questions that produce conflict and foster reflection. Dreyfus and Hillel (1998) reported on the use of *Maple* in tasks concerning the Gram-Schmidt procedure to obtain an orthogonal basis for the space of quadratic polynomials. The tasks of finding an approximation for cos(x) and for x^3, which can be carried away implementing an algorithmic technique in *Maple*, led some students to reflect about the theory. In particular, the researchers describe the event organized in episodes: (1) *Maple* at rest (where students exchange about the task without actually using the software), (2) *Maple* as a graphic calculator, (3) *Maple* as an investigative tool, and (4) beyond *Maple* (where the students, after the investigation, discuss about the concepts at play and their own understanding). This organization shows that, even without actually using the software, students can exchange to try to understand some tasks, use the technology in an algorithmic way, and then explore new questions that lead them to better grasp the underlying notions and results. Dreyfus and Hillel (1998) contend that the technology helped as a mediator, supporting students' reasoning even when they lacked the precise, technical vocabulary necessary. In this sense, the participants of the course in question agree to some extent that by taking care of the computations, the use of technology allowed for more time to be devoted to reflection. This matches one of the advantages of using technology highlighted by several studies and mentioned above.

We also wish to mention some recent advances in the use of specialized technology for the learning of proof, with two examples. In the first example, Sümmermann et al. (2021) discuss the role of simulations (in their case, for an algebraic topology course, using the dynamic topology environment ARIADNE) and their impact on the concept of proof. The authors propose that simulations can allow the construction of proofs, with the advantage of avoiding symbolic representations. This way, these activities can be seen as a gateway into proving, "giving an alternative *access* to proofs in a non-formal highly interactive setting" (p. 457). In particular, they study the necessary conditions that simulation-based environments need to fulfill to allow the construction of proofs while also identifying some challenges for simulation-based proofs to be regarded as genuine proofs. This leads again to considerations about the epistemological dimensions of mathematical activity. Due to the increasing number of mathematical simulations becoming available for students and teachers, more research is needed in this area. The second example concerns automated theorem provers. These environments are not, strictly speaking, ready-made technology as they require some level of coding activity (of a special kind). Very recent reports "suggest a positive impact [of using theorem provers] on students' understanding of the necessity of mathematics rigor and subsequent advantages for proof production and proof writing" (Thoma and Iannone 2022, p. 65). Thoma and Iannone (2022) report on the use of the theorem prover LEAN to support students' learning of proving. The technology was presented to students in a voluntary workshop, which ran parallel to a proof course in a first-year mathematics program. In their study, 36 students (7 LEAN users and 29 no LEAN users) tackled in the interviews one specific question (an unfamiliar result to prove: if $n \in \mathbb{N}$ is perfect, then kn is abundant for any $k \in \mathbb{N}$). Among the LEAN users, two main characteristics of their proofs are identified: (1) an accurate and correct use of mathematics language and symbols, together with the use of complete sentences and punctuation, and (2) a clearer structure, with an overt breakdown of proofs in goals and subgoals. This study illustrates how the use of ready-made tools – in particular theorem provers – is gradually including more theoretical areas that go beyond the classic uses of CAS in calculus and linear algebra, where ready-made tools are usually reduced to provide techniques for calculations and graphing. These very recent studies call for further research into how technology integration affects not only the learning of proof and reasoning but also their very nature while also call for studies identifying potential drawbacks.

Most of the studies cited in this section concern experiments where the use of technology is seen as a lever to improve student learning of mathematics. Reflections on the problems that the use of technology can provoke are also necessary. We finish this section citing one report that analyzes some challenges about the use of technology. Discussing the potential of CAS use in introductory university teaching, Winsløw (2003) uses the notions of semiotic and discursive activity to argue that the lever potential (allowing students to operate at a high conceptual level), usually put forward by studies, needs to be problematized. He also notes that

> . . .the use of a CAS – at least, a priori – facilitates neither *coordination* of registers nor the main *discursive* functions. The simple *representation* of objects and transformations is not simplified, either. On the contrary, we have an extra medium (the computer), an additional special code (depending on the CAS) for semiotic activity, and a kind of 'automatic semiotic agent' with a potential influence on discourse [. . .]. These additions may be particularly disturbing for novice users of CAS. (Winsløw 2003, p. 276)

Some of the potential risks of CAS use – identified in observations by Winsløw (2003) – and based on Brousseau (1997, pp. 25–27) are:

- The Jourdain effect: where students perform CAS-assisted semiotic actions and are then told "what they have done" in terms of a higher-level discourse that is essentially beyond their reach
- The animator effect: where the teacher's activity becomes conditioned by how much the students talk or use the CAS, losing focus on the actual (mathematical) aim of the activity
- The particularity problem: where the focus on particular examples may lead to an emphasis on inductive reasoning, hindering deductive reasoning which is common in advanced courses.
- The black-box effect: where the inaccessibility to the processes leading to a result promotes the (ab)use of trial-and-error techniques when confronted to an unexpected result instead of resorting to analyzing the reasons for the first result
- Conflicting intentions: when students tend to learn instrumented techniques with the aim of passing the course, rather than trying to understand the content at play

These potential risks are a consequence of the lack of well-planned integration of CAS in the mathematical activities that are taught and learnt. They contrast with the number of papers that report on successful experiments when introducing technology and only consider observed advantages. They also warn us about being aware that simply visualizing certain notions does not ensure, per se, an adequate grasp of these notions. Gyöngyösi et al. (2011) provide some examples of tasks that aim at going further, through activities that may lead from checking some examples to seeing the need of using theoretical results. However, more research is needed concerning the limitations and unwanted side effects of common use of CAS at university level, along with proven strategies to overcome these.

Programming and Links with Computer Science

It is by no means a recent position that computer programming should figure in the undergraduate mathematics curriculum, citing programming as a tool to develop powerful techniques for carrying out a wide range of mathematical tasks. As a recently appointed mathematics professor at Stanford University, Forsythe (1959) estimated that "there seems to be over 3,000 automatic digital computers now installed in the United States" (p. 651). This was of course supposed to be an impressively high figure at the time of writing. Forsythe defines "numerical analysis" as "a branch of applied mathematics" which covers "any type of problem which an automatic computer should be able to solve" as well as "techniques for coding" (pp. 654–655). For the undergraduate mathematics curriculum, he proposes "a special coding course for all students" and that "numerical analysis could otherwise be sifted in with the basic mathematical theory" (p. 658). He also outlines topics suitable for such sifting in as diverse domains as calculus, linear algebra, logic, differential equations, probability and statistics, number theory, geometry, and Fourier series. Forsythe admits that he is "making a recommendation which is almost two generations ahead of the current textbooks" (p. 661), and in fact he soon moved to other, perhaps more fertile, institutional terrains. From 1965 until his death, he served as chairman of the then new computer science department at Stanford – thus personifying how this new discipline broke away from mathematics. Before the creation of the department, he notes:

> The role of the Computer Science Division is likely to be increasingly divergent from that of Mathematics. It is important to acquire people with strong mathematics backgrounds, who are nevertheless prepared to follow Computer Science into its new directions. (Forsythe, quoted in Knuth 1972, p. 723)

It is interesting to dwell on a wider and older perspective in Forsythe's (1959) plea to integrate numerical analysis more or less throughout undergraduate mathematics. He cites a famous three-volume text written 50 years before, by none less than Felix Klein (2016). Klein distinguished "precision mathematics" and "approximation mathematics" (the third volume of his book treats the latter). According to Forsythe (1959), "Klein goes on to say that, while in research differentiation between pure and applied mathematics may be essential, such differentiation is not reasonable in teaching" (p. 653). However, while Klein focused on secondary school mathematics, Forsythe saw the new uses of computers and, in particular, coding, as means and reason for extending the integration to university teaching – arguing that university students must also experience computational and theoretical mathematics as deeply related.

Now, over 60 years later, we cannot say that this idea has fully materialized. As we discuss in the following, there are repercussions in undergraduate mathematics of both the institutional separation and the continued scholarly interaction, between mathematics and computer science. The use of programming for solving a wide range of mathematical tasks has certainly continued to be an area of vivid research and technological development. Still, we cannot say that undergraduate mathematics has become permeated, in general, by the use of computer programming and "numerical analysis" in the sense of Forsythe's early work.

Studies focusing on programming as a tool or subject in university mathematics education fall in two kinds: case studies of actual uses in individual institutions and courses (e.g., Lockwood and De Chenne 2020) and more general overviews and proposals (such as Forsythe's). A few studies also address how the use of programming in mathematical research is (or is not) reflected in UME.

We first consider the more global sort of studies – far from citing concrete types of tasks for which the students use or device techniques based on programming. Clearly, "university mathematics education" cannot be reduced to undergraduate and graduate programs in pure mathematics or to courses taught by faculty employed in a mathematics department (of some sort), but in our experience, relatively many studies focus more or less explicitly on these cases.

In Buteau et al.'s (2014) previously cited survey with 302 Canadian instructors, 18% indicated to have used programming in their teaching, while 42% said to have done so in their research. One could naturally speculate that, with the low response rate, the absolute figures might not be representative; in particular, mathematicians with little interest or experience with technology might not be inclined to reply to such a survey. The authors certainly note that "conclusions of the study are [. . .] somewhat limited by these various sampling issues" (p. 40) and, possibly as an instance of this, that only a minority of the respondents identified themselves as pure mathematicians (p. 39). Indeed, one would expect applied mathematicians to be more likely to make use of programming in their research while still undertaking their part of large enrollment, basic courses which involve no such use. Still, citing the difference in programming use within research and teaching, the authors consider that "reflection by mathematicians on the potential benefits of incorporating computer programming into mathematics research activities could, we feel, lead to an increased integration of computer programming in undergraduate mathematics instruction" (p. 53).

A similar survey, focused exclusively on the presence and character of courses involving programming in undergraduate mathematics curricula in the UK, was carried out by Sangwin and O'Toole (2017). Complete responses from about half of the identified mathematics departments were received, with partial responses from 63% of those departments. Among those departments who responded to the question, 52% indicated “all single and joint honors students take a compulsory course where computer programming is a significant learning goal” (p. 1139). The authors also found that “numerical analysis is currently the most common mathematical subject for compulsory courses which involve programming,” MATLAB being the most popular programming language in such courses (p. 1141). We note that similar trends were found in a recent study of European and Canadian undergraduate programs in pure mathematics (Bosch et al. 2021). As for the UK, Sangwin and O’Toole (2017) further found that “computing is disproportionally assessed by a significant course work component, compared to other university mathematics modules” (p. 1140), in which closed-book exams dominate. Programming may, indeed, lend itself more to forms of assessment based on course work (e.g., Buteau and Muller 2017); however, in a context where most course units are primarily assessed by traditional exams, one could speculate that the result could be that students and faculty consider programming-oriented units as somehow inferior and isolated. The authors also found that “no university let someone from outside the mathematics department teach the core compulsory programming modules” (p. 1144) and that, according to faculty members, a significant part of the students are “reluctant” to learn programming (p. 1146). The first observation could in part be explained by institutional mechanisms for distributing funding based on teaching and in part result from a desire to align such modules with the overall curriculum. The latter may reflect that such modules are, perhaps frequently, seen as a necessary evil by some students and faculty, as when the latter refer to programming as “a skill” (p. 1146).

Moving closer to the actual and potential roles of programming as a tool for students to create mathematical techniques, recent studies report on degree programs in which programming has been deliberately and extensively integrated in several core mathematics modules. Cline et al. (2020) report on how this was done for more than a decade in a liberal arts college in the USA, again with MATLAB used both as a programming language and as a CAS. In single variable calculus, for instance, students create a technique for finding zeros of a function based on Newton-Raphson’s method, as well as techniques to compute Riemann integrals to solve certain modeling tasks (p. 741). Later, in complex analysis, they use programming to study complex difference equations and their visualization through fractals (p. 745). In final projects, students use programming in their study of more advanced tasks, such as solving the Navier-Stokes equations with the finite element method (p. 746). Faculty members of Brock University, Canada (e.g., Buteau and Muller 2017), have presented a similar, long-lasting, and apparently successful experience in a series of papers. In both contexts, it is interesting to note a prevalence of project-based assessment and that mathematics faculties have fully taken on the curriculum, noting (in the case of Cline et al. 2020) that “the introductory computer science courses at our institution do not meet our needs, as the existing courses do not sufficiently demonstrate their relevance to mathematics” (p. 739). Both notes align well with findings from the more general UK study, outlined above.

Currently, there thus seems to be three overall options for teaching students to create mathematical techniques by programming (rather than simply using packages of techniques, such as CAS): (1) not doing so at all, (2) doing so in a few more or less isolated add-on modules, and (3) doing so in many or all undergraduate courses. The latter option, echoing Forsythe's (1959) vision, does not simply enrich an otherwise classical "precision mathematics" curriculum but also changes its mathematical contents, with more emphasis on applied and discrete mathematics. The choice is therefore largely between different mathematical contents and profiles: it is didactic, not merely pedagogical. Indeed, the third choice could have considerable potential and presence in mathematics courses taught, for instance, to engineering and business students. It is not merely a choice to make, as such transformations would require considerable work to prepare new mathematical and didactic praxeologies: new types of tasks and problems, assessment methods, explanations and other theoretical discourses, etc. It would also demand a significant change in our vision of what mathematical activities consist of.

Summary and Outlook

As we hope to have shown in the preceding sections, technology use in university mathematics education represents a research object with several distinct parts, and it is also in several ways different from technology use in primary and secondary level mathematics. The present chapter offers two main contributions to research on this topic:

1. A theoretical framework for distinguishing and analyzing the different didactic uses of technology (outlined in Table 1), based on ATD
2. A critical review of existing research, pointing out not only achievements and insights but also blind spots both in terms of the research objects considered and the questions that are explicitly addressed

We note that these two contributions are not independent. One can interpret (1) as creating a "map" of the research object, but the ATD perspective also implies a deliberate focus on the (praxeological) levels at which the different types of technology use affect students' and teachers' mathematical activities and how this depends on institutional conditions and constraints (not only on individual or cognitive features). The map aspect suggests that some parts of the research object are intensively investigated, most notably the use of CAS in basic university mathematics teaching, while others – such as programming as a tool for learning and doing mathematics – are only emerging as phenomena to be studied. Even for the more extensively studied case of how ready-made tools are or could be used in teaching, the critical and comparative viewpoint seems to be underrepresented in research. Studies mainly focus on the effect of technology resources in students' practices and learning outcomes but rarely connect these practices and outcomes to the choices made about the type of mathematical content and activities that are chosen and organized in the course.

The focus on various positions within institutions (here, universities) provides a framework for the critical (rather than merely synthesizing) reading of the literature. In particular, we can observe that a large part of the research conducted so far takes – more or less implicitly – the viewpoint of one or a few teachers within *one* institution and often one particular course, in which experimental practices with technology have been observed and documented. Then, this context – which acts as a more or less extensively described background – is rarely questioned, and very often this includes the mathematical praxeologies to be taught (possibly with very local variations due, for instance, to the use of CAS). In other words, the external didactic transposition (e.g., Bosch et al. 2021) is rarely visible or questioned, and the institutional relativity of the results is at best implicit. A major value of the institutional focus is to emphasize this relativity and to question its necessities beyond the viewpoint of teachers who face (but may not even fully realize) the constraints of one particular institution.

One can also note a certain tendency, particularly but not exclusively in early studies, to adopt a proponent perspective, in the sense that local designs are described along with observations of their benefits for students (according to implicit assumptions about the kind of mathematics that should be learnt), while problematic features or alternatives may be less emphasized. This, in fact, could also be seen as an effect of research done from a teaching development perspective, however useful that perspective may be for identifying and exploring new potentials.

We have also considered a few large-scale studies involving several institutions and questioning, for instance, the connection between technology use in mathematical research and in undergraduate or graduate teaching and the perspectives and possibilities of individual university teachers. We note that these studies concern mostly the use of technology. In relation to university mathematics, the use of digital library media by teachers and students – and how it is or could be institutionally conditioned – appears to be particularly underresearched. When it comes to research on digital interactive media, the pandemic situation from 2020 has certainly led to new waves of studies also in relation to the university level, and it is probably too early to summarize what this will add to existing research. Less well-understood aspects, such as the role of students' collaboration in the context of mathematics courses, may indeed come to appear more prominently due to the nonvoluntary and extensive use of digital platforms during this period. The same could be said more globally about how tools and media enable or enforce *changes* (rather than mere enhancement) in the mathematical praxeologies to be taught and consequently those that are actually taught and learned.

To conclude, the use of digital tools and media in university mathematics education remains a small but growing field of research. Due to the continuous development of the mathematical sciences and the way they use and contribute to digital technologies, the field requires a multiplicity of expertise and potential for collaboration between scholars and university teachers with different backgrounds. At the same time, it is important that didactics research develops more global viewpoints than explorative experimentation with new tools and media, to take into account the changes that digital tools operate not only in the way mathematics is taught at university but also in the very nature of mathematical praxeologies.

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Drawing on a computer algorithm to advance future teachers' knowledge of real numbers: A case study of task design

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Drawing on a computer algorithm to advance future teachers' knowledge of real numbers: A case study of task design

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ABSTRACT

In our investigation of university students' knowledge about real numbers in relation to computer algebra systems (CAS) and how it could be developed in view of their future activity as teachers, we used a computer algorithm as a case to explore the relationship between CAS and the knowledge of real numbers as decimal representations. Our work was carried out in the context of a course for university students who aim to become mathematics teachers in high schools. The main data consists of students' written responses to an assignment of the course and interviews to clarify students' perspectives in relation to the responses. The analysis of students' work is based on the anthropological theory of the didactic (ATD). Our results indicate that simple CAS-routines have a potential to help university students (future teachers) to apply their university knowledge on certain problems related to the decimal representation of real number which are typically encountered but not well explained in high school.

Keywords: CAS, ATD, real numbers, future teachers

INTRODUCTION

Klein (2016, p. 1) pointed out that young university students find it hard to use the mathematical knowledge, which they learnt at school as they deal with university level mathematics problems. Then, after they graduate from the university and go back to upper secondary school as mathematics teachers, it is also hard for them to find connections between teaching there, and what they learnt at university. This constitutes Klein's (2016) classical "double discontinuity" problem.

Durand-Guerrier (2016) studied the first discontinuity in relation to the specific case of real numbers. The author identified several gaps between the ways real numbers are conceived and taught in high school and at university. Along the lines of Durand-Guerrier (2016), González et al. (2019) proposed two challenging situations for the teaching of real numbers at high school. In this paper we show a proposal for university teaching that aims to address the second discontinuity in the special case of student's relation to real numbers.

So, the question is what should future teachers know about the real numbers and especially, what should they learn about real numbers at university? This question is almost as old as mathematics education research itself. Already, Klein (2016, p. 34-35) suggested that decimal notation was historically decisive to lead mathematicians onto the general arithmetic of "irrational numbers". Nevertheless, research on real numbers teaching or learning we can find is not much. González-Martín et al. (2013) believe that one of the difficulties in teaching or learning real numbers lies in the relevant definitions. For example, Zazkis and Sirotic (2004) found the obstacle to learn irrational numbers for students is their understanding of the equivalence of definitions. There is a "missing link" between the fraction representation and decimal representation of a real number. This kind of "missing link" is not only the negligence in teaching but also the unclear definition of real numbers in high school textbooks. González-Martín et al. (2013) studied the introduction of irrational and real numbers in Brazilian textbooks for secondary education. They found that the definition of real numbers always appears after the definition of irrational numbers and is expressed as the union of the set of rational numbers and the set of irrational numbers, i.e., $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$, where $\mathbb{R}$, $\mathbb{I}$, and $\mathbb{Q}$ are the sets of real, irrational, and rational numbers respectively. However, the definition of the irrational number is based on the assumption of the existence of real numbers, i.e., $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$. This seems to make sense but in fact, an independent definition of real numbers did not appear at all. This problem is easily ignored in textbooks by teachers and students, and they accept the definition in textbooks as "a transparent rule which does not need justification" (González-Martín et al., 2013, p. 239). Enhancing teachers' knowledge of real numbers might assist them in going beyond circular and unsatisfactory definitions as found in the textbooks mentioned above. Our present work is on the teacher knowledge about decimal representations of real numbers and focuses on those university students who intend to become teachers at secondary school.

Another question arises: How could we support them to develop that knowledge? Maybe today's widespread use of digital technology (in mathematics and other school subjects) offers and requires modified answers to this question. We can find some studies related to the use of computer algebra systems (CAS) in mathematics education (e.g., Gyöngyösi et al., 2011; Lagrange, 2005). In this paper we take into consideration an aspect of secondary school mathematics which has so far been neglected in most of the literature on the teaching of real numbers there: the use of calculators and computers, including more advanced uses such as graphing and programming, which is common at this level in many countries. In Denmark, CAS like *Maple*, *TI Nspire*, and *Geogebra* are commonly used in upper secondary schools. In schools and in society at large, real numbers and functions are increasingly accessed and handled through CAS and other mathematics software. However, in the teaching of mathematics at university, such use of tools appears at most in introductory Calculus courses and not in later, more theoretical courses. Future teachers of mathematics often take such more theoretical courses and find that they are far from what is taught at secondary school. In this paper, we particularly investigate the use of simple programming as one strategy to bridge the gap.

The paper is structured as follows. In the next section, we will recall how the anthropological theory of the didactic (ATD) works as a theoretical framework to reformulate Klein's (2016) second discontinuity problem and based on this formulate our research questions. Then, the mathematical context will be introduced, including the background on infinite decimal model of real numbers, the elaboration of the given computer algorithms and addition of infinite decimals. After mathematical context, we will introduce the empirical context for the study, and the methodology used to analyze the research questions will also be shown in this section. Then, we will present the results based on data collected in a "capstone course" called UvMat. In the end we will draw up conclusions along with perspectives for further research.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

ATD, initiated by Chevallard (2019), was used by numerous authors (Barquero & Winsløw, 2022; Winsløw, 2013; Winsløw & Grønbæk, 2014) to model Klein's (2016) "double discontinuity". Following Chevallard (2019), we denote by $R_I(p, o)$ the relation between a position p within the institution I, and the knowledge object o. In ATD, a knowledge object is modelled as a *praxeology*. A praxeology contains a *praxis* block and a *logos* block. There are two parts in the praxis block, a type of tasks and techniques used to solve the tasks. The logos block is composed by the technology part, which is a discourse about the techniques, and the theory part which justifies the technology part and explains its relation to praxis block. These notions are more thoroughly introduced in Chevallard (2019) or Winsløw (2011).

The praxeologies we study in this paper are related to real numbers as these appear in Danish high school and at university. Using the theoretical model of Klein's (2016) second discontinuity proposed by Winsløw (2013) we can represent the passage we are interested in, as

$$R_U(\sigma, \omega_{\mathbb{R}}) \rightarrow R_S(t, o_{\mathbb{R}}), \quad (1)$$

where $\omega_{\mathbb{R}}$ is any mathematical praxeology about real numbers worked by students σ in the university U, while $o_{\mathbb{R}}$ is any mathematical praxeology about real numbers supposed to be taught in the institution secondary school S by teachers t (the $\rightarrow$ of the passages above can be considered as a gap between university and high school). Future teachers' knowledge could be important for narrowing this gap. In particular, we can sometimes select some elements related to logos blocks from $\omega_{\mathbb{R}}$ to justify the praxis blocks from $o_{\mathbb{R}}$. In other cases, we need to add some mathematical elements to bridge the gap, ending up with a slightly larger object which we denote by $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$. Therefore, for those future teachers in university (represented by σ_{ft}), we aim at a new relation $R_U\left(\sigma_{ft}, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}\right)$, where $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$ is really connected by links known to σ_{ft}.

The first question is what kind of content should be included in $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$. Barquero and Winsløw (2022) have considered the decimal representations of real numbers such as constructing a decimal representation for a given number. They particularly investigated students' work on the graphs of a function by using different representations of $\sqrt{3}$ on *Maple* (a CAS the students know from the first semester courses). In this paper, we continue to consider real numbers as decimal representations, particularly about addition of infinite decimals operated on *Maple.*

In university, mathematics students who intend to become future teachers will acquire a certain amount of knowledge of real numbers. How students select the suitable theoretical elements from $\omega_{\mathbb{R}}$ to explain the real number problems in secondary school is a sub-discontinuity under the Klein's (2016) second gap, which can be formalized as the following transition:

$$R_U(\sigma, \omega_{\mathbb{R}}) \rightarrow R_U\left(\sigma_{ft}, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}\right). \tag{2}$$

How to support this transition is our second question, which is also the main work in this paper. We formulate these two questions as the precise research questions:

RQ1: How could the idea of 'infinite decimal' be related to university mathematics and taught to future secondary school teachers?

RQ2: In particular, how can working with a given computer algorithm support university students' use of (university level) mathematical knowledge to address secondary school level questions related to the infinite decimal model of real numbers? What new mathematical and didactical knowledge on decimal representations of real numbers can such work enable students to develop?

Note that RQ1 is a theoretical research question, asking for specific links between a school mathematical notion ("infinite decimals") and appropriate undergraduate mathematics, thus it is *about* the knowledge to be taught to future teachers. RQ1 is in general answered by several classical and newer texts (e.g., Sultan & Artzt, 2018), and we summarize one answer at the beginning of the next section. We address RQ2 by investigating students' work on a concrete assignment, also presented in the next section. Our answers to RQ2 are derived from analyzing students' written reports in response to the assignment, as well as interviews with selected students.

MATHEMATICAL CONTEXT

Real Numbers Represented as Infinite Decimals

In university, students will meet the completeness property of real numbers in one of the first courses in analysis. They may also be given various explanations of what real numbers are, from the number line to Cauchy sequences or Dedekind cuts (Bergé, 2010), although these constructions are seldomly treated in detail (so, they do not study equivalence classes of Cauchy sequences, etc.). The elements actually covered, particular various consequences of completeness related to convergence and compactness, then form part of $\omega_{\mathbb{R}}$, but with little direct connection to the earlier praxeologies $o_{\mathbb{R}}$ learnt at school. The completeness property, however, was coined at the end of the 19th century in order to formalize the idea that any real number can be represented by an infinite decimal (Bergé, 2010), an idea already met in school. Completeness–or, more intuitively, the decimal representation–can be used to explain that the set of real numbers satisfies the Archimedean axiom, i.e., for any $x \in \mathbb{R}$, there is an $N \in \mathbb{Z}$ such that $x \leq N$. It follows from this that, for any $x \in \mathbb{R}$, there is a unique $N \in \mathbb{Z}$ such that $N \leq x < N + 1$.

This kind of knowledge is thus formally related to logos block in $\omega_{\mathbb{R}}$, but will not be explicit at the secondary school level, although it is entirely compatible with the "number line" metaphor used there. It may not even be taught to university students although it is crucial to the connected extension $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$ (a metaphorical notation we use to designate the coherent praxeology aimed at future teachers). For RQ1, our aim is to elaborate from $\omega_{\mathbb{R}}$ the meaning of an infinite decimal as a more or less unique representation of a real number. An infinite decimal is denoted as $\pm(N.c_1c_2c_3 \ldots)$, where $N \in \mathbb{N} \cup \{0\}$ and $c_i \in \{0, 1, 2, \ldots, 9\}$. We mainly consider nonnegative infinite decimals here, as a negative infinite decimal can be defined as the additive inverse of a positive one. We use a part of chapter 8 in the book *The mathematics that every secondary school math teacher needs to know* (Sultan & Artzt, 2018) to outline a possible answer to RQ1.

An infinite decimal $0.c_1c_2c_3 \ldots \in [0, 1]$, where $c_1, c_2, c_3 \ldots \in \{0, 1, 2, \ldots, 9\}$ can be rigorously defined by $0.c_1c_2c_3 \ldots = \sum_{i=1}^{\infty} c_i \cdot 10^{-i}$. It is sufficient to consider the interval $[0, 1]$ because we can get numbers in other intervals by adding some integer. The two main results that should be established are (a) $\sum_{i=1}^{\infty} c_i \cdot 10^{-i}$ always converges and (b) every $x \in (0, 1)$ can be written as a unique infinite decimal which does not terminate in $\bar{9}$. Result (a) means that any infinite decimal makes sense because the infinite series $\sum_{i=1}^{\infty} c_i \cdot 10^{-i}$ always has a finite sum, due to convergence properties known from $\omega_{\mathbb{R}}$ (geometric series). Result (b) begins with the converse: any real number in (0, 1) can be written as an infinite decimal. The last part of (b) amounts to prove, again from $\omega_{\mathbb{R}}$, that $0.c_1 \ldots c_k000 \ldots$ and $0.c_1 \ldots c_{k-1}(c_k - 1)999 \ldots$ represent the same number. The two results are divided into three theorems (Sultan & Artzt, 2018) and the corresponding proofs based on $\omega_{\mathbb{R}}$ can be found there too. After this, we can define irrational numbers as infinite decimals that do not represent a fraction of integers. It is also proved that irrational numbers correspond exactly to infinite, non-periodic decimals. Several examples and exercises related to secondary school mathematics are provided by Sultan and Artzt (2018), concerning, for instance, how to find a fraction representation of a periodic infinite decimal representation. These praxeologies can, to some extent, eliminate the "missing link" of the two representations mentioned before. With this we have outlined the central elements of $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$, which forms the basis of our answer to RQ1. We now consider how to elaborate on this answer by considering RQ2.

Computer Algorithms

Before we answer RQ2, we need to introduce two things: computer algorithms, and more knowledge on infinite decimals related to (but exceeding) what is presented by Sultan and Artzt (2018). These two things are concretely developed in an assignment (see **Appendix A**) designed for students and the answer to RQ2 is based on students' answers to the assignment. This assignment contains two parts. The first part is the understanding of a given routine. This routine is implemented in *Maple* and can be used to find the first 10 digits of $\sqrt{2}$, one by one.

It is not a new idea that programming could be a possibly central tool to introduce in undergraduate mathematics. As early as 1959, Forsythe (1959) proposed to integrate coding in most of all introductory university mathematics. Our aim with the assignment was not quite as ambitious. The aim of introducing the *Maple* based routine is to show how to compute the decimals of certain well-known irrational numbers by elementary computations (relying only on the four operations with finite decimals, and on evaluating inequalities of rational numbers) that could in principle be carried out manually. The routine merely allows us to speed up the calculation. Producing the decimal representation of $\sqrt{2}$ is a secondary school task, carried out there with calculators, but seeing how it could be done concretely is not a common experience in secondary school. Thus, the new technique contributes to the praxis block in $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$, while considering its justification and consequences contributes to its logos block.

No matter which way we use, computer or pen-and-paper, we usually cannot specify all of the decimal representation of an irrational number. Therefore, we usually use an approximate decimal to represent an irrational number and this approximate decimal is a rational number. Let $x = N.c_1c_2c_3 \ldots$ be an infinite decimal where $N \in \mathbb{N}$ and $c_i \in \{0, 1, 2, \ldots, 9\}$ for all $i \in \mathbb{N}$. We denote by $x(n)$ the n digits approximation of x, so $x(1) = N.c_1 = N + \frac{c_1}{10}$, $x(2) = N.c_1c_2 = N + \frac{c_1}{10} + \frac{c_2}{100}, \ldots, x(n) = N.c_1c_2c_3 \ldots c_n = N + \sum_{i=1}^{n} c_i \cdot 10^{-i}$. Obviously $\{x(1), x(2), \ldots, x(n), \ldots\}$ is a monotone bounded sequence, so the limit of this sequence exists and is in fact equal to x. This is connected to the content in the previous subsection and justifies the praxis block.

In this routine we use the polynomial $f(x) = x^2 - 2$ whose unique positive root is $\sqrt{2}$. Since f is increasing on $(0, \infty)$, and we have $f(1) = -1 < 0$ and $f(2) = 2 > 0$, it follows from the intermediate value theorem ($\omega_{\mathbb{R}}$) that $\sqrt{2}$ is located in $(1, 2)$. Similar basic reasoning shows that $\sqrt{2}$ is in $(1.4, 1.5)$. Therefore, the first decimal of $\sqrt{2}$ is 4. The routine uses two loops to repeat this process until it finds all first 10 decimal digits of $\sqrt{2}$. First of all, the initial value (which is called K here) should be set to 1, the integer part of $\sqrt{2}$ according to the above.

In the routine, i is used to number the decimals that we aim to find, and this is the "external" loop. The number of the decimal digits to be produced can be modified by the users (in the assignment, we set it so as to find the first 10 decimal digits of $\sqrt{2}$). Considering that our focus is on the decimal part, we simplified the routine. Now, let us turn to the "inner" loop, where $j \in \{0,1,2,\ldots,9\}$ and $j \cdot 10^{-i}$ represents a potential ith decimal contribution to the sum. When $\left(K + j \cdot 10^{-i}\right)^2 - 2 \leq 0$, the computer will save the last value $p = K + j \cdot 10^{-i}$ and continue to increment j until $\left(K + j \cdot 10^{-i}\right)^2 - 2 \geq 0$. After the *if*- condition is satisfied, the value of the sum K will be updated for the next i and the computer will print the value $x(i)$ which is equal to $p = K + j \cdot 10^{-i}$ (*Maple* by default gives a fraction form of p, so we have to use the *evalf*-command to transform p to a (finite) decimal form; no "rounding" is done here. The number of digits produced by the *evalf*-command includes the integer part, so we need to use $i+1$). In general, for each i, the "inner" loop will produce a new K and an $x(i)$ (In order to only show $x(i)$ in the final result, we use ":" to make K invisible).

The main part of this routine is the "inner" loop (the "external" loop is mainly used to determine the position of the decimal digits). In this loop, we can get that for each $x(i)$, one has $x(i)^2 - 2 \leq 0$ and $(x(i) + 10^{-i})^2 - 2 \geq 0$. Therefore, by results known from $\omega_{\mathbb{R}}$, $\lim_{i \to \infty} x(i) = \sqrt{2}$. The proof of the limit is a part of what students could produce as an answer to question b) in the assignment. We also hope students could associate with the intermediate value theorem for continuous functions as they explain the working of this routine. A complementary visualized explanation from students to this routine is also asked for (question c)); indeed, using *Maple* to "visualize" difficult points forms part of what could reasonably be expected from teachers' relationship to instrumented techniques.

Addition of Infinite Decimals

The real numbers are not simply to be computed digit by digit; we also need to consider operations with real numbers, which leads to difficulties in the case of decimal representations. To add two integers or finite decimals numbers, the algorithm learnt in primary school is to add digits from the last position on the right (possibly "carrying over" exceeding digits). For some infinite decimals like irrational numbers, this way does not work. Can we add with approximate decimals of numbers? To be concrete, two irrational numbers, 0.1234 … and 0.8765 …, the sum of their first four decimal digits is 0.9999. If the sum of their 5th decimal digits is more than nine, the first four decimal digits of the new sum will turn out not be 9's but 0's. Some students may have become aware of this problem while studying decimals, or their teachers could have mentioned this in secondary school. The investigation about how students use the computer algorithms above to address this question from the new university level construction is also a part of our answer to the first part of RQ2.

We need to introduce some notation from the first part of the assignment. Let $\mathbb{D}_n = \{10^{-n}y : y \in \mathbb{Z}\}$ for $n \in \mathbb{N}$ and $\mathbb{D} = \cup_{n \in \mathbb{N}} \mathbb{D}_n$. We denote by $\mathbb{D}_\infty$ the set of formal expressions $\pm(N.c_1c_2c_3 \ldots)$ where $N \in \mathbb{N} \cup \{0\}$ and $c_i \in \{0,1,2,\ldots,9\}$, and by $\mathbb{D}_0$ be the set of formal expressions $\pm(N.c_1c_2 \ldots c_j0000 \ldots)$ where $N \in \mathbb{N} \cup \{0\}$ and $c_1 \ldots c_j \in \{0,1,2,\ldots,9\}$. Clearly we can interpret numbers in $\mathbb{D}_\infty \setminus \mathbb{D}_0$ as real numbers based on the above (the representation being furthermore unique). Let $x, y \in \mathbb{D}_\infty \setminus \mathbb{D}_0$. Through calculation (by computer or pen-and-paper), we can only get n digits approximation of x and y, denoted by $x(n)$ and $y(n)$ where $x(n), y(n) \in \mathbb{D}_n$. For each n we can form $z(n) = x(n) + y(n)$. When n goes to infinity, one has that $\lim_{n \to \infty} x(n) + \lim_{n \to \infty} y(n) = \lim_{n \to \infty} z(n)$ determines some number w in $\mathbb{D}_\infty \setminus \mathbb{D}_0$. But we have not, thereby, reduced addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$ to the addition on $\mathbb{D}$; but we cannot know if "carry overs" will lead to a failure of the following equation:

$$x(n) + y(n) = w(n), \text{ for any } n \in \mathbb{N}. \qquad (3)$$

This quandary is not treated in any depth by Sultan and Artzt (2018). We hope this quandary would be discovered by students through working with the example in e) and f) of the assignment in **Appendix A**. This will then develop students' technical and theoretical knowledge related to the non-trivial addition of two infinite decimals.

In the question e) of the assignment in **Appendix A**, students were concretely asked to discuss the sum of $\sqrt{2}$ and $\sqrt{3}$. If we regard $\sqrt{2} + \sqrt{3}$ as one number, which is still an irrational number, the above routine can be adapted to produce the first 10 (or more) decimal digits of $\sqrt{2} + \sqrt{3}$. The main job for students is to find a polynomial with $\sqrt{2} + \sqrt{3}$ as root, and with integer coefficients (in order to remain with finite decimals in the algorithm). The simplest candidate, found by many students (following a technique known from exercises in the textbook) is $p(x) = x^4 - 10x^2 + 1$. Now, let $x = \sqrt{2}$, $y = \sqrt{3}$ and $w = \sqrt{2} + \sqrt{3}$. Students were expected to use the above routine to find $x(n)$, $y(n)$ and $w(n)$ for n from 1 to 10. When $n = 6$, we will get $x(6) + y(6) = 3.146263$ and $w(6) = 3.146264$, which means $x(6) + y(6) \neq w(6)$. This shows concretely that the equation (3) does not always hold, and that is the point which students are asked to make in question f).

It is clear that the use of computers to answer the last question greatly saves time and in practice enables getting to the point above. Students' didactical knowledge gained from this assignment will be analyzed later.

CONTEXT AND METHODOLOGY

We conducted an experiment to investigate the two research questions presented above in a course called UvMat, which is taught for future mathematics teachers in secondary school at the University of Copenhagen (Denmark). UvMat is not a mandatory course, and it involves 20-30 students each year, a professor who plans the course and gives the lectures, and a teaching assistant (TA) who is in charge of exercise classes and of grading assignments. In Denmark, high school teachers have to qualify in two subjects, one is called "major" and is studied for about three years, the other is the "minor" and is studied for about two years. Most participants in UvMat study mathematics as a minor, in addition to their major subject (such as physics) and will then be authorized to teach these subjects in high school. Due to the shortage of authorized teachers, some of them already have some experience with part-time high school mathematics teaching. The aim of this "capstone course" is to help future teachers to consider high school mathematics from an advanced standpoint, according to Klein's (2016) expression. This kind of "capstone course" thus aims to strengthen the connections between praxeologies met in high school and at university. Throughout the course, students attend lectures and exercise classes (in Danish), following the textbook (Sultan & Artzt, 2018), and they also do mandatory written assignments in groups (in Danish), every week.

The fourth week of UvMat is based on the second part of chapter 8 of the textbook, dealing with decimal representation of real numbers. We designed a group assignment (called WA4) with six questions for this week (the full text is included here as an appendix which the original version is in Danish). Our study is designed to investigate RQ2 by analyzing data collected from students' written answers to WA4 and the interviews with students. Our answer to RQ1 (see the previous section) outlines what students could learn from lecture and textbook and is part of the background for WA4.

The first question in RQ2 is analyzed from the logos blocks that were used by students to explain the given computer algorithm (question b) and c) in WA4 and their observations related to the addition of infinite decimals with the help of this computer algorithm [question f] in WA4). Eight groups' answers to WA4 from students were received. First, we reviewed their answers to question b) and c) which asked students to explain the given routine in two ways: mathematical proof, and visualization. Our analysis focused on two aspects: what university-level knowledge the students applied, and how students made use of this knowledge. Secondly, we reviewed students' answers to question f). The analysis of students' answers considered their extent to which they can apply the mathematical theory of the course to explain observations from using the routine on the given concrete case. In addition, we also interviewed five students from five different groups who had volunteered to participate. The purpose of the first part of the interviews is to further understand the written answers given by the students. For example, some groups post similar visualizations to question c), but without a clear description, so the rationales and intentions behind those visualizations could still differ. In the interview, they got another chance to explain what they want to express through the visualizations and how it is, for them, connected to the given routine. In addition, there were also some errors that appeared in students' written answers to question f), where we could not decide whether they are due to calculation mistakes or more conceptual problems, as many groups did not show how they get to their answers. So, in the beginning of the interview, students were asked to elaborate on their answers to question c) and f), which was used to complement our previous analysis for the first question of RQ2. The last part of the interviews relate to more general questions relating to the purpose of the assignment and its relation to the rest of the course.

UvMat does not aim to impart didactical knowledge, but we still expected that students could develop some didactically relevant knowledge from WA4. Question f) was designed not only to investigate the addition between two infinite decimals, but also to identify the theoretical perspectives that students could develop from exploring this question (the second question of RQ2). To investigate the perspectives and views of students emerging from their work with WA4, we asked relatively broad questions in the interviews such as: what did you learn from working with WA4? and what is the relation between WA4 and mathematics teaching in high school? Our analysis for the second question of RQ2 was based on their answers to such questions (and follow up questions) raised during the interview.

RESULTS

Students' Work With the Given Computer Algorithm

In question b), we asked students to solve the tasks: Explain what the routine does, why $x(n) \in \mathbb{D}_n$, and why $x(n) \to \sqrt{2}$. The main challenge of question b) is explaining the convergence of $x(n)$. Therefore, in students' answers, the university-level notions involved were "sequence" and "limit of sequences". However, students use these notions in similar and rather informal ways in their answers. We found that seven groups did not give a formal argument (as hoped for) but explained the convergence informally, like merely summarizing what they observed from the numbers produced by running the routine. Such empirical reasons are common in secondary school but are not acceptable in most undergraduate mathematics courses. A typical example of an explanation is this answer from one group:

> In this way, the routine finds, for each decimal added, the decimal number that is closest to $\sqrt{2}$ from below. We thus form a sequence of numbers $x(n)$, where n is the number of decimals, and $x(n)$ converges towards $\sqrt{2}$. That is $\lim_{n\to\infty} x(n) = \sqrt{2}$ (translated from Danish).

Only one group solved this task as we explained in the former section by using the definition of limit:

> We will now show that $\lim_{n\to\infty} x(n) = \sqrt{2}$. By construction, $x(n) \leq \sqrt{2}$ for all n, as in the code, we only add decimals as long as the number is smaller than $\sqrt{2}$. Furthermore, the decimal which is added in the nth decimal is the largest number for which the output remains under $\sqrt{2}$, so if you add 10^{-n}, you will get above $\sqrt{2}$. Therefore, $x(n) + 10^{-n} \geq \sqrt{2}$. If we put this together, we can get $x(n) \leq \sqrt{2} \leq x(n) + 10^{-n}$, which is the same as $0 \leq \sqrt{2} - x(n) \leq 10^{-n}$. That means $|\sqrt{2} - x(n)| \leq 10^{-n}$. So, for any $\varepsilon > 0$, we have $|\sqrt{2} - x(n)| < \varepsilon$, for all $n > log_{10}(\frac{1}{\varepsilon})$. So, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|\sqrt{2} - x(n)| < \varepsilon$ when $n > N$. This means exactly $\lim_{n\to\infty} x(n) = \sqrt{2}$ (translated from Danish).

In general, students do have a strong tendency to prefer informal explanations based on secondary school mathematical conceptions even though the concept of sequence only appears at university in Denmark–not in high school. This situation can be attributed to students not putting themselves in the right position, as university students about to deepen their knowledge of high school mathematics. When they were facing what they perceive as a secondary school task, without being given specific directions for what to do, most students act as high school students s rather than preservice students σ_{ft} at university. They did not spontaneously consider that it is necessary or useful to draw on university level methods. In addition, this also reflects the lack of coherence, perceived by students, between logos blocks (from $\omega_{\mathbb{R}}$) and praxis blocks (from $o_{\mathbb{R}}$). Therefore, this type of exercise helps with the construction of $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$ and thus the transition (2).

We also asked students to explain the routine in another way (question c): use *Maple* to produce a visual explanation of how the routine works. Similar with question b), the pertinent university level knowledge in question c) is about sequences and continuity, and we focus on whether students' answers draw on such knowledge,. All groups used the point plot to show first 10 elements of the sequence $\{x(n)\}_{n=1}^{\infty}$, which can be done easily after running the routine. Students used this way to illustrate the convergence $x(n) \to \sqrt{2}$, as visualization of their explanations in question b). To explain this convergence, some groups combined the point plot with a function plot for $x \mapsto x^2 - 2$. Another common way is to superpose the point plot with a plot of the line $y = \sqrt{2}$ where we can see the 10 points are closer and closer to this line. One group superposed the graph of the function $y = x^2 - 2$ and the ten points $\big(x(1), y(x(1))\big), \dots, \big(x(10), y(x(10))\big)$, showing how they get closer and closer to the zero ($\sqrt{2}$, 0) of y. Even though this group had plotted the function $y = x^2 - 2$ on the interval (0, 1.42), after the fourth point it is no longer possible to see the changes between points. This problem was also indicated in the group's answer. In fact, this problem also occurred on figures with the added line $y = \sqrt{2}$. One of the groups solved this problem with three local zooms of the figure.

The visualizations above emphasize the sequence produced by the routine, rather than how the routine actually works (the two loops) and why (intermediate value theorem). Two of the groups made up for this by also presenting another kind of point plot (e.g., **Figure 1**), to explain how the "inner" loop works. **Figure 1** shows how the "inner" loop determines $x(1)$, by stopping as it "overshoots" and returning the previous value. In fact, **Figure 1** only compared $(K + j * 10^{-i})$ with $\sqrt{2}$ when $i = 1$, which simplified from the *if-command* in the routine (which compares, in fact, $\left(K + j * 10^{-i}\right)^2 - 2$ with 0). The "inner" loop stops when it finds some j such that $\left(K + j * 10^{-i}\right)^2 - 2 > 0$ and that is why in **Figure 1**, the sequence stops at $j = 5$.

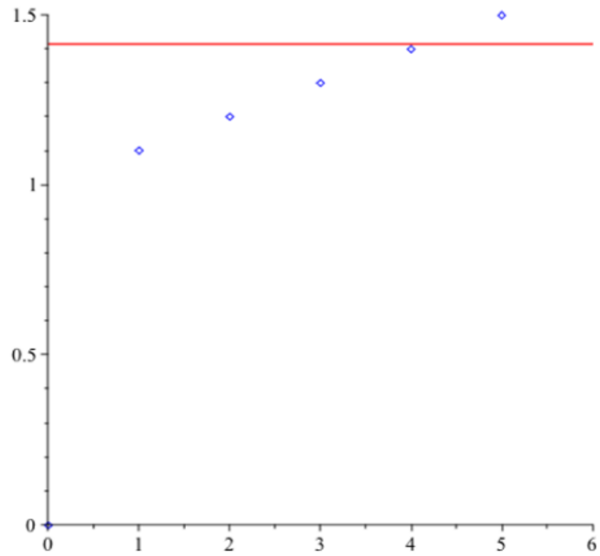

Figure 1. The point plot that shows how the "inner" loop of the routine determines $x(1)$ (Source: Author's own elaboration)

Although we think most visualizations fail to substantially address "how the routine works", students were quite confident about their work, when asked to explain it during interviews. They believed the figures match their explanation of the routine. One student described their figure in the interview as: *"... it made good sense..."* Many groups also, verbally, explained details in the coding of the routine, like why K starts from 1, but this was not visualized. From the explanations both in their answers and interviews, students thought the routine is used to create the sequence and thus, the convergence is, to them, the most important aspect of the routine. Therefore, the point plot which shows the subsequence of $\{x(n)\}_{n=1}^{\infty}$ appears sufficient for students. Still, we think that merely showing a visualization of the outcome or product of the routine does not really correspond to the normal understanding of "visualizing how the routine works" (the process that produces the outcome or product).

In addition to the students' capacity of reading the routine, question c) also requires students to use simple codes from *Maple* by themselves. Students' competence to use computer tools influenced their creation of the visualizations. In the interviews, all participants were satisfied with the content of their figures, but they still thought the figure itself could be improved. One interviewee said his group considered to use an animation to show the "inner" loop and the final subsequence, but they failed to realize that idea. Though all students had some experience with *Maple* before UvMat, the interviewees still recognized that they spent more time on tedious details of how to make a figure than on how to understand the routine. One student described the main difficulty in answering question c) as related to names of colors in *Maple*:

> "... I was not sure which colors *Maple* has. That was something I could not find easily, and I was not sure how to program the colors ..."

Questions d) and e) serve to extend students' work with the routine, and to prepare question f). The first half of question d) prompts students to activate their knowledge of the intermediate value theorem, which clearly belongs to university mathematics. This theorem, in fact, guarantees the success of the routine. The second part of question d) and e) asked students to use the routine to find approximate decimals of $\sqrt{3}$ and $\sqrt{2}+\sqrt{3}$ respectively. All groups were able to modify the routine to get correct answers. Question e) also examined students' knowledge of polynomial which typically presents at university. We will use the results obtained in question d) and e) by students and analyze their answers for question f) in the next subsection.

Students' Work on the Addition of Infinite Decimals

The last question of the assignment is: f) investigate what the results from b), d), and e) tell you about addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$. This is a very open question. It does not require new applications of the routine, but rather to reflect on previous results. As explained before, the point of this question is that addition and approximation to finite decimal do not commute: with $x=\sqrt{2}$, $y=\sqrt{3}$ and $w=\sqrt{2}+\sqrt{3}$, we may have $x(n)+y(n)\neq w(n)$. In fact, students computed $x(1)$ to $x(10)$ using the given routine in b), $y(1)$ to $y(10)$ in d) and $w(1)$ to $w(10)$ in e). Comparing, they could notice that $x(6)+y(6)\neq w(6)$.

To explain this phenomenon, the students can combine knowledge from $\overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}}$. On the one hand, we know from practice with calculators that something which ought to be 0 sometimes end up as something which is not zero. On the other hand, the theoretical model of real numbers as infinite decimals does not offer an easy way to describe the basic operations like addition: we cannot add "from the right" and adding from the left may lead to errors (as in the case above).

Six groups pointed out the "unusual" addition. Three of them showed the comparison of $x(n)+y(n)$ and $w(n)$ on *Maple* and found the special case when $n=6$ as we expected (e.g., **Figure 2**). However, the other three groups did not show any examples, but they all pointed out that two infinite decimals cannot be added decimal by decimal. We interviewed two students from these groups. One of them was not happy with this answer because they did not see what the question required from them. According to this student they compared the data obtained from the previous questions both by hand and *Maple*, and noticed $x(6)+y(6)\neq w(6)$, but then struggled to explain the phenomenon, considering if it might be related to the possibility of equivalent decimal representations which terminate in infinite numbers of 9's or 0's.

Fra b) har vi følgen $(x_n)\to\sqrt{2}$, fra d) har vi følgen $(y_n)\to\sqrt{3}$, og fra e) har vi følgen $(z_n)\to\sqrt{2}+\sqrt{3}$. Her er de fundende følger.

$x_1 = 1.4$	$y_1 = 1.7$	$z_1 = 3.1$
$x_2 = 1.41$	$y_2 = 1.73$	$z_2 = 3.14$
$x_3 = 1.414$	$y_3 = 1.732$	$z_3 = 3.146$
$x_4 = 1.4142$	$y_4 = 1.7320$	$z_4 = 3.1462$
$x_5 = 1.41421$	$y_5 = 1.73205$	$z_5 = 3.14626$
$x_6 = 1.414213$	$y_6 = 1.732050$	$z_6 = 3.146264$
$x_7 = 1.4142135$	$y_7 = 1.7320508$	$z_7 = 3.1462643$
$x_8 = 1.41421356$	$y_8 = 1.73205080$	$z_8 = 3.14626436$
$x_9 = 1.414213562$	$y_9 = 1.732050807$	$z_9 = 3.146264369$
$x_{10} = 1.4142135623$	$y_{10} = 1.7320508075$	$z_{10} = 3.1462643699$

Men bemærk at $x_6+y_6=3.146263\neq 3.146264=z_6$, og at vi har lignende uoverensstemmelser for senere n. Vi har altså at $(x_n+y_n)\neq(z_n)$

Figure 2. An example from one group that how to find the "unusual" addition by *Maple* (reprinted with permission of the students)

In fact, this concern is irrelevant because the question was restricted to the set $\mathbb{D}_\infty \setminus \mathbb{D}_0$, which avoids decimals with nothing but 0's in the end. The other student was very satisfied with the given answer and thought there was no need to put more effort into question f) because the conclusion they gave was obvious. In this group they did not compare the results from previous questions because they believed cases like $x(6)+y(6)\neq w(6)$ were not a surprise. As the student from this group said

> "… it makes sense, but we did not think we have to do it …"

In addition to the above-mentioned six groups, there were also two groups that produced entirely misleading answers. One of the groups presented their conclusion after comparing $x(n)+y(n)$ and $w(n)$ thus:

After looking at the different decimal numbers, it is the same whether you first take the different decimal expansions separately or you take it together …" (translated from Danish).

One possible reason for this answer is that students did not carefully compare the data. Another group did not pay attention to the comparison between $x(n) + y(n)$ and $w(n)$. They gave the following answer:

"… If we combine the three observations, these examples indicate that addition in $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is a closed operation, since the sum of two elements from $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is again an element in $\mathbb{D}_\infty \setminus \mathbb{D}_0$. This is consistent with that $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is isomorphic with $\mathbb{R}$ (question a)), which is closed during addition …" (translated from Danish).

We can see this group was trying to answer question f) at a theoretical level using ideas (about properties of operations) learnt at university, but there are two problems in their answer. Firstly, all results from the routine are finite decimals which means the addition between these decimals happened in $\mathbb{D}_n$. Students could not get any results about addition in $\mathbb{D}_\infty \setminus \mathbb{D}_0$ directly from such observations but they could consider the "cut off" map $\varphi: \mathbb{D}_\infty \setminus \mathbb{D}_0 \to \mathbb{D}_n$ given by $\varphi(x) = x(n)$. Then the question would turn into looking at whether $\varphi(x + y)$ equal to $\varphi(x) + \varphi(y)$. Obviously, from the same case ($n = 6$), the equation does not always hold, so that (in university language) φ is not a homomorphism. Secondly, the operation on $\mathbb{R}$ can be transferred to $\mathbb{D}_\infty \setminus \mathbb{D}_0$ because there exists a bijection between $\mathbb{D}_\infty \setminus \mathbb{D}_0$ and $\mathbb{R}$ (proved in question a)) so that, trivially $\mathbb{D}_\infty \setminus \mathbb{D}_0$ is isomorphic with $\mathbb{R}$ when endowed with this addition. But the results to be considered for question f) suggest that students should think about whether the addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$ could be defined directly using decimals. This, however, is far from straightforward, as the examples show. From the interview of one student of this group, we know that they did not really reflect on this difficulty. The way they thought about this question falls short of university level standards and also lacks links to praxis block of secondary school mathematics (as we will now detail).

Students' Mathematical and Didactical Knowledge From the Assignment

Finally, how could the knowledge have developed from students' work on this assignment support teaching–as integrated mathematical and didactical knowledge? One of the purposes of this assignment is to help students think about how computers and calculators handle infinite decimals. We all understand that infinite decimals cannot be completely displayed by a computer, so the computer has to somehow convert infinite decimals into finite decimals. In particular, irrational numbers are handled as a special kind of rational numbers. This transformation could cause computers to make apparent errors when they operate on irrational numbers (e.g., question f). Indeed, secondary school teachers in Denmark need to manage pupils' use of CAS in relation to real numbers. How can a teacher deal with infinite decimals on CAS when they teach real numbers? The routine given in the assignment allows students to take a look into a possible procedure for directly calculating the first 10 decimal digits of $\sqrt{2}$ (and, in fact, a wide range of zeros of other given functions). Although we do not consider the actual algorithms behind commands such as "sqrt" (square root) or "solve", the routine opens up the "black box" to some extent. This could help future teachers reflect on how computers may more generally handle real numbers. In addition, the discussion about addition on infinite decimals could also help teachers understand why this sometimes leads to strange-looking results.

However, not all students saw the didactical relevance of WA4 as we would like them to. When we asked students about the relation between this assignment and secondary school teaching during interviews, only one student's response directly involved the representation of infinite decimals on computers and suggested that secondary school teachers

"should be able to figure out when *Maple* is good to use and when it is not."

Whereas other interviewees gave a neutral answer like

"… no matter what you teach, it is always a good thing to know more than you actually need …"

or claimed that this assignment is beyond the secondary mathematics level,

"… the curriculum of high school students is very far from this …"

Students did not mention question c) about visualizing the algorithm when asked to identify the main points of the assignment. We consider this question may have been too open or technically demanding to really contribute to their relation of type $R_U(\sigma_{ft}, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}})$.

Students' responses in the interview did not reflect much awareness about the didactical implications of the assignment. It would require further research to evaluate if and how working with such interface tasks could nevertheless leave an impact on students' subsequent teaching practice.

DISCUSSION

How can a course like UvMat support the development of secondary school mathematics? More specifically, to what extent can achieved new relationships $R_U(\sigma, \overline{o_{\mathbb{R}} \cup \omega_{\mathbb{R}}})$ improve their future teaching? Our data does not shed light on this, but the relation between teachers' knowledge and school teaching is not a new topic. For example, Hill et al. (2005) found teachers' mathematical knowledge (close to the mathematics they teach) has a significant effect on their students' achievement at primary level in USA. From a more global point, the teacher education and development study in mathematics (TEDS-M) conducted surveys about teacher education within 17 countries, which included future teachers at secondary level (Krainer et al., 2015). Schmidt et al. (2011) reexamined the data from 2010 TEDS-M and focused on middle school teachers' course taking. Their results indicate that high performing teacher education programs include both general undergraduate mathematics courses and also courses which, like UvMat, focus more specifically on mathematics for teaching. But these studies all consider mathematical knowledge in very broad categories, and it is a completely open question how work with specific school mathematical themes like the real number system in practice contribute to teaching related to those themes.

CONCLUSION

Our brief analysis of the students' answers and interviews with them, suggests that problem situations involving simple CAS-routines could be a promising setting for applying university mathematics on high school level problems related to real numbers, but that this does not necessarily develop new, didactically relevant knowledge. Indeed, we found that when we do not clearly indicate the direction of problem solving, some students do not draw properly on university knowledge, but resort to informal or misguided explanations rooted in their high school experience. Further research is needed to explore how problems could be designed in order to reduce or eliminate this kind of regression.

To summarize our answer to RQ1, the main idea is to explore the definition of real numbers as 'infinite decimals' in terms of infinite, convergent sums of fractions. This turns out to be a challenging approach for students, in particular when confronted with special cases where these sums are computed step by step using a computer routine, and do not add up as expected. Indeed, our observations in relation to RQ2 suggest (as developed in more detail in previous sections) that students struggle to explain the subtle difficulties that arise with addition and, more generally, with the operations on real numbers in this approach. Nevertheless, this approach is highly relevant to understand the shortcomings of computers and calculators when it comes to numerical computations with real numbers.

Indeed, with the increasing use of digital tools in high school mathematics, it becomes problematic if teachers have no idea of the connections–and differences–between theoretical mathematics (in particular, real numbers and their operations) and the representations and operations which such tools offer. We should also note that the first answers by students, considered here, were not the end of their work with the assignment: incorrect or incomplete answers had to be reworked in order for the assignment to be accepted. Confronting students with inadequacies or insufficiencies in their initial answers, and prompting them to submit acceptable ones, certainly leads the students to realize how advanced mathematical viewpoints can be used to think about subtleties in what appears, initially, to be elementary and somewhat trivial. To make students realize the need to question and analyze outputs from digital tools–as well as to look into how they are or may be produced–is one important goal which assignments, such as the one considered, may contribute to achieve.

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APPENDIX A: WEEK ASSIGNMENT 4

For $n \in \mathbb{N}$ we define $\mathbb{D}_n = \{10^{-n}x : x \in \mathbb{Z}\}$, and we define $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$. Also define $\mathbb{D}_\infty$ to be the set of formal expressions $\pm N.c_1c_2...$ where $N \in \mathbb{N} \cup \{0\}$ and $c_k \in \{0, 1, ..., 9\}$ for all $k \in \mathbb{N}$, and finally let $\mathbb{D}_0$ be the set of formal expressions $\pm N.c_1c_2...c_k000...$ where $N \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ and $c_1, ..., c_k \in \{0, 1, ..., 9\}$.

a) Prove that there exists bijections $\varphi : \mathbb{D}_0 \to \mathbb{D}$ and $\psi : \mathbb{D}_\infty \setminus \mathbb{D}_0 \to \mathbb{R}$, but that no bijection exists between $\mathbb{R}$ and $\mathbb{D}$.

b) Consider the following routine in Maple (try it out!):

```
K := 1 :;
for i from 0 to 10 do
      for j from 0 to 9 do
           if (K + j * 10^(-i))^2-2 <= 0 then
                p := K + j*10^(-i);
           end if ;
end do;
K := p :;
print(x(i) = evalf(p, i + 1));
end do :;
```

Explain what the routine does, why $x(n) \in \mathbb{D}_n$, and why $x(n) \to \sqrt{2}$.

c) Use Maple to produce a visual explanation of how the routine from b) works.

d) Explain how a similar routine can be made for any continuous function f, to find a zero between $a \in \mathbb{Z}$ and $a + 1$, when $f(a)f(a + 1) < 0$. How does the intermediate value theorem come into play? How can you use this idea to approximate $\sqrt{3}$ by numbers from $\mathbb{D}$?

e) Find a polynomial p such that $p(\sqrt{2} + \sqrt{3}) = 0$, and use the idea from d) to approximate $\sqrt{2} + \sqrt{3}$ by numbers from $\mathbb{D}$.

f) Investigate what the results from b), d) and e) tell you about addition on $\mathbb{D}_\infty \setminus \mathbb{D}_0$.

(Source: Author)

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From global to local aspects of Klein's second discontinuity

Carl Winsløw, Rongrong Huo

FROM GLOBAL TO LOCAL ASPECTS OF KLEIN'S SECOND DISCONTINUITY

Carl Winsløw[*], Rongrong Huo[**]

Abstract – The global question of how to identify, develop and assess mathematical knowledge that is relevant to future secondary school teachers, has been central in the emergence of mathematics education research from early on. We review parts of this history from the viewpoint of the anthropological theory of the didactic, and in particular the notion of relationships to mathematical praxeologies that are held by certain positions within school and university institutions. We also consider a modern case, where the questions arise in a very practical sense: how to bridge the gap between standard undergraduate mathematics courses and a school relevant model of real numbers and functions? We show how both theoretical and practical aspects of this more local question arises in a so-called capstone course for students with about two years of undergraduate mathematics experience.

Key words: mathematics teacher knowledge; infinite decimal representations of real numbers; capstone courses; Klein's second discontinuity.

TITRE DE L'ARTICLE EN ESPAGNOL (STYLE RESUME TITRE)

Resumen – Texte espagnol du résumé (style resume texte)

Palabras-claves: en espagnol sans majuscule séparés par des virgules

LA DOUBLE DISCONTINUITÉ DE KLEIN : DE PERSPECTIVES GLOBALES A PERSPECTIVES LOCALES

Resumé – La question globale d'identifier, développer et évaluer les connaissances mathématiques qui sont pertinentes pour les futurs enseignants du secondaire, a été depuis les débuts un levier central dans l'émergence de recherches en didactique des mathématiques. Nous exposons des éléments historiques de cette question du point de vue de la théorie anthropologique du didactique, et en particulier la notion de

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rapport aux praxéologies mathématiques entretenu par certaines positions au sein des institutions scolaires et universitaires. Nous examinons aussi un cas moderne où ces questions apparaissent d'une manière plus pratique : comment combler le fossé entre une licence générale en mathématiques et des conceptions des nombres réels et des fonctions d'une variable réelle qui sera pertinente pour l'enseignement secondaire ? Nous montrons comment les aspects théoriques et pratiques de cette question plus locale apparaissent dans un cours de synthèse pour des futurs enseignants, qui ont passé deux ans de cours mathématiques universitaires.

Mots-Clés : connaissances mathématiques d'enseignant ; représentation décimale de nombres réels ; cours de synthèse pour enseignants ; seconde discontinuité de Klein.

INTRODUCTION

We consider that the following question indicates a central *raison d'être* of the Didactics of Mathematics (across all of its variations):

Q_0: *What knowledge must mathematics teachers have in order to deliver good teaching?*

The question is evidently broad and imprecise, most notably due to the undefined meaning of "good". It is also clear that more precision is needed to obtain a question that could have scientific answers. But both teachers, researchers and even broader groups would recognize some meaning in Q_0. They might also agree that the links between research in Didactics of Mathematics on the one hand, and mathematics teacher education on the other, are both strong and old, and come from the expectation that the former could produce knowledge that is useful to the latter and hence, at least in some sense, the kind of knowledge that Q_0 asks about.

Of course, teacher knowledge is in general a very complex object. Few would deny that it involves *professional* components that need to be acquired through practice. On the other hand, few societies assume today that teacher knowledge can be acquired exclusively through practice; in other words they establish some form of "initial" education. For the teaching of academic subjects like mathematics, this initial education almost invariably involves this subject matter in some form. And it is generally considered a truism that teachers should possess a solid knowledge of the subject they teach, *in casu* mathematics.

As we shall see later, it could be said that the Didactics of Mathematics was born from the realisation (or at least the conviction) that "mathematical knowledge" is part of the answer to Q_0, but that the following subquestions are non-trivial:

Q_1: *What mathematical knowledge is necessary (or just relevant) for mathematics teachers to deliver "good" teaching? How is it best acquired and certified?*

Q_2: *What other forms of knowledge (if any) are necessary? How are they best acquired and certified?*

Again, these questions clearly lack precision, but we can now formulate the overall aims and structure of this paper:

- first, provide a theoretical framework for the study (including more precise formulation) of Q_1, based on the Anthropological theory of the Didactic (ATD),

and then use this framework to:

- outline main trends of existing methods and answers for Q_1 that can be found in the international research literature,
- analyse more deeply the problem of task design for pre-service teacher education (as a partial way to answer Q_1)

illustrated by some cases of tasks developed at the University of Copenhagen, in relation to prospective secondary level mathematics teachers' knowledge about real numbers.

The last point constitutes the main part of the present paper, which is mainly theoretical (with the case illustrating and generating theoretical points). At the end we return to the meaning of this particular problem and case within the broader context of Q_1.

Note that in this paper, we consider only the (needs for developing) *mathematical* knowledge of *prospective* teachers. This is in no way to be construed as a denial of the relevance of other forms of knowledge or of the professional knowledge developed in and through teaching practice. We also recognise that it is not possible or productive to fully isolate or delimit "mathematical" components of mathematics teachers' knowledge within their theoretical and practical knowledge at large. Nevertheless, there are important and researchable problems related to Q_1 which are specific to the selection, delivery and assessment of mathematical knowledge within (initial) mathematics teacher education – and it is on some aspects of these that we focus here.

FRAMEWORK AND RESEARCH QUESTIONS

Any society that certifies individuals for teaching mathematics at a given level will furnish practical answers to Q_0, Q_1 and Q_2, at least (but not always limited to) the knowledge required at the entrance of the profession. We can consider these answers as collections of *relationships to* (knowledge) *objects* O_k to which the mathematics teacher t within a certain school institution S must hold a certain relationship $R_S(t, O_k)$ to, i.e. some collection of type

(1) $$\bigcup_{k \in K_S} R_S(t, O_k)$$

(cf. Chevallard, 1992), where K_S is a finite index set. This collection may in principle be empty, if no requirements are present; but even if no initial teacher education exists, other requirements (such as t having previously occupied the position p as pupil in some school institution S', where $S' = S$ is possible) with more or less specified relationships $R_{S'}(p, O_k)$ obtained to some objects O_k, could still be stipulated. Even in this case, a special institution I – which is typically, but not always, a teacher education institution of some sort - may be endowed with the power to decide whether or not an individual y has the relationships required to occupy the position t in S. In principle, the question is then whether the relationships of y to O_k are sufficiently near $R_S(t, O_k)$ for all $k \in K_S$. However, in practice, due to a great distance between I and S, it is common that I replaces $\{ R_S(t, O_k) | k \in K_S\}$ by $\{ R_I(y, \omega_k) | k \in K_I\}$, where $\{\omega_k | k \in K_I\}$ consists of objects used by I to certify officially that y satisfy the said requirements (and we use the letter ω, instead of O, to stress the change of institution). And then society will assume that if

(2) $$\bigcup_{k \in K_I} R_I(y, \omega_k)$$

is affirmed by I for some individual y, then that person can be admitted to position t within S, and we may assume, or merely declare, that (1) is then satisfied. Concretely, (2) is often determined by y passing a certain number of tests within I, each determining whether a certain number of relationships $R_I(y, \omega_k)$ are satisfactory. This is to some extent the case also for the tasks for future teachers, presented later in this paper, even if they are designed deliberately to relate to some O_k.

Anyone with any experience of current teacher education (or certification) systems will know that while (2) is more or less concretely specified by the regulations within I, the relation of (2) to (1), and also (1) itself, are often far from transparent. Moreover, (1) develops throughout the career of a teacher in position t, and this may well lead to initial inadequacies being remedied.

Nevertheless, we cannot assume or claim from the outset that (2) is completely arbitrary with respect to (1). In particular, when it comes to *mathematical* objects ω_k met by y within I, some are indeed likely to be closely related to mathematical objects O_k met by pupils p and teachers t within S. To identify and question such cases, at least locally, is the main idea of this paper when it comes to addressing Q_1 in practice, following up on our previous work (Winsløw and Grønbæk, 2014).

To do so, we need a less abstract way to describe the "objects" of type O_k and ω_k. In ATD, knowledge objects – in particular, elements of mathematical knowledge – are more recently modelled, within ATD, as *praxeologies*, consisting of praxis and logos blocks (Chevallard, 1999). We will also consider, from this point on, the frequent case (cf. below) where the institution I educating and certifying teachers is a university institution U.

Considering now the special case of mathematical praxeologies $\{\omega_k | k \in K_U\}$ for which U requires future teachers to hold relations $R_U(\sigma, \omega_k)$ in view of their pertinence to some school mathematical praxeology O_k, we may consider the passage (or rather, possible relations) of type

(3) $$R_U(\sigma, \omega_k) \rightarrow R_S(t, O_k)$$
where the arrow merely indicates the chronological order in which an individual may occupy the positions σ and t as student within U and teacher within S. The mathematical praxeologies, to which the individual relates in these positions, are in principle different. Even the mathematical praxis and logos required from the teacher in relation to some O_k, in which her pupils are to engage, may be quite different from the relation aimed at for the pupil, as when the teacher is supposed to pose and correct exercises for the pupils.

In other words, (3) can be used on very specific cases of the relation between (1) to (2), typically singled out because $R_S(t, O_k)$ is of some importance, and can be expected to be related to $R_U(\sigma, \omega_k)$, due to O_k and ω_k being somehow related mathematical praxeologies. That such impact and relatedness may be relatively absent – not only locally, but in a more general sense – is what Klein (1908) singled out as the *second discontinuity* afflicting modern organizations of mathematics teacher education (cf. Grønbæk and Winsløw, 2014): both the university student and the active teacher may perceive little or no impact or relatedness of the kind just defined.

On this theoretical background, we can now develop the initial question Q_1 into the following research questions which, although they are likely far from covering all aspects one could see in Q_1, are at least amenable to research, for fixed institutions S and U:

RQ1. Given a central praxeology O to be taught in S, how can some $R_U(\sigma, \omega)$ be used to build school relevant $R_U(\sigma, O)$?

RQ2. What needs exist to develop $R_U(\sigma, \omega)$ further (into what we shall later call $R_U^*(\sigma, \omega)$), in view of contributing to $R_U(\sigma, O)$?

These questions do not adopt the global viewpoint indicated by (1) and (2), as they focus on "central" instances of praxeologies. This implies a methodology of case studies (constrained by institutions and specific choices of the praxeological instances). However, some of the previous international research, which we review in the next (background) section, has in fact adopted the more global viewpoint. We discuss how these studies contribute to answer the questions above, or at least to motivate them.

We emphasize that this paper is essentially a theoretical paper, where cases are used to generate hypotheses and illustrate the more general research questions outlined above; by cases we mean instances of mathematical praxeologies, and to some extent concrete institutional contexts.

SYNTHESIS OF GLOBAL POSITIONS AND RESULTS

Klein's heritage

As exposed in some detail by Winsløw and Grønbæk (2014), Felix Klein was one of the first who problematized the passage from university studies of mathematics to teaching at what we would now call secondary level, beginning with his inaugural address as professor in Erlangen in 1872. The life, work and legacy of Klein – particularly within mathematics education – has been reviewed in larger depth within a recent book edited by Weigand, et al. (2019). Kilpatrick (2019, p. 215) notes, in his chapter within this book, that

> Klein's courses for teachers were part of his efforts to improve secondary mathematics by improving teacher preparation. Despite the many setbacks he encountered, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice.

We note here the strong link between teacher education and the birth of "mathematics education as a field of scholarship", also stressed in our introduction of Q_0 at the outset of the present paper. It was Klein's personal and institutional efforts to improve the preparation of secondary mathematics teachers that first led him to reflect, more broadly, on the needs and nature of mathematics education. His influence in this regard stretches far beyond his own environment and time, most famously through the foundation of the International Commission on Mathematical Instruction, for which he served as the first President from 1908 to 1920.

His ideas on mathematics teacher education have also exercised a more practical and longstanding influence, not least through the use of his lecture course for future teachers both in Germany and in other countries (for a recent translation into English, see Klein, 2016). The main idea of these lecture notes was interpreted, by Grønbæk and Winsløw (2014), in terms of (3): to develop prospective teachers' relationship $R_U(\sigma, \omega)$ with "higher mathematics" (Klein's term) through specific courses at the university, in view of becoming more useful for them as teachers, in other words, to enrich relationships of type $R_S(t, O)$. In Klein's work, Q_0 and Q_1 are not sharply distinguished, and while the possibility of ruptures between (1) and (2) is broadly recognized, Klein clearly saw it as a task for universities to bridge it through adaptations of (1), not so much to some inert version of (2) as to the service of a secondary school mathematics curriculum which would also have be updated in the light of recent developments of "higher mathematics". Klein clearly saw that such an endeavour would require a strong commitment of university mathematics teachers not only in teacher education, but also in contributing more directly to the development of secondary school mathematics; his own efforts in this direction were many-sided and influential as well, as documented by several chapters in (Weigand et al., 2019).

Now, a century later, we can notice both successes and apparent failures of this programme. "Higher mathematics", in the sense of courses whose content is roughly selected from what form the bases of current scholarship in pure mathematics – continues to be a main ingredient in secondary mathematics teacher education in many countries. Sweeping reforms of mathematics education curricula, both at university and in schools, were carried out in the 1960's and 1970's, under the label "New Math". The outcomes continue to be analysed and debated (see, for instance, Kline, 1973, for an early, and naturally controverted, contribution). While it is impossible to know what Klein's view on these later reforms would have been, it is certain that the fundamental distance between the mathematical sciences (not limited, by the way, to pure mathematics) and school mathematics has not ceased to grow. University mathematics curricula have remained surprisingly stable since the 1960's – notwithstanding later adaptations, most notably to include newer developments in statistics, computing and discrete mathematics (cf. Bosch et al., 2021). At the same time, reforms of school mathematics have been frequent, deep and strongly debated in many countries of the world, both before and after the period of New Math, and in many different directions.

Despite the necessary brevity of this outline, there is no doubt that the problem for mathematics teacher education which Klein identified, remains of strong actuality. It is, roughly, the non-trivial character of Q_1 in institutional set-ups where university mathematics is strongly involved, as it continues to be in most Western countries (OECD, 2014). Some of Klein's concrete proposals are also relevant to answering RQ1-RQ2, as we shall touch upon later for the special case of the mathematics surrounding the concept of real numbers.

Qualitative and quantitative research on Q_1 and Q_2

The more global questions introduced above have been the subject of both theoretical, qualitative and quantitative research, at least since the late 1960's. A famous early contribution was Begle's (1972) study of how teachers' knowledge of abstract algebra correlated with the knowledge on school algebra of their 9th grade students. Begle (1972, p. 14) concluded that:

> …teacher understanding of modern algebra (groups, rings, and fields) has no significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra. Teacher understanding of the algebra of the real number system has no significant correlation with student achievement in algebraic computation. However, teacher understanding of the algebra of real number system does have a significant positive correlation with student achievement in the understanding of ninth grade algebra. Nevertheless, while this correlation is statistically significant, it is so small as to be educationally insignificant.

These first results are still sometimes cited without the reservations and limitations that the author himself points out – such as the fact that the involved teachers were voluntary participants in a summer school on mathematics, and therefore not likely to be representative of 9th grade teachers at large. Nevertheless, these first results challenged the assumption that teachers' more extensive record of higher mathematics courses will automatically result in better teaching, reflected through the knowledge of their students in theoretically related fields of school mathematics. This largely confirms one of Klein's basic claims that the impact of academic courses cannot be taken for granted. Follow-up studies with somewhat less biased samples of teachers, such as Eisenberg's (1977), broadly confirmed this point, but also strengthened one of Begle's (1972) explicit hypotheses: that there might be "a lower bound of knowledge, below which the relationship between teacher knowledge and student performance does hold" (Eisenberg, 1977, p. 221).

This hypothesis, together with the possibility of other measures of "teacher knowledge of mathematics" correlating with student knowledge, was since examined further. An interesting study of the cited hypothesis – with a much more global scope than the case of abstract algebra and school algebra – was carried out by Monk (1994). He examined correlations between the number of academic mathematics courses taken by secondary level mathematics teachers, and their students' performance gains. Monk did in fact find a positive correlation with students having taken up to about 5 courses (a minimum largely exceeded by current undergraduate requirements in many countries). This suggests – with multiple caveats – that a minimal undergraduate mathematics background, formed by up to a year of full time academic mathematics study, does have a positive effect on the teachers' efficiency, but that anything beyond that may have little or no effect. Naturally, as with all quantitative studies of correlations, many other variables could possibly have significant explanatory value, and at least to some extent put the suggested "positive effect" into question.

The question of *how* to define, and possibly measure, relevant forms of teacher knowledge, is latent in Q_0, Q_1 and Q_2, and more explicit (and limited to mathematical praxeologies) in RQ1-RQ2. Quantitative studies will eventually make choices along these lines, as when items are formulated for use in a test (where a relationship $R_I(x, O)$ of some member of I to some O is assessed based on how x solves one or more tasks pertaining to O). The question then arises, especially for studies of more global categories of knowledge: what relation exists between the inventory of tasks proposed, and a qualitative or theoretical definition of the categories?

Indeed, inventories of items have recently been constructed and used in major international studies of how student and mathematics teacher knowledge correlate, along with categories of knowledge (relevant to Q_1 and Q_2) that are defined in careful, yet quite general terms. A major centre for research in this area has been the University of Michigan, where an elaborate theorization of *mathematical knowledge for teaching* (MKT) was used, at the dawn of this millennium, in a large scale investigation of primary school teachers' MKT and found strong correlation with their students' mathematical achievement, even when controlling for other plausible factors (Hill et al., 2005). Moving to international comparative studies, these ideas and methods were further refined and subsequently deployed in the "Teacher Education and Development Study in Mathematics" (TEDS-M) study, which involved 17 countries (Tatto, 2013). The results from this study are very rich and complex – including comparisons of teacher education programmes across and within countries – and cannot be subsumed in a few phrases. We shall however note two points, in the words of some of the main specialists:

> For secondary programs the most important influence on knowledge for teaching is the opportunity to learn university level mathematics (…) and the opportunity to read research in teaching and learning. (…) Teacher education programs' quality of opportunities to learn – as measured by their association with high levels of mathematics teaching knowledge, coherence on program philosophy and approaches, and internal and external quality assurance and accountability mechanisms, are all features that seem to contribute to increased levels of mathematics knowledge for teaching among future teachers (Kraineret al., 2015, p. 118)

Closer studies of the most successful mathematics teacher education programmes for (lower) secondary school, carried out by Schmidt et al. (2013, p.5), further identified course elements which these seem to share to a high degree; these include six standard undergraduate mathematics units (beginning calculus, calculus, multivariate calculus, differential equations, linear algebra, probability) along with three units on school mathematics education (math instruction, observing math teaching, functions). These programmes naturally all contain more elements; but this "core" is important to note. It is hard not to notice the consistence with Monk's early results and also with Klein's contention that school oriented complements to university mathematics are needed. The emphasis, in the previous citation, on "coherence" and "quality assurance", still leaves much room to fill in, in relation to (3) and the more specific questions RQ1-RQ2: how, in fact, can well-acquired elements of "basic undergraduate mathematics" be developed and tuned towards the needs of the future teacher in a coherent way? After a brief discussion of the experimental context, we shall turn to this question while, as already mentioned, focusing on some central mathematical objects.

A SPECIFIC INSTITUTIONAL CONTEXT

A considerable part of TEDS-M was focused on mapping out teacher education system at a global level, briefly explained above. We shall now delve further into local aspects related to RQ1-RQ2.

We consider these in the context of the largest mathematics programme in Denmark which offers teacher qualification for upper secondary school, offered at the University of Copenhagen. In Denmark, only upper secondary teachers receive their initial education in universities. After graduating from university, teachers have to pass a practical and theoretical course on pedagogy, while teaching; the subject specific parts of this course are quite limited, and as the various university programmes are quite different, the course has few if any concrete links to these.

From the list of courses listed by Schmidt et al. (2013), all of the general mathematics courses (and much more) are required for future teachers studying at the University of Copenhagen. Meanwhile, only two units specifically directed towards teachers are currently offered: a general course on didactics of mathematics labelled DidG (with some parts being shared with other science disciplines, due to the teachers having to specialise in two disciplines), and a course labelled UvMat (Mathematics in a teaching context). The first course corresponds roughly to the "math instruction" unit mentioned by Schmidt et al., while UvMat covers a relatively wide range of elementary school mathematics subjects (besides functions and equations, also number systems, discrete mathematics and statistics), all aiming at providing students with a deeper knowledge of these subjects in view of preparing them as future teachers with respects to how these domains appear in Danish upper secondary school.

Both DidG and UvMat deliberately draw on elements of the undergraduate courses, and thus aim at providing elements of the "higher standpoint" called for by Klein, as well as being capstone courses in the sense further described by Winsløw and Grønbæk (2014). The two courses are still quite different in the sense that DidG is focused on cases and methods of teaching, while UvMat is focused on mathematical content. Both courses involve (as other university courses) both lectures and extensive work with assignments or "exercises".

As in the study (Winsløw & Grønbæk, 2014) of "challenges" met by such a capstone course, we shall focus here on how UvMat attempts to tackle concrete instances of (3). In that paper, it was pointed out that UvMat does not attempt to address $R_S(t, O)$ directly, while students are still in position σ within U; this may also to some extent represent a difference with DidG. In our recent paper (Winsløw & Huo, 2023), we described a main strategy of the course as supporting students in a transition represented as

$$R_U(\sigma, \omega) \overset{T}{\rightarrow} R_U(\sigma, O)$$

through the design of tasks T that somehow link a university mathematical praxeology ω with a school mathematical praxeology O. As some of the university level praxeologies are also developed further within the course (rather than simply drawn from standard courses) a full representation of the course objectives is

$$(4) \qquad R_U(\sigma, \omega) \rightarrow R_U^*(\sigma, \omega) \overset{T}{\rightarrow} R_U(\sigma, O)$$

and with this extension, the course can be said to offer many concrete proposals related to RQ1-RQ2. In particular, the task design is used not only in the development but also in the assessment of $R_U^*(\sigma, \omega)$ and $R_U(\sigma, O)$, or combinations of these. We note that T itself does usually not belong to the types of tasks found in ω or O, but is designed to link these, while drawing on $R_U^*(\sigma, \omega)$ and enriching both this and $R_U(\sigma, O)$.

LOCAL MATHEMATICAL CONTEXT: REAL NUMBERS

In our recent research in the context of UvMat (within the frame of the second authors' thesis) we have focused on the students' knowledge about the system of *real numbers*. This system can roughly be described as a set, $\mathbb{R}$, equipped with arithmetical operations, an order structure, and a related topology. All of these are crucial to central domains of upper secondary mathematics, including calculus, analytic geometry, and vector algebra (over $\mathbb{R}$), among others. The real number system is of course linked to and based on subsystems, especially the systems of integers and of rational numbers. Nevertheless, there are considerable and general differences between how these number systems appear in university and school institutions. In this section we present these along with overall UvMat choices related to RQ2 (chiefly, at the level of theory).

Real numbers in undergraduate mathematics

Real numbers are especially fundamental to calculus and analysis, where university students will meet more or less deep treatment of some of their properties related to limits and more generally, order structure. Even at university, such properties may be simply claimed or presented as evident, especially in calculus courses. In more theoretical courses, students are presented with the notion of *Cauchy sequence*, and the fundamental property that Cauchy sequences of real numbers converge. Results related to limits and continuity of functions get a more rigorous foundation in axioms or claims about the real number system, and the related topology of the real number set. Also $\mathbb{R}$ itself may get to be defined in some way, most commonly as the completion of $\mathbb{Q}$. The latter number system – of rational numbers – is usually taken for granted in analysis courses, while a more formal treatment appears in abstract algebra (as the field of fraction determined by the integral domain $\mathbb{Z}$). However, as algebra and analysis courses operate independently in the undergraduate curriculum, these constructions will appear rather disconnected to students. Also, introductory analysis texts typically pass over the construction of $\mathbb{R}$, and present only some of the fundamental properties (like the supremum property) as an early stepping stone towards more technical results, such as the extremal value theorem for continuous functions on compact sets. Students are then exposed to a rapid succession of theorems and proofs, confirming and adding to what they learned in calculus at secondary or university level. Special functions (like exponential or trigonometric functions) still appear in examples, but they are (like the real numbers themselves) not treated any further. The end result, which we can write roughly as $R_U(\sigma, \mathbb{R})$, is then what students retain from these various expositions to the properties of real numbers, mostly within calculus and analysis courses. Naturally, many other mathematical objects than numbers strictly speaking – such as functions, operations on functions and various results on these – contribute to students' theoretical and practical conception of the numbers. But about these in isolation, students may actually know little more than what they learned in school.

Real numbers at primary and secondary school

The real numbers appear little by little, and in a much more fragmented and intuitive way, in primary and secondary school mathematics (see e.g. González-Martín et al., 2013), both within arithmetic of natural numbers, integers, and rational numbers, and also in geometry, school algebra, and (based on these) early calculus.

The idea that each point on the "number line" correspond to a number, also appears early on, with the representations of integers and fractions helping to view these as related and subject to a common order. Since digital technologies play a great role in calculation with numbers, both in school and society, *decimal* representations of numbers are likely to occupy a strong place in the pupils' relationship to the mathematical object $\mathbb{R}$, that is $R_S(p, \mathbb{R})$. Arithmetic operations are supported by handheld calculators from primary school on. Both these and the order structure are more straightforward with finite decimals than with fractions. Finite decimals also seem to exhaust the points that can be identified on a number line equipped with a scale or ruler.

The fact that fractions are needed both to define finite decimals, and that not all fractions correspond to finite decimals, is not really treated. Of course, periodic or otherwise strange "decimals" may be contended to be really somehow "infinite". If finite decimals are not carefully defined in terms of fractions, this new variety of decimals may also pass silently into $R_S(p, \mathbb{R})$ as a fact of life which does not require further explanation or questioning. Indeed, students will encounter "numbers" like roots of integers or the mysterious fellow π that are chiefly "real" to them as a consequence of being easily manageable on a calculator (where they work, indeed, as and with decimals).

At more advanced points in upper secondary school, the work with functions and equations is also heavily supported by graphical representations and (at least to produce these) by digital tools. This will then add more geometrical or visual elements to $R_S(p,\mathbb{R})$, mostly in a non-conflictual way: the intersections of curves can be both seen and calculated in consistent ways. Since the work is, at this point, also often heavily supported by algebra, many pupils struggle even when calculating tools are proposed as means to overcome some of the more technical points of the tasks they are assigned; but these tasks are frequently constructed so as to limit these struggles through the use of standard techniques. Pupils are rarely or never exposed to tasks that challenge their intuitive notion $R_S(p,\mathbb{R})$ of the real numbers as points and decimals.

Real numbers in UvMat

We now present how UvMat addresses RQ2 at the level of mathematical theory on $\mathbb{R}$, in view of the discontinuities outlined in the preceding subsections. The main idea is to formalize the idea of real numbers as infinite decimals *on the basis of* theory from the undergraduate analysis courses.

During the fourth week lectures of UvMat, the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$ is rapidly reviewed and institutionalized, including the existence of suprema for non-empty subsets of $\mathbb{R}$ with an upper or lower bound (presented as an "axiom" in a prerequisite analysis course). The starting point is thus the existence of a complete ordered field $\mathbb{R}$ containing the integers. From the supremum axiom, we derive the Archimedean property: for every real number x there is a unique integer m, such that $m \leq x < m + 1$.

From this point, the lectures follow Sultan and Artzt (2018, pp. 335-353) to show the existence of decimal representations of real numbers x, through an inductive construction of a sequence d_k of finite decimals such that $0 \leq |x - d_k| < 10^{-k}$ for all k. By the definition of limit, which is well known to the students, this means $x = \lim_{k\to\infty} d_k$. It is also shown that sequences of the form $d_k = \sum_{j=1}^{k} \frac{c_j}{10^j}$, where $c_j \in \{0, \dots, 9\}$, always converge, and that if two such sequences have the same limit – say $\sum_{j=1}^{\infty} \frac{b_j}{10^j} = \sum_{j=1}^{\infty} \frac{c_j}{10^j}$ – then either $b_j = c_j$ for all j, or one of the sequences of decimals becomes eventually 0 (say, $c_j = 0$ for $j > N_0$) while the other becomes eventually 9 ($b_j = 9$ for $j > N_1$), and moreover if N is the least natural number that realizes both properties, we have $c_j =$

b_j for $1 \leq j < N$ and $c_N = b_N + 1$. This, with some minor details added, proves that real numbers *are* in fact "infinite decimals" $\sum_{j=1}^{\infty} \frac{c_j}{10^j}$ in the sense that all real numbers do have an infinite decimal representation, that every infinite decimal representation corresponds to a real number, and that this representation is unique *except* if it terminates with 0's or 9's (in which case there are exactly two such representations).

Naturally, other properties, such as the rational numbers being exactly all real numbers with an eventually periodic decimal representation, are also added (for some students recalled) to enrich this formalization. The lectures also address, briefly, whether $\mathbb{R}$ could be simply *defined* as the set of formal infinite decimals, and point out some difficulties related to arithmetic operations.

Many students have certainly become aware – often in school – of facts like that finite decimals such as $1.02 = 1.02\overline{0}$ also have an alternative infinite decimal representation (here, $1.01\overline{9}$). But it is clearly new to them that they can be derived from university material on $\mathbb{R}$. We thus have a theoretical extension $R_U^*(\sigma, \mathbb{R})$ of $R_U(\sigma, \mathbb{R})$, which formalizes crucial elements of $R_S(p, \mathbb{R})$, with a potential of strengthening a future $R_S(t, \mathbb{R})$ – at least in the sense of denaturalizing, for the teacher, the intuitive idea of infinite decimals, as a way to think of general real numbers. Also crucial *practices* of the teacher – such as relating to the way computers handle real numbers – could be prepared by it, as we shall argue in the next section, when considering some elements of the tasks students engage in to build $R_U(\sigma, O)$.

A CASE OF TASK DESIGN IN THE LOCAL CONTEXT

University mathematics courses (such as UvMat) present students with some praxeological elements of a more theoretical nature during lectures, while devolving assignments and other tasks to students in view of strengthen their relationship with the praxeology at large. Especially in more theoretical courses, one may seek to engage students in tasks which build or extend theory (e.g. Grønbæk & Winsløw, 2007) and UvMat does so in at least through mandatory weekly assignments which develop some theoretical point, often starting from examples. They can be considered concrete proposals for the aims explicit in RQ1: build new, school relevant relationships of type $R_U(\sigma, O)$ while drawing on $R_U(\sigma, \omega)$ or a possible extension $R_U^*(\sigma, \omega)$. As RQ1 suggests, the design work departs from some school relevant

$R_U(\sigma, O)$, and seeks a relevant $R_U(\sigma, \omega)$ or $R_U^*(\sigma, \omega)$ that could be used to build $R_U(\sigma, O)$.

Among the crucial new objects introduced in upper secondary school are exponential, logarithmic and trigonometric functions, whose importance in mathematics and other disciplines need no defense. We wish to strengthen $R_U(\sigma, O)$ related to these, while drawing on the extension $R_U^*(\sigma, \mathbb{R})$ outlined above. In particular we consider that knowing an algorithm which computes a function "from the decimals of the input to the decimals of the output" could reinforce the students' relationship to the (school) model of the real numbers and relate it to a non-trivial function. We shall now consider a proposal for how to do so in the case of logarithms.

An algorithmic approach to logarithms

The assignment is based on an algorithm proposed by Goldberg (2006) for the computation of logarithms "digit by digit". The algorithm is most easily introduced by way of an example.

Consider $x = 432.1$; the idea to compute $\log_{10} x$ is to determine the decimals of a real number $y = N.d_1 d_2 \cdots$ satisfying $10^y = x$. We should thus have

$$(*) \qquad 432.1 = 10^{N+\frac{d_1}{10}+\cdots} = 10^N \cdot 10^{\frac{d_1}{10}+\cdots} = 10^N \cdot m_1$$

where $m_1 = 10^{0.d_1 d_2 \cdots}$. As $1 \le m_1 < 10$, the right hand side has $N+1$ digits before the comma, so from the left hand side we get $N = 2$. Dividing (*) by 10^N we get

$$4.321 = 10^{0.d_1 d_2 \cdots}$$

and as we now wish to proceed to determine d_1 we rewrite this as

$$4.321^{10} = (10^{0.d_1 d_2 \cdots})^{10} = 10^{d_1.d_2 d_3 \cdots} = 10^{d_1} \cdot m_2.$$

Again $1 \le m_2 = 10^{0.d_2 d_3 \cdots} < 10$ so d_1 is one less than the number of digits in the integer part of $4.321^{10} \approx 2269042.7$, that is, $d_1 = 6$. Note here that the computation of $4{,}321^{10}$ requires nothing more than basic multiplication and results in a finite decimal. We continue with

$$(2.269042671)^{10} = 10^{d_2} \cdot m_3$$

to find d_2, and so on.

With this procedure, all it takes to compute $\log_{10} x$ is to be able to count the number of digits in the integer part of a given number, and multiply the number by itself (10 times). It can thus – for a given *finite* decimal – be done with only the four basic operations. In particular exponential functions are not required to carry out the algorithm. Of course such knowledge is required to *verify* that it computes an inverse of $x \mapsto 10^x$ – or as above, to develop the algorithm for this purpose.

In the assignment developed for the UvMat students we chose to focus on the verification issue, and on the possibility of computer implementation. Details are given in the next section. The motivation for these choices was that the detailed construction and properties of exponential functions were already treated in the lectures, with only brief remarks about logarithms as the inverses of these (cf. Winsløw, 2013). This builds only on $R_U(\sigma, \mathbb{R})$ from university courses, and not on $R_U^*(\sigma, \mathbb{R})$ more directly connected to school models of $\mathbb{R}$. However, as students always compute and graph transcendent functions with digital devices, the algorithmic or decimal approach, along with computer experiments, explains what that "black box" may contain. At a deeper level, it formalises the intuitive idea of $\log_{10} x$ as "the number of times 10 divides x" (Weber, 2016).

Goldberg (2006) developed the same ideas with other bases (both for the logarithms and the representation of real numbers); indeed computation is simpler in base 2. In the course we did not include expansions of real numbers in bases other than 10, as this further extension of $R_U^*(\sigma, \mathbb{R})$ has much less relevance to $R_S(t, \mathbb{R})$ than the formalisation of decimal expansion.

An example of a student assignment

The assignment begins with a preamble, explaining its purpose as "developing a method to compute $\log_{10} M$ for a given $M > 0$ (…) in the sense that we compute the decimals of $\log_{10} M$ successively, using only basic arithmetic operations." The assignment had six tasks:

a) Prove from properties of $y \mapsto 10^y$ that for any $x > 0$ there is a unique $y \in \mathbb{R}$ so that $10^y = x$. If we have a method to compute such y for any $x > 1$, how can we do it for $0 < x < 1$?

b) Assuming $x > 1$, show existence and uniqueness of $c(x) \in \mathbb{N} \cup \{0\}$ such that $10^{c(x)} \leq x < 10^{c(x)+1}$.

c) Explain how to determine $c(x)$ from the decimal representation of x. Give a couple of examples.

d) Given $x > 1$ and letting y be as in a), we wish to find the decimal representation $y_0 + \sum_k y_k 10^{-k}$ of y. Show that this can be done by : $y_k = c(x_k)$ when we define, recursively: $x_0 = x$ and $x_k = \left(\frac{x_{k-1}}{10^{c(x_{k-1})}}\right)^{10}$ for $k \in \mathbb{N}$.

e) Use d) to compute $\log_{10} 57.64$ with four decimals.

f) Interpret a given *Maple* routine as implementing d).

The aim of this assignment is primarily that students work on a non-trivial case of an algorithm that computes the values of a function "digit by digit", and could therefore be thought of as the mathematical basis of a "calculator button" (or command) to compute that function. In other words, the primary point is more on the decimal representation of real numbers, and less on the concrete example ($\log_{10}$).

In task a), students need to use the property that $y \mapsto 10^y$ is a bijection from $\mathbb{R}$ to $\mathbb{R}^+$. The rest of the assignment is about the algorithm to actually compute (the decimals of) y for a given x. Tasks b) and c) were designed to define the auxiliary function $c: [1, \infty[\to \mathbb{N} \cup \{0\}$ which is central to the algorithm. In d) students must then explain how the given algorithm allows us to find the decimals of y (from x). The task e) allows students to try out the algorithm on a concrete number (like we did above) and task f) provides students with a piece of code which they should recognise as implementing the algorithm from d), and try out in Maple.

Now, we consider briefly how students draw on $R_U(\sigma, \mathbb{R})$ to answer the assignment, based on the students' answer sheets and interviews conducted with them in view of gaining further insight into what they learned from working with the assignment.

In task b), there was a considerable diversity among student answers. More than half of students considered that what is to be proved is equivalent to existence and uniqueness of $c(x)$ such that $c(x) \leq y < c(x) + 1$, where $x = 10^y$, but without explicitly referring to the result from a). Moreover the existence of the "integer part" is treated as obvious by students, while in the course it was proved to be a consequence of the Archimedean property of $\mathbb{R}$. So at this point students continue to treat properties of $\mathbb{R}$ with the same informality as is usual in high school.

Another way some students take is to divide $[1, \infty[$ into segments $[10^k, 10^{k+1}[$, observing that $[10^k, 10^{k+1}[$ and $[10^{k+1}, 10^{k+2}[$ are disjoint for all $k \in \mathbb{N} \cup \{0\}$. Therefore, as students explained: "It is then true that the union of all these disjoint sets corresponds to $[1, \infty[$, and it is then true that a number will always lie in just one of the sets". Some students used a proof by contradiction to show that a given x could only be in one interval of type $[10^k, 10^{k+1}[$. These arguments are more similar to what students will have met at university, relying explicitly or at least implicitly on the equality $[1, \infty[= \bigcup_{k=0}^{\infty} [10^k, 10^{k+1}[$.

In task c), students learnt how to determine $c(x)$ by only looking at the decimal representation. It was not difficult for students to find that $c(x)$ is the number of digits in the integer part of x minus one. There were, curiously, still some groups who did not give any examples, as asked for by the task. This task is used to help students to solve the core part – task d).

In task d), students were asked to explain the algorithm, where one really needs to use explicitly that a decimal representation of a real number is a kind of sum, and also the property $10^{a+b} = 10^a 10^b$. However, some students mixed formal and informal representations. For example, one group used $y = \lfloor y \rfloor + 0.c_1c_2c_3$ ...($\lfloor y \rfloor$ is the integer part of y) as the representation of infinite decimals when they were solving the task although y is represented as $y_0 + \sum_k y_k 10^{-k}$ in the task. The informal representation is of course closer to high school practice. To explain infinite decimals as infinite series is on the other hand a main point in this part of the course. Although those informal expressions did not affect the essence of students' final proof, we still observe that some students are somewhat limited by high school conceptions when faced with common high school notions. Their reluctance to use formal reasoning in relation to such notions is a very general experience in the course.

Task e) asked students to try out the algorithm with a concrete example and all students succeeded by following the steps in task d). Task f) tested whether students could relate the code to their own explanation in d). These two tasks are follow-up questions to the task d) which are hoped to increase the students' grasp of the point of the assignment.

We interviewed 8 students after they got the revision comments, mainly to learn what they saw as the point of this assignment. All students agreed that the assignment showed them another way to calculate logarithms where they got new insight into logarithms, beyond or behind its status as a "button" on calculators. One students described "*...I knew that the logarithm was the inverse to the exponential but I never quite figured out how to calculate them. But now we learned a little bit about that with this approximation and then of course something about how maple works...*" Some students also felt it was very surprising that they actually could calculate logarithms by hand: "*I think I learned how to easily calculate logarithms by hand without using Maple."* Most students did not focus on the relation between the infinite decimal representation of real numbers and this assignment, even though this was the main focus in that course week. Only one interviewee talked about the computation of decimals "digit by digit": *I think it* *is to develop a method to actually calculate the logarithms sequentially one decimal at the time.*

The mixed student impression of what this task was for, illustrates a general challenge with assignments in the course, namely that students may succeed with carrying out certain technical steps (drawing on some $R_U(\sigma, \omega)$) without seeing how the steps, together, support a major point in relation to high school mathematics. It visibly does not suffice to state the overall point in a preamble. One point that needs more attention is how to formulate "summary questions" which allow students to reflect on more general points of the assignment, without these questions being perceived as of the type "write your opinion" (or worse, "guess what the teacher wants"). In this case, the meaning of an explicit or computational specification of a function is extended from "algebraic formulae" (thoroughly known from school) to a recursive algorithm that makes explicitly use of the decimal representation of real numbers. Another possibility is to institutionalise such theoretical points in a follow-up lecture, referring explicitly to the assignment. Naturally, the whole set-up with lectures and exercises could be questioned, however in a relatively traditional institutional context, there are also strong conveniences by keeping the formats that students are used to.

CONCLUSION

It is interesting to observe that while modern research into the nature and effects of mathematics teacher knowledge has adopted relatively global categories and viewpoints – corresponding to what might be represented as $R_S(t, M)$ where M is in some sense "school mathematics" – the original point of view of Klein was much more local, considering for instance how future teachers' relationship $R_U(\sigma, \mathbb{R})$ to the real numbers could be developed based on the "advanced standpoint", of type $R_U(\sigma, \omega)$, developed at university. The global viewpoint is certainly important when considering policy issues related to institutions and international comparison, which in some cases even goes beyond considering the single school discipline $M = \cup_i O_i$. Still, the more local viewpoint needs to be recovered in order to address the didactical question of how to actually develop and assess relationships of type $R_U(\sigma, O \quad)$, while drawing on some $R_U(\sigma, \omega)$. Even some policy issues – like what contents to include or reinforce in study programmes for future teachers – depends on what we know at this level.

Klein's concrete proposals to this end were given in the form of notes from a lecture course. In this paper, we have developed and exemplified an alternative and altogether more student oriented approach related to task design. At the same time, we have exemplified the general scheme (4): with $R_U(\sigma, \omega)$ being given by an undergraduate mathematics programme that is not specifically designed for teacher education, it may be necessary to develop such relationships further to what we have denoted $R_U^*(\sigma, \omega)$, in order to create viable tasks that can lead students to didactically relevant new relationships to school praxeologies, such as a deeper understanding of decimal representations of real numbers, special functions and so on. We emphasize that developing such tasks requires simultaneous and up-dated knowledge of both the undergraduate prerequisites and high school mathematics. Moreover, previous studies of how pupils and teachers at large relate to a given high school praxeology could be invested in the selection of problematic local contexts and in the design process. Other simple aims and methods that one can pursue in the design of such tasks were proposed by Huo and Winsløw (2023).

We do not claim that task design is the only or even a sufficient means to achieve, for instance, a relationship to the real numbers which is relevant to how these appear at secondary level, and in other contexts where digital tools are more dominant than in scholarly mathematics. In fact, our case also suggests that just like regular undergraduate courses, capstone courses may benefit from a vigorous dynamics between students' work with challenging aspects of high school mathematics and lectures which focus on extending deepening their theoretical knowledge in directions that are relevant to such student work. Further research is needed to estimate the effects of such courses on actual relations of type $R_S(t, O)$, and effects of $R_S(t, O)$on the relationship to O of the students of t.

Thus, from a modern point of view – where the gap between the standard undergraduate mathematics programme and mathematics in secondary school and society has certainly increased – RQ1 cannot be seriously considered without also taking RQ2 into account. In the case considered, the rigorous approach to infinite decimals requires revisiting and extending previous work on properties related to completeness. In many contexts, identifying such needs could lead to renegotiating key elements of the external didactic transposition at university, with the possibility of enriching the general undergraduate programme.

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Paper IV: Submitted to For the learning of mathematics and currently under revision

Secondary teacher knowledge of real numbers and functions as handled by computers: the critical notion of computability

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1 What are real numbers?

Numbers are central mathematical objects throughout a student's mathematical journey. They make their initial appearance in primary school as natural numbers and later expand to include fractions and decimals. Upon entering secondary school, students are introduced to rational numbers, represented by fractions, before encountering irrational numbers, which are distinct from rational ones. Eventually, they come to real numbers, which are typically defined as the union of rational and irrational numbers (González-Martín et al., 2013). With the introduction of transcendent functions like logarithms, the utilization of real numbers extends to non-arithmetical calculation. Although these computations can be intricate, especially when involving irrational numbers, calculators or computers often replace manual calculations. The results are typically displayed on calculators or computers as decimals, accurate to a specific number of decimal places, depending on the task's requirements. This transition to digital computation is undeniably convenient and widely accepted by both students and teachers. However, the underlying reasons for certain errors (cf. Barquero & Winsløw, 2022) in computer calculations are often neglected by students and teachers or rarely contemplated by them along with how a calculator or a computer performs these calculations.

As students transition into university, they often carry with them the naive model of real numbers characterized as the first level of conceptualization of real numbers according to Bergé (2008) from their high school experience. However, they quickly realize that this familiar model no longer applies in a university setting. In university, students are introduced to a newly defined concept of real numbers, often beginning in their first analysis classes. One of the most common approaches to constructing real numbers at this level is through Dedekind cuts. While some textbooks may introduce infinite decimal representations of real numbers, these topics are often briefly covered in class or assigned as supplementary reading. Real numbers in the university context extend beyond being mere numerical values; they become a set or a space that serves various mathematical concepts and theorems. The practical use of real numbers, as encountered in high school, becomes less relevant in advanced university-level mathematics. At the same time, computer-based calculations, which dominate the use of real numbers in high school, disappear in the university mathematical landscape.

The understanding of real numbers can be somewhat unclear for students both in high school and university (cf. González-Martín et al., 2013; Bergé, 2010). In high school, real numbers are often introduced individually, and they are also depicted as points on a number line. However, questions regarding the mathematical meaning of the number line and in what sense there are no gaps or discontinuities in the number line are typically left unanswered. On the other hand, university-level mathematics, which places a strong emphasis on principles and foundations, typically does not answer the specifics of how real numbers are processed or represented on computers. This lack of insight into the computational aspects of real numbers can create a significant challenge for individuals who have completed their university mathematics studies and are on the verge of becoming high school mathematics teachers (called future teachers following). They face a substantial transition between different models of real numbers, which can make the task of teaching real numbers in high school more challenging. Given this significant transition, it becomes essential for prospective secondary teachers to learn about mathematical connections between the real number models encountered in university and high school mathematics (Branchetti, 2017b; Branchetti, 2017a).

In this paper, I will present a (well-known) intermediate model of how real numbers can mathematically be defined as infinite decimals, serving as a bridge between the two previously mentioned models. Based on this, I will introduce the concept of computability, delving into how computers handle infinite decimals. This exploration of computability seeks to establish a connection between the non-computer-based university-level real numbers context and the computer-dependent high school-level real numbers environment. Furthermore, I will outline a possible approach to efficiently convey this idea to future teachers. To illustrate these two points, I will provide two examples drawn from a specific course taught at the University of Copenhagen.

2 Theoretical framework and research questions

This paper is based on the Anthropological Theory of the Didactic (cf. Chevallard, 1992), abbreviated ATD. In ATD, a piece of knowledge is modeled as a *praxeology*. Each praxeology is composed of two components: a *praxis* block and a *logos* block. The praxis block encompasses practical problems or specific tasks, along with the corresponding techniques employed to address these tasks, and the logos block comprises discourses that provide a theoretical elaboration of the praxis block.

While every praxeology has these two blocks, they are not always directly apparent in teaching and learning. In high school, the praxis block, denoted as Π_{HS} is relatively straightforward to identify. For instance, students use a calculator to find the value of $f(x) = \ln(x)$ when $x = 2$ (which can amount to one or more techniques within the praxis block) but without any explanation from teachers regarding how the function works on the calculator. Students may become proficient in operating the calculator to obtain correct results without a deeper understanding of the function or real numbers. Exploration of how a function works on a calculator is rarely an objective in high school teaching and learning but it belongs to the corresponding logos block. Teachers might not even possess this knowledge themselves. Hence, crucial elements of a potential logos block often remain invisible within the context of high school mathematics education.

Conversely, teaching and learning at the university level place greater emphasis on logos blocks denoted as Λ_U. Students delve into various theoretical aspects of real numbers, including concepts like completeness. However, this theoretical knowledge often lacks a direct application to concrete problems. Additionally, calculators and some mathematical software become less prominent in university mathematics. The focus shifts towards understanding the underlying principles and theories, fostering a deeper comprehension of mathematics. Hence, future teachers may possess substantial theoretical knowledge concerning real numbers but they might still encounter difficulties in determining which theorems or theories to apply when faced with practical questions like "how a function works on the calculator".

Future teachers, therefore, might perceive a substantial disparity between what they learned at university and what they are expected to teach in secondary schools. This gap between high school and university for a future teacher, as described by Klein (2016), is often referred to as Klein's second discontinuity. One potential explanation for this discontinuity is the disconnection between the two blocks of praxeology in both academic institutions. In this paper, my objective is to construct a new praxeology that serves as a bridge, forging a connection between high school mathematics and university mathematics for future teachers. This new praxeology should incorporate a praxis block derived from Π_{HS}, denoted as Π^*_{HS} and a logos block featuring components from Λ_U, which can be used to elaborate the high school mathematics-related logos block, denoted as Λ^*_U.

Taking into account the wide use of digital tools within high school mathematics, and building upon the challenges in teaching and learning real numbers mentioned in the previous section, the following two questions emerge:

> **RQ1**. How to expand a praxeology of real numbers by incorporating a logos block Λ^*_U that provides a mathematically meaningful description of Π^*_{HS} on computers?
>
> **RQ2**. How can the idea from RQ1 be prepared and disseminated to future teachers in a pure mathematics study programme (concretely: in a teaching-oriented capstone course within such a programme)?

To answer RQ1, we first need to define real numbers as a set of infinite decimals. The construction of Π^*_{HS} essentially signifies the utilization of a computer to find the decimal representation of a real number. For instance, concrete tasks like calculating the first 10 decimal places of π or $\ln 2$ are where the solution is normally achieved through computers by high school students. The key to addressing RQ1 lies in the construction of Λ_U^*, to determine what corresponding university-level knowledge concerning real numbers needs to be included and how to establish its relevance to Π_{HS}^*.

The core challenge in comprehending how computers calculate decimals pertains to the ambiguity surrounding the computer commands responsible for generating these decimals. For instance, deciphering how the computer interprets a command like "ln" remains elusive. An understanding of these internal mechanisms would enable us to demystify how computers produce real numbers as decimal representations. Although it is an impossible task for future teachers to fully open the "black box", they can explore potential solutions based on their existing knowledge. One approach is to devise some computer algorithms that can substitute commands within the "black box". Based on this idea we can integrate programming with high school exercises, where Π_{HS}^* could be reformed as obtaining the first n digits of a decimal representation for a given real number by executing a designated piece of coding on a computer. Hence, the purpose of Λ_U^* at this point is to provide a theoretical explanation of the given computer algorithms. In this paper, I will illustrate how a computer produces real numbers as decimal representations through two examples with two different computer algorithms from a computability perspective. It is necessary to emphasize that computability is not typically part of the standard university mathematics curriculum; rather, it serves as a guiding framework here to aid future teachers in connecting segments of university-level knowledge and high school computation practice, and forging connections between them.

The extended praxeology, asked for by RQ1, can be likened to an unpacked parcel awaiting collection and delivery to future teachers. To address RQ2 effectively, we can divide this question into three distinct components:

1. Parcel Preparation: How should we "pack" the praxeology in a comprehensible and pedagogically effective manner?

2. Delivery Context and Method: In what context and through what methods should we deliver this prepared praxeology?

3. Confirmation of Delivery: How can we ensure that the packaged praxeology reaches future teachers and is duly affirmed by them?

In this paper, "packing" the praxeology involves the development of a structured lesson plan which includes designing the lecture content, formulating exercises as practice, and creating an assignment to assess future teachers' understanding. The execution of this plan serves as the delivery context and method. The confirmation of the extended praxeology's delivery is assessed through future teachers' performance in completing the assignment.

3 Computability

What is computability? In a broad sense, as described by Lucier (2022), "an object is computable if there can exist a computer program that is guaranteed to compute it in a finite number of steps." This section is not intended to provide a complete mathematical exploration of computability but it focuses on how to describe real numbers as decimal representations on computers from the perspective of computability (RQ1), specifically within the *computer algebra system* (CAS). Lucier (2022) did not give a clear definition for "computer program". Therefore, an interpretation of the "computer program" outlined is demanded here. In the context of this paper, a branch of coding is called a *computer program*, when provided with an input and a natural number n, which could produce a decimal with n decimal digits. A computer program has to have an end condition and be restricted to basic arithmetic operations — addition, subtraction, multiplication, and division - of integers. In other words, a computer program does not function like a "magic button"; its algorithm should facilitate a process that can also in principle be performed manually with paper and pen, which we refer to as a "manual operation".

3.1 Computable numbers – square root 2

To produce a decimal representation of irrational numbers like $\sqrt{2}$ is a common task in high school, and students usually use calculators or CAS to get an answer (for example, the command "evalf" on Maple). I will explore this example within this subsection. I will use the following slight formalization of Lucier's definition of computable real numbers (Lucier, 2022, p330).

Definition 3.1. Let $\mathbb{D}_n = \{10^{-n}y : y \in \mathbb{Z}\}$. A real number x is called *computable* if there exists a computer program that, for any given $n \in \mathbb{N}$, computes a number $x(n) \in \mathbb{D}_n$ that satisfies $|x(n) - x| \leq 10^{-n}$ for all $n \in \mathbb{N}$.

A collection of mathematical results related to this definition has been presented by Ménissier-Morain (2005).

In high school, both teachers and students place great emphasis on obtaining a specific value through calculation, often seeking a unique result. Because of this, Definition 3.1 needs to be strengthened. For example, when we take $n = 2$, both 1.41 and 1.42 can represent $\sqrt{2}$. In this setup, a real number can be represented by two different decimals for a given level of approximation you specify; it depends on the computer program which of the two is produced (for instance, it may round up or down, always give the smaller one, and so on).

A critical requirement for the decimal representation of real numbers is uniqueness, which is evident in the description provided by the following theorem (Sultan & Artzt, 2018, p. 348, Theorem 8.65).

Theorem 1. *Every nonnegative real number x between 0 and 1 is represented by a unique infinite decimal, except those numbers whose decimal representations terminate in an infinite number of zeros or an infinite numbers of 9's. These and only these decimals can also be represented in two ways.*

The unique infinite decimal presentation of a real number that does not end in an infinite sequence of 9's is called the *canonical decimal representation* of the real number in this paper.

If the number to be computed is 1, then the computer program satisfying the requirements in Definition 3.1 may produce, for a given k, either $0.9\ldots9$ (k 9's) or $1.0\ldots0$ (k 0's), and we may in this case not know if even the first decimal produced is correct (since at an unknown distance, something different from 9 may appear in the "true" number). This inability to produce the canonical decimal representations (every time, not just sometimes) is frequent with real-life calculation devices (cf. example in Figure 1). It often confuses students when they expect a certain result (say 0, with the canonical decimal representation 0.0...) but get something else (see Figure 1 for an example). We, therefore, consider it an important part of teachers' knowledge that computer programs sometimes do not produce the canonical decimal representation, that they may even get all the decimals wrong, and that in actual practice it may be hard to predict when that happens.

$$\mathit{evalf}(2/3) - \mathit{evalf}(1/3) - \mathit{evalf}(1/3)$$

$$1. \times 10^{-10}$$

Figure 1: An example of calculation error on Maple.

Here, I introduce a new concept "absolutely computable".

Definition 3.2. A real number x is called *absolutely computable* if there exists a computer program that, for any given $n \in \mathbb{N}$, computes the first n decimals of the canonical representation of x, namely $\hat{x}(n) = \frac{\lfloor 10^n \cdot x \rfloor}{10^n}$ for all $n \in \mathbb{N}$.

As an example, we can easily construct a computer routine (see Figure 2) that can be run on Maple, to find the first 10 decimal digits of the canonical decimal representation of $\sqrt{2}$. In this routine, the function $f(x) = x^2 - 2$ is an aid to produce $\sqrt{2}$ because $\sqrt{2}$ is one of its zeros; as the function is only based on arithmetic (multiplication and subtraction) it is much less of a "black box" than $\sqrt{2}$. The routine can produce as many canonical decimal digits as you want by changing i, thus it is a computer program that demonstrates $\sqrt{2}$ is absolutely computable. This routine can also be modified to show the absolute computability of any other algebraic number such as $\sqrt{3}$, by changing the function and the initial value K.

```
K := 1 :;
for i from 1 to 10 do
      for j from 0 to 9 do
           if (K + j * 10^(-i))^2-2 <= 0 then
                p := K + j*10^(-i);
           end if ;
end do;
K := p :;
print(x(i) = evalf(p, i + 1));
end do :;
```

Figure 2: A computer program to compute first 10 decimal digits of $\sqrt{2}$ one by one.

Let $x = \sqrt{2}$. The sequence $\{\hat{x}(i)\}$ ($\hat{x}(i)$ is written as $x(i)$ in the routine for coding convenience) that can be produced up to some point by the routine, is convergent, with limit $\sqrt{2}$. More specifically, for each i, one has $\hat{x}(i)^2 - 2 \leq 0$ and $(\hat{x}(i) + 10^{-i})^2 - 2 \geq 0$, which implies $\lim_{i\to\infty} \hat{x}(i) = \sqrt{2}$. Huo (2023) presented a detailed explanation of the routine.

There are a couple of important points to highlight here. Firstly, it is crucial to note that the function employed in this routine must have only integer coefficients. For instance, if we use the function $f(x) = x - \sqrt{2}$, which also has zero $\sqrt{2}$, the routine can still run and obtain the same results. However, it loses its purpose in exploring and showcasing the decimal representation of $\sqrt{2}$.

Secondly, the addition of two canonical decimal representations may not yield another canonical decimal representation. For example, letting $x = \sqrt{2}$, $y = \sqrt{3}$ and $z = \sqrt{2} + \sqrt{3}$, the equation $\hat{x}(n) + \hat{y}(n) = \hat{z}(n)$ does not hold with $n = 6$ ($\hat{y}(n)$ and $\hat{z}(n)$ can be obtained by modifying the routine in Figure 2).

Thirdly, it is essential to provide a more detailed explanation of the "evalf" command. As mentioned earlier, this command can easily cause errors. However, in this specific routine, I opted to use it because p is a finite decimal represented as a fraction in Maple. Therefore, employing evalf(p, $i + 1$) ensures that the output is presented as a finite decimal.

3.2 Computable functions – logarithm

In the previous example, we focused on the real numbers themselves – to find the canonical decimal representation - but in fact, the decimal representation is often found in high school when calculating the value of some transcendental function as some decimal approximation. It is not complicated for polynomials, for instance, to find $f(x) = x^2 - 2$ when $x = 1.414$; this, students can do by hand. However, it is normally impossible for high school students to compute transcendental functions without computers or calculators (except for very special cases, like $\sin 0$) – for example, to calculate $f(x) = \log_{10} x$ when $x = 1.414$.

But what exactly do these symbols represent? To high school students, these symbols might resemble buttons on a calculator or commands in computer software. What do these symbols signify to a calculator or a computer, and how do they perform these calculations? This exploration leads us to broaden our understanding of computability from real numbers to functions.

Definition 3.3. Let $\mathbb{D} = \bigcup_{n=\mathbb{N}} \mathbb{D}_n$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *absolutely computable* if there exists a computer program that, for any given $x \in \mathbb{D}$ and $n \in \mathbb{N}$, computes $\widehat{f(x)}(n) \in \mathbb{D}_n$ for all $n \in \mathbb{N}$.

It is important to distinguish between absolutely computable functions and absolutely computable real numbers. For instance, let $\log_{10} a$ and $\log_{10} b$ be absolutely computable with computer program A and computer program B respectively, where $a, b \in \mathbb{D}_n$ and $a, b > 0$. When we assert that $\log_{10} a$ and $\log_{10} b$ are two absolutely computable numbers, it means that computer program A and computer program B can either be the same or different. However, if we claim that $f(x) = \log_{10}(x)$ is an absolutely computable function, computer program A and computer program B must be the same. In essence, for computable functions, the computer program must accommodate all possible inputs.

Therefore, if a function $f : \mathbb{R} \to \mathbb{R}$ is absolutely computable, then for any $x \in \mathbb{D}_n$, $f(x)$ is also an absolutely computable number.

I will take Figure 3 as an illustrative example to elucidate a possible way a computer computes the function $f(x) = \log_{10} x$. The idea behind the algorithm (further discussed by Weber, 2016) is that since $a^{\log_a M} = M$ for $a > 0$ and $M > 0$, we can informally think of $\log_a M$ as the "number of times" (not necessarily integer) that we need to multiply a by itself to get M, or as the number of times M contains a multiplicatively. Incidentally, this establishes an analog with division, since M/a can, similarly, be thought of as the number of times M contains a additively.

```
compLog := proc (M, d := 10, n := 4);
    local a, i, res, Mnew;
    Mnew := M;
    res := 0;
    for i from 0 to n do
        for a from 0 by 1 while d^(a+1) <= Mnew do
        end do;
        Mnew := evalf((Mnew/d^a)^d);
        res := res+evalf(a*d^(-i));
    end do;
    return res;
end proc:;
```

Figure 3: A computer program to compute first 4 decimal digits of $\log_{10} M$ one by one

Notice that the routine only works for $M \geq 1$, but if $0 < M < 1$ we can use $\log_a M^{-1} = -\log_a M$. To better explain the routine, we take $M = 57.64$ as an example. If $\log_{10} 57.64 = N.c_1c_2...$ where N is a non-negative integer and $c_1, c_2... \in \{1, 2, ...9\}$, then we have

$$57.64 = 10^{N+\frac{c_1}{10}+\frac{c_2}{10^2}+\cdots} = 10^N \cdot 10^{\frac{c_1}{10}+\cdots} = 10^N \cdot m_1$$

Because $1 < m_1 < 10$, we have $10^N \leq 57.64 < 10^{N+1}$ and we see $N = 1$. Therefore, by dividing 10^N (10 here), the above equation can be rewritten as $5.764 = 10^{0.c_1c_2\cdots}$ which can be rewritten as

$$M_1 = 5.764^{10} = 10^{c_1} \cdot 10^{\frac{c_2}{10}+\cdots} = 10^{c_1} \cdot m_2$$

where $1 \leq m_2 < 10$. We can now find $c_1 = 7$ (as $M_1 = 5.764^{10} \approx 4 \cdot 10^8$ can be found using primary school techniques). By repeating the process, we could find as many c_k as we want to, where $k = 1, 2, 3, ...$. In fact, the decimal approximation we get from the routine is the canonical decimal representation of $\log_{10} 57.64$ because there does not exist $c'_k \in \{1, 2, ...9\}$ with $c'_k \neq c_k$ such that $10^{c'_k} \leq M_k < 10^{c_k+1}$. Thus, $N.c_1c_2...$ is the canonical decimal representation of $\log_{10} 57.64$.

I must bring attention to a minor flaw in the design of the routine illustrated in Figure 3. While M_k is indeed a finite number, the "evalf" command displays only 10 digits, subject to rounding, if specific accuracy requirements are not stated. This rounding does not affect the value of c_k, but it can influence some M_l $(l > k)$, resulting in a wrong c_l. However, this issue can be corrected by adjusting the commands within the algorithm.

4 Dissemination of computability: a case study

Before delving into the response to RQ2, it is essential to clarify what constitutes the "unpacked" praxeology that will be delivered to future teachers in both examples. In the case of computable numbers, Π^*_{HS} involves the execution of the computer program detailed in Figure 2 using Maple to obtain the initial 10 decimal digits of the canonical decimal representation of $\sqrt{2}$. On the other hand, Λ^*_U encompasses the explanation provided in the previous section, elucidating how the computer program operates. Similarly, in the example of computable functions, Π^*_{HS} pertains to acquiring the first 4 decimal digits of the canonical decimal representation of $f(x) = \log_{10} x$ for a given value of x, such as 57.64. This is accomplished by running and analyzing the computer program on Maple outlined in Figure 3. Correspondingly, Λ_U^* includes the explanatory content related to the computer program. Hence, the ribbon of the parcel in both examples is the knowledge of real numbers as decimal representations, which should be the key during the preparation before delivery. It's worth noting that some might argue that knowledge of coding is also a crucial component in understanding computability. While this is a valid point, it is assumed that most mathematics students at university possess some degree of knowledge and experience with coding. In this paper, the primary focus is on the pure mathematics aspect.

A suitable context for delivering the praxeology must contain two points: a group of students who have finished university mathematics study, and individuals or a designated entity capable of guiding these students with a planned curriculum that encompasses mathematical knowledge. In my specific case, I have chosen the UvMat course at the University of Copenhagen. UvMat is an elective course designed for individuals who have completed at least two years of pure mathematics study at university and intend to pursue careers as secondary school teachers in Denmark. This course is offered annually and enrolls 20 to 30 participants. Its primary objective is to guide future teachers in revisiting high school-related mathematical concepts from a university standpoint, a model referred to as a "capstone course" according to Winsløw and Grønbæk (2014). Notably, the course's curriculum includes a topic dedicated to real numbers as decimal representations during its fourth week each year, which serves as a suitable testing ground for the praxeology created in this paper.

During the fourth week of this course, students are required to attend two lectures that are prepared and presented by a professor. These lectures are based on Chapter 8 of the book titled *The Mathematics That Every Secondary School Math Teacher Needs to Know* authored by Sultan and Artzt (2018). Following each lecture, a teacher assistant takes on the responsibility of guiding the students through a series of exercises that have been selected by the professor. These exercises, which are often drawn from the book itself (Sultan & Artzt, 2018), serve as practical applications and opportunities for the students to practice the concepts covered in the lectures. Towards the end of the week, students are presented with a challenging assignment that is closely tied to the content they have studied throughout the entire week. This assignment is designed to apply the knowledge and concepts they have acquired. The teacher assistant reviews the assignments submitted by the students and provides feedback based on their responses. Students are then given the opportunity to revise and resubmit their assignments, taking into consideration the feedback provided. The revised assignments serve as their final submissions. In Sultan and Artzt's book, an infinite decimal $\pm 0.c_1c_2c_3....$ is rewritten as a series $\sum_{i=1}^{\infty} c_i \cdot 10^{-i}$, where $c_1, c_2... \in \{0, 1, 2, ..9\}$. A key component of Chapter 8 in the book is to facilitate the understanding of the fundamental theorem such as Theorem 1, which also serves as the central theme for the lectures. The proofs of these theorems necessitate the application of knowledge acquired in university-level mathematics. For example, students are required to prove that $\sum_{i=1}^{\infty} c_i \cdot 10^{-i}$ always converges.

In addition to covering the fundamental theorems and exercises from the book, the assignment is intentionally designed to encourage students to explore concepts beyond the book's scope, utilizing the knowledge they have gained during the course. The assignments often prove to be quite challenging for many students. These assignments essentially can be seen as a method to deliver the praxeology. During the academic year in 2021, I crafted an assignment for this course week that focused on the infinite decimal representation of the square root. In this assignment, students were asked to explain the routine presented in Figure 2 and use it to explore and discuss the addition of two infinite decimals (the whole assignment is presented by Huo (2023)). Students were expected to go through the whole idea I presented in the previous section about how a computer deal with real numbers as infinite decimals representation. Certain questions in the assignment were framed from a high school perspective, such as the request to explain the coding visually.

Upon reviewing the initial submitted versions of the students' answers and considering the feedback provided, two key issues related to the assignment's design have been identified. Firstly, the use of terms like "explain" and "why", led students to adopt an informal style of response, resembling the way high school exercises are typically answered. For instance, when asked to explain why $x(i) \to \sqrt{2}$ as $i \to \infty$, some students simply stated that $x(i)$ gets close enough to $\sqrt{2}$ without providing a formal proof using the limit definition. However, when the feedback pointed out the need for a formal limit proof, all groups eventually attached a complete proof. As a result, students faced challenges in proactively applying the concepts from Λ_U^* in their responses, leading to connect Π_{HS}^* and Λ_U^*. Secondly, the assignment was designed with the intention of placing students in the role of a teacher, prompting them to contemplate how a computer produces decimal numbers digit by digit and why it sometimes produces unexpected results.

We now consider a similar assignment for the fourth week of the UvMat course in the 2022 academic year (all the tasks on this assignment were shown by Winsløw and Huo (2023)). This assignment commenced with a review of how infinite decimals are presented in the textbook and lectures and explicitly stated that the purpose of the assignment was to enable the calculation of $x = \log_{10}^M$, for any $M > 1$, using only basic arithmetic operations. The computer program shown in Figure 3 was introduced after students had completed five purely mathematical tasks. Students were eventually required to employ the solutions they had derived for the previous questions to elucidate how the routine functioned, with $M = 57.64$. The assignment shifted all attention towards comprehending the provided routine. I view this assignment as a successful delivery method because it compelled students to engage with Λ_U^* first, prompting them to uncover the corresponding Π_{HS}^*.

After students revised their assignment answers, I conducted interviews with voluntary participants in both years. In the interviews, participants were asked to explain what they had learned through the assignments. In 2021, the majority of students struggled to articulate the overall significance of the entire assignment. Only one student pointed out, "I believe that a teacher should understand when to employ computers and when not to." Another student went as far as to question the relevance of this assignment to high school teaching. Through the students' responses in these interviews, it became evident that the prepared praxeology was not being entirely delivered to these future teachers in 2021. In 2022, I improved the interview process. Each group of students had a representative for the interviews, and I conducted interviews with individuals before the lectures to assess their comprehension of logarithms. All interviewees were initially familiar with logarithms, viewing them as the inverse of exponential functions. However, during the follow-up interviews after revising their assignment answers, they expressed surprise and demonstrated that they now comprehended how to manually calculate logarithms and understood them as separate from exponential functions. One student mentioned, "I never quite understood how it worked. Now I understand after this course a little bit more." This shift in their understanding marked a successful delivery of the praxeology.

5 Conlusion

The newly proposed praxeology indeed constructs a bridge, connecting the model of real numbers in university study with the model of real numbers to be taught in high school (RQ1), and it can be disseminated to future teachers through a capstone course like UvMat (RQ2). However, how future teachers could use this praxeology to autonomously explore real numbers as infinite decimal models on computers is not addressed in this paper. In the examples discussed in the previous section, as students engaged with the assignments, their primary focus remained on completing the designated tasks. Despite our efforts to clarify the purpose of the assignments, most students continued to prioritize task completion without fully comprehending the assignment's broader significance within the context of the week's theme (i.e. the representation of real numbers as infinite decimals on computers). Only a small minority might contemplate the potential value these two assignments held for their future teaching roles (such as the student who said "I think a teacher needs to know when to use computers and when not.") One possible explanation for this phenomenon is that the UvMat course has not sufficiently emphasized the importance of integrating computability as a central element in the exploration of real numbers as infinite decimals on computers. In fact, the UvMat course itself did not explicitly cover the concept of computability, despite its incorporation into the task design. Expecting future teachers to independently explore real numbers as infinite decimals on computers from the perspective of computability merely through completing the assignment may be an ambitious proposition. Therefore, extending RQ2, another question emerges: once future teachers receive this "packaged" parcel, how should they proficiently unpack and harness its contents? I hypothesize integrating the concept of computability as a mathematical concept in the course could potentially enhance future teachers' ability to unpack this praxeology effectively. The limited existing literature and research on computability in mathematics education contribute to the difficulty of predicting the full extent of the potential improvement that this computability-focused approach could bring to future teachers. Further investigation and empirical studies conducted within real course settings may provide valuable insights into the practical implications of this approach.

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