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KØBENHAVNS UNIVERSITET

Didactics of Mathematics - The French Way

Texts from a Nordic Ph.D.-Course at the University of Copenhagen

Editor:
Carl Winsløw
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DIDACTICS OF MATHEMATICS – THE FRENCH WAY

TEXTS FROM A NORDIC PH.D.-COURSE AT THE UNIVERSITY OF COPENHAGEN

Editor: Carl Winsløw, CND, University of Copenhagen

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FOREWORD BY PROFESSOR BARBRO GREVHOLM

The Nordic Graduate School of Mathematics Education started to exist in 2004 and at once invited all of the 37 participating research environments to create and develop courses, which could be given to Nordic and Baltic doctoral students. A number of courses that already existed in different places were offered and attended by students. The first really new and original course was given by Professor Carl Winsløw at the University of Copenhagen. The board of the Nordic Graduate School of Mathematics Education welcomed this course very much and wanted it to take place. The board found it highly valuable to introduce the French didactics to the Nordic audience. As it was not possible to get as many participants as required by the University, I volunteered to be present during course time and add another voice to the discussions.

It has been most interesting for me to follow the development of the course and experience how doctoral student conceptualized the four French theories that have been treated in the course. The students were very satisfied with the course and Carl Winsløw carried it out in an excellent way. His engagement gave the participants a good sense of how challenging and valuable it can be to interpret one's research from different theoretical perspectives. The varied and important ways of working with a mixture of reading, reflecting, writing, presenting and discussing the literature have entitled the participants to really work through the theories. The process of learning through writing seems to have been very important for all the students. Carl has led the sessions with great success and given thoughtful reflections and feed back that were accessible and highly adapted to each contributor's work. The fact that the final paper in the course was based on the student's own research added an extra dimension to the examination. The final papers that course members have written and that are presented in this booklet clearly give a good picture of the outcome of such a course.

The board of the Nordic Graduate School expresses its gratitude to Carl Winsløw for his work and cooperation with us.

On behalf of the board

Barbro Grevholm

Director

Professor at Agder University College

Norway

INTRODUCTION:
A GRADUATE COURSE ON FOUR FRENCH FRAMEWORKS FOR RESEARCH ON DIDACTICS OF MATHEMATICS

Carl Winsløw

CND, University of Copenhagen

A personal introduction: how this course came about

As a newly appointed assistant professor of mathematics at the University of Copenhagen, I was obliged to take a course on university pedagogy. Teaching quickly became an important and enjoyable part of my new job, and the course was a valuable opportunity to discuss teaching experience with other young colleagues and to reflect systematically on my own attempts. Yet I was somewhat disappointed with the theoretical aspects of the course, for it seemed to me that the texts we studied in the course contained mostly trivialities about teaching methods and descriptions of fancy practices which had little to do with the lecture-type courses I was supposed to deliver. More seriously, it appeared to me that those texts did not address the crucial issues of *mathematics* teaching. Indeed, when contents were at all visible in the discussions about teaching practice, it tended to be either from the humanities or from engineering education.

In short, I was wondering if it could really be true that nothing was known about the teaching of *mathematics*. Then I discovered the existence of a whole discipline about this subject, and some of the first texts I read on it were by Guy Brousseau, a name totally unknown to me. This was 1995, ten years ago – but of course, many young mathematicians are in a state of similar ignorance even today.

It was a pure coincidence that my own readings on didactics soon led me to Brousseau. My university studies had included a minor in French, and this auxiliary academic interest made me naturally inclined to pay particular attention to references in French that occurred during my initial explorations of the literature (including the anthology *Didactics of Mathematics as a Scientific Discipline*, Kluwer 1994). And as I familiarised myself with the field, it appeared to me that Brousseau's *theory of didactical situations* was among the most deep and challenging ways of thinking about mathematics *as a subject for teaching*. Here is one of my favourite quotations (taken from the English translation of Brousseau's book):

> Mathematicians don't communicate their results in the form in which they discover them; they re-organize them, they give them as general a form as possible.

> Mathematicians perform a “didactical practice” which consists of putting knowledge into a communicable, decontextualized, depersonalized, detemporalized form.
>
> The teacher first undertakes the opposite action; a recontextualization and a repersonalization of knowledge. She looks for situations which can give meaning to the knowledge to be taught. But when the student has responded to the proposed situation (…) she will, with the assistance of the teacher, have to redepersonalize and redecontextualize the knowledge which she has produced so that she can see that it has a universal character, and that it is re-usable cultural knowledge.

(Brousseau, 1997, p. 227)

So, very roughly, the teacher’s task is to arrange situations for the student to discover knowledge (necessarily this will be first *personal* knowledge linked to the situation) and then depersonalise it. The students’ work then consists in a double process of personal discovery followed by depersonalisation. It is, in a sense, *similar* to the work of the mathematician (with obvious differences as well). By contrast, the *teacher’s work* – which, by the way, is that of the teaching mathematician as well – is *foremost* to produce transformation of established knowledge *into* situations of learning and discovery. Even if the teacher is a lecturer, the theory of didactical situations is useful to understand this necessity for the student to acquire knowledge in a personally meaningful – and, perhaps initially, idiosyncratic – context. Even in lectures, such a process could be supported.

Over the years, the field of didactics of mathematics became my own field of research. And increasingly I found that the French work in this field was not only of particular value, but also surprisingly ignored in the Scandinavian countries. No doubt a significant part of the reason for this has been the fact that almost all the research works of French didactics (at least up to 1995) were published in French, a language which only a small minority of researchers in Scandinavia are comfortable with reading. However, over the past 10 years or so, several existing and also new works from the French school of didactics have been published in English. Still, these works are not common knowledge, and certainly not in common usage, among Scandinavian researchers. Of course the language barrier was only part of the explanation. More importantly, the French school is based on rather difficult theoretical frameworks – among them the theory of didactical situations – and it does not suffice to read an exposition of such a theory, or a few instances of its applications, in order to familiarise oneself with its use. At the same time, didactics in Scandinavia has developed its own theories and perspectives, in close interaction with traditions developed in the English-speaking world.

Part of my own work – particularly over the past 5 years or so – has drawn heavily on works from French schools in didactics. I have participated with increasing enthusiasm in meetings and conferences in France, especially the biannual *summer schools* arranged by the ARDM (Association pour la Recherche en Didactique des Mathématiques). I increasingly felt that the knowledge and insight which I gained from this should be made available to other researchers in Scandinavia, particularly beginners in search for new and fruitful approaches. And so, although my experience and knowledge of the field is certainly quite limited, I decided to offer the graduate course 4F, the products of which this booklet is documenting. The Nordic Graduate school of Mathematics Education, in the person of its director, professor Barbro Grevholm, kindly offered its support of the course, financially as well as morally.

In the rest of this introduction, I give an outline of the idea, structure and development of the course. Although – and because – the papers that follow are indeed independent, self-contained pieces of research work of considerable quality and originality, I find it useful for the reader to bear in mind the conditions under which they were produced: most of the participants had no previous knowledge of French didactics; at the end of the course, all of them were required to produce a paper using these frameworks. Also, it should be remembered that the participants – and authors of this booklet – are graduate students at various stages of advancement in their ph.d.-studies: two were in their first semester, two were just a few months from finishing their dissertation; the remaining two were *in medias res*.

The idea, structure and results of the course

As indicated in the title and description (appendix 1) of the course, four main "frameworks" – identified with major French researchers within in the field – were to be introduced and worked on, and this was done in three phases corresponding to the three "sessions" of the course (each consisting of 12 hours over two days, cf. appendix 3):

A common introduction to the main ideas, concepts and results of the four frameworks, based on texts read in advance (cf. appendix 2); the first session was much like an ordinary student course, with lectures and short accompanying exercises, but with little extensive discussion.

A familiarisation with one or more works based on one or more of the four frameworks, and on topics within the individual doctoral students' domain of interest. The texts assigned for the second session (appendix 2) were assigned to individual participants according to their interests, even if everyone got and could read all the papers; it was the responsibility of each student to choose among her/his papers which one(s) were relevant to her/his interests, and to present this selection at the second session. At the second session, an extensive discussion was done about the relation to the individual projects; one-page descriptions of the participants' projects were distributed together with the programme of the second session. The discussion also included reflections on what to focus on in the final essay.

Finally, as mentioned in the course description (appendix 1), all participants were requested to write a 10-page essay to be presented and discussed at the third session. The aim of this essay was to relate and discuss the participants' own area of interest to one or more of the four frameworks, and ideally to produce a genuine research paper based on one of them. These essays were finished and distributed to all participants one week before the third session, and each participant was assigned two papers by other participants to read, in order to produce a reaction during the third session (cf. appendix 3). The third session was thus designed as a mini-conference with paper presentations, reactions and discussions.

To sum up, the course was designed to present basic and general ideas of the four frameworks, to help students to relate these ideas to their interests, and based on reflecting on these relations, to help them consider (or simply pursue) the use of the frameworks within their own research project.

The requirement of the final paper was meant to emphasise the product aspect of reflections which would, otherwise, remain rather abstract and perhaps remote from the students' own preoccupations. This is a crucial point of the design; exactly when the topic of the course is a set of theoretical frameworks, it is necessary to proceed to *concrete, relevant studies using the theories* without which one cannot make sense of them. For doctoral students, an immediate usage could of course be at a perceptive level: the literature using the frameworks becomes (more) accessible. Several participants went to the European CERME-4 conference in February (between the second and the third sessions of the course) and reported that a concrete benefit of the course was to facilitate their understanding of lectures given by researchers using the French frameworks. But the course aimed deliberately at the (more advanced) *productive* level: to try out the frameworks for oneself, and to report on the results in the "depersonalised" form of a paper. In a sense, the various forms of *personalising* and *depersonalising* the students' work with the theories were designed in the spirit of the quotation of Brousseau in the previous section.

The format of the course seemed novel to participants, in particular its emphasis on the individual students' work on the one hand, and the definite (given) choice of theoretical frameworks on the other. Judging from the final evaluation, it can be described as a success. The participants ascribe the success to the structure of the course, the personalised approach with individual texts and presentations, and the quality of communication (at meetings and in between, via email and the course web site). A major factor they quote was also the friendly and supportive atmosphere during the meetings, in particular that discussions and reactions presented critique in a constructive way. As an organiser of the course, I can only agree that this last factor was quite important, and it is due to the impressive work each participant did during the course. In fact, the success of any teaching depends crucially on the participants, and this seems to be particularly true with a format like the present one, with its high demands in terms of autonomous and individual work between the sessions.

One of the frameworks – the theory of conceptual fields – was only marginally used by one of the participants in the final works. During our final discussion of the course, we agreed that including all four frameworks in the first session was nevertheless appropriate, given that they were all referred to in the literature of the second session, and in view of the broader formative aims of the course. But it is clear that more concrete examples of studies using the theory of conceptual fields would have improved the chances that more participants would choose to use it.

Almost all of the participants indicated at the end of the course their intention to make use of the work done in this course – and, more broadly, one or more of the frameworks – in their future research work. Those who are active in teacher education also mentioned the possibility of including, for instance, examples from the theory of didactical situations in their work as teacher educators. It therefore seems plausible that the consequences of this course may reach farther than to the papers presented in the booklet: to further international communication and collaboration in the didactics of mathematics between Scandinavia and the francophone world. If this should become the case, it would clearly be in itself an important result of our work during these dark winter months.

It was not a part of my initial planning to publish the papers produced during the course. However, as the third session was devoted to the critical discussion of the "first version" of these papers, it seemed natural to make use of all the comments and ideas generated during this session to revise the papers into a more final form. This revision work was not part of the required course work, as all essays presented did indeed, in their first version, fulfil the requirements stipulated in the course description. And so, to stimulate and motivate this revision work, and to generate a material product of the course, the idea of the present booklet came into being.

To be honest, I am surprised and very impressed of how far the participants got with the four frameworks. The papers reflect the different domains of interest represented among participants: discourse and authorship in teacher education (Heidi Måsøval), advanced mathematical writing (Morten Misfeldt), teaching of elementary algebra (Claire Berg), computer algebra systems in upper secondary school (Mette Andresen), university level teaching (Stine Timmermann), affect and mathematics (Kirsti Kislenko). And they demonstrate that each of these domains could be related to one or more of the French frameworks and how these can be used to shed light on the more specific questions and data from the authors' on-going studies and research.

I warmly thank the authors of this book – who were also participants of the course – for playing the games proposed, for investing a lot of hard work, and for bearing with the shortcomings of the course organiser.

Reference.

G Brousseau (1997). *Theory of didactical situations in mathematics*. Kluwer: Dordrecht.

Appendix 1: Description of the course (as advertised to prospective participants)

This course aims to introduce participants to four related theoretical frameworks for research in the didactics of mathematics, to foster reflection and discussion among participants on the nature of the discipline in the light of these theories, and to enrich the participants' own work by relating it to one or more of these frameworks. The four frameworks are:

- The theory of conceptual fields (due mainly to Gérard Vergnaud)
- The semiotic approach (due mainly to Raymond Duval)
- The theory of didactical situations in mathematics (due mainly to Guy Brousseau)
- The anthropological approach (due mainly to Yves Chevallard)

We shall read both "basic theory" and related, more specialised research articles. All of the required readings will be in English, but some additional (and optional) texts may be in French. The course language will be English or, if all participants indicate this preference *at the time of registration*, Danish/Norwegian/Swedish.

Scientific organiser: Professor Carl Winsløw, Centre for Science Education, University of Copenhagen

Timeline of the course.

0. November 2004: The first texts are sent to the participants, along with questions to reflect on while reading.
1. January 6-7, 2005: First session, with the following agenda:
   - Short introductions by participants of their projects and interests
   - Presentation and discussion of the four frameworks based on the first texts.
   - A new package of texts (research articles based on the frameworks) will be distributed at the end of this session. These will, as far as possible, be related to the projects and interests of participants.
2. February 3-4, 2005: Second session, treating the texts of the second package and their relations to the projects of the participants. Participants are expected to prepare a short oral presentation of their ideas regarding their own project, which will be discussed among the whole group.
3. March 10-11, 2005: Third and final session, a "mini-conference" where each participant will present a paper on aspects of their own projects which can be usefully related to one or more of the four frameworks (this paper must be finished and distributed via email to all participants before the session according to a deadline fixed at the second session).

Prerequisites: Participants are expected to be doctoral students in the didactics of mathematics with some general acquaintance of the field as such, and a clearly formulated research project which could be in any stage between beginnings and almost finished. We shall also assume some familiarity with cognitive psychology (of Jean Piaget) and its use in educational theory. A good and sufficient preparation for this latter area can be found in Chap. 8 of Svein Sjøberg: Naturfag som allmenndannelse – en kritisk fagdidaktikk (ad Notam Gyldendal, Oslo, 1998 and later).

Workloads and credits: The course will require about 200 hours of work, corresponding to a course credit of 7.5 ECTS points (pre-approbation of course credits must be obtained from home institution). The work includes readings, oral presentations and final essay (of about 10 pages). Course assessment will be based on the final essay.

Appendix 2: Readings for the first two sessions of the course

Readings for the first session, sent to participants in November.

1A. Yves Chevallard: Didactique? you must be joking! A critical comment on terminology**.** *Instructional Science* 27 (1999), 5-7.

1B. Guy Brousseau: Research in Mathematics Education: Observation and ... mathematics**.** In: I. Schwank (Ed.), *Proceedings of CERME 1, European research in mathematics education*, Osnabrueck 1999. Vol. 1. p. 35-49.

1C. Guy Brousseau: *Theory of Didactical Situations in mathematics*, Dordrecht, Kluwer, 1997, Introduction and Chap. 1 (pp. 3-75)

1D. Joaquim Barbé, Marianna Bosch, Lorena Espinoza and Josep Gascón: Didactic restrictions on teachers practice - the case of limits of functions at spanish high schools. *Ed. Stud. in Math.* (to appear).

1E. Gérard Vergnaud: The theory of conceptual fields. In: L. P. Steffe, P. Nesher, P. Cobb, G. Goldin, B. Greer (eds), *Theories of Mathematical Learning*. Lawrence Erlbaum,1996, pp. 219-240.

1F. Raymond Duval: The cognitive analysis of problems of comprehension in the learning of mathematics. Manuscript prepared for a research seminar in Copenhagen, 2002.

In addition, optional texts were provided for each of the four frameworks.

Readings for the second session, distributed at the first session in January.

NB: These texts were, except for the first one, assigned to one student (1-3 papers per student) and selected because of their bearing on the individual student project. Subsequently, each student chose from the assigned papers one or two to present at the second session.

2A. Michèle Artigue: Didactical engineering as a framework for the conception of teaching products. In: R. Biehler et al., *Didactics of mathematics as a scientific discipline*, 27-39. Dordrecht: Kluwer, 1994.

2B. Michèle Artigue: Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *Int. J. of Computers for Math. Learning* 7 (2002), 245-274. (Appendix: comment, pp. 293-299, same J.)

2C. Raymond Duval: L’apprentissage de l’algèbre et le problème cognitif de la désignation des objets. In: J.P. Drouhard et M. Maurel, eds, *Actes du 13ème séminaire de didactique de l’algèbre*, IREM de Nice, 2000, pp. 67-94.

2D. Michèle Artigue et al. : Teaching and learning algebra: approaching complexity through complementary perspectives. In: H. Chick et al. (eds), *Proceedings of the 12th ICMI Study Conference on The Future of the Teaching and Learning of Algebra*, vol.1, 21-32, University of Melbourne, 2002.

2E. Régine Douady: Relation function/al algebra: an example in high school (age 15-16). In: I. Schwank (ed), *Proceedings of CERME 1, European research in mathematics education.* Vol. 1, pp. 113-124.

2F. Michèle Artigue: The teaching and learning of mathematics at the university level. *Notices of the AMS*, Dec. 1999, 1377-1385.

2G. Carl Winsløw: Research and development of university level teaching: The interaction of didactical and mathematical organizations. *Proceedings of CERME 4, European research in mathematics education*, to appear.

2H. Virginia Warfield: Calculus by scientific debate. Unpubl. Manus.

2I. Isabelle Bloch: From academic mathematics to mathematics to be taught: situations for mathematics teachers’ education. Preprint 2004

2J. Alain Mercier et. al.: How social interactions within a class depend on the teachers’ assessment of the pupils’ various mathematical capabilities: a case study. In: I. Schwank (ed), *Proceedings of CERME 1, European research in mathematics education*. Vol. 1, pp. 342-353. Osnabrück, 1999.

2K. Nicholas Balacheff: A modeling challenge: untangling learners’ knowing. Paper presented at Journées internationale d’Orsay sur les sciences cognitives (2000).

2L. Jean-Philippe Drouhard et al.: Necessary mathematical statements and aspects of knowledge in the classroom. In: I. Schwank (ed), *Proceedings of CERME 1, European research in mathematics education*. Vol. 1, pp. 320-330. Osnabrück, 1999.

2M. Luis Radford: On heroes and the collapse of narratives: a contribution to the study of symbolic thinking. In A. Cockburn et al., *Proc. of PME 26*, vol. 4, 81-88. Norwich: UEA.

2N. Heinz Steinbring: What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction. *Ed. Stud. in Math.*, to appear.

<u>*Readings for the third session*</u>

(distributed via the course web site at the beginning of March; the titles in capital letters are, in revised version, the contents of the present booklet).

3A. Heidi Måsøval: WHEN THE NEGOTIATION OF MATHEMATICAL MEANING IS REPLACED BY THE STRIVE FOR THE FULFILMENT OF THE DIDACTICAL CONTRACT

3B. Morten Misfeldt: SEMIOTIC REPRESENTATIONS AND MATHEMATICAL THINKING: THE CASE OF COMMUTATIVE DIAGRAMS

3C. Claire Berg: WORKING WITHIN A LEARNING COMMUNITY IN MATHEMATICS: AN ANALYSIS OF THE RESEARCH DESIGN AND OF TEACHERS' REFLECTIONS USING FRENCH THEORETICAL FRAMEWORKS

3D. Mette Andresen: INSTRUMENTED TECHNIQUES IN THE DUALITY BETWEEN TOOL - AND OBJECT PERSPECTIVE

3E. Stine Timmermann: UNDERGRADUATES' SOLVING STRATEGIES AND TEACHERS' PRACTICE

3F. Kirsti Kislenko: STUDENTS' BELIEFS ABOUT MATHEMATICS FROM THE PERSPECTIVE OF THE THEORY OF DIDACTICAL SITUATIONS

3G. Marianna Bosch, Yves Chevallard and Josep Gascòn: Science or magic? The use of models and theories in didactics of mathematics**.** *Proceedings of CERME 4, European research in mathematics education*, to appear.

Appendix 3: Schedules of the three working sessions.

4F – programme for the first session.

Thursday, January 6.

10.00-11.00. *Round of presentations*. Each participant will provide a short (about 5 min.) presentation of her/him-self and her/his project. Possibly a few practical matters.

11.00-12.00. *A general introduction to the French school in didactics of mathematics* (interactive lecture relating to texts 1A and 1B, and to the history and current practices of the "school").

12.00-13.00. Lunch.

13.00-13.45. Cognitive theory and didactics : the legacy of Jean Piaget, the theory of conceptual fields (lecture).

13.45-14.45. *Discussion and questions* pertaining to the lecture and text 1E, and potential links to participants' projects.

14.45-15.15. Coffee break (faculty lounge, building D), continuing discussion.

15.15-16.00. Semiotics and didactics of mathematics (lecture).

16.00-17.00. *Discussion and questions* pertaining to the lecture and text 1F, and potential links to participants' projects.

Friday, January 7.

9.30-11.00. *The theory of didactical situations* (lecture and exemplary work with a situation).

11.00-12.30. *Discussion and questions* pertaining to the lecture and text 1C, and potential links to participants' projects. Your 'concept maps' might become handy here!

12.30-13.00. Lunch.

13.00-14.00. The anthropological approach (lecture).

14.00-15.00. *Discussion and questions* pertaining to the lecture and text 1D, and potential links to participants' projects.

15.00-16.00. *Café discussion* (faculty lounge, building D). Over a cup of coffee/tea, we talk more informally on the connections and differences among the four frameworks, and of the organisation of the second session. In particular, I will distribute the texts to be studied before and at the second session. Each text is chosen to be pertinent to one or more of your own projects, and each of them will be accordingly assigned to a participant for in-depth study and presentation at the second session (together with reflections pertaining to the participant's project).

4F – programme for the second session.

Thursday, February 3

10.00-10.15 Short introduction to the second session and its role in the course.

10.15-11.45 THEME 1: Students conceptions of the nature of mathematical statements. Presentation of text 2L (Kirsti), discussion of links to project etc.

11.45-12.30 COMMON THEME: Didactical engineering as a framework for research. Presentation (Carl) based on text 2A et al.

12.30-13.30 Lunch.

13.30-15.00 THEME 2: Research on teaching and learning mathematics in universities. Presentation of text 2G and 2H (Stine), discussion of links to project etc.

15.00-15.30 Coffe/tea

15.30-17.00 THEME 3: Computer algebra systems for teaching differential equations. Presentation of text 2B (Mette), discussion of links to project etc.

Friday, February 4

9.30-11.00 THEME 4: Difficulties of teaching and learning elementary algebra. Presentation of text 2C (Claire), discussion of links to project etc.

11.10-12.40 THEME 5: The nature of mathematical writing processes. Presentation of texts 2M and 2N (Morten), discussion of links to project etc.

12.40-13.30 Lunch at Kroghs cafeteria.

13.30-15.00 THEME 6: Authority in mathematics classroom discourse. Presentation of text 2J (Heidi), discussion of links to project etc.

15.00-16.30 ORGANISED DISCUSSION over coffee/tea: *How to write a good paper.*

4F – programme for the third session.

For each of the paper 3A-3F, the author first gave a 30 minute presentation. This was followed by three reactions of about 10 minutes by two other participants (designated beforehand) and the course teacher, and about 30 minutes discussion of the paper and its theme.

Thursday, March 10

10.00-10.45 PAPER 3G. Introduction (by Carl, short) and discussion (by everyone).

10.45-12.30 PAPER 3A (see appendix 2).

12.30-13.30 Lunch.

13.30-15.00 PAPER 3B (see appendix 2).

15.00-15.30 Coffe/tea

15.30-17.00 PAPER 3C (see appendix 2).

Friday, March 11

9.30-11.00 PAPER 3D (see appendix 2).

11.15-12.45 PAPER 3E (see appendix 2) .

12.45-13.45 Lunch.

13.45-15.15 PAPER 3F (see appendix 2).

15.15-16.00 Final remarks. Evaluation of the course.

WHEN NEGOTIATION OF MATHEMATICAL MEANING IS REPLACED BY STRIVING FOR FULFILMENT OF THE DIDACTICAL CONTRACT

Heidi S. Måsøval

Sør-Trøndelag University College

The object of this paper is to present the analysis of an observation of student teachers' small-group work on a generalization problem in algebra. I begin my analysis by looking at the student teachers' attention to the teacher educator's thinking, at the cost of their own interpretation of the problem. Further analysis deals with the difficulties in changing representation from natural language to mathematical symbols. The work is part of my ongoing PhD study. The analysis is based on Brousseau's theory of didactical situations in mathematics, and a semiotic approach to the problem of algebraic reference, informed by Radford.

INTRODUCTION

The processes of generalizing and justifying in mathematics are often perceived as problematic to students. The research reported in this paper aims at examining a generalization process carried out by three student teachers, who are collaborating on a task designed by a teacher educator in mathematics. When I observed the small-group lesson presented in the paper, I perceived the interaction over lengthy periods as not being productive. Through the close examination of the interaction of the student teachers and the teacher, I got insights into the nature and complexity of the interaction. The objective of the paper is to show how the goal of the mathematical activity for the student teachers becomes the fulfilment of the didactical contract, and how this focus constrains the student teachers' sense making from a mathematical point of view. A better understanding of the phenomena related to the didactical contract is important knowledge for student teachers and teacher educators, as well as for pupils and teachers in school.

THEORETICAL FRAMEWORK

In the episode to be discussed the students[1] are supposed to go from the particular to the general and then to justify a formula for a given pattern using processes of specializing, generalizing, and justifying as elaborated by Mason, Burton, and Stacey (1984). It is relevant that Mason (1996) has pointed, further, at teachers' and students' different comprehension of examples which are intended to illustrate a generalizing process. While a teacher might understand specific numbers and items in an example as placeholders, generic examples, the students interpret them as complete in themselves.

In students' investigation of the general term of a sequence, two main strategies can be identified (Mason, 1996). The first one focuses on the relationship between some terms of the sequence, usually a relation between consecutive terms. In this strategy perception and natural language play an important role. The relation between two consecutive terms can be seen and expressed in natural language, even if not in a stringent way concerning the naming of the terms. The general term is then represented by an implicit or iterative relation. The second strategy aims at an explicit representation of the general term. Here, perception is much less helpful. The production of a symbolic expression for the general term requires that a point of reference is chosen. This point of reference is related to the position of the term in the sequence, which is unperceivable. Radford (2000) refers to this as the positional problem. The analysis of the episode in this paper indicates that the students focus on an implicit relation between terms, while the teacher focuses on an explicit representation of the general term of a sequence.

The students in the actual episode are not driven by the need of justifying a conjecture. The teacher has revealed the connection between the sum of odd numbers and the square numbers, and the task involves representing this relation in terms of mathematical symbols. The students are concerned with answering the questions, and ensuring the use of the teacher's stated connection. Their motive for doing the task is interpreted in terms of fulfilling the didactical contract (Brousseau, 1997).

> "In a teaching situation, prepared and delivered by a teacher, the student generally has the task of solving the (mathematical) problem she is given, but access to this task is made through interpretation of the questions asked, the information provided and the constraints that have been imposed, which are all constants in the teacher's method of instruction. These (specific) habits of the teacher are expected by the students and the behaviour of the student is expected by the teacher; this is the *didactical contract*." (Brousseau 1980, in Brousseau 1997, p. 225)

According to Freudenthal (1973) the goal for mathematics education should be to support a process of guided reinvention in which the students can participate in negotiation processes that, to some extent, parallels the deliberations in the development of mathematics itself. Brousseau (1997) explains what such a process involves and requires when he writes that

> "[a] faithful reproduction of a scientific activity by the student would require that she produce, formulate, prove, and construct models, languages, concepts, and theories; that she exchange them with other people…The teacher must therefore simulate in her class a scientific microsociety if she wants the use of knowledge to be an economical way of asking good questions and settling disputes…" (pp. 22-23)

Brousseau (ibid., p. 30) defines an *adidactical situation* to be a situation in which the student is enabled to use some knowledge to solve a problem "without appealing to didactical reasoning [and] in the absence of any intentional direction [from the teacher]". The teacher's enterprise is to organize the *devolution*[2] of an adidactical situation to the learner. The negotiation of a didactical contract is a tool for this purpose. When the devolution is such that the learners no longer take into account any feature related to the didactical contract but just act with reference to the characteristics of the adidactical situation, the ideal state is accomplished.

A classroom can be said to have an institutionalized power imbalance between the teacher and the students. The analysis of the episode indicates how the students' enterprise is funnelled by the teacher's utterances. Cobb et al. (1997) claim that the teacher's authority can be expressed by initiating reflective shifts in the discourse, such that what is said and done in action can become an explicit topic of discussion. In order to make this possible, the teacher has to have a deep understanding of what is going on in action.

When learners mathematize empirical phenomena differently than expected by the teacher, the didactical contract is threatened. Such a situation may cause a conflict, which can not be solved by pure inferences. Voigt (1994) claims that "[t]his is one reason why mathematical meanings in school are necessarily a matter under negotiation" (p. 176).

It is necessary to ensure that mathematics learners don't restrict their thinking to empirical evidence which is obvious to them. They should develop familiarity with mathematical rationality.

> "Through processes of negotiation of what counts as a reason, the teacher can stimulate the students to develop a sense of theoretical reasoning even if empirical reasons are convincing and seem to be sufficient". (Voigt, 1994, p. 176)

Considering the enterprise of the students from the perspective of the didactical contract, their task is to give a solution to the problem given to them by the teacher, a solution which is acceptable in the classroom context. In this situation the learner acts as a practical person, for whom the priority is to be efficient, not to be rigorous. The aim is possibly to produce a solution, not to produce knowledge. Balacheff (1991) argues that beyond the social characteristics of the teaching situation, we must analyze the nature of the target it aims at.

> "If students see the target as 'doing', more then 'knowing', then their debate will focus more on efficiency and reliability, than on rigor and certainty." (Balacheff, 1991, p. 188)

METHODOLOGY

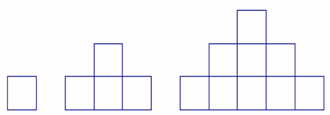
The participants in the research reported are three female students in their first year of a programme of teacher education for primary and lower secondary school, and a male teacher in mathematics. The students are medium-achieving in mathematics. The three students are constituting a practice group, which is a composition of three or four students being grouped together to have school-based learning in a particular class in primary or lower secondary school. At the time the data was collected, they have been collaborating on several tasks in different topics during the five months they have been on the programme. Along with his colleagues, the teacher (who teaches mathematics to the group of students) is concerned about development of relational understanding (Skemp, 1976) for students in mathematics.

The episode described is a video recorded small-group work session at the university college, in which the students are supposed to collaborate on a generalizing problem in algebra. The teacher has designed the task aiming at developing competence in conjecturing, generalizing, and justifying. The data is collected in order to answer the research questions of my PhD project, which is about how mathematical knowledge is authored by the learner, and how mathematical meaning is negotiated through collaboration. I will analyze the episode from the perspective of the didactical contract and of a semiotic approach to the problem of algebraic reference.

DESCRIPTION AND ANALYSIS OF THE EPISODE

Three students, Alise, Ida and Sofie (pseudonyms), are sitting in a group room adjacent to a big classroom, in which the rest of the students are working in groups on the same task for 75 minutes. There are two teachers present in the big room, observing the work of the students, helping them, and participating in dialogues with them. Only one of the teachers, the one who has designed the task, is in contact with the students during the episode described. The teachers are colleagues of mine, and I have also been involved in mathematics teaching in the class. I observe and video record with a handheld camera the work of Alise, Ida, and Sofie. My role is to be an observer and neither to interfere with their work nor to help them. This role is justified and explained to the students as necessary because the data collection should be in as naturalistic a setting as possible. Although my presence in the room, and the video recording, is indeed a disturbance, I try to minimize this as explained.

The first part of the task handed out is printed below (in italics).



Here the first three figures in a pattern are illustrated. You might use centicubes to illustrate.

How many cubes are there in the fourth figure? In the fifth? How many do you think there will be in figure number 10? And in figure number n? What kinds of numbers are present in these figures? In each stripe, and totally in the figure? Can you express, as a mathematical statement, what the figures seem to show? - With words? - With symbols?

The mathematical problem interpreted as fulfilment of the didactical contract

The students have been collaborating for 6 minutes. They have found out that the stripes in the pattern consist of odd numbers, and that each whole figure consists of a square number. They have agreed on $F(n) = n^2$ as a representation of the general term of the sequence of staircase towers, but haven't revealed any connection between odd numbers and square numbers. There is uncertainty connected to the concept of 'a mathematical statement'. Sofie has focused on an implicit relation between consecutive terms in the sequence, asking Alise and Ida if they were supposed to show the increase (from one figure to the next). When the teacher enters the room, they ask him for help.

Excerpt I from transcript[3]

100. Alise: What is this…what is that which you are thinking about…(to the teacher)

101. Ida: A mathematical statement. We have made a formula for…but how do you make a mathematical statement?

102. T: Yes, but a formula is a mathematical statement if it…

103. Ida: Yes, because we have made it with symbols actually.

104. Alise: Yes, that is what we made here (points at her notepad), but with words – shall we tell what it is then?

105. Sofie: Is it for the increase, or is it for one here (about one figure) for instance?

106. Ida: It has to be for the increase.

Twelve turns between the teacher and the students.

119. T: So that…mmm…odd numbers which we build up – and what are we doing, and what is the result? It is such a connection here now. (pause 5 seconds)
Results in *n* squared, as you have said, it results in square numbers, but what do we do in order to make these square numbers appear?

120. Sofie: What do we do? We just square the figure? Or the number of the figure.

121. T: Yes, yes, you do that, when you…but that may not be the most obvious, visual (character) of these towers.

122. Sofie: That you add a line.

123. T: Yes.

124. Alise: That you increase at the ends with one at each side.

125. T: Indeed.

126. Alise: This in order to have this staircase pattern.

127. T: And then we build it line by line…So we are concerned with adding some numbers…in order to get the total number (of cubes)… (hesitantly). Figure number two is one plus three…and the next figure is one plus three plus five…

128. Alise: So you…just increase all the time, so if – there is one (T: mm) – one plus three (T: mmm) – one plus three plus five (T: mmm) – one plus three plus five plus seven (T: mmm) – one plus three plus five plus seven plus nine.

129. T: Yes, exactly.

130. Alise: And like this the whole way upwards.

131. T: And instead of saying this, what could you say that you are doing, in this adding process? …Now you have said it with examples, one plus three plus five plus seven, but what is it you are adding here now? (Ida looks at Alise, then at Sofie's notepad, then she looks at her sweater, before she gasps discretely)

132. Alise: The odd numbers in this series (she has a cheerless facial expression).

133. T: Yes, it is so. Adding odd numbers. And what numbers do you get as an answer? (teacher in an excited voice, Alise strokes her eyes). What kind of numbers do you get as an answer?

134. Alise: Square…What kind of numbers I get as an answer…?

135. T: When you are adding the odd numbers in this way?

136. Alise: Square num…(hesitantly) (Sofie and Ida look down in their notepads)

137. T: Then you get a square number, yes. This is almost a little discovery… (pause 5 seconds)

Which is, at least, such that…if I have asked: What happens if I add – what kind of numbers do you get if you add the ten first odd numbers? (Alise: mmm) If I had asked you this question this morning, then you sure couldn't have answered: Then I get the tenth square number (Alise shakes her head and says: no). So this is nothing which is quite obvious, which you know without any more fuss. This, you can say, is the idea of a mathematical statement; that nobody knows it without any further thinking – it has to be done a piece of work. And that is the process which has been going on here now, which – which is resulting in (the formulation): It actually is like this, that if I add the three first odd numbers I will get the third square number. (Alise and Sofie nod and say: mmm. Ida leans her head in her arm, watching down at the table). Yes. If I add the four first odd numbers I get sixteen, which is the fourth square number, yes oh – connection in the world of numbers in a way. (Ida looks down at the table, nods) It looks like this – as if it's going to be like this.

138. Alise: []

139. Sofie: [] is it just this we are supposed to write, in a way?

Sofie follows up her concern with an iterative formula. In turn 105 she asks if it (the mathematical statement) should be about the increase, and is supported by Ida in turn 106. There are multiple interpretations of what a mathematical statement in this context might look like, and an iterative formula would be appropriate. A mathematical statement could then be formulated for instance as "Square number (n+1) equals square number n, plus odd number (n+1)".

The teacher's intention with the task is the formulation of the fact that the sum of the n first odd numbers equals n^2. The evidence which indicates this is the 'funnel pattern of interaction' (Bauersfeld, 1988) in turns 121-137. The (from the teacher) expected notion 'square number' from Alise in turn 136 brings the teacher to the presentation of the solution of his interpretation of the task in turn 137, in which the teacher accomplishes a monologue lasting for one and a half minute. Here he reveals that the sum of the ten first odd numbers equals the tenth square number, representing in natural language an explicit relation between the position and the representation of the general term of the staircase tower sequence. When the teacher in turn 137 characterizes the outcome of the dialogue as "almost a little discovery", it is an example of the Jourdain effect, which is a form of what Brousseau (1997, p. 25) calls Topaze effect. The Jourdain effect is characterized by the teacher's disposition to "recognize the indication of an item of scientific knowledge in the student's behaviour or answer, even though these are in fact motivated by ordinary causes and meanings" (ibid., p. 26).

Now the task implicitly is reformulated or narrowed, so that the problem is interpreted to be what the teacher originally had in mind when setting up the task: The students are supposed to represent with symbols what the teacher has stated generally; that the sum of the n first odd numbers equals the n-th square number. This process of step by step reduction of the teacher's presumption of the students' abilities and self-government is "quite opposite to his intentions and in contradiction even to his subjective perception of his own action (he sees himself 'providing for individual guidance')" (Bauersfeld, 1988, p. 36).

Excerpt II from transcript

The students are collaborating. The teacher is not present in the group room.

191. Sofie: I'm lost (smiles). It should be n squared and we have to find out what we must do to n in order to achieve this, which fits a number. It is just taking a number then and write it as n squared. And then add and subtract till we are there…(all three laugh).

192. Ida: What is n and what is n squared?

Five turns between the students. Alise writes in her notepad:

$$n+(n-1)=n^2$$
$$3+(3-1)=3^2$$

Alise finds out that the sum on the left side of the ‘equalities’ is the increase from the previous figure to the figure represented by the square number on the right hand side of the ‘equality’.

198. Alise: This is a “two-in-one formula” (they all laugh loudly). Because (by the formula) you find both the increase and how many (cubes) there are totally. Actually it doesn’t make any sense that 5=9 (all three laugh very load). But he (the teacher) *did say* that it was one of the sides! He said that n squared is the right hand side. But this formula of ours doesn’t say anything about adding the odd numbers.

Turn 198 indicates that Alise understands well what is being asked for but not how to get to it. The table below gives an overview of the numbers and variables included in the actual pattern, and might have been helpful for the students when dealing with the problem. Alise considers the terms in the second and fourth column, and until turn 198 she denotes the terms of the same row to be identical.

n	$2n-1$	$1+3+5+\cdots+2n-1$	n^2
1	1	1	1
2	3	1+3	4
3	5	1+3+5	9
4	7	1+3+5+7	16
5	9	1+3+5+7+9	25

The teacher has designed an adidactical situation which presupposes the students’ mastering of a technique of expressing a general odd number. Because the students don’t master this technique, there is a didactical problem. The teacher has not “contrived one [adidactical situation] which the students can handle” (Brousseau, 1997, p. 30); the devolution of the problem specific to the construction of the target knowledge has not worked well.

In turn 119-137 the teacher’s interventions can be seen to follow a kind of Socratic teaching method under the constraints of a didactical contract which faces the teacher with a paradoxical injunction: the more precisely the teacher tells the students what they have to do, the more he risks provoking the disappearance of the expected learning. A paradoxical injunction is also faced by the student: if she accepts that the teacher teaches her the result (according to the didactical contract), she does not establish it herself and therefore does not learn mathematics. If on the other hand she refuses all information from the teacher, then the didactical relationship is broken (ibid., pp. 41-42).

When Alise in turn 100 asks for the teacher's thinking, she takes into account features related to the didactical contract. This, together with the funnel pattern of interaction mentioned above, and the fact that the teacher doesn't succeed in hiding his will and intervention as a determinant of the students' focus and action, causes the collapse of the adidactical situation. Sofie illustrates in turn 191 an important aspect of the didactical contract: It is even more important to come up with the solution expected by the teacher, than making sense of the mathematics involved in solving the problem. The laughter and their facial gestures indicate that they are aware of the detrimental effect of such an attitude. Even if Sofie's utterance expresses despair, it can at the same time be interpreted to be ironic, which can be seen as an effect of the paradoxical injunction faced by the students.

Symbolic narratives and the problem of a point of reference

The challenge of the 'reformulated'[4] task is to transform the mathematical statement represented by natural language into a representation by mathematical symbols. An aspect of the complexity of this transformation is made evident through the problems of representing generality.

Excerpt III from transcript

156. T: This is square number two. (Alise: mmm) If I take the three first ones (writes $1+3+5=9=3^2$), I come out with square number three, and so on. Then I take the *n* first ones, but how should I give the message about this. Then I continue here, and I shall up to…odd number *n*. How shall I express odd number *n*? (he has written $1+3+5+...+\quad =n^2$ in Alise's notepad)

(pause 7 seconds, the students look at Alise's notepad)

157. Sofie: Can't we just take *n* (for the *n*-th odd number)?

158. T: Yes, and then I ask: What is the fourth odd number?

159. Sofie: Eeemm

160. Alise: Seven.

161. T: One three five seven. Seven, yes. Odd number four is seven. Mmm. The fifth odd number is…?

162. Alise: Nine.

163. T: Nine. So odd number *n* is…?

164. Ida: $n+2$?

165. T: Ok, but it is exactly such a structure you have to try to search for now. Odd number *n* can't be *n*, because then odd number four would have

been four. If you say n twice in a sentence, it has the same meaning.
(pause 5 seconds)

166. Ida: But it has to be something with "plus 2" (T: ok?) because it increases with two each time.

In turn 157 Sofie suggests that they represent odd number n by n. The teacher offers a contradiction, but in turn 164 Ida follows up Sofie's suggestion by proposing (n+2) as a representation of the n-th odd number. For Sofie and Ida, the symbols n and (n+2) respectively, function as nouns in a referencing act, not as variables in the pattern. The symbols n and (n+2) appear as narratives, what Radford (2002b) calls symbolic narratives. Ida takes Sofie's narrative as a starting point, and develops it in accordance to the fact that we have to add 2 when we go from one odd number to the next. Ida's point of reference is here seen to be what I call *local*. Her narrative (n+2) is related to the sequence of consecutive odd numbers, which are the focus of attention at the moment. She explains the choice of the narrative in turn 166 when she says: "But it has to be something with 'plus two', because it increases with 'two' each time." The indefinite pronoun 'it' appears twice in the quote, and refers to an arbitrary odd number, the general term of the sequence of odd numbers, which is at stake of turns 158-163. The incompatibility is caused by the choice of n as the number of the n-th figure in the sequence of square numbers built up from sums of odd numbers. This point of reference is what I call *global* and is chosen by the students as they have let $F(n) = n^2$ refer to the n-th figurate number in the sequence, a choice which is followed up by the teacher. The students fail to take this point of reference into account when symbolizing the odd numbers. Therefore their suggestions, n and n+2, remain without link to the general term of the sequence of square numbers.

The above interpretation of the symbols n and (n+2) as symbolic narratives, and a chosen point of reference being local or global, informs the interpretation of Sofie and Alise's responses in turns 122 and 124 in excerpt I from transcript. Turns 121 and 127 indicate that the teacher is concerned with the global act of (constantly) adding odd numbers, building a developing sequence of staircase towers. In turn 127 he is aiming at an explicit formula for the general term of the staircase towers sequence. This focus is not in line with Sofie's local act of expressing the relation from one staircase tower to the next. Sofie's attention is on an iterative formula for the staircase tower sequence, considering one term of the sequence known and then getting the next term by adding a line (odd number). Alise's point of reference is local in a different meaning than Sofie's point of reference. Alise's attention is on an iterative formula for the sequence of odd numbers, considering one term of the sequence known and then getting the next term by adding "at the ends by one at each side" (turn 124).

The different points of reference which the teacher and the students have, is important in understanding the lack of success in the interaction between the interlocutors. The teacher puts a lot of effort in revealing the functional features of the objects in action, and he offers concrete examples aiming at the students' own conjecturing. But the students seem not to be sensible to the teacher's contributions due to the different focus in the generalizing. Their focus is on the iterative relationship between the terms, manifested through the attention to the concept of 'increase', an attention which seems to be ignored by the teacher.

The interpretation of utterances in which generality is represented by natural language, has been informed by insights offered by the analysis of utterances in which generality is represented by mathematical symbols. This indicates the different effects of the two semiotic registers (Duval, 2002), natural language and mathematical symbols. When algebraic reference is manifested in the use of natural language it is more difficult to express and perceive nuances and exactness. The discrepancy between the teacher's and the students' point of reference was easier to perceive when generality was represented by mathematical symbols, as for instance in excerpt III.

Theoretical versus empirical reasoning

The interpretation of the situation from a reasoning point of view is that the students consider the statement that the sum of the n first odd numbers equals the n-th square number, to be truth, and not to be a conjecture. The starting point of the task is the empirical phenomenon of the staircase towers in the task. The statement about the connection between the sum of odd numbers, and the square numbers appears for the students to be an empirical statement, not a theoretical statement. These two types of statements have different rational bases, and this explains why the students don't feel the need to justify the statement, which for them is based in the empirical phenomena. The students mathematize the empirical phenomena differently than expected by the teacher, hence the didactical contract is threatened. Because the students are convinced by the empirical reason offered in the form of illustrations and hands-on material, they don't feel the need to justify the statement in a theoretical sense. This neutralizes the need of generalizing in terms of mathematical symbols, because the motivation of a general statement in terms of mathematical symbols is likely to be driven by the need of the justification of a conjecture (e.g. proof by induction). The teacher wants the students to experience and deal with mathematical rationality, and he suggests theoretical reasoning when he says:

> "If I add the four first odd numbers I get sixteen, which is the fourth square number, yes oh – connection in the world of numbers in a way. It looks like this – *as if it's going to be like this*." (Excerpt I of transcript, turn 137, my emphasis)

This utterance points at the uncertainty of the empirical reasoning. If the students were challenged to negotiate about what would "count as a reason"

(Voigt, 1994, p. 176) in this situation, they were likely to experience familiarity with mathematical rationality.

CONCLUSION

The situation referred to in this paper points at the necessity for the teacher to make an *a priori* analysis of the problem he gives to the students. An a priori analysis of the actual problem could have pointed at the necessity (for the students) of mastering a technique of symbolizing a general odd number. In addition it might have pointed at different interpretations of the request to make a mathematical statement based on the figurate numbers in the pattern (exposing that an implicit formula would be in conformity with an explicit formula).

Negotiation of a didactical contract takes place in a metadidactical situation (see Brousseau, 1997, p. 248), outside the didactical situation, in which the teacher reflects on and prepares the sequence (lesson) he must construct, and the student looks at the teaching situation from the outside. In teacher education the actual episode could be used at a metadidactical level to reflect on the didactical situation; the devolution of the learning responsibility to the students, and the validation and institutionalization of knowing and meaning. Reflection on the didactical situation would contribute to a better understanding of the didactical phenomena (ibid., p. 247) related to the didactical contract (e.g. the Topaze effect), and reflection at the metadidactical level could predict expected outcomes of the didactical contract.

The paper also points at the necessity of paying attention to the fact that a point of reference is to be chosen when handling generalizing problems in algebra. Awareness about and the dealing with the referential problem would probably have improved the negotiation, in the sense that it could have been more productive from a mathematical point of view. Thus, the paper indicates the importance of implementing considerations about the referential problem in an a priori analysis of generalizing problems in algebra.

Notes

[1] I will refer to the student teachers as ‘students’, and the teacher educator as the ‘teacher’.

[2] Devolution is the act by which the teacher makes the student accept the responsibility for an adidactical learning situation or for a problem, and accepts the transfer of this responsibility (Brousseau, 1997, p. 230).

[3] The transcripts are translated from Norwegian by the author.

[4] 'Reformulated' denotes that the task is narrowed to be what the teacher originally had in mind (an explicit formula), excluding the possibility of working on an implicit relationship.

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Appendix (Transcription codes)

B interrupts A or speaks at the same time as A:

A: What I have experienced in the practice field, is that the pupils are so used to achieving one answer, they are so concerned about: Is this right? Is this right?

While/

B: yes yes

A: we perhaps try to focus on the method.

/	indicates that A's utterance continues even when interrupted (or during simultaneous speech) and spans more than one line of transcript
[]	inarticulate utterance
…	pause
italics	emphasis
(text in brackets)	representation of action, explanation of nonverbal action, or comment on utterance or action
T:	the teacher

SEMIOTIC REPRESENTATIONS AND MATHEMATICAL THINKING: THE CASE OF COMMUTATIVE DIAGRAMS

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This paper investigates the role of semiotic representations for mathematical thinking by analysing commutative diagrams. The purpose is twofold; to describe and compare different theoretical perspectives on the use of semiotic representations in mathematics, and to understand how this specific kind of representation support mathematical thinking.

INTRODUCTION

One could think that the role of semiotic representation in mathematical work is first and foremost communicative, because mathematical work is mental work concerned with structures and relations. Nevertheless the archetypical image of mathematical work is a person doing calculations using pen and paper. In this paper I use the framework of Raymond Duval (2000, 2002) together with ideas from Heinz Steinbring (2005) and Luis Radford (2001, 2002) to describe what role semiotic representations play when working with mathematics and for the development of mathematics as such.

The discussion focuses on a special kind of representations, namely commutative diagrams. The reason for focussing on this type of representations is that it combines iconic aspects of drawings with a precise syntax and hence a-priori occupies a place between discourse and drawings, a place that could be especially interesting in connection to mathematical writing.

COMMUTATIVE DIAGRAMS

Commutative diagrams are used to denote mappings between sets in mathematics. So for instance the diagram in figure 1 denotes that f is a mapping from X to Y, g a mapping from A to B, φ one from X to A and ψ a mapping from Y to B. Stating that the diagram is commutative is saying that $g \circ \varphi = \psi \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

Figure 1: A commutative diagram

For many mathematicians commutative diagrams functions as a convenient way to denote systems of maps and spaces. In some mathematical disciplines (e.g. homological algebra, algebraic topology, and other functorial disciplines) this notation is used extensively, not only for establishing an overview of a system of maps and spaces, but also to introduce mathematical objects and to perform calculations. See Hatcher (2002) or Bredon (1993) for further introduction to commutative diagrams.

SEMIOTIC REPRESENTATIONS IN MATHEMATICS

Raymond Duval describes "the paramount importance of semiotic representations" (Duval, 2001, p. 2). His main claim is that the only access we have to mathematical objects is through semiotic representations but that these objects should not be confused with any semiotic representation of them (Duval, 2001, p. 7), because any mathematical objects has several different semiotic representations and these representations are qualitatively different. Duval points to that many leaning difficulties in mathematics can be analysed focusing on transformation of semiotic representations. Duval describes two qualitatively different types of transformations of semiotic representations, *treatments* and *conversions*.

1. Treatments are transformations inside a semiotic system, such as rephrasing a sentence or isolating x in an equation.
2. Conversion is a transformation that changes the system, maintaining the same conceptual reference, such as going from an algebraic to a geometric representation of a line in the plane.

A semiotic system that allows for treatments inside the system and for conversion of sign to signs in another system referring to the same conceptual object is designated a "register".

Duval shows empirically (2000 figs. 4 & 5) that conversions in some situations are very difficult for students. The conversions that seem to be easiest for students are the ones that are congruent, meaning that the representation in the starting register is transparent to the target register (Duval, 2000, pp. 1-63), an example of congruent conversion is when a sentence in natural language can be translated into an algebraic expression keeping the order of signs just translating each word to its similar algebraic symbol.

The relation between semiotic representations and conceptual understanding can then be described with the following figure.

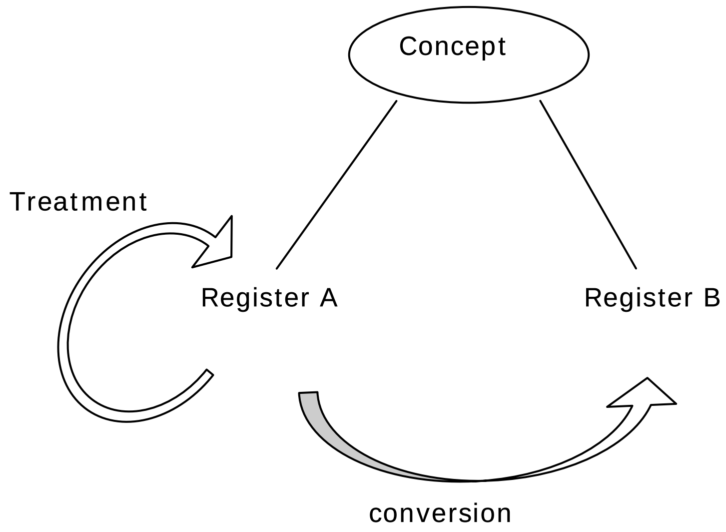

Fig 2: from Duval (2000) figure 6

Conceptual understanding can then be described as a person's degree of freedom towards various semiotic representations of the same mathematical concept (Winsløw 2003).

Duval divides registers into four categories depending on if they are Discursive, non-discursive, multifunctional or monofunctional. Discursive Registers are registers where it is possible to perform valid deductions, and for all practical purposes these types of registers are linear in some form. Non discursive registers are typically geometrical. Multifunctional registers are register used in many fields of culture whereas monofunctional registers are more technical registers used for a narrow purpose. The following table, taken from Duval (2000, figure 7) exemplifies these types of registers.

Empirical grounding of mathematical signs

One could see the framework of Duval as controversial in the sense that mathematical concepts seem to be connected only to semiotic representations. Duval does not discuss references of non semiotic nature.

Michael Otte (2001) explains that 'The ultimate meaning or basic foundation of a sign cannot be a sign itself; it must be of the nature either of an intuition or of a singular event' (Otte, 2001, p. 1), but simultaneously he stresses that 'A mathematical object, such as 'number' or 'function', does not exist independently of the totality of its possible representations, but it must not be confused with any particular representation, either' (Otte, 2001, p. 3).

	Discursive representations	Non-discursive representations
Multifunctional registers (processes cannot be made into algorithms)	Natural language Verbal (conceptual) associations Reasoning: -arguments from observations, beliefs… -valid deductions from definitions or theorems	Plane or perspective geometrical (configurations of 0,1,2 and 3 dimensional forms) Operatory and not only perceptive apprehension Ruler and compass construction
Monofunctional registers (most processes are algorithmic)	Notation systems: Numeric (binary, decimal, fractional…) Algebraic Symbolic (formal language)	Cartesian graphs Changes of coordinate systems Interpolation, extrapolation

Table 1: types of registers (Duval, 2000 fig 7).

This conflict is addressed by Heinz Steinbring (2005) with the introduction of what he describes as the *epistemological triangle* connecting concepts not only to the signs that represents them but also to a 'reference context'. The reference context consists of the actual processes and concrete objects, but Steinbring notes that there in some cases is exchangeability between the reference context and the sign/symbol. This is obvious because many/most advances mathematical problems and results are formulated purely symbolic since the objects in play are only perceivable through semiotic representations. What Steinbring points out is that signs can serve as objects constituting a reference context in some cases and as representation for a mathematical concept in other.

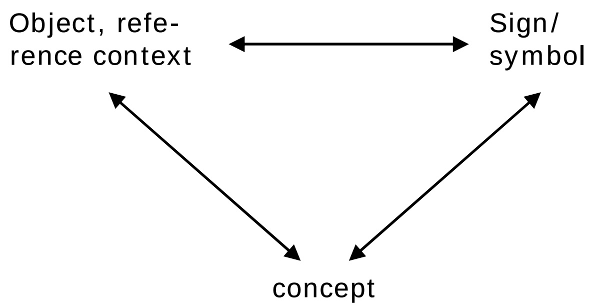

Fig. 3: The epistemological triangle (Steinbring 2005).

In recent work Luis Radford (Radford, 2002) has focused on the creation of meaning by looking at the role of the physical environment such as inscriptions, technologies, utterances and gestures in mathematical objectification. He takes a semiotic outset in his exploration and he describes an important aspect of the environment with the concept *semiotic means of objectification* that describes the important role of deictic signs in objectification. The idea is that the process of designating and pointing to abstract object is crucial to conceptual understanding.

Background and focus in mathematical symbol manipulation

Luis Radford (2001) considers the interplay between the grounding or designation of mathematical concepts and concrete operations carried out on the symbols designating these objects. The context of his investigations is lover secondary school children working on word problems.

He describes the kind of 'flexible' thinking that is needed to move back and forth between mathematical registers but also inside mathematical registers, as he states in the following quote.

> "Indeed, in the designation of objects, the way signs stand for something else is related to the individuals' intentions as they hermeneutically unfold against the background of the contextual activity. In the designative act, intentions come to occupy the space between the intended object and the signs 'representing' it. In doing so, intentions lend life to the marks constituting the corporeal dimension of the signs (e.g. alphanumeric marks) and the marks then become signs that express something, and what they express is their meaning. The possibility to operate with the unknown thereby appears linked to the type of meaning that symbols carry." (Radford, 2001)

In order to compare these ideas with the ideas from Raymond Duval we can represent them in a diagram similar to figure 2. The point with the diagram is that the designation of meaning is connected to the entire process consisting of both the translation from natural language to algebra and the work in the algebraic register.

Luis Radford (2001) describes how students interact with word problems of the type:

> "Kelly has 2 more candies than Manuel. Josée has 5 more candies than Manuel. All together they have 37 candies."

Given this narrative the students should express the number of candies that each of the children have. The students solve the task three times designating *x* to the number of candies the different children have.

In his essay Radford uses the narrative metaphor of 'heroes' to describe the focus point of the students activities, and how it evolves when the students move from working with a story about three children that have some candies to working with solving equations. In the story the apriori heroes are the three children. Persons are the usual heroes in stories, and in this case what we are interested in is connected to the children in the sense that it is the amount of candies assigned to each of them. Nevertheless in the move to algebra there is a problem. The "hero" changes from being the persons (or their candies) to being the relations between the numbers of candies described in the story. While the symbol, *x*, is assigned to one of the old 'heroes'. Radford makes several interesting observations.

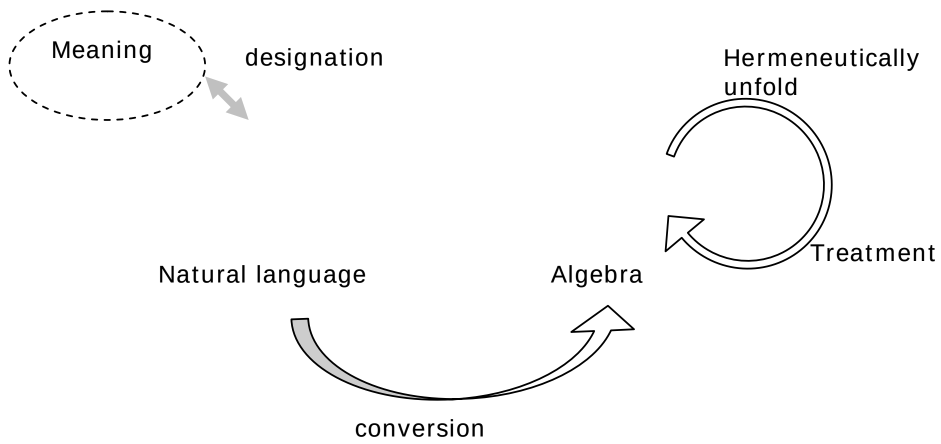

Fig. 4: An attempt to describe Radford's idea of designation of meaning, compare to fig. 2.

Some students translate the problem to algebra directly more or less maintaining the person heroes. This leads to misunderstanding or miscalculation. Radford describes one example where a student wrongly translates "Kelly has two more than Manuel" to x+2=Manuel, where x is supposed to describe the number of candies for Kelly. As a narrative it actually makes some sense to translate "Kelly has two more" to "$x+2$" if x is Kelly.

In the language of Duval this misunderstanding can be described as the student's attempt to do a conversion not respecting the non congruent nature of this specific conversion.

From the more narrative viewpoint taken by Radford other problems are apparent, for instance that the designation "x is Manuel" (introduced by the teacher) strongly influences some of the children's abilities with respect to treatment in the algebraic register. An example is one student having very strong emotional response to the innocent treatment going from: (x+2)+(x+5)+x=37 to 3x+7=37. This treatment totally collapses the narrative. In the first formulae one can see the relations between the number of candies the children have and in the second one this is entirely gone. (x+2), (x+5), and x becomes signs designating each of the three children. Of course it is an important and unavoidable aspect of mathematical work that you do this transformation (in this case from question to answer), but what Radford's analysis points out is that in order to use the flexibility of the mathematical system one has to be able to temporarily let go of the meaning assigned to symbolic expressions, or collapse the narrative.

The point here is that the cognitive struggle with the many representation forms in mathematics is not only an inherent feature semiotic systems that is in play, but the meaning that students assign to the symbols, and with that the narrative framing the problem, is crucial for understanding students struggles with mathematics.

Radford's work explains us several things; first and foremost the heroes change when the representation form is changed, and secondly the meaning that different persons designates semiotic representations in mathematical discourse is affects what treatments and conversions that are possible for that person.

MATHEMATICAL THINKING AND COMMUTATIVE DIAGRAMS

In this discussion I will use the framework of Raymond Duval together with some ideas from Luis Radford to describe the cognitive advantages, and challenges, of using this representation. It should be mentioned that the use of commutative diagrams is considered hard to master in the mathematical community, at the same time these diagrams are considered powerful for some aspects of mathematical work.

The register of commutative diagrams

It is worth noting that commutative diagrams actually constitute a register, as defined in the section "semiotic representations in mathematics". This implies checking whether it is possible to perform Treatment inside the system of commutative diagrams and if conversions to another register reveal qualitatively different aspects of the conceptual objects involved.

One example of a Treatment taking place inside the systems of commutative diagrams is to state that there exists a 'lift' in a diagram like fig 1. That is; there is a mapping *h* from A to Y, such that the new diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ A \arrow[r, "g"] \arrow[ur, "h"] & B \end{tikzcd}$$

Fig 5: a lift in the diagram

That commutative diagrams reveal qualitatively different aspects of the systems of maps they describe is actually evident from the same example, since the property that a mapping is a 'lift' is not a natural property in a purely algebraic description of the system of maps. Actually, the lifting property has a geometric origin (see for instance Bredon, 1993) but the nature of the problem is still reflected in the more formal diagrams, and this is an example of the strength of this register.

In order to study the conversions between commutative diagrams and other registers we can attempt to express the mathematical content of the diagram in another way. The relations in the diagram in figure 5 can be expressed as:

$f{:}X\to Y,\ g{:}A\to B,\ \varphi{:}X\to A,\ \psi{:}Y\to B,$ and $h{:}A \to Y,$ such that $g\circ\varphi= \psi\circ f,$ and $h\circ \varphi=f.$

The target register for a conversion from a commutative diagram is a mixture between logical uses of natural language and symbols, and I don't think that the conversion between these two registers is congruent in either direction. The main reason is that the diagrams uses the non linear nature of a two dimensional figure very active, and for instance there is no obvious order of the propositions in the linguistic/symbolic register, the target for the conversion could just as well be:

$f{:}X\to Y,\ \varphi{:}X\to A,\ g{:}A\to B,\ \psi{:}Y\to B,$ and $h{:}A \to Y,$ such that $h\circ \varphi=f$ and $g\circ\varphi= \psi\circ f.$

Conversely the above linguistic/symbolic expression does not show exactly how to create the diagram in fig 5.

If commutative diagrams constitute a register, it should be possible to describe if this register is monofunctional or multifunctional and also if it is discursive or non discursive.

To check if the register is monofunctional is the same as determining if all processes/treatment can be made into algorithms. One can say that this register is as monofunctional as a symbolic register denoting the spaces and mappings.

If the register is discursive or not is more peculiar because on one hand commutative diagrams have many of the properties of discursive registers; there are rules and syntax, and they can definitely play an important part in valid deductions, but nevertheless this register is definitely not linear. Commutative diagrams uses two dimensions as well as for instance algebraic notations do (fractions is an example of that) but I commutative diagrams this two dimensional properties are in a sense more radical. A fraction can be "spoken aloud" or written in a linear form in a simple, congruent form. As we saw above conversions from commutative diagrams to a linear register tends to be non congruent.

The heroes of commutative diagrams

The diagrams are designed to show relations between sets or spaces. This means that this notation introduces a new hero in the study of maps between sets or spaces. This new hero is not an element of the sets, as they are transformed by the described maps, neither the maps f, φ, g, ψ or the sets A, B, X and Y but the entire amount and shape of relations between the sets in the diagrams.

In the middle of last century a formal theory for describing the "new hero" was developed (Eilenberg & Mac Lane, 1945). This new theory is called "Category Theory" and takes its outset in diagrams – forgetting what the elements and arrows in the diagrams designate (Marquis 2004, Mac Lane 1971).

An educational point from Luis Radford (2001) is that the experienced educational challenges with introducing commutative diagrams could be due to this change of hero. It is necessary to move back and forth between many different heroes; sometimes it is crucial to know and focus on concrete aspects of a mapping (as for instance to calculate what is $f(a)$) but at other points it is equally important to be able to forget such issues.

Commutative diagrams as a grounding context for mathematical objects

Having a way to designate systems of maps gives a new symbolic context. Following Steinbring (2005) this should allow for new concepts to develop. Without performing a larger historical analysis this definitely seems to be the case. In the years after the introduction of notation based on diagrams there was an enormous development in algebraic topology. The ability to express a large, or even infinity amount of relations, for instance between maps and spaces, allows for other phenomenon's to be designated. A very simple example is the lift from above, the introduction of Category Theory another, but there are many other examples.

CONCLUSION

In this paper I have looked at commutative diagrams as an example of a semiotic representation used to support mathematical thinking. I have shown that commutative diagrams constitute a register in the sense given by Duval. This register seems to be monofunctional and discursive, but not linear. Furthermore conversions between commutative diagrams and linear registers generally seem to be non congruent, and hence difficult to make.

I have also speculated to what extend commutative diagrams gives a new focus in working with relations between sets and maps between them, and that this new focus can allow for other objects to be designated.

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WORKING WITHIN A LEARNING COMMUNITY IN MATHEMATICS: FOCUS ON TEACHERS' REFLECTIONS EMERGING FROM A MATHEMATICAL TASK

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This article presents the main features of my ongoing research concerning the possibility to enhance the teaching and learning of elementary algebra. It introduces the notion of a learning community and proposes an analysis of teachers' reflections, when engaging in a mathematical task, using Duval's semiotic approach. Some preliminary results of this research show that teachers' awareness of the complexity of the learning situation and of mathematical reasoning is increasing.

INTRODUCTION: RESEARCH FOCUS

Recent research shows a growing interest in the possibility to develop mathematics learning and teaching through the creation of *communities of inquiry* (Jaworski, 2004a; Cestari et al., 2005). Furthermore the development of learning and teaching is closely related to the development of teachers' thinking and to the opportunity of engaging teachers in reflections concerning their practices (Calderhead, 1987; Jaworski, 1994; Jaworski, 2004b). In this study a group of three teachers (lower secondary school) and a researcher (myself) are working together and reflecting on mathematical tasks related to algebra and on the possibility of implementation of teachers' reflections in the classroom, following a six-level developmental and analytical model as presented below. The aim of the research is to look at the way in which developing a learning community can lead to a deeper understanding of algebraic thinking. More precisely I am looking at the creation, the development, and the dynamics in our group. The notion of *learning community*, rooted in Wenger's (1998) notion of *community of practice*, is introduced in the next paragraph. Through the data presented in the article the reader can follow particularly one teacher, Paul, in his reflections and the way he makes sense of a mathematical task concerning even and odd numbers. In relation to the analysis of these data, I particularly see Duval's (2000, 2002) cognitive approach as a fruitful theoretical approach. In the following a brief account of the theoretical perspective underpinning the notion of learning community and Duval's approach is introduced. The research project is then introduced and finally an example from my own research shows in more detail the usefulness of Duval's theoretical framework.

The notion of *learning community*

The notion of learning community, central to this work, is rooted in Wenger's (1998) notion of *community of practice* which is characterized by the following three dimensions: 1) mutuality of engagement, 2) accountability to the enterprise and 3) negotiability of the repertoire. Building on Wenger's community of practice, the notion of *community of inquiry* offers the possibility to explore the reflexive relationship between inquiry and development, where development implies both learning and deeper knowing (Jaworski, 2004a). The distinction between the notions of *community of practice* and *community of inquiry* is, according to Wells:

> However, there is a further feature of a community of inquiry that distinguishes it from most other communities of practice. And that is the importance attached to metaknowing through reflecting on what is being or has been constructed and on the tools and practices involved in the process. (Wells, 1999, p. 124)

I see a *community of inquiry* as a particular type of a *learning community*, in which the focus is placed on reflection and development, as defined in Jaworski, for the four participants (the three teachers and the researcher) of this community. In this project, the nature of the learning community can be defined in the following way: certain tasks are created or found by the researcher, through which the community can think about and address algebra. My work, as a researcher, is to study the development of the four participants emerging from the way the community is engaged in some activities.

Duval's semiotic approach

As explained in the introduction a cognitive approach allows the researcher to follow the process of teachers making sense of the mathematics. Duval's semiotic approach (Duval, 2000) is based on the acknowledgment that a cognitive approach is necessary in order to understand the problems and difficulties that many students have with comprehension in mathematics. It is not possible to give here an extensive insight in this theory (for more detail see Duval, 1995; Winsløw, 2004) but I will particularly look at the notions of *semiotic register* and at two types of transformations of semiotic representations: *treatments* and *conversions*. A semiotic register is a semiotic system that allows first the representation of an object (related to mathematics the object can be a circle, or a function), second the transformation of the representations following the rules specific to that system which is referred to as treatment, and third the conversion of representations in one system to an other system (Duval, 1995, p. 20-21). Using Duval's words:

> Treatments are transformations of representations which happen within the same register: for example, carrying out a calculation while remaining strictly in the same notation system for representing the numbers, solving an equation or system of

equations, completing a figure using perceptual criteria of connectivity or symmetry, etc.

> Conversions are transformations of representations which consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc. (Duval, 2002, p.4)

The data presented later in this article give more details concerning the notion of conversion. According to Duval the characteristic features of mathematical activity are:

> …. the simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another. (Duval, 2002, p.3)

and

> …. comprehension in mathematics assumes the coordination of at least two registers of semiotic representation. (Duval, 2002, p. 3)

The research project

In the study a six-level developmental and analytical model is described with different layers of teachers' reflections emerging from this model. The model consists of the following steps: 1) at the first level, mathematical tasks related to algebra are proposed to the teachers and the researcher observes the way they cooperate solving these. 2) during the second level, the emphasis is on the reflections emerging from this process. 3) the next step is to observe how teachers plan what kind of tasks they can offer to their pupils in their respective classes in order to foster the same kind of reflections that they experienced in level 2. 4) at this level, the researcher follows each teacher in his/her class in order to observe the possible implementation of the reflections in the practice of each teacher. 5) here the focus is on teachers' evaluations of and reflections on the teaching period. 6) finally, the last level gives the possibility to the three teachers and the researcher to exchange reflections in common after the observation in class. One of the central features of the design of this research is the creation and development of tasks which may provoke teachers' reflections and enhance their awareness about the teaching of algebra.

As research questions, I ask, what is the nature of teachers' reflections, and how do these relate to the creation of a learning community? How are these reflections implemented in the teaching of algebra, and what issues does implementation raise for the teachers? Data from one of our workshops address the question of the nature of teachers' reflections. Issues concerning the implementation of these reflections in the teaching of algebra will not be addressed in this article, but data related to this question will be analysed at a later stage.

METHODOLOGY AND DATA COLLECTION

This study follows the design-based research paradigm and according to Kelly (2003) research design can be described as *an emerging research dialect* whose *operative grammar* is both *generative* and *transformative* (p. 3). It is both generative by creating new thinking and ideas, and transformative by influencing practices. This new research approach addresses problems of practice and lead to the development of *usable knowledge* (The Design-Based Research Collective, 2003, p. 5). According to Wood and Berry (2003), design research can be characterized as a process consisting of five steps: the creation of physical/theoretical artefact or product; an iterative cycle of product development; the deep connection between models and theories and the design and revision of products; the acknowledgment of the contextual setting of development and the fact that results should be shareable and generalizeable; and the role of the teacher educator/researcher as an interventionist rather than a participant observer.

In the six-level model as described above, mathematical problems are found/created by the researcher as a means to provoke teachers' reflections concerning algebra, and at the same time the tasks allow the community to work together. In other words the problems are an instrument both to the development of algebra and to the building of the community. The focus in the iterative cycle is on the development and refinement of teachers' reflections and on the possible implementation of teachers' reflections in their respective practices. The plan in this study is to follow the teachers during one year. All the workshops and classroom observations are audio-recorded. Interviews with teachers both before and after class are also audio-recorded.

THE MATHEMATICAL TASKS: AN *A PRIORI* ANALYSIS

One of the central aspects of the research consists of the choice of the mathematical tasks which I present to the three teachers. The choice of the tasks relies on the following criteria:

the task is easily understandable in order to motivate and engage all participants

the task can be solved by using different approaches, at least in an algebraic way

the task offers the opportunity to widening and deepening of mathematical understanding with focus on algebra

the task may offer some insight in the history of mathematics and in this way it encourages the community to see mathematics as a continuous process of reflection and improvement over time, and provides an opportunity for developing participants' conception of what mathematics is.

Items 1 and 2 are fundamental and present in every task. The third item is fulfilled in almost all the tasks that have been presented until now. The last item has been taken into account in a few tasks.

After some workshops where I took the responsibility for the problems, the teachers were invited to present some task for the group. In this way the teachers are invited to take over the responsibility for the preparation and the fulfilment of some workshops. I consider this step as an important development within the community building and it allows to progress to equality between participants.

A mathematical task concerning Odd and Even numbers

Our group have meetings approximately once a month during the evening. We meet at one of the school of the teachers, use the teachers' meeting room, and work together for about two hours. The three teachers (Mary, Paul and John) and I sit around a table and we have the possibility to use a flip-chart. Mary and Paul work at the same school with pupils at grade 9 (13-14 years). John works in another school with grade 10 (14-15 years). All the dialogues during our workshop are audio-recorded. Our meetings have the following pattern: first a mathematical task is proposed and we try to solve it. Second the teachers are invited to reflect on the way the task has been solved and on the possibility to use similar tasks in their class or to implement similar reflections during their teaching. The task, as exposed below, was presented during our second workshop.

The following mathematical task is proposed to the teachers:

- What happens when we add even and odd numbers?

After some clarification concerning the meaning of the task, like do we have to take only the sum of even and odd numbers or can we also add even and even, odd and odd numbers, the teachers took some minutes to write down some examples with specific numbers.

The following dialogue occurs shortly after the teachers tried several numerical examples:

Paul: it depends on how many numbers you take, if you have two or three

Researcher: two or three what?

Paul: yes, either even numbers or odd numbers, what ever it is, then the result will change

Researcher: can you go a little deeper?

Paul: yes, so if you put just together even numbers, so it will be, you will never see odd numbers, but if you put together odd numbers then it depends on how many numbers you take, if you take even of odd numbers (laugh) to put it that way

Researcher: an even number of odd numbers?

Paul: yes, (several are laughing) and for an odd number of odd numbers, the result will then be influenced!

Researcher: ok, then you got this (result), (to the other participants) do you agree, disagree?

John: at once, it seems that this is the pattern that …

The teachers were engaged to reflect on a particular task and they first tried out some special numerical cases or specialised the task probably in order to get some sense of what the task is about and by doing this they experience that concrete numbers can be manipulated with more confidence than the words used in the question, as the need for some clarification of the question shows. On this other hand, the conclusion expressed after this individual exploration is a generalisation of their examples. By trying several examples a specific pattern emerges which is expressed in their dialogue. Nevertheless the words and expressions needed to formulate this generalisation become quickly unclear and sound more like words-game (even number of odd numbers). Looking at this short excerpt from Duval's perspective, the analysis shows the transition between the representation of the task in one semiotic register (the use of natural language when the task is proposed) to the representation of *the same task* using another semiotic register (the use of natural numbers in the phase of specialisation). This illustrates the type of transformation referred as *conversion*, as exposed earlier. Finally the conclusion is expressed through a new conversion from natural numbers back to natural language. Two semiotic registers have been activated: the natural language and number systems. However a third register consisting of formal notation is needed to express the conclusion: the fact that the teachers are laughing while they try to formulate their results shows that the meaning of the conclusion disappears in a kind of phonetic game and it has to be re-formulated in another way. In the following excerpt one of the teachers, Paul, writes on a flip-chart, trying to re-formulate the generalisation by using his own formal notation. These notation are written in parentheses and are inserted in the dialogue as they were written during the workshop:

Paul goes to the flip-over and writes:

(e. n + e. n = e. n)

Researcher: yes, this is the first one

Paul: so, if you have, hmm,

John: odd numbers

Paul: odd numbers, yes, then it's depends on the number

(o. n + o. n = e. n)

Mary: yes, but the way you wrote it now, then it is …

Researcher: yes, and I think you said something about an odd number of odd numbers

Paul: yes, then I have to look at …

Researcher: yes, you can write it down, for example

Paul: yes, if you, (pause), if you write, yes, …

Researcher: if it is difficult to write in general, take three (odd numbers) as an example

Paul: yes, then, (pause)

(o. n + o. n + o. n = o. n)

Paul: and if you take this, like this, and we can write, if you say plus two, then (unclear)

(o. n + e. n = o. n)

Researcher (to the other participants): do you agree?

Mary: yes, if you had odd number, yes, let me see, even number plus odd number plus odd number …

Researcher: even number plus odd number plus odd number?

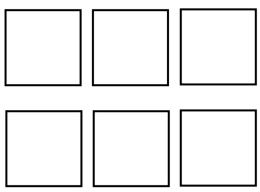
Mary: isn't it even number (laughing)?

The aim of introducing formal notation was to clarify the previous conclusion stated in natural language. Two aspects of this transition between these different semiotic registers have to be underlined. The first one concerns the notation introduced by Paul. Even numbers are designed as "e. n" and odd numbers as "o. n". This means that his notation consist of taking the first letter from the two words and in this way they function more like abbreviations then as formal notation. The nature of his notation is a mix of natural language and formal notation. Therefore the efficiency of the introduced notation is reduced and they may cause misunderstandings. The use of letters to refer to unknowns is also addressed in Jaworski (1988) where pupils try to generalize activities with Cuisenaire rods and experience the limitation of their own notation. Paul's notation may be understood as "letter used as an object" which means "the letter is regarded as a shorthand for an object or as an object in its own right" (Küchemann, 1981). The second aspect concerns the difficulty to "translate" the sentence *an odd number of odd numbers* into notation. Here the notation has to pinpoint both what kind of numbers is considered and how many of them we have to add. As mentioned above the notation "o. n" corresponds to the kind of numbers under consideration, but to re-formulate, in general, how many we can take is too difficult and therefore an alternative is to consider three odd numbers as a particular case of an odd number of odd numbers (o. n + o. n + o. n = o. n). Here the transition from one semiotic register (natural language with the words *an odd number*) to another (formal notations as for example *2n +1*) is not fulfilled and this fact underlines the problems related to the step of conversion. These difficulties concerning the process of generalisation are resumed by the teachers in this way:

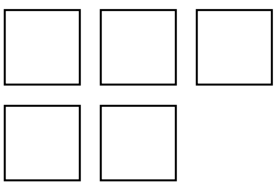
> Paul: then you touch what is called a mathematical proof and mathematical (unclear) and here I must admit that I am not good at all, to make it clear, I can't deduce general results from things that seems to be like this
>
> Mary: I think this was terribly difficult

This task has until now involved three different semiotic registers, natural language, number systems and a kind of formal notation, and the analysis has shown both the possibilities and the limitations of each of the semiotic registers that have been used.

The possibility to use a fourth register involving geometric figures is offered by the researcher with the use of manipulatives. Even and odd numbers can be represented by using small squares in the following way:



An even numbers is represented with this kind of arrangements of manipulative. The aim is to focus on the *shape* of this arrangement. All even numbers *can* have a rectangular shape.



An odd number is represented with this kind of arrangement. In this case the shape looks like a rectangle with one extra square on the top or bottom row. Odd numbers *cannot* have a rectangular shape.

The goal of this activity is to illustrate how geometric figures can be used to deal with some problems involving even and odd numbers and to discover properties of these numbers under addition. In this part of the workshop the possibilities to connect numbers and figures was examined:

> John: yes, but what we really talk about here is the transition from being a (number) seven to being a odd number, this is this transition here
>
> Researcher: to say that all odd numbers can be represented in this way?
>
> John: yes
>
> Researcher: yes
>
> John: and then, yes, if we have an odd number here, then we can begin to count, and then perhaps we have another here, isn't, here it is, and so on isn't? Then they (the pupils) can see that this get together to an even number, but from here to get a long series (of numbers) and have an understanding of it, this … (pause) They will probably see it with these manipulatives which they can count, and then take some elementary connections, oh, yes, an odd number will look like this

This short excerpt shows another level in teachers' reflections. Their experience and sensitivity to the students becomes visible and they question the possibility and the consequences of introducing this task in their class. A way of doing this would be to focus on the *shape* of the number:

> John: to take them [the pupils] away from counting! Just look at the shape, is it an even or odd number? This is what we are looking at now, look at the shape, look at the shape!
>
> Mary: but then it is that they are so preoccupied in mathematics that everything is about numbers, if it was in Norwegian, yes, if it was a noun or a verb, then they have a name on it, but in mathematics it is an answer they want, something concrete, to sum many times
>
> Paul: they are very obsessed with not the values or numbers, but what, when you talk about, about algebra, but when there is an "a", what is that "a"? It symbolizes a number, I say, yes, but what number then, no, I don't know that, yes, but how can we calculate something when we don't know what it is? So they are very dependent on, in a way, to know that it is like this, this, this.

Issues concerning the transition between numbers and geometrical figures and between arithmetic and algebra are underlined here. John's first utterance "to take them away from counting" shows the difficulty to introduce different tasks in the class. The students are used to count and have done this activity in many years. Now the goal of this activity is to look at the shape of a number in order to decide if it is an even or odd number. Teachers' reflections concerning this shift in attention reveal that it could be difficult to plan this kind of activity without stressing that the focus is on the shape no longer on counting. Through this discussion the teachers develop their awareness of the complexity of both introducing different tasks and the learning situation. The introduction of this fourth semiotic register has to be done with stressing the reason for it (look at the shape) and the goal (geometric pattern with addition).

THE MATHEMATICAL TASKS: AN *A POSTERIORI* ANALYSIS

As I see it the problem" what happens when we add even and odd numbers?" fulfil the first three criteria from the *a priori* analysis. It is easy to understand, it can be explored from different approaches (with specific numbers, algebraic notation and geometrically), and it offers the opportunity to widening and deepening mathematical understanding. The analysis shows that Paul's notation was inappropriate to generalization, but the use of geometrical representation is justified as a means to fulfil the shift from informal notation, which is not generalizable, to conventional notation, which is. In other words, it is possible to move from the informal abbreviations notation, through the geometrical notation where the pattern $2n$ and $2n+1$ is actually visible, to a more conventional algebraic notation that will apply more generally.

Conclusion

This article presents the main features of my ongoing research in which the notion of learning community is deeply connected to teachers' development of algebraic thinking. The data present a group of three teachers and a researcher working together on a mathematical task concerning even and odd numbers. The focus is on one particular teacher, Paul, and the way he works through the task, his way of making sense of it, and the difficulties encountered with the articulation of the conclusion. Furthermore the analysis reveals the use, gives an explanation, and shows the limitations of four different semiotic registers introduced during the workshop. Here Duval's notion of conversion is useful in the identification, analysis and understanding of Paul's difficulties. In this way Duval's cognitive approach offers a powerful theoretical framework to analyse individual teachers' sense making of the proposed task.

The analysis also reveals the different layers in teachers' reflections, from reflections related to the mathematical task and ways of exploring it, to reflections connected to the possible use and the consequences of the implementation of the task in their own practices. These results give evidence for teachers' sensitivity for their students and underline at the same time the simultaneity of the different levels in their reflections.

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INSTRUMENTED TECHNIQUES IN THE DUALITY BETWEEN TOOL - AND OBJECT PERSPECTIVE

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This paper analyses an episode from a case study of students' use of instrumented techniques for solving differential equations in a CAS environment. The aim is to study what role use of the actual techniques plays in the students' learning process. The analysis demonstrates how tasks can be designed for encouraging the students to change between the perspective of tool on the mathematical conceptions and the perspective of object on the conceptions. Finally, it is outlined how changing between these two perspectives may support the instrumental genesis as well as the conceptual development.

INSTRUMENTAL GENESIS AND INSTRUMENTED TECHNIQUES.

The French theory of instrumental genesis is based on the idea that an artefact, for example a CAS calculator, does not in itself serve as a tool for the student. It becomes a tool, referred to as an 'instrument' in this French notion, only by the student's formation of (one or more) mental utilisation scheme(s). Such utilisation schemes connect the artefact with conceptual knowledge and understanding of the way it may be used to solve a given task. Thereby, the schemes contribute to the formation of instrumented action schemes. The term 'instrumental genesis' denotes the process in which the artefact becomes an instrument. (Drijvers & Gravemeijer 2005 pp 165-169). The formation of utilisation schemes and the building up of instrumented action schemes proceed through activities in 'The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking'. (ibid. p 190). In the notion of instrumental genesis the process directed toward the student is called 'instrumentation' and the process directed toward the artefact is called 'instrumentalisation'. Drijvers and Gravemeijer suggest the notion of 'shaping' and 'being shaped' instead. I agree that such change is very likely to promote the accessibility of the instrumental approach.

This outline of the theory of instrumental genesis reveals the underlying French framework: according to Luc Trouche the scheme concept, encompassing utilisation schemes and instrumented action schemes, was introduced by G. Vergnaud as 'an invariant organization of activity for a given class of situations. It has an intention and a goal and constitutes a functional dynamic entity. In order to understand its function and dynamic, one has to take into account its components as a whole: goal and subgoals, anticipations, rules of action, of gathering information and exercising control, operational invariants and possibilities of inference within the situation. (Trouche 2005 p 149)

Since the utilisation schemes are mental, they are not directly accessible for study and analysis. Therefore, the concept of 'instrumented techniques' is of special interest: the term 'technique' is included in Chevallard's French notion of 'praxeologies' composed by the four components: i) type of task, ii) technique to solve it, iii) technology to explain and justify the technique and iv) theory to form the basis for the technological discourse. An 'instrumented technique', which I see as the external, visible and manifest part of the instrumented action scheme, then is a *set of rules and methods in a technological environment that is used for solving a specific type of problem*. (Drijvers & Gravemeijer 2005 p 169).

An instrumented technique includes conceptual elements as far as the technique reflects the schemes.

This leads to two crucial points:

- A student's development of an instrumented action scheme can be studied by inquiry of the student's development and use of instrumented techniques related to the scheme.
- Development of mathematical conceptions cannot be studied if use of technology is considered separate from the student's other activities.

The first point stresses the importance of empirical studies of students' work. The second point opposes my research to the standpoint, that teaching may be performed independently of what tools the students have at their disposal.

In line with this, Jean-Baptiste Lagrange stresses, that '*the traditional opposition of concepts and skills should be tempered by recognising a technical dimension in mathematical activity, which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill.*' (Lagrange 2005 pp 131-132).

TOOL – AND OBJECT PERSPECTIVES.

The notion of a 'tool perspective' on mathematical conceptions is opposed to a 'pure skill' understanding of mathematical activity and the notion includes the technical dimension mentioned by Lagrange. Like Anna Sfard's process – object duality (Sfard 1991) is useful to frame aspects of learning mathematics, especially in the case of algebra, a duality composed by a 'tool perspective' on a mathematical conception and an 'object perspective' on the same conception seems to be appropriate in problem-solving settings. The term 'tool', then, denotes mathematical processes, carried out to serve a concrete purpose. Accordingly, the term 'tool perspective' of a given mathematical conception denotes aspects of the conception, which are in accordance with Régine

Douady's definition: 'We say that a concept is a tool when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems, several tools may be adapted to a given problem.' (Douady 1991 p 115)

For example, a tool perspective on the conception of 'derivative of a function' could be the derivative seen and used as a tool for finding out how the function changes over time, whereas a corresponding object perspective could be the derivative, characterised or categorised by its merits and demerits as a tool for solving a specific problem. In contrast, a process perspective of derivative, parallel to the case of algebra, could be focusing on the actual determination of the derivative in question, whereas a corresponding object perspective here could be the derivative, characterised or categorised by its qualities within in a structure of functions.

Mathematical activities, then, must be considered from a tool perspective when they are parts or elements of a 'technique'. The generation of the instrument is in a crucial way linked to the change to object perspective. This issue is subject of further study in my Ph.D. project.

CHANGE OF PERSPECTIVE TO SUPPORT LEARNING

The case presented in this paper is part of the data underlying the inquiry in my Ph.D project. Basic to the inquiry is the key idea that learning is supported by alternating diving into the process of solving a problem and taking a distant look upon the activities and efforts. Edith Ackermann presents this idea in (Ackermann 1990) as a mean to integrate, roughly speaking, Jean Piaget's and Seymour Papert's views on children's cognitive development: 'Along with Piaget I view separateness through progressive decentration as a necessary step toward reaching deeper understanding. I see constructing invariants as the flipside of generating variation. (..) we need to project part of our experience outwards, to detach from it, to encapsulate it, and then reengage with it. (…) I share Papert's idea that diving into unknown situations, at the cost of experiencing a momentary sense of loss, is a crucial part of learning. (…) My claim is that both 'diving in' and 'stepping back' are equally important in getting such a cognitive dance going.' (ibid p 6). This basic idea led to focusing my inquiry on dualities of perspectives on given mathematical conceptions. To form a duality, a pair of perspectives should be mutually exclusive and relate, respectively, to the situated point of view, referred to as Papert's in the above mentioned, and to the distant point of view, referred to as Piaget's. The aim of the inquiry in the Ph.D. project, then, was to discover how the teacher can provoke and support the students' change of perspective in both directions within these dualities, and to interpret the role of such changes for the students' ongoing mathematical activities.

In the actual case, an interpretation is offered of the role in the students' learning process of using several instrumented techniques in 3 episodes.

CASE

A group of three students were working with a differential equation model of the transformation of cholesterol in the human body. The students were in third year of an experimental class in upper secondary school, where they had their own laptops at their disposal from first year on. The CAS software Derive was installed on the laptops. This case is based on group's work during one lesson in the teaching sequence which was video recorded. The students' written report and the teaching materials were examined in relation to the analysis of the case.

The students were preparing a written report on a series of tasks. The tasks were rather closed even if they appeared in a problem based setting. At the concrete level, the tasks concerned exploring the model for transformation of cholesterol. For example, twin brothers with different habits of living were compared. In this part of the group work, the students were not supposed to alter or revise the differential equation model of the cholesterol transformation. The tasks aimed to stimulate the students' learning about equilibrium point and general as well as specific solutions to differential equations. Further, the tasks concerned relations between general and specific solution and connections between analytic and graphic representation, both mediated by computer language. These second aims may be interpreted in terms of change from a 'model of'- perspective on the problem of cholesterol to the perspective of 'model for': the 'model of'-perspective was concerned during exploring activities within the differential equation model, whereas a 'model for'- perspective emerged during the students experiences of solving that type of problems, using differential equations. The notion of 'model of' and 'model for' perspectives is in accordance with Gravemeijer & Stephan's use of this terminology concerning the referential- and general levels of activity, respectively (Gravemeijer & Stephan 2002 p 159-160). Though, the issue is only touched in the case, presented in this paper.

In the case, the group was in an early phase of their work, concentrating on this text from the teaching materials (Hjersing et.al. 2004):

> ... another handy form is:
>
> $$\frac{dC}{dt} = 0.1(265 - C) \qquad (8.2)$$
>
> (Bubba changes his diet at $t_0 = 0$, with $C_0 = 180$ mg/dl, the new daily cholesterol intake is E = 250 mg/day.)

If we let t0 = 0 be the time where Bubba starts eating at the grill and if Bubba's level of cholesterol at that time is supposed to be C0 = 180 mg/dl, then Bubba's cholesterol level is expressed:

$$\frac{dC}{dt} = 0.1(265 - C)$$
$$C(0) = 180 \qquad (8.3)$$

Tasks

1. Find the equilibrium point for (8.2) and analyse the variation of the sign of the right side.

Is the equilibrium point a sink or a source? Use the answers to sketch (in hand) more solutions to this differential equation.

2. Find the general solution to the differential equation (8.2) (Show calculations)

3. Find the specific solution to the initial value problem (8.3)

4. If Bubba keeps this high cholesterol level diet for a very long time (one year or more), at what level will he end? Explain how you reach the conclusion?

During the case, the students used several instrumented techniques: First, in episode 1, they used the Derive command RK (stands for the 4.order Runge Kutta method of numerical solution) to obtain a graph of the solution to the differential equation (8.2) with the initial conditions t0=0, C(t_0)=180. In episode 2, they used their compendium of formulas supplied by paper and pencil techniques to find the general solution to the equation. The solution was typed into the computer and the students used the Derive command VECTOR to get a family of graphs of solution curves, as kind of an intermediate between general and specific solution. To answer the next question, they substituted the initial values in the formula for the general solution, calculated the constant d (determined by the initial values) and substituted it into the expression. To answer question 4. in episode 3, the students again graphed the same solution curve as they did in the first episode, but this time from the expression obtained from the preceding answer. Their answer to question 4., then, was based on visual inspection of this later graph.

Episode 1

To answer question 1., the students wrote:

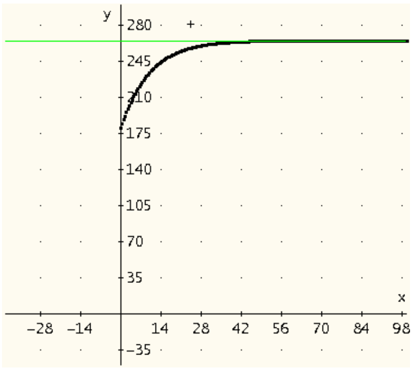
Sketching the graph:

Has to pass through (0,180) because this is Bubba's cholesterol level before he takes in food. [graph and expression omitted]

```
#1:   dC/dt = 0.1·(265 − C)

#2:   RK([0.1·(265 − y)], [x, y], [0, 180],
        0.5, 200)

#3:   y = 265
```



followed by this text:

> ‘ The function nears 265, so, 0.1 is the rate of growth and 265 is the point of equilibrium.
>
> The right-hand side is positive if his start C is below 265 and negative if it is above. The equilibrium point is a sink, that is, a stable equilibrium.

The students made at least one guess before they reached this result: their first try in the written report was a RK command, which was impossible to graph because the capacity of the computer-memory was exceeded. So, their strategy implied a trial-and error use of an instrumented technique that can be described as follows: 1) substitute the left side from the differential equation into the RK command, 2) type in the names of the independent and the dependent variables, 3) type in the initial values and 4) try to find values for the x-increase and the number of tangent-segments, which allows for: 5) graph the solution. Apparently, the students identified the horizontal asymptote by inspection of the graph and then graphed the function y=265 to verify the result visually. Afterwards, the equilibrium point was identified with this horizontal asymptote. So, since the graph with its asymptote was used to determine the equilibrium point, the graph with asymptote was in this case seen in a tool perspective and it was obtained using the instrumented technique sketched above.

The second part of the answer must be obtained from analysis of the differential equation. Therefore, the graphic method used in the first part of the solution serves to link graphic and analytic representations closely.

Episode 2

To answer question 2, the students wrote:

> General solution:

The equation for cholesterol is of the type dy/dx=b-ay and may be solved as follows: (b is a constant)

#1: $b = 0.1 \cdot 265$

#2: $b = 26.5$

#3: $a \cdot y = 0.1 \cdot c$

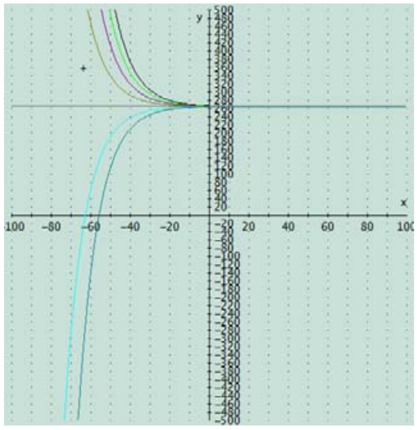
#4: $f(t) = \frac{b}{a} - d \cdot e^{-a \cdot t}$

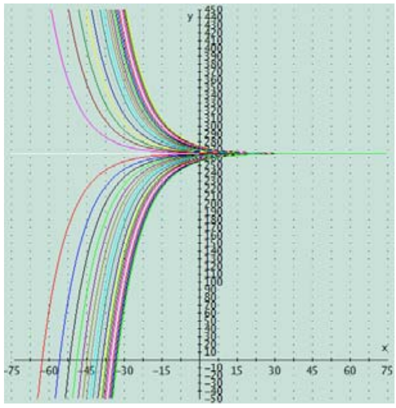
#5: $C := \frac{26.5}{0.1} - d \cdot e^{-0.1 \cdot t}$

#6: $C = 265 - d \cdot e^{-0.1 \cdot t}$

#7: $\mathrm{VECTOR}(C = 265 - d \cdot e^{-0.1 \cdot t}, d, 0, 2, 0.5)$

#8: $\mathrm{VECTOR}(C = 265 - d \cdot e^{-0.1 \cdot t}, d, -2, 2, 0.5)$





#9: $\mathrm{VECTOR}(C = 265 - d \cdot e^{-0.1 \cdot t}, d, -10, 10, 0.5)$

First, the students used paper and pencil and they looked in their compendium of formulas to find the general solution. They tried to identify the type of equation:

P1:It was because we compared with that one from earlier

P2: Model of growth

P1: Yes

P2: You don't know what it is called, that one?

P3: What is it called? It isn't called a name? (…)

P2: (looks in the compendium) they are not called any names – then leave it! (…)

Apparently, all three students tried to find an anchor point, which could launch an algorithm for solving the problem. They did not identify the type of equation by name, but the report shows that they found the type in their compendium.

P3: A general differential equation or solution

P2: We can take this one, can't we? (Points at the screen)

P1: Yes

P2: And then it is.. and then solve it. So that it.. and then we have made a graph

P1: One could do that… then one could draw it like this. Yes and then it was this (points at the screen, follows the curve) it is, you see, the same one it has just been displaced down there

P2 may have referred to the equation (8.3), as it was substituted into the formula from the compendium. P1 probably referred to the instrumented technique of graphing a family of functions, using the command VECTOR.

The paper and pencil technique implied to 1) identify the type of equation, 2) recognise it in the compendium, 3) identify and substitute the actual values of the constants in the expression for the solution. The students typed the results into the computer stepwise, as they were asked to show the calculations. Apparently, they then wanted to graph the result, which is, obviously, impossible. The students used the command VECTOR to graph a family of solution curves, which could be seen as kind of an intermediate between general and specific solutions. The report reveals, that they did not completely manage this instrumented technique at that stage of their work so they must have made more than one trial: The command VECTOR(C = 265-….) would not result in graphs as shown, as far as 'C = 265…' is evaluated logically. To succeed in graphing that family of curves it is necessary to delete the 'C='.

Intermezzo

The students answered question 3 by 1) substituting the initial values in the formula for the general solution, 2) calculating the constant d and 3) substituting it into the expression. Though, the dialogue in the group revealed no clear signs of having developed a general perspective of solution to the differential equation (Andresen 2004).

Episode 3

When starting to answer the last question in this task, question 4, it was clear from the dialogue in the group that the students did not try to estimate the result, based on the preceding answers:

P1: Here again, we are just supposed to find another curve. More or less. (..)

P1: When he starts at that level, we are supposed to find out if it increases or decreases (…)

P2: OK what is next?

P1: We are supposed to graph it

P2: It [the computer] should be able to do that, shouldn't it?

The students spent some time in the group discussing how long time they had to take into account. Two of the three refused to consider the fact, that they found an asymptote:

P1: It has no influence, then. But it must be his approximate cholesterol level. It has to be so because here you have the day and it has reached 50 days, which is one year so…

P2: 50 days is one year?

P1: Oeh weeks. You know what I mean

(they laugh)

P1: What so ever. What I said was right. It gets closer asymptotically. (…)

P3: Yes but now we do not have a year, we have 50 days

P1: Yes

P2: No it is weeks

P1-3: No it is days

P1: but the farer out towards unlimited you go…

P2: I can easily change it to one year, that is no problem

P1: It is completely irrelevant

Apparently, the fact that the students found equilibrium for the general solution earlier in the lesson did not ‘ring a bell’ when they were asked to argue for their latest result:

P2: We found out that the equilibrium point does not depend on d

P3: Yes but we have to explain how we reached to that conclusion (…)

P2: We substituted into the one we created ourselves - we did create one on our own (…)

P1: But we did create it, based on something

P2: We stuffed it into the general one

P1: This one did we stuff into it and created…

P3: But does it matter, which one it is?

P1: That is, what I try to find out. That is why I ask

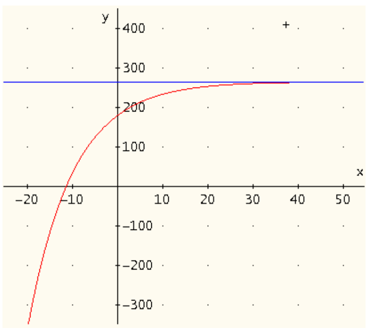
P1: This one. Here, we found a general one..

P2: Then we all agree!

These intermediate results are deleted from the final report, where the students wrote:

#1: $C := 265 - 85 \cdot e^{-t/10}$

#2: $y = 265$



‘Based on the graph we conclude, that the equilibrium point does not change even if the starting point is different. The general as well as this solution therefore near to the same equilibrium point and whatever long he keeps the high level, the equilibrium point does not change.’

In the final version of the report, the students simply graphed the specific solution from episode 1 once more. As the window was changed it is obvious that they re-graphed it. The written comment reveals, that the students did not expect the coincidence between the equilibria points for the general solution and the specific one in question. This fact questions the students' adoption in advance of the general perspective on solution to differential equation. In line with this the last statement, in my interpretation, reveals unfamiliarity with the conceptions of asymptotic behaviour and of equilibrium.

CONCLUSION

In the case, the students' work with the task concentrated on two mathematical conceptions, represented by the example of one differential equation: 1) equilibrium point for differential equations and 2) solutions to differential equations. The equilibrium point was closely connected to asymptotic behaviour of the solution curve. So, an instrumented technique of solving and graphing the solution curve, encompassing the RK command and seeing the curve with its asymptote in a tool perspective, was used by the students to build and strengthen their conception of equilibrium point. Determination of the general solution was carried out with a combined paper & pencil- and computer-instrumented technique, where the last part concerned change to graphic representation. Especially, the computer-instrumented part of the determination served to link between a family of solution curves, acting as a pseudo-graphing of the general solution on the one hand, and the specific solution curve, examined earlier, on the other hand. The experiences of asymptotic behaviour and of coincidence between the asymptotes of these solution curves, provoked by the task, support the students' change of perspective on the two conceptions in question: Realising that 'whatever long he keeps the high level, the equilibrium point does not change' is one step into an object perspective on 'equilibrium interpreted by horizontal asymptote'. Likewise, the family of graphs are visually convincing about the fact, that the general solution should encompass the specific solution.

The case illustrates genesis of Derive-commands as an instrument in an ongoing process. The first use of RK had the character of 'trial and error' in episode 1 (omitted from the data presented in this paper). Changes to graphic representation were not carried out with full routine, as the report reveals in the case of VECTOR. But it was very clear, that especially the possibilities of graphing shaped the students' thinking. So, the tool influenced: 1) Their strategy, which implied to choose asymptote as the tool for finding the equilibrium, 2) Their thinking of general solution by making it tangible by pseudo-graphing into a family of solution curves and 3) Their idea of verifying the asymptotic behaviour by visual inspection and comparison with the graph of y=265.

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UNDERGRADUATES' SOLVING STRATEGIES AND TEACHERS' PRACTICE

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This paper discusses the possibility of using the Anthropological Theory of Didactics to analyse the possible relationship between how the teacher conducts his teaching and the solving strategies that undergraduate mathematics student acquire. A short outline of the theoretical framework is presented and two authentic solving situations are being analysed within the notions of the framework.

INTRODUCTION

A good mathematician is a person whom among other things is good at solving mathematical problems. The vast amount of time students spend solving exercises at basically all levels of the mathematical educational system is a consequence of this property of a mathematician. In order to get the students to do better in problem solving situations it is natural to investigate the solving strategies of the students.

In the literature researchers in the didactics of mathematics proclaims that mathematics students even at university require and use solving strategies based on superficial arguments, for instance arguments copied from examples, and not on mathematical knowledge [Artigue, 1999], [Lithner, 2001]. When the teaching and learning of mathematics primarily is based on the solving of exercises it is of great importance, that students learn strategies based on mathematical arguments, because it is, roughly speaking, through exercise solving that the student acquire the mathematical knowledge.

Undergraduate' solving strategies have for instance been examined and categorised in a research study by Lithner (2001). The study is based upon observations of Swedish students' struggle to solve calculus exercises. He puts forward several categories of solving strategies. One of the main conclusions of the study is that students don't use the mathematical knowledge they (are supposed to) have learned but instead use more superficial strategies, even though the use of the mathematical knowledge could have led them in a more efficient way to the right solution. Lithner emphasises that the connection between the students' choice of strategies and the way they were taught hasn't been made in his studies.

I am interested in this connection. In order to be able to help the students to choose the right strategies and to use the mathematics that they are being introduced to in the lectures it is necessary to investigate *which* elements in the teaching practice that makes the students choose the superficial strategies. So on the long run this research interest has also a practical didactical purpose. The purpose of this paper is to examine the possibility of using the anthropological approach to investigate these described research interests.

As any other human academic activity, mathematically activity is a mixture of practice and knowledge. The main goal for a mathematics student is to acquire and learn the knowledge (decided by the mathematics society) and the means to do it is to practice mathematics. These two concepts are precisely the main concepts in the Anthropological Theory of Didactics developed by Yves Chevallard. In what follows the basics notions of the theoretical framework will be introduced and explained in so far as it is necessary to understand the rest of the paper. The introduction is based on [Artigue, 2002], [Barbé, 2004] and [Winsløw, 2005] and is by no means meant to be an exhaustive presentation of the anthropological theory.

The basic notion in the framework is the notion of a *praxeology*, a word that combines both praxis and knowledge (~logos). A praxeology consists of two parts or blocks; a practical block and a knowledge block. As implied above the main assumption of the framework is that every human activity can be characterised by *tasks* and the solutions to the tasks are the main point of the activity. In mathematical activity it is easy to understand that the solution of mathematical tasks is one of the main points and it is not difficult to give examples of representative tasks that mathematics students are meant to be able to solve, for instance to find the derivative of a given function. To find a solution to a task require the use of some mathematical *technique* associated with the given task. In the case of the derivative of a function the technique could for instance be to use the differentiation rule of a product, depending on the function in question of course. The mathematical task and the associated technique are placed in the practical block.

In the knowledge block on the other hand we find the *technology,* a notion which covers the theoretical discourse in which the applied techniques are embedded. The use of the word technology can be somewhat confusing, because we usually mean something quite different by the word (it has nothing to do with technical equipment of any kind). The technology provides the explanation and justification of the applied techniques. Close connected to the technology is the *theory* providing the ground and structure on which the tasks, techniques and technologies lie. The technology and the theory are placed in the knowledge block.

Most human activities consist of a number of different tasks, e.g. a collection or a family of praxeologies and then one talks about *an organisation*.

The investigation of the teaching and learning of mathematics can, according to the anthropological approach, not be done without considering the institution in which the practices and knowledge are being carried out and diffused. That is, one has to consider the institutional context in which the teaching takes place in order to fully understand what is going on.

The notion of praxeologies or organisations can be used to describe mathematical activity and then one talks about a mathematical praxeology and mathematical organisations, abbreviated as MO. But when we want to address the question of how to describe the teaching of mathematics another notion, namely the notion of a didactical praxeology or a didactical organisation (DO), is needed. In the didactical praxeology the tasks could for instance be *how to establish* the mathematical praxeology defined by the task of determining the derivative and the techniques could be to introduce and prove the different differentiation rules, provide examples where the rules are being used and it could be to give geometrical interpretations of the function and its derivative.

The teacher is the one who creates the DO in order to engage the students in a certain MO. But when the MO is being presented through the DO some transformations takes place. The MO that the teacher is actually going to engage the students in is no longer the same; it has been altered or transposed. The alternation of the MO through the DO is called the *didactical transposition*.

Given this short description of the framework what could be said about the justification for using this approach to analyse the connection between the teacher's practice and the acquired strategies of the students? If one wants to investigate a proposed relationship between the students learning of mathematics, defined by the acquired task solving strategies, and the teacher's practice it is important first of all to actually consider what the teacher is doing in the classroom. In contrast to other theoretical framework the anthropological approach emphasises the action of the teacher and the institutional constrains that the teacher is submitted to through the notions of the didactical transposition and the didactical organisation, so the approach seems to be appropriate to describe and analyse the teacher's practice.

Besides that the framework includes the institutional settings, it also allows a description and a possible explanation of the difficulties mathematics students experience during their studies at university [Winsløw, 2005]. The difficulties are connected to the two transitions in the MO the students are exposed to. From high school the students are primarily engaged in praxeologies consisting of only the task and associated techniques, while the technology and theory is absent. In the first elementary courses at university level, for instance algebra and calculus, the students are engaged in solving tasks, which resemble the tasks and techniques they know from high school but now the technology and theory associated with the task is presented as well, and the students are expected to be able to choose among different techniques and to justify their choices [Winsløw, 2005].

The second transition which occurs in the MO at the university level is a change in the task type. The task example given above is a good example of a very common task in first year calculus courses at university and the associated techniques to solve the task are well defined and easy to describe. Later in the mathematical study program the task type changes dramatically and becomes tasks presenting a conjecture for which a proof has to be constructed (one seldom sees that the students are being asked to give a proof that a given conjecture is wrong). The techniques associated with these kinds of tasks are not easy to define or to describe.

It is the hypothesis of this paper, that one of the reasons that the students use superficial strategies not founded on mathematical reasoning, could be that the students hasn't properly adjusted to the new situations occurring after the two transitions. For instance, in the research study by Lithner (2001) the students are in the middle of the first transition eo ipso they attend a first year university course and it seems that they try to use strategies or techniques which worked successfully for them in high school. The students are not used to make arguments using the knowledge block of the MO and the solving strategies therefore seem to be based on very superficial and inadequate reasoning.

TWO EXAMPLES OF SOLVING STRATEGIES

To illustrate this hypothesis and to use the notions of the framework I present two authentic situations, where students in pair have tried to solve a task. The task could be characterized as a task very likely to occur in an analysis course where the second transition has taken place. Both pair of students knows, in principal, the required mathematics needed to solve the task. Besides the illustration of the hypothesis, the given examples also show two different solving strategies with very different rate of success.

Team A consists of two below average achieving students (this characterisation is partly based on the students own saying and partly on informal evaluation done by the teacher). When the task solving situation took place the students had just finished their first real analysis course (following the traditionally calculus course at the beginning of their mathematics study).

The two students in team B come from another university than the students in team A. They were both characterised as very well achieving students by the instructor in charge of the exercise sessions they were attending. They had nearly finished their second real analysis course (in addition to the first calculus course) which had included the definition of continuity in a general metric space. So compared to the A-team the B-team had had one more course in analysis and could be considered as being more mathematically mature.

In reference to the mentioned transitions which students at university go through, the students in both teams had attended a course where the shift in types of exercise had taken place, e.g. the second transition (in the case of team-B they had attended two courses with the new types of exercises).

The task (the solution is to be found after the list of references)

Find all functions $f: R \to R$ which satisfy $f(x) - f(y) \leq (x-y)^2$ for all $x, y \in R$.

The two teams got 30 minutes to try to solve the task and they did not have any available textbooks. Neither of the two groups managed to give a completely satisfying answer to the question during that period. There were two reasons for setting a deadline. The first was to prevent a situation where I had to make a decision to end the session if the students could not solve the task and the second was very pragmatic: It was necessary in order to get some volunteers to do the exercise (they didn't want to spend hours trying to solve some incidental task). The students were audio taped and the transcripts have been translated from Danish. The notation in excerpts is as follows: Text in ( ) are comments about what the students are doing and the text in [ ] are comments from the author. The following is illustrative excerpts from the discussion between the two students in the A-team (named A1 and A2).

The A-team

The two students began to read the exercise and after some time one of the students starts to talk:

A1: It smells of the triangle inequality. Maybe we shall start by stating what we know [which he doesn't do] … The question is, that we have to determine all functions which satisfy this (he points at the inequality). That is, we are going to find some general expression.

A2: Hm … determine all functions … I mean functions are a very diffuse concept, in my opinion.

A1: … one could calculate what is on the right side. Is it not possible to write it as … you know the "slick" thing we always do. Does it bring us anywhere?

A2: No, it is not the difference between two squared numbers.

A1: Oh, it could otherwise have been clever. What if we make a sketch? Can we do that? Sometimes it can help. We can say something about the difference between the function called f(x) and the other function … or is this the same function?

A2: Yes, it is the same mapping, it is just two values.

A1' statement about finding "a general expression" indicates that he has an idea of what kind of answer they are looking for, but they seem very confused about the more precise nature of this "expression" and how to find it. A1 doesn't know if there are one or two functions present, a confusion which surfaces again and again. They try to use familiar methods (A1 suggests that they should operate on the expression on the right side); methods they have experience using when they encounter an algebraic expression. In the next excerpt the interviewer (e.g. the author!!) decides to intervene in lack of progress.

I: Can you think of any functions that satisfy the inequality?

A1: It concerns the fact that the functions are growing. Whether we talk about functions which converge or something like that … if this (points at the expression $(x-y)$ on the right side of the inequality) becomes less than one, we have a small problem because the difference is raised to the second. I don't know if that is a problem.

A2: No, but it says something about the structure that we need to have.

A1: We have a function which does something. It has to satisfy something between zero and one because we know that something inverse happens, here the squared term becomes smaller and if it is bigger than one it will explode. Which functions satisfy this? We can guess a function. There is x raised to the second, for instance. It's something between zero and one. What do we called this? It converges.

I: Maybe it will be a good idea to get an overview over which functions satisfy the inequality.

A1: Yes, it is functions which have to approach each other and it doesn't matter which side you come from. The distance between x and y, if it is small, then the functions will approach each other.

The excerpt shows that they are totally confused about different mathematical concepts. A1 keeps on forgetting that they are only dealing with one function and not several and he is using the word "converging" in a very unclear way. They try to get a grasp of which conditions the unknown function should fulfilled, but they are not able to express it in a useful way.

The two students were asked if the function $f(x)=x^2$ worked. They don't know. They insert values for x and y in the inequality and realize that it doesn't fulfil the inequality. By encouragement from the interviewer they try the function $f(x)=x$ and find out that it doesn't work either. Then the student A1 remembers the existence of bounded functions, like $f(x)=\sin(x)$ and they discuss if this could be a characteristics for the unknown functions. The interviewer intervenes again, now deciding to give them the answer and to guide them toward a proof:

I: What about the constant functions? Do they work?

A2: Oh, (he inserts some numbers). It holds for the constant functions.

I: What do constant functions satisfy which no other functions satisfy?

A2: It's that $f(x)=f(y)$ for any x, y.

A1: They are not differentiable, one could say.

A2: Yes, they are. If you differentiate a constant you get zero. They are infinite often differentiable.

A1: Yes, you could say that.

I: Do there exist other functions, where the derivatives are always zero?

A2: No, I don't believe there is. Is there?

I: Well, then it could be a characteristic for constant functions that you could use.

A1: So you could differentiate the expression. That would be … a possibility, right?

A2: It's difficult. There are two variables. I can't say that it's not possible to differentiate with respect to two variables, but I haven't seen it yet.

Here one of the students, A1, reveals a great miscomprehension of the notion of differentiability, because he doesn't regard constant functions as differentiable and when the other student corrects him, he talks as if he had never heard before that constant functions are infinitely differentiable. But the other student also reveals a lack of understanding of functions and derivative, not sure that the only functions with derivatives that are always zero are the constant functions.

The B-team

After reading the exercise one of the students in the B-team quickly states that the function must be continuous in order to satisfy the inequality:

B1: Yes, clearly it is the continuous functions. If we make this (points at the right side) less than delta, then this will be less than epsilon (points at the left side)… But are they also differentiable? This is difficult to say.

B2: I am just sitting here thinking … what if you calculate the quadratic term, then you get something in two variables, but that is probably not very exciting. Because it is not a function in two variables.

B1: No. I am just sitting and trying to rule out things … for instance the constant functions. They don't work. Because it is supposed to work for all x and y.

B2: They do work!!

B1: Yes, of course.

B2: Does it work for all continuous function?

B1: This is less than delta, but this is not necessarily less than …

B2: Can we find a continuous function were it doesn't work?

The students in the B team also reveals uncertainty about the interpretation of the inequality and B1 gets suddenly confused about the constant functions. But they are very quick to correct their mistakes. They leave the idea and start discussing whether they are dealing with functions in one or two variables. Not getting anywhere with this, B1 tries a specific function $x^2 + x$, substitutes the equation (she is able to do it without using numbers) and realises that this function doesn't work. She returns to considerations about a more general property:

B1: It would … be satisfied for all isometric functions?

B2: Not if $x - y$ is less than one, then the quadratic term will be less than …

B1: Would it even work for $f(x) = x$? (B1 checks it out) It won't!

B2: I was just wondering, if it is the uniformly continuous functions … oh no, we have just…

B1: It wouldn't even work for x^2.

At this stage the two students have already mentioned three different kinds of function properties; continuity, uniformly continuity and the isometric property and they have discovered that there exists continuous function for which the inequality is not satisfied. So they have an idea of what kind of solution they are looking for, what kind of "general expression" they seek.

B2: It wouldn't work for $x = y$, it wouldn't work for functions with a slope which goes this way (draws on the paper a function crossing $x = y$ in $x = 1$) or this way (draws a function looking like x^2). We must be able to give some sort of a geometrical interpretation.

B1: So, if you have some function, if you had x^2, $f(x) - f(y)$ is that distance there (points at the distance on the y-axis). So, it must be a constantly growing function. For instance, cosine. [It is not clear what she is referring to here]

B2: What you mean is that it should be one-to-one.

B1: Yes, it has to be.

I: What if it is the constant functions. Could you show it?

B1: We could show that the constant functions satisfy it. Maybe we could use Pythagoras?

B2: Yes, we could do that. When $f(x) - f(y)$ has to be less than $x - y$ to the second for all x and y, then it doesn't matter how much the values approaches.

B1: When x goes to y …

B2: Then we must be able to look at $f(x) - f(y)$ divided by $x - y$ … this is the slope of the secant line. It should be less than $x - y$. Well, let's see. The slope of the secant …

B1: $f^{'}(x)$ … no it is $f^{'}(y)$. Now we have defined $f^{'}$.

(On the paper they have written: $\lim_{x \to y} \dfrac{f(x) - f(y)}{x - y} \leq x - y$ and just below is $f'(y)$ replacing the left hand side)

B2: Can we use that for anything?

The two students don't realize the importance of the reached result, because they only have taken the limit on the left side, forgetting the right side, and they turn back to try to use a geometric interpretation. The interviewer intervenes:

I: You could look at what really stands here (points at the inequality including the limit of the difference quotient)

B2: It's $f^{'}(y)$. Oh, well then it is solved!! It has to be less than or equal to zero.

The interviewer points out that they are not quite finished because the derivative still can be negative (they realise that they have divided the inequality with a term that can be negative). They try to "fix" it, but don't succeed completely.

Comparison of strategies

First of all the mathematical level of the two teams are different, as mentioned. This fact becomes evident in the way the teams treat the task. The B-team is from the beginning aware of what kind of answer they are looking for. They seek a property by which they can define the functions, for instance they try to see if the function is continuous or isometric. This difference between the two teams could be explained in terms of the above description of the transition that takes place later in the mathematics study, where the tasks shift from the more operational kinds, which solution demand a concrete method or algorithm, to tasks of a more theoretical kind, where the associated techniques moves towards the technological and theoretical part of the praxeology. The students in the B-team have therefore done more tasks, where they have been training the technological and theoretical part of the praxeology and they are therefore better equipped to understand and use the mathematical definitions.

In the analysis course the students in the B-team attended they have done a lot of proof and exercises where they should apply the definitions of continuity and uniformly continuity, so they are very familiar with the epsilon-delta definition and notation. The definition of the derivative is more difficult for them to operate with, so it seems. When asked afterwards why they in the beginning gave up on checking for differentiability B1 said that it was too difficult; at first climbs they could only see that the functions had to be continuous. But near the end of the half hour it becomes clear that they knew the definition of the derivative as the limit of the difference quotient (or the slope of the secant line, the terminology B2 uses), but using the notation of the limit seemed more difficult for them than using the epsilon-delta notation.

This could also be seen as a consequence of the kind of task the students have been doing. The kind of task and associated techniques the students have been practicing in the course affects the way they approach an exercise.

The students in team A have very little knowledge of the technology and theory connected to the differentiability of one-variable functions. They have only a vague idea of what kind of solution they are looking for and they seem very confused about the interpretation of the inequality (whether it is a question of one or several functions or function of one or several variables). They start to use a technique which is familiar to them when they encounter an algebraic expression: They start to do algebraically manipulations in the hope that something might reveal itself. So even though they have attended an analysis course, where the mentioned task type transition has occurred, it seems that they have not had sufficient amount of training in the new kind of tasks and techniques.

The teacher's practice

The students in team B have attended a course consisting of a lecture session given by a professor and a subsequent exercise session maintained by an instructor (an older student). In the lecture session full praxeologies is provided, while the students in the exercise session practice on the more reduced praxeologies (they focus on very specific techniques, for instance epsilon-delta arguments). The point of these sessions is not to facilitate discussions among the students of the mathematical concepts and their internal relations.

The students in team A have attended a smaller course (in number of participants) and the lecture session and exercise session are maintained by the same person, in this case the professor. Rarely the teacher or the students exhibit the solution to a given task on the blackboard, and in the exercises sessions the teacher often helps them individually. So the students have only seen few examples of solutions of task presented in a stringent way on the blackboard. This could be a reason why they try to use old strategies relating to the kind of task of their calculus course.

Within the anthropological approach it could be possible to give a thorough description of how the teacher conducts his teaching. How does he transpose the mathematical organisation, e.g. what is the didactical organisation and which resulting mathematical organisation are the students engaged in? This could be linked to the strategies or techniques which the students use. If the teaching emphasises the required techniques (as in the case of the students in the B team) this will be revealed when the students try to solve tasks relying on the more technological and theoretical parts of the mathematical praxeologies.

Concluding remarks

The notions of the anthropological approach have in this paper been used to describe the solving strategies of undergraduate students. It seems to be possible to analyse and interpret solving situations using this notions and especially has the proposed transition in the task types (described by Winsløw (2005)) which takes place between the first year calculus course and the following analysis courses in the mathematics study shown to be applicable.

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Appendix: Solution to the task

Assume the inequality is satisfied for all $x, y \in R$. By symmetry we also get the inequality $f(y) - f(x) \leq (x-y)^2$ for all $x, y \in R$. Combination of the two inequalities gives:

$$\left|f(x) - f(y)\right| \leq (x-y)^2 \Rightarrow \frac{\left|f(x) - f(y)\right|}{\left|x-y\right|} \leq \left|x-y\right| \Rightarrow \left|\frac{f(x) - f(y)}{x-y}\right| \leq \left|x-y\right|.$$

Taking the limit $x \to y$ we get that the derivative of f exists and that the numerical values is less than or equal to zero, e.g. $f'(y) = 0$ for all y.

STUDENTS' BELIEFS ABOUT MATHEMATICS FROM THE PERSPECTIVE OF THE THEORY OF DIDACTICAL SITUATIONS

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INTRODUCTION

Four French frameworks: the theory of conceptual field, the theory of semiotic representations, the theory of didactical situations and the theory of didactic transpositions are all related with the teaching of mathematics. Aforementioned theories are well-developed but it is not obvious how to use them to study students' affective domain, for example students' beliefs about mathematics. What link can be made between these theories and students attitudes? This paper is an attempt to relate the theory of didactical situations to the investigation which will be carried out for exploring students' beliefs and attitudes towards mathematics teaching and learning. In the part of the epistemological background to my study I explain the basic notions and their relationships which are related with the belief theory. In the section about the methodological concepts there is a short description of the methodology used in the study. The theory of didactical situations emerges in the part where observing and changing students' beliefs are discussed. There I give the theoretical aspects for constructing the change of the attitudes (didactical engineering and theory about different games) in the paper and in the last section more practical suggestions for constructing these kinds of games is presented.

The aim for writing this essay is to improve the study and its methods about students' beliefs and attitudes in mathematics which will be carried out in Norway and in Estonia. The research questions are:

- What do Estonian and Norwegian students think about mathematics as a school subject?
- Why do Estonian students feel about mathematics as they feel?
- What kind of variances/analogies can be found comparing Estonian and Norwegian students' beliefs and attitudes? Why have these kinds of variances/analogies emerged?
- How are the students' performances related to their beliefs?

All these questions have been taken into account during writing this paper to complement the research. But as the process of changing students' beliefs is not concerned into the aforementioned study the following ideas are mainly illustrative and support the research in the cognitive and not in the methodological level.

AN EPISTEMOLOGICAL BACKROUND TO MY WORK

One possibility to divide the domains inside of human person in two is affective and cognitive. The affective part can be a set of beliefs, attitudes and emotions (McLeod, 1989) and cognitive is formed of knowledge and knowing (or *savoir* and *connaissance*). One can acknowledge that these domains are inseparable and in complex connection. The question how to distinguish knowledge from beliefs has still not been clarified (e.g. Thompson, 1992). The further discussion tries to make the schema of the aforementioned notions and their relationships.

There are two words in French: *les savoirs* and *les connaissances* which are 'both translated as "knowledge"' (Brousseau et al, 2004, p. 2). According to Brousseau (2004):

> All school learning is an alternation of *savoirs* and *connaissances*. Isolated parts are acquired as *savoirs* connected by *connaissances*. Without the *connaissances*, the *savoirs* have no context and are swiftly mixed or lost. Without *savoirs*, the *connaissances* are more touristic than useful.

Reading different articles one can find both notions - *knowledge* and *knowing* and mostly there is no reference of the "real" French word so it impossible to ascertain which word has been used in original paper. My interpretation is that the notion *savoir* represents the word knowledge. The idea of *connaissance* is more related with the concept of knowing. The editor of the Brousseau's (1997) book notes the same idea and points out that knowledge is 'socially shared and recognised cognitive constructs, which must be made explicit' (p. 72) and knowing is 'individual intellectual cognitive construct' (p. 72). Brousseau (1997) finds it significant not to confuse these two notions because they have different meanings and makes explicit the relationship between these words: 'the distinction between knowledge and knowing depends primarily on their cultural status; a piece of knowledge is an institutionalized knowing' (Brousseau, 1997, p. 62). Going to mathematics, what can be institutionalised in mathematics discussions? In other words, which kind of mathematical knowing can be institutionalized into mathematical knowledge? Drouhard et al (1999) pose the hypothesis that at the end of the mathematics lesson the institutionalized knowledge is what the students have been exposed to; but they reject the idea with the notion of *resistance*. To feel the resistance of mathematical objects is crucial in mathematics discussions. As important as resistant, are *experiences* and *contradictions*. They acknowledge that mathematical discussion can not be some usual debate, like political debates, it makes individuals experience private contradictions which arise from the fact that others are convinced of the opposite ideas that can not be rejected with the authority (Drouhard et al, 1999).

This points back to the concepts of knowledge and knowing because in definitions of them both the contradictions play an essential role. According to the theory of didactical situations, ‘knowledge is constructed through adaptation to an environment that, at least in part, appears problematic to the subject’ (Artigue, 1994, p. 29). The sources of knowing are problems so ‘a knowing is characterised as the state of dynamical equilibrium of an action/feedback loop between a subject and a milieu under proscriptive constraints of viability’ (Balacheff, 2000, paragraph 1). Thus, the constraints and contradictions are the basis of building of mental schemes. It will be discussed further with the notion of *errors*.

But are the word belief and the notion of knowledge related to each other? For clarification, in here, both terms - knowledge and knowing - will be used because it is not substantive to make the distinction. I am aware of the epistemological complication which arises using these words but the aim of this paper is not to go more deeply into this issue.

According to the English-Estonian dictionary, *belief* is something that one believes in strongly, connected with faith, something that does not change easily. *Attitude* can be the mentality, the view of some phenomenon and *emotion* is unsteady blind impulse which can vary to a great extent. ‘Beliefs and attitudes are generally thought to be relatively stable and resistant to change, but emotional responses to mathematics change rapidly’ (McLeod, 1989, p. 246). For example, during struggling in difficult mathematics task one can feel “I hate mathematics”, and after solving the same problem emotions can be the opposite – “I love mathematics, mathematics is fun”. It is leading to the problem which arises in measuring the beliefs and attitudes because there is no guarantee that the outcome from questionnaires which should reverberate students beliefs is really about **the students’ beliefs** and not about the moment impulse for instance**.** The answers given by the students are influenced by several factors – environment, mood, health, time, peers’ presence, etc.

It is hard to draw a line between beliefs and knowledge because there can not be made visible groups with titles “knowledge” / “knowing” and “belief”. Drouhard et al (1999) proposes the idea to consider beliefs as a part of the taught knowledge. They see three aspects of taught knowledge: ‘content’, ‘the rules of mathematical game, the know-how’ and ‘the most general believes about mathematics’ (p. 326) and point out that ‘these components could hardly be defined and studied separately one from another’ (Drouhard et al, 1999, p. 326).

Underhill (1991) agrees with the idea and is in the position that ‘knowing is believing’ (p. 20). Whatever one knows or does not know is simply the same that one believes or does not believe.

A. whenever we say we know something, we are simply asserting that we believe something, whether it is about quality called “red” or the being called “God” or the relationship “3+4=7”;

B. whatever we do not know, we may not known passively (we have no belief; we have never been exposed to it!), or negatively (we believe its opposite or some anti-belief or substitute belief; we believe something other than that);

C. all that we know reflects our beliefs based empirical data or reason or faith; these might be thought to exist continuum.

(Underhill, 1991, p. 21)

Thompson (1992) points out two ways to make a distinction between beliefs and knowledge which are 'degree of conviction' (p. 129) and consensus. First, beliefs can be held weakly or strongly. Beliefs can compartmentalize as something uncertain or certain, important or not so important. But one can not say that one knows the fact weakly or strongly. Secondly, it is possible to believe something despite the awareness that the others do not agree with it and think about it differently.

There can be found the notion of *conception* related both with the beliefs and with the knowledge. In one case the beliefs are understood as a subclass of conceptions (Thompson, 1992) and on the other hand the conceptions are seen as a subset of beliefs (Pekhonen, 1994). Pehkonen (1994) explains that 'conceptions are higher order beliefs which are based on such reasoning process for which the premises are conscious' (p. 28). But not only exists the relationship between conception and belief, there is a direct connection between the conception and knowing. One can interpret Balacheff's text (2000) as Venn-diagram (Figure 1) and build the following mental image:

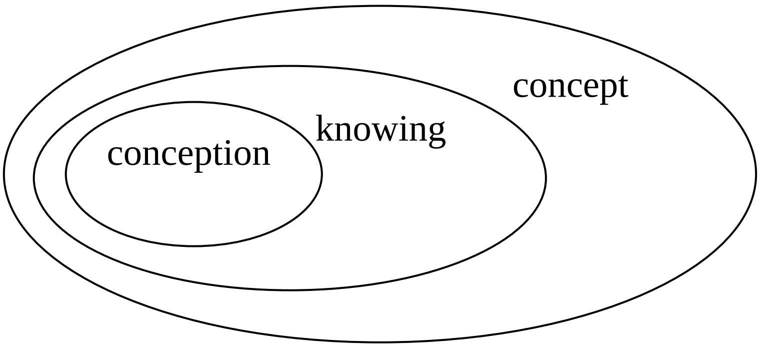

Figure 1

It simply means that knowing is a set of conceptions and concept is a set of knowings. This is not the same as Thompson's (1992) claims: '"conceptions" viewed as more general mental structure, encompassing the beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like' (p. 130). The aim is not to take a straight position about these notions. It is described in here because it once more convinces that there can not be drawn a distinction line between belief and knowledge and the conjecture to watch a person's both domains as the whole is justified in this case. '*All knowledge is a set of beliefs*'

(Underhill, 1991, p. 5, emphasis original). In further discussion the beliefs are seen as a special form of knowledge - ‘an internal knowledge’ - like Lester Jr. (2002) points out.

ABOUT THE METHODOLOGICAL CONCEPTS

Research

It has already been pointed out that the aim of the study is to investigate students’ beliefs towards mathematics teaching and learning. I will use both - quantitative and qualitative - methods. The questionnaire which is based on Likert-scale method has been already developed in year 1995 for the project called KIM (Kvalitet I Matematikkundervisningen, translated as Quality in Mathematics). Using the same questionnaire allows the comparison between students’ beliefs in year 1995 and in year 2005. Whilst the study will take part in two countries – Norway and Estonia – it makes possible another comparison part. In Estonia the lesson participations, the interviews with teachers and students for getting the answer to the question “why these kind of beliefs are held” form the qualitative part of the research. As one of the research questions is “how students’ mathematical performance is related to their beliefs and attitudes about mathematics” then the mathematical tests constitute one element of the study. Previous studies have shown that there is significant connection between the performance of the mathematical test and self-confidence in mathematics (e.g. Streitlien, Wiik & Brekke, 2001).

The theory of didactical situations ‘aims to model teaching situations so that they can be developed in a controlled way’ (Artigue, 1994, p. 28). It may lead to the thought to use the ideas of this theory for observing students’ beliefs during the didactical situation, for example game. Another possibility is to use these ideas as a didactical tool for changing students’ beliefs. Both of these concepts are discussed in following sections.

Observation

In the theory of didactical situations a key postulate is ‘that each problem-situation demands on the part of the student behaviours which are indications of knowing’ (Balacheff, 2000, paragraph 1). It is simply that students’ behaviour visualise aspects of their knowing, thus the aspects of their beliefs. In problem-situation students’ beliefs should be activated and made explicit (this is a hard part!). While lesson participations are planned in the study then the observation is possible but to work out such games that will reveal students emotions and make the attitudes traceable is difficult. It is not easy to think out the games which could stimulate to emerge the beliefs but these are necessary for getting any information. ‘It is indispensable to create conditions that probably won’t occur naturally, in order to study them. A system delivers more information when it reacts to well chosen stimulation' (Brousseau, 1999, p. 40). The conclusions which can be inferred from observations have to be dealt with the highest cautiousness because this is a situation where emotions are more included than beliefs.

Change

The thought to use the games - didactical situations - as tools for changing students' beliefs carries an idea. But starting to talk about changes incurs a lot of sensitive questions. A quote from Erkki Pehkonen is relevant at this point:

> Such an education which aims to involuntary change one's beliefs is called indoctrination – the common name "brainwash" is a very fitting description for it. Therefore, speaking about changing beliefs should be understood as "offering change opportunities".
>
> (Pehkonen, 1999, also in Pehkonen, 2003)

Pertinent questions are: is variation necessary at all? In which direction should the changes be leaded? Can others (teacher, milieu) change persons (students) thoughts, ideas, and beliefs? To which extent?

The last question, related to the extent of the change, touches the issue of frustration. There can be found the ideas that some tensions inside of the student are good and leading the child to progress, like according to the associationism the emotions (both positive, like fear, and negative, as hope) are seen as *secondary reinforcement* and they are part of motivation (Richardson, 1985). Goldin (2002) talks about the *meta-affect* and sees that in some cases the fear (for example, fear of mathematics) can be joyful and pleasurable. On the one hand, the errors made during the process (for example during solving the mathematical task) can cause negative attitude towards the subject if these occur continuously. On the other hand,

> errors witness the inertia of the instrumental power of knowledge, which has proved its efficiency in enough situations. … the trajectory of the student may have to pass by the (provisional) construction of erroneous knowings because the awareness of the reasons why this knowing is erroneous could be necessary to the understanding of the new knowing.
>
> (Balacheff, 2000, paragraph 2, section 2)

Errors can help to increase the mathematical understanding which is one of the aims in mathematics lessons. Actually, mistakes do not show what one does not know but they show what one knows. Even the knowledge which can not be internalized as mathematical knowledge (the purpose for not using the notion "false knowledge" here is that there is not the "true knowledge" out of "somewhere" because it is contradictory with the idea of constructivism wherein the theory of didactical situations lie (Artigue, 1994)) is a knowledge. Counterexamples can in some situations act very effectively and lead towards to the higher understanding than any other teaching tactic.

Despite the importance to experience some quantity of frustrations and errors (and through errors the resistance of mathematical statements, compare with the discussion earlier) the general idea is that mathematical beliefs should be changing towards a positive direction because motivation to learn will rise together with positive belief (e.g. Streitlien, Wiik & Brekke, 2001) and not on the contrary.

The idea is to discuss with the notions from the theory of didactical situations some possibilities how to change students' mathematical beliefs and attitudes towards more positive. The aim of the theory of didactical situations is

> to develop the conceptual and methodological means to control the interacting phenomena and their relation to the construction and functioning of mathematical knowledge in the student.
>
> (Artigue, 1994, p. 29).

Applying the idea in here, it means that 'mathematical knowledge' is seen as student's positive attitudes towards mathematics. Thus, the theory helps to elaborate the tool for controlling the situation where students' beliefs are 'constructed' (changed) in a more positive direction.

THEORETICAL ASPECTS OF THE CONSTRUCTION

Didactical Engineering

Negative belief toward mathematics can be an unsurmountable obstacle in the process of teaching and learning mathematics. Using the idea of didactical engineering (Artigue, 1994), one can create a plan for changing students' attitudes in mathematics. The first phase of didactical engineering is preliminary analysis. 'The first, unavoidable phase consists in analysing the teaching object as it already exist, in determining its adequacy, and in outlining the epistemology of the reform project' (Artigue, 1994, p. 31). This approach is similar with the one-year teaching experiment carried out in a 12th grade classroom of the school in Naples (Di Martino & Mellone, 2005) where the first phase of the research for changing students beliefs in mathematics was observation of the class in initial situation in order to make clear the starting point. The conception and prior analysis will take part during the phase two. The researcher forms his/her research hypotheses, designs the process of data collection. Implementation takes part in the phase number three and there the theory of didactical situations will stand out. For changing students' beliefs in mathematics one can use different games (some of the examples of the games are illustrated in the next sections). The periods when the students play these games form the phase of implementation. The situation (game) where the changes should be constituted has to be characterized before making any comparison.

> …the theory of didactical situations is based on the principle that the meaning, in terms of knowledge, of a student's behaviour can only be understood if this behaviour is closely related to the situation in which it is observed, this situation and its cognitive potential have to be characterized before comparing this a priory analysis with observed reality.
>
> (Artigue, 1994, p. 35)

In the phase four *a posteriori* analysis and validation is expected. *A posteriori* analysis comprises the comparison part between the data and the posed hypothesis. To carry out the validation it needs the way back to the *a priori* analysis and may demand the changes in the phase two.

Game theory

Teaching situation comprises three components: students, teachers, and the milieu (Brousseau, 1997). Students are interacting with other students, with the teacher and with the milieu. The target knowledge in this particular case of teaching situation is more positive beliefs about mathematics.

In the first part of the game which is called 'situation of action' (Brousseau, 1997) the milieu is defined as all what influences the student and what the student influences by himself/herself. 'Within a situation of action, everything that acts on the student or that she acts on is called *milieu*' (Brousseau, 1997, p. 9, emphasis original) in which the student starts to develop his/her own strategies how to find the solution. He/she starts to make decisions about the process of the settlement. 'The sequence of "situations of actions" constitutes the process by which the student forms strategies, that is to say, "teaches herself" a method of solving her problem' (Brousseau, 1997, p. 9). One can draw a parallel and say that this is the part where students' *peripheral beliefs* (Green, 1971) are formed. The notion is related with the psychological centrality of beliefs; peripheral beliefs can be changed more easily. They are not being held strongly in psychological sense. The student starts to develop his/her conscious (by Kaplan (1991) defined as *surface* beliefs) and subconscious (*deep* according to Kaplan, 1991) beliefs during the process of solving a mathematical problem. Because 'to learn mathematics (I and II), is also to learn what are mathematics (III)' (Drouhard et al, 1999, p. 328) where I is meant to be the mathematical content, II marks the rules of mathematics and III are the general beliefs about mathematics.

As peripheral beliefs are quite mutable then the emphasis of other is essential in this part of the game because in the first part children play against each other, two by two. The effect of the peers on student' beliefs are significant. Research has shown that high attainers in mathematics were more likely to agree to “my friends take school seriously” (e.g. Tinklin, 2003). Friends can be seen as strong influence to students’ motivation too. The idea of transporting the motivation makes possible the affection of motivation through other persons (for example, other students, teacher). ‘Social situations can present worthwhile advantages in the domain of student motivation – motivation that is quite often transferable’ (Brousseau, 1997, p. 71). And through motivation students’ beliefs and thoughts which play a predominant part in behaviour are being influenced according to Williams, Burden & Lanvers (2002). It means that in the first episode of the game peers’ presence can be one of the circumstances which will evoke the change in the students’ belief system.

In the second part of the game groups start to play against the group. Again, the importance of other students plays a great role. The ideas of solution must be defend even stronger, first inside of the group and after that against the other group.

The third part of game carries the idea of ‘establishment of theorems’ (Brousseau, 1997, p. 13). This is the episode of the institutionalization. The teacher is supposed to “start” with students’ knowings and “end” with the socially constructed mathematical knowledge. It is possible that during this period the *central* (Green, 1971) beliefs are formed. Cental beliefs have strong psychological significance, they are held most sturdy. It is important to pay attention to the development of students’ attitudes because when already formed it is hard to change the central beliefs. In this part teachers’ influence is most recognizable and teachers’ beliefs start to influence students ones. Even the way how the teacher poses the theorems and strategies on the blackboard is one way of showing his/her mathematical beliefs.

The notion *didactical contract* is one of the aspects that will be briefly discussed further because of its relation with the students’ beliefs. The didactical situation (for instance, game) starts with the teachers intention that the student has to learn the target knowledge. ‘The didactical contract is the rule of the game and the strategy of the didactical situation. It is the justification that the teacher has for presenting the situation (Brousseau, 1997, p. 31). The didactical contract regulates the meta-game between the student and the teacher, it is implicit and vital. Significant is the breaking of the contract and the contract appears explicit only when it has been broken (Brousseau, 1997).

> At the moment of such breakdown, everything happens as if an implicit contract were linking the teacher and the student; surprise for the student, who doesn’t know how to solve the problem and who rebels against what the teacher cannot give her the ability to do – surprise for the teacher, who reasonably thought that she performed sufficiently well – revolt, negotiation, search for a new contract which depends on the new state of knowledge, acquired and desired.
>
> (Brousseau, 1997, p. 32)

The exercise (example 1) with the title “What is wrong?” is presented next to show how the didactical contract between the students and the teacher has been interrupted.

Example 1: the teacher starts with the hypothesis “I will prove that 2=1” and indicates that there is something awkward with the task. Hereupon, the teacher gives the following solution. “Let us make the assumption that

$$A = B \Leftrightarrow A^2 = AB \Leftrightarrow A^2 - B^2 = AB - B^2 \Leftrightarrow (A-B)\cdot(A+B) = B(A-B) \Leftarrow A+B = B$$

$$\textit{Since} \quad A = B, \quad \textit{then} \quad B + B = B \Leftrightarrow 2B = B \Leftrightarrow 2 = 1,$$

(Grevholm, 1988, p. 3:2, own translation).

These kinds of crises involve different reactions amongst the students. The question arises: which kind of emotions? In which way they can be harmful/useful for the students’ beliefs towards mathematics? On the one hand, crises are the impelling force which can evoke the will to overcome the barrier and when the impediment is crossed, it brings alone the feeling of satisfaction and raises students self confidence in mathematics what is in direct connection with the attitude. On the other hand, students need some stability in mathematics lessons. Too many experiments during the class incur the hesitation which in turn decreases the motivation of learning especially among the students who are not in high level in mathematics. Even if the concept of the *didactique* is not the contract itself but the process of finding the contract (Brousseau, 1997) then this process is quite laboured and time-consuming which often disconcerts students’ expectations to the teacher and to the learning process. When the student does not feel secure in mathematics lesson then his/her beliefs towards mathematics will more likely appear in negative than in positive direction. Even if the breakdowns are significant the teacher has to be extremely careful because too much confusion about the contract will influence the process of learning and not always toward better way.

EMPIRICAL ASPECTS OF THE CONSTRUCTION

All these games considered in this section can be played in different parts as illustrated before (one to one, group to group, establishment of the theorems). But how do these games which should change students’ beliefs have to be designed? The answer depends largely upon the content of the belief. If one holds the attitude that mathematics is boring then games should be constructed to evoke the feelings that mathematics is fun and enjoyable. When student thinks mathematics as something very difficult then helpful would be the play where tricky and complicated exercises can be solved easily and in the nice way. Let us take for example the algebra. Many students believe that algebra is useless. So they have to be convinced that the new knowledge which will arise from working with the algebraic exercises is needful and interesting. There can be found nice tasks to show that algebra is a powerful tool for solving complex exercises.

Example 2: “it is given three numbers, for example 8, 11, 3; the construction of the questions which will lead to the solution is: will the sum of these numbers be odd number or even number?; in which the answer depends upon?; what will determine the parity of the bottom number (33 in this example, look at the Figure 2)?; how can one change the parity?” Other possibility is to hypothesize different hypotheses and control their validity. Going further with the solving, teacher should take four numbers, five numbers, etc.

The following image (Figure 2) is given as an illustration for the solution of the example 2.

$$\begin{array}{ccccc} 8 & & 11 & & 3 \\ & 19 & & 14 & \\ & & 33 & & \end{array}$$

Figure 2

This example can be called as the example of the “power of the algebra”. And the power can be felt most explicitly during the third part of the game where the theorem will be established. It is necessary for the students to solve the exercise (example 2) in algebraic way to fix on the bottom number. It is impossible to reach to the answer of that task without using the algebra. Solving the algebraic exercises (e. g. example 2) should lead the students to the path of the understanding how influential is proving and further, proving something in algebraic way can lead to the understanding of the beauty of generalisation. The second example is taken from the seminar given by John Mason in this spring in Agder University College for showing again the power of the algebra. And the power of generalisation.

Example 3: “it is given:

$1+2=3$

$4+5+6=7+8$

$9+10+11+12=13+14+15$

$16+17+18+19+20=21+22+23+24$

…

The question is: how does the row number “t” look like?”

Therefore if the student gets good emotions and really feels “the power of algebra” from this solution processes then his/her central beliefs about algebra will move towards positive way. And hopefully not only the attitude will be more positive but the algebraic knowledge will improve as well.

If one holds the belief that mathematics is less to do with real life and “algebra can not be applied for the real life situations” then next example will hopefully convince him/her on the opposite. It is found from the real life situation among gypsies bargaining.

Example4: the exercise (see Appendix A) how to multiply the numbers between 5 and 10. The teacher should demonstrate the following schema: “The question is: how much is 6·7? Both hands have five fingers. Now, lift up one finger at the left hand what means 5+1=6. Lift up two fingers at the right hand for symbolising 5+2=7. Multiply the sum of the fingers up by 10, it will be 3·10=30. Now multiply the fingers down at the left hand (4) by the fingers down at the right hand (3) and the answer is 4·3=12. Sum up these results 30+12=42. And the final answer is 42!” (Grevholm, 1988, p. 19:2, own translation).

The following questions can be: will it work in every case taken numbers between 5 and 10? Can it be extend to the bigger numbers? Which will be the algebraic interpretation of this exercise?

This is the example of the exercise which is close to the real life situation, has a practical effect for using the method in everyday activities, has the ability to “translate” this task into mathematical language and prove it in algebraic way. Thus, it is an assignment of several possibilities and solving that task should uncover different ways of interpreting it. Like, it can be related to the computer science using for example the program Microsoft Excel for verification of the “finger method”. In turn it will show how computer science and mathematics and real life situations are related to each other and should bring up the bounds one can see between the mathematics and everyday life situation. And expands students’ knowledge in algebraic sense. To solve different kind of exercises students need algebra because there are tasks in mathematics which can not be proved without using algebra.

It is quite obvious that the observation of changing beliefs and making conclusions from these observations is a challenge to every researcher because this is a long and complicated process. Attitudes are not explicit to the observer, one can investigate these through questionnaires, interviews, lessons observations but because of their complex nature beliefs still remain partly undiscovered. The notion of change always brings alone the course: in which direction the change should be conduced? Positive or negative – how to make a distinction? Who has the power to decide the directions?

However, there is still a hope that students’ beliefs and attitudes in mathematics teaching and learning are changeable during mathematics lessons, using different games for example. One can find a proof from the research carried out by Di Martino & Mellone (2005) ‘… there is no longer fear for mathematics …. collective discussions characterising the activities, together with the fact that often everyone played a role in the lesson construction, made many students’ focus shift from the assessment of results (product) to the assessment of contributions (process)' (p. 8).

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Appendix A

Handy multiplication

WITH THE NUMBERS BETWEEN 5 AND 10

EXAMPLE 6x7.

LIFT UP ONE FINGER AT THE LEFT HAND, THIS SYMBOLISES 5+1=6. LIFT UP TWO FINGERS AT THE RIGHT HAND FOR SYMBOLISING 5+2=7.

MULTIPLY THE SUM OF THE FINGERS UP BY 10.
3x10=30.

THEN, MULTIPLY THE NUMBER OF THE FINGERS DOWN AT THE LEFT HAND (4) BY THE FINGERS DOWN AT THE RIGHT HAND (3).
4x3=12.

ADD THESE NUMBERS: 30+12=42.
THUS, THIS IS 6x7=42.

WHY IS THIS RIGHT?
TRY AGAIN WITH THE OTHER NUMBERS.

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Matematikkens didaktik er et relativt ungt forskningsområde i de nordiske lande, og der er i de senere år iværksat en række initiativer i de nordiske lande for at styrke området bl.a. gennem etableringen af forskeruddannelsesprogrammer og forskerskoler. Dette hæfte indeholder tekster fra et phd-kursus som blev afholdt på Københavns Universitet med støtte fra Den Nordiske Forskerskole i Matematikkens Didaktik, som er etableret af NorFa (Nordisk Forskeruddannelsesakademi).
Kurset "Four French Frameworks for Research in Didactics of Mathematics" drejede sig om en række centrale teorier og metoder fra den franske skole i matematikkens didaktik. Teksterne i hæftet er deltagernes afsluttende "opgaver", som gik ud på at producere en selvstændig brug af kursets indhold i forhold til egne forskningsprojekter.

'Naturvidenskabsdidaktik' er det teoribaserede arbejde med at vinde indsigt i naturvidenskabelig tænkning, læring og undervisning. Centrets mission er, gennem forskning, undervisning og formidling, at bidrage til denne indsigt og dens udmøntninger i kvalitetsløft på alle tre felter for det naturvidenskabelige fakultets videnskabelige medarbejdere og studerende.

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