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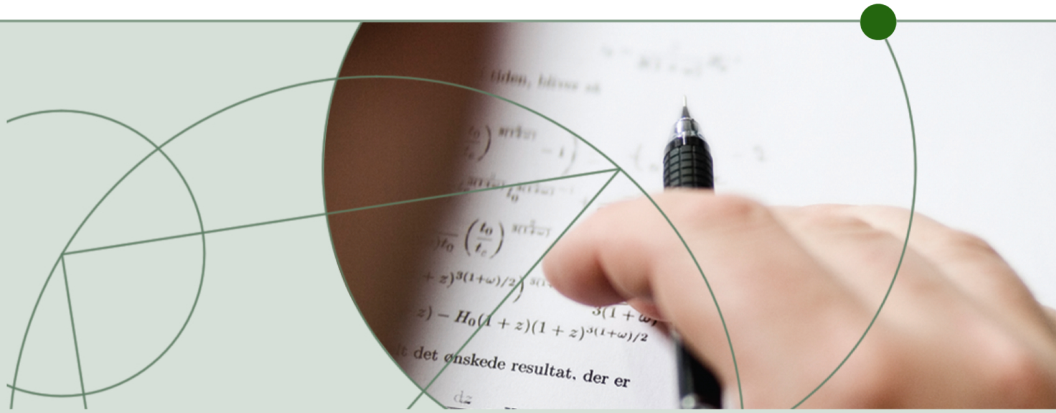

The Anthropological Theory of the Didactical (ATD)

Peer reviewed papers from a PhD-course

Marianne Foss Mortensen
Carl Winsløw

February 2011

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Peer reviewed papers from a PhD course at the University of Copenhagen, 2010

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Contents

A graduate course on the anthropological theory of didactics

Carl Winsløw and Marianne Foss Mortensen

This book presents the products of a doctoral course held in Copenhagen from January to June, 2010. Its aim was to introduce a new and exciting direction in research on educational subjects: the anthropological theory of didactics, founded by the French didactician Y. Chevallard (cf. 1985). While the aims of the theory have developed and expanded over the years (cf. Bosch and Gascón, 2006), the institutional perspective on knowledge has remained central to its objective. This means that the theory aims to study human practices and discourses as phenomena profoundly linked to the institutions that frame, enable, and constrain them.

Teaching is at first sight a practice which appears so familiar to us all that it may even seem trivially derived from the knowledge and capacity it exists to transmit. This, however, turns out to be a rather naïve viewpoint. A closer look reveals that teaching institutions tend to develop coherent and autonomous cultures, which shape not only their practices as schools but also the practices and the knowledge they transmit. An evident example, which is also quite present in this book, is school mathematics: it cannot be said that what is practiced and explained there simply repeats what we find in books or practices elsewhere, such as in universities or in the antique origins of the subjects. A school discipline transforms and develops its practices, ideas and justifications on its own, precisely as a culture does – notwithstanding its more or less lively interactions with the surrounding world.

The anthropological theory of didactics (ATD) is not a theory *about* scientific didactics but one which is tailored to study didactic phenomena, and thus to serve *in* the science of didactics. The French name, *théorie anthropologique du didactique*, means literally "anthropological theory of the didactical", where "the didactical" refers to objects of a didactic nature: teaching, text books, regulations, institutions and any other entity set up to teach something to somebody. Even the word "teach" must be understood rather broadly here. The essential of being "didactic" is *the intention* of someone to enable someone else to know or do something; didactic phenomena include everything done in such an intention. Thus, didactic phenomena are found in many other places than school institutions – for instance also in enterprises, television, exhibits, sports fields and concert halls.

A doctoral course itself fits the definition of didactic phenomena, albeit of a strangely particular kind: what the participants should become able to know and do is to engage autonomously in the pursuit of knowledge (within a certain field). The present course was quite broad in its definition of the field but very sharp in designating ATD as the theoretical perspective to learn – in the sense just mentioned, that is, in a process of becoming autonomous researchers.

We initially had 11 participants from 4 different countries and a similar number of research fields (didactics of mathematics, didactics of physics, didactics of music and general education). The course was organized in three two-day sessions, as follows (cf. Appendix 1):

1. Introductions and first experiences with ATD
2. In-depth discussion of participants' own projects and ideas for putting ATD into use in relation to them;
3. Conference-like presentation and discussion of participants' papers, resulting from the work in and after the second session.

Before the first session, participants were requested to read a certain number of basic texts (listed in Appendix 2). The session itself benefited from the presence of the founder of ATD, Yves Chevallard, who along with Marianna Bosch was present during all of the first session. They both gave three hour workshops, Marianna on "levels of didactic co-determination" (which is one of the newer developments of ATD), and Yves on more fundamental aspects of ATD, which he is currently working to collect and organize in what he terms a "dictionary". Shorter lectures on general principles and further examples of ATD-based research were given by us, as well as by one experienced participant (Finn Holst).

The second session had a more individual flavor. Every participant had been given, at the conclusion of the first session, one or more papers on ATD-based research with a more or less evident relation to the doctoral project of that participant (a list of these texts can be found in appendix 3). Participants had prepared a short exposition of that paper along with some first ideas of how to use it, and other elements of the course, in a small research project that could lead to the final paper (based on which the course was assessed). These expositions formed the basis of intensive discussions at the session, with inputs from both us as course teachers, and from the other participants. The second session was held in an especially festive atmosphere, due to the announcement (around Feb. 1, 2010) of the award of the Hans Freudenthal medal to Yves Chevallard, in acknowledgement of his world class achievements related to ATD.

Finally, the third session saw the presentation of the papers produced by each participant, as at a regular conference. We notice in passing that the organization of the course let the participants experience most of the forms of work involved in research (particularly at the level of generating, sharpening and criticizing ideas). This aspect of the course was further strengthened by the revision of the texts after the course through a genuine peer review process (other participants acting, naturally, as "peers").

A total of nine participants completed the course (and thus produced a paper of the kind mentioned). Three of them, however, did not go through with the final peer review process to publish their paper in this booklet (lack of time being the most frequent reason). Thus, we can here present a part of the work resulting from the course, and we are happy to be able to say that it includes genuine research accomplishments, using a difficult and novel theoretical approach which was also new to the authors.

To them, as well as to Yves and Marianna, we extend our sincere and heartfelt thanks for having collaborated with us in the setting of this course. We are confident you will be pleased with the result of your efforts, as presented here – and that you will take the projects begun here further on, to complete studies and eventually journal papers. We sincerely hope that every other reader of this book will find material and inspiration for pursuing the study and development of the research programme which ATD really is.

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Appendix 1. Schedules of the three course sessions.

Session 1 – day one (February 11, 2010)
10.00-11.00: Welcome and introduction by participants.
11.15-12.00: *ATD and this course*. Introduction by Carl Winsløw.
13.00-16.00: Workshop by Marianna Bosch
16.30-18.10: Presentation and discussion of two almost finished PhD-projects related to ATD (Marianne Mortensen, Finn Holst)

Session 1 – day two (February 12, 2010)
9.30-12.30: Workshop by Yves Chevallard.
13.30-14.30: *ATD as an anthropological theory of mathematics.* Lecture by Carl Winsløw.
14.30-15.30: First discussion of participants' individual projects, selection and distribution of (individual) texts for 2nd session.

Session 2 (April 11-12, 2010)
Each of the 10 participants had an individual session (1 hour) during which their ideas for an ATD-based paper were discussed by all.

Session 3 (June 10-11, 2010)
Each of the nine remaining participants presented an ATD-based paper which was then discussed by the whole group.

Appendix 2. Reading list for the first session (for all participants).

1. Bosch, M. and Gascón, J. (2006). Twenty five years of the didactic transposition. *ICMI Bulletin* 58, 51-65.
2. Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the IVth Congress of the European Society for Research in Mathematics Education* (pp.21-30). Barcelona: Universitat Ramon Llull.
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level. To appear in *Proceedings of the VIth Congress of the European Society for Research in Mathematics Education.*

5. Rodríguez, E., Bosch, M. and Gascón, J. A networking method to compare theories: metacognition in problem solving reformulated within the Anthropological Theory of the Didactical. *ZDM Mathematics Education* (2008) 40:287–301.

Appendix 3. Reading list for the second session (individual assignments).

1. Bourg, A. (2006). Analyse comparative des notions de transposition didactique et depratiques sociales de reference. Le choix d'un modèle en didactique de la musique? *Journal de Recherche en Education Musicale* 5 (1), 79-116.
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13. Hansen, B. and Winsløw, C. (2010). Research and study course diagrams as an analytic tool: the case of bidisciplinary projects combining mathematics and history. To appear in *Proceedings of CATD-3.*
14. Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning* 7, 245–274.

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Revisiting groups of students' solving process of realistic Fermi problem from the perspective of the Anthropological Theory of Didactics

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Abstract. *This paper reports on the first attempt to use the notions of 'Research and Study Course' (RSC) and 'praxeologies' within the Anthropological Theory of Didactics (ATD) to analyse groups of students engaged in the mathematical activity of solving realistic Fermi problems. By considering so called realistic Fermi problem as a generating question in a RSC the groups' derived sub-questions are identified and the praxeologies developed to address these are discussed.*

Introduction

Working with mathematical models and modelling is a central part of the national written official intended curriculum for the Swedish upper secondary mathematics education (Skolverket, 2000). Indeed, these notions have successively been both more emphasised and made more explicit in the last two curricula reforms from 1994 and 2000 respectively (Ärlebäck, 2009a). Nevertheless, research indicates that teachers have difficulties in formulating and explaining their conceptions of these notions (Ärlebäck, in press) and in a study of 381 upper secondary students across Sweden 77 % stated that they never had encountered the notions during their upper secondary education (Frejd & Ärlebäck, submitted). However, it has been suggested and concluded that the introduction of the notions and the students' initial conceptualisation of mathematical modelling at the upper secondary level adequately and efficiently can be done using so called *realistic Fermi problems* (Ärlebäck, 2009b; Ärlebäck & Bergsten, 2010). This conclusion is drawn using the so called *MAD* *framework* which provides a macroscopic analytical tool originating from Schoenfeld's 'graphs of problem solving' (Schoenfeld, 1985) applied to the work of groups of students solving realistic Fermi problems. This paper reports on the first attempt to use notions from the Anthropological Theory of Didactics (ATD), founded and foremost developed by Yves Chevallard, to revisits the data used in the analysis to provide a microscopic analysis of the work of the students with the aim to add a more detailed and nuanced picture of the problem solving process. In addition, the microscopic analysis also aims to in more detail highlight the possibilities and limitations of realistic Fermi problems in connection with the teaching and learning of mathematical modelling as a curriculum goal in it self as well as a mean for teaching and learning mathematics more generally.

Background

In the research literature in mathematics education there are many different perspectives on and ways to approach mathematical modelling (e.g. Blum, Galbraith, Henn, & Niss, 2007; Lesh, Galbraith, Haines, & Hurford, 2010). Concepts and notations used are for instance those of *competencies* (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006); *modelling skills* (Berry, 2002); and, *sub-processes* or *sub-activities* (Blomhøj & Højgaard Jensen, 2003). Normally these focus on the descriptions of, relations between and/or the transitions of phenomena in the real world and their mathematical representations. From an ATD perspective García et al. (2006) have presented a conceptualisation of mathematical modelling which basically equates all mathematical activity with mathematical modelling. The perspective on mathematical modelling adapted in this paper however is inherited from Ärlebäck (2009b) in line with the aim to in a coherent and natural way complement and deepen this previous research. This perspective is based on how mathematical modelling is described in the Swedish upper secondary curriculum (e.g. Skolverket, 2000), here illustrated in Figure 1. A similar interpretation of the modelling process from the Swedish context has been presented by Palm et al. (2004).

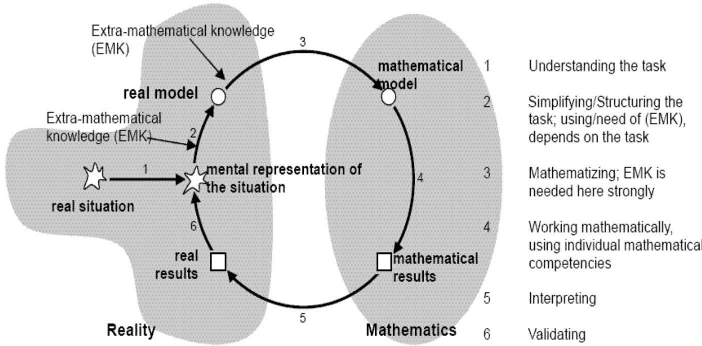

Figure 1. The modelling cycle as presented by Borromeo Ferri (2006, p. 92) after adaption from Blum and Leiβ (2007).

Ärlebäck (2009b) and Ärlebäck and Bergsten (2010) report on an investigation of the potential of using so called *realistic Fermi problems* to introduce mathematical modelling at the upper secondary level. Realistic Fermi problems are characterized by (I) their *accessibility*, meaning that they can be approached by all individual students or groups of students as well as be solved on both different educational levels and on different levels of complexity. Normally, any specific pre-mathematical knowledge is not required to provide an answer; (II) their clear real-world connection, to be *realistic*; (III) the need to *specify and structure the relevant information and relationships* to be able to tackle the problem. In other words for the problem formulation to be open in such a way that the problem solvers not immediately associated the problem with a know strategy or procedure on how to solve it, but rather urge the problem solvers to invoke prior experiences, conceptions, constructs, strategies and other cognitive skills in approaching the problem; (IV) the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities; and (V) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities (e.g. respectively (III) and (IV) above).

The realistic Fermi problem the groups of students solved used in Ärlebäck (2009b) and Ärlebäck and Bergsten (2010) was *the Empire State Building problem (ESB-problem)*:

***The Empire State Building problem*:**

On the street level in Empire State Building there is an information desk. The two most frequently asked questions to the staff are:

- *How long does the tourist elevator take to the top floor observatory?*
- *If one instead decides to walk the stairs, how long does this take?*

Your task is to write short answers to these questions, including the assumptions on which you base your reasoning, to give to the staff at the information desk.

The data from three groups of students working on the ESB-problem was analysed using a developed analytical tool called *Modelling Activity Diagram* (*the MAD framework*) inspired by Schoenfeld's 'graphs of problem solving' (Schoenfeld, 1985), the view adapted on mathematical modelling briefly mentioned in the beginning of this section, and the five characteristic features of realistic Fermi problems.

The MAD framework, see Figure 2, picture the different types of activities the groups engage in during the problem solving process in terms of the categories *Reading*, *Making model*, *Estimating*, *Verifying*, *Calculating*, and *Writing* depicted on the *y*-axis, and elapsed time on the *x*-axis (see Ärlebäck (2009b) for details). However, this schematic macroscopic representation does not provide any detailed information about what kind of discussions, topics and questions the groups address and investigate. In order to get a more nuanced and circumstantial picture of the problem solving process involving realistic Fermi problems in these respects this paper aims to provide a more microscopic analysis focusing on what actually is discussed within the groups, especially in connection to mathematical topics and content.

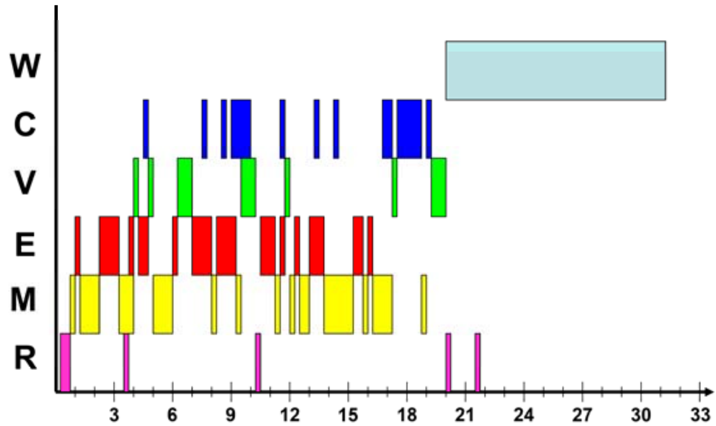

Figure 2. An example of a Modelling Activity Diagram of one of the groups solving the ESB-problem (Ärlebäck, 2009b, s. 346).

Theoretical framework

This paper uses the notions of *praxeologies* and *Research and Study Course* (RSC) from ATD. Within this framework *praxeologies* are used to describe any human activity in terms of two 'blocks': a *praxis* block ('know-how' or 'practical-part') containing both a designated type of *tasks* and the *techniques* used/needed to complete/perform these; and a *logos* block ('know-why' or 'knowledge-part') containing the *technologies* that explain, justify and describe the techniques as well as the formal justification of these technologies, the *theory*. As the name praxeologies suggests the praxis- and logos blocks are to be regarded as inseparable (Barbé, Bosch, Espinoza, & Gascón, 2005; Rodríguez, Bosch, & Gascón, 2008).

The notion of *Research and Study Course* (RSC) introduced by Chevallard (2004; 2006) is a general model that can be used for both designing and analyzing learning and study processes. A main emphasis of a RSC is on the *generating question*, Q_0, which should be intriguing and of genuine interest to the students as well as 'rich enough' to encourage the students to derive, pursue and answer dynamically raised and related (sub-)questions in the quest of trying to arrive at an answer to the question *Q0*. In addressing these questions the students have to invoke, use and/or develop one or more praxeologies. The derived sequence of sub-questions *Qi* and their respective answers *Ri* are often represented and illustrated in a 'tree-diagram' which illustrates the relationships between the different studied questions *Qi*; see Figure 3 for an example.

Given the notions briefly introduced above and interpreting the ESB-problem as a generating question in a RSC, the research question(s) studied in this paper can be stated as: *What sub-questions are addressed by the participating groups of students and what (mathematical) praxeologies developed and/or used?*

Methodology and Method

In terms of ATD the study reported on in Ärlebäck (2009b) and Ärlebäck and Bergsten (2010) can be conceptualised as an investigation of the *didactical praxeology* with the *task* to introduce mathematical modelling to students at the upper secondary level using the suggested *technique* presented by realistic Fermi problems. The issues addressed in these papers, as well as in this one, are concerning the (underdeveloped) *logos block* of this *didactical praxeology*, especially the *technology* part addressing issues of justifying the use of realistic Fermi problems.

To address the research question, widening and deepening the analysis of the groups of 2-3 students solving realistic Fermi problems, the recorded video and transcribed data from Ärlebäck (2009b) was revisited and re-analysed. The approach taken was in line with Hansen and Winsløw (2010) who make use of the RSC as an analytic model. Although there exist an a priori analysis in Ärlebäck (2009b) of some of the questions the problem solvers need to address, this paper only focus on the empirical questions actually addressed by one of the groups of students during their problem solving session. In other words, the idea is to consider the students work on the realistic Fermi problem as the generative question *Q0* and to see what (sub-)questions *Qi,j,...* this led the students to investigate, and in addition to link these questions the MAD representation of the problem solving process of the studied group. Note that in the ESB-problem the generating question, *Q0*, actually is two questions:

Q0,1: *How long does the tourist elevator take to the top floor observatory?*

Q0,2: *If one instead decides to walk the stairs, how long does this take?*

Results

The questions *Qi,j,…* the students derived from the generative questions and examined are presented below in the order in which they were raised and posed during the problem solving session. The formulations below are translated but in principle the students' own wording; some minor alterations have been made in order make the actual question intelligible and more concise. Basically the questions *Q1…* are concerned with the ESB's physical appearance, *Q2…* address *Q0,1* (taking the elevator), and *Q3…* address *Q0,2* (taking the stairs):

Q1: How tall is the Empire State building?

Q1,1: How many floors are there in the Empire State Building?

Q1,1,1: How high is a floor?

Q1,2: How tall can a general building be?

Q1,3: How tall was the World Trade Centre?

Q2: How fast is an elevator?

Q2,1: What is the weight of the elevator?

Q2,1,1: How much work is being done by the elevator?

Q2,1,1,1: Given the work done by the elevator, can we then calculate its velocity?

Q2,2: How long does it take to ride the elevator to Michael's [a friend] apartment?

Q2,2,1: On what floor is Michael's apartment?

Q3: How tired does one get from walking the stairs?

Q3,1: How longer does it take walking up one floor?

Q3,2: How much longer does it take for each consecutive floor?

Q3,2,1: How long does it take to walk up the first floor?

Q3,2,1,1: How fast is normal walking pace?

Q3,2,1,2: What is the inclination of the stairs?

Q3,3: My [one of the students] mother lives on the fifth floor – I wonder how long it takes walking up the stairs to her place?

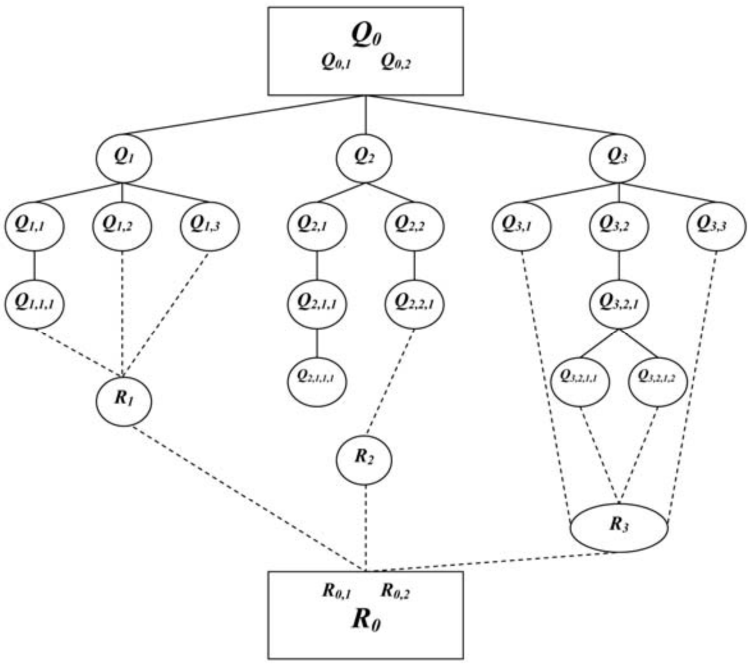

Figure 3. A tree-diagram illustrating the relationship between the questions addressed by one group of students solving the ESB-problem

Figure 4 illustrates the dynamic aspects of the addressed questions added to the MAD representation of the students' problem solving process. The first time a specific question is explicitly addressed it is preceded by an asterisk (*).

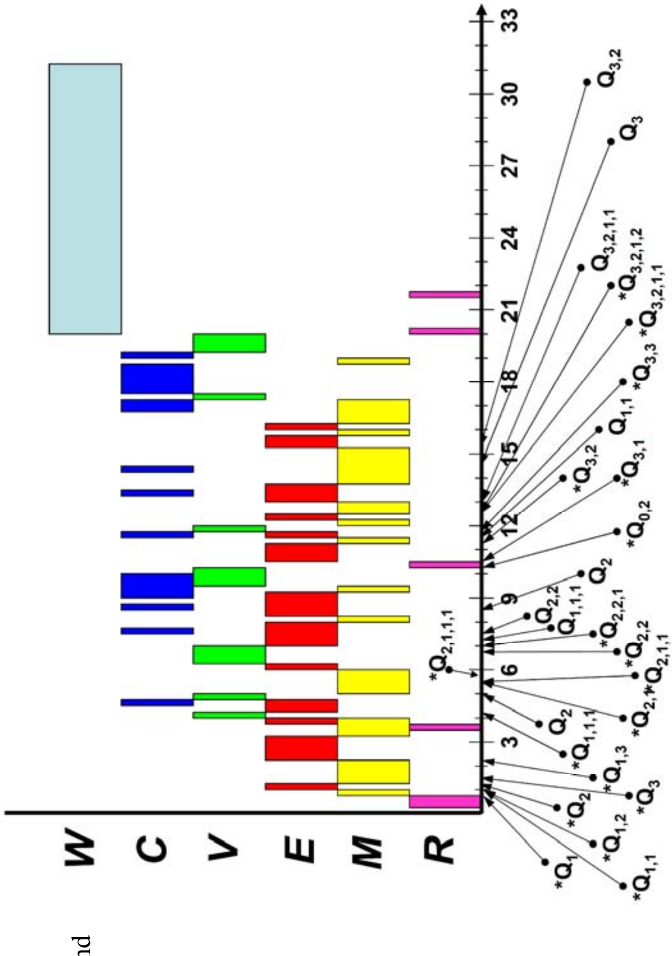

Figure 4. The Modelling Activity Diagram (Ärlebäck, 2009b, s. 346) extended with the order and dynamics of the derived questions in the RSC.

The relationships between these (sub-)questions are illustrated in Figure 3. Note that the doted lines in the tree-diagram display the dependence of the answers *R1*, *R2*, and *R3* respectively with respect to previously answers to questions in the tree. However, due to space limitations, and the fact that the focus of this paper is on the derived questions, these details are omitted here to be discussed elsewhere.

All branches except *Q2,1,…* represent questions which answers contributed to the solving of the ESB-problem. The branch *Q2,1,…* is about the classical mechanics concept of work, which the students briefly discuss as one possible strategy to get an estimate for the velocity of the elevators in the ESB.

After about having spent about 15 minutes on the problem the group starts working on details concerning their suggested model on how to take the physical exertion into consideration in the *Q0,2* question. They continue to do this in approximately 4 minutes, before the writing of the letter instructed in the problem formulation begins.

Conclusion and Discussion

One can notice that the actual modelling in terms of discussing, structuring and determining central variables and relationships important for solving the problem is something that is made implicitly and silently throughout the problem solving session. The praxeologies developed to address the questions (tasks) *Q0,1* and *Q0,2*, all three groups in Ärlebäck (2009b) used the mathematical model $t=s/v$ (t being the time, s the distance, and v the (average) velocity) as the basic technique to approach the questions. However, the decision to use this model is not explicitly uttered, or in any other way directly communicated, within the groups; it seem that all the participating students took it for granted that this was the model to use to solve the problem. In other words, the logos of this praxeology is kept hidden. It is possible that this 'choice' narrowed the groups' possibilities to go beyond this model and come up with more elaborated models.

A majority of the praxeologies the students developed made use of estimation as the technique to resolve the tasks originating from all (but *Q2,1,1,1* and *Q3*) of the derived questions the students engaged in. All these praxeologies have underdeveloped logos and the technologies and theories invoked to justify and verify the estimates are based on personal and often anecdotal experiences. This is due to the feature of realistic Fermi problem to not provide the students with any explicit numbers to work with. It should be noted that one of the technologies applied and made use of to validate the estimate in some of these praxeologies are the same mathematical model as used as the technique in addressing *Q0,1* and *Q0,2*; *v=s/t*.

The result suggests that there are some often used mathematical models, here exemplified by *v=s/t*, which are taken for granted used without second thought and reflection on underlying assumptions, limitations or alternatives. An explanation might be found in the different institutional conditions and constrains where these models are taught, learnt, practiced and applied. In particular, it would be interesting to study the didactical transposition of the notions and use of mathematical models and modelling to see where these conditions and constrains arise.

Though it has proven productive and useful to use realistic Fermi problems for the introduction of mathematical modelling at the upper secondary level (Ärlebäck, 2009b), the challenge for the future is to design generative questions in the RSC so that also more advanced mathematical praxeologies are invoked and developed. The RSC 'allows' for the teacher to intervene, comment and make suggestions during the course of study, and this present a possibility to achieve more, and perhaps specific, advanced mathematical content

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2

Proportion in mathematics textbooks in upper secondary school

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Abstract. *Proportional reasoning and knowledge of proportion are prerequisites for success in higher studies in mathematics. The aim of this paper is to investigate what possibilities Swedish upper secondary school textbook tasks offer students to develop knowledge about proportion during the first course in mathematics. The five textbooks investigated in this paper show a high degree of variation in "know – how" of proportional reasoning but less variation regarding knowledge about proportion.*

Background

Knowledge of proportion is one of the learning goals for grade nine in Swedish compulsory school (Skolverket, 2001). However, results from TIMSS 2007 show that 50% of the students in grade eight have difficulties solving tasks about proportionality (Mullis, 2008). But we do not know how the students at upper secondary school are managing proportions. International research also shows a predominance to use the linear model in solving proportion tasks in upper secondary school (De Bock, Vershaffel, & Janssens, 1998). My ongoing study attempts to shed light on various aspects of how proportion and proportional reasoning are exposed in textbooks at upper secondary school level in Sweden. According to several studies (e.g. Johansson, 2006), Swedish mathematics teachers rely on textbooks mainly in terms of exercises, making the textbook a critical factor in the classroom to study.

The aim of this paper is to present the first results from an investigation into what possibilities Swedish upper secondary school textbook tasks offer students to develop knowledge about proportion during the first course in mathematics.

Theoretical framework

This section will present the theoretical background from which the analytical tool used for this study was developed by first discussing how the notions of proportion and proportional reasoning were interpreted, then shortly outlining the theory linked to the tool, and finally presenting the tool as constituted by its four main parts.

Proportion and proportional reasoning

The term proportion is used when two quantities x and y are related by an equation $y=kx$, where k is a constant. Then y is said to be (directly) proportional to x, which may be written $y \propto x$ (*The Concise Oxford Dictionary of Mathematics*, 2009). It is also common to use the term proportion for some specific relations such as *direct* proportion $y = k \cdot x$, *square* proportion $y = k \cdot x^2$, *inverse* proportion $y = \frac{k}{x}$, *inverse square* proportion $y = \frac{k}{x^2}$ and *inverse square root* proportion $y = \frac{k}{\sqrt{x}}$.

I am also investigating proportional reasoning tasks though it is difficult to set up a general definition of proportional reasoning, maybe because proportion is such a complex concept. I will here use the description of proportional reasoning found in Lamon (2007, p. 638).

According to Cramer and Post (1993) and several other studies there are three central types of problem situations in proportional reasoning: *numerical comparison*, *missing value*, and *qualitative prediction & comparison*. In numerical comparison problems, the answer does not call for a numerical value. The student compares two known complete rates, as in Noelting´s (1980) well known orange juice problem. Lybeck (1986) among others, found that there exist two different main solution strategies: the A-form or the so-called Within Comparison, where quantities of the same unit are compred, and B-form, a Between Comparison across different units. In missing value problems three objects of numerical information in a proportion setting are specified with a fourth number to be discovered. A popular such task is the tall-man short-man problem (Karplus, Karplus, & Wollman, 1974). The third problem situation, qualitative prediction & comparison, does not demand memorized skill. These types of problems force the students to gain knowledge about the meaning of proportion with qualitative thinking (Cramer & Post, (1993).

Knowledge and know-how related to proportion tasks

As this study is focused on how a specific mathematical notion is treated in the school institution in terms of types of tasks and strategies, The Anthropological Theory of Didactics (ATD; see e.g. Bosch & Gascon, 2006) offers a useful approach. The ATD postulates an institutional conception of mathematical activity, starting from the assumption that mathematics, like any other human activity, is produced, taught, learned and diffused in social institutions. Mathematical work can be described in terms of mathematical organisation. A mathematical organisation (MO) is constituted by two levels, the *know-how* (task & techniques) and the (discursive) *knowledge* (technology & theory) related to a given task (Chevallard, 2006). Task – different kinds of tasks to be studied, Techniques – how to solve tasks, Technology – justification and explanation of the techniques, Theory – founding technology and justification of technology. In this study, there are influences from two MO's, one where proportion is defined as a 'dynamic' notion MO_1 and one where proportion is defined as a 'static' notion MO_2 (see below).

In order to study a phenomenon a Reference Epistemological Model (REM) should be created by the researcher (Bosch & Gascón, 2006). Otherwise it is difficult to be independent in relation to the educational institutions under study and the result may be a model that is implicitly imposed by the educational institution. The REM is a corresponding body of mathematical knowledge that is continuously developed by the research community and connected to the different steps of the didactic transposition. The transposition process describes how the mathematical knowledge is transformed from the institution of knowledge production through the educational system to the classroom (Bosch & Gascón, 2006).

The knowledge of proportion

As a REM and theory category, the two MO's of describing proportion in textbooks was used. MO_1 was observed in a pilot textbook study (Lundberg & Hemmi, 2009), where it was found that a frequent way to present proportion is by the relationship $y = k \cdot x$, where *y* are dependent of *x* and *k* is a fixed constant. This has been named a *dynamic notion of proportion* (Miyakawa & Winsløw, 2009), as we have different values of *x* as input producing specific outputs as y depending on the value of *k*.

Another way to describe proportion is *static* (Miyakawa & Winsløw, 2009). It is possible to identify this phenomenon in Euclid's definition of proportion (Euklides & Heath, 1956), where it is regarded as static in nature because it deals with pairs of "magnitudes" rather than numbers. A magnitude could be a length, like the diagonal of a square. For the Greeks it could not be measured in centimetres, but nevertheless multiplied in a geometric sense (e.g. enlargement). The static way of defining proportion is more general in comparison with the dynamic notion because it can be defined in n-tuples of real numbers and does not constrain proportion to pairs.

An example from a Swedish textbook (Gennow, Gustafsson, Johansson, & Silborn, 2003, p. 314) will serve as an illustration of static and dynamic definition:

"An electric radiator influences by power *P* (the thermal energy emitted per second) of voltage the *U* that the radiator has been connected to. The table shows some values of *U* and *P* that belong together. Check if there is a relation between P and U represented by $P = k\cdot U^2$ and if so calculate *k*. The power has the unit Watt (W) and the voltage Volt (V).

U (V)	120	160	200	240
P (W)	144	256	400	576

Solution: To investigate $k = \dfrac{P}{U^2}$ we put a new row in the table.

P/U^2 (W/V^2)	0,010	0,010	0,010	0,010

We obtain the same result for all pairs. This implies that the relation can be written as $P = k\cdot U^2$ and $k = 1{,}0\cdot 10^{-2} W/V^2$. In electricity the unit W/V^2 is denoted *S* (Siemens). The relation can be also be written $P = \dfrac{U^2}{R}$ where $k = \dfrac{1}{R}$ and R has the unit Ω (Ohm)."

Auth transl.

In the beginning of the example the static definition is found in the table where the given data is n-tuples of *U* and *P*. In the solution it is necessary to switch to a dynamic definition because it is eligible to calculate *k* in order to check if there is a proportionality that can be expressed by the general formula $P = k \cdot U^2$.

The know-how in calculating proportional tasks

To investigate solution techniques for proportion tasks a study by Hersant (2005) was used because it had a lot of similarities with this study. She found six different types of techniques in her analysis of solved proportion examples in French textbooks. To show the differences between these categories of solution techniques (1-6 below, auth. transl.) I will use the following task provided by Hersant:

> *If 18 meters of fabric costs 189 francs, how much will 13 meters cost?*

1 Reduction by unit

If 18 meters cost 189 francs, 1 meter will cost 18 times less or $\frac{189}{18}$, and 13 meters will cost 13 times more than one meter or $\frac{189}{18} \cdot 13$ where $x = \frac{189 \cdot 13}{18}$. The answer will be 136.50 francs. In context of the theory of proportionalities, it will be justified by the characteristic of property proportion expressed here in as part of quantities. Two quantities U and V are proportional so when U is multiplied by 2, 3, 4…λ (and λ real), V is multiplied by 2, 3, 4,…. λ.

2 Multiplication by a relationship

Now consider the following solution: If 18 meters cost 189 francs, then 13 meters will cost $\frac{13}{18}\cdot 189$. Pay attention to that neither ratio nor proportion are used here. The technique of within measures proportion is used here.

3 Use of proportion

If we let the price for 13 meters of fabric be *x francs* then the price should be in proportion to the length of the fabric $\frac{189}{18}=\frac{x}{13}$ so, $189\cdot 13=18\cdot x$ and $x=\frac{189\cdot 13}{18}$. This technique differs from the earlier two when the proportion involves two different measures, length and price (between measures proportion).

4 Cross multiplication

Now consider the resolution that follows: Let *x* be the price of 13 meters of fabric. (1) $\frac{189}{18}=\frac{x}{13}$ then (2) $189\cdot 13=18\cdot x$ and $x=\frac{189\cdot 13}{18}$ that can be summarized as follows in a table:

Table 1. Cross product table

189	x
18	13

So $189\cdot 13=18\cdot x$ and $x=\frac{189\cdot 13}{18}$. In the spirit of this technique, the equality (1) does not match proportion but more rather a technique with a formal setting of the magnitudes and detached from the theories of proportions.

5 Use of coefficient

Another way of arguing is as follows: The fabric costs $\frac{189}{18}$ francs/meters, so 13 meters of fabric cost $13\cdot\frac{189}{18}$ francs. Here again a reduction technique to unit is used but with no connection to proportion.

6 Other possible techniques

It is also possible to solve the task with a graphical solution method. However, if only one value is calculated it seems to be a waste of energy to use graphical technique unless you don't have access to a graphic calculator then it is very easy to sketch a graph.

Research methodology

To investigate what possibilities textbooks offer upper secondary students in Sweden to develop knowledge about proportion during the first course in mathematics, the following research questions were set up (in terms of the ATD): What types of textbook tasks involve proportion? What techniques are used in the given solutions of proportion tasks? What explanations and justifications (technologies/theory) are presented in the proportion tasks? To answer these questions, an analytic tool was developed to investigate a selection of textbooks.

In a study about proportion in textbooks, da Ponte and Marques (2007) used the Pisa Assessment Framework as an analysing tool. In the pilot study (Lundberg & Hemmi, 2009) this tool was evaluated but for my purpose the categorisation at a cognitive level was problematic, so there was a need to develop an analytic tool better suited for a text analysis.

The textbooks selected for this paper are from the A-course at the Swedish upper secondary school. The A-course is a special case because it is mandatory for all students at upper secondary school (Skolverket, 2001), selected here because it is the beginners' course for all further studies at both the theoretical and the vocational programs of upper secondary school. In Sweden there is an open market for textbooks without regulations from the authorities. There are several textbooks on the market for this course, among which I have selected the five most commonly used in my region (three municipals).See table 2.

Table 2. The five analysed Swedish textbooks

Title
Matematik 4000 kurs A blå lärobok (Alfredsson, Brolin, Erixon, Heikne, & Ristamäki, 2007)
Exponent A röd: Matematik för gymnasieskolan (Gennow et al., 2003)
NT a+b: Gymnasiematematik för naturvetenskaps- och teknikprogrammen: Kurs A och B (Wallin, Lithner, Wiklund, & Jacobsson, 2000)
Matematik från A till E: För komvux och gymnasieskolan (Holmström & Smedhamre, 2000)
Matematik A (Norberg, Viklund, & Larsson, 2004)

The book chapters analysed were those where proportion was expected to be one of the key notions: percent, geometry, equations and functions. There was also a limitation in the equation and geometry chapter. Only tasks in the problem solving part in the equation chapter were analysed and in the geometry chapter only tasks with scale were analysed. The textbooks were investigated concerning both the knowledge and the “know-how” of proportion. The textbooks were analysed to determine what type of tasks were given (missing value, numerical comparison and qualitative prediction & comparison) and what kinds of proportion were used (direct proportion, inverse proportion, square proportion, and square root proportion). Finally, solution techniques presented in the textbooks and the knowledge of proportion (theories and technologies) related to the tasks found were investigated. Thus, in terms of the ATD, the analytic tool used for this study was comprised by the following categories:

- Task – missing value, numerical comparison, qualitative prediction and comparison, static or dynamic proportion, direct proportion, square proportion, inverse proportion, inverse square proportion, and inverse square root proportion
- Technique – how to solve tasks, the six categories by Hersant are used here
- Technology – justification and explanation of the techniques

- Theory – the two definitions of proportion static and dynamic are used as categories here

First results

The study is still ongoing but this paper will report some first results from the textbooks that have been analysed. In this section, the first most significant observations are presented, quoting selectively from the textbooks to illustrate the main findings. In all the textbooks, the definitions of the notions are introduced by solved examples and the examples presented in this section will therefore be structured by taking the technique used as the overarching categorisation principle, before type of task and knowledge (justification) are identified.

The number of examined tasks in total for all five textbooks were 3474 (one of the books had significantly more tasks (*1195*, 859, 652, 496 & 272) than the others). The preliminary data indicate that missing value tasks occur twice as much as numerical comparison and qualitative prediction & comparison tasks. The static definition was used only in the geometry chapter and the dynamic definition of proportion was only found in the chapters about percent and functions. The equation chapter was a mix of static and dynamic proportion. There were only a few justifications found in the textbooks in the geometry section. The most prevalent type of proportion was direct proportion. However, all types of techniques described above were found, as shown by the following illustrative examples.

1 Reduction by unit

In Swedish textbooks this solution strategy is easy to find in the chapter about percent. The following example is taken from Alfredsson et al, (2007, p. 45):

In a municipality the number of citizens is increasing by 8% in one year to 70 200. How many citizens were there in the municipality before the increase?

108% is equivalent to 70 200.

1% is equivalent to $\frac{70200}{108}$.

100 % is equivalent to $\frac{100 \cdot 70200}{108} = 65000$ *Auth. Transl.*

This example is found in all the textbooks. This task is categorized as a proportional reasoning task called missing value. The MO represented is dynamic (MO_1). Direct proportion.

2 Multiplication by a relationship

This solution technique is found in the chapter about percent in several Swedish textbooks, here Liber Pyramid (Wallin et al., 2000, p. 43):

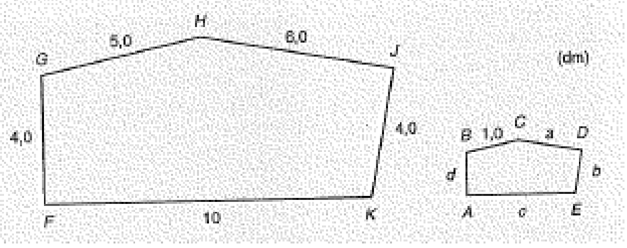
> Anna has a salary of 17 250 *SEK*. She got a rise in salary with 4 %. How much is her new salary?
>
> The new salary is 100% of the old salary and the salary rise of 4 % of the same salary. The new salary will be: 104% of 17 250 *SEK* and that will be $1{,}04 \cdot 17250\ SEK = 17940\ SEK$. *Auth Transl.*

This is a typical example in all five textbooks. I interpret this solution technique to use the same technique as in Hersant's example but here different data is used. The tasks is categorized as a missing value task and the notion of proportion is dynamic (MO_1). Direct proportion.

3 Use of proportion

This special solution technique is to be found in general in the geometry chapter. An example from a Swedish textbook (Wallin et al., 2000, p. 122):

> The pentagon *ABCDE* is similar to the pentagon *FGHJK*.



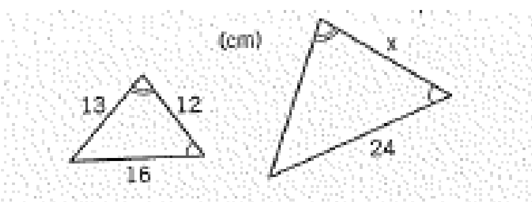
Calculate the length of the sides *a, b, c*, and *d*.

From the similarity it follows, $\frac{a}{6}=\frac{b}{4}=\frac{c}{10}=\frac{d}{4}=\frac{1}{5}$ From the first and last equality we get $\frac{a}{6}=\frac{1}{5}$, $a=\frac{6}{5}=1{,}2$. In the same way we get b = 0,8, c = 2,0 and d = 0,8. *Auth. Transl*.

This category is not very common, it is only found in two textbooks. This task is analyzed as a missing value task and the notion of proportion is static (MO_2). Direct proportion.

4 Cross multiplication

This particular solution technique is found in the chapter about geometry (Alfredsson, et al., 2008, p. 153):



In the figure the angles are marked with the same sign if they are in the same size. Calculate the length of x. The triangles are equal in two angles then they are similar and the ratio between two sides is equal. *Auth. Transl*

$$\frac{x}{12}=\frac{24}{16}\,.$$

$$x=18$$

A very unusual solution strategy, only to be found in one textbook. This task is analysed as a missing value task and the notion is represented as a static notion (MO_2). Direct proportion.

5 Use of coefficient

This category can be found in the function chapter (Gennow et al., 2003, p. 301):

In a physics experiment, the students were measuring mass and volume for different amounts of aluminium tacks. First, the students weighed the tacks and then they poured them into a graduated measuring glass with water. The findings from one of the groups were:

Volume (cm^3) (V)	12	17	22	29	38
Mass (g) (m)	32	46	59	78	103

Determine the density (i.e. mass/volume) of aluminium if it is in proportion.

For this proportion to be valid *k* have to be $k = \frac{m}{V}$. We chose a pair of numbers far away from the origin of coordinates to increase the accuracy, draw lines to *x* and *y* axis.

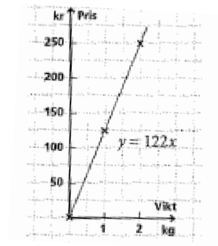
Reading gives

V= 40 cm^3, m = 108g, $k = \frac{108\ g}{40\ cm^3} = 2{,}7\ g/cm^3$ *Auth. Transl.*

This category is found in all the textbooks. This task is categorized as a missing value task and the notion is static (MO_2). Direct proportion.

6 Other possible techniques

An example of this solution technique comes from the chapter about functions (Norberg et al., 2004):



Anton buys prawns for the cost of 122 SEK / kg. Express the cost for the prawns that he buys as a function of the weight in kilograms.
x = kilograms of prawns, y = the total cost, 1 kilogram of prawns costs 122 SEK. If he buys x kilograms then the cost is $k = 122 \cdot x$. Put in some different x in a table of points and sketch a graph. Auth. Transl.

This type of solution is found in all analysed textbooks. This task is categorized as a missing value task and the notion is dynamic (MO_1). Direct proportion.

Discussion

The five textbooks investigated offer rich variation in types of tasks and techniques. The two notions of proportion (dynamic and static) are both represented but justifications are rare. Thus two MO’s are presented in different chapters (percent, functions, equations and geometry) with no link pointed out between them, which can be misleading for both teachers and students in their practice and might result in a predominance of the dynamic notion. It appears that the static notion is represented to a higher extent in the chapters about geometry and the dynamic notion more in the chapter about percent and functions whereas the equation chapter is a mix of both MO’s. The theoretical description of proportion appear similar in all the textbooks and not presented by way of different approaches in parallel. Proportion is also represented mainly as direct proportion with a few exceptions, which may be problematic as for example also inverse proportion is important for the further mathematics studies. Justifying technologies are very often missing. The explanation might be that justification is not a learning goal in the curriculum for this first basic course (Mathematics A). This study has also illustrated how the particular analytical tool developed for investigating tasks can be used as an instrument for what types of “knowledge” and “know-how” are represented in mathematics textbooks. This might be a benefit also for teachers in their practice by providing principles for the selection of tasks. However, if the students really take up the techniques presented in the textbook is another research issue which will be investigated in a follow-up paper about students' solutions of proportion tasks.

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3

ATD and CoP in a framework for investigating social networks in physics classrooms

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Abstract. *The article presents a tool for analysing transcribed and annotated video recordings. The tool relies on a network representation of the data, where the nodes derive from categories of activities. Following a summary of the observed learning situation, it is suggested how anthropological theory of the didactical (ATD) and communities of practice (CoP) can be incorporated in the network representation in order to investigate student discussion networks in physics classrooms.*

Introduction

One of the major concerns for researchers in physics education is whether the students of a given class learn physics or not. Indeed, it is not clear what it means to have learned physics. Learning physics may consist of mastering different modes/forms of representation (Dolin, 2002), or it may amount to refining and organising conceptual structures to be retrieved as appropriate (diSessa, 1993). One view is that the cognitive development of the individual learner needs to be in focus when investigating science (and therefore physics) learning. A complementary view is that learning in general, and therefore the learning of physics, is the development of shared repertoire by a group over time (Wenger, 1998). However, what the dynamics of physics learning actually is seems to be an underdeveloped area.

One strategy for shedding light on how physics is learned is to observe learning situations in physics classes where the students engage in an appropriate assignment. The assignment - and the related tasks needed to complete it - may be designed or initiated by researchers or more naturally occurring (designed or initiated by teachers or even students) in the course of a typical physics class. The aim in either case is to make a detailed analysis of a particular learning situation, in order to identify the processes by which physics is learned.

One opportunity for achieving data for such analysis is to video record students participating in group discussions about a particular physics problem. The video recordings may serve as documentation for some of the processes going on in the learning situation, and they may be analysed qualitatively or as a mixture of qualitative and quantitative data (Johnson & Onwuegbuzie, 2004). With the proper use of qualitative methods and theory, such as thematic analysis (Braun & Clarke, 2006) and ATD/CoP, one should be able to identify categories consistent for the theory for a further quantitative treatment. In this paper, the quantitative treatment utilizes network theory as it is named in physics or graph theory as it is named in mathematics.

The *anthropological theory of the didactic(al)* (ATD) and *communities of practice* (CoP) are both theories aiming at understanding the activities of humans. ATD focuses on the strategies and justifications (praxeologies) for solving problems which can be abstracted from a non-individual oriented analysis of teaching, learning situations, and the materials used in them (Rodriguez et al., 2008, Artigue, 2006).

CoP on the other hand, as applied to an educational perspective, has four aspects, namely: participation/reification, designed/emergent, local/global, and negotiability/identification (Wenger, 1998), all pertaining to individuals or sets of individuals. Wenger denotes these pairs of concepts dualities, because they exist as two parts of a coin: without participation from learners existing reifications become meaningless and on the other hand reifications such as (but not exclusively) teaching materials, are the basis for participation.

Some research (Shaffer et al., 2009) suggest that observational data from a framework resembling CoP, can be described with a graph theoretical approach where appropriate categories of action in the learning situation are recorded as well as the order in which they occur. In this paper, a notion of representing observational data is pursued with the ultimate aim of making relations, similarities, and differences between ATD and CoP explicit. This is a challenging task, since the two theories have different foci.

In this work, the video recorded *activities* of four Danish high school students, one male and three female, which serve to develop a network representation. The network shows how the dynamics of a conversation about physics can be illustrated using a set of categories. The "data" is a 400 second recording of the students engaging in a physics problem on predicting the ranking of currents at different places in an electrical circuit.

This particular situation has been selected from a larger data set (approximately 1 hour with different students), because it is suitable for a first attempt of describing a student discussion using networks, ATD, and CoP. First, the students actually engage in discussion with each other. Second, as will be shown, one student arguably changes her problem-solving strategy for a specific physics assignment, while three other students do not. Third, the technical quality of the material has enabled the researcher to clearly code the recording.

The focus in this paper is on describing learning situation as a network of categories. Also, this paper investigates the question of how physics praxeologies of ATD and shared repertoire of CoP are connected and may be captured by a network representation of the situation. Ultimately, the questions I wish to answer is, (1) what change in behaviour (utterances, gestures, actions) do the students display when about electrical current in a specific system for each of the participants and (2) what processes of communications are observable in the analysed data set?

Background

To learn science and in particular physics can be viewed as learning to work with different kinds of representation (Boulter & Buckley, 2000, Dolin, 2003). Dolin (2002) argues for seven distinct types of representation, namely the *phenomenological*, *conceptual*, *mathematical-symbolic*, *mathematical-graphical*, *experimental*, *pictorial*, and the *kinaesthetic*. To master a subject in physics is to master all the representations to some level given by the educational system.

It can be argued that at least some of the knowledge inherent in using these forms of representation is tacit. For example, an undergraduate student working with mechanics might not be aware that a specific conceptual schema is at work, when he explains how a moving object is affected by an applied force (diSessa, 1993). At some point however, the tacit knowledge in play in an observed teaching situation should be expressed by the learner in a way discernable to the observer, if this observer is to claim that learning has taken place. Otherwise, claims of learning become just that: claims.

However, one must acknowledge that there can be many signs of learning taking place or of a student possessing knowledge, which are not stated verbally or explicitly written down. Gestures may play an important role in student communication, as has been shown by Roth & Lawless (2002). They found that students working with physics develop gestures which start out as actual depictions of the physical processes or objects, where the gesturing start before the actual words they utter, while they end up being short hand metaphoric versions of the same processes or objects, which are performed almost exactly at the same time as the words accompanying them. This also serves as an example that when learning, people change their behaviour in an observable way. Their actions change.

In order to quantify actions using a network approach [1], Shaffer et al. (2009) used a framework called the *epistemic frame,* claiming that a *culture* has a *grammar* composed of *skills*, *knowledge*, *identity*, *values*, and *epistemology*. These categories form the basis of a coding scheme, where if a subject in an activity seen from an observer's perspective as belonging to the category *skill* then the activity is labelled accordingly. Thus, these categories can be named activity categories. This framework is reminiscent of the work of Wenger (1998) where a *community of practice* has a *shared repertoire*. One difference is that the framework of Shaffer et al. has been operationalised in their work, whereas Wenger does not operationalise his theory in a specific research design. Further similarities and differences are beyond the scope of this paper, and will not be pursued further. However, their translation of the framework into a time based network description can be helpful for the present work.

From their framework Shaffer et al. (2009) make a number of categories, and put their data in to these categories. Then they analyse data one time segment, δt, at a time. If two "activity categories" are performed within δt, then a non-directed link is drawn between the two. As time passes, some nodes become more connected than others resembling the fact that some types of action are performed more often than others. Looking at the whole network, as time passes may give information on what actions are tightly linked, and what actions are more isolated.

Transferring information about learner activities from the qualitative framework of epistemic frames to the quantitative network descriptions allows for mathematical analyses of the network, which can then be reinterpreted into the qualitative framework. This allows for an evidence based assessment of the learning situation rooted in very direct way on a qualitative theory. In the case of Shaffer et al. (2009), where they analyse the actions of persons playing a learning game called *Urban Science* (see Schaffer et al. (2009) for details), they hypothesise that learning progression can be connected to the structural properties of a developing network.

Shaffer et al.'s (2009) approach is interesting to this paper. They define relevant categories on the basis of a theory and code some research material (qualitative data), and transfer these codes to a mathematically rooted network representation (quantifying the data). From the networks different structural properties of the activities are then extracted and re-interpreted in to the theory.

The theoretical frameworks used in this work are different, because they are not based on the assumptions of epistemic frames, but they also have their categories. In ATD categories describing learner activities should be discernable from relevant praxeologies (Tetchueng et al., 2008). Praxeologies consist of a theoretical block (justifications) and a practical block (the activities). The theoretical block is subdivided into a theory and technology section, while the practical block is divided into tasks and techniques (Rodriguez et al., 2008).

Students solving problems in physics as analysed by ATD requires the notion of a praxeology. To solve a problem requires the activation (or development) of an appropriate praxeology or set of praxeologies. A problem can be decomposed into a set of tasks, each of which are dealt with by some techniques, each of which are again justified by a technology, which is a way of employing theory in practice (Tetchueng et al., 2008).

Tetchueng et al. (2008) made a learning management system for teacher education where they created *predefined* task-technique pairs for solving problems. The work relies on experts defining the correct task-technique pair, and learners acquiring them. In this work, I will go the other way around. From the data, praxeologies (task-technique-technology(-theory)), should be abstracted in order to see, what problem solving strategies the students subscribe to and why.

In ATD, the term "why" refers to the disciplinary justification for using a given technique (Chevallard 2006). So, if we ask why a student enacts a given technique, we are not expecting an answer based on the student's social situation or how the student feels at that time. However, in CoP it would be a question of the shared repertoire in the community and the identity of the student (Wenger, 1998). Thus, in CoP asking why *also* entails social and identity factors.

From this it seems it seems likely that the praxeologies from ATD may fall into the shared repertoire of a community of practice, while the praxeological equipment (Rodriguez et al., 2008) of the student would be part of the students' identity. Here, the praxeological equipment is interpreted as the praxeologies available for a student to enact in a given context.

However, this author believes that disciplinary praxeologies are not exhaustive when dealing with the dimensions of a learning situation (see Wenger, 1998, chapter 12) as seen from a CoP point of view. This is because the CoP aspects of learning include more than the disciplinary aspect, for example student motivation and student-student relations. The following summary aims both at describing the disciplinary part of the students' discussion, but with a focus on the individuals and at highlighting some of the student-student interaction.

Short description of the learning situation

Four students are working on an assignment where they are to rank the currents (predict the read-outs of four ampere meters) in an electrical system. The electrical system is represented on-line in an interactive computer model. The assignment is on a piece of paper given to the students by the teacher. The setup is shown in Figure 1.

The assignment is to predict the ranking of the ampere meters, observe if the ranking is correct by consulting the computer model [2], and finally they should explain the result. However, in the process of predicting, the students explain their thinking to each other. Thus, they rather suggest predictions to each other, discuss and explain their predictions, and finally check that they have the correct answer.

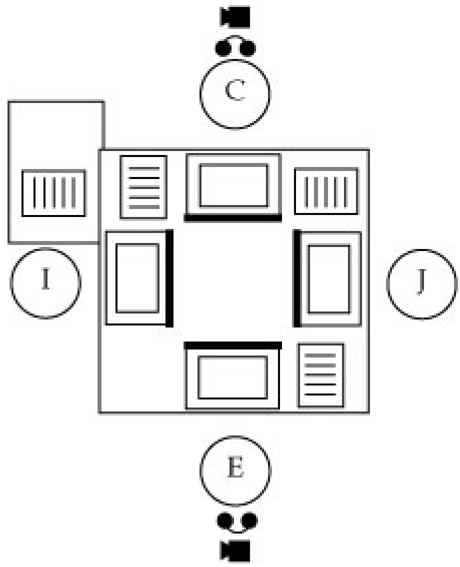

Figure 1. The student setup. Catherine (C), and Elin (E) are wearing glasses with built-in video and audio recorders, while James (J) and Idun (I) are not. They are sitting around a table, with laptops and assignment papers in front of them. Idun has a small table on the side. The resolution of the video recording is such that one can see what a student writes on a piece of paper provided that the lighting is suitable.

The circuit given in the assignment is described in Figure 2 along with an explanation of the solution.

The two video cameras deserve mentioning. They are integrated in the frame of a set of sunglasses along with a microphone. The actual glasses have been removed leaving only the frame. The resolution of the camera is 800x600 pixels; it adjusts to lighting conditions, and records at 30 frames per second. In good lighting it is possible to see what a student writes on a piece of paper, or how she uses the mouse on a computer screen. The microphone makes it possible to hear what the wearer says clearly, and to some extend what a person a meter away is saying. In the configuration shown in Figure 1, it is possible to synchronize the two cameras audio output to enhance the sound quality. This makes it possible to hear almost everything being said around the table.

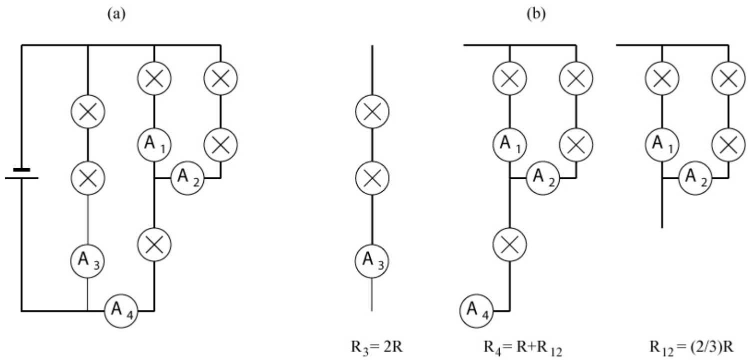

Figure 2. (a) The circuit given in the assignment. The circles with X's in them represent light bulbs with equal resistance, and the A's represent the ampere meters. One solution to the assignment relies on use of Ohm's law, knowledge on how to calculate the resistance in a parallel connection, and Kirchoffs current law. (b) A division of the circuit into it's constituent parts.The resistance of each part is given below the drawings. Since the resistance of $R_4 < R_3$ it follows that $A_4 > A_3$. The resistance of R_{12} is calculated as $\frac{1}{R_{12}} = \frac{1}{R} + \frac{1}{2R} = \frac{3}{2R}$. Note! In (b) the first two parts will be referred to as the main parts .

Here is a summary of the main points in the discussion as derived from the transcript and annotations: In the beginning of the clip Catherine states the ranking: $A_3 > A_4 > A_1 > A_2$, where the first inequality is wrong since $A_4 > A_3$ (see Figure 2). In the proceeding discussion James argues (correctly) for switching A_3 and A_4. Catherine's approach is initially to find the ranking by looking at pathways for the electrons to travel, while James has calculated the resistance of to two main loops. James convinces Catherine, and explains - to both her and Elin - in words and with extensive use of gestures, how to perform the calculations he has used to solve the problem.

During the discussion, Elin tries to understand what's going on, while Idun is almost absent in the discussion. At the end of the clip, Elin asks Catherine to explain the calculations, and Catherine does this.

With respect to the forms of representations (Boulter & Buckley, 2000, Dolin, 2003) Catherine uses, they can be regarded as a mix of conceptual, visual, and kinaesthetic (gestural). The conceptual form involves the theories, concepts and generalizations she uses to explain her thinking. She frequently uses the visual representation on the paper and on the screen as a support for her suggestions, and she does this mainly by gesturing with her hand/pencil. Because the visual and kinaesthetic forms seem auxiliary in this case, her approach is denoted *conceptual*.

While a conceptual approach is not wrong, Catherine does make explicit a limited conceptual understanding of the nature of electrons in a circuit. She says that the electron(s) go where there *are fewer light bulbs*. When applied to the ampere meters A_3 and A_4 this understanding yield wrong predictions. Later she changes her explanation by using a different and more correct statement: *where there is least resistance*.

James makes explicit during the discussion how he has calculated the resistances: He identified the two parallel connections in the drawing (Figure 2). He calculated the resistance of the smaller of the two parallel connections, the one labelled R_{12} in Figure 2. Then he added this resistance to the resistance of the remaining light bulb in that part of the circuit, yielding R_4. In the discussion, he argues that since R_4 is smaller than R_3, the current running through A_4 is larger than the current running through A_3.

From the perspective of the "learning" which can be observed, James does not show (much) evidence that he has learned how to solve the problem. He has already learned how to do the calculations, has done them, and thus the task he saw and the technique activates during the clip are already developed.

His contribution is to tell the others (mainly Catherine) how to perform calculations and how to transfer the pictorial representation to a set of mathematical representations. Thus, he may have learned how to communicate his knowledge about how to perform the calculation of currents in the circuit shown in Figure 2.

Since Elin needs to ask Catherine how to solve the problem at the end of the clip, she doesn't seem to have learned how to solve the problem. Catherine begins the clip with a partially wrong statement.

She has seen another task than James and uses another technique to engage with it. She not only changes her own explanation, but can explain why to Elin towards the end of the recordings. Therefore, she shows that she can enact the task-technique pair which James explained to her.

Of course, one can argue that Catherine just needed a nudge; that James actually learned something new from Catherine's representation; and that Elin is in the process of learning. These claims however, need to be substantiated either by a more extended observation period where they solve new problems, and/or by a more detailed analysis of the data. Another possibility is pursued in the following.

Network description of the learning situation

The goal with a network description is to gain knowledge about the learning situation, which is not as apparent from a linear representation of the transcripts and annotations. In the annotation process, three categories have been predefined for each participant, namely *spoken words*, *actions*, and *gestures*.

These three categories represent what is observable on the video: a participant speaks and/or performs an action. An action may be a (non-facial) gesture, a facial gesture, or some other action. While spoken words, gestures, and actions are described below, facial gestures have been left out for simplicity of analysis. As will be discussed later, these categories could be further subdivided in to more specific ones, such as the gesture for the fraction 2/3.

A *spoken word activity* begins when the first word is uttered, and ends with a natural pause from the student, for example if the student stops for a brief moment of time to think or when the student is interrupted by another student. The difference between action and gesture is not well defined, but as a rule of thumb an action here is some sort of movement which is not linked to linguistic meaning, while gestures are. Gestures may be for example pointing at a drawing of the circuit in question while talking, drawing a fraction in the air, or accentuating one's words with the hand. All these examples relate to something linguistic, while a repositioning of the hand or grabbing a pencil does not. Also, changing the direction of ones vision is regarded as an action.

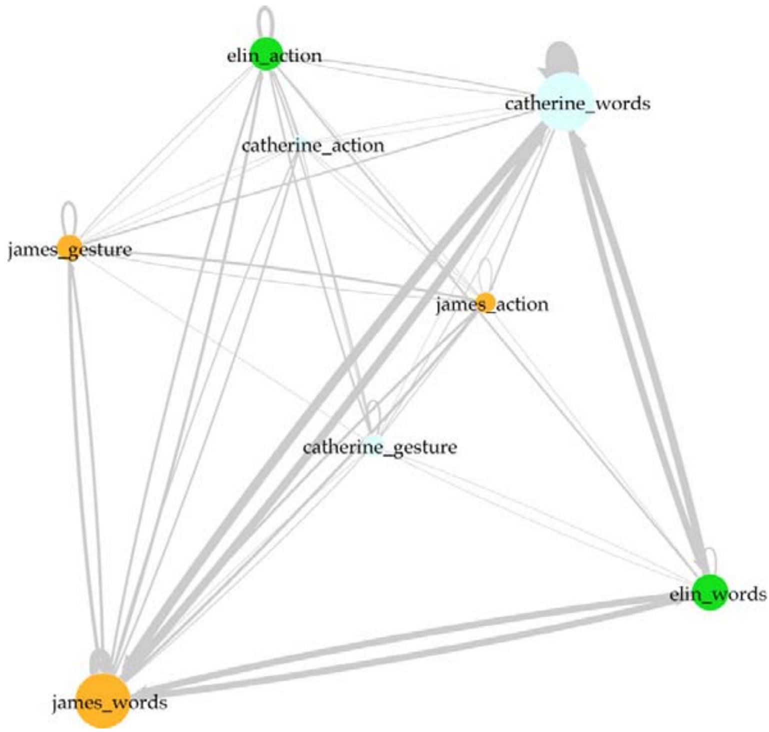

Figure 3. A crude network representation of the transcribed and annotated data. Each of the students, have been assigned three activity categories: words, gestures, and actions. The difference between gestures and actions is not clear. Actions are movements, which are not tied to linguistic meaning. Thus pointing with a pen while explaining how to do calculations is a gesture, while turning ones head, or repositioning the hand are actions. The size of a node is proportional to the time spent doing the activity the node represents. The thickness of the links, are proportional to the number of times one activity follows another other. The loops signify how many times one activity is followed by the same type of activity. Note that the links are directed. The positions of the nodes have no immediate significance. They are the result of an algorithm which tries to minimize overlaps between links.

Using a video annotation program [3] capable of frame by frame analysis of both video and audio makes it possible to very accurately decide what type of activity begins first. It is thus possible to decide for example whether a gesture comes before spoken words or vice versa. Meticulously marking all spoken word, gesture, and action activities from their beginning to their end also yields the total time spent for each student on each activity. Figure 3 shows a network representation of the transcribed and annotated data. It is a very crude representation, which quantifies how many times different actitivites follow each other (speech, action, and gesture), and how much time is used on a given type of act. Even though it is a crude representation, it still reveals information about the situation. First of all, Idun is not present in the representation. This is because she barely says anything, so her nodes would be markedly smaller than all the others. Second, words are most abundant, and Catherine and James utter most words. James uses the most gestures, followed by Catherine. Elin does not use any gestures during the period of observation. Elin performs most actions, a lot of which is switching between looking at James and Catherine.

Integrating ATD and CoP in the network

As seen from the perspective of ATD the important information is what praxeologies are under development and which ones survive the learning situation. If a network description is to be relevant to ATD, it needs to capture the different parts of the praxeologies as they play out. Also, one can view the learning situation above as a battle between two praxeologies. During the time of observation, two main techniques are in play, namely Catherine's conceptual technique and James' mathematical-pictorial technique. See Table 1 for a description of them.

A lot of time is used explaining and discussing James' technique. In fact, most of the conversation represented by the large nodes of James' words and Catherine's words and the arrows between these two categories, is devoted to the discussion of James' technique. It appears to be desirable to the other students to learn, even though Catherine at some point expresses a more correct version of her original technique. This is apparent in the video clips where Idun asks the others to explain their thinking later, and Elin asks Catherine to explain the technique to her.

The network depicted in Figure 3 does not capture these praxeologies, but may serve to overview some of the dynamics of the situation. Clearly, most of what is going on is a discussion between James and Catherine, with some input from Elin. Due to the fact that James uses a lot of gestures while explaining, one can also see a notable connection between his use of words and gesture. The gestures are usually short handed, meaning that they occupy a small amount of space and time (Roth & Lawless, 2002). Since the time used on gesture is small compared to words, gestures are not represented by a large node in the network.

Also, Elin does not use any gestures, but has a fairly big action node. This is because she spends a lot of time looking back and forth between James, Catherine, her notes, and to a limited extent, her computer. While this can also be discerned from watching and annotating the video material, the network representation makes the time spent on action/gesture/spoken words very explicit in a quantitative way. As such, this *particular* network representation can give an overview with some of the qualities of graphs, histograms and number tables. But it is also a graphical representation of the structure of the communication, which may serve qualitative purposes.

The question to ask is what observational categories would allow us to capture the praxeologies in play or under development in the learning situation (for example the ones shown in Table 1). First, students' words and gestures are important indicators. That is, what the students explicitly say and do, tells us both about their techniques and their justification for these techniques. James, for example seems to use a circling gesture a lot while explaining how he identified the parts of the parallel circuit. That is, he circles the parts of the circuit he is talking about. Catherine argues, as mentioned that *electrons want to go where there is least resistance*. The light bulbs are always referred to as *R*, that is resistors, and not light bulbs. These three examples could be converted to categories in a network representation with a focus on the disciplinary strategies and justifications of the students, because they are used many times by the student to express their thoughts.

The dynamics by which the praxeologies are developed by students is a different area than the description of praxeologies. It can be argued that one has to go into the dynamics of groups to see how this happens. These dynamics may be particular for the individuals in the group, but like disciplinary praxeologies exist beyond the individual student, so might group dynamics (social praxeologies?) exist which can be abstracted from the particular.

As seen from a CoP perspective, praxeologies could be a part of the shared repertoire of some community of practice, since the shared repertoire includes (but is not limited to) routines, words, tools, symbols, gestures, and concepts (Wenger, 1998). These are all part of

Table 1. A list of the three praxeologies which the students enact to answer the assignment. The formulations are re-wordings of the students own utterances. The middle one is a more correct version of Catherine's praxeology. In ATD literature Θ denotes the theory part, θ the technology, τ the technique, and T the task (see e.g. Tetchueng et al., 2008).

Catherine (initial)	Conceptual	James
(Θ^-: N/A)	(Θ: Current as a fluid, conservation of charge, Kirchoff's current law)	(Θ: Ohm's law, Kirchoff's voltage and current laws.)
θ^-: Current (movement of many electrons) is equivalent to the path of a single electron.	θ: Current consists of many electrons. Offering many pathways limits queueing.	θ : Constant voltage for two main parts. Ohm's law used on each main part.
τ^-: Follow the lines of the diagram. At a junction, evaluate which path leads to fewest light bulbs.	τ: Follow the lines of the diagram. At a parallel connection, electrons have more paths than at a serial.	τ: Mark the parts that are parallel (serial). τ: Use appropriate formula
T^-: Follow the current and identify path with fewest light bulbs.	T: Follow the current and identify path of least resistance.	T: Identify the parallel and serial connections. T: Calculate the resistance.
Assignment: Rank the ampere meters in the circuit shown in Figure 2. Then, observe the correct prediction. Explain your thinking.		

the negotiation of meaning, which is a crucial concept in learning, as seen from Wenger's point of view. It may be that a set of praxeologies in use in a certain institution overlaps with the shared repertoire of communities of practice embedded in the institution. The analysis at hand can be seen as the clash of two praxeologies. One of them looses terrain during the clip, but the two do not inherently exclude each other. They could both be part of the praxeological equipment of the students.

Say that the four students in this clip formed a short lived community of practice. It may never be assembled again, if the four students never work in a group again with the shared enterprise of solving physics problems. However, one may expect that James for example is part of a larger and more lasting community of practice – *the boys in the class* [4]. If James were to learn the conceptual praxeology, he could bring it with him to the community of practice consisting of the boys in the class. If it was used by the other boys as well, it would then be part of the shared repertoire of their community of practice.

In Wenger's frame work, the negotiation of meaning relies on whatever means for mutual engagement and reifications are available to the learners. As seen from this perspective, the paper, the computers, and other objects are possible nodes in the network, but also the fraction $\frac{2}{3}$, which the students refer to many times, both in words and in gesture.

A possible extension of the categories in order to gain a new perspective on student interactions would be to use Wenger's (1998) four aspects relevant to the learning situation. For each of these, one could find relevant categories to put in the network description, thus expanding one's analysis of the student interactions.

Summary

A 400 second long learning situation was analysed with the purpose of developing a network description of the dynamics and praxeologies in play. A crude network representation was presented, along with both a description of some of the praxeologies in play and a short-hand qualitative description of the dynamics of the situation. These descriptions were used to produce suggestions for node categories in further analysis of this and other learning situations. Further work should focus on extending these categories and on relating them to what the students learning progress.

End notes

1. Networks consist of a number of nodes (circles) and links (lines between circles). Links can be either directed or non-directed. One use for networks are social networks, where

the nodes represent people, and the links resemble interaction. See Sneppen (2006) for further details.

2. An English version of the software can be found here:

   http://www.phy.ntnu.edu.tw/ntnujava/index.php?PHPSESSID=7095d39b86bb20a46ce8195c88c63e2c&topic=1500.msg5680#msg5680

3. The software used here is ELAN, which can be downloaded here: http://www.lat-mpi.eu/tools/elan/

4. Incidentally the boys in this class where tightly knit together. They sat close to each other in other class room situations and when asked who they work with in physics, they always named each other.

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A Study of Problem Solving Oriented Lesson Structure in Mathematics in Japan

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Abstract. *This paper presents and analyses "Mondaikaiketsu no jugyou", the problem solving oriented – approach which is a set of didactic techniques with the aim of motivating the students' positive attitude toward engaging in mathematical activities and fostering mathematical thinking. As an analytical tool, The Anthropological Theory of Didactics will be applied.*

Introduction

Teaching methods were developed differently in Japan compared to other industrialized countries. Hiebert, Stigler and Manaster (1999) analysed, by studying videos from the material in the TIMSS study, differences in teaching methods and interaction in the classrooms between Japan, Germany and the United States. They argue that Japanese teachers emphasize mathematical thinking, in particular, the development of the pupils' attitudes toward and ability to communicate mathematics, rather than mathematical skills as a goal for the lessons and choose problems starting the lesson that can be solved by varying methods developed during previous lessons. This goal is reached by having the students discuss with the teacher and peers on the settlement options. I will call this type of didactic techniques, where students work on whole-class problem solving, for *problem oriented lesson structure* (POLS).

A basic problem in mathematics education, and in the training of teachers, is to find ways how to organise the classroom work so as to make the students active learners of mathematics, without losing the focus on the mathematical content. Japanese teaching methods, like the ones described, have attracted attention in Sweden lately (Dagens

Nyheter, 2009), and it has been discussed as a possible model to develop in Swedish school system.

Kazuhiko Souma is one of the teacher educators/researchers who has proposed and introduced POLS. He calls his method "*The problem solving oriented*" approach (shortened to PSO; the author's translation; "*Mondaikaiketsu no jugyou*", in Japanese). The PSO approach is a set of didactic techniques, with motivation based on psychological aspects of learning. Its purpose, like POLS, is to enhance the students' attitude towards engaging in mathematical activities in the classroom. In Japan, there is a tradition of publishing practical books for mathematics teachers as the target group. This inspirational literature, often written by practising teachers, aim to present ideas and concrete lesson plans, based on well-constructed mathematical problems, that connects to the proposed teaching methods (Souma, 1995; Kunimune & Koseki, 1999, Tsubota, 2007), There are number of practical books for teachers in Japan showing examples of possible lesson plans. Souma has written and edited a number of such books and his method is actively and widely used by teachers in service in Japan, but has received little attention from the academic community, perhaps because of its practical attribute and the lack of clear theoretical base. The lack of theoretical overhead is perhaps part of the appeal, but is certainly also a problem when one is to describe and assess such approaches. As a tool, I will use the anthropological theory of didactics (ATD) developed by Y. Chevallard (2006). ATD supplies a framework for the analysis of how the didactic process relates to and transforms the mathematics taught in educational institutions and the didactic process is described as an organised collective work aiming to construct a *mathematical organisation* (MO).

I conjecture that PSO could make a powerful impact during pre-service teacher education due to its distinct didactical structure, but the lack of theoretical base is a hinder. The aim of my paper is therefore to present and analyse PSO in relation to other Japanese POLS approaches, where I incorporate ATD as a tool for the epistemological description, with the assumption that, this is useful for the purpose of didactic planning. To illustrate, I will analyse an episode of practical application of PSO in a Japanese classroom.

Background to the PSO approach

The Anthropological Theory of Didactics

The anthropological theory of didactics (ATD) approaches learning as institutional issues. Mathematics learning can be modelled as the construction, within a context of social institutions of interlinked *praxeologies* of mathematical activity, which is also called a mathematical organisation (MO) (Chevallard, 1999 in Barbé, et al., 2005). A praxeology is described by its *tasks* and *techniques (praxis),* together with its *technology* and *theory (logos)*. Technology constitutes the tools for discourse on and justification of the techniques and the theory provides further justification of the technology and connections to other MOs.

The process, under which a mathematical praxeology is constructed within educational institutions, is called the *didactic process* (ibid.). Chevallard proposes to describe it as being organised in six "moments" that can be thought of as different modes of activity in the study of mathematics. The moments are: (FE) the moment of *first encounter* (or re-encounter) of tasks associated to the praxeology, (EX) the *exploratory* moment of finding and elaboration of techniques suitable to the tasks, (T) the *technical-work* moment of using and improving techniques, (TT) the *technological–theoretical* moment in which possible techniques are assessed and technological discourse is taking place, (I) the *institutionalisation* moment where one is trying to identify and discern the elaborated MO, (EV) the *evaluation* moment which aims to examine the value of the MO.

To organise the work of achieving an appropriate MO, control the didactic process, the educator develop a *didactical organisation* (DO) with techniques to design a didactic process. It is possible to use ATD to describe a didactic organisations, in terms of praxis and logos block, independently of whether the studied DO's have ATD incorporated as an epistemological model, but in this paper I will attempt to use ATD to describe and motivate the techniques in the DO proposed by Souma. Thus, in a way, propose an *extension* of the DO named PSO, with a technological-theoretical block from ATD.

Souma, like most Japanese writers of this genre, often points out the need for general didactic techniques, like giving generosity with positive feedback in order to handle the long-term didactic goals, such as "fostering the students to active learners of mathematics". The motivation is usually taken from a technological-theoretical block, which could be referred to as "didactical common sense", where the epistemological model is usually a concrete description of the mathematical situation. In this description I will incorporate the "motivational" technology as "qualitative measures" on the didactic process, like the degree of *participation* in the didactic process. A more problematic, but central, recurring term is that of "mathematical activity", which is a concept that measures the degree of participation, interest, independence and motivation with which students are carrying out the mathematical work. A technological term I will use is "*invigorate* the didactic process" to mean, "increasing the *activity*" of the didactic process.

The PSO lesson template and technological terms

I will here describe PSO in the form of a "lesson plan template". Souma states that it is instrumental that the PSO approach is applied with the same basic form regularly. The motivation is that familiarity with the situation makes the students feel more secure in participating in the discourse and engaging in the didactic process.

According to Souma's example from his book (1995), a typical POLS lesson starts with a teacher giving a problem, for instance, "Show that the difference of the squares of two integers that follow each other is equal to the sum of the two numbers ($5^2 - 4^2 = 9 = 5 + 4$, $24^2 - 23^2 = 24 + 23$ and so on)." The students try to solve the task and some students who have solved the task write their answers on the blackboard. Then the students explain their solution orally. Souma wonders (pp. 103-104) if the students in this situation will feel a "necessity" to reflect upon the task. Furthermore, some students might not get any ideas on how to solve the problem and will therefore become alienated from the discourse. As an alternative, he proposes the following variation: The teacher writes down expressions on the blackboard without any comments;

$$5^2 - 4^2 = 9,\ \ 24^2 - 23^2 = 47,\ \ (-9)^2 - (-10)^2 = -19$$

and asks the students what they can observe. All students are supposed to be able find such observations, perhaps working in groups. Students may answer "It becomes odd numbers", "The differences equal the sum of the integers", "The differences equals the first integer times two minus one", "The last integer times two plus one". After the response of the students, the teacher then controls that all proposals are correct on the blackboard and says; "Now we try to prove each of the statements". Ideally, the formulated problems have many possible roads to solutions: Several students may use the formula for expanding the square of a sum; and several others, using x to the first integer and y for the second integer, the rule of the conjugate.

Souma proposes to use a didactic technique, which I refer to as *guessing*. One should, regularly, let all students guess an answer, state hypotheses or formulate questions about the phenomena. It is implied that the "guess" is something that all students can participate in. In the example the students are not, strictly speaking, guessing an answer, but they are invited to, discover patterns by themselves, make hypotheses about the phenomena and by implication set their own tasks. By committing to make a guess or a hypothesis, especially in the social context of the class, the student will have a stronger motivation to study the task and follow it up. Thus using guessing will *invigorate* the didactical process.

In his book (1997), Souma declares that he is inspired by John Dewey's theory of reflective thinking. Dewey (1933) presents five cognitive phases of problem solving. 1. Recognize the problem. 2. Define the problem. 3. Generate hypotheses about the phenomena. 4. Use reasoning if the hypotheses are viable to solve the problem. 5. Test the most credible hypotheses. Dewey's theory has a general scope and is applicable to any problem context. It is also concerned with the cognitive dimensions, rather than the didactic process as such. Souma states that educators in mathematics may have a tendency to hurry up to address the later phase to "use reasoning". In this way, the development of reflective thinking and motivation may be impeded. Souma thus feels that it is necessary to pay attention especially to the first three phases. He expresses that (1997), from Dewey's theory, we may infer that it is important that we should "(a) have an aim for why we solve the task, (b) feel a necessity to solve the task and (c) have made hypotheses before starting the reasoning process." (p. 34) Souma also refers to Polya's (1957) cognitive theories on problem solving and, in particular, Polya's insistence on the importance of guessing. Polya states that our hypothesis may of course be wrong, but the process of examining the guess should lead to improved hypotheses and a deeper understanding.

The focus on motivation on the first encounter and the exploration, together with the insistence on a well defined mathematical content, is perhaps the point that, most distinctively, sets PSO apart from other proposed DO's in the POLS tradition. Souma states that the teacher much take care to plan how the problem is presented and how students are supposed to act in relation to the presented problem. Souma names (1987) the type of tasks a teacher should aim at, *"open-closed" tasks*. It means that the tasks, apart from stimulate conjecture and application of guessing, should lead to multiple methods of solution etc., be constructed so as to later stimulate a discourse on theory that should stay somewhat focused on the well-defined subject that the teacher aims to cover. In ATD terms one can say the task should be “closed” so as to give a well defined and *controlled* vector from the (FE), the moment first encounter, to (EX) and (T), and also a predictable outcome during the following discourse, which usually would concern the establishing of the technological-theoretical environment (TT). The task should also be “open”, by giving the student a chance to make individual choices during the exploring, and later give ample material for discussion, so as to invigorate the didactic process.

Souma means that, starting from standard tasks in the ordinary textbooks of mathematics, the teacher can modify parts of the tasks or change the way of stating them as in the example we saw. If the tasks presented during a sequence of lessons, are carefully constructed, it can lead to conjectures, new problems and methods that productively connects the local MO’s covered to more global ones and inspire to technological and theoretical discourses on higher-level MOs. This type of didactic design can be compared to the ideas proposed in Garcia, et al.. (2006) of designing the didactic/study process so that it constructs, in the end, ”integrated and connected” MOs.

This insistence on open-endedness of the task is common with the “open approach method” (Nohda, 1991) is a proposed variant of POLS. The open approach method is used and analysed by Japanese educators (Hino, 2007). Open-ended problems often take the form of formulating a mathematical model and will therefore lead to multitude of, problem formulation solutions and answers. The intent is to let students develop and express different approaches and to let them reflect on their own ideas by seeking to grasp those of their peers (Miyakawa & Winsløw, 2009). Souma (Personal Communication, 2010) judges the open-approach method as something that can not be used in everyday school mathematics. Souma states that POLS lessons applying too ambitious open-ended problems might be isolated from ordinary lessons that, for instance, aim to train students’ basic mathematical skills, but Souma (1987) acknowledge this type of projects at the end of a course. Nohda also notifies that “We do the teaching with the open-approach once a month as a rule” (Nohda, 1991, p. 34). Bosch et al. (2007) have discussed the danger with open-ended activities, which are often introduced at school without any connection to a specific content or discipline. They state that this type of didactic technology suffers the risk of causing the construction of very punctual mathematical organisations, since this is what students are trained to study.

If we return to the lesson template and the example, the teacher should let students who have different types of solutions present their problem in class. The teacher then leads the class to discuss the reason behind each method and have the students determine which of the techniques they have used and why. This is the didactic technology of *whole class discussion* of solutions, which PSO has in common with POLS in general. The discussion of alternative solutions gives an opportunity to establish and reinforce technological and theoretical components of the MO studied, like in this case, the expansion of the square, the rule of the conjugate and the different use of variables, i.e, steering the didactic process into (TT), where new methods and techniques are approved. The class discussion also serves the purpose of increase participation and invigorating the didactic process.

After this, Souma recommends that the students have an opportunity to reflect upon the mathematical theory. The teacher can point out what they have learned by having a student read out from the textbooks explanations of the theory relevant for the lessons. During this the theoretical reflection, the teacher can steer the didactic process towards, say, (I) institutionalisation or (EV) evaluation. Souma states firmly (Souma 1997) that studies in mathematics should be organised and based on a well-written textbook that gives a clear explanation of the mathematical definitions and theories. The classroom discourse is only one form of the study process, the study of mathematics will always entail individual studies and individual problem solving inside or out of school. Moreover, the textbook allows the students to recognize and get familiar with the theory, which the textbooks usually explain in more full detail. In other words, the *textbook* technique is proposed, for the purpose of further covering of the moments (TT), (I) and (EV).

A mathematical problem oriented class in Japan

The following episode illustrates a mathematical problem oriented lesson where the teacher practices the PSO approach. This study take place during a lesson study in grade eight at a lower secondary school affiliated to the School of Education in Asahikawa, Japan, 2009. The teacher is a former Masters student of Souma. The number of students in this class is 40. The lesson is about how to solve a system of linear equations and is the third lesson on this topic. The students have already studied the addition method by solving linear equations obtained from word problems with an everyday life character. The lesson plan was written and distributed by the teacher to us observers beforehand. Posing the mathematical tasks and problems presented during the lesson is common with the POLS based lesson plans, but distinct to PSO is, that it is always written "students possible conjectures" and "students possible solutions", so that teachers always prepare different didactic responses depending on which act students take (Souma, personal communication, 2010).

In the guidebook of Japanese national curriculum standards "The curriculum guidelines" (2008) for mathematics for Japanese secondary school, a system of linear equations with two variables is described (p. 90) as follows: "Solving a linear equation with two unknowns is to make clear that this can be done by using a method that eliminates one of the two variables and then solve equations with one unknown, which is a method students already know". Thus, the didactic transposition of the praxeology "System of linear equations" to the knowledge to be taught in class (Chevallard, 1985 in Bosch & Gascón, 2006), focuses here on the technique of elimination; reducing the pair of two variables equations to one equation with one unknown. Techniques and technological terms present are substitution, row operations, isolation, coefficients, variables, etc. which are collected from the theoretical base of "Elementary algebra".

The lesson

As the first step, the teacher shows the problem by verbally reading out a system of linear equations; $\{7x + 3y = 30, x - 5y = 26\}$ and the students are asked to copy this in writing. He asks: "There are two boys, Taro and Jiro, who both solved this problem. Taro said, "I eliminate *x*". Jiro said, "I eliminate *x as well*". Their answers were the same, but their methods of the solutions are different. Today's task is to consider how they solved the problem differently". The teacher does not show the techniques; the students must consider the possible techniques, which obviously is not only one.

The teacher gives them a few minutes ("individual thinking activity" –according to the lesson plan) and encourages them to find as many solutions as possible. He states in lesson plan that this is especially meant for the gifted students who find solutions quickly. The teacher picks up two students who have obtained different techniques and lets those two students write their solutions on the blackboard. The teacher asks the class how many of them used the technique one of the two students has used. The students raise the hands and it is 37 of them. The teacher asks what is the name of this technique and gets the answer "the addition method" which the class already learned at the previous lessons. The teacher asks the class how this technique works. A student answers “Change the coefficient to the same and erase one of the variable”. The student who has written the solution on the black board explain her reasoning how she has “changed the coefficient”. She says, “x’s coefficient must be changed, so I multiplied it by 7”. The teacher responds, –“OK, you multiplied by 7 and got the same coefficient for all the x:s”. He changes his voice tone a little and then asks “And then, (looks around the class) *what can you do* with the x?” Several students respond, “We can *eliminate* the x”.

They later discuss the other solution technique called the “substitution method”, He inquires again how many of the students came up with an example of that technique (17; –many of them used both methods), and asks for the name of the technique, and then lets the students explain how the technique works. (Some students might already have learned about the technique at “Juku” – a private school offering special classes held on weekends and after regular school hours.) The teacher later asks if there are any students who found variants of the addition method, with an intention to let the students be aware to variation of techniques of the addition method. One student presents his solution by multiplying with 1/7 to $7x + 3y = 30$, instead of multiplying by –7 to $x - 5y = 26$. This presentation awakes a big discussion in the class if it is not a bit too complicated. The teacher concludes the discussion encourage the student with: “But it worked? Didn’t it?”

After the class has had this look at the two different techniques, the teacher lets one student read out loud a passage from the chapter in the textbook, explaining the substitution method. The students work out three to five textbook problems using the substitution method from the book. Afterwards, the teacher asks the class “In which types of problem do you use the addition method and in which types do you use the substitution method?” He lets the students write down their reasoning. The students are then encouraged to *create* several examples of problems they think fit each technique and different proposals are then later discussed.

ATD analysis of the lesson

The purpose of this lesson is to introduce the substitution method and compare it with it to the addition method and to show that both methods reduce the system to the one-variable one-equation case. In his lesson plan, the teacher writes that “The aim of the task” is to “make the students *find out* there is an another method than addition method through mentioning that two boys use different methods”. He asks how to reconstruct the solution of two boys, instead of asking them "Solve this system of linear equations using substitution method". This is an instance of the Souma's guessing technique, since all students are assumed to be able to use the addition method and students are requested to make proposals rather than fixed answers. This is also an example of an "open–closed" tasks; with alternative solutions, but a limited number of possible outcomes. As intended, the task steer the didactic process from (FE) to (EX) and (TT), since it is about finding a new technique, where (TT) is mainly covered during the whole class discussion. The task will also entail (T), technical work, since the students should solve the system with the chosen method. The teacher stimulates *participation* by having all students report which method they have followed. In the discourse, the teacher takes care to make the students use the correct technological terms, like "addition method", "substitution method", and the use of "eliminate" rather than "erase". Much of the same holds for the final task when they are asked to construct suitable problems for each method. None of these tasks are intended in the MO *to be taught*, but are the result of a didactic transposition with the intention to reinforce the *actually taught* MO (Barbé, et al., 2005). As proposed by Souma, reflection on theory is carried out when one student read out loud from the textbook. This steers the didactic process to the moment of (I), so the class verifies now what they have done during the lesson. More (T) is covered when the students work on problems in the textbook.

Discussion and Conclusion

One can summarise Souma's approach as one firmly grounded in the POLS tradition: He argues for the didactic techniques of presenting problems followed by whole-class discussion, theoretical reflection and the use of textbooks. The main difference with POLS in general is the technique of guessing and that Souma stress the need for *open-closedness* when it comes to task construction. Souma's theoretical motivation is pedagogical and based on Dewey's model for the "reflective thinker" and Polyas' cognitive theory. They both focus on the cognitive process of individuals of improved hypotheses and a deeper understanding, rather than the construction of knowledge in a social context.

Like ATD, the PSO takes the organisation of the mathematical content of the study process seriously. I make an attempt to describe PSO using the descriptive potential of ATD to describe the DO proposed by Souma and also to use ATD as en epistemological model to describe the learning object and the learning process: The didactic techniques proposed in the POLS tradition suggest a way to *invigorate* the classroom discourse and PSO, in particular, focuses on how to start up the didactic process using the guessing technique by adding the elements of conjecture, construction and choice from the start, stimulate students' curiosity to tackle with the mathematical tasks. The guessing technique allows *all* the students in class to *join* the lesson. *Open-closed* tasks have the dual purpose of both *invigorating* and *control* the didactic process. Some qualitative predicates, like "participation", "activity" and "invigorate", regarding the didactic process had to be introduced to cover the motivational issues. Further investigation on how such qualifications properly should be handled is needed. Still, I think that the epistemological components of ATD, fits well as an extension of the *logos* block supplied by the DO proposed by Souma.

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5

Mathematical modelling in the Swedish national course tests in mathematics

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Abstract. *One competence assessed in the Swedish national course tests in mathematics (NCT) is mathematical modelling. This paper presents, with empirical data from the NCT, constraints and conditions for developing mathematical modelling in tests viewed from the Anthropological theory of didactics (ATD). The institutional restrictions are numerous and a reformulation of the test problems is suggested.*

Introduction

Many researchers in the field of mathematical modelling and applications are interested in introducing mathematical modelling activities into mathematics education, supported by curriculums that emphasize the focus on "real world problems" in mathematics teaching (Barquero, Bosch & Gascón, in press). Barquero et al. also state that several research studies focusing on implementing modelling activities in schools under suitable conditions have shown success, but "the problem of the large-scale dissemination of these processes has recently been addressed as both an urgent and intricate task" (p. 1). These statements could also illustrate the present situation in Sweden, where on the one hand the official curriculum guidelines for upper secondary school stress the role of mathematical modelling in teaching and learning mathematics, and this has been made more and more explicit since 1965 (Ärlebäck, 2009). On the other hand, in the study by Frejd and Ärlebäck (2009) only 23% of upper secondary students (n=381) stated that they had heard about or used mathematical models or modelling in their education before. A possible way to make modelling into a more standard activity in school is, according to Niss' (1993) premise: "What is not assessed in education becomes invisible or unimportant" (p. 27), to use more modelling activity in assessment tasks. One of the six competences that are being assessed in the national course tests (NCT) in mathematics is mathematical modelling.

The aim of this paper is, in the light of the Anthropological theory of didactics (ATD), to describe and analyse constrains and conditions that would allow development for mathematical modelling to be assessed in the Swedish course tests in mathematics (C and D-course) [1].

There has been a growing international interest in the Anthropological theory of didactics (ATD) and its founder Yves Chevallard has recently received the 2009 Hans Freudenthal Award. The theoretical perspective of ATD is broad and some of its key notions will be explained in the next section. For other details concerning ATD see for instance Chevallard (1999) and Bosch and Gascón (2006).

Modelling and mathematical knowledge in ATD

The notion of mathematical modelling is not unambiguously defined and is depending on the theoretical perspective adopted (Frejd, 2010). Garcia, Gascón, Higueras and Bosch (2006) use ATD in order to formulate a meaning of mathematical modelling. They claim that mathematical modelling is *not* another dimension or another aspect of mathematics, instead they propose "that mathematical activity is essentially modelling activity in itself" (p. 232). However, this view of modelling is only meaningful if one defines mathematical activity and if modelling is considered to include both extra mathematical modelling ("real-world problems") and intra mathematical modelling ("problems related to pure mathematics", such as different representations of algebraic notions). The effect on the statement above will be that the problem situation is not the most important aspect, but the problem itself (*a generative question*) will be the key-point in order to develop and create new, wider and more complex problems (ibid.). A generative question is "a question with enough `generative power´, in the sense that the work done on it by the group is bound to engender a rich succession of problems that they will have to solve- at least partially- in order to reach a valuable answer to the question studied" (Chevallard, 2007, pp. 7-8). These generative questions also named *crucial questions* or *productive questions* (Garcia et al., 2006) should also be of real interest to the students (Rodriguez, Bosch & Gascón, 2008). A prerequisite to create and tackle these new and wider problems is analysis of mathematical knowledge in terms of mathematical praxeology.

The notion of praxeology is one of the most central notions in ATD (Garcia et al., 2006). A knowledge or a body of knowledge is defined as “a praxeology (or a complex of praxeologies) which has gained epistemic recognition from some culturally dominant institutions, so that mastering that praxeology is equated with mastering a “true” body of knowledge” (Chevallard, 2007, p. 6). The praxeologies are described by the two main components *praxis* or “know-how” and *logos* or thinking and reasoning about the praxis. These two main components can be divided into four sub-components, see Figure 1 below.

Praxis	Tasks within a specific activity
(know-how)	Techniques to accomplish the tasks
Logos	Technology that justifies the techniques
(know why)	Theory that justifies the technology

Figure 1. The four sub-components of praxeologies

The praxis part refers to the types of tasks and techniques that are available to solve the tasks and the logos part refers to technology that describes and explains the techniques and the theory that explains the technology. In addition, the praxeologies of mathematics can be analyzed as *global*, *regional*, *local* and *point* praxeologies (Bosch & Gascón, 2006). A *point* praxeology is characterized by a specific type of problem and an appurtenant specific technique in a technology, a *local* praxeology is characterized by a set of point praxeologies that are integrated within the same technology, a *regional* praxeology is characterized by connected local praxeologies within a mathematical theory and a *global* praxeology is characterized by linked regional praxeologies (see e.g. Rodriguez, Bosch & Gascón, 2008). I will illustrate *global*, *regional*, *local* and *point* praxeologies by examples inspired from Artigue and Winsløw (2010). A *point* praxeology is for instance the specific technique to solve “x–3=0” by moving the -3 to the other side of the equal sign and change minus to plus. A *Local* praxeology may be seen as the discourse relating to solve polynomial equations and a *regional* praxeology may be an algebraic theory for solving equations. Finally *global* praxeology may be a unified theory of equation solving including nu numerical theory, algebraic theory etc.

The process of refining or constructing mathematical praxeologies is a complex activity called *the process of study* (see e.g. Rodriguez, Bosch & Gascón, 2008). *The process of study* is classified into six didactic moments (non chronological): (1) *first encounter*, (2) *exploration*, (3) *constructing environment for technology and theory*, (4) *working on the technique*, (5) *institutionalization* and (6) *evaluation* (ibid.). I will describe the process of study with the modelling example used by Ruiz, Bosch and Gascón (2007) about selling and buying T-shirts. The students in the investigation were given a chart with the number of sold t-shirts, the total costs, the total incomes and benefits for three months (May, June and July) and a corresponding question about the possibility to earn 3000 euro in August by selling a reasonable number of T-shirts (*first encounter*). Based on the given conditions the students started to create a model and did some calculations and estimations in order to develop a technique (*exploration*) and then continued to improve this technique to set up other models (*working on the technique).* For instance, the students had to find connections between numerical and functional language as well as investigate about the roles of parameters and variables (*constructing environment for technology and theory*) in order to discuss the question. Finally an identification of praxeologies regarding institutional demand is done (*institutionalization*) and students reflect over the value of those praxeologies (*evaluation*) (Rodriguez, Bosch & Gascón, 2008).

To specify the aim and get an initial indication on the current situation of modelling activities in the national course tests, I will investigate the following question based on the theoretical discussions in the section above: *What generative questions are presented in the Swedish national course tests in mathematics (C and D- course)?*

Methodology

The national course tests (NCT) in mathematics in Sweden are designed to be an instrument for assessing competencies (including mathematical modelling) according to the curriculum guidelines and to stimulate teachers and students to discuss about goals and content in the curriculum (Palm et al., 2004). The NCT are institutional inventions because they are based on curriculum guidelines (institution), developed by a department of educational measurement (at Umeå University; institution) and used in schools (institution). The reason to choose ATD to describe and analyse constraints and conditions of modelling in NCT (institution) is, in line with the second handbook of research (Silver & Herbst, 2007) that ATD attempts to explain and describe institutions.

In order to describe and analyse the present situation of modelling tasks in the NCT, I have, in line with Bosch and Gascón (2006), developed an epistemological reference model to be able to adopt an external viewpoint. The reference model concerns modelling as well as types of praxeologies (point, local, regional, global). I have adopted the definition of modelling activity used by Barquero, Bosch and Gascón (2006), who claim that "the modelling activity is a process of reconstruction and articulation of mathematical praxeologies which become progressively broader and more complex. That process starts from the consideration of a (mathematical of extra-mathematical) problematic question that constitutes the rationale of the mathematical models that are being constructed and integrated" (p. 2051). In my view this definition focuses on (possible) *generating questions* [Q_0] and the corresponding generated sub-questions [$Q_1,\ldots,Q_n$]. These *generating questions* may be identified (thus it is operational) by studying and analysing tasks and problems, written solutions (answers [$R_1,\ldots,R_n$]) and assessment guidelines (identify praxeologies that are assessed as rewarded to use). The reference model that I will use, based on the discussions above, will be that a *generating question* will be defined as: a question (a problem or a task) [Q_0] that generates at least one corresponding sub-question [Q_1] which is needed to ask in order to solve the initial question. An example of a generating question (discussed before) used by Ruiz, Bosch and Gascón (2007) is "In the given initial conditions, is it possible to obtain a benefit of 3000 € in August by selling a reasonable number of T-shirts?" (p. 6). To evaluate possible types of praxeologies (point, local, regional, global) I will use the following definitions: a *point* praxeology is used when only one isolated technique is involved, i.e. a routine operation according to the assessment guidelines (such as solving an polynomial equation of second order by using a formula, using an instrumented technique to evaluate a maximum value, differentiate an exponential function by using some stated rules etc.); a *local* praxeology involves at least two point praxeologies which are justified under a technology, i.e. when the assessment guidelines emphasize more than one technique to get to a solution; a *regional* praxeology connects at least two local praxeologies under a theory, i.e. that the assessment guidelines stress different technologies are needed to solve the question; a *global* praxeology is linking together at least two regional praxeologies ,i.e. more than one theory is needed according to the assessment guidelines.

I have used the reference model to get an initial indication on the current situation of possible generative questions in NCT, by analysing (see next section) the last four freely available [2] NCT (C (2009), C (2005), D (2005) and D (2002) [3]). The available NCT are supposed to give the students (as well as teachers and researchers) a representative picture of the general design about the test as well as give information about the mathematical content being asked about. The tests are divided in two parts, one part with and one part without the possibility for the student to use a calculator. Another condition is that the expected answers according to the NCT authors are categorized. The three types of categories are *short answers* (one sentence of explanation or a numerical calculation), *long answers* (extended explanations about the solution) and *essay answers* (performance assessment, where the students are supposed to write some paragraphs in order to explain a situation which includes to describe and use some method, draw conclusions based on mathematical reasoning and to do a distinct and clear presentation of the problem with mathematical language). The time limitation for the C and D-course tests is 4 hours and it is recommended in the instructions to work at most 90 minutes (C, 2009) or at most 60 minutes (C 2005, D 2005, D 2002) on the first part without the calculator, and that the performance assessment may take an hour to execute. To every NCT there are also teacher guidelines for assessment with examples of students' answers that are supposed to help the teachers to assess the test as uniformly as possible across the country. I have used these guidelines and the reference model in the analysis in the next section.

The analysis and the results

The main result in this study is that there exist no generative questions with respect to the reference model in the investigated national course tests. This main result will be illustrated below along with excerpts from the teacher guidelines in order to explain how the analysis has progressed.

The solutions to the first part of the test (no calculator allowed), are characterized by mostly *short answers* and only a few *long answers* (two applied minimum and maximum problems). Below in Figure 2 are two examples from the first part.

Calculate the integral $\int_0^3 (x^2 + 4x)dx$ (D, 2002)	Is lg9 larger or less than 1? Please motivate your answer. (C, 2005)
Short answer, point praxeology, no generating question	*Short answer, point/local praxeology, no generating question*

Figure 2. Translated tasks from the first part of the NCT D (2002) and C (2005) (my translation).

The task to the left is supposed to be a routine question in the D-course and according to the guidelines it emphasizes the algorithm (technique) of finding the antiderivatives (i.e. a technique to use the fundamental theorem of calculus, without the necessity to refer to the theorem). This type of question at this level (D-course) is testing a technique and is not a generative question and could be seen to test a single point praxeology. The task to the right is no generating question either, there are no obvious sub-questions needed. However, the technique to solve the task is not totally clear from the outset. The three students' solutions (C 2005) in Figure 3 below may illustrate the situation.

Student A uses a technique based on two special cases to and the same technique is used by student B, but in addition student B declares in words that the logarithmic function is increasing for the chosen interval. Student C uses a technique based on the definition of the logarithmic function. In the assessment guidelines to the teacher student C receives most credits, because he/she uses the definition of logarithmic function to prove the statement. The solutions above may be seen as *local* praxeologyrather than point praxeology within the technology of increasing/decreasing function.

Student A	lg1=0 lg10=1 0<lg9<1	
Student B	Mindre än 1 eftersom den logaritmiska kurvan är växande mellan lg1 och lg10 och lg10=1 och lg1=0	Less then 1 because the logarithmic curve is increasing between lg1 and lg10 and lg10=1 and lg1=0
Student C	lg9 betyder 10 upphöjt i vad å är 9 Alltså : $10^x=9$ Vi vet att : $10^1=10$ Därför borde lg9 eller x vara mindre än 1	lg9 meaning 10 raised to what power is 9. Thus: $10^x=9$, we know that $10^1=10$, Therefore lg9 or x should be less then 1

Figure 3. Student solutions from the NCT C (2005).

Many of the tasks from the first part add to the dominant view of algebra as *generalised arithmetic,* which is one of many obstacles for implementing mathematical modelling activities, where algebra may be seen as a modelling tool (Ruiz, Bosch & Gascón, 2007, Bolea, Bosch & Gascón, 2004). An example of algebra as *generalised arithmetic* is for instance a problem in the C-course (2009), where the student is given a formula for the (least) total number of "handshakes" that are needed in a group for everyone to shake every others hand. The student is supposed to use this formula (in a general way, no numbers are given) to write an expression (as short as possible) for the difference (i.e. test arithmetic) of "handshakes" between two different groups, with the condition that there are twice as much people in one of the groups.

For the second part of the tests the students are allowed to use calculators (both graphical calculators and CAS-calculators are allowed). The focus is moved from *short answers* to more or less *long answers*, and the last problem in each test will fall under the category *essay answer.* To provide a picture of the second part of the test some examples will be discussed and analysed in Figure 4 on the next page.

Find *the number* of solutions to the equation sin2x=x^2/10-1, x is measured in radians. (D 2005)	A thermos fills up with hot coffee and is placed outdoors where the temperature is about zero degrees. The temperature of the coffee decreases exponentially over a time period. After 4 hours the temperature is 76 degree Celsius and at that point of time the temperature decreases with a rate of 4.1 degrees per hour. What was the temperature of the coffee when it was poured into the thermos? (C 2005)
Short answer, point/local praxeology, no generating question	*Long answer, local praxeology, no generating question*

Figure 4. Translated tasks from the second part of the NCT D (2005) and C (2005) (my translation).

The task to the left is no generating question (the assessment guidelines do not stress the use of any sub-questions) and is supposed to test an instrumented technique (or drawing a graph by hand). To solve this equation one may say that a single point praxeology is needed according to the reference model. However, there are several different possible instrumental techniques (solve, graph, trace) to solve the equation and a solution may be seen as a connection of single point praxeologies (solve, graph, trace) under the technology of instrumented equation solver; if so than the solution will refer to a local praxeology. The task to the right demands a longer answer, including setting up and solving an equation system with exponential functions, based on the two conditions in the text, but still the restrictions are still too many with the respect to the ideal of a generative question. However, in a less restricted situation this question could be turned into a generative question (see section conclusion and implications). The different connected point praxeologies within the technology of linear first order differential equations make it a local praxeology.

Finally, the *essay answer* will be illustrated by the following situation; a patient gets equal intravenous doses of medicine (10 mg) at repeated occasions (C-course, 2009). The student is given a model ($y=10e^{-8t}$) which is supposed to describe the amount of medicine in the blood y (mg) after a certain time t (hours). Two questions are then given: How much medicine is there in the blood after 5 hours after the first dose? After 8 hours the patient gets a second injection of medicine, how much medicine does the patient have in the blood immediately after the second injection? A graph is given after the two questions in order to visualize a simple model of the total amount of medicine M (mg) the patient has in the blood after five injections (see Figure 5).

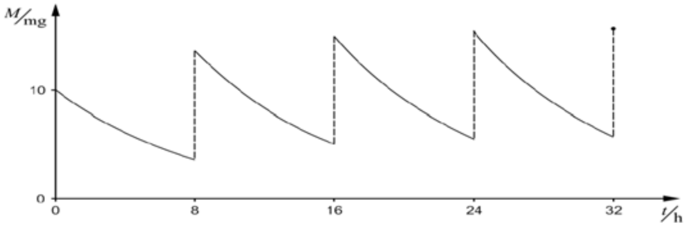

Figure 5. A graph from the NCT C (2009) describing the amount of medicine in the blood after five injections.

Two more questions are given; first, write an expression and calculate the amount of medicine the patient has in the blood after five injections; and second, suppose the patient gets more injections by the same model, try to find the limit for the total amount of medicine in the blood (it is explicitly given that there is an upper limit).

In this essay answer question, one may see that the author of the test is "guiding" the student with sub questions towards a solution. The last question, according to my point of view, is not a generative question but it has some similarities. I will explain by analysing some answers, parts of the answers are displayed in Figure 6 below.

Student D

Student E

$10 + 10e^{-\frac{8}{8}} + 10e^{-\frac{16}{8}} + 10e^{-\frac{24}{8}} + 10e^{-\frac{32}{8}} \approx 15{,}71\ \text{mg}$

6:e dosen, Totalt ≈ 15,780

7:e dosen, Totalt ≈ 15,805

8:e dosen, Totalt ≈ 15,814

9:e dosen, Totalt ≈ 15,818

Den övre gränsen blir 15,8 mg (ungefär)

Student F

- grafen av funktionen ser ut så här

$$\boxed{M = \frac{10((e^{-1})^x - 1)}{e^{-1} - 1}}$$

Student G

Generellt uttryck för medicinen $M = 10\left(\frac{(e^{-1})^n - 1}{e^{-1} - 1}\right)$

$M(100) = \frac{10((e^{-1})^{100} - 1)}{e^{-1} - 1} \approx 15{,}82\ \text{mg}$

forts.

Ett högre n resulterar i att e^{-1} närmer sig 0. Då närmar sig parantesen −1.

$M = \lim_{n\to} 10\left(\frac{(e^{-1})^n - 1}{e^{-1} - 1}\right) = \frac{10\cdot(-1)}{e^{-1} - 1} \approx 15{,}82$

Ett högre n gör att medicinmängden närmar sig 15,82 mg, men kommer aldrig att överstiga det.

Figure 6. Four students' answers from the essay answer question, C (2009).

Student D uses a technique of proportions by measuring in the given model and concludes that the total amount of medicine after 5 injections is 15.94 mg. An iterative technique is used by Student E, who concludes that after 9 iterations (injections) the total amount of medicine is about 15.8 mg. Students F and G get the same result (15.82 mg) and use a technique which is justified by geometric sums (technology), where student F sketches a graph (an assumption could be that the student uses his/hers calculator) to find the limit and student G evaluates the limit at infinity by the theorem of limits for exponential functions (even though the students do not write $n \to \infty$ and that the parenthesis moves closer to -1 instead of $(e^{-1})^n$ moves closer to zero or $(e^{-1})^n$-1 moves closer to -1). This may be a *regional* praxeology according to the reference model connecting the technologies of exponential functions and geometric series within the theory of limits. One may also say there are some similarities to a generative question, depending on what sub-questions the students give themselves (if they do), and these sub-questions may generate a specific technique (proportion, iterative process, limit definitions, etc) and a corresponding answer. In other words, this pattern of questions and answers $(Q_1, R_1), \ldots, (Q_n, R_n)$ is similar (or close to similar) to the pattern generated by a generative question used in so called *research study courses* RSC, see for instance Barquero, Bosch and Gascón (in press), Ruiz, Bosch and Gascón (2007) or Garcia et al. (2006) for more details and explanations. The next section will sum up the analysis and discuss the result of the current state of modelling activities in national course tests in Sweden.

Discussion

The ATD has been a useful tool to use in order to show that the institutional restrictions are numerous (time limit and other institutional constrains). A problem I found with using the ATD for analysing NCT is that it was hard to identify types of praxeologies based only on problems and solutions: the point, local, regional or regional praxeologies depend much of the students' background which is not seen in a solution. The only opportunity (condition) I see to implement a generative question is in the *essay answer* question, which means that the students will have approximately one hour to perform a modelling activity by themselves. The restrictions are several. For instance, it is not possible during the NCT test situation to work in a group and modelling activities often take place in a group where the members of the group help each other to raise the sub-questions. The group members in a modelling activity usually have access to computers and other powerful tools as well as the access to *go out* to validate an extra modelling situation, which is not acceptable while taking the NCT. The institutional history of test making of the NCT is also a strong restriction, as the authors of the tests are used to create the NCT the same way, with the same interpretation of the curriculum. The curriculum aspects of modelling are not interpreted as a generative question by Palm et al. (2004), instead modelling is viewed as a cyclic process starting with a real-life problem (for details see Palm et al., 2004). However, according to Garcia et al. (2006) this view does not contradict a view on modelling from an ATD perspective. Nevertheless, I found no example to compare the different views and I have some problems to see that they do not contradict, especially the intra modelling, and I argue for more research comparing the different views.

One way to introduce modelling activity in the test could be to reformulate the essay answer question previously discussed with realistic data and take away the present sub-questions, the mathematical model and not explicitly explain that there is an upper limit. Instead, students could try to find the model themselves by some given data in a table like in Ruiz et al. (2007). The students could then use for instance regression in order to set up a model that later can be examined like the present question.

However, in my point of view the main restriction for implementing modelling activity in the NCT is the demand of constant learning/ constant testing (time factor) and the fact that modelling activities is a *long-term objective* (Bolea et al., 2004). I propose, therefore, for the use of the problems in the NCT to be an inspiration in order to reformulate them into modelling activities (generating questions) that can take place under less restricted conditions, within ATD called a research study course (RSC). I will give two examples, first the problem with the cooling coffee and the thermos. A generating question Q_0 could be: “Given the temperature of the coffee at some points of time, can we predict the temperature after different lengths in time? Is it possible to predict the long term behaviour of the temperature? What sort of assumptions on the thermos, initial temperature, the surroundings etc. should be made? How can one create forecasts and test them?” These questions are similar to the one used on population growth by Barquero et al. (2007). The second example deals with medicine, as in the essay answer question. The *first encounter* for the students could be to read on the back side of a box with headache pills (which may be familiar to the students). One may read on the box that “grownups and children above 12 years: 1

pill up to 1-3 times per day, the maximum dose of 3 pills should not be exceeded and the pills are only for short term use maximum 5 days. 1 pill includes 400 mg Ibuprofen." A generating question Q_0 may be "how can one make a prediction when it is time to take a new pill? What happens with the amount of medicine in the body while using the medicine over a longer time? To what extent is it possible to test our hypothesis (models)?" This could be one way to introduce or work with mathematical praxeologies involving exponential functions, geometric series, the limits of infinity, but it could also start other discussions about other aspects in life and in society, such as knowledge about addictive substances like drugs, alcohol and tobacco and what effect they have on the body.

Conclusion

The analysis of tasks and problems in the NCT reveals that the focus is on assessing different techniques. The solutions to the *long* and the *short* answer questions require point or local praxeologies. Even though the modelling competence is supposed to be assessed, no generating questions are found with respect to the reference model. However, the *essay answer* questions with reduced restriction have a potential to be transformed into something close to a generating question. To transform the essay answer question into a generative question, the essay question could well be discussed in terms of *the process of study* with the six didactic moments: (1) *first encounter* (the essay question), (2) *exploration* (what technique should I use), (3) *constructing environment for technology and theory* (discuss why use this technique), (4) *working on the technique* (use the chosen technique), (5) *institutionalization* (what praxeologies have the students used and are they in line with the curriculum?), and (6) *evaluation* (what is the outcome of the used praxeologies, is the chosen technique optimal?). This could be tested in a future research study with an aim to provide more information about generative questions and their potential use also in national course tests.

End notes

1. Example of content in: Mathematics A-arithmetic, algebra, statistics; Mathematics B-probability theory, geometry, equations of second order; Mathematics C-differential calculus; Mathematics D-Trigonometry, integral calculus; Mathematics E-complex numbers, more advanced differential and integral calculus, differential equations.

2. Open meaning there is no secrecy on the test and it is free to download from the internet.

3. The NCT from the C course (2005, 2009) and the D course (2002, 2005) are retrieved from: http://www8.umu.se/edmeas/np/information/np-tidigare-prov.html

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Institutional practices in the case of the number *e* at upper secondary school in Sweden

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The aim of this study is to follow institutional introductions of the number e *with the framework of The Anthropological Theory of Didactics. To simplify the process of analysing these introductions two mathematical organizations are constructed. The first organization focuses on the introduction and the other one on the value of the number* e. *This article shows some hinders for students learning and familiarization with the number* e *within and between the observed institutions.*

Introduction

Mathematical studies at upper secondary school level in Sweden have usually been influenced by some very important, irrational numbers. The number π , for example, students bring with themselves from the previous educational phase to gymnasium. Although they have continually dealt with the number in many different situations at comprehensive school, my teaching experience unfortunately documents the lack of student understanding of the nature of the number π and its uses in mathematics education. Students have initial difficulties to explain our mathematical needs for such numbers in school education. The absence of the set of theory which could facilitate and improve student understanding of numbers at both levels of education has lead to inadequate perceptions. Students' interest in how the number π has come to be defined and what kind of number π is appears to be very vague. Their encounters with the number π are entirely connected to geometry and some of the trigonometrical functions. On the one hand, to deal with this type of task students have just been taught to use this number π in some appropriate ways which probably lead to the right answers without any demands on understanding what lies behind these uses. On the other hand, the current-day introduction of the number π does not appear to cause the students any calculating and operating difficulties. But in general, proving the irrational nature of the number π can cause problems for the students.

To go a bit further and mathematically account for the number e and its irrational nature requires a carefully prepared introduction strategy which could help students in their understandings of this number without loosing sight of its mathematical nature. Students' better understanding of the number e should improve not just their mathematical implementation of the number but even some mathematical definitions, processes and algorithms in connection to those. Definition of the natural logarithms, differentiations of common and composed exponential functions and some appliances in the field of the complex numbers are just few of those worth mentioning initially.

To investigate present-day introductions, appliances and applications of the number e into the upper secondary school, this study uses the frameworks of the anthropological theory of didactic with the process of the didactic transposition created by Yves Chevallard. Some crucial elements of this theory are briefly described in the following.

Theory

The anthropological theory of didactic (ATD) and the theory of didactic transposition by Yves Chevallard (Barbé, Bosch, Espinoza, & Gascón, 2005, p 236) are initially rooted in the theory of didactic situations from Guy Brousseau (1997). A set of knowledge (usually called a *body of knowledge*) created somewhere outside of school is

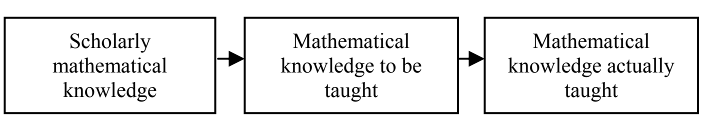


Figure 1. Reference mathematical knowledge

transferred to the school both with and without any social, cultural or educational reasons (Bosch & Gascón, 2006). The didactic transposition process is constituted by the four following steps:

scholarly knowledge, knowledge to be thought, taught knowledge and learned knowledge. *The scholarly knowledge* is the kind of knowledge often produced at higher educational level, by mathematicians without any deeper considerations regarding appliances at lower educational school level. The mathematical needs of mathematicians can be quite different from the needs that characterise mathematical education at the upper secondary level. To address these latter types of needs the scholarly knowledge must be transformed into what is called *knowledge to be taught*. Here, even society enters into the picture with its own needs and reasons in producing and creating that knowledge, of course with a cultural and national signature, which is supposed to be of crucial interest in the future. Among others, syllabi, curriculum, textbooks and other educational materials for different purposes are located here. The next step in the didactic transposition process is *taught knowledge*. Teachers together with their students are always trying to find some more appropriate ways of teaching which lead to improved student understanding and acquirement. The focus is on what actually is done in classrooms and in which ways. The students learning are usually most influenced of this type of knowledge, which is presented in the classrooms by the teachers in some kind of interaction with the students. Both consciously and unconsciously, teachers and students shape and create the educational material that is important to know according to various social and educational reasons. As the last part in this process *learned knowledge* appears as knowledge that is actually acquired by the students, which can be available for appropriate situations.

These steps from the didactic transposition process provide the possibility of creating a new reference model which is called ”*reference epistemological model*” and can be produced by data from the aforementioned three institutions: the mathematical community, the educational system and the classroom. To study mathematical problems without this point of view can cause many problems related to the institutions in which they are shaped, taught or learned.

This is precisely what the proposal for these paper is, namely to investigate both different aspects and approaches to the introduction of the number e regarding these institutions and eventual constraints for the learning which can be caused in the field within and between these institutions. It is important to mention that the classroom institution is excluded in the present project.

According to the ATD (Hardy, 2009) any mathematical knowledge can be described and investigated in terms of a *mathematical* *organization* or a *praxeological organization*, sometimes called mathematical *praxeology*. “A didactic praxeology is used when a person of group of persons want to have an appropriate MO available (the mathematician’s or student’s didactic praxeology) or to help others to do it (the teacher’s didactic praxeology)” (Barbé, Bosch, Espinoza, & Gascón, 2005, p. 239).

The mathematical praxeology consists of four main parts divided into two blocks. The first block is denoted as a *practical* one and includes *type of task* and a *technique* needs to solve it. This block is often connected to the phrase *know-how*. The other block, which justify the practical block is named as *theoretical block* and is constituted of *technology* and *theory*. The technology is meant as a type of discourse or *logos* about the technique while the theory is a broader and deeper notion, which justifies some different technologies (Barbé et al., 2005).

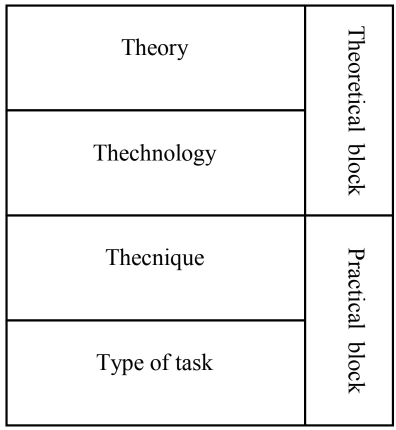


Figure 2. Praxeology

With the introduction of the number e as a task in the institutional educational practice it should be important to investigate what kind of mathematical organizations are considered within the mathematician community and educational system and the relationships between them which are also a complementary proposal of this paper.

The aim of this project is consequently to investigate both different approaches to the introduction of the number e regarding the mathematical community and the educational system and also eventual constrains for the learning, which can be caused with these introductions in the field within and between these institutions.

Analysis and Results

Domar et al (1969, p 127) begin with a look at the graphs of the functions

$f(x) = 2^x$

and $g(x) = 3^x$:

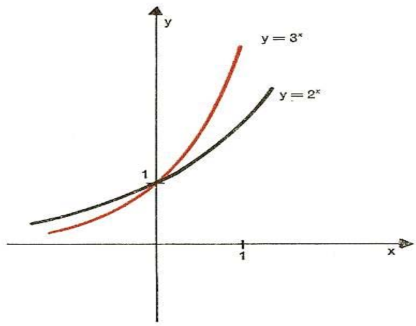


Fig. 32.

Through ascertaining that the functions are derivable, according the graphs above, the authors give derivatives of these functions for $x = 0$, that is $f'(0) \approx 0{,}7$ and $g'(0) = 1{,}1$. The question if there is any value $x = a$ so that $f'(a) = 1$ emerges. This leads to that:

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1 .$$

The absence of some appropriate method that could prove the expression above quite easily heads the authors to assume the existence of such a number. They denote it as usually with the number e, that is:

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 .$$

To get round this problem the authors propose another approach through the definition of the function:

$$\ln x = \int_1^x \frac{1}{t} dt \quad for \quad t \succ 0 ,$$

that is an area between the curve $y = \frac{1}{t}$, the x axis and the lines $t = 1$ and $t = x$, for $x \geq 1$. If $0 \prec x \prec 1$ then the area is taken as negative.

To prove that this function is continuous for all positive x numbers and strong increasing becomes less difficult compared with the former approach. This means that there is a value for x so that: $\ln x = 1$ which is known as the number e, that is:

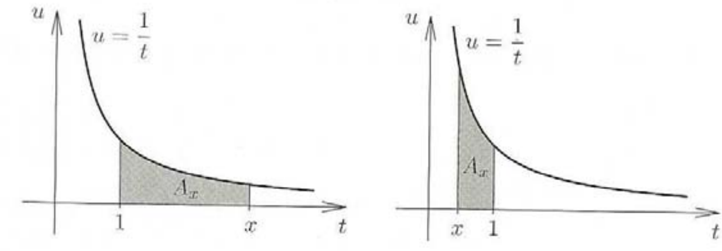
$$\ln e = 1.$$

Hyltén-Cavallius and Sandgren (1968), in another approach to the number e, choose initially to keep away from precisely defining of this number. They begin instead with the definition of the natural logarithm $\ln x$ through the claim of the number e`s existence and it's approximately value 2,718. The function $f(x) = e^x$ becomes subsequently defined as the inverse function to the function $f(x) = \ln x$. Finally, the authors prove the theorem: $\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x$ which for $x = 1$ leads to:

$$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e$$

and creates the possibility to define the number e via the series of the numbers: $\left(1+\frac{1}{n}\right)^n$, $n = 1, 2, 3, \ldots$

Forsling and Neymark (2004, p 80) begin like Domar et al (1969) with the definition of the function $\ln x$:



as the area below the graph of the function $u(t) = \frac{1}{t}$ which shifts from the positive to the negative one depending on where the variable x is located, to the right or to the left of the number 1 respectively:

$$\ln x = \begin{cases} A_x & \text{for } x \geq 1 \\ -A_x & \text{for } 0 \prec x \prec 1 \end{cases}$$

That is:

$$\ln x = \int_1^x \frac{dt}{t} \quad \text{for} \quad x \succ 0$$

there the area is connected to the integral above.

The exponential function $y = e^x$ is then defined as the inverse function to the function $x = \ln y$ under some appropriate circumstances, which qualify the existence of this inverse.

The number e becomes thereafter confirm for a special value of x, that is x=1 and e^1=e. The way to enquire the exactly value of the number e goes via Maclaurin series:

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \ldots + \frac{f^{(n)}(0)}{n!} \cdot x^n + r(x) \quad \text{there} \quad r(x) = O(x^{n+1})$$

and Langrange´s theorem:

$$r(x) = \frac{f^{n+1}(\xi)}{(n+1)!} \cdot x^{n+1} \quad \text{for} \quad 0 \prec \xi \prec x.$$

All this leads further that:

$$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + r(x) \quad and \quad r(x) = \frac{e^{\xi}}{(n+1)!} \cdot x^{n+1}$$

.

For x = 1:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \ldots + \frac{1}{n!} + r \quad and \quad r = \frac{e^{\xi}}{(n+1)!}.$$

An easy approximation:

$$0 \prec r = \frac{e^{\xi}}{(n+1)} \leq \frac{e^{1}}{(n+1)} \prec \frac{3}{(n+1)!}$$

and

n = 9

gives:

$$e \approx 2{,}7182815\ldots \quad and \quad 0 \prec r \prec \frac{3}{10!} \prec \frac{1}{10^6} = 10^{-6}$$

So that we get: $e \approx 2{,}71828$

Some of the textbooks that teachers usually wok with at the upper secondary school in Sweden try to solve this problem of the number e more and less similarly. The textbooks don't hesitate to assault this introductory problem without going deeper in mathematical reasoning about the continuity of the exponential functions in a given interval or about the existence of the number e, for example. The existence of the number e is just shown, by the ratio $\frac{y'}{y}$ there the function y varies between a particular $y = 2^x$ and a general case $y = a^x$ (Björk & Brolin, 2000; Eriksson, Sjunnesson, Jonsson, & Gavel, 2008; Gennow, Gustafsson, & Silborn, 2004; Szabo, Larson, Viklund, & Marklund, 2008). The presentations shift as well between a graphical one to some algebraic. With almost nonexistent differences between all these approaches end up with the expression: $\frac{y'}{y} = k$ and the question if there is such a number a that $\frac{y'}{y} = 1$ and the discovery of the number begin with the differentiation of the general function $y = f(x) = a^x$:

$$y' = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h}$$

,

which resembles Domar et al (1969). The question if there is such a number a that $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$ is the same as previously. There are after that two different advances to the number e. The first one (Björk & Brolin, 2000; Gennow et al., 2004) examines the last expression for different small h values close to the zero. The other one develops the last expression (Eriksson et al., 2008; Szabo et al., 2008) as $\frac{a^h - 1}{h} = 1 \Rightarrow a = (h+1)^{\frac{1}{h}}$ before doing the same thing which resembles Hyltén-Cavallius and Sandgren (1968) if you change h with $\frac{1}{h}$ and of course $h \to 0$ with $h \to \infty$:

$$\lim_{h\to\infty} \left(1 + \frac{1}{h}\right)^h = e$$

The notion of a"continuous expression" has been mentioned just in the one of the examined textbooks (Eriksson et al., 2008), and mostly informative.

The derivative of the function

$$f(x) = e^{k\cdot x}$$

derives either algebraically (Björk & Brolin, 2000; Eriksson et al., 2008; Szabo et al., 2008) or graphically (Gennow et al., 2004) and becomes:

$$f(x) = k \cdot e^{kx}.$$

This doesn't just facilitates further differentiating of the functions $f(x) = \ln x$ and $f(x) = a^x$ but plays even quite important role in student mathematical learning in the field of the complex numbers.

Mathematical organizations

Different approaches that are done and described above in the mathematical community, by mathematicians, compared to those in the educational system might be treated dissimilarly in terms of mathematical organizations. The first of these approaches, **MO$_1$**, deals with the existence of the number e while the second, **MO$_2$**, with the value of the number e.

Mathematical organizations	
MO$_1$ The existence of the number "e"	**MO$_2$** The value of the number "e"

Figure 3. Mathematical organizations

The obvious disparities, which predominate between the mathematical and educational institutions, call therefore for the creating more than one of mathematical organizations. They are signified as above with **MO$_1$** and **MO$_2$**. The question is now how these corresponded praxeological organisations are both built and functioning.

MO$_1$:s task **T$_1$** deals with justifying the existence of the number e. The techniques **τ_i** within this praxeology, which facilitate managing of that, are closely related to their own technologies **θ_i** that both justify and theoretically present them on the same time. As we could notice from above there are some quite similar techniques within the mathematical community.

The praxeology MO_1					
The Mathematical Institution					
$\tau_{1'}$	Exponential function $y = a^x$	τ_4	The existence of the number "e"	τ_3	The definition of lnx
$\theta_{1'}$	At different places	θ_4	The notion of function	$\theta_{2,3}$	The notion of function, inverse area and integral
$\tau_{1''}$	The problem $\lim_{h\to 0}\frac{a^h - 1}{h} = 1$		The definition of lnx		The inverse function
$\theta_{1''}$	Not mentioned at all		The inverse function		
τ_2	New function $\ln x$	τ_5	The theorem $\lim_{t\to\infty}(1+\frac{x}{t})^t = e^x$		Maclaurin series
$\theta_{2,3}$	The notion of function, inverse, area and integral	θ_5	Limits of functions, algebra and series		

Figure 4. The Praxeology MO1 within the Mathematical Institution

The first of them τ_1 starts with general functions $y = a^x$ and two different attempts $\tau_{1'}$ and $\tau_{1''}$ to justify the existence of the value of a for which $y' = a^x$. The first one uses the limits and the second discourse about properties of the function. Two of the other techniques τ_2 and τ_3 end initially up in the definition of a new function, $\ln x$, which is once close connected to the notion of the area with quite vague explanation or justification. The exponential function is after that defined as an inverse to $\ln x$ and the number e is received for $x = 1$. The fourth technique τ_4 begins with the assumption of the existence of the number e and go over to defining the function $\ln x$. The number e is then acquired from the equation

$\ln x = 1$. The fifth technique $\boldsymbol{\tau}_5$ proves the following phrase $\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x$ and takes e as $\left(1+\frac{1}{n}\right)^n$, $n = 1, 2, 3, \ldots$.

The technologies $\boldsymbol{\theta}_i$ constituted of different parts of mathematics are either behind these techniques, not mentioned at all $\boldsymbol{\theta}_{1''}$ or at the different places in the treated textbooks $\boldsymbol{\theta}_{1'}$, with and without adequate references. The notion of the area below a graph of a function, continuity of functions, inverse function, limits, integrals, series and their convergences are some of them. The technologies overlap each other in some of the techniques. The technologies $\boldsymbol{\theta}_{2,3}$ are about the notion of the function, inverse function, integral and the area. The notion of the function belongs to $\boldsymbol{\theta}_4$. The discourses about limits of functions, algebra and series constitute $\boldsymbol{\theta}_5$.

The theory $\boldsymbol{\Theta}_1$ that justify these praxeologies must include a big part of the theory of calculus considered functions, limits, derivates, integrals and infinite series with adequate algebra.

What is **MO$_2$**:s task constituted of? The task $\mathbf{T}_2$ determines the value of the number e. There are three techniques within the mathematical and educational institution.

The praxeology MO$_2$					
The Mathematical Institution					
	Exponential function $y = a^x$		The existence of the number "e"		The definition of lnx
	The problem $\lim_{h\to 0}\frac{a^h-1}{h} = 1$		The definition of lnx		The inverse function
			The inverse function		
	New function $\ln x$		The theorem $\lim_{t\to\infty}(1+\frac{x}{t})^t = e^x$	τ_{21}	Maclaurin series
				θ_{21}	Algebra and series

Figure 5. The praxeology MO2 within the Mathematical Institution

The first technique τ_{21} uses Maclaurin series while the second τ_{22} decides the value of the number a so that $\frac{y'}{y} = 1$ there $y = a^x$. Here there are two different sub-techniques. The graphical one and the calculating one. The last technique τ_{23} applies the limit of the function $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$, to calculate the value of the number e.

The technologies $\theta_{21,22,23}$ includes discourses about series, derivatives and algebra. Algebra and series make up θ_{21}, derivatives and graphs θ_{22} and limits of function, algebra and calculus θ_{23}.

Θ_2 are made up of the theories about series derivatives and algebra.

		The praxeology MO_2		
		The Educational Institution		
τ_{22}	The ratio $\frac{y'}{y} = k$ $y = a^x$		$y = a^x$	$\frac{a^h - 1}{h} = 1$ leads to $a = (h+1)^{\frac{1}{h}}$
θ_{22}	Derivatives and graphs			
	Presentations: graphical and algebraic	τ_{23}	Then: $y' = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h}$ that`s: $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$	and $\lim_{h\to\infty} (1 + \frac{1}{h})^h = e$
		θ_{23}	Limits, algebra and calculus	

Figure 6. The praxeology MO2 within The Educational Institution

A look at the introduction of the number e at the upper secondary school level begins with the presentations of some exponential function $y = a^x$, the quote y/y' and the question if there is such number e that this ratio becomes equal with 1. The next step becomes the algebraic or the graphical one. The aim in the first one is to answer the question of the existence of the number e with the notion of differential quotient according to its own definition. This leads to the problem of the solving, which is described above. The solution is created through the testing of this difficult expression for quite small numbers, which leads to the approximation of the number e. The graphical one does the same thing of the testing but with help of calculator and illustrations of the graphs to the ratio y/y` directing towards to the value 1. The technologies of these two very similar approaches are not well satisfied from the mathematical point of view. There are apparent lacks of mathematical explanations of the notions of the functions and continuity of functions and the existence of this number e but thanks to the learning practice that usually is conformed at the upper secondary school it doesn't bring on any introductory problems for pupils with already acquired learning habits, that is how things go on at schools and what continually happens. This technology justifies some kind of reasons for the pupils. The pupils are usually both without demanded mathematical knowledge to react against the introduction and without really wish to do that. Learning studying practices don't impact the student's endeavours if they realize this "theoretical" kind of knowledge, which usually is perceived as to much theoretical and difficult to learn by. The reasons are about that between the numbers 2 and 3 should or shall exist a number that the value of the ratio $\frac{y'}{y}$ or of the limit mentioned above converge to the number 1. This is repeated time after time in the direction to get better and better approximated value of the number e. Theory of calculus is vague or not at all presented.

Solution

To begin instead with the well-known expression $S = P(1+r)^t$ to calculate the amount of money S with r percental interest rate during the time t if initially the investment is P can be a good solution. It leads to $S = P(1+\frac{r}{n})^{nt}$ by n times a year. The expression becomes $S = (1+\frac{1}{n})^n$ for $P = r = t = 1$. The table for different values on n shows directly quite well approach to the value of the number e (Maor, 1994, p 26). Continually the expression $(e^x)' = e^x$ can be easily proved while the proof for the existence of the number e goes via the theory of number series, which would stay out of the treatment at upper secondary school level.

Discussion

The anthropological theory of didactic with its epistemological and didactic model, which in this paper facilitate analysing mathematicians and educational literary practices in the case of the number *e* by two of the mathematical organizations seems even here to be a very practical investigating tool as in the article of Barbé et al (2005). The reference model or reference mathematical knowledge helps to follow up and explore in details the procedures of the introduction of the number ”e” in these two institutions and their connections within and between the institutions.

The problem of the introduction of the number e stems not only from the transposition process from the scholarly to the mathematical knowledge but as well from the problem within these institutional forms. There are consequently two kinds of difficulties. The first one has to do with the framework of organisational practices and the second one with the process of the transposition of the knowledge. Inside the organisational practices there are both the problem of the existence of the number and the problem of its value, which more and less demanding on which level they are treated on, are connected with, mathematical constrains. These mathematical constraints, which have clearly a mathematical theoretical source in their nature, are well known as for example the continuity and the limits of functions, the convergence of series and area problems with the notion of the integral. The students require this mathematical knowledge to understand the indigence and introduction of the number e in order to be able to manipulate with it. If they experience just constraints and failure in their struggling with mathematics already at the introductory level the possibilities for a good and successful mathematics leaning and understanding detract. The other constraints, which arise from the process of the transposition, are a lot unwieldy. Their treatments are at upper secondary school level and the problems, mentioned above, already exist at the higher level. This nevertheless there are transparencies between the institutional levels, which are protecting in the process of the transposition of this knowledge.

Barbé et al. (2005) point difficulties in the didactic process, which form mathematical organizations, regarding integration of its six moments in the case of the punctual mathematical organizations, which are not linked to each other. The moment of the first encounter, the exploratory, the technical, the technological-theoretical, the institutionalisation and the evaluation moment are included.

More studies both within institutional practices and within the process of the transposition are required not only in this case of the number ”e” but also in the other cases of the irrational numbers at special and mathematical knowledge in general.

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