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Mathematics and Science:

The relationships and disconnections between research and education.

Papers from a doctoral course at the University of Copenhagen

Edited by Marianne Achiam and Carl Winsløw

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The relationships and disconnections between research and education

Papers from a doctoral course at the University of Copenhagen

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Introduction

Marianne Achiam and Carl Winsløw

Department of Science Education, University of Copenhagen

Introduction

This book presents the products of a doctoral course held in Helsingør, Denmark in August 2014. The course was entitled *Mathematics and Science: The relationships and disconnections between research and education*, and involved the participants in a series of discussions and reflections about the intricacies of those relationships. Although the teaching and dissemination of mathematics, physics, biology, chemistry and other sciences in educational institutions is fundamentally based on the research disciplines that give them their name, the connections and disconnections between the research disciplines and those of teaching are never trivial. The complex process by which science (including mathematics) knowledge, values and practices are apprehended from their place of production - the domain of scientific research - and deconstructed and reconstructed in order to become teachable in the domain of education is known as didactic transposition. Understanding this process and the phenomena related to it is the fundament of the research programme entitled the Anthropological Theory of the Didactical (or simply ATD) and was the focus of this course.

ATD proceeds from the observation that didactic transposition is inevitable and indeed, necessary, in any kind of educational undertaking. However, the process is regulated by a number of factors, not all of which correspond to didactic intents. On one hand, the role of school is not simply to transmit as large a portion of the scientific disciplines as accurately as possible, but to form citizens who can function in today's and not least tomorrow's societies. On the other hand, the distance between 'living science' and 'school science' is not always explained by this basic role, but simply by the distance among research institutions and schools, as when the school discipline perpetuates the teaching of knowledge which has become more or less obsolete in scientific research and, eventually (sometimes even quicker!) also in society. Thus, didactic transposition carries with it the risk of delay (Quessada & Clément 2007), the introduction of notions in isolation from their origin and thus concealment of the 'true' functioning of science (Brousseau 1997/2002*a*), or ultimately, pathological substitutions (Chevallard 1991). These phenomena have implications for variety of educational contexts; from teacher education (Winsløw 2013) to the design of teaching-learning sequences (Levrini & Fantini 2013), quality of textbooks (Quessada & Clément 2007) and development of museum exhibits (Mortensen 2010).

In sum, didactic transposition must remain 'alive' and alert to new potentials of interactions between 'science in the making' and, more broadly, 'science in society' on the one hand, and the teaching and dissemination of science in schools and museums on the other. The societal and institutional conditions for maintaining these links differ from one society to another, and so an international perspective can also help to identify possibilities and obstacles in a particular society. It is the role of ATD to frame and structure inquiries into didactic transposition phenomena, and to contribute to explaining them. ATD thus provided the overarching conceptual and theoretical scaffolding of the doctoral course.

We initially had 17 participants from six countries, representing almost as many research fields. The course was arranged over five days of intensive work (Appendix A), including lectures, workshops, participant presentations, and intensive feedback sessions. Prior to the course, the participants were requested to read a number of basic texts (listed in Appendix B), and on the basis of these readings, to formulate a five-page paper outlining their own research and how they proposed to use the theoretical frameworks given in the readings in their own work.

During the course itself, the participants presented their ideas and actively participated in the discussions and reflections to further develop their proposal, under the supervision of the course teachers and guest lecturers.

Finally, participants were required to submit a revised and expanded ten-page version of their initial paper on the use of the theoretical framework in their own research, based on the course discussions and reflections and on the feedback from the course teachers. This aspect of the course was further strengthened by the revision of the texts after the course through a peer review process (with the course participants acting as peers). A total of fourteen participants went on to complete the course and produce papers that subsequently underwent peer review and revision. We are happy to present these fourteen papers in the following sections of this booklet.

The course benefited from the presence of several international scholars, including Professor Marianna Bosch from Ramon Llull University, Spain, Professor Tetsuo Isozaki from Hiroshima University, Japan, and Associate Professor Niklas Gericke from Karlstad University, Sweden. These scholars have worked with the phenomena of didactic transposition in various ways in their research, and contributed their unique insights to the discussions in the course through a series of lectures, workshops, and intensive feedback sessions.

To them, as well as to all seventeen course participants, we extend our sincere thanks for having worked so constructively with us. To the participants: we are confident that you will be pleased with the results of your efforts as presented here, and that you will take the projects you commenced here on towards complete studies and eventually, journal papers.

A Schedule of the course

Day 1 (Monday, August 18, 2014)

9:00-10:00 Welcome and introduction to the course

10:00-11:00 *Didactic transposition as a key research tool*. Lecture by Marianna Bosch

11:00-12:00 *Why to use the didactic transposition theory*. Lecture by Niklas Gericke

13:00-16:00 Participant presentations by Kaj Østergaard, Yukiko Asami-Johannson, Heather Bort, David Johnston, and Aerin Benavides

16:15-17:45 Seminar led by Marianna Bosch and Niklas Gericke

Day 2 (Tuesday, August 19, 2014)

9:00-10:00 Group work led by Niklas Gericke

10:00-11:00 *Studying exhibit design as a case of didactic transposition*. Lecture by Marianne Achiam

11:00-12:00 *Klein's double continuity and the institutional contingency of knowledge*. Lecture by Carl Winsløw

13:00-16:00 Participant presentations by Søren Witzel Clausen, John Andersen, Britta Jessen, Louise Meier Carlsen, Dyana Wijayanti, and Kerstin Bäckman

16:15-17:45 Seminar led by Marianne Achiam and Carl Winsløw

Day 3 (Wednesday, August 20, 2014)

9:00-10:00 *Teachers' knowledge and its relation to scientific knowledge*. Lecture by Tetsuo Isozakie

10:00-11:00 *From REM to didactic infrastructures and teacher education*. Lecture by Marianna Bosch

11:00-12:00 Group work led by Niklas Gericke

13:00-14:00 Seminar led by Tetsuo Isozaki and Marianna Bosch

14:30-17:45 Excursion to the Maritime Museum and Kronborg Castle

Day 4 (Thursday, August 21, 2014)

9:00-10:00 *Science versus school science*. Lecture by Niklas Gericke

10:00-11:00 *Constraints and conditions for didactic transposition in museums*. Lecture by Marianne Achiam

11:00-12:00 Group work led by Tetsuo Isozaki

13:00-16:00 Participant presentations by Klaus Rasmussen, Gary Williams, Steen Grode, Radka Stepánková, Nadia Azrou and Miguel Perez

16:15-17:45 Seminar led by Marianne Achiam and Niklas Gericke

Day 5 (Friday, August 22, 2014)

9:00-10:00 Group work led by Marianna Bosch

10:00-11:00 *Professional and scientific knowledge: In-service cases from Japan school science*. Lecture by Tetsuo Isozaki

11:00-12:00 *The interaction between research and teaching in university institutions*. Lecture by Carl Winsløw

13:00-14:00 Seminar led by Tetsuo Isozaki and Carl Winsløw

14:00-14:30 Closing and assignments

B Course readings

1. Achiam, M., & Marandino, M. (2014). A framework for understanding the conditions of science representation and dissemination in museums. Museum Management and Curatorship, 29(1), 66-82. (Excerpt)
2. Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. ICMI Bulletin, 58, 51-65.
3. Chevallard, Y., & Bosch, M. (2013). Didactic transposition in mathematics education. In S. Lerman (Ed.), Encyclopedia of Mathematics Education: SpringerReference. Berlin Heidelberg: Springer-Verlag.
4. Gericke, N. (2009) Science versus School-science. Multiple models in genetics - The depiction of gene function in upper secondary textbooks and its influence on students' understanding. Doctoral dissertation. Karlstad: Karlstad University. (Excerpt)
5. Isozaki, T. (2014). How have Japanese rika (school science) teacher traditionally formed their own cultures and improved their teaching competencies through research and practice? 16 p.
6. Isozaki, T. (2014). How do teachers develop a professional knowledge base through their in-service teacher training? Two case studies of developing scientific knowledge in Japan. 13 p
7. Mortensen, M. F. (2010). Museographic transposition: The development of a museum exhibit on animal adaptations to darkness. Éducation & Didactique, 4(1), 119-137. (Excerpt)
8. Winsløw, C. & Madsen, L. M. (2008). Interplay between research and teaching from the perspective of mathematicians. In D. Pitta-Pantazi & G. Philippou (Eds), Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education, pp. 2379-2388. Larnaca: University of Cyprus.
9. Winsløw, C. (2013). The transition from university to high school and the case of exponential functions. In: Behiye Ubuz, Çiğdem Haser, Maria Alessandra Mariotti (Eds.), Proceedings of the 8th Congress of the European Society for Research in Mathematics Education, pp. 2476-2485.

Participant list, Mathematics and Science: The relationships and disconnections between research and education.

Name	University / UC
Søren Witzel Clausen	Institut for Uddannelse og Pædagogik samt Center for Science Education, Aarhus Universitet
Steen Grode	RUC
Klaus Rasmussen	Department of Science Education, University of Copenhagen, Denmark
Kaj Østergaard	IMFUFA Institut for Natur, Systemer og Modeller, RUC
John Andersson	University College Zealand (UCSJ) Teacher Education in Vordingborg
Aerin W Benavides	Department of Teacher Education and Higher Education The University of North Carolina at Greensboro (UNCG)
David Johnston	Montclair State University, College of Education and Human Services, NJ, USA
Britta Eyrich Jessen	Department of Science Education, University of Copenhagen, Denmark
Gary Williams	76 Portland Pl, London W1B 1NT, UK
Louise Meier Carlsen	Department of Science Education, University of Copenhagen, Denmark
Yukiko Asami – Johansson	Högskolan i Gävle, Sweden
Nadia Azrou	Ecole Nationale Préparatoire aux études d'ingénieur. Département de mathématiques et informatique
Miguel Perez	Linnéuniversitetet, Sweden
Kerstin Bäckman	University of Gävle
Radka Stepankova	Department of Mathematics, Faculty of Education, University of South Bohemia, Czech Republic
Dyana Wijayanti	Department of Science Education, University of Copenhagen
Heather Bort	Department of Mathematics, Statistics and Computer Science Marquette University, WI, USA

1

Designing mathematics lessons using a Japanese problem solving oriented lesson structure - A Swedish case study

Yukiko Asami-Johansson

University of Gävle, Sweden

Abstract. This paper reports a case study of a lesson "Introduction of algebra" where a Japanese lesson plan, based on the "Problem solving oriented" approach was adapted in a lower secondary school in Sweden. The analysis based on the Anthropological theory of didactics explains the mathematical and didactical organisation of the implicated lesson. In this paper, it is also examined how the discrepancy between the Japanese and the Swedish curricula influence the adaptation of Japanese lesson plans to a Swedish classroom. The study showed that the epistemologically well-structured Japanese curricula in Geometry limited the adaptation of lessons plans in Geometry. Such limitations were less obvious in Algebra.

Introduction

Teaching methods, centred on problem solving, so called "structured problem solving" has been developed in Japan with an emphasis on making the students active participants in the mathematics lessons, without losing the focus on the mathematical content (Stigler & Hiebert 1999). Generally, the didactical technique of this approach is to start up the lesson by presenting problems that can be solved by varying methods and later having a whole-class discussion on the settlement options. Kazuhiko Souma developed "The problem solving oriented" approach (Souma 1997) (shortened to PSO; the author's translation; "Mondaikaiketsu no jugyou", in Japanese).

It is a variation of Japanese structured problem solving, where teachers emphasize the process of mathematical thinking and also focus on how to enhance the students' attitude towards engaging in mathematical activities. The PSO approach has a quite specific routine:

1. State a problem in a way to incite a guess or immediate observations on part of the students.
2. Let all students state a guess or hypothesis.
3. Discuss the viability of guesses and let students motivate their guesses.
4. Let students, individually or in groups, work on the problem, or, perhaps, on a derived reformulation or a sub-problem.
5. Let students present their solution in class and discuss the different solution techniques.
6. Turn to the textbook for an outline of the theory.

The points where PSO differs from other structured problem solving approaches is the emphasis on having students making conjectures or guesses. Souma stresses the need for *open–closedness* in the design of appropriate tasks; the student response – the answers – should be predictable but give rise to variation of the methods. The PSO methodology also contains elements for establishing motivation, a didactical contract (Brousseau 1997), as a necessary condition for establishing a successful discourse.

From 2010 to 2011, I carried out classroom observations in a lower secondary school in Sweden, where the teacher applied lesson plans according to PSO's basic lesson structure. One practical reason for choosing to employ Souma's PSO approach, among several variations of the Japanese structured problem solving approaches, was that it supplied a lot of material ready to use in lessons. Souma has written and edited a number of practical books, including textbooks (2012), in which he proposes lesson plans using the PSO approach and which also contain collections of problems, suitable for the different sections to work with. His second book, "The problem solving oriented approach" has reached its 13th edition since 1997 and his task collection is now the 9th edition since 2000. This means PSO is quite successful with regards to publishing books on mathematics education in Japan. The fact suggests that teachers in service in Japan actively use the PSO approach. Also, as I mentioned above, PSO has very clear basic structure which is quite easy to follow even for Swedish teachers.

In a case study that covered the introduction of negative numbers, I discussed how the PSO approach influences the students' attitudes towards participation in lessons (Asami-Johansson 2012). In this long-term empirical study, I found that Souma's lesson plans in geometry were difficult to adapt to our Swedish lessons, while the sections on arithmetic and introduction of algebra were comparatively easier to work with. One could assume that it was due to the discrepancy between the Japanese curriculum, which the PSO lessons largely follow, and the Swedish counterpart.

The aims of this paper are twofold. First, to report a case study of a lesson in "Introduction of algebra" and estimate the mathematical and didactical organisations of the PSO adapted lesson. Subsequently, I will describe the students' activities during a lesson and how they become encouraged to inquire the knowledge of algebra. The second aim is to find out to what extent the discrepancy of Japanese and Swedish curricula influence the adaptation of Souma's lesson plans and to show what kind of epistemological difficulties that rise when one try to adapt Japanese lesson plans in Sweden.

Theoretical Framework

The didactical transposition theory

Chevallard developed the concept of didactical transposition – the content of the knowledge adapted for the purpose to be taught within a given institution. It means that transposition from *scholarly knowledge* (Bosch & Gascón 2006), which was produced in the community of mathematicians, into the knowledge for teaching, for different levels within the teaching system. Bosch & Gascón (2006) illustrated the steps of didactical transposition process through different institutions as *Scholarly knowledge* → *Knowledge to be taught* → *Taught knowledge* → Learned, available knowledge (ibid, p. 56). The "knowledge to be taught" is, designed by institutions, which Chevallard calls *noosphere* – sphere of those who think (Greek word, *noos* – mind), which is a non-structured set of experts, who have a big influence within the education system, like educators, curriculum developers, politicians, text book authors and so on (Chevallard 1992*b*). The "taught knowledge" is created by the institution of the teachers' praxis. The "learned, available knowledge" is the knowledge formed by the learners in the classroom.

The anthropological theory of didactics

Chevallard's anthropological attempt to study the mathematical knowledge in an institutional context extended into "*the anthropological theory of* *didactics*" (ATD) (Bosch & Gascón 2006). There, mathematics learning is modelled as the construction within social institutions of *praxeologies* (ibid). A praxeology supplies both methods for the solution of a domain of problems (praxis) and a framework (*the logos*) for the discourse regarding those methods and their relations to a more general setting. A praxeology describing mathematical knowledge is also called *mathematical organisations* (MO) (Barbé et al. 2006). The praxis can be described by the set of *tasks* and *techniques*, which is applied to solve the tasks. The logos constitute of a *technology*, which justifies the technology and a *theory*, which in turn, justifies the technology. A *didactical organisation* (DO) is a praxeology developed by teachers to organise the work of achieving an appropriate mathematical organisation (ibid) and consists also of *tasks*, *techniques*, a *technology* and a *theory*.

The act of the didactical transposition

Japanese curriculum

Japanese teachers in lower secondary school (grade seven to nine, students' ages are between 13 to 15) are supposed to refer and mind "The Commentary to the Course of Study" ("Gakushu shido yoryo kaisetsu" in Japanese, Ministry of Education, Culture, Sports, Science and Technology, shortened to MEXT, 2008). There, the goal and significance of learning the domains are described in detail. In the domain "Numbers and Expressions", the goal is to cultivate students' ability to the use of algebraic expressions to represent constituting relations between quantities. The understanding of such representations with expressions, together with skills calculating with algebraic expressions is also stressed.

The focus in the description of the domain "Geometrical Figures" is about training students' skills of observation and their skills in making correct proofs. The content of "Geometrical Figures" for grade 8 consists of: 1. parallel lines and angles, 2. property of polygons, 3. congruence of polygons, 4. mathematical assumption, 5. necessity and significance of using proof, 6. properties of triangles and parallelogram, 7. reading a proof and finding out a new property. "The students begin by studying to deduce mathematical proof coherently and logically by studying the property of parallel lines and vertical angles" (ibid, 92). The intention of the first paragraph is that, through learning the properties of vertical angles, parallel lines and transversal lines and corresponding angles, students will be able to draw the conclusion of properties of alternate angles and vice versa.

The commentary explains that it is a core mathematical strategy to reason on things using previous knowledge, for instance – the triangles angle sum – to deduce corresponding result on the sum of polygons' inner angles. The goal the paragraph *Congruence of polygons* is to deepen the students' perception of geometrical figures, by understanding the notion of congruence of polygons. One starts from the "conditions of congruence of triangles". In this way, one develops the students' abilities of logical thinking and their ability to express their reasoning. The significance of understanding congruence of polygons and triangles is emphasized. Similarly, the necessity of proofs and methods for proofs is stressed. Considering the basic properties of triangles and parallelograms by studying conditions for congruence is also emphasized (ibid, p. 93).

Geometry to be taught

The Commentary shows clearly that the focus is on deductive proofs, where one should learn to apply previously established theorems. Miyakawa (2014) has made a study on the use of mathematical proofs in school geometry for lower secondary schools. There, he discussed the Japanese *knowledge to be taught* in the domain of mathematical proofs and discussed why some part of it is present in the curriculum and others parts are not, using the notion of an *ecology of mathematical praxeologies* (Chevallard 1992*b*, Bosch & Gascón 2006). Miyakawa points out that there are two characteristics in *geometry to be taught*; first, that it emphasizes generality, second, that one works with a "quasi axiomatic" geometry (2013).

Miyakawa explains that the treatment is usually general, where one set out to prove various universal propositions. In the Commentary for the Course of study, the definition of proof is stated as "a method to clarify that a proposition is always possible to be applied without any exception". The phenomenon of emphasizing generality is notable within the domain of proofs (ibid, p. 350).

The second issue "quasi axiomatic geometry" is about the system of geometry. Miyakawa means that, similar to the axiomatic system of Euclid's Elements, the system of Japanese school geometry is roughly (that is why the term *quasi* is used) axiomatic. The quasi axiomatic geometry is a specific theory within the regional praxeology of geometry constructed in lower secondary school. It is a simplified version of Euclid's geometry for use in the school mathematics. This theory contains deductively approved theorems, which are also approved one by one, and derived from a few accepted axioms (ibid, p. 350).

Regarding the *geometry to be taught* in the Swedish lower secondary school, presented in the curriculum from 1994 (Skolverket 1994), no descriptions for learning axiomatic proof is found. To learn and state geometrical theorems and to implement deductive proofs of such theorems is not required in lower secondary school. This discrepancy constituted the main limitation for us, in our attempt to adapt Souma's lesson plans in geometry in Swedish mathematics lessons.

Knowledge to be taught in arithmetic and algebra

On the other hand, the connections, between different sectors in arithmetic and algebra in the Course of study, are not as strong as in geometry. A deductive proof is not obligatory in arithmetic and algebra: one does not start from the Peano axioms and the definition of the order of natural numbers, to prove that the number two is greater than one. Such statements are already generally accepted for cultural reason. Theory based on axioms and deductive proofs is not indispensable for the fitness of the mathematical praxeologies constructed in arithmetic and algebra.

That the sectors in the Japanese arithmetic and algebra are less explicitly connected reflects on the sequence of Souma's lesson plans. Hence, we could adapt the lesson plans from arithmetic and algebra comparatively easier than those from geometry. There are many more commonalities between the Japanese and Swedish course of study in arithmetic and algebra. Both include an extension of the number concept to the negative numbers and rational numbers and respect for acquisition of basic skills of arithmetic operations with symbolic expressions. Both curricula treat the use of algebraic equations as a powerful tool for modeling. Both use identities between algebraic expressions for stating arithmetic laws and formulas and, similarly, to state equivalences, as in the cancellation laws for equations.

A case study in Swedish classroom

Praxeology in Souma's lesson plan

In Souma's lesson plans for grade seven, the section of "Introduction of variables" is located in the chapter "Algebraic expressions" (Kunimune &

Souma 2009). The goal of the chapter is “to express quantities and the relations between different quantities simply, clearly, and generally by using algebraic expressions” (ibid, p. 26). The focus of this section is to introduce modelling using variables, before starting to train students’ skills of solving problems with linear equations. The weight is put on making the students realise the convenience of using formulas; for instance, the formula for the area of a triangle. The goal of the lesson is described as “to let the students understand the significance of using variables instead of numbers” and “to develop students’ ability to explain their thinking process to each other with the use of images/figures and formulas” (ibid, p. 27). To realise that goal, the task is formulated so that the students should generate a formula that determines the total number of stones in a square. Firstly, one needs to find a solution (in several different ways) for the case of 5 stones on each side. Later, the students use the same arguments for the case of 20 stones on the each side. These activities will lead to the finding and statement of a formula in the variable n, defined as the total number of n stones one side of the square.

The task in the lesson plan is posed as follows: “We will make a square putting stones as the picture below. If one of the sides consists of 5 stones, how many stones are used in total?” (ibid, p. 27) (See Figure 1.1)

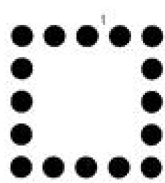


Fig. 1.1: Illustration of the task; “How many stones are used in total?”

The lesson plan assumes students might find out the following methods (see Figure 1.2) to determine the number of the stones (ibid, p. 28):

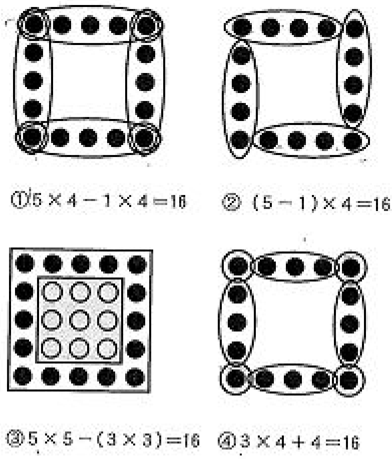


Fig. 1.2: Illustration of students' possible solutions.

When the students have suggested those techniques, the teacher states a next task: "If one of the sides consists of 20 stones, how many stones are used in total?", "If one of the sides consists of n stones, how many stones are used in total?".

The goal here is, letting the students to notice that it is possible to apply the same set of methods to determine this task. According to the figures above, the techniques (τ), of the lesson plan suggests the following expressions for the result: $\tau 1 : (n \cdot 4) - (1 \cdot 4), \tau 2 : (n-1) \cdot 4, \tau 3 : (n \cdot n) - (n-2)^2, \tau 4 : (n-2) \cdot 4 + 4$. The distributive law can then be used to solve the sub-task to show equality of these expressions.

The technologies (θ), which are not explicitly mentioned in the lesson plan (except the $\theta 1$"figures") but are noticeable, which are: $\theta 1$: the use of figures, $\theta 2$: the additive and inclusion-exclusion laws for the cardinality of sets, $\theta 3$: the distributive law, $\theta 4$: the squaring law for the cardinality of a squared set.

The theory (Θ), which justifies those technologies are: $\Theta 1$: arithmetic operations and $\Theta 2$: algebraic expressions, and $\Theta 3$: cardinality of sets.

When we planned the lesson of introduction of algebra for the grade 8, we had the aim to illuminate the concept of modelling, using variables, and to develop students' skills to explain the process of their reasoning by using figures and formulas. In a sense, we adopted the lesson plan as it was from Souma, without adding additional techniques or technologies. Two lessons were implemented for the introduction of algebra. One lesson using the lesson plan mentioned above and another lesson that compared different algebraic expressions, in order to find out which expressions are actually identical. Thereafter we started to implement lessons on modelling with linear equations. We followed the sequence of Souma's lesson plan for the sector "Algebraic expressions" for the domain "Algebra" in our lessons.

The lesson

This is a lesson we have initiated as the introduction of algebra to the class of grade 8. The data collection method I applied in this study is classroom observation with video recordings and written protocols. In total 18 students attended this lesson. They have learned to solve basic linear equations like $2x+x=15$ at grade 7. The teacher starts the lesson by writing the word "general solution" on the whiteboard. She asks the class if anyone has heard the word "general solution" before. A student answers it may mean "something common". The teacher states that it means "a solution, which is applicable for all values". Then she draws the first figure of the square with stones and asks the class what kind of figure she has drawn. A student answers "a square" and another student "four equal sides". Then the teacher asks how many stones totally are in the picture. Nirma immediately answers it is 20. But when Alex answers shortly after it is 16, Nirma corrects her conjecture to 16 as well. When the teacher asks Nirma why she thought it was 20, she answers:

Nirma: Well, you said, 5 stones on each side. And there are 4 sides, then 20. I was little too quick.

Teacher: Ok, I understand exactly how you thought. (turns to the class) Do you understand how Nirma thought? . . . Alex? Why did you think it was 16?

Alex: (Points at the squarer) 4 stones on the bottom . . . then disappears [He means actually "shifts"] one stone on each side

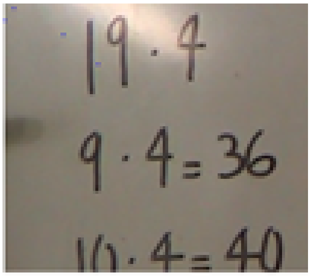
T: You mean. . . (circles first, the four stones on the bottom, then all the other sides) here? [see figure 1.2, nr. 2]

In this episode, Nirma notices that something must be done to calculate the total number of the stones. It does not work with simply using 5 times 4. She understood it as soon as Alex started to explain. The teacher states the main task for today. Nirma wants to explain.

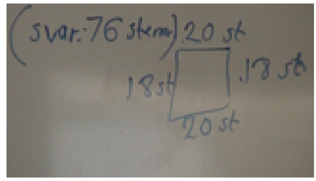
Nirma: One takes away one stone from each corner.
T: You mean these? (circles the stones on each corner) [see figure 1.2, nr. 4]
Nirma: Yes, then three stones from the two sides … six stones … and another six stones from another two sides. 4 + 6 + 6. 16 stones.

Here, Nimra suggested actually $\tau 4 : (5-2)\cdot 4+4$. She has probably focused on the number 4 and the total number 16 and automatically counted the rest 12. She might somehow have associated 6 as a factor of 12, than 3, then added two 3s and made 6. The teacher later modified her expression to $3+3+3+4$ and furthermore to $3\cdot 4+4$ through asking Nimra and the class. Vincent suggested $2\cdot 5+2\cdot 3$. It is not shown in Souma's lesson plan, so we call it $\tau 5 : 2n+2(n-2)$. It is a variation of the $\tau 1 : (n\cdot 4)-(1\cdot 4)$. Oliver's idea is $\tau 2 : (n-1)\cdot 4$, which is same solution Alex described in the beginning.

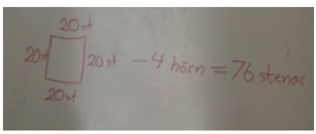
Now the teacher steers the lesson to the next step. The students are supposed to work in pair and consider the ways to determine the total number of the stones when there are 20 on each side. The teacher gives the students several minutes and walks around between the groups. She listens to the different groups discuss and asks some groups to present their solutions. Vincent and Umit have found $\tau 2 : (n-1)\cdot 4$ and write directly $19\cdot 4$ (see figure 1.3a). Muhannad and Bint have chosen $\tau 5 : 2n+2(n-2)$. They write a square with "20" by the two horizontal sides and 18 by the vertical sides (see figure 1.3b). Nimra and Rachel use $\tau 1 : (n\cdot 4)-(1\cdot 4)$. They write a square with "20" on all sides and added "– 4 corners" (see figure 1.4a). Oliver and Magda have found $\tau 2 : (n-1)\cdot 4$. They write on the whiteboard $(20-1)\cdot 4$ (see figure 1.4b) which shows the process of their solution directly.



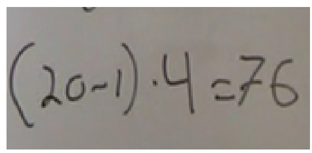
(a) Vincent and Umit's solution.



(b) Muhannad and Bint's solution.



(a) Nirma and Rachel's solution.



(b) Oliver and Magda's solution.

Then the teacher asks the class in which way one can express the total number of dots using n.

T: For instance, if the n is one million, how we can express the total number of the stones?

Ahmed: One million times 999,998.

T: Ok, you mean this one (points Muhannad and Bint's solution $2 \cdot 20 + 2 \cdot 18$) [see figure 1.3b]? Anyone else?

Oliver: If we use our example, it will be n minus 1, times 4.

T: (writes down $(n-1) \cdot 4$ below Oliver's solution $(20-1) \cdot 4$). Is that correct (asks the class)? Put in 5 in n and see if it will be 16 (points the formula). Is $(n-1) \cdot 4$ must be as it looks like? Can we move 4 to in front of $(n-1)$?

Students: We can bend the place.

T: Really? Ok. (writes $(n-1)\cdot 4 = 4(n-1)$).

Ahmed's answer shows that he has an idea to show how the technique works. But he has not really understood to express his reasoning correctly. On the other hand, Oliver has already seen how the technique works. Thereafter, the teacher takes up Vincent and Umit's solution $19\cdot 4$. She asks the class to consider how it could be expressed using n. Vincent explains that 19 came from $20-1$. The class establishes that $20-1$ can be expressed also $n-1$, then this solution is also $4\cdot(n-1)$. Then the teacher asks the class to express Muhannad and Bint's solution, $20+20+18+18$ and the students confirm that it will be expressed $2n+2(n-2) = 4n-4$, as well as Nimra and Rachel's expression.

It might still be that not everyone in the classroom has grasped the algebraic operations. However, the intention of this lesson was to connect the different techniques τ_1, τ_2, τ_3 and τ_4 in a common technological discourse. Three of the four proposed techniques; τ_1: $(n\cdot 4)-(1\cdot 4)$, τ_2: $(n-1)\cdot 4$ and τ_4: $(n-2)\cdot 4+4$, were suggested by the students. These techniques are technically justified by the technologies the use of figures, θ_2: the additive and inclusion-exclusion laws for the cardinality of sets, θ_3: the distributive law.

Conclusion

The Japanese curriculum provides a relatively detailed fundament for the large mathematical (and also didactical) organisations, at least in comparison to the Swedish curriculum proposed by government-level agencies. As a consequence, the transposition from the knowledge to be taught to the taught knowledge shows less variation in Japan. This precondition of a large and integrated mathematical organisation makes it hard to adapt Souma's lesson plans in geometry in Swedish mathematics lessons.

One significant finding from the analysis of the empirical study is the power of the tasks. The task looked quite simple but not so apparent to be able to guess correct answer directly. It also generates several techniques, which was justified by several technologies and gave an opportunity for the students to explain their reasoning.

The other finding is the power of the guessing technique. Nirma's case shows that, the guessing moment played an important role to initiate the didactical process and thereby giving her an opportunity to participate in the lesson.

2

Teaching mathematics and difficulties with proof at undergraduate level

Nadia Azrou

University Yahia Farès of Médéa, Algeria

Abstract. This paper presents our work in progress, which is about the difficulties of university students with proof and proving in a course of Complex Analysis. We analyse students' productions through tests administered during a third year Complex Analysis course in an outstanding, selective School of Engineering in Algeria. The analysis uses the tools of the anthropological didactic theory to evaluate the teaching of mathematics.

Introduction

Mathematic proof activity is one of the problems that face students at all levels, at the university level; research has documented that many students still struggle with proof regarding many aspects (Harel & Sowder 1998, Weber 2001, Moore 1994, Selden & Selden 2003, 2007*a*, Dreyfus 1999, Pedemonte 2007). The situation in Algerian universities doesn't provide an exception for this question, at the undergraduate level; proof and proving becomes not only a recurrent activity for students, but also the only means for students' evaluation. Moreover, it has been reported by many teachers that this might be one of the principal causes of students failing in mathematics which led to the disaffection towards mathematics and engineers disciplines for many years.

Our analysis, in this paper, would show how the institutional choices have been set regarding teaching and learning proof and how do they affect the proving ability of students at the undergraduate level.

We use for this purpose analysis of students' copies for given tests with some chosen questions emphasising on the anthropological didactic theory elements.

Proof at university level

Proof is the essence of mathematics at tertiary level, if we don't prove, it is just not mathematics. Mathematical activity goes beyond formulas and calculations; it's all about constructing objects, results and even conjectures. The transition to proof at tertiary level occurs at many levels, students' conception of proof does not correspond do what they are required to produce at the university level (Selden & Selden 2007*a*, Harel & Sowder 2007, Weber 2006). They are first faced with the unclear and inexplicable necessity to prove, then, they confuse argumentation with proof (Balcheff 1999, Duval 1995, Douek 1999, Thurston 1994, Boero et al. 1996) and assume a proof to be one of some situations like: verifying a property for some examples, giving a diagram or a graph, a proof is true if given in a book or by a teacher. The objectives and the focus is not the same as in school mathematics, as pointed out by Winsløw (2008). Proofs with university mathematics are different from those made at school, they are more precise, concise, and more formal, they are also more complex; moreover, they require a deep larger knowledge base (Selden 2010). The didactic contract changes from school to university, "from describing to defining, from convincing to proving in a logical manner based on definitions" (Tall 1991, p. 20). The logical formal language using quantifiers and implications along with logical connectives in definitions, theorems and results is a major obstacle to students (Epp 2003, Selden & Selden 2007*a*). University teachers, who know a little about mathematics courses in high school, assume that students are familiar with this language (Clark & Lorvic 2009, pp. 763-764). Students are required to have more autonomy and flexibility between mathematical registers (e.g., algebraic, graphical, or natural language); some results can serve as partial results on a path toward an important theorem (Larson as cited in Selden (2010)). Students need to have meta-mathematical knowledge and establish links between concepts. "Sometimes, a proof requires not only applying directly a theorem in a particular case, but also to adapt or even to transform a theorem before recognizing and/or using it (Guzman et al. 1998)."

Proof in the curriculum

Let's first set an overview of proof teaching from elementary school to the university level (in Algeria) to examine the whole process which is close to some countries' one. In elementary school, children do not get (or rarely) any idea about argumentation, they always trust the teacher who has the last word (the correct answer). In middle school (or lower secondary), proof activity appears first in geometry (angles, triangles...). Even if students encounter proof for the first time at this level, they do not seem to pay attention to what really a proof is or what does this proving process teach. During the next years (four to five) till the end of high secondary level, proof doesn't appear at all in mathematics activity as it did in geometry. When students reach the university and if they choose mathematics or any engineer's discipline they have to deal with proof every day. This creates a non suitable situation for the students to learn proof. Not only students do not know what a proof is and what are they required to do when they are asked 'prove that...', but they lack many important skills that allow them to do well with their proofs, unfortunately many of these skills are not taught to them.

Theoretical framework

We will use in our analysis the didactic transposition theory. It's the whole process made to any knowledge to be ready taught to students. The process begins with a selection, by some scholars and other people concerned about teaching and learning of some specific information related to some discipline called scholarly *knowledge*, then the *knowledge to be taught* is designed in curricula and institutions programmes, the *taught knowledge* is the effective practices, subjects and activities taught in classrooms, and finally the *learnt knowledge* is the one the students got at the end of the learning process. In doing so, any knowledge is allowed to be 'taught' and transposed to another situation that is concrete and applicable to be used by learners. In our case, we'll discuss the teaching of mathematics regarding the teaching of proof and proving activity. It's not a mathematical content per se, but we'll examine where (in which case of the previous steps) it is introduced and how during the university level. Our analysis will be also supported by the anthropological didactic theory of Chevallard. In order to study the institution practices related to an object of knowledge, Chevallard proposes modeling the mathematical activity into ‘praxeology’ as an organisation into different steps.

It consists of the four following elements: [**T**,τ , θ, Θ] where:

- **T** is a *task* that is introduced by any order like: show, prove;
- τ is a *technique* which is the way or means to realise, accomplish or solve the task ;
- θ is a *technology* which is the discourse explaining and justifying the use of the previous technique τ;
- Θ is a *theory* which is the technology of technology θ, it’s the main source of the principles supporting the justification of the technology.

[**T**, τ] is called the *pratical block* of the praxeology which stands for the doing-how or the practical knowledge, and [θ, Θ] is called the *theoretical block* which stands for the theoretical knowledge.

Teaching proof

When considering research about proof and proving in undergraduate level, it’s not hard to see the constant position that presents this activity as an obstacle for students regardless the culture, the country or the mathematical contents, which might be translated as a meagre learned knowledge of students about how to make proofs and about what a proof is. Looking back at some steps in the didactic transposition process, merely ‘taught knowledge’ and ‘knowledge to be taught’; we can see many mathematical concepts and contents in books and teachers’ courses. Teaching mathematics is reduced to a classical and traditional exposition of definitions, theorems and their proofs. These proofs are more presented as evidence or reason that these results have been proved, that is to be trustful and taken for granted, rather to teach proofs and show some examples about proving process. The focus in teaching mathematics at the undergraduate level is more about the concepts, their understanding and their uses; these aspects are more emphasized during proofs given by teachers. A clear difficulty with this model of mathematics called definition-theorem-proof (DTP) is that it doesn’t explain the source of the question (Thurston 1994). Another aspect that might act as a veil to see in depth the core activity of proof and proving is the high level of formalism in mathematics at the university level. Algerian teaching at this level that occurs in French has inherited the Bourbaki aspect of writing mathematics whose strong focus is on propositional logical symbols. With this kind of teaching, students learn that mathematics' statements, results and especially proofs are reduced to a set of syntactic rules more than what could be expressed behind, as ideas and links. An additional difficulty for students is to unpack the logic of mathematical statements to be able to make proofs (Selden & Selden 1995). But proof is more about substance, conjectures, discovery ideas and communicating insights than about the formal shape (Thurston 1994). At the university level, students are supposed not only to be engaged with proving, but also to construct proofs using theorems, definitions, and multiple other results and techniques (Winsløw 2008), in other words, the theoretical block should be assumed by them.

Research findings indicate the need for teaching proofs and proving process (Dean 1996, Hanna 1989, Mariotti 2006, Moore 1994, Selden & Selden 2007*b*) that would provide students with a clear conception of a proof and proving process, and how it should be done and written. Many researchers (Hanna 1989) question the way proof is presented by mathematics teaching and propose other ways that show both the exploratory-discovery and structural sides of the proving process focusing on discussing the arguments more than on the formal side. Contrarily to what most mathematics teachers assume, mathematical thinking does not automatically emerge from a study of mathematics (Burton 1984). As cited in (Dean 1996), they (proofs) do not provide insight into the logic of the discovery. The question is not, what is the connection between concepts shown in the proof, but rather, how does one come to make the conjecture in the first place, or how does one then construct the proof, or how does one come to understand the proof.... The actual history of proof conflicts with the presentation of a proof as an ahistorical certainty that is typical of classroom presentations. Richards (1991).

Complex analysis course

The complex analysis course that supports our study is a course for students of third year, either are mathematics majors or engineers. The course whose description will be detailed is for engineers students in a high competitive school in Algeria. It's a school preparation of three years, after this period, students will leave to another school in which they will choose some speciality. Students are selected by an exam at the entrance to the school in the first year; only a limited number are accepted to maintain a quality of education and organization. The programmes in the school are all covered, and the exams are of a good level, more time and organisation is devoted to exercises sessions. All these elements are about to indicate that the students are hard workers, whose prerequisites are complete and possible difficulties are not a result of unfinished programmes or low achievers students.

The course is rich gathering contents in algebra, geometry and basic analysis along with topology. I was one of the two teachers who taught it, during the exercises sessions. The course curriculum is covered during 28 weeks with two course sessions (one hour and a half for each) and two exercises sessions (one hour and a half for each) per week. It's important to notice at this level that the course contents are too many to cover in a short time, which indicates a quantitative approach rather than a qualitative one of the constitution, this situation doesn't help for learning and teaching proofs. The time is mostly devoted to present contents definitions, results and theorems, the objectives are more about 'explaining' the concepts and the theorems than working and practicing in proving other results. The courses are given in a magisterial way where the teacher writes and explains, no much time is devoted to questions, and particularly proofs are exposed just after the theorems. Knowing that they will never get proving a theorem in their exams questions, students do not read again these proofs. The exercises' sessions are about solving some exercises given generally ahead of time to the students. Teachers expect students to 'try' to solve them and then expose their successful or unsuccessful attempts. Not all the exercises are solved in a formal way, some of the proofs are explained, main ideas and plan are shown, and finally tiny details that seem obvious are left to the students along with the final proof text, that are rarely checked later.

Methodology

Our study examines the written personal proof texts of the students. We have given four tests (in French) for the students every two months during the academic year. We have chosen to give each test just before an official exam (there are four exams), to be sure that students have revised, worked and mastered the contents in question. Each test contains three questions and lasts thirty minutes. Students had to respond on the test sheet where a blank of some lines has been left under each question. To help students not giving blank sheets in case they are stuck, which is not helpful for our study, finally we added a notice in the sheet inviting students to give a reason or a remark in case they don't know the response or are stuck, making clear that this test is meant to know more their difficulties to overcome them rather than judging them. The analysis will use the anthropology theories tool and will be focused on how proofs presented in the teaching affects students' ones.

A-priori analysis

We'll examine the test 3, and analyse only the first exercise along with two interesting students' productions.

> Exercice 1:
> Let $f(z) = z+1, z \neq 0$ a holomorphic function. Let γ be the upper half circle of the unit circle. Calculate $\int_\gamma f(z)dz$.

The exercise is about calculating an integral, along a familiar path which is the upper half of the unit circle, of a holomorphic function. Students are required to use the integral on a closed path theorem along with the parameterisation of the half circle which requires knowledge in both algebra and geometry. This kind of exercises is very usual to the students who are very familiar with calculating integral using a suitable parameterisation. Therefore, this exercise uses a technical knowledge already mastered by the students. Besides a clear understanding of the results about calculating an integral through a closed path, we expect students not only to calculate formally the integral but also to give the necessary justification that sustain their ideas and claim.

A-posteriori analysis

Among almost one hundred students (98), only 52 students calculated correctly the integral and gave the final result which is $i\pi$. The rest of the students whose answers are wrong, 29 one calculated the integral using a new closed path in order to use Cauchy integral formula and 17 students made varied mistakes about the chosen angle of parameterisation and the primitive function. Let's recall the Cauchy's integral formula.

Theorem (Cauchy's Integral Formula): Let γ be a simple, closed, positively oriented contour. If f is analytic in some simply connected domain S containing γ and Z_0 is any point inside γ, then $f(z_0) = \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{z-z_0}dz$. Which is the same as giving the integral value $\int_\gamma \frac{f(z)}{z-z_0} = 2i\pi f(z_0)$.

Most of the students who used this formula, although impossible to use as the path is open, considered the path closed by adding to the half circle the segment [-1,1].

Then, transforming the function to be $f(z) = \frac{z^2+1}{z}$, they denote the numerator by $h(z)$ to obtain the same form as the theorem's one $\frac{h(z)}{z-0}$ such that the function h is holomorphic in the whole domain delimited by γ, then the integral formula follows: $\int_\gamma f(z)dz = 2i\pi h(0) = 2i\pi$.

These students used a technique that is supported by a theorem that cannot be applied in this case. It's a difficulty at the theoretical block. The misuse of the theorem tells about how and what the students remember about the theorem. They remember this theorem more by the formula of the integral and the function form to be integrated more than by the conditions within which this theorem should be used.

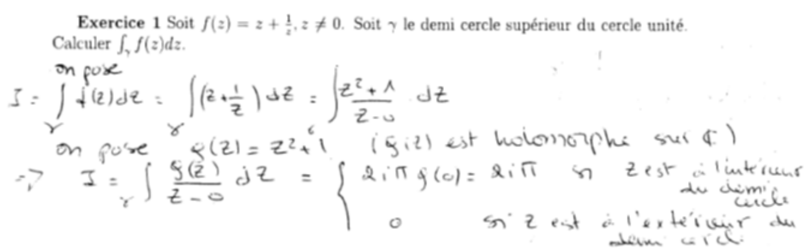
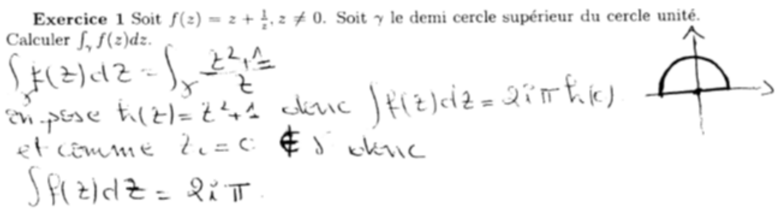


Fig. 2.1: Production 1

In this production, the student works the form of the function to make it look as the one used by the theorem $\frac{function}{z-critical z_0}$, then she gives the two possible results according to the position of the critical point (which is correct). But, even though she denotes the critical point by 0, she writes z in the conditions to refer to 0. We think that she meant 0 and didn't pay attention to this mistake. However, the response doesn't give the final value of the integral as the student doesn't decide which case the question is about. Moreover as the path is not indicated, we might think that the students either assumes it close or didn't care about it which means that she doesn't know the conditions of the theorem. This response seems to be more reciting a theorem than applying it. The student shows that she knows the theorem to apply, but not mastering it fails to let her know how to apply it.


Exercice 1 Soit $f(z) = z + \frac{1}{z}, z \neq 0$. Soit γ le demi cercle supérieur du cercle unité. Calculer $\int_\gamma f(z)dz$.

$\int_\gamma f(z)dz = \int_\gamma \frac{z^2+1}{z}$

on pose $h(z) = z^2+1$ donc $\int f(z)dz = 2i\pi h(c)$

et comme $z_c = c \notin \gamma$ donc

$\int f(z)dz = 2i\pi$

Fig. 2.2: Production 2

In this response, the student shows a new considered close path (the upper half circle along with the segment) rather than the given one, which is a change of the exercise's hypothesis, she made this change certainly to be able to apply the theorem. For this student, the conditions of the theorem are clear. Denoting by h the numerator function (as stated in the theorem formula) and considering 0 as to be the critical point, she gathers the principle elements of Cauchy's formula. Then, the students remarks that because 0 is not in γ, she can finally find the result $2i\pi$. This last step is not clear for us, the students gives a conditions that is not stated in the theorem, besides it's not clear what γ stands for, is it the upper half circle or the closed path (with the segment) showed in the picture. According to the theorem, if the critical point is not in the domain, the integral value is 0. If this is the situation the student refers to, she should give 0, but the result given suppose that the point is included in the domain, two opposite facts that do not explain what the student meant by saying that 0 is not in γ. We suppose that, by adding the segment to the path and 0 becomes on the path edge, the student gave a very unusual situation which is neither treated by Cauchy, neither taught by the course programme (the students know it already), this pushed probably the student to confuse the cases of the theorem.

Conclusion

These two productions show problems at the theoretical block of the anthropological theory elements. This problem leads to mistake the practical block. A wrong technique is used because students mistake the theorem to use. Remembering and memorizing a theorem by its final result and failing to state the conditions first, suggests a wrong technique. On another hand, teaching a theorem doesn't go till mastering its use, students show that they know which theorem they might use by reciting it like they remember it (not necessarily mathematically), not complete, without questioning its conditions. This kind of teaching, at the university level, might reflect a choice that does not match with proof and proving activity where students are required to adapt the theorem to a new situation and establish the possible solutions or cases after deep understanding. It also indicates how the students do in mathematics by findings tricks (even not available) that would make the situation sound true, which tells about their poor meta-knowledge (or a culture) about mathematics and especially about proof that is not taken into consideration in the didactic transposition. Learning proofs doesn't occur by learning mathematics contents; it's about developing skills related to many components as (concepts understanding, mastery of mathematical language, strategic knowledge, meta-mathematical knowledge, mastery of different registers and techniques), but not limited to these. Proofs as presented in books and by teachers, already ready in their final version, are shown to support a validity of a result. The teaching doesn't tell how these proofs have been constructed, developed and validated, and how results have been conjectured. Moreover, the process of constructing proofs (exploratory phase, argumentation phase, text writing phase, Arzarello (2000)) is nowhere revealed to the students, to let them see that proofs are not linearly made and written. The didactic transposition along with the teaching of proving process need more work and research to be adapted and shaped in order to overcome students' difficulties with proof.

3

Teaching and learning of shapes in preschool didactic situations

Kerstin Bäckman

University of Gävle, Sweden

Abstract. This study is about the teaching and learning of mathematical phenomena in preschool. Drawing from the Theory of Didactic Situations, the concepts 'milieu', 'didactic' and 'a-didactic' situations are illustrated in a play situation oriented towards teachable moments. An analysis of the didactic transposition is used to investigate the process that takes place when the knowledge to be taught is applied to the preschool. The findings show the teaching design that is carried out in didactic and a-didactic situations; a design that is examined from the perspective of mathematical praxeology.

Background

As a child's early learning in mathematics is essential for further learning processes, education in preschool is a contentious issue. Research on early mathematics shows two strong approaches to education in the preschool. The first is an informal approach based on play, where the idea is that children learn mathematics through play (Bishop 1992). In the second and more formal approach, the teacher chooses the content in advance and plans the learning situations together with the necessary instructions (Claessens & Engel 2013, Clements & Sarama 2009, Ginsburg & Amit 2008).

In this study, a third approach is suggested that is carefully planned and goal-oriented and makes use of play. The approach to play and the content used in teaching and learning are inspired by Brousseau (2002*b*) and involve the concepts 'milieu' and 'didactic' and 'a-didactic' situations. This consists of a play-based approach and the content is carefully planned by the teacher in advance. Furthermore, the environment, manipulatives/material and questions are taken into consideration when the preschool teacher plans the milieu.

The central *problematique* of the paper is to examine the didactic transposition of knowledge to be taught in preschool teaching, i.e. the specific knowledge to be taught and the taught knowledge. Here I take the teachers' actions into consideration and evaluate them in terms of didactic praxeology and the resulting mathematics praxeology (Bosch & Gascón 2006, Chevallard 1992*a*). The aim is to study how preschool teachers make use of play in education and how the knowledge to be taught is transformed into taught knowledge in the preschool context.

Theoretical frame

Teacher' teaching and children's learning

When preschool teachers design their teaching, play is an important aspect (Ginsburg & Amit 2008). The potential of play in teaching has great importance because it is integrated into the learning process. In play children can think hypothetically and follow rules. They can participate and develop in play and games with more or less formalized rules that have to be followed. In play children can predict, guess, estimate or assume what might happen. They can also explore numbers, shapes, dimensions and positions.

The Theory of Didactic Situations, TDS (Brousseau 2002*b*), also stresses play and games as important in the mathematical learning milieu. Brousseau argues that the teaching should be carefully planned (engineering) and based on what the children are expected to learn. The central TDS concepts used in this study are milieu and didactic and a-didactic situations. The milieu should be carefully planned and rich and consist of enough suitable material for the children to solve the set problems through their own play. The manipulatives are artefacts that support children's solutions. The milieu in preschool situations is less strongly structured than in other school forms with older children.

Didactic situations, like planned play, include phases when the teacher is active and the children listen. They also include moments of joint reflection and children working on their own. In the design of didactic situations the teacher constructs a milieu in which children have to act and engage in order to solve a given problem. Such engagement is similar to that experienced by children when playing a game, which is to understand the rules and seek winning strategies. The situations also work without too much intervention from the teacher and are called a-didactic situations. Children receive feedback from the milieu, although the teacher has to ensure that all children have sufficient opportunity to solve the problem, which means that the teacher sometimes has to change the rules of the game. When teachers plan didactic situations in preschool their use of teaching materials or manipulatives are important aspects of the milieu. Here, the questions and materials/manipulatives are tools that support children's discernment of a specific feature or content.

Knowledge to be taught

Shape is part of geometry and children encounter different kinds of shapes daily but need to become aware of similarities and differences between shapes to understand their environment. Children's ideas and knowledge about shape are part of their spatial thinking. Van-Hiele (1959) describes different levels of children's geometric thinking and identifies the first level as the visual level. This means that children have an early experience of an object when they see its structure or form. At this level of their spatial thinking children assess figures or shapes that belong to the same category. For example, a rectangle could be a door or a table. The second level is the descriptive level. At this level children examine the properties of the shapes, rather than their appearance. Children can verbally describe that triangles have three corners and three sides and that a circle is round. This means that at this descriptive level language is a major factor. The third level is the deductive level and means that children are able to formulate definitions for shapes like triangles and rectangles.

According to the Swedish Curriculum for the Preschool (*The Swedish National Agency for Education* 2010) teachers should give children opportunities to develop their understanding of shapes, location and direction. They should have chances to develop their ability to investigate, reflect over and test different solutions to problems raised by themselves and others, and also develop their ability to distinguish, express, examine and use mathematical concepts and their interrelationships.

Children interpret the environment by building, designing and constructing using a variety of shapes. For example, Clements & Sarama

(2009) emphasize that children develop their spatial thinking by exploring the shapes and features of the different materials they use. Children can recognize and create patterns with different shapes in play. Van-Hiele (1959) argues that children develop their spatial thinking through play, and that this can be carefully planned by the teacher. Spatial thinking can also occur in spontaneous play.

Claessens & Engel (2013) highlight the importance of early mathematical knowledge and skills, because mathematics predicts other content areas such as language. They also suggest that if children focus on mathematical content in the early years of schooling it will benefit their future learning. The authors maintain that teachers' instructions are necessary for children's outcomes in mathematics. Clements & Sarama (2009) argue that teaching and learning take place in a context, and that teachers need to understand mathematics and have a specific goal with their teaching. Teachers also need to know how children think and learn about maths, and how best to support children's learning. They argue that a set of instructional activities in mathematics can help children along their developmental path (ibid.).

Didactic transposition

The process of didactic transposition (Chevallard 1992*a*, Bosch & Gascón 2006) consists of different steps (Fig. 3.1).

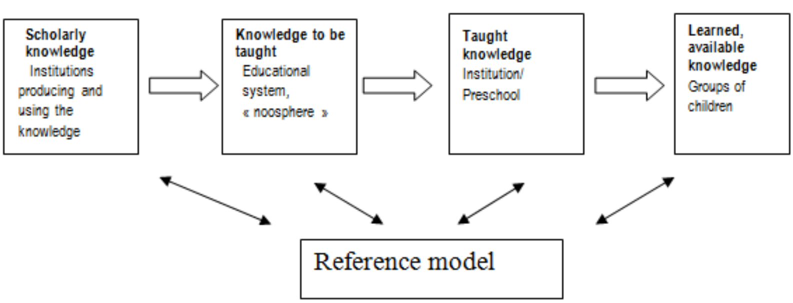


Fig. 3.1: The process of didatic transposition (after Bosch & Gascón (2006))

Teachers need to transform the knowledge goals stated in the curriculum into didactic knowledge; the knowledge to be taught. How, then, can a knowledge of shapes be taught in preschool?

Play is the context for the didactic transposition of teaching and learning mathematics in preschool, i.e. the taught knowledge. The knowledge that is available for children to learn depends on children's earlier experiences, how the teacher designs and didactifies the content and whether the situation is teachable.

Chevallard (1992*a*) describes mathematical praxeology as mathematical and didactical organization and the result of the construction in the process. The knowledge needed includes the theory, techniques and technology to solve a given task. The praxeology is affected by culture and interaction in the preschool. The intent in this paper is to understand how the knowledge to be taught is transformed into taught knowledge. How knowledge is taught in the preschool context is supported by the reference model together with the presented praxeology.

Methodology

The research methodology organizes the relationship between the knowledge to be taught in the curriculum and the knowledge that is taught in preschool. Here, the focus is on teaching and learning in didactic situations in preschool. I use the didactic transposition (Bosch & Gascón 2006, Chevallard 1992*a*) as a tool with which to examine "the subject matter" in an example of a mathematical activity in preschool. The reference model in this study is based on experiences of scholarly knowledge and noosphere, such as the national curriculum and research on mathematics in the preschool years. It shows that praxis (know how) involves play that gives children an opportunity to observe and examine the various properties of shapes and to describe them in words. The logo is knowledge about shapes. Research has shown that preschool children's spatial thinking begins with the discernment of figures and shapes and their experiences of positioning and locating culturally conveyed concepts.

In the analysis, praxeology is used to understand the knowledge about shapes in preschool and how the knowledge taught is designed and carried out. A praxeology consists of a task and a technique, or "know-how", to solve the task. It also includes technology and theory "knowledge" (Bosch & Gascón 2006, Chevallard 1992*a*). The praxeology is affected by the culture in the preschool. *The set task is to stack all the differently shaped blocks. The techniques used are positioning and stacking of the blocks. The technology involves figures and composite images. The theory is two- and three dimensional figures*. Observations from video recordings are the main data in the analysis. Video is used because it facilitates observations of a situation from different perspectives and at different times. The video recordings are transcribed. From the data, one learning situation/episode from one preschool is selected and analyzed using TDS and didactic transposition. The preschool teacher's intention is that the activity will stimulate the children to play, help them to look for winning strategies and thereby learn the intended content. The questions formulated to support the planning are: *How does the teaching design support children's spatial thinking and discernment? What should the children learn about the shapes? Which questions should I ask the children? Which task is important? How can I help the children to discern and describe the shapes? What kind of techniques should I use? How will the activity motivate the children to explore and look for other strategies?* The intended learning outcome is that in their play the children will discern the differences and similarities between the shapes and the importance of specific shapes when stacking. The children's expressions and actions during the a-didactic situations are focused on in the analysis. The data drawn from a larger study in Sweden (Bäckman 2015) and the children are 4 years of age. The teaching material/manipulatives are differently shaped wooden building blocks. One particular child as the subject in an a-didactic situation commonly found in the preschool, which is one child playing with artefacts.

Findings

In the didactic situation the knowledge to be taught is about shapes. According to the preschool curriculum (*The Swedish National Agency for Education* 2010), children should have opportunities to distinguish shapes and examine their interrelationships. Children should also use mathematics to investigate, reflect on and test different solutions. The teacher has planned a didactic situation and the knowledge taught concerns shapes and their similarities and differences.

Milieu

The TDS planning consists of a careful analysis of the content and the construction of questions that will help the children to solve the set problem. When planning the task, the preschool teacher considers which material should be used to support the children's learning of shapes. In the example chosen here, she selects differently shaped wooden building blocks. The milieu, including the play, the task, questions, reflection and material, gives the children the opportunity to discern different shapes and features when they reflect together in the didactic situation and to solve the problem in the a-didactic situation.

Didactic situation

The teacher gathers six children together in a circle. She has hidden differently shaped wooden building blocks in a cloth bag, which is also placed in the circle. The teacher lets one child at a time choose a block from the bag, which they then reflect on together by responding to questions like: How many corners does each block have? Do any of the blocks have a rounded side? How many sides does the block have? Are they the same length? The children answer the questions.

In this didactic situation the teacher *leads the game* by using planned questions to scaffold the children to direct their focus to specific features. She gives each child an opportunity to *feel, turn and twist* the chosen block and to *imaging and reflect on its features without looking at it*. After these reflections the child put the block in the middle of the circle so that the children can continue the reflection together. The teacher gives the different blocks names, such as "cube" and "cylinder". When all the differently shaped blocks have been placed in the middle of the circle the teacher asks the children questions about their similarities and differences, which they then reflect on and reason together.

The next step is when the teacher presents the task: *Try to build with all blocks! Can all the blocks be put on top of each other? Can you solve the problem?* She tells the children that they can build with all the blocks to try to solve the problem. In this didactic situation the teacher ask questions about the blocks' features. The teacher focuses on the characteristics of the different shapes and reflects on them with the children. The children are given an opportunity to discern and reason about the similarities and differences of the shapes. After the reflection the teacher gives the children a task to perform and indicates techniques that could help them to solve the set problem.

A-didactic situation

It is early morning in a Swedish preschool and Erik, 4, is playing with building blocks in the hall. One of the preschool teachers is standing beside him talking to a parent. Erik doesn't seem to pay any attention to the conversation between the adults. He is building with blocks that have different shapes.

> Erik quickly builds a high stack of eight blocks. The blocks consist of seven cubes in three different sizes, as well as a tetrahedron. Every second block is a small cube and every second is a larger cube. On top is one tetrahedron. Erik starts then building a lower construction, consisting of six half-cylinders. He is totally focused on the construction and is trying different ways of placing the half cylinders. He seems to aim at building both horizontally and vertically. When the blocks fall down he solves this problem by twisting and turning the blocks, perhaps in order to find a sustainable way of placing them on top of each other. At the same time, there also seems to be a desire for symmetry in the building work. He starts with a half cylinder and builds towers with identical half cylinders on each side of the half cylinder in the centre. When the blocks are in place, and they haven't fallen down, he picks up a tetrahedron and puts it on top.

Erik explores the shapes by placing them on and next to the other blocks in different ways. He distinguishes the qualities and features of the various blocks as he twists and turns them. The episode shows that the discernment of critical aspects of the geometric shapes and characteristics takes place in a context. He seems to have an idea or intent with the construction, and experiences how the differently shaped blocks can or cannot be stacked. Erik concentrates on his construction work, and sometimes looks up to check his surroundings. It appears as though he has goals in mind with his building work and is not unduly disturbed by the adults talking to each other beside him. Erik continues building his construction:

> Erik points to two cubes in the tall tower and says,

E: "It's over and it's over."

Then he points to the two top blocks and says,

E:"Those should be removed."

He takes the top two blocks from the highest tower and places them on the floor. He then picks up a pyramid from the floor and places it on top of the tall tower. He looks at the tall tower and says,

E: "There you go."

The positioning seems important to Erik and his building appears to give him an idea of the similarities and differences between blocks. He does not talk about the shapes or the features, but looks very busy when building the construction. He is both interested in and motivated to work on the given task and, as instructed by the teacher, uses all the blocks. The winning strategies in this game are to determine which blocks can be stacked on top of each other and to use all the blocks in the construction. The boy has to find out which critical aspects in the shapes that make the blocks as sustainable buildings.

Didactic and mathematical praxeology

In the analysis of the milieu with didactic and a-didactic situations, the knowledge to be taught is shapes and the teacher has carefully planned how she can direct the children's attention to specific features and how to didactify the content using building blocks. The didactic transposition from the knowledge to be taught to taught knowledge is from ideas about geometry and more specific knowledge about shapes to knowledge about shape in the preschool (*The Swedish National Agency for Education* 2010). The teacher has planned which questions she can ask and how play can be used to help the children discover the shapes' specific features. A praxeology consists of tasks, techniques, technologies and theories (Chevallard 1992*a*, Bosch & Gascón 2006).

The knowledge taught is different shapes and their specific features and the aim of the task is to encourage the children to explore and reflect on different shapes. The technique concerns how the children use the building blocks in constructions. The child in the presented observation investigates, reflects on and tests different possible solutions to problems encountered in the construction work. He does this by remaining totally focused on the task in hand and working on his own. The technique he uses is the same as that suggested by the teacher. The technology and theory explain and justify the used technique to solve the actual problem. The design, with play and the milieu with didactic and a-didactic situations and concrete material, is based on the teacher's assumption of how children develop their understanding of shapes and how this is done. The design, ideas and assumptions that make up the praxeology form the "theory".

Discussion and conclusions

The findings illustrate the transposition and the relation between the knowledge to be taught and the taught knowledge. The preschool teacher designs and uses the milieu to direct children's attention to the similarities and differences of shapes. She formulates a task with the goal that the child should use all the blocks in the construction and thereby explore their different shapes. The questions asked aim to encourage the children's exploration of and reflection on the different features of the shapes. The teacher's careful planning and the taught knowledge seem to support the children's discernment in the way that Claessens & Engel (2013), Clements & Sarama (2009) and Ginsburg & Amit (2008) highlight as important. The authors argue that teachers' instructions are necessary for children's outcomes in mathematics. The outcomes in the planned situation are that children should be given opportunities to discern and describe shapes and use all blocks in the construction.

According to TDS (Brousseau 2002*b*) the teacher has to design a problem situation in which children should discover possible solutions with the material. The designed situation with specific material should guide the children's explorations and reflections without the need for a teacher to be present. In an a-didactic situation the possible solution and winning strategy is that the children use all the blocks and discern the positions and locations of the various shapes. In a didactic situation the children are given an opportunity to describe their actions, but in the a-didactic situation the boy is quiet. This paper stress that the boy discerns the shapes as a result of his actions, which reflects the well known fact that young children express their knowledge through their actions.

The knowledge to be taught and the differences and similarities between the shapes are, in the didactic situation, transformed into steps with tasks and techniques that give children opportunities to observe and examine the properties of shapes. In the didactic situation they discern and describe shapes.

The teacher uses play in didactic and a-didactic situations. Play is a common context in Scandinavian preschool teaching, and teachers usually want children to explore, reflect on and test different solutions to problems in their play. She begins the teaching with a game in which she invites the children to pick one block at a time from a cloth bag and feel, twist, turn and reflect on the features of each block. The children have an opportunity to focus and reflect individually and together. According to Clements & Sarama (2009), children develop their spatial thinking by exploring shapes and the features and qualities of different materials. When children are given opportunities to feel shapes and twist and turn them, as in the planned didactic situation with the blocks, the teacher is able to support their spatial thinking. When reflecting together, children have a chance to reason and argue and describe what they have observed. Van-Hiele (1959) argues that children develop their spatial thinking through play that is carefully planned by the teacher and that geometric thinking can also occur in spontaneous play. The presented situation with the building blocks includes both carefully planned play and spontaneous play.

The theory of didactic situations (Brousseau 2002*b*) is complex, and in this paper consideration is given to the concepts of ‘milieu’ and ‘didactic’ and ‘a-didactic’ situations in relation to the preschool institution and culture. The milieu concerns the play, the questions, the task and the techniques. The given observation shows that when the boy builds a tower with the blocks the teacher is able to observe his actions when standing beside him. Play situations like this are common in preschool and allow the teacher to observe and ask questions about the construction. On the other hand, in the a-didactic situation children can play and explore and learn about different shapes. These visual and bodily experiences can form the basis for further teaching and learning about the similarities and differences of shapes at a more descriptive level. On the one hand, spontaneous play can have a mathematical content and make use of materials like building blocks so that children can discern shapes and their various features. On the other hand, the learning opportunities increase if the teacher asks questions that make children reflect on the content, and according to Ginsburg & Amit (2008) teachers planning and the use of teachable moment promote learning. Play with both didactic and a-didactic forms is beneficial for teaching/learning in preschool settings.

Conclusion

The approach to play and the teaching and learning content is inspired by Brousseau (2002*b*) and makes use of the concepts 'milieu' and 'didactic' and 'a-didactic' situations. This means that a play-based approach that is carefully planned by the teacher in advance can promote children's learning. Furthermore, materials and questions are taken into consideration when the preschool teacher plans the milieu. The findings show the didactic transposition from the knowledge to be taught to the taught knowledge. Teacher's knowledge of the content and how to didactify in didactic situations seems to be important. Children's play can be understood as a-didactic situation if they continue to work on the construction and try to solve the problem themselves.

4

Using Didactic transposition to formulate a reference framework for a better understanding of elementary outdoor science lessons taught by explore teachers receiving environmental education training in North Carolina

Aerin W. Benavides

The University of North Carolina at Greensboro

Abstract. We live in an era of critical global biodiversity losses (Dirzo et al. 2014) and childhood experiences in nature are recommended to encourage strategic behaviour (Chawla & Cushing 2007) to help prevent devastating impacts of projected environmental changes. Science learning outdoors has been shown to benefit learners (Dillion et al. 2006, Wals et al. 2014) but science knowledge alone does not lead to enlightened behaviour (Slingsby & Barker 2003). I seek to examine and understand North Carolina (NC) elementary school teachers' (n = 6) outdoor science teaching using Didactic Transposition as a theoretical framework to better understand their meanings of knowledge construction. This qualitative study will examine how elementary school citizen science lessons are posited in relation to broader more meaningful scholarly knowledge and who the agents are that de-construct this scholarly knowledge and build teachable knowledge for young people (ages 5-12). This study may address teachers' lack of success in implementing outdoor science lessons for children in the United States, which should include critical thinking for environmentally sustainable action, and allow us to better understand teachers' meanings of learning to transpose scholarly knowledge to teachable knowledge for use in science fieldwork at the elementary school level.

Introduction

Life as we know it on planet Earth is currently undergoing destructive change, and as scientists Dirzo, R., Young, H., Galetti, M., Ceballos, G., Isaac, N., & Collen, B., (Dirzo et al. 2014) described it, we are fully immersed in the sixth mass extinction on the planet, in the current Anthropocene era, and committing critical 'defaunation' on a global scale. Dirzo, et al. use the term 'defaunation' to "... denote the loss of both species and populations of wildlife, as well as local declines in abundance of individuals, [and this] needs to be considered in the same sense as deforestation..." (p. 401). This phenomena is under-recognized according to Dirzo, et al. Without scientific knowledge, and knowledge of wildlife, how will children today who have been described by Richard Louv (2006) in his book, Last Child in the Woods, as suffering from 'nature deficiency', truly know what there is that needs saving, the science behind the causes for destructive change, and be environmentally literate before it is gone?

As Chawla & Cushing (2007) reported in their article *Education for Strategic Environmental Behavior* an important antecedent for taking action for the environment is experiences in nature as a child, mentioned by 80% of respondents in large studies. A common theme throughout research on this topic is the importance of role models. Both time outdoors and a role model could be provided if the elementary school teacher were to designate an area of the schoolyard (Wals et al. 2014) in which to teach science by using the natural world as a platform for learning, as a tool for teaching, including environmental education. Wals, et al., recommend a 'convergence' of science and environmental education. I argue using the live natural world as a teaching tool for environmental education can make a huge difference in addressing fundamental changes needed in society. Authentic science instruction outdoors to foster understanding of urgent and highly complex environmental issues increases the quality of science education for elementary age children. This is needed at a time in their life when such instruction makes a difference.

I propose qualitative descriptive case studies of teachers (n=6) participating in the citizen science and environmental education program Project EXPLORE (Experiences Promoting Learning Outdoors for Research and Education). EXPLORE is an outreach program to local schools designed and implemented by The North Carolina Arboretum. Participating teachers teach science outdoors, including Environmental Education in the lessons, on a regular basis as they collect data for citizen science programs in their schoolyard. Environmental Education for these children teaches them how to investigate and learn about their environment, and how to make informed decisions about how to take care of it[1]. These citizen science programs address the need for scientific data on current climate change and defaunation trends (Dirzo et al. 2014). Knowing what meanings they make of their learning and knowledge construction will help the field understand how better to move science outside for children, using The North Carolina Arboretum program as a model. Arboretum educators (some are North Carolina EE Certified) in this study could provide an example of a program that has trained teachers to teach outdoors and this could be used as a model of concrete methods for teachers to use in teaching science outdoors by participating in citizen science. Teachers may need concrete references of outstanding practices in order to enact a convergence of environmental education and science education while teaching science lessons in the outdoors. Describing examples of outstanding EXPLORE lessons can help the field establish evaluation criteria. Evaluating the quality of an elementary outdoor science lesson[2] is especially challenging. *The Association for Science Education Outdoor Science Working Group* (2011) included an important issue arising in their seminars—the ability to evaluate the quality of outdoor science lessons. "It is crucial to develop success criteria to enable teachers to evaluate the quality of their fieldwork[3]..." (p. 10). The purpose of this study is to describe Project EXPLORE and explore teacher meanings of learning. As part of this examination "Didactic Transposition" (Bosch & Gascón 2006, p. 53) will be used as a reference framework for the description of public elementary school teachers' classroom outdoor science

[1] EE lessons here indicates teaching activities that should provide hands-on, interactive experiences for the audience and should support the definition of environmental education (http://www.eenorthcarolina.org/about-us–what-is-ee.html)—Environmental education (EE) is defined as education that teaches children and adults how to learn about and investigate their environment, and to make intelligent, informed decisions about how they can take care of it (http://www.naaee.net/what-is-ee).

[2] Elementary outdoor science lesson here indicates all science educational activities from kindergarten through grade 5, for students ages 5-12, which take place outside the classroom and make use of the outdoor natural and built environments.

[3] Fieldwork here indicates all educational activities from early years through to post-16, which take place outside the classroom and make use of the outdoor natural and built environments.

lessons used by teachers participating in The North Carolina Arboretum's Project EXPLORE.

Tal & Morag (2009) found it is more difficult to teach science outdoors than to teach indoors for participants in their study of preservice teachers. While research supports the benefits of learning elementary science through outdoor lessons (Rickinson et al. 2004), findings in a recent study by Carrier et al. (2013) in North Carolina revealed that despite elementary school support, teachers find it very difficult to utilize outdoor activities for science instruction. Teachers in their study felt traditional [indoor] methods were a more efficient way of meeting heavy science content demands. Tal & Morag (2009) discovered that one of the major obstacles teachers face to teaching outdoors in nature is insufficient preparation for teaching in the outdoors.

The North Carolina Arboretum state Certified Environmental Educator prepares and supports teachers with expertise and materials in Project EXPLORE, and follows up on teachers with twice annual visits to take their class out to collect schoolyard ecosystem and citizen science data. North Carolina was the first state in the United States (US) to offer professional development to Environmental Educators through a state EE certification program (see http://www.eenorthcarolina.org/index.asp). EE certified educators teaching in North Carolina school systems are referred to as *formal* educators (and there are relatively few of them). These formal educators are trained to teach outdoors and to use the natural world as a platform for student learning as part of their EE professional development program. EE certified educators at the Arboretum are referred to as *informal* educators. The North Carolina Arboretum educators often adapt EE lessons to teach science to specific age groups in or out of schools and align their lessons with state required curriculum standards.

All three of the citizen science projects that teacher participants in Project EXPLORE can choose to participate in have an environmental component to data collection and analysis. The citizen science project eBird tracks bird populations and species ranges and data can show biodiversity gain or loss as well as population increase or decline. This data can show bird habitat shift due to changing climate as well. The citizen science Project Squirrel tracks populations and species range for squirrels. Data from Project Squirrel can show biodiversity gain or loss as well as species population increase or decline. Biodiversity losses and species population declines are now referred to as defaunation (Dirzo et al. 2014). Nature's Notebook data are collected on trees in Project EXPLORE, and can show changes in the dates each year that trees in each class's study area sprout leaves, bloom and when they loose their leaves. Scientists, by comparing data year to year, can discern and compare dates of seasonal changes from year to year, as well as how changes in season's dates affect different species of trees in different ways. This data also tracks different effects of season's dates in different geographic locations.

Some teachers design their own data uploading sheets for their students and all supervise their own lessons outdoors, except when Arboretum educators are visiting. Adaptations of teachable knowledge at the elementary school level by EE educators and teachers in schools is a process of taking science knowledge from a prepared EE lesson, from public school curriculum or textbooks, citizen science programs, or directly from scientific research, and de-constructing this knowledge to then transform it into a teachable form for young children ages 5-12. "Deconstruction and a rebuilding of the different components of knowledge with the aim of making it teachable" (Achiam, from Chevallard (1991)), called 'didactic transposition', begins "far away from school, in the choice of the bodies of knowledge that have to be transmitted" (Bosch & Gascón 2006, p. 53). However, the theory of Didactic Transposition posits that different actors participate in this transpositive work, including teachers. Teachers can enact Didactic Transposition on their own and I propose to examine and describe EXPLORE teachers' act of doing this as reflected in their elementary outdoor science lessons that I observe and in their reflections from interviews. Thus Didactic Transposition will serve as an analytical framework to help me examine and describe the element of knowledge construction in the elementary outdoor science lessons in my proposed study.

Using Didactic Transposition as a framework for analysis of EXPLORE outdoor lessons with a focus on citizen science broadens the span of recognized agents of determination of the contents to be taught. A range of agents, from policy makers to the teachers of the lesson, are actors along the way who affected the content of the lesson observed. "Researchers often took for granted the specific delimitiation [established boundaries] of [lessons'] contents that is given by scholarly or educational institutions" (Bosch & Gascón 2006, p. 61).

But using Didactic Transposition as a framework for analysis, I as a researcher would not take established boundaries for lessons for granted; it forces a broader view of the complexity of how knowledge becomes a teachable lesson through transposition. This encourages me as the researcher to look at whether or not the taught knowledge has lost the rationale behind it after the transposition process. One danger in transposition is that school instruction may ignore the questions that motivated the creation of knowledge. An example of this would be if in my study of citizen science projects outdoors I were to find that the data is not being discussed by the class or examined and analysed after data collection. In such a case the teacher's instruction would ignore the larger questions being posed by the scientists who will be using the data the class collected. Students would then be left to figure out bigger scientific ideas, or make improvisational extrapolations of knowledge from the scientific data.

A danger of improvisational extrapolations of knowledge is that they may ignore the validity field of models of instruction (Bosch & Gascón 2006). In practice elementary school students are taught that anything above a pH of 7 is base, and anything below a pH of 7 is acid, and that a pH of 7 is neutral. They are also taught that a pH of 7 is optimal, or normal, for drinking water. Students may go down to the stream and test the pH of the water and find that it has a pH 6.5, or is acidic. While a pH of 7 may not be optimal (normal, historical) for the specific stream ecosystem, the students may think the stream is 'too' acidic because their point of reference is what is normal for drinking water for them. There is a restricted field of validity to what they were taught was a normal pH of 7, for drinking water. Without learning broader knowledge that is valid in a variety of circumstances, something present in the natural world, they may be left with a misconception and assume that a stream pH of 6.5 is not optimal, or normal for the stream, when scientific evidence may support otherwise

In the United States we have the added danger that science instruction after transposition is irresponsible due to teachers' limited content knowledge (Borko 2004), or invalid altogether. An extreme example of this in the US schools is teaching creationism as a scientific alternative to the theory of evolution. Responsible instruction goes beyond offering valid science knowledge; it endows students with tools to think, encouraging their reasoning with learned knowledge (Chevallard 2004).

Theoretical framework

Knowledge transformation in the science education process

The theory of Didactic Transposition (Chevallard 1991) although originating in mathematics, has been extended to other disciplines. An important assumption of this theory is that we cannot limit analysis of any didactic situation to how the learner learns, we must consider the process which takes an object of knowledge and makes it into an object to be taught (see Figure 4.1), that is, teacher knowledge. It is a process of deconstruction and rebuilding, Didactic Transposition (Mortensen 2010). Teachers have been shown to be essential to improving education in our schools (Borko 2004) and can make a huge positive difference through enacting Didactic Transposition at the classroom level as Didactic Transposition is part of the teaching process whether acknowledged as such or not (Winsløw 2007).

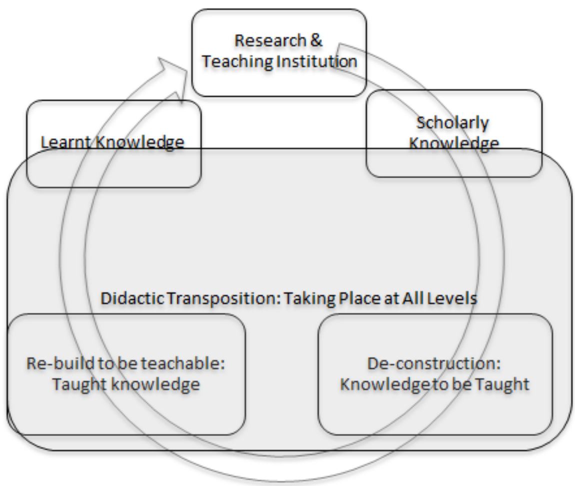


Fig. 4.1: This is my graphic representation of Didactic Transposition using the epistemological model by Chevallard (1991), adapted from Winsløw in the *Encyclopaedia of Mathematics Education*-Article ID: 313227- Chapter ID: 48, showing the cyclical process of education, including the creation, teaching and learning of knowledge. Here Didactic Transposition is an alternative epistemology with knowledge existing independently outside of the knower (Kang & Kilpatrick 1992). In this view, Didactic Transposition does not violate much of the constructivist epistemological position on learning (Bosch & Gascón 2006), which is shown in this graphic representation as 'learnt knowledge' 'research' and the generation of 'scholarly knowledge'.

In the Didactic Transposition theoretical model, knowledge is considered a changing reality embodied in human practices in the process of education. There exists a gap between research and what is taught, especially between university research and what is taught below the university level (Winsløw & Madsen 2008). There are many different actors that take part in this transformative work, de-constructing and building knowledge into a teachable form. Teachers take part in what Winsløw (2007) describes as internal didactic transposition. Realized teaching practices may depend upon the teacher, or upon strong didactical traditions within their school systems, can be based upon their curriculum, and thus can cause a fundamental "incoherence" between knowledge at the university level and school curriculum, through no fault of the teacher. But, scholarly knowledge has to be transposed to the classroom context and for the child (Chevallard 1999), and a teacher can potentially bridge the aforementioned knowledge gap.

Knowledge transformation from EE instruction to the outdoor science lesson

The North Carolina Arboretum EE certified instructor in charge of Project EXPLORE identified one of the objectives for the program was to help local elementary school students improve their end of year science standardized test scores. Another goal of the program is improving environmental literacy through the hands-on experiences of conducting citizen science data collection on school grounds. The authentic science questions pursued by these citizen science programs become an authentic science activity for students, increasing direct links from educational to scholarly knowledge. The chosen citizen science projects involve students in actual scientific questions about habitat loss, environmental hazards, species loss, populations decline, as well as about seasonal and climate shifts over time. The North Carolina Arboretum carefully chose these three citizen science programs (eBird, Project Squirrel, and Nature's Notebook) to offer teachers programs to choose from that could be easily adapted for all grades (K-12). The Arboretum developed data collection sheets with pictures for children as young as kindergarten age to use tally marks for sightings of animals, to identify weather conditions, and to identify characteristics of the habitats needed for these animals to survive. Arboretum EE twice annual presentations to schools as part of Project EXPLORE include adapted EE lessons and materials.

Adaptation of the activity of teaching science outdoors to fit specifically to each school is required of both the teacher and the Arboretum educator. They select local areas that would be better habitats for citizen science project data collection. For example, if one teacher chose Project Squirrel, the science lesson would be conducted in an area where the trees on the school grounds best supply squirrels with what they need to survive. Arboretum educators design valid descriptive sentences or engaging true stories for each animal, tree or plant that exist on the school grounds; choose a picture of the trees or plants for students' Arboretum field guides; as well as further develop their own knowledge about the local place. For example, the Arboretum educator and the teacher will walk the school grounds on the first visit by the Arboretum to the school, before deciding where it is best to have children collect citizen science data. In one case a teacher chose a wild overgrown area to collect data in, but the Arboretum educator suggested a grove of trees that supplied things squirrels like to eat instead in which to search for squirrels. In this way the lesson the teacher planned in one area was adapted in alignment with the scientific knowledge the Arboretum educator shared with her. My study of this innovative professional development program could help us define references for Didactic Transposition in such a context. There are many merits to teachers connecting their lesson to scholarly knowledge, as the lesson is authentic scientific data collection to be used by scientists nationally. And by enacting the process of selecting knowledge and constructing a custom lesson in collaboration with the Arboretum educator for their ecological area in order to teach the citizen a science lesson, each teacher may be better able to instruct their students as to the rationale behind these lesson adaptations. Thus, by doing so teachers may be able to improve their own teaching.

Method

As part of a qualitative case study method design (Creswell & Clark 2011), I plan to observe and describe public elementary school classroom teachers' (n = 6) meanings of learning to teach outdoor science lessons using a method of analysis derived from the theoretical framework of Didactic Transposition. To better understand and describe lessons I will explore if and how the lessons include a broader more meaningful scientific process and valid scholarly scientific knowledge. Who are the agents along the way that provide this deeper meaning to the lesson and perhaps bring the students closer to the actual research and scholarly knowledge?

One of the limitations of this study will be that I will be conducting the study alone. I will be influenced in my observations and analysis by my personal context and experience. I am both a North Carolina certified Environmental Educator, and experienced in teaching outdoor science lessons at the elementary school level. My use of Didactic Transposition as a framework for analysis is new to me and perhaps new to the field of science education in the United States, so I will have few similar models or references to guide me. I will need to determine how to best describe lessons, and materials used in the lessons, using this framework for analysis. I will also have limited time and resources.

I may be able to conduct as many as six case studies, and be able to observe up to six science outdoor lessons for each participant (this is observation of a 15 minute or so data collection period outdoors with preparation in the classroom beforehand and follow up afterwards in the classroom). I will conduct at least one interview with as many of the 15 public elementary school teachers participating in EXPLORE as is possible (this depends upon school district support for my research). Due to time constraints I will only be able to analyse limited amounts of data from a few participants. In order to better understand the teachers' meanings of their teaching of lessons in practice, I will conduct short interviews asking for teacher reflections after each observation. I will describe lessons with the Didactic Transposition theoretical framework in mind (see Table 4.1) in my observation notes, and will research the scholarly knowledge behind the lessons tracing back from research or theory to the actual observed lesson, asking the teacher to retrace their own lesson preparation process, to determine the transposition 'track', and the part various actors played in transposition.

In order to discern how the teacher enacted Didactic Transposition, I will have to have read the original lesson plan, observe how the teacher deviated from the plan, know where it came from, interview the teacher, and do some research on the original intent of the lesson plan the teacher used as a guide, or the source the teacher used as a guide for writing the lesson. I will also examine through teacher reflection where they feel their prior knowledge came from. I think this is a very important part of observing and interviewing if I am to use Didactic Transposition as a reference framework.

Observation of Didactic Transposition in Outdoor Science Lessons:
What is the specific scientific knowledge (and values and practices) the teachers seek to disseminate with their activities? Did the teacher de-construct knowledge to build a teachable lesson? Describe.
Does the teacher require students connect thinking about this disseminated knowledge to other knowledge? (Scholarly knowledge, funds of knowledge, new knowledge of the school yard) Describe.
Where does disseminated scientific knowledge (and values and practices) come from, and in what ways is it related to knowledge in the environmental/scientific research domain?
How has the knowledge been transformed to fit the stated learning objectives of the teacher, what are these learning objectives, and how are they related to environmental literacy objectives?

Table 4.1: Observation of Didactic Transposition in Outdoor Science Lessons

Discussion

This study will help formulate a reference model for Didactic Transposition more specific to this context of teaching science outdoors through the use of citizen science projects in the United States, in the public schools system. Although success criteria for quality outdoors science lessons are needed (*The Association for Science Education Outdoor Science Working Group* 2011), this study will be descriptive rather than evaluative in nature. By looking not only at teacher meanings of their experiences as learners, but by looking at their meanings behind the way the Arboretum and they construct teachable knowledge as well; we expand the description of outdoor science lessons to include a broader more meaningful science process in the lesson. This study, while not explicitly evaluating Project EXPLORE or teacher participants' outdoor science teaching, could help us better understand how to develop an evaluation of such a program.

A detailed description of Didactic Transposition at all levels of this innovative teacher education program, Project EXPLORE, should help us to better understand not only teacher meanings of learning, but their reasoning for inclusion of research and scholarly knowledge at the teacher level, even when curriculum has been scripted and predetermined as is done in many urban schools in the United States (Milner 2013). To improve education by educating teachers (Borko 2004), we need to better understand how teachers learn to use scholarly knowledge in their teachable knowledge. Project EXPLORE is a system of teacher education that fosters expansive transformation in teachers, and promotes teaching science outdoors to counter 'nature deficiency disorder' (Louv 2006). In the end if the system overall does not promote teaching science outdoors or environmental education, teachers can show us how they do it. Our world will be affected by destructive environmental change in the next generation, and children today if taught could care, and be able to know how to find a way to address these challenges as adults (Dirzo et al. 2014) in ways yet unknown.

5

Combining CAS based teaching with lesson study - A Theoretical paper

Louise Meier Carlsen

Department of Science Education, University of Copenhagen
Department of Education, University of the Faroe Islands

Abstract. There are many challenges when using technology in the teaching of mathematics ranging from redesign of traditional tasks to developing new teaching techniques. However there is some evidence that the implementation of technology in teaching can enrich the learning of mathematics in middle and higher levels of mathematics education by giving the students the tools to skip otherwise time consuming calculations. Many of these positive examples contain a very carefully planned teaching course with carefully designed activities but with continuously development of tasks designs, activities and evaluation forms. Therefore it seems natural to combine technology rich teaching with lesson study which has a strong and successful history in the development of and research on actual teaching in schools.

Introduction

The initial research question is: How to improve (develop, evolve, progress, sustain, etc.) spontaneous didactic praxeologies to integrate technology in the teaching of algebra (or mathematics) with technology?

Research on "spontaneous" technology use has demonstrated that teachers manage student work with technologies in very different ways, even within the same subject and teaching environment, with similarly varied results (Doerr & Zangor 2000, Lagrange & Erdogan 2009, Monaghan 2004, Sensevy et al. 2005). The use of CAS (computer algebra systems) in lower secondary school is very mixed and very dependent on the individual teachers' views on mathematical learning and teaching e.g. a teacher often using whole-class discussion of what is happening on the screen with the goal of enhancing the collective technological genesis finds interaction in the classroom very important and sees ICT as a means to stimulate this. (Drijvers et al. 2009).

Research experiments with CAS technology has been found to encourage the use of general mathematical reasoning processes and to improve student learning and interests: "It allows for generating, testing, and improving conjectures, it allows for developing awareness and intuition, it leads students to explore their own conjectures, it provides non-judgmental feedback and it develops the learner's confidence." (Kieran & Saldanha 2008). But how to get these results without the researcher being the controlling factor?

In Japan "a teacher as a professional is required to experience sustained professional development to improve their teaching and students' learning" (Isozaki & Isozaki 2011). To aid the teachers in this endeavour the teachers engage in lesson study which in Japan can be dated back to the beginning of the 19'th century. "Lesson study functions as a means of enabling teachers to develop and study their own teaching practices" (Isoda et al. 2007) and helps the teachers develop, experiment or do research on teaching but also get input from other educators in their area and share their projects. Hence it seems logical to try and use the main features of lesson study to address some of the challenges in teaching-with-CAS and linking theory and practice.

Theoretical framework

In the anthropological theory of the didactics, practices in any given situation can be described as praxeologies which again can be described by type of task, the techniques, the technology and the theory. *The tasks* can be of didactical nature: How to teach fractions to a 4'th grade? Or more theoretical: When I use 1.000 steps to go to work how big a portion of the daily recommended 10.000 steps have I done walking to and from work? *The techniques* are the approaches in which the task is tackled. *The technology* is the discourse for the explanation and justification of the techniques used to solve the task. *The theory* provides a basis for the technology i.e. the explanation and justifications used and if desired so can be considered as a technology of a technology.

Instrumental orchestration

Furthermore in order to describe the use of CAS in the classroom I adopt the metaphorical notions of instrumental orchestration introduced by Trouche back in 2004. *Instrumental orchestration* is defined as: given a mathematical task the teacher's intention and systematic organisation and use of various artefacts in order to guide students instrumental genesis (Drijvers et al. 2009). Instrumental orchestration has three parts: a didactical configuration, an exploitation mode and a didactical performance. *The didactical configuration* is the substantial frame and arrangement of the materialistic objects involved. This can be the technological tools but also the tasks. *The exploitation mode* of a didactical configuration is the teachers' decision to the way s/he introduces the tasks, the approaches used to work on the tasks, the performance of the artefacts and techniques to be developed or refined by the students. *The didactical performance* is the teachers' impromptu decisions in the didactical configuration on how to play the elements of the didactical configuration and the exploitation mode e.g. the teachers' spontaneous decision to bring forward a student result or problem for the entire class to consider on the smart board or similar. Hence the instrumental orchestration model is two layered. Not only does it focus on enhancing the students' instrumental genesis but also the teachers since the students and the teacher often do not share the exact same artefacts.

CAS based teaching

There are many challenges and obstacles when involved in CAS based teaching both for the teacher and for the students.

The following list of challenges and obstacles for CAS based learning is composed based on Drijvers (2012), Drijvers (2004) and Lagrange (2005).

Problems for the teachers: general confusion and fear of CAS, redesign of tasks in order to pose fruitful questions to the students, use of new teaching techniques, new didactical contract between by-hand work and machine work, and between numerical-graph methods and algebraic methods, applying pedagogical knowledge to a new teaching environment. The teachers limiting themselves to the theoretical framework due to integration of digital tools in mathematics education, integration of CAS techniques into teachers own content knowledge, CAS being inflexible with notation and syntax and other instrumentation difficulties, the mathematics becomes too far from the official mathematics i.e. the official curriculum, instructional activities requires time and cannot be rushed, evaluation of the students is difficult due to the lack of link between what students write on the screen and their mental tackle of the tasks.

When considering our theoretical framework there seems to be challenges in every box: didactical configuration, exploitation mode and didactical performance, mathematical or didactical organization: tasks, the techniques, the technology and the theory. Thus there are much more research to be done and new knowledge to be acquired.

Problems for the students: lack of link between what students write on the screen and students mental tackle of the tasks, students perception that CAS already contains all the algebra, procedures are not transparent and CAS therefore becomes a black box, generalization without gain of content knowledge, CAS as a micro-world not real-world, paper and pencil or mental mathematics, linking of a concept to closely to the corresponding technique. One example of this is a group of students used to solving equations with calculators are struggling with a pen-and-paper assignment: make one equation from the following in which y does not appear: $y = a - z$ and $x^2 + y^2 = 10$. One student answers "I don't know how to do it without calculator" but ends up solving the posed problem anyway because he can explain what he would have done with the calculator and understand how the calculator operates. Other students are not able to make the same transition because they don't know how the calculator works with the input in order to produce the output. (Drijvers 2003)

The above mentioned research articles where mostly based on the students performances and lessons had been planned by the researchers. One concluded that thorough introduction to students concerning the commands involved in the CAS programme and letting students spend time at home tackling the tasks using CAS where very beneficial for the success of the lessons. Studies more general conclude that the teachers' technological knowledge, content knowledge and pedagogical knowledge revolved around CAS are crucial and give this a new name: Technological Pedagogical Content Knowledge (TPACK).

However more things need to be said: Since the integration of CAS in teaching modifies the mathematical praxeological techniques it also changes the types of tasks to be taught and the praxoelogical technology and theory. I.e. the knowledge to be taught has changed. But what has appeared/are appearing? Thus it also becomes a problem of didactical transposition and therefore new productions are required.

Lesson study

It's origin in Japan

In this section I will give a brief introduction to lesson study which has recently been adopted by other countries. Lesson study in Japan is "an important part of Japanese teachers' continuing professional development" (Isozaki 2013) but also "foster the learning or professional community in and between schools" (Isozaki 2013).

Lesson study appears in different settings with different compositors and different facilitators but they all have joint commonalities. A lesson study has three parts: the preparation, the research lesson and the reflective meeting.

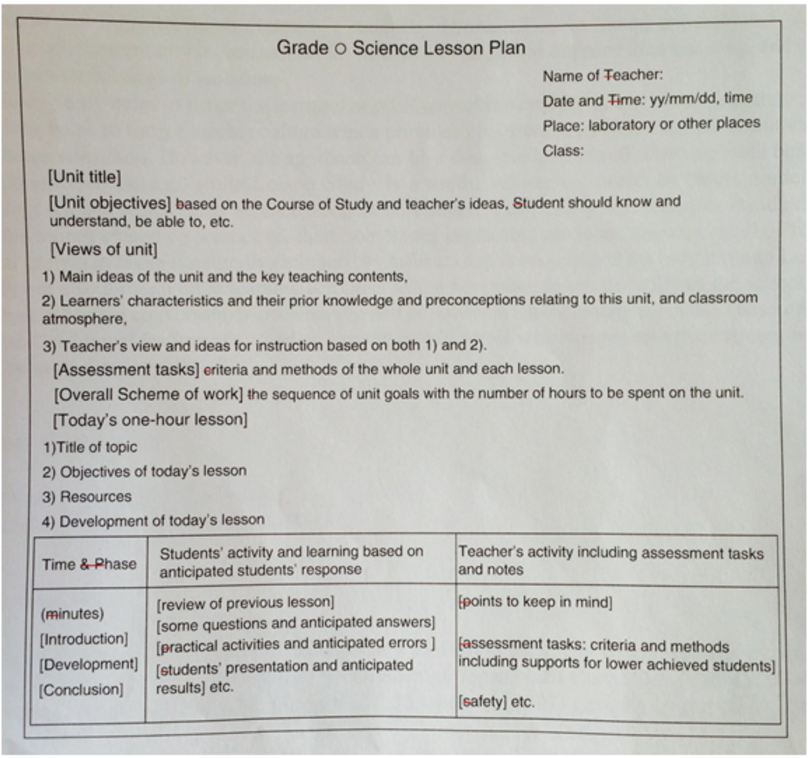
The preparation can be done with a group of teachers or an individual teacher with advisor and consists of two parts. The first part starts with defining and discussing a research question or goal for the research lesson. There after the conduct of the research lesson is planned. This is carefully done with focuses such as: the type of lesson, the content, the focus, the teaching material, the practical activities, predictions about the students' difficulties and how to help them, how to assess the students' performance, etc. The second part involves the finale development and adjustments in order for the planned research lesson to fit the specific class and then writing down the lesson plan.

The second part: *the research lesson* is a lesson where the lesson plan is tested in a classroom with participating students and data collected. During the research lesson observations of the students' interaction and work will be carried out by other teachers and invited guests for example university professors. Also the research lesson can be recorded.

The reflective meeting is held shortly afterwards with all previously involved peopled present. The reflective meeting is focused on assessing the research question or the goal for the research lesson but also elements of both the mathematical and the didactical praxeology. Discussions of the reflective meeting will be written down in the finale report which will also contain notes for improving various aspects of the lesson.

A fourth part of lesson study can be added but is not always present: *sharing the results*. This can be done either locally at the school on a bulletin, as part of a book or in conferences devoted to sharing lesson study reports.

Example of standard lesson plan template taken from Isozaki (2013):



Grade ○ Science Lesson Plan

Name of Teacher:
Date and Time: yy/mm/dd, time
Place: laboratory or other places
Class:

[Unit title]

[Unit objectives] based on the Course of Study and teacher's ideas, Student should know and understand, be able to, etc.

[Views of unit]

1) Main ideas of the unit and the key teaching contents,

2) Learners' characteristics and their prior knowledge and preconceptions relating to this unit, and classroom atmosphere,

3) Teacher's view and ideas for instruction based on both 1) and 2).

[Assessment tasks] criteria and methods of the whole unit and each lesson.

[Overall Scheme of work] the sequence of unit goals with the number of hours to be spent on the unit.

[Today's one-hour lesson]

1)Title of topic

2) Objectives of today's lesson

3) Resources

4) Development of today's lesson

Time & Phase	Students' activity and learning based on anticipated students' response	Teacher's activity including assessment tasks and notes
(minutes) [Introduction] [Development] [Conclusion]	[review of previous lesson] [some questions and anticipated answers] [practical activities and anticipated errors] [students' presentation and anticipated results] etc.	[points to keep in mind] [assessment tasks: criteria and methods including supports for lower achieved students] [safety] etc.

Implementation of lesson study in the US

In Japan lesson study is well integrated both in the teacher's education but also in the environment around the teachers. The teachers therefore have a well established knowledge of and the ability to use lesson study in their professional development.

Hence adapting the Japanese lesson study is not problem free and there are many challenges.

The following list of challenges one can encounter when trying to facilitate lesson study is taken from an empirical study that explored the feasibility of lesson study in the US. There where foreseen challenges but also new more prominent obstacles occurred during the implementation.

The, by the researchers, foreseen challenges for the teachers was “finding the time and interest for lesson study”, “overcoming the fear of making one’s teaching public”, “finding ways of creating a curricular common ground for joint lesson planning” and “overcoming the knowledge limitation many groups are likely to encounter”.

However during the implementation of lesson study more fundamental challenges occurred: “posing rich, researchable questions”, “designing a classroom experiment”, “specifying the type of evidence to be collected” and “interpreting and generalizing results”. For example a group of teachers struggling to specify the type of evidence to be collected: (Fernandez 2002)

T1: I want to look at the questions from the beginning to end.
T4: But what are you studying.
T5: What about their questions?
T3: What aspect are you studying?

The problems such as “finding the time and interest for lesson study”, “overcoming the fear of making one’s teaching public”, “finding ways of creating a curricular common ground for joint lesson planning” where overcome by the prospect of the benefit of lesson study. The rest and more severe problems such as “overcoming the knowledge limitation many groups are likely to encounter”, “posing rich, researchable questions”, “designing a classroom experiment”, “specifying the type of evidence to be collected” and “interpreting and generalizing results” where tackled coincidentally by having teachers in the working groups that had been educated in Japan and therefore took on a mentor role in the working group. An example of this is a group of teachers struggled to interpret and generalize the results from the research lesson. One of the Japanese teachers explains: “When I teach, I always look to see the solution method that the majority of students use. I believe this method is probably what they have learned from their mathematics education up to that point. Since most students in this lesson counted the boxes on the grid paper, that’s probably their level of understanding of area. That may be something you need to approach differently for students to learn better skills to solve this problem.” (Fernandez 2002).

Tackling CAS based teaching with lesson study

I theorize that many of the present challenges teachers and students encounter in CAS based teaching can be reduced with the use of lesson study with a special focus on the preparation part especially the lesson plan. However new obstacles are likely to surface.

Often when doing lesson studies the working groups have a standard lesson plan template to work with and from. Even though the standard lesson plan template leaves room for the teachers own designs and focuses to come forward the standard lesson plan template is not designed to deal with much of the praxoelogical didactical or mathematical theory block, since this is considered already known. With this in mind and since the content of the praxeological theory and practice blocks both mathematical and didactical are changing when implementing CAS based teaching I propose that the current lesson plan templates should be rethought and redesigned.

In order to aid the teachers I propose that the lesson plan template starts with a section focussing first on the mathematical praxeology containing both the practice and the theory block for the knowledge actually taught in the lesson. For example a section on quadratic equations starting with the definition, the notions involved and different types of quadratic equations followed by a list of smaller results approachable for students and proofs or verifications thereof. Second a subsection focusing on the CAS praxeology containing a list of CAS techniques (commands) intended for the students to use working with the tasks later described in the lesson plan. Succeeded by an explanation of the praxoelogical CAS technology and theory for the tasks (How the chosen CAS works with the techniques, what is the output of the commands, the syntax and the restrictions of the CAS, how do we define a variable? How does the program store the variable? etc.).

Secondly I propose a section focusing on the didactical praxeology. For instance it could first consist of a praxeological practice block with descriptions of the different types of orchestration intended to be used during the research lesson: technical-demo, explain-the-screen, link-screen-board, discuss-the-screen, spot-and-show and sherpa-at-work. Followed by a description of the praxeological technology and theory justifying the discourse of the didactical techniques such as minimising the gap between on screen mathematics and mental mathematics, reducing the black box perception by the students, minimising the risk of generalization without gain of content knowledge.

This redesign of the lesson study template can help set a focus on the lack of knowledge centred around teaching with CAS and help the teachers in the lesson study working group focus and deal with the new praxeologies required one step at a time. Despite the new composition of the lesson plan template this does not instantly help the teachers since they will still need to attain a lot of new knowledge. Also the preparation part of lesson study will now take a lot longer time.

A way to tackle this time-consuming knowledge quest and reduce the timeframe for the preparation step can be done in several ways. One is to do the composition of the lesson study working group with a blended team to make sure that a person with for example CAS expertise is present. Another more resource-demanding is a start-up lesson study CAS based teaching seminar where the different aspects of the didactical and mathematical and CAS praxologies around CAS based teaching is presented. This can easily be done if the region already is hosting a lesson study conference.

Also we need again to look at Japan where often in lesson studies 'knowledgeable other' (a university professor or similar) is involved in parts of the lesson study. This use of mentors is very resource-demanding but can with the aid of today's modern society be done using the internet and thereby reducing some of the costs. Involving a person with academic content knowledge in the preparation mode might help the teachers to quickly master the mathematical praxeology and get an overview of the involved content knowledge. Also the person with academic content knowledge can point the teachers in the direction of fruitful question for the students to examine.

Some teachers are reluctant to use CAS in their teaching some on the grounds that it is an entire new world to them and they therefore have inadequate knowledge not just on how to use the technology themselves but also how to use it in their teaching for more than just fun for the students. Being involved in a lesson study one way or the other the teachers could benefit. If they are part of the lesson study working group the teachers might be helped by their colleagues or simply reading the lesson study reports or attending a research lesson and the reflective meeting can be a start.

Later on in the fourth part of the lesson study if the lesson study report is being shared other teachers could benefit from the new acquired knowledge in a manner of practical experiences that new teachers can use in their own classroom.

Many teachers feel the pressure from the community in order to use technology in their teaching and having lesson study projects focusing on

CAS circulating on schools and teaching communities would give other teachers examples of well thought thorough lesson plans and the considerations needed in order to make a successful lesson plan.

The lesson study format could also be a very productive tool in approaching and determining the new occurring mathematical and didactical praxeology for researchers, mathematicians involved in education, policy-makers, textbooks authors, etc. in cooperation with teachers.

Deficit of lesson study for CAS based teaching

Though lesson studies bring forth a suggestion on dealing with many aspects of the challenges and obstacles with CAS based teaching they do not tackle all challenges and also create new hurdles.

One of the more prominent obstacles is time since lesson studies are only for a single or two lessons and is not traditionally meant to consider the students outside the classroom. However in order for the students to be comfortable with the CAS tool it is beneficial for the students to work at home sitting on their own trying and exploring different commands while working on assignments.

Another challenge is the lack of generalization since the lesson study working group has a specific class of students in mind and therefore also a specific constellation of artefacts such as the classrooms environment with computers, monitors and screens. This can limit the adaptation of the lesson plan to other schools with different classroom environments.

6

Geography teachers' pedagogical content knowledge and internal didactic transposition of the topic weather formation and climate change

Søren Witzel Clausen

PhD. Student, Institute of Education and Pedagogy, Aarhus University
Senior Lecturer of Geography, VIA University College Teacher Education

Abstract. This paper represents a part of a PhD project and will put emphasis on eight lower secondary Geography teachers, and how their Pedagogical Content Knowledge might influence their internal didactical transposition of the topic of weather formation and climate change. There are conducted semi-structured interview with the teachers. An analysis of the results implicates that there is a connection between the teachers' topic specific Pedagogical Content Knowledge, especially their subject matter knowledge, their educational profile, and how the internal didactic transposition is carried out.

Geography profession in the Danish school

Geography is a minor subject in the Danish Lower secondary school. The subject is one of the natural sciences subjects along with: Math, Biology, Physics/Chemistry and Science/Technology. In the Scandinavian countries the discipline of Geography is placed at different Faculties and has different content dimensions (*EUGEO* 2004, Molin 2006). In Denmark in the lower secondary school it consists of both Physical Geography and Human Geography, unlike in higher secondary school, where the subject is called Natural Geography mainly focusing on Physical Geography. Therefore in the lower secondary school Geography can be characterized as an interdisciplinary subject (Møller 2001) containing elements from different scientific disciplines: natural science, social science and humanities. It opens up a very varied terminology and a variety of teaching methods with the ability to motivate students. (eg. Heer et al. (2002), Ehlers & Volkers (2008), Mikkelsen & Sætre (2010)).

In addition, parts of the content in Geography is more or less reflected in Wolfgang Klafki's´ epoch typical key issues (e.g. Feierabend & Eilks (2010), Møller (2001)). Insight into these key issues can be expected to be of great relevance to the problems that students later in life are going to hear about and maybe work with. But working with these complex geographic content areas requires varied didactics and a detailed technical language that serves both cognitive and affective elements (Slater 1994, Nielsen 2000), and that the teacher is able to work with Socio Scientific Issues (Rattcliffe & Reiss 2006) in a qualified way.

Looking at the teachers´ formal qualifications, understood as educational background, there is in Denmark a significant difference between lower secondary school teachers and high school teachers. To teach physical geography in high school, you must have a University degree either in the subject of Geography or Geology. In the Lower secondary school, there has so far been no requirement that Geography teachers´ should have a formal education to teach lower secondary Geography. In the context of the new school reform (*Ministry of Education* 2014) it is however required that the teachers in the coming years must have a specialisation from a Teacher College in their teaching subject or have acquired equivalent qualifications. At present approximately 2/3 of the Geography teachers have those acquired competencies (*Uni-C* 2013).

Weather formation, climate change and geography

In the case of the subject Geography, the topic of weather formation and climate change is quite central in the curriculum of the Danish lower secondary School. In the last 10-15 years the public, politicians, NGO´s and scientists´ have increasingly emphasized the importance of students' insight into weather formation and climate change. All those different agents might be part of the external didactic transposition (Winsløv 2006, Borsch & Llull 2014), and that is probably the reason why the topic of weather formation and climate change has become an even more central part of the curriculum of Geography. You might go a step further, and say that the topic of weather formation and climate change have caught those agents interest, while it has become a vital question to keep a kind of status quo to the modern western life style.

In many ways, weather formation and climate change is a very exiting topic, not only of vital interest to the society, but also a key topic in Geography containing aspects from both Physical Geography and Cultural Geography. Furthermore the topic might be seen as quit essential, while it is about the interaction between mankind and nature. On one hand, mankind are influencing the global temperature by the emission of greenhouse gasses e.g. CO2 and CH4. On the other hand, mankind are also under influence of natural oscillations in the global climate caused e.g. by sunspots and the interaction of the Earth and the Suns´ tracks being of very different timescales (11 - 100.000 years).

And for the reason of that, becoming one the epochal key issues to the society, it should therefore be a part of the curriculum, according to Wolfgang Klafki (Møller 2001). From a motivational point of view, Rattcliffe & Reiss (2006) argues that students are very interested in working with so called Socio Scientific Issues (SSI) with emphasis on contemporary issues presented in the news and with science content that are relevant to students' lives. The topic of weather formation and climate change fulfil all of these aspects.

Pedagogical Content Knowledge (PCK)

International research shows that the teacher is the single most important factor in relation to strengthen students´ learning (Hattie 2008). Hence it is interesting to study Geography teachers´ teaching professionalism in the sense of Pedagogical Content Knowledge (PCK). Lee Shulman (1986) was the first to use the concept of PCK were he pointed out that teachers´ content knowledge consisted of three categories: subject matter content knowledge, pedagogical content knowledge and curricula knowledge (Shulman 1986). Since Shulman the concept of PCK has been developed being an overall emergent concept that integrates three knowledge areas: Subject Matter Knowledge (SMK), Pedagogical Knowledge (PK) and Knowledge of Context (KofC) (e.g. Van-Driel et al. (2014), Abell (2007), Magnusson et al. (1999)) see figure 6.1. In this study PCK is used as an overarching analytical framework to study Geography teachers´ teaching professionalism.

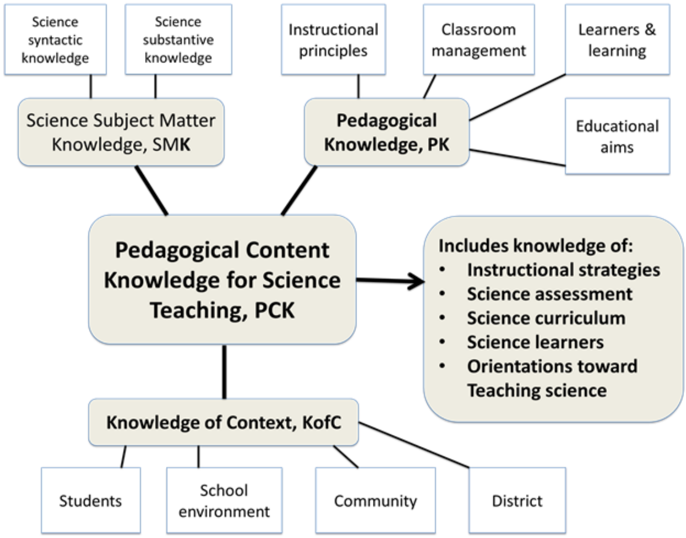


Fig. 6.1: Pedagogical Content Knowledge (after: Abell (2007), Magnusson et al. (1999)).

It is implicit in both Lee Shulman´s and Sandra Abell´s understanding of PCK, that it is a personal knowledge which the teacher holds - strongly connected to the teachers´ formal education and personal experiences. But it can both be developed individually as well as in professional collegiate communities (Van-Driel & Berry 2012).

Didactic transposition

The process of transforming a sudden content of scientific knowledge, accumulated and refined over the years, into something the students actually learn, is a long and hard process - difficult to describe and analyse. In the recent years some authors have used the concept of didactic transposition as a framework for describing and analysing this process (e.g. Winsløv (2006), Borsch & Llull (2014)).

Here, the didactic transposition as a process separated into four different stages called: 1) scholary scientific knowledge, 2) scientific knowledge to be taught, 3) taught scientific knowledge, and 4) learned knowledge, see figure 6.2. This illustration shows that the scientific knowledge, which consists of scientific concepts, theories and models goes through a transformation where different actors, based on what they think is important and valuable, select parts of the scientific content to be passed through the educational system. On the other hand, what student, teachers and other actors of the society think is important and valuable knowledge may also influence what the scientific establishment find important to study. In figure 6.2, this double-process is indicated with a double-arrow. The process of selecting scientific knowledge for the curriculum, called scientific knowledge to be taught, is called the external didactic transposition. The processes of transforming the curriculum into actually classroom teaching (taught scientific knowledge) and transforming this into learned knowledge is called internal didactic transposition (Winsløv 2006, Borsch & Llull 2014).

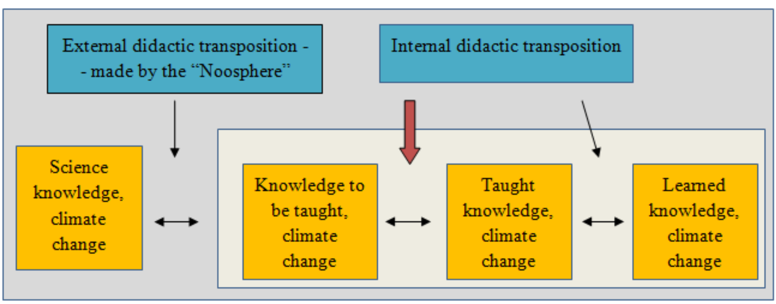


Fig. 6.2: The bold arrow indicates the analytic focus of didactic transposition in this study (after: Borsch & Llull (2014)).

To understand the didactic transposition it is highly relevant to study teachers´ PCK. Therefore this study will put emphasis on the characteristics of eight Danish Geography teachers´ PCK according to the topic of weather formation and climate changes (the content dimension) and how it might influence the teachers´ ability to implement the internal transposition from the knowledge to be taught to the taught knowledge. This process is indicated with the bold arrow in figure 6.2.

The "Noosphere" and climate change

In the didactical transposition model in figure 6.2, the "Noosphere" consist of different actors doing the transpositive work (Borsch & Llull 2014). Teaching Geography, climate change has become central in the curriculum of the Danish lower secondary School. In the last 10-15 years the policy level, NGO´s and scientist have increasingly emphasized climate change as an important topic (e.g. *IPCC* (2013), Hayhoe et al. (2011), Andersson & Wallin (2000)). All those different actors might be part of the Noosphere in the external didactic transposition (Winsløv 2006, Borsch & Llull 2014). Having this in mind, this is probably one of the reasons why the topic has become a central part in the Danish curriculum of Geography. Going a step further back climate change might have catched those actors interest, while it is of vital interest to the modern society trying to reduce the impacts´ from the climate change and maybe keep a kind of status quo to the present way of living.

Aim and research question

What is of primary interest in this paper is the analysis of the teachers´ internal transposition of the curriculum of weather formation and climate change to taught knowledge, and what might influence this. The Noosphere is of secondary relevance while it constitutes the knowledge to be processed in the internal transformation between the knowledge to be taught and the taught knowledge. Therefore it is the aim of this study to explore what may have significance for Geography teachers´ internal didactic transposition in relation to the topic of weather formation and climate change.

Methodology

There have been conducted semi-structured interviews (Cohen et al. 2010, Kvale 1997) with eight Geography teachers at lower secondary school, coming from four different schools. The schools are geographically evenly distributed, so they represent towns and schools of different sizes in the Region of Middle Jutland. They also represent schools with children from different socio-economic background (Larsen 2013). In addition, all four schools are participating in the QUEST project[1], which have made interventions to promote science teacher collaboration to improve the science teaching.

The eight interviews were conducted during the spring of 2014. Before the first interview, an interview-guide was developed concerning four different items which were: *background information, PCK concerning climate changes, general development of the Geography teachers own PCK, The specific influence of the QUEST project on the PCK*. Before the start of each interview, the guide was passed on to the teachers´, so they were informed about the questions. They were also informed about the purpose of research and promised full anonymity. The interviews were transcribed and different categories were developed through an iterative process reading the data. From this material, central statements according to purpose of this study were selected - these can be seen in table 6.1.

Results

In table 6.1, a short description of the involved schools and teachers can be seen in the two first columns. In the last column, statements from the interviews emphasising the teachers' thoughts about potential factors that affect the didactic transposition in relation to the topic of climate change, are shown.

Discussion

As can be seen from table 6.1, only Karen and Rasmus have not taken Geography as a subject from Teacher College. Grete and Henrik (school A), Nellie (school B) and Michael and Rasmus (School D) have more than one science subject from Teacher College, and therefore presumably rather strong science profiles. On the contrary Annette (school B) and Karen and Erik (school C) have some more humanistic orientated academic profiles.

Several of the teachers mention the importance of "the book" when being a new teacher (Grete, Henrik, Nellie and Erik), some of them thinking of the Geography book from the College others thinking of the Geography book for their students. Whether it is the first or second type of books, the

[1] See: www.questprojekt.dk

Schools	Informants and formal education	Statements from interviews
School A with 570 students in a small town	**Grete**: Teacher for 30 years and Geography teacher for 8 years. Geography, Physical Edu., Math.	*"when I started as a new teacher I stuck close to the book".* *"Now - instead of having a topic about water. Well - what is it actually the students need to learn about water?"*
	Henrik: Teacher and Geography teacher for 7 years. Geography, Biology, Danish, Physical Edu.	*"My Geography book at the college, I think it was an excellent book, [..] I have started from scratch, and then built a professionalism up..."* *"At first I thought a lot of - I will achieve all these goals in the curriculum. I hardly give it a thought anymore ..."*
School B with 1.050 students in a town with app. 60.000 citizens	**Nellie**: Teacher and Geography teacher for 12 years. Geography, Science and Technology (gr. 1-6), Danish, Physical Edu.	*"I think Geography is a big subject, and I might find it difficult to be equally competent in all the topics [..] so to be safe, so to say, as new teacher you start with the book".*
	Annette: Teacher and Geography teacher for 6 years. Geography, English, Danish, Hand craft + BA Anthropology	(Søren): "What put you emphasise on when teaching the topic of weather and climate? - *"I think it's more the social science discussion ..."* *"I get something out of talking to colleagues [..] it is more the didactic things - the activities. That's where we help each other".*
School C with 810 students in a city with app. 300.000 citizens	**Karen**: Teacher and Geography teacher for 12 years. Danish, Science and Technology (gr. 1-6)	(Søren): Is climate change a difficult topic to teach? - *"Actually, I do not know, because I know too little about Geography".* *"The old-fashioned understanding of the subject Geography with atlas [..] everything I learned when I was a young student".*
	Erik: Teacher for 11 years and Geography teacher for 3 years. Geography and Social studies	*"I use the web a lot. And I use my professional knowledge and the books we have".* *"Teacher trainees have ways such annoying question - then you are forced to reflect about the things you usually do".*
School D with 330 students in a village	**Michael**: Teacher and Geography teacher for 5 years. Geografi, Physics/Chemistry	*"Having a good colleagues is very important [..] so you can invent all sorts of things, because it is often easiest to get ideas through others".* *"This teaching session* (climate change, Søren) *maybe I have tried it many times, how can I do better? I know it's says this and this in the book - but how can it be invented"?*
	Rasmus: Teacher for 11 years and Geography teacher for 9 years. Biology, Physics/Chemistry	*"I do not think you can teach climate change in geography without and have some threads in biology and physics/chemistry".* *"I put emphasise on uncertainties connected to climate change, to encourage the students to consider the matter".*

Table 6.1: The schools (A-D) and associated informants together with selected statements from the interviews

writers and editors of Geographic literature might have a great influence on new teachers´ didactical transposition. All these Geographic textbooks are based on the curriculum of Geography in either lower secondary school or the teacher education. The writers and editors of textbooks then become parts of the so called Noosphere in the process of didactic transformation.

On the other hand Grete and Henrik (school A), and Michael (school D) are expressing confidence to their own judgement about what is of relevance teaching the topic of climate change. It is also of relevance to study the collegial aspect of didactic transposition, where several of the teachers´ mention the importance of collegial interaction as being of importance to their didactical transposition. Eric, as the only one mentions teacher trainees as being important asking "annoying question". These questions make him reflect about his own practice in the classroom.

Nellie, Annette and Karen all having different academic profiles also express that the subject of Geography have a huge curriculum, and parts of the curriculum will be downgraded when teaching. Especially the physical part of weather formation having low priority teaching the topic of weather formation and climate change. According to Annette and Nellie they like to teach in a more discursive manner with focus on Human Geography and might leave the more "heavy science part" of the curriculum to other colleagues. Erik too (school C) has a discursive approach to the students learning. This focus might influence the students´ learning outcomes. On one hand the students might become citizens having action competences. This implied as students being able to express their points of view according to Socio Scientific Issues (Rattcliffe & Reiss 2006). On the other hand, their students may have a lack of scientific knowledge to make proper scientific based argumentations on such questions.

In the case of Grete and Henrik (school A) and Michael and Rasmus (school D) their subject matter knowledge is high and the story of internal transposition of the curriculum of climate change into taught knowledge, is different. All of them cover all parts of the curriculum connected to climate change trying to have a great deal of variation in their teaching methods. During the interviews they emphasise that they give ordinary Geography lessons with presentations and hand-on activities, combined with student centred project work.

In case of Karen (school C) the Subject Matter Knowledge is especially low, which highly influences her internal didactic transposition and therefore her ability to teach the topic of weather formation and climate change in a satisfactory manner. She neither poses the complex Subject

Matter Knowledge or Pedagogical Knowledge of the topic. This may affect the students´ learning outcome.

Conclusion

This preliminary study of eight Geography teachers' PCK and their internal didactic transposition according to the topic of weather formation and climate changes shows some characteristic results. The rather complex topic contains both elements from Physical and Human Geography, but teachers with an affinity for Human Geography and a humanity profile leaves out elements from the Physical Geography which they find difficult to deal with. It seems as though the cooperation and stronger science SMK on school A and D, promote another internal didactic transposition, with emphasis on greater variation in content and didactics in the geography lessons. This kind of close collegial cooperation is in focus in the large scale development project QUEST focusing on in-service science teachers´ cooperation. It might be an idea to involve pre-service teachers in such actions.

This small-scale project can only give some preliminary ideas about the connection between Geography teachers PCK and the internal didactic transposition. To have some more reliable results there is a need for a large-scale project.

In the spring of 2015 this PhD project will study the PCK in-action where the internal transposition will be observed during the Geography teachers´ lessons in the topic of climate change. It might give another, more precise and nuanced, picture, than these statements from the teachers´ themselves.

7

What role do study and research activities play in the relation between research and education?

Britta Eyrich Jessen

University of Copenhagen and Frederiksborg Gymnasium & HF

Abstract. This paper presents a study organising the teaching of exponential functions in an attempt to re-establish a connection between research and education (or school mathematics) based on the anthropological theory of the didactics (ATD). The teaching aims at students developing new knowledge like scholarly knowledge, which is developed from open-ended questions and is justified by mathematical argumentation. The design is based on research and study activities and requirements for upper secondary mathematics in Denmark. In this study the emphasis is put on the media-milieu dialectics and how this can be orchestrated in a fruitful manner. The study shows improvements in student's relation to proofs and the student's ability to read and write mathematics.

Disconnection in the teaching of exponential functions

It has been pointed out that Danish high school does not treat exponential functions rigorously and that there is no raison d'être given to the notion a^x where $x \in \mathbb{R}$ (Winsløw 2013, p. 5). The curriculum phrases requirements for the working with exponential functions as: "*students should be able to [...] use relations between variables for modelling purposes of data, predict how the modelled system evolves and be able to discuss how good the model fits with the system*" and further the students should work with "*equations describing [...] exponential relations between variables [...]*" (*Ministry of*

Education 2013*a*). There are no explicit requirements regarding real number exponents. The ministerial guidelines for teaching the curriculum says that "*it can be an advantage to put focus on [...] arithmetic rules for calculation with exponents while working with exponential relations [...]*" and further it is suggested at the introduction phase of the course to look at: "$\sqrt{2}, 10^{100}$ - *how to explain what these symbols stands for?*" (*Ministry of Education* 2013*b*, pp. 5-7). In the last sentence there is an opening for introducing real number exponents but the sections ends by saying it should be done through experiments and games – not in the rigorous manner as treated in the capstone courses described in (Winsløw 2013). Most textbooks present rules for calculating with exponents based on the definitions of a^n and $a^{\frac{1}{n}}$ and natural numbers. It is noted that this can be extended to the $n \in \mathbb{R}$, but it is too advanced to give the details in the book or it is calculated using CAS-tools (Winsløw 2013). Some books do not even treat the extension with a single word (Clausen et al. 2010, p. 67f). Moreover there is a disconnection in the content between scholarly knowledge and school mathematics regarding exponential functions. This mean, that there are theoretical elements and reasoning in the treatment of exponential functions high school students do not develop.

Looking at a widely used textbook one finds an introduction of the notion $f(x) = a^x$ and $f(x) = b \cdot a^x$ and the possible graphic representations, then the book presents how the coefficients a and b are calculated from coordinates of two points on the graph and after that two examples on how to use the formulas are given. Next, the doubling constant is presented graphically, how to calculate it and examples are explained. All examples are followed by suggested exercises where students are supposed to copy the techniques as they are presented in the examples (Clausen et al. 2010). These pages covers the requirements of the curriculum mentioned above though it might not bring reasoning for the techniques to the students.

Another important difference is how mathematical knowledge is developed among students in secondary school and how mathematics is developed by mathematicians concerning the study process and the reasoning behind the newly developed techniques or answers in the two institutional settings (Bosch et al. 2005) and (Winsløw & Madsen 2008). In scholarly mathematics questions serves as generators that drives the development of new knowledge in terms of praxeologies;

> "Doing mathematics consists in trying to solve a problematic question using previously available techniques and theoretical elements

in order to elaborate new ways of doing, new explanations and new justifications of these ways of doing." (Bosch et al. 2005, p. 5).

Research like activities are proposed as ideals for how students should work with school mathematics (Winsløw & Madsen 2008, p. 1). At the same time it is reported that it is not the questions research and education shares at university level but it is the production of answers, as a teacher points out: "we are after them thinking about what are the problems involved in this task (...) just they explore this problematique we feel they have achieved a lot." (Winsløw & Madsen 2008, p. 6). The aim of the study of this paper was for the students would develop new praxeologies covering the techniques from curriculum based on (for the students) open questions using their existing praxeological equipment. This is not an original idea but the orchestrating of the activity and the creation of a suitable milieu in order to create the process described by the Herbartian schema is new.

Therefor the research question of this study was how to design teaching for upper secondary mathematics ensuring the students potential of developing praxeologies as answers to questions – including the raisons d´être of the praxeologies?

Teaching design based on students' study process

To design teaching fulfilling this, tools from the ATD were used. In this section study and research activities (SRA) are presented as a design tool for teaching build on questions, the dialectics of media and milieu and didactic moments that characterises the different activities during the didactic process. Firstly the ATD perception of a teaching and learning process is given. ATD characterises the teaching and learning process by so called Herbartian schema. The process requires a question Q to be studied by a group X assisted by another group Y. The groups can consist of only one element. X, Y and Q form a didactic system, which develops the answer A♡, where ♡ denotes that it is the answer developed by the specific didactic system. The development of an answer happens when the system interacts with a milieu M, which consists of existing answers or praxeologies A_i♦ and other resources (the work of others). X and Y can act as part of the milieu. The semi-developed Herbartian schema represents the study process:

$$[S(X;Y;Q) \to M] \hookrightarrow A\heartsuit$$

This study process leads to the development of a raisons d´être of the praxeology, even if the praxeology is considered in a decontextualized way (Kidron et al. 2014, p. 157). The process described by the Herbartian schema is dynamics of the design tool given below.

The inspiration for the teaching design is taken from the idea of study and research paths (SRP). These start with a generative question Q_0 real and understandable to the students and strong enough to lead to derived question Q_i, Q_i', Q_i'' and so forth. The studying of all these questions should lead to the development of a set of praxeologies covering a body of knowledge, or in terms of ATD an intended mathematical organisation (Barquero et al. 2007) and (Barbé 2005). New mathematical praxeologies are developed as it is described above through answering new questions through adjusting existing praxeologies and by studying the works of others.

It is crucial in SRA that students do engage themselves in the process of studying the work of others being books, web sides, videos on the internet and likewise. The research process covers reorganisation of the pieces of knowledge and techniques found in the media consulted in the study process in order to (partly) answer the question at stake. The latter corresponds to the students acting with a milieu in the sense of the Theory of Didactic Situations (Winsløv 2006, p. 135f) and (Kidron et al. 2014, p. 157). The working process of the students can then be characterised by what have been called the dialectics of media and milieu (Barquero et al. 2007) and often X, computers and internet can function both as media and milieu in these processes (Kidron et al. 2014, p. 159). To further this process in the particular study a "resource room" were proposed to the students at the beginning of each question. This was simply a list of web sides with text, illustrations, video links and pages in different textbooks.

Because of the constraints and conditions for this study; namely the students should be able to present their work at an oral exam, they should be able to find the exponential function through two given points and be able to talk about characteristics of exponential growth and the doubling constant. The generating questions Q cover previously established praxeologies and are not open in the sense of study and research paths (see for instance (Barquero et al. 2007)). In this study, there do exist an optimal combination of pre-established techniques for solving the questions not known to the students. This is the difference between SRA and SRP. The SRA are related to the paradigm of visiting monuments and SRP are related to what Chevallard denotes the paradigm of questioning the world (Chevallard 2012). The study process of the study of this paper is to some extend similar to the one of SRP's from the perspective of the students. Different papers presenting realised SRP (Barquero et al. 2010) and (Garciá & Higuares 2005) points out that there is still research to do on the realisations in order to secure the full theoretical potentials being activated in these activities. This paper attempts to give one possible answer to (part of) this challenge.

One can argue that there are similarities between working with open questions – especially in the case of study and research paths – and inquiry based mathematics education. It was not an explicit intention for the design to be IBME, but the theoretical relations are given in (Artigue & Blomhøj 2013) and (Winsløw et al. 2013). The last paper noting the importance of the study process and that inquiry cannot stand alone.

How the activity is managed in the classroom is described by the notion of the moments of didactic organisations. According to ATD there are six moments of the didactic process: the moment of first encounter, the exploratory moment, the technical moment, technical-theoretical moment, the institutionalisation moment and the evaluation moment (Barbé 2005). First encounter means the first encounter with the organisation (MO), which can be in the form of a task or question that belonging to the organisation. The moment of exploration covers the development of techniques, which can solve the task. This can be modification of existing techniques or the studying of new ones or a combination of both. The technical moment explores further what the technique can answer and is strongly related to the technical-theoretical moment which is the students development of the discourse regarding the explored techniques belonging to the organisation. The moment of institutionalisation seek to elaborate on the answer found and what it is. This moment is strongly linked to the moment of evaluation, where the value of the developed praxeology is checked. It is further concluded that "a 'complete realisation of the six moments of the didactic process must give rise to the creation of a MO that goes beyond the simple resolution of a single mathematical task" (Barbé 2005, p. 239). In the next section a presentation of how this is carried out is given.

How is the teaching managed in the class room

The study was conducted in a class of 24 students at first year of upper secondary school in Denmark (15-16 year old students) studying language and social sciences (i.e. not interested in mathematics, and at the lowest level C). The author is both the researcher behind the design and the regular mathematics teacher of the class. The teaching was conducted in the spring 2014 and colleagues (mathematics teachers) were recording and observing the teaching.

The teaching was organised as group work. The class was divided into 8 groups of three or four students. The criteria for being in the same group, was the teachers expectation that the students in each group had approximately the same praxeological equipment. The reason for this is a wish for the student to develop new praxeologies as answers to the open question developed from their existing praxeologies, i.e. maximize the potentials in the study process for each student according to the media-milieu dialectics described above. This means that in the first encounter with the organisation formulated as an open question the students in the group activates to some extend the same praxeologies. A single group member is not likely to give the solution of the problem to the group. Instead the entire group must study some media and reconstruct the content of the media to create the praxeology, which answers the question.

The number of groups was chosen, so that no student would be alone in her group if one other student were missing. And presumably at least one student would be able to generate an idea to answer the open question from media or existing praxeologies. The groups had 5 or 7 minutes per question for first exploration. After these few minutes every group must present their preliminary thoughts in writing on the whiteboard. The board was divided into eight spaces and each group should be able to present their entire solution in this small space without erasing anything. It was always the group with the poorest praxeological equipment, which started, and it ended with the best equipped. This ideally should ensure that the students were presenting their thoughts without comparing themselves to a better solution – since solutions were presented according to ascending quality. It was not allowed to say that one group had done the same as the previous one, since often two groups presenting the same strategy emphasised different elements or reasoned differently. All of this was crucial parts of the media-milieu dialectics and assisted the development of mathematical language among students. The groups were working in the technical moment when showing what answers to the question certain techniques could give them. Some groups were even able to discuss reasons why they had chosen different techniques compared to previous groups and were therefore in the beginning of the technological-theoretical moment.

When all groups had presented their thoughts the teacher organised a discussion and comparison of the solutions suggested, asking the class to formulate these. Since the time schedule was tight the students most often would not have completed the answer to the question and the class agreed which method seemed the most promising and pursued this, or if no group had given a reasonable method a derived question was posed by the teacher according to the a priori analysis. Evaluating the preliminary answers as the sum of techniques and the technology presented by the group was part of the institutionalisation moment of the MO at stake in the particular question. In the context of media and milieu this moment adds to the process that students in the class served as media to each other. When students were questioning the new ideas posed by classmates they actually formulated a Qi' assisting the development of A♡. This dynamic of generating questions to explore new techniques and develop new technologies for the mathematical organisation of exponential functions were the intention of the teaching design.

Homework between lessons were not reading a number of pages in a book but writing a thematic report. A thematic report is a written production presenting a body of knowledge covering theoretical presentations and resolution of different tasks or type of tasks. The reports are used as synopsis for the oral exam (*Ministry of Education* 2013*b*). Writing these reports further served as technical moment for the teaching of the MO. They were based on each group work but also new elements brought into light in the whiteboard presentations of the other groups. This meant that after the sharing sessions each group adjusted their answers by testing strategies presented by the other groups. This might call for students to further study their own notes or media and reconstruct it as new answers. Or simply, the group had to adjust their answer with respect to questions and ambiguities, which became clear during the presentations and class discussions.

The groups presented their homework writings on the whiteboard in the next lesson, which served as further evaluation of their work. Finally, the students handed in the reports, which were corrected by the teacher. The outline for the report was to make a more rigorous presentation of the exponential functions and the techniques related to solving problems involving these, exemplified by the questions solved during lessons.

A SRA on the exponential function

In the following some of the questions used for the SRA are presented. Most students knew (or were able to recall) from lower secondary how to calculate the amount of money on a bank account with annual rate of interest of 4 %. The students used the formula $K_n = K_0 \cdot (1+r)^n$. This is a discrete form of the exponential function $y = b \cdot a^x$, where $a = 1+r, b = K_0$, $f(x) = K_n$ and $x = n$.

The intention of the design was that the students enlarged their praxeological equipment on the notion of exponential functions not being restricted to bank accounts, but to give them the ability of modelling much more different setups. Implicitly the intention was for the students to see the need of proofs and see those as exact arguments. These aims are not explicit in the curriculum but it is said that the "*students should be able to carry out simple mathematical reasoning*" (*Ministry of Education* 2013*a*). More important was the potential of developing a technological-theoretical dimension of the techniques in order to connect the secondary teaching of mathematics with the scholarly development of mathematical knowledge. The first question posed to the students was the following one:

> Q1: Grandparents starts a saving account for their newborn grandchild by putting 5000 dkk. on an account at an annual rate of interest of 2.5%. Bank regulations say that the amount of money cannot exceed 50,000 dkr. Will that be a problem if the money is being paid to the child at its 21st birthday?

The students were supposed to predict the amount of money after 21 years using their knowledge from lower secondary. Some students were assumed to choose to calculate the amount of money by calculating K_{21}, others to use the graphical representation to find the corresponding y value to $x = 21$. Others again could use a somehow iterative method multiplying 5000 by 1.25 and continue doing that 21 times. This moment of first encounter with the MO of exponential functions as it is described in curriculum and guidelines is an example of giving the students the potential of re-encounter the type of task: "calculate K_{21} when given $K_0 = 5000$ and $r = \frac{2.5}{100}$". For other students it called for studying media again. Next the following question is posed:

> Q2: At the neighbours the grandparents also created an account for their grandchild starting at 5000 dkk. After 10 years the amount of money has grown to 5947.22 dkk. How much money do the two children have on the respectively accounts after 18 years?

The students could partly answer the question using praxeologies from lower secondary, using CAS-tools or not. Since they might recognise

K_0,K_{10} and n=10. From this they could formulate an equation, which could be solved taking the 10th root on both sides. Their calculators were able to perform this if they knew how to use them. Most likely the students had to study the textbook examples and recognise the coordinates of the two points given in the text and calculate a. From this the rate of interest was deduced. This meant that the solution to this question called for a combination of techniques and the discourse regarding the combination of these, which led to the development of a technological level of a new praxeology. Q2 was an example of a question the students not were supposed to answer during the first 5 or 7 minutes. Hypothesis and arguments for combination of techniques were expected but not a complete answer. Finally the students had two saving models and they were supposed to use the previous established technique to calculate the amount of money on the accounts to compare the models.

Q2 was followed by asking the students how they could be certain that the formulas $a = {}^{x_2-x_1}\sqrt{\frac{y_2}{y_1}}$ and $b = \frac{y_1}{a^{x_1}}$ gave them the model they were looking for. This led to another study and research process developing their mathematical reasoning using existing arithmetic techniques in new context.

In a following lesson the students were asked how long it takes before the amount of money is doubled on the account. This calls for the students to use the technique $T_2 = \frac{log2}{loga}$, which is a requirement for mathematics at this level though logarithms are not required (*Ministry of Education* 2013*b*, p. 5). The textbook says "*On a calculator there are two useful buttons, log and ln. These are two functions, which can be used on the positive numbers. [...] Logarithms were earlier an important technical tool for making calculations. Now they are replaced with calculators except from special usages*" (Clausen et al. 2010). This means students are given no reasoning for the technique. The design offers the potential of the students to explore some of the reasoning behind this technique.

An entire analysis of the design requires a praxeological analysis of the classroom observation and the written reports compared to the a priori analysis (for how this can be done see for instance (Jessen & Winsløw 2011) and (Barquero et al. 2007)). Comparing these tree diagrams will show what praxeologies have been developed and how the potentials of the design were explored. Though some findings are given below.

Findings

The study of this paper was carried out after a pilot design on the re-encounter with linear functions. This meant that students did not question the sharing of responsibilities in the class meaning who provide answers, questions and who institutionalise and evaluate the answers elaborated by the students.

When asked for reasons for the formula $a = \sqrt[x_2-x_1]{\frac{y_2}{y_1}}$, the students did not question a proof as the way to argue in mathematics. Not all groups did provide equally rigorous version of proofs. Some groups could point to the proof but only carry out parts of the reasoning and therefore they used the other group presentations as media for further study for their written reports. Without explicitly asking for a proof for the doubling constant several groups presented the definition of $\log_{10}x$ and used this for proving the doubling constant, a "quadruple constant" and "eight times constant" (translated by the author from the students writings). These self-invented techniques were used to characterise the growth of an exponential function for $a > 1$. This work went beyond curriculum and was closer to scholarly development of mathematical praxeologies than usual teaching. Not all groups were able to deliver this and some of the first groups did not include the proof of T_2 in their written reports. They only used the technique correctly and phrased that it could be proved using the rules for calculating with logarithms.

An even more clear advantage of this teaching design was that the students improved their writing and reading of mathematics. In the beginning it was very hard for the students to figure out what writings were needed on the board. They thought that it depended on the teacher, and if the teacher got the point it was good enough. But as the students started the written reports they saw the need of understanding ideas and methods presented by other groups. This meant that they asked the other groups to elaborate on their presentations, put more details on a sketch of a graph or write more details in proofs on the board. This again affected the student's ability to talk mathematics. In the end, students combined notes with what they read or saw online or read in textbooks in order to complete their written thematic reports. As a result of this teaching the reports were formulated in a more mathematically correct way and at the same time more independent of the media used by the students.

Students were asked to put in writing their thoughts on the teaching. This showed that they felt they "did a lot of mathematics during every lesson”, that “they taught each other instead of being taught by the teacher” (not a positive comment from all of them), they said they had to participate - nothing was served to them. Finally student found it boring listening to all 8 presentations. The author agree that there is a dilemma that the design is time consuming but it seems important that every group experience the moment of institutionalisation and evaluation of their developed praxeologies.

Concluding remarks

Since the design is based on a monumentalistic presentation of the knowledge to be taught given by curriculum, ministerial guidelines and textbooks there are parallels to what Chevallard have named the teaching paradigm of visiting monuments (Chevallard 2012). Conversely the design aims at students questioning and studying of these pre-established pieces of knowledge. The students did generate good questions such as the prolonging of the doubling constant as part of the characteristics of the growth of the function. In this case some connection between research and education is established. And just as important they all achieved the intended techniques for solving problems involving the exponential function.

Still to be investigated further is how we can support the generating of derived questions through the media-milieu dialectics in feasible teaching designs. But also the advantages of students developing a written and spoken language of mathematics.

8

Addressing the tension between constructivist theories of learning and inquiry practice in science education through didactic transposition

Dave Liam Johnston

Montclair State University

Abstract. The current climate of science education reform in the United States is driven by policy which aligns strongly with a constructivist theory of learning. Constructivism supports an inquiry-based approach to the teaching of science content, which is demonstrated in the learning standards in the Common Core State Standards and Next Generation Science Standards. However, in an education context in which quantitative standardized test scores are highly valued, many science teachers encounter dissonance between their constructivist oriented theories of learning and their daily pedagogical practices. I propose didactic transposition as a theoretical framework that can be used to examine this dissonance and identify the ways in which science content is transformed from teacher knowledge to teacher practice.

Introduction

The current climate of science education reform favors educational approaches that support constructivist theories of learning. This trend is consistent with the decline of positivism in science education in the 1970's and 1980's, which led to the popularity of Piagetian and Vygotskian theories of constructivist learning (Tobin & Tippins 1993). This shift away from "teacher as transmitter of knowledge" and towards "teacher as facilitator of learning" saw the rise of inquiry based instructional strategies that are often associated with constructivist theories of learning (Gallagher 1993). However, given the "epistemological anarchy" created by the numerous types of constructivism (Geelan 1997), many unclearly defined, the nebulous working definitions of inquiry-based practices, and the lack of explicit instructional modelling in teacher education, it is no wonder that science teachers struggle with aligning constructivist theories of learning with their everyday practice (Klein 2001, Windschitl 2002).

Inquiry-based science education is sometimes conflated with "constructivist practices", as if they are one in the same. In the United States, the push for inquiry-based instruction within the Next Generation Science Standards (*Next Generation Science Standards* 2013) and the Common Core State Standards (*Common Core State Standards* 2010) draws up on constructivist theories of learning. However, without explicit descriptions of what inquiry-based practices science teachers should use nor how they are related to constructivism as a theory of learning, teachers often create learning environments directly opposed to that from which they draw. The changing school climate, being influenced by policy reform, creates shifts in practice, creates a tension between theory and practice (Russell, 1993) and brings about classroom situations in which teachers adopt "constructivist practice" without understanding constructivism as it relates to learning (Russell 1993, Shaw & Etchberger 1993).

Driven by the need to better support science teachers as they begin to implement constructivist practices in their classrooms, teacher education, as a discipline, needs to recognize that such teachers are learners similar to the students in their classroom (Kirschner et al. 2006, Russell 1993, Shaw & Etchberger 1993). Similar to how students must acquire, integrate, and apply information, teachers as learners must do the same in order to better implement teaching practices that truly align with constructivist theories of learning (Gallagher 1993). Teacher education programs should, therefore, prepare teachers in a constructivist manner similar to that of which they espouse (Klein 2001). These concerns are particularly applicable given that there is a line of research that criticizes constructivism as an appropriate learning strategy outright (Good et al. 1993, Kirschner et al. 2006, Nola 1998). It is in this context that I propose didactic transposition as a potential approach for teacher educators to take in order to explicitly support novice science teachers as they attempt to reconcile dissonant beliefs about constructivist learning and science teacher practice. Looking at this problem through the lens of didactic transposition, one can identify the places within the transpositive process in which teachers are making deliberate choices about which concepts to teach and the instructional strategies they select to teach those concepts, with the goal being towards alignment between learning theory and pedagogical practices.

Framing literature

Constructivism in Education

Constructivism as a theory of learning generally traces its roots back to two individuals: Piaget, who articulated a cognitive or process-oriented view on learning, and Vygotsky, who advocated a socially-mediated view on learning. Those who subscribe to a Piagetian perspective would argue that the acquisition of knowledge is moderated by the ability to adapt to new information and to organize new information according to existing schema in order to assimilate that knowledge into the learner's world view (Piaget 1972). In contrast, those who subscribe to a Vygotskian perspective would argue that the acquisition of knowledge is socially mediated and is particularly effective when a "more knowledgeable other" can help facilitate the process for the learner (Vygotsky 1978). It is from these two perspectives that the various constructivist orientations have emerged.

Regardless of the particular orientation of constructivism to which one subscribes, there are several fundamental understandings that are common across constructivism as a theory of learning. Windschitl (1997) identifies six core components of constructivism, including: teachers providing students with opportunities to engage with new material before instruction; students having multiple opportunities to engage in problem-based activities; and students having options as to how they express what they have learned. From a practical perspective, constructivist pedagogy requires taking into account that learners bring experiences with them that affect their understanding (Tobin & Tippins 1993).

Of particular interest to a didactic transposition approach to teacher education could potentially be radical constructivism. This constructivist orientation is similar to a Piagetian cognitive constructivist stance, but emphasizes that the individual learner is responsible for constructing knowledge by linking their pre-existing schema to their context, in order to draw meaning from new knowledge (Von-Glasserfeld 1993). From a radical constructivist perspective, knowledge cannot be viewed as a commodity to be transferred from teacher to learner, as the learner's interpretations about newly encountered content are drawn from his or her past experiences (Bettencourt 1993). Radical constructivism is focused on the externally influenced construction of knowledge within a larger social context.

Inquiry-Based Instruction

Inquiry-based instruction has been promoted as a pedagogical approach that aligns well with traditional scientific approaches. However, the term inquiry-based instruction is often confusing to teachers, particularly given the many definitions associated with the term. Coburn (2000) suggests that one possible explanation for this confusion is the fact that inquiry, as defined by the U.S. National Science Education Standards describes inquiry in two ways: the teaching of science and the doing of science. He then describes numerous examples of inquiry in the current literature, including structured inquiry, guided inquiry, and open inquiry, each representing a very different pedagogical approach. Coburn goes on to suggest that this confusion is a possible contributing factor as to why some teachers fail to implement inquiry-based practice into their classroom.

In reality, inquiry is hardly a new concept in education. Dewey (1938) discusses the definition and role of inquiry in learning. Dewey specifically references the importance of knowledge transformation for learners and this serves as a fundamental aspect of his conceptualization of inquiry. To modern academics, inquiry-based instruction is sometimes conflated with problem-based learning, which arose from medical education. Problem-based education holds several tenets that are similar to inquiry: students holding responsibility for their own learning, a foundation in the pedagogical - as opposed to didactic - curriculum (Savery 2006). However, problem-based learning is an example of one type of inquiry-based instructional approach. In this case, learners are expected to solve real-world problems through collaborative, reflective learning exercises intended to draw forth analysis that will be helpful in similar real-world experiences (Savery 2006). These characteristics *could* be included in all inquiry-based practices, but are not necessary for a pedagogical strategy to be considered inquiry-based.

So the question begs – What is inquiry-based instruction in science education? For the definition of this term for my purposes, I draw from the work of Minner et al. (2009). In a comprehensive literature review of inquiry-based science instruction, they identified six characteristics of inquiry-based instruction:

(1) Learners are engaged by scientifically oriented questions;
(2) Learners give priority to evidence, which allows them to develop and evaluate explanations that address scientifically oriented questions;
(3) Learners formulate explanations from evidence to address scientifically oriented questions;
(4) Learners evaluated their explanations in light of alternative explanations, particularly those reflecting scientific understanding;
(5) Learners communicate and justify their proposed explanations;
(6) Learners design and conduct investigations

It is instructional approaches that include these characters that are considered to be inquiry-based for the purposes of this essay. Lord & Orkwiszewksi (2006) give examples of what this does and does not look like in a science classroom. Foremost in their discussion is the "cookbook method of lab instruction", in which students follow a prescribed set of instructions in a laboratory setting. According to Lord & Orkwiszewski, this is an example of a common practice that is not inquiry-based. Using Minner, Levy, & Century's criteria, it would appear to fail to be inquiry, particularly seeming to violate criterion 6, by not allowing student to design their own investigations. To contrast this, Lord & Orkwiszewski describe a lesson where students work in small, cooperative learning teams to identify examples of a scientific concept. These students were then given items and asked to design a way to test a question around that concept using those items. Students followed through by designing and conducting the experiment and then analysing the results. This, according to Lord & Orkwiszewski, would represent an inquiry-approach and, using the Minner, Levy, & Century's criteria, appear to do so. However, despite the apparent benefits for student learning (Lord & Orkwiszewksi 2006), many teachers continue to use the "cookbook method of lab instruction". The question becomes, why?

Didactic Transposition

The process of didactic transposition was originated by Chevallard (1985) in the field of mathematics education. It describes the process in which information is transformed from its origin, through its use, and ultimately to the teaching of the concept. Of particular note is the fact that there are numerous participants in the transpositive work, as the information is transported through various iterations from the institution in which it originated to the educational facility in which it is taught. As information progresses across this continuum, individuals at each step offer their unique interpretation on the information, which ultimately shapes the understanding that the next participant will have (Bosch 2014). Bosch (2014) offers a diagram of the “journey” that information takes, which demonstrates the cyclical nature of the transposition, as learnt knowledge ultimately informs scholarly knowledge (See Figure 8.1).

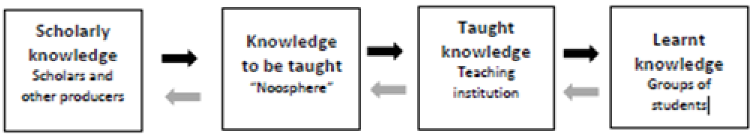


Fig. 8.1: Process of didactic transposition (Bosch 2014).

Another concept related to the model of didactic transposition, praxeology, recognizes that all knowledge has multiple related parts: practical knowledge and theoretical knowledge. For the purposes of this paper, there are two types of knowledge being discussed: science disciplinary knowledge and science didactic disciplinary knowledge. When framed praxeologically, the former consists of science concepts and facts (theoretical knowledge) and the practical applications of those concepts and facts (practical knowledge). On the other hand, when framed praxeologically, the latter consists of the science concepts and facts included in the noosphere (theoretical knowledge) and the practical applications of how to teach those concepts and facts (practical knowledge). From a DT framework, science teachers would be responsible for both aspects of knowledge as they determined how to best teach science concepts to their students.

Related to this concept is what has been described as a ‘double transition’ of knowledge, where a learner must not only negotiate the transition of knowledge from one context to another, but also from an academic to practical purpose (Winsløw 2013). This second transition is a common area of concern for novice teachers, as they struggle to put their content knowledge into practical purpose (Winsløw 2013). Could it be from this transition, from theory to practice, that novice science teachers fail to align constructivist theories of learning with inquiry-based practices? Given that inquiry is often poorly conceptualized and/or defined as a construct within the field of science education (Crowther 2005), it could stand that science educators are unsure as how to identify appropriate inquiry based practices that align with the content that they are teaching.

One could easily draw a connection between the idea of knowledge transformation from a didactic transposition lens and constructivism as a learning theory with that of Dewey's understanding of inquiry. Dewey discusses the concept of controlled inquiry, which is the way in which people create structure of ideas during learning. Dewey discusses how people use controlled inquiry in both scientific learning, as well as in everyday 'common sense' (Baskerville & Myers 2004). Is Dewey discussing a form of praxeology in relation to the construction of knowledge in inquiry-based science education?

Bridging the gap between constructivism and inquiry

Constructivism and Inquiry Teaching

In both the Common Core State Standards (CCSS) and the Next Generation Science Standards (NGSS), learning standards are supplemented with student action-oriented objectives which describe how students can go about meeting the learning standard. While these objectives are not mandated and serve primarily as suggestions for teachers who are planning lessons, many times teacher rely upon them in order to visualize what students might be doing during a particular lesson. In many cases, these objectives are inquiry-oriented in nature (*Common Core State Standards* 2010, *Next Generation Science Standards* 2013). For example, in the NGSS, to meet the learning standard about natural selection and adaptations, students must be able to "Construct an explanation based on evidence for how natural selection leads to adaptation of populations" (*Next Generation Science Standards* 2013).

Running counter intuitively to this notion of constructivist oriented teaching, current education reform efforts focus on the use of standardized testing approaches to measure student understanding. These approaches privilege rote memorization and traditional positivist approaches to teaching through transmission model instruction (Sacks 2000). It is no wonder that teachers who hold constructivist oriented theories of learning struggle to implement standards which seemingly conflict with an inquiry approach to pedagogy.

Didactic Transposition as a Mediator

Didactic Transposition offers an interesting approach to the study of this dissonance between theory and practice in science teachers. One could view the transformation of content knowledge (CK) to pedagogical content knowledge (PCK) as a form of didactic transposition. Shulman (1986) discussed how content knowledge is distinct from pedagogical content knowledge. According to Shulman, CK refers to the specific pieces of information within their content expertise that a teacher knows. For example, a biology teacher's understanding of the biological mechanisms that lead to natural selection would constitute a portion of her content knowledge. Contrastingly, PCK represents an understanding of how to teach particular content knowledge. For example, the biology teacher's understanding of how to best teach natural selection content would constitute a portion of her pedagogical content knowledge.

With regard to a didactic transpositive framework, CK would be the teacher's disciplinary knowledge, whereas PCK would be the teacher's didactic disciplinary knowledge. Indeed, given that PCK is often considered to be the result of a learned process (Smith et al. 2013), the connection between didactic disciplinary knowledge and the transpositive process between boxes 2 & 3 in the DT framework (see Figure 8.1) is particularly relevant. It is this process that is internalized within an individual, as it is the process in which an individual teacher determines the knowledge from the noosphere that will be taught in their classroom and how they will teach it.

In science education, it has been suggested that dissonance between disciplinary knowledge and didactic disciplinary knolwedge is, at least partially, caused by a difficulty in capturing and understanding PCK (Loughran et al. 2004). By utilizing a didactic transpositive framework, a focus would be placed on the transformation of knowledge from a theoretical purpose to a practical purpose. This explicit focus on the transformation of knowledge would, potentially, allow PCK (or didactic disciplinary knowledge) to be more readily identified and, therefore, could be a vital step in alleviating this dissonance. One possible concern is the previously mentioned poorly defined terms of inquiry. Likely, a specific definition of inquiry, such as Minner et al. (2009) six learner focused characteristics, would be used as criteria to determine if inquiry is, indeed, being practiced in the classroom. It is through this lens that I suggest approaching further investigation of constructivist learning theory and inquiry based instruction for secondary science teachers.

Implications for Research

To address this question, one must identify that the alignment of learning theory with pedagogical practice is the transpositive process represented as the arrows between the 2nd and 3rd boxes in Bosch (2014) Process of Didactic Transposition continuum (see Figure 8.1). In this transpositive process, teachers are identifying which information from the noosphere will be taught in their classroom. This transpositive work is focused not just on learning or doing, but a combination of both. Specifically, it is not just focused on the content knowledge (CK) or pedagogical knowledge (PK), but on the pedagogical content knowledge (PCK), as teachers determine which knowledge is most effectively taught using inquiry-based instruction.

It is important to recognize that Bosch's continuum represents a linear 'journey' that information makes as it progresses through the continuum. That is, information must travel through the noosphere in order to be transposed from scholarly knowledge into knowledge to be taught. A modified version of this continuum suggests that information can be shaped and transposed at all steps in the process (Gericke 2014). That is to say, it would be important to recognize that a teacher's understanding of scholarly knowledge of a scientific concept could influence the knowledge to be taught in their classroom, despite the effect that the noosphere might have on that knowledge. Similarly, it should be expected to see that learnt knowledge (i.e. how students perform on academic examinations) would heavily influence a teacher's decision about what knowledge is taught and how that knowledge is being taught to students.

To examine this question, I recommend an empirical study in which several case studies of teachers who have been identified as holding constructivist theories of learning and who successfully implement inquiry-based instruction in their classroom are examined using a didactic transposition theoretical framework. Analysis would focus on an examination of the ways in which information is transposed as it teachers identify scientific concepts from the noosphere to the knowledge to be taught by the teachers. For the purposes of this study, the noosphere would represent the national and state level science standards, the Common Core State Standards, and individual school teaching policy with regard to the science curriculum. What types of information are being selected by the teachers? Around which content are teachers choosing to implement inquiry-based practices? How are teachers identifying and explaining the decision making process as they transpose information? Implications can be drawn between learning theory, academic research on pedagogical practices, and didactic transposition in science education. Additionally, recommendations may be made about teacher preparation practices in order to better prepare teachers to engage in thoughtful transposition as they reconcile constructivist learning theory with inquiry-based pedagogical practices in the science classroom.

9

Supporting teachers' instrumental genesis

Miguel Perez

Linnaeus University, Sweden

Abstract. In this paper we unpack some of the critical issues when supporting teachers in the appropriate and effective use of the affordances provided by ICT in mathematics classrooms. Our findings suggest that the integration of ICT may need to be accompanied by resources that could aid the teacher in the process of transforming ICT into effective instruments for teaching and learning.

Introduction

The difficulties associated with the integration of ICT in teachers everyday practice is still an unresolved issue (Cuban et al. 2001, Hew & Brush 2007). Even if the affordances of well-designed ICT could meaningfully and beneficially transform instruction, this is far from automatic, and a significant teacher professional development investment must be made (Trouche & Drijvers 2014). But a significant problem is that we don't know what teachers learn from professional development or how it actually changes their pedagogy, we just know what they think about professional development (Lawless & Pellegrino 2007).

This paper contributes to research by unpacking some of the critical issues when supporting teachers in the appropriate and effective use of the affordances provided by ICT in mathematics classrooms.

Theoretical perspective

In this section we present an outline of the theories that we have used in this work. A key concept is the notion of praxeologies (Chevallard 2007) and the notion of instrumental genesis (Trouche 2004).

Praxeologies

The Anthropological Theory of Didactic (ATD) proposes a conceptualization of a body of knowledge, a praxeology, as consisting of two inseparable blocks, the *praxis* and the *logos*. The praxis block refers to the kind of given tasks that you aim to study and the different techniques used to face these problematic tasks. The logos block provides a discourse that is structured in two levels with the purpose to justify the praxis. The first level of the logos is technology, which provides a discourse about the technique. A technological element around a type of tasks could appear as a description of the different techniques associated with the task or as a description of properties of the task itself. The second level of the logos is theory, which provides a more general discourse that serves as explanation and justification of the technology itself (Chevallard 2007).

The notion of praxeologies can be applied to any form of human activity. Including teacher's practice in term of didactic praxeologies (Barbé et al. 2006). In this sense, teaching is a didactic type of task that teachers can solve by using a set of available resources, both external resources (curriculum, textbooks, tests, ICT-tools, colleagues, manipulatives, etc.) and teachers' internal resources. Furthermore, the teachers didactical praxeologies do not only condition and constrain the mathematical praxeologies that are made available in the classroom. They may also be a constraining factor for teachers enacting new pedagogical approaches and practices. Existing didactical praxeologies are likely to influence how teachers implement different ICT tools in their classrooms as teachers may be reluctant to alter or deviate from well-established teaching routines (Drijvers et al. 2010, Guskey 2002). Furthermore, the teachers didactical techniques may not necessarily be applicable in ICT supported environments in a straightforward way. Thus, a successful integration of ICT might therefore require the teachers to enhance their techniques in order to make use of the full potential of ICT to support students in the learning of mathematics (Drijvers et al. 2010). The instrumental approach described in the next section provides a framework to address this complex issue.

The instrumental genesis

A tenet in the instrumental approach is the fundamental distinction between an artifact and an instrument. The artifact refers to the object, material or abstract, that is used as a tool. The instrument refers both to the object and to the techniques and the thinking that the user develops while using the tool in order to make the tool a functional extension of the body. The instrument is build from the artifact in a complex process, called instrumental genesis that is dependent on both the users praxeologies and the properties of the tool (Trouche 2004).

Although difficult to separate, the instrumental genesis can be considered as a bidirectional movement targeting both the user and the artifact. Instrumentation is the movement where the artifact itself modifies the behavior by allowing different forms of activities afforded by the tool. The other movement, called instrumentalization, is a process of differentiation directed towards the artifacts themselves where different users may develop different understandings on how to use the tool (Trouche 2004).

Furthermore, within the process of instrumental genesis the notion of instrumental orchestration is used to describe how teachers use different strategies to regulate the students' instrumental genesis. A triplet that is partly created a priori and partly created on the actual moment of teaching defines the instrumental orchestration: a didactical configuration, an exploitation mode and a didactical performance. The didactical configuration concerns the settings and the artifacts involved in the teaching situation. The exploitation mode concerns the way the teachers decides to exploit a didactical configuration for the benefits of his or hers didactical intentions (Trouche 2004). Finally, the didactical performance involves interactive decisions while teaching, e.g. what questions to pose and how to respond to students answers, how to deal with unexpected aspects of the mathematical task, of the tool or to other emerging goals (Drijvers et al. 2010).

Rationale and research question

The purpose of this case study is to deepen our understanding about the conditions and constraints related to the teachers' instrumental genesis. This is an important aspect that has received less attention compared to the students' instrumental genesis (Kieran & Drijvers 2006). The notion of instrumental orchestration will be used to describe how we have staged a certain orchestration with a particular didactical configuration in order to support the teachers' instrumental genesis and the development of instrumented didactical techniques. An important characteristic of this design is the decision to include both ICT and theoretical concepts in the teachers' instrumental genesis.

Furthermore, the affordances of a tool are not just the physical properties of the tool but what behavior it offers to the user of the tool in the current environment - the *praxis*. Thus, the affordances are relative to the user and constrained by the environment. In other words, when we speak about the teachers' instrumental genesis we refer to the affordances provided by the tool and not on the tool itself. The research question is the following:

RQ: How can the teachers' instrumental genesis be supported in order to transform ICT into effective instruments for teaching and learning mathematics?

Didactical configuration

The setting

This work involved collaboration with three experienced mathematics teachers from a lower secondary school in Sweden. The school administration invited a researcher to take part of an already ongoing project at the school. The aim of this project was to increase students' motivation to engage with challenging and creative mathematical activities by targeting students' self-efficacy. With respect to the overall character of the already ongoing project, problem solving supported by ICT was suggested as learning objectives.

After an initial meeting, the school administration selected three voluntary teachers to participate and to form a team together with a researcher. Our part in this project was originally planned to last for one semester and to encompass approximately 30 hours of physical meeting with the teachers. This is still an ongoing project.

The structure of the teachers professional development

A central part of the configuration is the iterative work involving the teachers similar to other approaches related to professional development, e.g. the lesson study approach. The purpose was to engage the teachers in activities addressing both *praxis* and *logos* near the teachers' own classrooms. This approach was based on the models for teacher change proposed by Guskey (2002) and Clarke & Hollingsworth (2002).

According to Guskey (2002), teacher change is primarily an experiential learning process that begins when there is evidence of changes in classrooms. Therefore, in order to establish any sustainable change in the teachers' didactical praxeology there has to be evidence of classroom change and changes in student learning outcomes prior to teacher's change. Adopting this perspective implies that the design should focus on creating improvements in the classroom by e.g. introducing a new instructional approach, new material or curricula or by modifying the teachers' didactical techniques. A challenge is that teachers do not easily abandon teaching routines that they have developed in the demanding environment of their own classroom (Drijvers et al. 2010, Guskey 2002). Therefore, in order to become committed to the new practices, it is essential to give teachers opportunities to enact new practices in their own classrooms (Guskey 2002). This challenges the researcher to engage the teachers in meaningful discussions how new tools and new pedagogical approaches will be used and also provide teachers with routines and didactical resources that support this new practice (Garet et al. 2001, Gellert 2008, Guskey 2002).

In addition, the "interconnected model for teacher growth" (Clarke & Hollingsworth 2002) was used for designing feedback to the teachers. This empirical founded model proposes that teacher change occurs in four interconnected domains analogous to the domains proposed by Guskey (2002):

- Personal (changes in teachers attitudes)
- Practice (classroom experimentation, e.g. using ICT)
- Consequence (e.g. salient outcomes such as student engagement)
- External (input, e.g. workshops, lectures)

Two mediating processes, enactment and reflection, models the interconnection between different domains located within as well as outside the teacher's personal and professional world of practice. By supporting the mediating processes, change in one domain may be translated into change in another domain (Clarke & Hollingsworth 2002). The didactical configuration includes elements (e.g. lectures, workshops, classroom experimentation, resources, etc.) that address all of these domains. Therefore, the intention was to support the mediating processes by giving feedback on the basis of all four domains. In other words, while Guskey's model provided an outline and a sense of what to focus on, the model proposed by Clarke & Hollingsworth (2002) was used as a complementary model to design feedback to support the teachers' professional development.

Before continuing with the exploitation of the configuration, two additional elements in the didactical configuration need to be presented. As mentioned previously, there were two parallel and interrelated processes of instrumental genesis. One process relates to the instrumental genesis of a pedagogical tool, i.e. the notion of high-level and low-level evaluation (described below). The other process is related to the instrumental genesis of the affordances of the dynamic geometry software GeoGebra (www.geogebra.org), of which the teachers had no previous experience in using. The pedagogical tool was used by the researcher together with the teachers to monitor and guide the teachers' instrumental genesis of GeoGebra through the enactment of a theoretically underpinned lesson in problem solving. The following presentation will focus on the pedagogical tool and the lesson in problem solving as the other elements in the didactical configuration were imbedded in these elements.

The pedagogical tool: High-level and low-level evaluation

Teaching requires didactical configurations that involve communication and the way teachers exploit these configuration can be very decisive. Practices that give students the opportunity to e.g. interpret, generalize, justify, prove their ideas or participate in other forms of mathematical argumentation can greatly enhance the development of their mathematical thinking (Kieran & Drijvers 2006, Walshaw & Anthony 2008).

A typical pattern for teacher initiated communication is a three-part pattern commonly known as the IRE sequence (Initiate, Reply, Evaluate) where the teacher asks a question, the students reply, and the teacher evaluates or gives feedback (Leinhardt et al. 1987, Mehan 1979, Shavelson & Stern 1981). In its most basic form the teacher initiates the sequence by posing a question to a student to which the teacher already knows the answer. The student then replies and the teacher evaluates by using phrases such as "that's fine" and continues with the next question or next problem. If the student response is not what the teacher anticipated the teacher may ignore and continue by calling on other students until the teacher receives the correct answer (Leinhardt et al. 1987). Simplifying the complexity of a question is another form of evaluation that sometimes is driven to the extreme where the content disappears completely and students no longer are expected to provide any answers other than repeating already available information (Brousseau 1997, Mehan 1979). In these cases the teacher do not stimulate a discourse that could bring out students mathematical thinking regarding the task at hand. In other cases the teacher engage students in a more substantial exchange by e.g. posing follow-up question, pretending not to understand, asking for clarifications, asking for examples or by giving hints or providing other scaffolds based on students' replies (Chi 2009, Leinhardt et al. 1987). The IRE sequence could be attributed to a didactical technique and a characterization of the technique could be made based on the kind of behavior the technique affords student in the didactic process. For this purpose we differentiate between students being *passive*, *active*, *constructive* and *interactive* (Chi 2009)). Being *active* refers to overt activities such as students copying the solution from the board, manipulating or measuring and solving routine problems. Being *constructive* includes being *active* with the addition that the learner is producing some additional outputs that go beyond and are not explicitly presented in the learning materials. It includes mathematical valued activities such as self-explaining, posing problems, asking questions, providing justifications, formulating hypotheses, comparing and contrasting, reflecting, monitoring and other self-regulation activities (ibid.). It is not the activity per se that guarantee or defines a constructive activity but the nature of the produced outputs. A student asking shallow question by rephrasing available information is not being *constructive*. Instead, the student is merely being *active* by engaging with the learning material (Chi 2009).

When being *Interactive*, both partners have to make substantive contributions (being *constructive*) on the same topic or concept and also consider each other's contributions. This mode is defined by the produced outputs meaning that not all conversations between two persons are necessarily *interactive*. Furthermore, being *constructive* and *interactive* not only stimulates students to participate in the construction of a mathematical praxeology including a well developed *logos*, there is also evidence that being *interactive* is better than being *constructive*, and *constructive* is better than *active* in terms of students learning outcomes. All these modes are better than *passive* that focus on students receiving instruction for example by reading, listening or observing a teacher presentation (Chi 2009).

In the classroom, student can be *interactive* with peers or with the teacher. In the latter case the dialogue often proceed according to an extended IRE sequence where the teacher engage students in a substantial exchange. For our purposes, we define high-level evaluation as teacher actions within the IRE sequence that afford students to be *constructive* or *interactive*. Low-level evaluation only stimulates being *active* or *passive*.

The lesson in problem solving supported by GeoGebra

The lesson in problem solving supported by GeoGebra is organized in two sections, an introduction and a geometrical problem. In the introduction the students are presented with some ideas (heuristics), concepts and the mathematical content that the students need in order to solve the problem (i.e knowing how to compare the area of triangles). The introduction also serves as a scaffold for the students to formulate a mathematical question to explore in the second section. The geometrical problem is illustrated below (Fig. 9.1). Inside the larger rectangle, a point (denoted M) connects two smaller rectangles (blue and red) and by moving the point inside the larger rectangle the area of the two smaller rectangles changes dynamically.

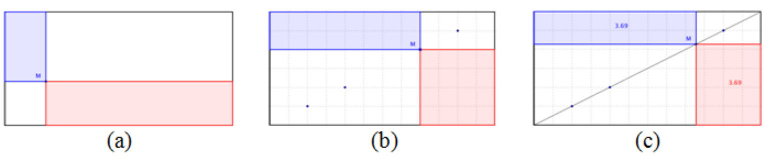


Fig. 9.1: Sequences from the geometrical problem implemented in GeoGebra.

The teacher introduces the students to the context of the geometrical problem by using the dynamical affordances of GeoGebra. Moving on to the next step, the purpose is to formulate a mathematical question to pursue. Instead of the teacher formulating the problem (students being *passive*) students are invited to formulate their own questions (being *constructive*). Students' responses are of course not always meaningful and relevant but by requesting several suggestions and by using built-in scaffolds in the lesson (from section one) and high-level evaluation students can be stimulated being *constructive/interactive*. If the intended question still is not formulated the teacher has no choice but to formulate the intended question or even to pursue another question. Thus, the teachers had the opportunity to deviate from the lesson plan if the situation in the classroom would call for it. The intended question is where to position the point so that the size of the two areas coincides.

Once the question is posed the teacher guides the student through the remaining part that consists basically of the following five steps: Making initial "guesses" (random or based on symmetrical properties); Refining the initial guesses and making new ones ("guessing" systematically in order to find a pattern); Finding a pattern and hypothesizing that there are an infinite number of solutions that could be represented by a straight line (diagonal); Controlling the hypothesis; Proving the hypothesis.

The dynamical affordances are used to support students' initial "guesses" (Fig.9.1a). A built-in grid in GeoGebra provides affordances for comparing areas by counting the number of squares within each rectangle. Thereby, additional solutions may be found (Fig.9.1b). Once a pattern is discovered and a hypothesis formulated, the computational affordances are used to control the hypothesis. In GeoGebra areas can easily be calculated and dynamically updated as the rectangles are manipulated (Fig.9.1c). To complete the evaluation process the hypothesis is proven by deductive reasoning (geometrically or/and algebraically).

The focus of the lesson is not necessarily on introducing new concepts but on students participating in a structured problem solving process and engaging them in mathematical activities such as posing questions, making guesses, hypothesizing, reasoning and proving (being *constructive/interactive*). Activities such as posing questions, planning and evaluation are not always considered by teachers as a mathematical task but what has to be done to organize the process of study (Rodríguez 2008). But, a gradual shift of responsibility from teacher to students is a key aspect in mathematics as many regulatory processes may be considered as a part of a mathematical activity (Rodríguez 2008). Furthermore, the use of specific strategies instead of unstructured discovery to solve problems is a capability that students can learn effectively in instruction to enhance their problem solving skills (Zimmerman & Campillo 2003).

The didactical exploitation and performance

The didactical exploitation is illustrated in Figure 9.2. The teachers were provided with different types of scaffolds such as lectures, workshops and also the lesson in problem solving supported by GeoGebra. First, the lesson was demonstrated to the teachers and afterwards there was a discussion about the purpose and the underlying theoretical components of the lesson. In this meeting the IRE sequence and notion of high-level and low-level evaluation was introduced to the teachers. Furthermore, they were provided with a manuscript of the lesson explaining the underlying theories and also a video from the demonstration.

The three teachers were engaged in an iterative process of enactment and reflection encompassing the teachers conducting the lesson in problem solving together with a follow-up discussion where they were provided with feedback based on these lessons. In the first iteration, we wanted the teachers to enact the lesson within our team (the teachers and the researcher) and in the following iterations with their own students. The teachers received immediate feedback after the first iteration but in the following iterations the feedback was provided at a separate occasion and based on the analysis of the video recordings from the enacted lessons. After each iteration the lesson was challenged in terms of usability and the teachers were invited to suggest any modification. This iterative approach enabled us to monitor the process of instrumental genesis and to adapt the didactical performance to the emerging needs.

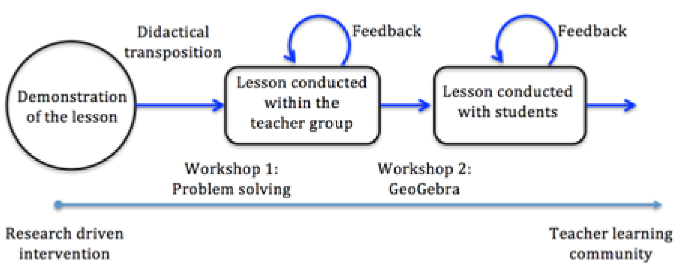


Fig. 9.2: Overview of the didactical exploitation.

The didactical exploitation could be understood as a collaborative regulatory process related to some aspects of the teachers' didactical praxeologies and the knowledge taught (didactical regulation). The didactical exploitation involved important opportunities for self-reflection. For example, the analysis of the videos was initially made by the researcher and presented to the teachers, but gradually the teachers assumed some responsibility for the analysis. In other words, even if the researcher initially assumed the regulatory responsibility there was a specific aim of gradually transferring agency in order to shift ownership (Coburn 2003, Penuel et al. 2007). The common language, supported by the pedagogical tool, allowed the researcher and the teachers to explore how the resources that we had introduced could be used to support institutionally valued forms of mathematics instruction. Especially in terms of improved learning gains by focusing on students' interactive-constructive processes.

Results and analysis

As the teachers became more accustomed to conducting the lesson we could see some progress related to the instrumental genesis of the two different but intervened artifacts. Initially the teachers found it difficult to separate between students being interested or motivated and students being *active*. Between students "not doing anything" and being *passive*. Another issue that was discussed was that the student's outputs had to be assessed in relation to the learning objectives in order to decide if the students were being *constructive* or not. We used video sequences from their own practices to address these issues and to study how different forms of evaluation affected their students' behavior.

The teachers had the freedom to deviate from the manuscript if necessary, which they did at several occasions. We noticed an increased use of high-level evaluation resulting in situations where less important issues were discussed at the expense of moving on to more essential parts of the lesson. In other situations, the teachers avoided all forms of explicit evaluation even when the students' responses were obviously incorrect. This indicated that the teachers interpreted the different forms of evaluation in terms of good and bad evaluation. The inputs that were provided to the teachers were reviewed but a decisive cause for this emerging interpretation was not found other than the possibility that the duality of the notion had induced this interpretation. Our iterative approach allowed us to discuss this issue with the teachers and a potential misunderstanding was avoided.

From the beginning the teachers had some problem in managing some of the features of GeoGebra but this became rapidly a minor issue for the teachers. The real struggle was the use of GeoGebra to engage the students in the process of problem solving without jumping directly to the concluding moments. This was especially critical in the second part of the lesson but the teachers tended to use the computational affordances to reach the solution of the problem instead of stimulating the students' mathematical thinking when searching for the solution. The computational affordances were intended to be used once a hypothesis was formulated but the teachers used them with the students to generate new solutions (step two: refining the initial guesses and making new ones) and value them either as correct or incorrect. The computational affordances were also used to find additional solutions in search of a pattern merely by tracking the numerical values calculated by GeoGebra. In other words, the computational affordances were used to provide low-level evaluation by in terms of "right" or "wrong" and by significantly simplifying the complexity of the task of searching for additional solutions.

This specific sequence was shown to the teacher and they were asked to give an analysis in terms of identifying the IRE sequence, analyzing different forms of evaluation and also the students' behavior. The teachers could, as a group, identify the sequence, which extended for several minutes. They were also able to recognize how the teacher used GeoGebra to perform the act of evaluation and to provide low-level evaluation in a situation where the students' interest and attention toward the task was relatively high. In this case, the level of student motivation was not confused with students being *constructive/interactive*.

The progress made by the teachers suggests that the instrumental genesis of the notion of high-level and low-level evaluation was successful to some extent. The teachers were able to use the pedagogical tool to analyze the implication of their use of GeoGebra in relation to how the students were engaged in the construction of the mathematical praxeology. In addition, the teachers have reported that the pedagogical tool has helped them to reflect on how they communicate with students in their everyday practice.

The pedagogical tool allowed the teachers to recognize the ineffective use of the computational affordances in terms of engaging students in constructive/interactive processes, but still we do not know to what extent the teachers have developed appropriate and effective didactical techniques associated with the GeoGebra. In this sense, we consider the instrumental genesis of the tool to be partial at the moment. On the other hand, the notion of high-level and low-level has increased the teachers' self-awareness and of how different actions affects students' possibilities to participate in effective and mathematically valuable activities. With this support, the teachers could now continue the commenced work of developing effective didactical techniques appropriate for ICT supported learning environments.

Conclusions

In this paper we have highlighted some important aspects of the teachers' instrumental genesis. As we have seen, teachers can use ICT to support any existing practice. Thus, we cannot expect that the integration of ICT itself will necessarily produce any improvements on teachers' practices. Teachers need time and support to be committed the new practices but we need to remember that this is a process of change that is difficult for the teachers. To support teachers in this process of change, we propose that designers of professional development should include pedagogical resources that could aid the teachers in their continuing mission to transform tools into effective instruments for teaching and learning.

10

Didactic transposition of mathematics and biology into a course for pre-service teachers: A case study of 'Health - Risk or Chance?'

Klaus Rasmussen

Metropolitan University College and the Department of Science Education, University of Copenhagen

Abstract. This paper represents work in progress on the didactic transposition resulting in an interdisciplinary course for pre-service teachers. It is shown how knowledge for teachers can be described using the anthropological theory of the didactic, and the first steps of an analysis of the course's genesis is presented. Firstly, the 'knowledge to be taught' chosen for interdisciplinary consideration is seen to be determined by the "noosphere" outside the teaching disciplines. Secondly, a more internal transposition takes over, attempting to supply, and create a meaningful bi-disciplinary connection. The resulting course description is presented, and furthermore an 'a priori' analysis of a Study and Research Path is briefly discussed as a means to satisfy many of the constraints put on 'taught knowledge'.

Introducing the problematic

Pre-service teachers are to acquire didactic knowledge that enables them to help pupils acquire mathematical and biological knowledge in lower secondary school. This is a well-known plight of prospective teachers if we regard only one discipline at a time, and those who educate teachers have for many years' expounded and enacted didactic knowledge that helped pre-service teachers do this. The challenge for those who *educate* teachers is thus on a meta-level; it is a nested challenge, which inherent intricacies they are not free to dismiss (Chevallard 1989). This challenge has in Denmark traditionally been meet by providing pre-service teachers with greater mastery of the single (scholarly) discipline, and, over the last 20 years, increasing amounts of pedagogic and didactic knowledge, have been integrated into the teacher education curriculum. However, in the last ten years, increasing demands have been put directly on teachers in secondary education, to combine different disciplines into coherent teaching which is interdisciplinary in some form or another (See fx Hansen & Winsløw (2011) regarding upper secondary eduation, and ; *Undervisningsministeriet* (2009*a*) regarding lower secondary). On the other hand, those who educate teachers have not been expected to do the same to any great extent. Therefore newly graduated teachers have largely been left on their own to do the interdisciplinary synthesis. Furthermore it is worth noting, that the majority of existing research is concerned with the direct design and implementation of interdisciplinary education, or the evaluation of interdisciplinary teaching compared to ordinary teaching (See e.g. Berlin & White (2010)). Very few consider the nested problem of interdisciplinarity in *teacher education*.

I begin by presenting the theoretical framework of the Anthropological Theory of the Didactic (ATD) to express the challenge of helping others acquire interdisciplinary teacher knowledge. And then I provide a case from a Danish project (named ASTE) developing a new teacher education program, where selected knowledge from "the teaching of mathematics" and "the teaching of biology" has undergone transposition into an interdisciplinary course under the heading 'Health - Risk or Chance?' The subsequent analysis considers two parts: a) the transposition process from existing national curricula to the specific course curriculum and course description. b)The design of a Study and Research Path (SRP) (Chevallard 2006) for pre-service teachers. Part a) is done "a posteori" while part b) provides an 'a priory' analysis of the SRP design, and a discussion regarding its appropriateness to instil, in pre-service teachers, didactic knowledge regarding the combined disciplines.

Framing the problematic in ATD

In ATD, knowledge is modelled using the notion of praxeological organisations (Chevallard 1999). Praxeologies are the combination doing and knowing. When presented with a task, humans can employ a technique to handle that task. Task and technique is called *praxis* and what reasoned discourse and theorizing can be done relating hereto is called *logos*. One cannot exist without the other, although either may be very simple or underdeveloped. When considering teacher knowledge (Huillet 2009), a distinguishing is made between knowledge to be taught and the knowledge to help others acquire that knowledge. The former I will abbreviate KO, where K stands for some “declared Knowledge” (cf. (Chevallard 1989, p. 8) and O for praxeological Organisation, and the later I abbreviate DO: “Didactic Organisation” The knowledge is usually declared in a form belonging to an established discipline (in this case mathematics or biology).

In an interdisciplinary course pre-service teachers are to engage in a number of lessons (very broadly understood), where some kind of disciplinary synthesis is apparent, and the intention is that the didactic organisation of the teacher educator (DO_E) will give rise to (inter)disciplinary and didactic organisations among the pre-service teachers (KO_{PS} <u>and</u> DO_{PS}). It is important to stress that KO’s and DO’s are intimately connected, and therefore DO do not refer to general pedagogical issues.

Thus is the challenge for the teacher educator: To build DO_E which helps pre-service teachers to build their KO_{PS} and DO_{PS} which again will help pupils build their own praxeologies regarding the involved disciplines (KO_P). The general research question then becomes: *How is knowledge (KO_{PS} and DO_{PS}) for an interdisciplinary course for pre-service teachers compiled, and what conditions do the characteristics of such a course put on DO_E?*

The following paragraphs pave the way for a reformulation into the case-specific research question, and each of the two parts of the analysis will be preceded by a short section detailing further theoretical tools needed to deal with the specificity of each.

The case context, data and methods for the two parts of the analysis

Teacher education in Denmark takes place at University Colleges (UC) where pre-service teachers study the “teaching-disciplines” of lower secondary school. It is an education directed specifically towards the profession as a teacher, and the courses can roughly be divided into school-discipline specific ones and general pedagogic ones. I will concern myself only with the school-subject specific ones, which feature an integrated study of the related “scholarly knowledge” (KO) and its didactics (DO).

To become a certified teacher of e.g. mathematics, the pre-service teacher needs to take four ‘math’-courses valued ten ECTS[1] each (Which together with other courses makes a total of 240 ECTS for the entire degree). The norm is for a pre-service teacher to study three *mono-disciplinary* school-disciplines, thus becoming a certified lower secondary teacher of e.g. mathematics, biology and history. The course under consideration in this paper is situated in the special program *ASTE*, which offers pre-service teachers to become certified in *four* school-disciplines: mathematics, physics/chemistry, biology and geography. The central idea is to make a recombination of elements from existing mono-disciplinary teacher education courses, utilizing synergy between the mentioned, very much related disciplines, and thus making “room” for the extra school-discipline certification. The recombination resulted in four courses named: “Energy and Climate” (covering elements from geography and physics/chemistry), “Sustainability” (biology and physics/chemistry), “Nature playing dice” (mathematics and physics/chemistry), and “Health - risk or chance?” (mathematics and biology). It can be gleamed from the course-names that those involving mathematics, has stochastic elements as focus, the reasons for which will become apparent during the analysis of the didactic transposition below. It should also be remarked that the math-perspective has been given a slight precedence over the bio-perspective due to limitations of space.

Data for part A, regarding the transposition of knowledge for the course selected as case for this paper, comes in the form of audio recordings of meetings, as well as documents, both external (e.g. curricular guidelines) and those produced by the participants internal to the development of the special educational program. Both recordings and documents have been inventoried and coded using the qualitative data analysis software NVivo. Excerpts offering insight into the transposition process have been coded according to its (disciplinary) position or lack thereof, and impact on the realized course curriculum and course description.

Data for part B, the design of the SRP, is a conglomeration of background reading of textbooks, research papers, web-based “information material” and discussions during a ph.d. course (*Mathematics and Science: The relationships and disconnections between research and education.*, 2014). Data for the design proposal is necessarily diffuse, as it seeks to describe paths possible and desired, intended to satisfy a number of design requirements. Not the paths actually taken by pre-service teachers.

[1] European Credit Transfer System

Part A: Analysing the didactic transposition resulting in "Health - Risk or Chance?"

In this part I answer the case-specific research question: *How is knowledge (KO_{PS} and DO_{PS}) from teacher education curriculum in mathematics and biology selected and combined to produce the course: 'Health - Risk or Chance?'*

The didactic transposition of knowledge can be divided into four steps:

Fig. 10.1: The didactic transposition process (Bosch & Gascón 2006)

Knowledge is created at some point, here called "scholarly knowledge" and then it is selected by the educational system to be taught to some students. In this process the knowledge changes. Although the process most certainly goes from left to right, there is a definite feedback indicated by the arrows going in the other direction. In this analysis I take a closer look at the 'knowledge to be taught', which to a large degree is defined within the educational system, or the institutions entrusted to provide education. Nevertheless society in a much larger sense has a say in what should be taught. The "noosphere" is all actors having a say regarding what to be taught. (Politicians, researchers, public and private interest groups etc.) Tracing the genesis of "Health - risk or chance?" it is seen that the transposition process inside the educational system can be divided into more detailed steps:

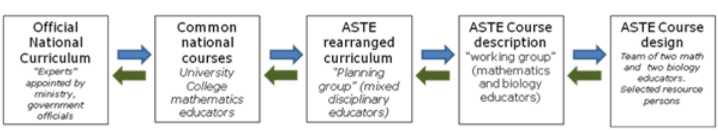


Fig. 10.2: Expanded 'Knowledge to be taught'

The Official National Curriculum (ONC) for teacher education describes mathematics using what is called, four "areas" of competence: *Topics, Ways of Working and Thinking, Math Didactics and Didactic Methods*. The ONC is permeated by the idea that you can describe curriculum in terms of "competencies" (Blomhøj & Jensen 2007). E.g. *Math Didactics* is described as: '*The scientific domain encompassing the study of actual mathematical teaching and learning, as well as development of a theoretical basis for math teaching.*' and it has the goal of enabling the pre-service teacher to "*describe, analyse and asses teaching and learning of mathematics with the support of didactic theory.*" (*LU13* 2013, Annex 2)

To further specify what knowledge these competences actually cover, the ONC lists around forty paired elements specifying "target knowledge" and associated "target skill". E.g. under *Math Didactics* it is stipulated that the pre-service teacher must know "*how math curricula changes with time and how it is related to societal and scientific challenges*." Associated hereto the pre-service teacher can skill wise "*relate to existing curricula for mathematics education in relation to mixed ability instruction*" (*LU13* 2013, Annex 2). Some readers may find the example pair somewhat non sequitur, but as all legal documents do, the ONC requires interpretation: What does it mean to 'relate' to curriculum?

It is noteworthy that the "competency" description does not stand alone. The specification into terms of 'knowledge and skills' render "competency"-description somewhat superfluous, and more significantly, the pairs of "knowledge and skills" lend themselves readily to praxeological analysis: The "target skill"-requirements all point to something the pre-service teacher has to be able to do, which is the same as solving a task using a technique (this is *praxis*). Likewise the "target knowledge" can be interpreted as a specification of the *logos* the pre-service teacher should have, making this an example of DO_{PS}. Elaborating the example it is evident that 'to assess whether an exercise from an old math-textbook is relevant for high ability pupils', is a didactic task and technique for the pre-service teacher, which can be informed by knowledge of curricular change. In terms of the presented ATD framework I argue that ONC declares teacher knowledge in the form of praxeological organisations (DO_{PS}) to be taught by teacher educators (using DO_{E}).

'Knowledge to be taught', as written in the ONC, is considered by an assembly of mathematics teacher educators[2] (step two in Figure 10.2),

[2] "Den Nationale Faggruppe"

one from the math department at each University College. At this step roughly half of the praxeologixal organisations are selected to be taught in two *Common national courses* for all math teacher education programs (A parallel process takes place regarding biological knowledge to be taught). The remaining praxeological organisations are considered *locally* by the math educators at each University College, and arranged into another two courses (This is a 'semi-step' not depicted in Figure 10.2). This is where the ASTE project enters into the transposition process: Principally ASTE could choose to make any *rearrangement* of the praxeological organisations left for local determination and combine them with appropriate ones from biology. (Appropriate in the sense of being well suited for synergy between the two teaching disciplines) This did not happen due to institutional organisational constraints: The ASTE courses could only replace one of the locally determined math courses; "Special needs pupils and mathematical aids" or "Evaluation and stochastic processes", because one of them had already been written into the local curriculum. (The above mentioned 'semi-step' of the transposition process) Which one it actually was, took a while for the ASTE-developers to figure out, as the local curriculum development process ran alongside the ASTE endeavour.

> "*Uhh, we have just solved it! ... Yes it fits, so it is stochastics which is the common topic and then some different competencies...*" (ASTE Curriculum Planning, May 2, 2013, Time index 2:09:5–2:10:21)

The developers had only briefly considered mixing stochastic and biological topics beforehand:

> "*The use of genetics in connection with biotech could be reserved for the bi-disciplinary course ... then you could also, if they [the pre-service teachers] had already had a little about genetics, develop the aspect of probability in mathematics ... also combinatorics, like with the colour of eyes*" (ASTE Curriculum Planning, May 2, 2013, Time index 1:56:40-1:57:18

Even though the mathematical praxeologies regarding "modelling", especially "functions", had been the favourite connection between math and biology, the development was now bound to combine statistical aspects of mathematics with the biological elements:

> "*It just has to be very special kinds of models, it should not be function-models, and it must have something to do with gathering*

> *data, that is, statistics or probability, right? At least if we have to cover "our" [praxeological organisations]...*" (ASTE Curriculum Planning, Mathematics Educator, May 2, 2013, Time index 2:53:35-2:54:00)

> '*'What kind of truth is it when we say carrots are healthy? What does it really mean? What is the data founded on? Here you can work with all the mathematical methods and models you like in order to understand. ... Uncle Sofus smoked cigars all his life and did not die, ergo tobacco is not lethal. It is this kind of problematics...*" (ASTE Curriculum Planning, Biology Educator, May 2, 2013, Time index 2:54:42-2:54:00-2:54:56)

This resulted in the following 'knowledge to be taught' to the pre-service teachers (Table 10.1) which can be viewed as very general, but none the less paired descriptions of the knowledge and praxis block of intended praxeologies.

Logos: The student has knowledge of:	**Praxis:** The student can:
science education, research in didactics of science	develop teaching in biology on the basis of science didactics
natural sciences use in social, technological and business contexts and didactic knowledge of the involvement of the outside world in education	include examples of the application of science and technology in society in the teaching of biology
relationships between structure and function at all levels of organization in living organisms, especially human anatomy and physiology and health concepts	plan, implement and evaluate lessons that promote students 'understanding of the relationship between structure and function of human anatomy and physiology, as well as allowing students to actively use the acquired knowledge and insight in relation to their own and others' health
recent research in the natural sciences	include examples of recent scientific research in education
Science educational resources such as textbooks, multimodal and web-based learning resources, science centers, outdoor spaces, businesses, museums, etc.	Using different teaching resources
Food and biotechnological production: biological basis for the opportunities and consequences for health, environment and nature, including knowledge of educational resources in this topic, such as science centres, companies and research institutions	Plan and conduct courses that make pupils able to explain biological processes in production and put them in perspective in relation to the ecological, developmental and social perspectives.
Statistics, systematic collection, description, analysis and evaluation of data, including IT and the use of descriptors of location, spread and correlation as well as statistical tests and their use.	*Analyze systematically collected data using statistical descriptors and diagrams. Use tests as a basis for the teaching of statistics including using ICT.*
mathematical problem solving	*handle problems by detecting, formulating, defining and solving mathematical problems by systematic choice of strategies and tools.*
mathematical representations.	*use mathematical modes of representation to understand, use, select and translate various forms of representation, including understanding their inter-relationships, strengths and weaknesses.*

Table 10.1: Biological and mathematical 'knowledge to be taught' ("math" indicated by italics)

Following the selection of praxeologies by the planning group, a "working group" of ASTE developers was formed (not identical to the planning group). This group produced through a series of meetings and document exchanges, a course description carrying over a great deal of the ideas hinted in the last citation above:

The module includes basic knowledge of the human body's physiology, anatomy and health, both individually and in the larger context. Working with risk assessment, distribution and prediction of health-issues include basic knowledge of statistics and statistical tests. The module is built around three themes:
Theme 1: Normality Concepts in relation to e.g. weight, height, fitness and longevity will be treated physiologically as well as mathematically. This will include statistical investigations with data collection, statistical descriptions, representations and processing, for example in relation to physiological training. IT will play a key role in these investigations, which basically will be problem-oriented.
Theme 2: Analysis of the spread of various diseases and health conditions, both in time (a historical angle) and space (north-south dimension). Including measures for population growth and efforts to limit growth. In relation to the analysis, statistical tests will be executed in order to assess the risk of e.g. development of certain diseases based on genetics
A historical analysis of biology books for the Danish school for an assessment of the presented health problems, health didactics and sex education. In relation to this is also the study of changing curricula, together with the current mathematics curricula in elementary school in order to see connections to social and scientific - especially health - challenges.
Theme 3: Selected human physiological themes

Table 10.2: "Health - risk or chance" course description

Concluding remarks on part A

This brief account of the transposition process show that the deciding factors for KO$_{PS}$ and DO$_{PS}$ selected for the course "Health - risk or chance?" resides primarily outside the ASTE project. Decisions taken at the first two steps of the expanded transposition process (depicted in figure 10.2), and in at another "semi-step" by local teacher educators, severely narrowed down the possible combinations. Regarding the last two transposition steps, it can be said that the presented course-description expresses the ASTE-working group's attempt at turning "left over" 'knowledge to be taught' into a meaningful course, and it still remains to be seen if the course designers will be able to make it meaningful for the pre-service teachers. The transposition process has thus put quite daunting conditions on DO$_E$ to be developed. It was not clearly perceived prospects of synergy, either from the perspective of mathematics education or biology education, which dictated the combination of statistics and health to be an interdisciplinary field in which teacher education could flourish. The teacher educators are thus required to 'break new ground', and in the following part I present the SRP design, which the educators will employ to, at least partially, handle this challenge.

Part B: Design a bi-diciplinary SRP

A Study and Research Path is didactic mechanism, or teaching proposal, where pupils set out to answer a grand question in a semi-autonomous fashion (See e.g. Chevallard (2009), Winsløw et al. (2013) for more details).

The central requirement for a SRP is the necessity for pre-service teachers to have enough means to start the study and deal with the initial grand question. The fundamental objective for the teacher educator is to secure that responsibility to answer the question is assumed, as well as responsibility for the majority of decisions of the study process. The SRP is initiated by a question with strong generating power, capable of imposing numerous derived questions leading to various elements of 'knowledge to be taught'. Instead of starting from "classic content" established in the strict framework of a discipline or a body of knowledge, the proposal consists in ideally "covering" curricula with a (or several sets of) SRP(s) without a specific connection to classic content. The study of these SRP's should cause the encounter with the intended knowledge, and other knowledge, and thus features a high degree of widening compared to the majority of study processes(Rodríguez et al. 2007). It is important to note that SRPs are in this manner "naturally" interdisciplinary.

When implementing a SRP, it provides a framework for DO_E and at the same time suggests a possible answer to the required interdisciplinarity. The central challenge is to design a generating question for the SRP which will spawn sub questions that will make pre-service teachers become knowledgeable about both KO_P and DO_{PS}, and in the process also develop their own KO_{PS}. The idea is that:

> *"Setting up a "scene" in the [...] classroom, with a crucial "opening question" in the beginning, may provide a rich field to initiate a dialogue and give the opportunity for knowledge conflicts and negotiation of meaning"*(Patronis & Spanos 2013, p. 1997).

Design requirements for the SRP dictates that it must cover a theme which is exemplary for the teaching of "the nature of science and mathematics" (Gericke 2014, pp. 7-8) and exemplary for the teaching of similar themes related to biology and mathematics. The SRP theme must fit into the overall descriptions and requirements of "Health – risk or chance?" and the theme must be clearly related to descriptions and requirements (curriculum) for the math and biology disciplines in lower secondary school. (That is, its relevance must be obvious to pre-service teachers). Furthermore the generating question must give rise to investigations of both mathematical and biological organisations, as well as their associated didactic praxeologies, as related to the teaching of children in lower secondary school.

Proposed design and the first steps of analysis

In this paragraph I only just present the generative question Q_0 (actually used in the course implementation) and its 'a-priori' derived questions revolving around the illness *diabetes*, which disrupts human blood sugar regulation. The result is given in the "tree-diagram"-form (Hansen & Winsløw 2011)

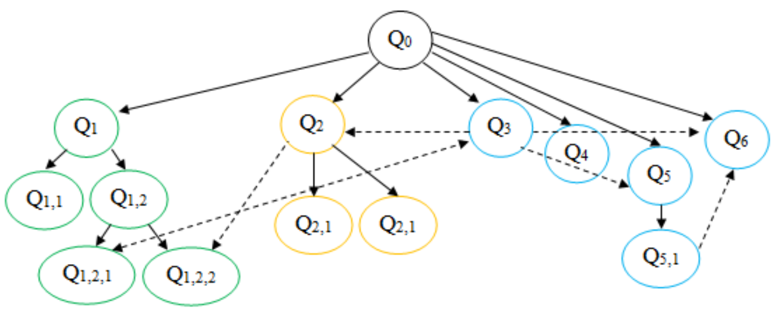


Fig. 10.3: Tree diagram showing the generating question and the first few sub questions. Green and yellow colour respectively indicates questions, and therefore disciplinary organisations, from biology and mathematics, whereas blue colour indicates questions to generate didactic organisations. Dashed arrows indicate that the receiving question is expected to draw on knowledge from the answers to originating questions.

Q_0 Why is diabetes a problem for school and society?
Q_1 What is diabetes?
Q_2 How to describe and investigate the distribution of diabetes?
Q_3 What is written about diabetes in school texts? (Curriculum, textbooks, etc.)
Q_4 How are known school lessons on diabetes? (e.g. in published lesson studies)

Q_5 Why choose the theme "diabetes"? Why should it be a concern for schools?
Q_6 What could be suitable *settings or scenarios* wherein to learn about diabetes?
$Q_{1,1}$ What is the consequences of diabetes?
$Q_{1,2}$ What is the cause(s) of diabetes?
$Q_{1,2,1}$ How has the causes of diabetes changed over time?
$Q_{1,2,2}$ How is possible causes identified?
$Q_{2,1}$ Who is afflicted with diabetes? (Where do they live (geographic distribution), how old are they age-distribution, socio-economic distribution...)
$Q_{2,2}$ How many gets diabetes? (Distribution in time)
$Q_{5,1}$ What do children know about diabetes? What are their experiences? Will diabetes be a motivating theme?

It is of cause central to elaborate why this diagram presents important and likely features of the SRP, and in particular why some questions are foreseen to make connections across the disciplines, but due to the limited space of this paper, I have to let the network of questions speak for itself, save for the considerations mentioned below

Considerations regarding SRPs as didactic organisation

The generative question is of paramount importance as it is likely to be the only part of the SRP that the educators have any direct control over. The chosen form is actually two questions in one, and point the pre-service teachers towards making a disciplinary investigation and a didactic investigation. In fact they are directed to consider the first two (or two and a half) steps in the didactic transposition process (Figure 10.1). They need to review the scholarly knowledge and the knowledge to be taught while thinking ahead to actual lessons. It is ideally necessary for pre-service teachers to concern themselves with all parts of the transposition process; in that respect pre-service teachers and teachers in general do not differ from researchers of didactic transposition processes. The (perhaps only) difference is that the former has to act out the final steps of the transposition; at some point have to perform actual teaching.

Suitably framed via the generating question, SRPs has the potential as part of DOE to force pre-service teachers into doing on their own, what they are indeed expected to do, on their own, as full members of the teacher profession. Teacher educators who seek to use SRPs, are obliged to perform an a priori analysis, using consecutive reformulations of the generative question, constantly evaluating the potential for realisations of 'knowledge to be taught'. This could also be the case for other forms of "project"-work in teacher education, but the inquiry-process of SRP's, which are based on questions, rather than works to be visited, is helpful in overcoming a too narrow "focus" on single parts of the transposition.

Concluding remarks

Part A of this paper showcases the peculiarities of didactic transposition which bring disciplines together in unexpected combinations. Not arbitrarily, but because forces in the 'noosphere' had priorities which left only certain elements for the course 'Health – risk or chance?' and a course description is possible to form. This part answers the research question while Part B touch upon the beginning of the internal didactic transposition of turning the course description into 'actually taught knowledge', for which a study and research path is presented as an element in the didactic praxeology of the teacher educator.

11

The new paradigm in teaching of stochastics methods

Radka Hájková

Department of Mathematics, Faculty of Education, University of South Bohemia, Czech Republic

Abstract. If we look into Czech curriculum from mathematics, we find one very interesting thing. It is completely lacking teaching combinatorics and theory of probability at lower secondary school. This is a completely different trend than in neighbouring countries. This article deals with the introduction of an experiment of teaching combinatorics and theory of probability in one 8th class of lower secondary school. There are described phase preceding the experiment – situation in the rest of OECD countries, especially in our Slavic neighbours, or history of teaching this part of mathematics in our country. It includes suggestions on further procedure and changes in the Czech curriculum. Of course, this opens up further questions and problem of integration combinatorics and theory of probability into the curriculum.

Introduction

Nowadays, the emphasis is usually placed on independent thinking of students, on their creative activity and scholarship. Problems are designed to be linked to real life (to be clear their applicability in practice).

This trend is evident in all subjects, is no exception in mathematics. But we focus on mathematics in Czech curriculum - Framework Education Programme. If we look at the educational content of Mathematics and its applications, we find that even here the trend is evident. As an example may be mentioned expected outcomes: students transfer simple real situations in math using variables; formulate and solve real-life situations using equations and their systems; analyze and solve simple problems, model specific situations in which they use mathematics tools in the whole and rational numbers and other outlets. (more in Jeřábek & Tupý (2007)).

It is certainly correct that there is a connection with the real world and mathematics, thus this subject is closer to students. Mathematics is an important part of everyone's life and everyone operates with it every day in everyday life and many of us need it at work (shop assistants, waiters, accountants, etc.). It is therefore very important that pupils in lower secondary school properly formed the basic mathematical concepts, which will be able to work and develop. One area of mathematics in our curriculum of basic education is lacking - combinatorics and theory of probability is not included in our Framework Educational Programme.

However this substance is for practical life very important. We often meet with the statement that something is "unlikely", to determine "how many options there", etc. We use this knowledge in evaluation surveys, but also for example in games, not only gambling. For pupils would therefore be beneficial to meet with these concepts and basics of probability and combinatorics at lower secondary school and formed at least a basic idea.

We can see we encounter with an unusual phenomenon in our country. The Czech curriculum does not include a content which is taught in the surrounding countries and in almost all other places. Why is it so? How can we explain it? This paper looks for answers for these and other questions.

A didactical transposition phenomen on teaching of combinatorics and theory of probability

As stated in Greer & Mukhopadhyay (2005) - Achieving greater understanding of probability in the population facing great resistance, or as noted Fischbein (1990) - we are afraid of probability.

About the importance of theory of probability Rubel writes: Inclusion probability and data analysis as one of the five NCTM standards is a tool that reflects the growing use of social data and the capabilities required to derive conclusions based on these data. Probabilistic reasoning is a key aspect for a wide range of professional activities and (in a broader sense) is part of everyday life. However, it is not only theory of probability, as mentioned Freudenthal (1973) - simple combinatorics is the backbone of basic probability, and our teaching should it be taken into account. Therefore, we cannot in any case combinatorics ignored. It is an integral part of the teaching of probability and should be prevented.

Finding solutions how to include teaching this part of mathematics in our curriculum is difficult. If we see a problem from a view of didactical transposition (Bosch & Gascón 2006), we can see a lot of work, which is necessary to the reintroduction of combinatorics and probability at lower secondary school. Didactical transposition is composed from four parts. The first one is Scholarly knowledge, which represents institutions producing, and using knowledge. Knowledge to be taught is the second part and it is based on educational system, “nooosphere”. The third is Taught knowledge, which takes place in a classroom. The last one is learned, available knowledge and is associated with community of study (you can see fig. 11.1)

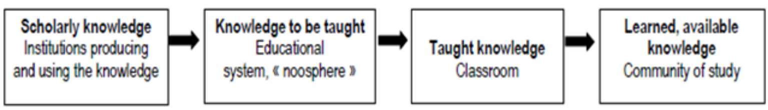


Fig. 11.1: Schema of didactical transposition process

Evolution of the knowledge to be taught

We can compare the results of the international research of TIMSS and PISA. Our country is usually comparable with rest of the OECD countries in mathematics. However one fact has attracted us - ‘It is interesting to take a look at how successful pupils from other countries were when solving this particular problems. If we compare our students’ level of success to the average of OECD countries the picture changes. All five problems the pupils had trouble to solve belong to the area of probability theory and statistics.’(Hejný et al. 2012).

And we can compare our curriculum with our neighbours too. The countries which are very similar linguistically and culturally, Slovakia and Poland, have this topic included. Combinatorics and probability theory in Slovakia are taught in the 6th, 7th and 8th grade in more or less 25 lessons (since 2003 when “Štátny vzelávací program” (*Štátny vzdelávací program* 2011) was settled, there is not a strict time schedule anymore). In Poland (Pazdo 2010), the assumed number of lessons is slightly lower but the content is nearly corresponding to the Slovakian. In both of these countries, pupils are introduced to these topics even earlier – the propaedeutic takes place latest during the 5th grade using dice or other games. If we take a look at other countries, the researches show that in 35 from 42 chosen developed countries this topic is taught during the school years that correspond to our lower secondary schools (Jones 2007). A similar conclusion made also Jelínek & *Šediv* (1982), who did a survey in 22 countries (Australia, Belgium, Canada, England, France, Finland, Hong Kong, Hungary, Ireland, Israel, Ivory Coast, Japan, Luxembourg, Netherlands, New Zealand, Nigeria, Scotland, Sweden, Spain, Thailand and the USA). They found that the theory of probability is taught in 20 countries and combinatorics in the 14 at lower secondary school.

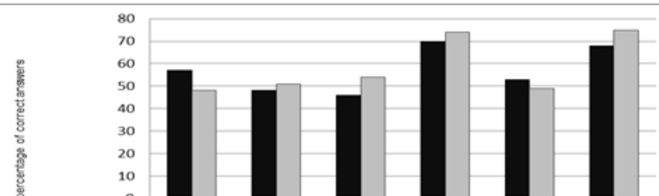
But on the other hand it is interesting that we taught combinatorics and theory if probability at lower secondary school 30 years ago (more in Mikulčák (2007)). The curriculum was changed in 1980 and our pupils learnt combinatorics in 6th grade and theory of probability in 7th grade of our school. There were major changes in the nineties - he political system has changed and the education system was loosened because of this. Directors in schools got more power in deciding which curriculum will be included and which omits and in 1996 Combinatorics and theory of probability disappeared from our curriculum again (*Zpráva o vývoji českého školství od listopadu 1989* 2009).

Today, it is difficult to decide what the main cause of the end of teaching this part of mathematics was. Maybe it was the change of the political regime. Another possibility is greater authority of school principals. Finally, it was a poor preparedness of teachers to teach it. All aspects are described in the aforementioned literatures. We can therefore assume that the combination of these factors caused a retreat from teaching of combinatorics and theory of probability. But the next question is why there is no change to these days? Why Czech textbooks contain this part only as a supplementary or hobby? Why are these examples classified in our standards as “non-standard”?

Learned, available knowledge

I started my experiment from the last part – Learned, available knowledge. Pupils’ knowledge usually come from everyday life. Pupils do not often meet combinatorics and probability at lower secondary school. However, I examined how the students are. The results can be seen in table one, details were published in Štěpánková & Tlustý (2014). The questions tested were selected from the international researches TIMSS and PISA (Frýzkoá et al. 2006, Tomá*šek* 2009, *Úlohy z matematiky a přírodních věd pro žáky 8. ročníku: třetí mezinárodní výzkum matematického a přírodního vzdělávání: replikace 1999* 2001) to make sure that the results can be compared to the national standards from a few years ago.

I analyzed probability and combinatory skills of 143 8th grade students. Pupils got 5 problems from combinatorics and theory of probability. In this test, we wonder if he results of our tests will be similar to the national results in the PISA and TIMSS researches.



	1.	2.a	2.b	3.	4.	5.
results of this experiment in %	57	48	46	70	53	68
resuls from our country in %	48	51	54	74	49	75

Table 11.1: Comparison of results of our experiment and national results in international researches

As Table 11.1 shows, the results are usually fractionally worse than the national average standard. The reason can be caused partially by the accumulation of combinatory and probability theory problems. Hence the test was quite difficult for the pupils and we do not consider the different results to be too important.

I therefore proceeded to the assessment of pupils' strategies, especially the estimation and guessing. As I expected, the most frequent method was calculus and schematic solving (see Fig.11.2). This type of problems is not usually taught so pupils are not familiar with any algorithm they might use. That is why their solutions are of inquiry based learning nature like drawing schemes, guessing and estimating.

2) Emil likes skateboarding. He went to the shop SKATER to learn about the prices. You can buy a readymade skateboard here. You can also buy a board, set of 4 wheels, set 2 suspensions and set of joining elements so that you can complete your own skateboard. These are the prices:

Equipment	Prices in Zeds
Skateboard	82 or 81
Board	40, 60 or 65
Set of 4 wheels	14 or 36
Suspension	16
Joining set (bearings, rubber layer, screws and nuts)	10 or 20

a) They have three types of boards, two types of wheels and two types of joints in the shop. There is only one type of suspension. How many different skateboards can Emil make?

A) 6 B) 8 C) 10 D) 12

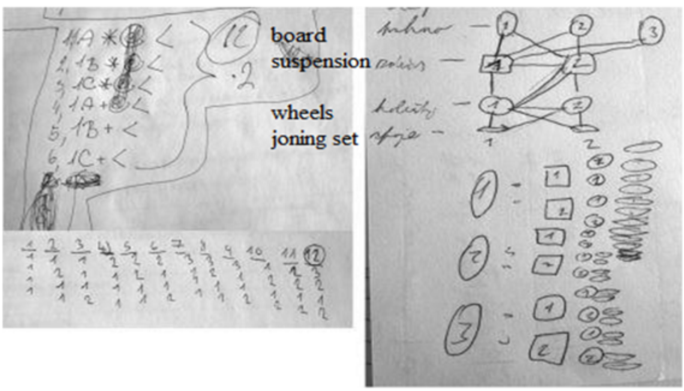


Fig. 11.2: Example of pupils' solutions

Results from my test confirmed me that would not be a problem pupils' understanding. They were able to solve some tasks without previous knowledge of school.

Change in the knowledge to be taught

Because of the results from my test I could precede to the next part. Next phase meant connection between Knowledge to be taught and Taught knowledge. The first step was the production of learning materials and finding schools where the teaching of combinatorics and probability could take place. So I created textbooks for pupils and a guidebook for teachers.

The textbook starts from the simple tasks of combinatorics on listing all possible phenomena and proceed to the selection of all favourable. The following part allows discovering rule of sum and rule of product. Last part of this chapter is topped charts and moving them and their applications in these tasks.

Marek went from Lhota to visit his friend Mirek in Lom. But he forgot the map and he hoped that he can find the way. Paths chose at random. Look at a map and tell how likely they hit the first time and came to a friend?

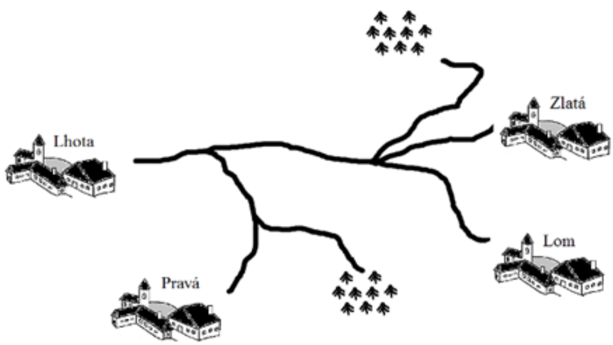


Fig. 11.3: Task of the role of probability

In these applications, therefore pupils have to analyze previous skills and at the same time find new rules. This chapter reveals to pupils law of large numbers and the classical definition of probability.

After consulting with several teachers, I managed to find a suitable class to try. It took place in the eighth grade for 7 lessons - the pupils wrote tests in two lessons and learned in five lessons. This time is quite sufficient, however, optimum time would be ten lessons, in the case of interest on more simulations longer time. I participated mainly as an observer, the lessons led mathematics teacher of this class. He got precise instructions and also had methodological guide before lessons.

Pupils received lessons generally positive; most of them actively engaged automatically, some pupils needed more time, but everybody worked in the final phase. The teacher confirmed that the class did not behave differently from other lessons and I, as unknown person, was not a disruptive element in the class.

The first three lessons we spent with combinatorics. Pupils got teaching materials, each task we read aloud and then everyone had some time on their own solutions (Fig.11.4). Then we started discussion and children who had correctly solved task, helped to find solutions to others. Thus, we worked with all the tasks, pupils got some time to find their own solution of problems which were solved in the final stage by using computers and simulations too. The first three lessons children studied combinatorics, in the remaining two we focused on probability calculus. Pupils wrote the second test after it and then I did semi-controlled interview with selected children.

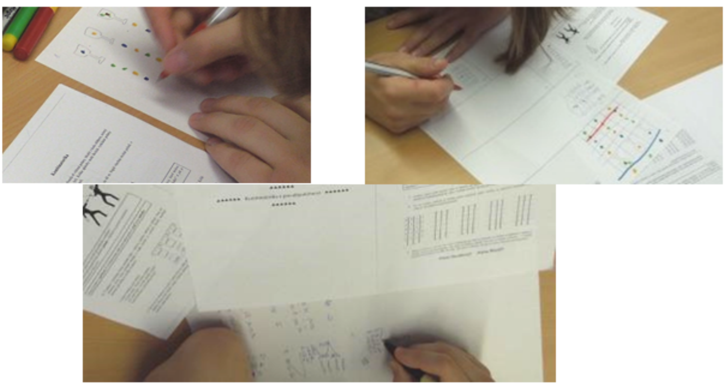


Fig. 11.4: Example of pupils‘ solutions

Nevertheless, the data from this pilot study have not yet been processed, I can already make a few notes I recorded during research and preliminary evaluation.

The reaction of pupils to teaching was generally positive. Very often repeated response was that this topic is “logical”, that “it was not mathematics.” Pupils further appreciated the support illustrative and schematic solutions and in independent work - “everything we could discover ourselves.”

Pupils, who has in mathematics generally low scores, could solve many tasks independently. The only advice we gave them was to use crayons, outline everything and write notes. Weakest pupils worked independently and found some solutions themselves and they often refer it as “fairly easy” and “logical”.

For pupils who achieve good results in mathematics, this substance was quite simple. Rules and patterns usually reveal a group of pupils, which is generally successful in mathematics. Some tasks, however, were problems for them too and one task we had to deal with our help significantly to make it for pupils manageable and understandable.

The group, which we would describe as average, worked alone for most tasks. They usually waited to help their classmates with more demanding tasks, which uncovered new rules.

This part is able to be considered as successful. Next step which waits me is a questionnaire survey with teachers of mathematics. I would like to know their opinion about:

- teaching combinatorics and probability at lower secondary school
- teaching without algorithm and formulas
- solving using schemes, charts, logical judgment etc.
- using simulations, technologies etc.

I hope all this things (tests, teaching and their result, opinion of teachers etc.) can help to include in our curriculum.

Scholarly knowledge of this part of math is incontestable as mentioned above. I think that the introduction of this topic will be beneficial for better understanding of this topic and hence to scholarly knowledge too.

Conclusion

Combinatorics and probability is usually not taught at lower secondary school in our country. However it is not a rule, it is possible to teach it and first textbooks contain several tasks there.

Another important step is that we have new standards from 2012 (in Fuchs (2012)), which specify the absolute minimum of what have to be achieved at the end of the 5th and 9th grade of basic school to know and learn. The thematic area “Non-standard application tasks and problems” has only two sample tasks and one is directly from combinatorics.

As mentioned (Bosch & Gascón 2006, p. 53), we have to overcome many obstacles. The transpositive work is done by a plurality of agents (the ‘noosphere’), including politicians, mathematicians (‘scholars’) and members of the teaching system (teachers in particular), and under historical and institutional conditions that are not always easy to discern. It makes teaching possible but it also imposes a lot of limitations on what can be and what cannot be done at school. I still hope that standards, textbooks and not least my work will lead to the reintroduction of this part of math to our schools.

12

Introducing proportion in arithmetics domain: The case of current and historical textbooks

Dyana Wijayanti

Department of Science Education, University of Copenhagen

Abstract. One of the ways to analyse textbook is to trace back how such a mathematical content was explicated in a current and historical textbooks. The purpose of this study is to compare the explanation of proportion in three different textbooks from two different periods. By using praxeology, I analyse a current open online US textbook and two historical US textbooks. I describe how proportion is explained and appeared in examples and in exercises. The results show that each textbook has its own typical type of task and technique. I also observe that the way proportion is explained in current textbooks draws on what we could call 'daily life situations' while the historical textbooks use a more formal definition. Furthermore, I find that the historical textbooks provide more algebraic approaches to discuss property of proportion while the current textbook gives advantage for students to have more than one techinques. The results of these analyses might be of interest in order to analyse future potential textbooks.

Keywords: arithmetics proportion, current textbook, historical textbooks

Introduction

Proportion is one of antique topics in mathematics. We can see how influential this topic because in old time this topic emerges in every mathematics textbook, from primary school to university. More recently, this topic has begun to be absent from the scientific mathematical world. For example, we can not find proportion terminology in the encyclopaedia of math (http://www.encyclopediaofmath.org). However, in some countries, such as the US, proportion is still discussed in lower secondary school.

Proportion is about relation between two couples of numbers or more. Mostly, the idea of proportion refers to Euclid's element in book 5 as a proportion that deals with magnitude and book 6 as a similar figure (see (Fitzpatrick 2008, p. 130 & 155). Recently, proportion also can be found in today's lower secondary school. National Governors Association Center for Best Practices & Council of Chief State School Officers (2010) states that proportion in the US lowers secondary textbooks is also appeared as ratio and proportional relationship topic that mainly use arithmetic task in grade 6 and as special linear equation form and similarity in grade 8. These discussions drive me to conclude that proportion can appear not only in arithmetic domain but also in geometry and algebra domain. Furthermore, I am curious by how proportion appear in textbooks.

Textbooks can be used by teachers to plan and to implement their teaching. Therefore, a good quality textbook should be developed to support teachers. One of the ways to understand what makes a good quality textbook is to analyse a previously used textbook, because a historical background of certain concept can be useful in understanding their present-day explication.

The purpose of this study is to compare the way proportion in arithmetic is introduced in two different periods of textbooks (a current textbook and two historical textbooks). By seeing how proportion is expressed at two different times, we can better analyse and understand the potential of future textbook content.

Theoretical approach

This comparative work is inspired by Clément (2007) who analysed the concept of a cell, as it is expressed in biology textbooks, becomes a potential didactical *obstacle*. This phenomenon is shown to have a pedagogical explanation, a historical explanation and a sociological explanation. In particular, Clément showed that the historical explanation gives a strong influence in current-day textbooks, even though it has lost its pertinence to today's knowledge. Thus, in this research, I also try to capture a comparative work between a current textbook and two historical textbooks.

Regarding the textbook analysis, I am inspired by the work by González-Martin et al. (2013) who carried out a textbook analysis for real number and found an un-integrated mathematical organization. However, in González-Martin et al. (2013) work's, they focus on general type of task in which means that this type of task can be interpreted in to more than one type of task . Therefore, it needs a research that focus on a specific type of task.

In this study, I use the anthropological theory of didactic (ATD) by Chevallard (1999, 2002). Based on this theory, knowledge cannot be considered as individual knowledge but rather it depends on selecting, designing, communicating, and learning knowledge in an institution. Therefore, there is a relationship between the scholarly institution which produces the knowledge and the institution of school which disseminates the knowledge. This relationship is governed by *didactic transposition* (Chevallard & Bosch 2014). In this research, I do not attempt to capture the *didactic transposition* of the topic of proportion, but I analyse the textbooks as part of the institution in which they are used as the 'knowledge to be taught'. This focus of analysis gives me opportunity to build a wider framework of proportion itself.

To build such explicit result in this study using ATD, I use *praxeology* as the main tool. Etymologically, *praxeology* refers to *practice* and *knowledge* and it consists of a four tuples: type of task, technique, technology and theory. When study mathematics, student often faces a type of task (T). This type of task can be solved by particular technique (τ). Also, a technologies (θ) s needed to explain the reason why a student choose the technique. Then a theory (Θ) is needed to justify the technology. The collection of type of task, technique, technology, and theory is called *mathematical praxeologies*. For further definitions, I refer the reader to (Winsløw 2011).

Context and methodology

According to common core state standard (CCSS) for mathematics in the US, ratio and proportional relationship is located in grade 6 (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010). However, I also realize that every state in the US does not have obligation to adopt that common core standard. Nevertheless, a report shows that forty five states and the District of Columbia have adopted the CCSS in mathematics (Achieve 2013).

Regarding to the textbooks regulation, a local school have an authority to approve the selection of each grade (U.S. Department of Education, International Affairs Office 2008). A current open online textbook from www.ck12.org is used for this research. CK-12 Foundation is a non-profit organization with a purpose to reduce the cost of textbook in the U.S. and worldwide. Specifically, I used a book by Mergerdichian et al. (2014) for grade 6 where ratio and proportion are located. I consider this online textbooks because it can be easily accessed by students (or might be for the readers) as a resource of the study. For a comparison, I use two the US historical textbooks by Adams (1848) and Hopkins & Underwood (1908) that can be found on https://archive.org/index.php.

I realize that only one current textbook and two historical textbook are chosen and these three textbooks do not represent a general condition of how arithmetic's proportion in the US textbooks. However, by using the case of these textbooks, I can apply a new approach in textbooks analysis using ATD.

The focus of this research is on type of task and technique that are provided by textbooks in the examples and exercises. I also realize that there is no given technique in the exercise. Thus, I consider how proportion is explained in the textbooks as a reference to adopt what technique that student might be used to solve the task. The main result of data analysis is a reference model that contains of a collection of type of task.

Result

Proportion can appear in many mathematical domains, e.g. arithmetic, algebra and probability. However, due to the scope of this study, I only focus on proportion in arithmetic domain. In the following discussion, I will discuss proportion theme in arithmetic domain in three parts: 1. How proportion is introduced historically by Adams (1848) and Hopkins & Underwood (1908), 2. How proportion is introduced currently by Mergerdichian et al. (2014), 3. How a reference model can be applied as textbooks analysis.

Proportion in historical mathematics textbooks

I consider discussing two historical textbooks from the year 1848 & 1908 in the same section because these two textbooks have the same structure: ratio, proportion, and compound proportion.

These two textbooks explain proportion using a formal definition: *proportionality is the equality of ratio*. The authors also gave an example using numbers such as '$6 : 8 = \frac{6}{8} = \frac{3}{4} = \frac{15}{20}$'. Thus, by this example, the authors conclude that '6 : 8' proportional to '15 : 20'.

Additionally, proportion can be generally expressed as a:b::c:d. It is interesting to note how the symbol of proportion is written using double colon (::) in Adams (1848). However, in Hopkins & Underwood (1908), proportion is already expressed with sign of equality (=) and they also explain that using double colon as a sign of equality is now rapidly becoming rare. These two textbooks also provide a formal algebraic explanation of cross product properties. Consider the cross product formal definition by Adams (1848) below:

> In every proportion the product of extremes is equal to the product of the means (Adams 1848, p. 61).

By this he means when we have then it is can be algebraically written, as $a \times d = b \times c$. a and d are the first and the fourth terms of proportion that is called the extremes and the d, b are second and third terms that is called the means. However, only Hopkins & Underwood (1908) who provide the proof properties of proportion algebraically: If $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{abd}{b} = \frac{cbd}{d} \Leftrightarrow ad = bc$. In this case, the authors consider a tuple as fraction and multiply each of these fractions by the product of their denominator.

Proportion in current mathematics textbooks

Proportion in the current textbook contains ratios, rate, proportion, percent, decimal and fraction. In this discussion I only focus on ratio and proportion. Different from historical textbooks, the authors provide a picture and a story using daily life situation to discuss proportion. Consider figure 12.1 and explanation by Mergerdichian et al. (2014) below:


8.8 Problem Solving Strategy: Use a Proportion

Introduction

The Frog Problem

Fig. 12.1: A discussion about proportion (Mergerdichian et al. 2014, p. 591).

> Tim loves to read about frogs.... . He amazed to read that frog can jump twenty times its body length. That means if a frog is three inches long, it can jump 60 inches. (Mergerdichian et al. 2014, p. 591).

Then, the authors formalise that proportion is a two equal ratio. They also use numbers to explain. However, in this current textbook, there is no discussion about how to prove property of proportion algebraically. Moreover, the authors do not discuss about means and extremes and do not use double colon as equal sign.

Reference model

We use *mathematical praxeologies* to describe the examples and the exercises in the textbooks. I categorize type of tasks and techniques to express the *praxis*. However, I do not use theory, due to limited source in lower secondary textbooks. In arithmetic, proportion is defined as the relation between two couples of numbers or more: if $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. There are six types of tasks ($T_3 - T_6$) belong to proportion from three textbooks that are categorized as follows (table 12.1).

	Types of tasks
T_1	Given (x_1,x_2) and y_1 find y_2 so that $(x_1,x_2)\sim(y_1,y_2)$.
T_2	Given (x_1,x_2) and y_1 find y_2 so that (x_1,x_2) & (y_2,y_1) are indirect proportion.
T_3	Given (x,y) determine r so that $(x,y)\sim(1,r)$.
T_4	Given x and r determine y so that $(x,y)\sim(1,r)$.
T_5	Given $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ calculate $\frac{x_n}{y_n}$ and decide which one is bigger.
T_6	Given (x_1,x_2), (y_1,y_2) and z_2 determine z_1 so that $((x_1,x_2),(y_2,y_1))\sim(z_1,z_2)$

Table 12.1: Types of tasks

In T_1, students are given a tuple (x_1,x_2) and one element (y_1) of a tuple and are asked to find remaining element of a tuple when these two tuples are similar. Later, I use similarity, proportion and direct proportion as the same meaning. While in T_2, students are faced with a modification of proportion that called indirect proportion. Here, students are asked to find the remaining element of a tuple by giving a tuple and one element of a tuple when these two tuples are indirect proportion.

In the 'ratio' type of task task (T_3), students are asked to find ratio by two given numbers. Algebraically, this type of task can be expressed by giving two variables (x,y) and determine the ratio (r) in order to prove (x,y) is similar to (l,r). Authors in T_3 give two numbers and students are asked to find the ratio of these numbers. While in T_4, students are given a ratio and a number and are asked to find a number so that these two numbers are in the given ratio. Often, a ratio task is called unit value task because students deal with a unit rate.

Students are given two tuples in type of task T_5 and are asked to decide which one is bigger. I acknowledge that this type of task is very far from definition of proportion that we have discussed before. However, I see this type of task as development of proportion application in daily life situation.

Compound ratio (T_6) at least contains three ratios. These are called compound ratio because it consists of several simple ratios to be multiplied together for a new ratio.

To illustrate these six types of tasks, I take examples from three textbooks as follows in the table 12.2. Additionally, I use a small 't' for task and I use capital 'T' to symbolise type of task.

	Examples of tasks
t_1	If 3 apples cost Rp. 4.500,00, how much money does 5 apples cost?
t_2	If 15 workers do a piece of work in 26 days, in how many days will 13 workers do it?
t_3	Laquita picked 12 peaches in 6 minutes, how many peaches were picked in one minutes by Laquita?
t_4	Mrs. Harris' class went apple picking. Each student picked eight apples. At this rate, how many apples were picked by seven students?
t_5	It takes Chase 15 minutes to stocks three shelves of canned goods, and it takes Marc 45 minutes to stock nine shelves. Who is faster?
t_6	If 24 men build a house in 18 day of 10 hours each, how many men will it take to build the same house in 30 days of 8 hours each?

Table 12.2: Examples of tasks

To solve those task in the table 12.2, students need to use a technique (τ) to solve (see table 12.3). Therefore, there are some tasks in the exercise that does not have the answer. Thus, I determine the techniques based on my interpretation from introduction section.

	Techniques
τ_1	If $\frac{x_1}{x_2}=\frac{y_1}{y_2}$ then $x_1.y_2=x_2.y_1$ so that $y_2=\frac{x_2.y_1}{x_1}$
τ_2	If $\frac{x_1}{x_2}=\frac{y_2}{y_1}$ then $x_1.y_1=x_2.y_2$ so that $y_2=\frac{x_1.y_1}{x_2}$
τ_3	$r=\frac{y}{x}$
τ_4	$y=rx$
τ_5	calculating and comparing $\frac{x_1}{y_1}<\frac{x_2}{y_2}$ or $\frac{x_1}{y_1}>\frac{x_2}{y_2}$
τ_6	$\left(\frac{x_1}{x_2}=\frac{y_2}{y_1}\right)=\frac{z_1}{z_2}\Leftrightarrow(x_1.y_1:x_2.y_2)=z_2:z_1\Leftrightarrow x_1.y_1.z_1=z_2.x_2.y_2\Leftrightarrow z_2=\frac{x_1.y_1.z_1}{x_2.y_2}$

Table 12.3: Examples of tasks

We also found in Mergerdichian et al. (2014) that the authors provide another technique for τ_1 called unit value technique 'τ_1':

τ_1: If $\frac{x_1}{x_2}=\frac{y_1}{y_2}$ then $\frac{x_1}{x_2}=\frac{1}{m}$ so that $y_2=my_1$ or $\frac{y_1}{x_1}=\frac{1}{m}$, so that $y_2=mx_2$.

Technique τ_1 gives students opportunity to use more than one technique to solve T_1. Students are asked to find the external ratio of some tuples and to multiply a given number of another tuple with this ratio. Besides using external ratios, student also can use internal ratios $(\frac{x_1}{x_2})$ to find remaining number by multiplying a given number of another tuple.

Discussion

All of textbooks have the same typical type of tasks (T_1 and T_3), namely the missing number and ratio. However, the technique in each textbook also has a different characteristic type of task. For example, current textbook has T_4 and T_5 and in the historical textbooks there are T_2 and T_6.

In the type of task T_2 (indirect proportion), students have opportunity to develop missing number technique, where it somehow has different concept from direct proportion. Student in the T_3 and T_4 can develop their idea about modification of proportion that use unit value or ratio. While in T_5 and T_6, students have to develop a new idea using proportion concept. By knowing these varieties of type of task, it can be enrich our reference to construct proportion task. Thus, students are challenged by having varied type of task.

Both current and historical textbooks define proportion using ratio equality. However, the two textbooks use different approaches. The historical textbooks directly define proportion with formal definition by using number and discuss specific terms like *means* and *extremes* which do not appear in current textbooks. Moreover, one of the historical textbooks use 'double colon' as sign of equality. Today, most of textbooks are using 'normal' sign equality (=) to express proportion. This situation shows the evolution of the using of mathematics symbol. In current textbooks, the authors use daily life situation to engage student and finish with formal definition.

lso, both current and historical textbooks discuss the property of proportion called cross product property. In historical textbooks, the authors also provide a mathematical proof of this property. However, authors in current textbook do not discuss about the proof. Students only apply the property of proportion. Moreover, in the current textbook, students more focus on using what the textbooks call 'ratio' (τ_1). In this situation students are asked to find unit rate and use it to find unknown number. Thus, students are given advantage by having more than one technique. The proof can support student to understand mathematically where the formula come from. However, by having more than one technique it helps students to see a task from more than one perspective.

Conclusion

The current and historical mathematics textbooks have the same kind of common types of tasks, but at the same time, they have a certain typical task that can only be found in each particular textbook. Also, we found that current textbooks have more techniques than historical textbooks.

Even though, both textbooks define proportion as equality ratio and discuss cross product properties, they have different approaches to explain. Daily life situation in the current textbook influences the number type of tasks and techniques. While formal definition approach in the historical textbooks influence mathematical understanding.

Different approaches to explain proportion and the properties of proportion and the variety of types of task can be used to develop future mathematical textbooks. Also, a general model of analysis for textbooks can be used to develop the analysis textbooks outside the theme of proportion or even outside the discipline of mathematics.

13

A challenge for physics continuous professional development in the UK based on didactic transposition theory

Gary Williams

Institute of Physics, UK

Abstract. The Institute of Physics Teacher Network is described and a direction for its future work presented along with ideas that have influenced this position. The aim of this paper is to propose a possible tool for more effective CPD provision and supported self-development, with two analytical frameworks combining to outline a possible driver for systemic change.

Background

Starting in 2001 the Institute of Physics (IOP) in the UK has funded and managed a network of coordinators, the IOP Teacher Network, who run continuing professional development (CPD) events for those teaching physics in secondary schools (*IOP Scotland* 2013). In the period 2009-2011, over 97% of teachers attending workshops run by the Teacher Network stated that the workshop would have a positive impact on their classroom practice (*IOP Scotland* 2013). From 2005 until 2009 the total number of teacher-days of CPD delivered by the Teacher Network was over 10,000. From March 2013 until December 2014 the total number of teacher- hours delivered by the Teacher Network was over 27,000. Currently 70% or more of state-funded secondary schools in England attended at least one IOP/Teacher Network event[1]).

[1] Monitoring of numbers attending workshops and their perceived quality is ongoing. Not yet published by IOP but available on request. Data for England has been gathered regularly due to funding arrangements.

The success of the model used by IOP, a centrally supported network of field workers who go into schools, has influenced other developments. The IOP identified a need for more targeted support during the early stages of the Teacher Network and developed the Stimulating Physics Network (SPN). The Perimeter Institute in Canada also set up a network of coordinators (Lambert 2010), as have the *Royal Academy of Engineering* (2014). Both organisations having initially interacted with the IOP. One of the aims of these networks is to help support and retain those teaching physics, as retention has been a problem for some time (*Centre for Education and Employment Research* 2008). This may not have been stated explicitly but it is an underlying principle of the model, a part of community building. It is bottom-up, grass roots, by teachers for teachers, as opposed to top-down.

There are problems with content being "delivered" via networks, with CPD workshops primarily being topic-based sets of practical activities, for example how to make a compressed air rocket launcher or use a Van de Graaff generator. They are very effective workshops, with many teachers taking up these ideas to make their lessons more inspiring, but they are a specific type of workshop and have limitations. (This is not a criticism of the networks but more a property of the whole; increasing demand is outstripping capacity, and some teachers justifiably want knowledge delivered). There is a lack of critical inspection and much centres on the passing on of standard ideas and "this is how I teach it", rather than looking at research evidence or evaluating impact or improving decision making. A possible reason for this is the belief by some that there is **one** best way of teaching a topic. This is not in agreement with the views of many of the Physics Network Coordinators (PNCs, the people running the Teacher Network) who tend more towards a craft interpretation of teaching (Winch 2008), in that the best method depends on the students, teacher and social context, prior knowledge and experience, and a host of other factors (Leach 2007). It is also not in agreement with the ideas developed by didactic transposition:

> "thus shattering the illusion of a unique mathematical knowledge already defined and for which the best teaching method was to be found" (Bosch & Gascón 2006).

These lines of investigation led to the area of pedagogical content knowledge (PCK). This had been looked at in the early days of the Teacher Network but seemed to be making little useful progress. The work of Loughran et al. (2012) has produced tools that help identify, document and develop

PCK, namely content representations (CoRes) and pedagogical and professional experience repertoires (PaP-eRs). CoRes are linked to PaP-eRs which are rich descriptions concerning an individual teachers approach, these often explain more about the reasoning the teacher followed to justify using a particular approach. These seem a useful device for sharing and developing teaching ideas and may have moved PCK on, at a level useful for teachers (Loughran et al. 2007*a*).

Brighouse (*Research Machines* 2006), quoting American researcher Judith Little, asserts that an outstanding school can be identified when four factors are visible. Teachers:

- talk about teaching,
- observe each other's practice,
- plan, organise, and evaluate their work together rather than separately,
- teach each other.

The first point covers something that happens and is encouraged at all Teacher Network events. The second is much more difficult to initiate due to the way that the Network operates. The final point is a cornerstone of the Teacher Network, which leaves the third point. The ideas outlined in this paper will hopefully address this point and introduce some systemic change, moving away from delivery and towards a more critical, analytical and reflective community.

Didactic transposition theory and cores

Didactic Transposition Theory (DTT) looks at the original intention of what should be taught and why and how it changes as institutions act upon it, see fig. 13.1 below (Bosch 2014).

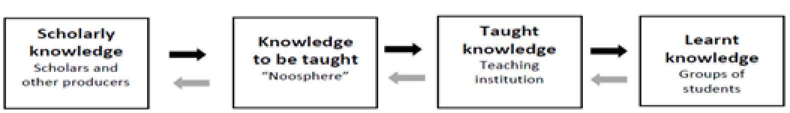


Fig. 13.1: Diagram of the process of didactic transposition (resource: Chevallard & Bosch (2014))

A CoRe is a table with column headings that list the teaching outcome in the sequence in which they are to be taught. Big Idea A being followed by Big Ideas B and C and so forth. The row headings relate to different aspects of teaching. The appealing thing about CoRes is their flexibility, simplicity, and their use as a tool for developing understanding as well as documenting and analysing practise. They do not define PCK but highlight important factors that are part of it.

The two approaches place emphasis in different areas. A CoRe places more emphasis on student difficulties. DTT places an equal emphasis on the initial transpositions. For practical reasons this may be an unfruitful research area in the UK. Despite these problems and differences DTT, and CoRes together with PaP-eRs seem useful tools following similar approaches. They do not appear to be at odds with each other.

It is informative to compare CoRes and PaP-eRs to didactic transposition theory. The first action needs to be the removal of the CoRe row heading “Other factors that influence your teaching of this idea” as this refers to practical issues, specific to a particular situation so of little use here (for instance whether the lab has sinks).

The CoRe row headings can be split into three areas: what you are doing, why you are doing it, and the implicit and explicit transpositions. There is also an added box – why evaluate in this way? (This wording needs further thought).

In the CoRe column heading we have the “Big Ideas” (which is a misleading label), the Scholarly Knowledge as DTT describes it. Teachers without a good understanding of this then need to initiate a subject specific study.

This CoRe is for (*topic and age range*)	**Big Idea A** (e.g., *electric current*)	**Big Idea B** etc→
What you intend students to learn about this idea		
Why it is important for students to know this		
What else you know about the idea (that you do not intend students to know yet)		
Difficulties/limitations connected with teaching this idea		
Knowledge about students' thinking which influences your teaching of this idea		
Other factors that influence your teaching of this idea		
Teaching procedures (and particular reasons for using these to engage with this idea)		
Specific ways of ascertaining students' understanding or confusion around this idea (include likely range of responses)		

Table 13.1: The beginnings of a content representation. (Resource: Loughran et al. (2007*b*))

There is then the first transposition. This is partly the curriculum level to which the subject knowledge has been allocated. Taking a part of physics as an example, the electromagnetic spectrum doesn't have a label on it that says "11-14 years". Hence putting it into a position in the curriculum is the first transposition.

The row heading "What you intend students to learn about this idea." has two possible sources. One source is the teachers interpretation (as teachers don't always stick to the curriculum and may by-pass it to deliver what they see as Scholarly Knowledge), which may be coloured by the prescribed curriculum or it may be an interpretation of Scholarly Knowledge by those in the "noosphere" (Chevallard 1989) producing a curriculum statement. Let us think of this simply as "what are we trying to teach?"

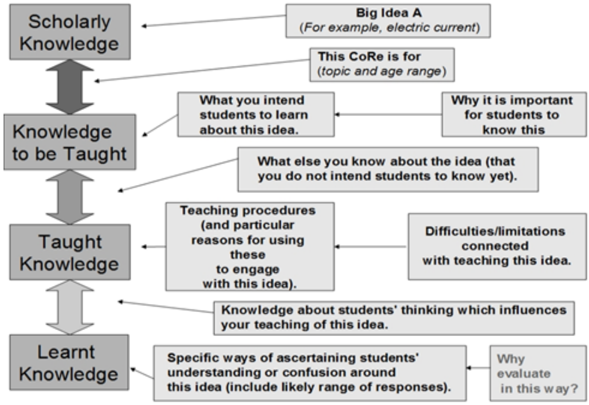


Fig. 13.2: Comparing two frameworks for analysis

For a new teacher the questions they often want answered are indeed "what am I trying to teach?" followed by "how do I teach it?", so we may link Teaching Procedures to Taught Knowledge. Fig. 13.2 suggests together with this the idea of evaluating progress is also important, be that summative or formative assessment.

Hence, a tool for directing CPD might start by making sure these areas are covered:

- What am I trying to teach?
- How do I teach it?
- How is success measured for both teachers and students?

These areas are shown in fig. 13.3, which we shall call *Praxis*[2].

Once teachers can answer these questions, they might be expected to start analysing their teaching and justifying decisions. Looking back at fig. 13.2 let us now examine the CoRe row headings that have been placed towards the right hand side. Figure 13.4 shows a more thoughtful set of processes – if fig. 13.3 is Praxis then fig. 13.4 is *Logos*. It covers more of the reasons why these actions were taken:

- Why are we trying to teach this?

[2] The terms Praxis, Logos and Transpositions are used here as labels in preference to terms like "stage 1" etc.

- Why are we teaching it this way?
- Why evaluate in this way?

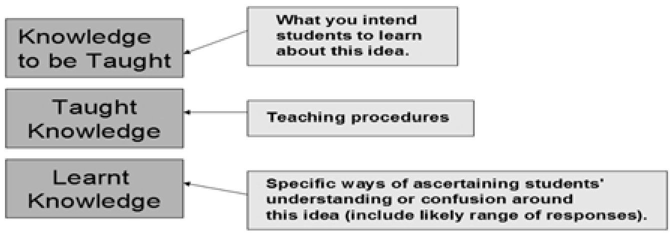


Fig. 13.3: A starting point, or "Praxis".

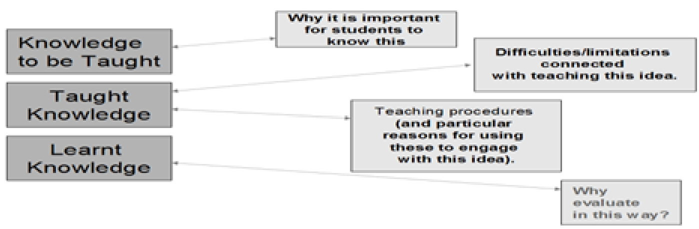


Fig. 13.4: "Logos".

Finally there are Transpositions, the remaining row headings from figure 13.2. This is the domain of the experienced teacher, those who see the flaws in the curriculum in comparison to Scholarly Knowledge and those aware of the bias they may have in their own teaching, see figure 13.5.

It could be argued that a certain row heading is incorrectly connected to the chain of didactic transpositions. Given the multifaceted nature of how teachers work, interact with students and rework the knowledge they hope students will come to understand, this isn't an issue that needs dealing with (Ogborn et al. 1996). What is being shown here is that two communities are arriving at similar conclusions. Their common conclusions can be used to direct progress.

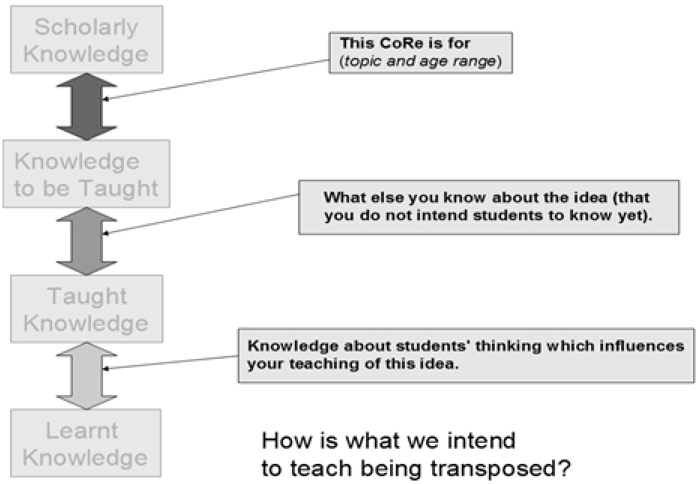


Fig. 13.5: "Transpositions".

What might this look like in practice?

The ideas described here have been presented to a sub-set of the Physics Network Coordinators. An attempt was made to put the "Praxis" step into practice, with the aid of a large piece of material and some A5 post-it notes, see fig. 13.6. PNCs built partial CoRes looking at the topics of particle physics and rockets – both of these topics are connected to workshops currently run by the Teacher Network. The material, which had a grid pattern drawn on it and the row headings from the "Praxis" stage written in, was laid on a table. PNCs then filled in these boxes by writing relevant information on the post-it notes and placing them appropriately. PNCs are experienced teachers, usually with a minimum of ten years of teaching behind them, so this happened quickly; their mastery obvious. The process of applying this analytical tool to PNC workshops raised several issues:

- A CoRe should be a dynamic tool. The point is not to complete a CoRe, but to catalyse a worthwhile discussion and improve decision making. The quality of the discussion while building the CoRe was at a high level and tightly focussed - it was all relevant to the classroom.

- Despite having different curricula (as PNCs from England and Scotland were present) a worthwhile discussion was still possible.
- As the Network Coordinators are all very experienced teachers the boxes for the "Praxis" stage were quickly filled in. The discussion then moved on to look at "Transpositions" with evidence about misconceptions and research occurring naturally in the normal flow of the conversation.
- The "Logos" stage was dealt with in the course of the discussions.

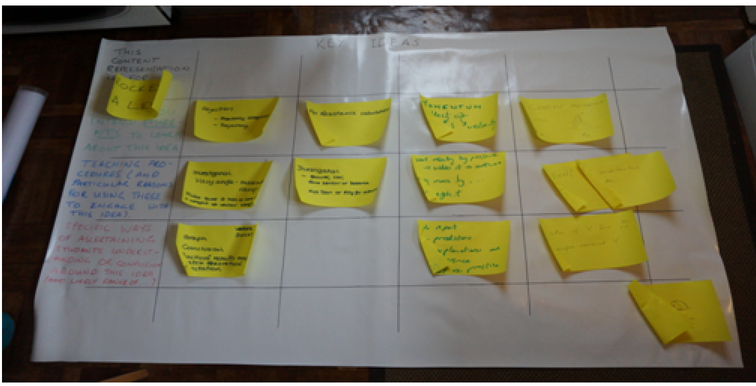


Fig. 13.6: A partial CoRe completed by the PNCs.

One outcome of the discussion during this small trial was a consensus about the direction the Network should pursue in future. A number of skeleton CoRes (complete frames but only partly filled in with suggested discussion prompts that relate to teaching) will be prepared and workshop participants will use these to build partial CoRes (incomplete frames being fully filled in if possible). The theory behind the idea and its use as a more powerful analytical tool will be made clear, but within the workshops it will be used to complement the usual topic based approach. It will hopefully be taken up in schools as it becomes more widely used and understood, the value of identifying problem areas becoming evident.

Systemic change and further work

While the IOP Teacher Network and SPN have been very successful, they could be more effective in improving teaching practice in schools. There are a number of practical requirements for what would constitute a systemic change:

- It must involve working with groups of teachers, or a department in a school.
- The workload cannot be the sole responsibility of the teachers.
- It must be flexible enough to work with outside support but not rely on it, and work across the curricula of different nations.
- It should continue after an initial set-up period.
- It must produce measurable results.

The proposal is to take some of the ideas from CoRes and PaP-eRs and DTT and use these to develop tools to help a department develop its own CPD program. Teachers would be empowered to assess their own needs and helped to find resources to improve teaching or subject knowledge where possible. The beauty of DTT when compared to CoRes is that it allows for the possibility of there not being a solution to a teaching problem at a school level. Teachers need not feel like they have failed when they are unable to solve a problem.

Step 1: The intention is to try and add a CoRe/DTT based tool into workshops already run by the Teacher Network. This will start to familiarise the physics teaching community with the idea. Teachers will also be pointed in the direction of material to allow them to take this further.

Step 2: A small pilot project will be run using a CoRe/DTT based tool to guide departments in their CPD provision. This will identify a curriculum area and age range and go through the praxis, logos and transposition stages, identifying areas of need and problems, and finding solutions, if necessary in conjunction with their local PNCs. Feedback will involve the use of Talkphysics - an online community for teachers of Physics and their supporters at www.talkphysics.org.

Step 3: Having provided a tool for in depth analysis of a scheme of work, for example, teachers also need support and nurturing should they want to take this further. One possibility could be summer schools where experienced researchers support teachers in doing this with the whole serving as a gateway through to broader research. The Teacher Network should explore this option with universities.

Conclusion

CoRes and PaP-eRs and DTT have been compared and used to develop ideas for a tool for self-direction of CPD, and more effective CPD provision. A proposal has been made to take this forward in the context of the IOP Teacher Network.

14

An ATD-model of theory - Practice relations in mathematics teacher education

Kaj Østergaard

Roskilde University and VIA University College, Denmark

Abstract. The paper presents and discusses an ATD based model of theory-practice relations in mathematics teacher education. The notions of didactic transposition and praxeology are combined and concretized in order to form a comprehensive model for analyzing the theory-practice problematique. It has been shown that the model can be used both as a descriptive tool to analyse interactions between, and interviews with, student teachers and teachers and as a normative tool to design and redesign learning environments in teacher education, in this case a lesson study context.

The theory-practic problematique

Establishing coherence between theory and practice is one of the main challenges in mathematics teacher education. In Denmark more than four out of ten student teachers experience a lack of coherence between the teaching of general educational science and didactics taking place at the university college and the practice of teaching in schools (Jensen et al. 2008). Throughout the last decades teacher education has become increasingly academic - which can be seen as positive – but concurrently, the practices at schools have become much more challenging due to increasing social and ethnic segregation, which affect schools particularly in disadvantaged neighborhoods. Therefore, many student teachers tend to focus on practical teaching tools rather than academic theories. This development causes a risk of a widening of the gap between theory and practice in teacher education.

The theory-practice divide can be regarded (1) from theory to practice or (2) from practice to theory. As regards (1) the questions are: How can theoretical knowledge be utilized to analyze and develop teaching practice in schools and how do we create a shared frame of reference from teaching practice to interpret the theory? Subject matter knowledge, pedagogical knowledge and pedagogical content knowledge are taught separately at university colleges but are in reality inextricably entwined with each other. The challenge is how to create interplay between the academic theories of mathematics and pedagogy and teaching practice in teacher education. It is crucial to create this interplay in order to legitimize the theoretical education and to place school knowledge in a wider context.

As regards (2) the teaching practice must be made transparent and treated as the main object of discussion and theorization in teacher education. This is necessary in order to ensure that student teachers' learning in and from teaching practice is connected to theoretical education and brings about a critical view on theory and research from a practical point of view.

These complex theory-practice relationships in teacher education call for a model which can be used to describe and analyze the interplay between mathematical and didactical knowledge; teaching practice and learning in both teacher education and mathematics teaching in school.

The aim of the research project behind this paper is to answer the following two research questions:

1) What different kinds of theory-practice problems appear in mathematics teacher education - according to student teachers?
2) How can these theory-practice problems be conceptualized and analyzed within an ATD based model?

In this paper the focus is on how the model can be used as a tool for analyzing empirical data from a lesson study project with teachers and student teachers. However, at first, the model will be presented and discussed. The paper is rounded off with a discussion of the benefit of the model in analyzing theory-practice relations in mathematics teacher education and on how such analyses can inform the design and use of lesson studies in teacher education.

A model of mathematical teacher education

The Anthropological Theory of the Didactic (Chevallard 2012) provides an epistemological framework for mathematical knowledge. In ATD mathematical knowledge, regarded as a human activity including teaching and learning mathematics, is modelled by mathematical and didactical *praxeologies* (Winsløw & Madsen 2008).

Praxeologies consist of a practice block (*praxis*) regarding the questions *what to do* and *how to do it* (know-how) and a theory block (*logos*) regarding *why to do it* (know-why). In addition to this, ATD models *the didactical transposition* of mathematical knowledge from *scholarly mathematics* mainly evolving at universities to *knowledge actually taught* and learnt in schools (Bosch & Gascón 2006, p. 56). The didactical transposition is divided into two steps. The first step is the *external didactical transposition* from *scholar mathematics to knowledge meant to be taught* - the mathematical knowledge as it is described in e.g. curriculum and textbooks. This step is often performed by people outside the school. The second step is the *internal didactical transposition* from *knowledge meant to be taught to knowledge actually taught* – this step is every day work for teachers.

The two concepts, praxeology and didactic transposition, both bring central theory-practice relations into focus – the first one inside an institutional frame (e.g. the school) and the second in a broader context between institutions. Together they provide a comprehensive picture of teacher education in mathematics, which can be used to point out and analyze problems and constraints as the theory-practice problematique. In the model below the two concepts are combined to form a model for analyzing the theory-practice problematique in teacher education (figure 14.1). In my research the model is intended to be a tool for both descriptive and normative analyses. At first, the model is used descriptively to analyze different kinds of empirical data from two lesson study projects in connection with teacher education. On this basis, the model will be used normatively to propose new ways to organize teaching practice, preparatory education and the theoretical education at the university college to improve coherence between theory and practice in teacher education.

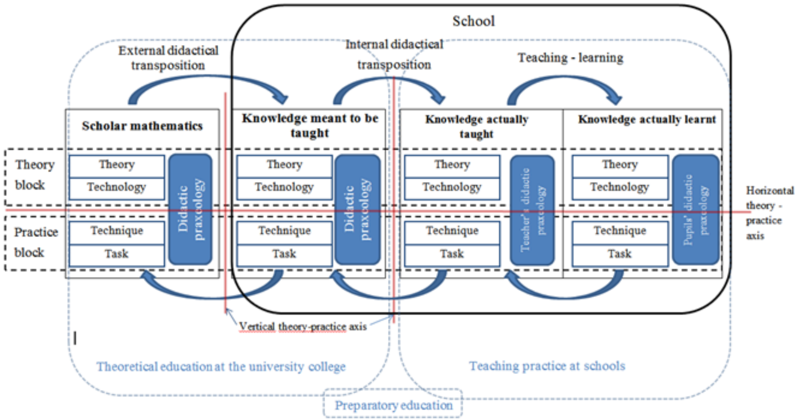


Fig. 14.1: Teacher education model based on ATD

he model consists of four columns containing the four kinds of knowledge in the didactic transposition. Each kind of knowledge is described by a mathematical praxeology with theory, technology, technique and task (see (Winsløw & Madsen 2008) for further details) depicted with white boxes and a didactic praxeology, also with a theory and practice block, depicted as “blue‘” boxes in figure 14.1. By collocating the model and teacher education practice three different, pivotal theory-practice problems can be located – occurring in different forms. These are emphasized by three red axes – two vertical and one horizontal axis.

The horizontal axis is dividing the practice blocks and the theory blocks. This axis stresses the divide between practical, procedural mathematics with emphasis on techniques to carry out tasks and theoretically *doing mathematics* by combining techniques and concepts, arguing, proving etc. The transcendence of this barrier is a crucial point for mathematical education – the higher level of abstraction in the theoretical block is a necessity but also a very difficult barrier to almost all pupils. Consequently, this axis is a significant problem area for teacher education both with regard to student teachers learning scholar mathematics and pupils learning mathematics at school and the relation between practice and theory block is an appropriate model in both cases.

The two vertical theory-practice axes are dividing, respectively, the scholar mathematics and knowledge meant to be taught and knowledge meant to be taught and knowledge actually taught. The divide in the first axis is treated at the university college. Comparison of scholar mathematics and knowledge meant to be taught is again highly relevant in teacher education to analyze what and why specific content is or is not selected for curriculum. It is pivotal for student teachers to be critical to this selection and to question the decisions in curriculum or textbooks. The arrows at the base of the model pointing "back" e.g. from knowledge meant to be taught to scholar mathematics stresses that knowledge meant to be taught or actually learned can be taken as a starting point for analyzing the mathematical knowledge on the previous levels in the system. The latter of the vertical axes is dividing the theoretical education taking place at the university college and teaching practice at schools. To combine these two, university colleges often organize preparatory education as a special forum, depict as a small box in the bottom of the model. The *internal didactical transposition* from knowledge meant to be taught to knowledge actually taught is everyday work for teachers and thus obvious content in mathematical teacher education. Again, the arrow is pointing both ways stressing that it is fruitful to change perspective and analyze knowledge meant to be taught on the basis of student teachers observations or descriptions of knowledge actually taught or knowledge actually learnt in teaching practice. This point is an example of the normative aspect of the model.

The two columns to the right are a little different compared to the other kinds of knowledge. The relation between knowledge actually taught and knowledge actually learnt cannot offhand be described as a theory-practice problem because both are a part of the teaching practice at schools – the knowledge *actually* taught and learnt. Off course, teaching and learning can be described and analyzed by theoretical tools but the interplay at schools is a practice matter. As the transposition takes place inside school it is a part of the internal transposition but knowledge actually taught and knowledge actually learnt are closer connected and appears in a more direct interrelationship than the other kinds of knowledge. Student teachers are supposed to react to pupils' communication and learning e.g. during a dialogue in the classroom and adapt the teaching to the individual pupil or the specific class. Knowledge actually taught and knowledge actually learnt can be theoretically analyzed separately but are intertwined in practice. Therefore, the two kinds of knowledge are not separated in the model, but have a common borderline regarded as the interplay between the pupil's knowledge and the knowledge presented by the teacher in the form of the teaching environment presented.

Analysis of a lesson study project

The next section is an analysis of a group of two teachers and three student teachers' learning outcome from a lesson study project on the basis of the ATD-model. The lesson study was conducted in autumn 2013 in two classes grade 6 and 7 and the title was "Similar – what does it mean?" It was a part of a bigger lesson study project with the title *Trigonometry and inquiry based learning* involving 29 student teachers and 17 teachers. The empirical data from the lesson study consists of a lesson plan, video recordings of the two completions of the lesson, two 45 minutes interviews with one of the teachers and one of the student teachers and an article written by the student teachers. Because of the limited space in this paper, lesson study will not be described and only the lesson plan, the video recordings and the interviews will be analyzed. For further information about lesson study reference is made to Lewis (2002), Hart et al. (2011) and Stigler & Hiebert (1999).

Lesson Plan

The lesson plan is divided into three sections: First, some practical information concerning the participants, who taught the lessons, the name of the school, dates for completion of the lesson and the classes involved. The second part encompasses the tittle and aims of the lesson, competencies involved and working method. The last section is a detailed plan of the lesson containing mathematical focus point and learning goals of the lesson, a timetable, key question, teaching resources and useful tips for the teacher.

The lesson starts with a 10 minutes introduction to *geometric similarity* on the basis of every day examples of similar and not similar objects like a golf ball and a football, different sizes of Toblerone packaging (chocolate), enlarging/reducing in a photocopier and a pony and an Arab horse (not similar). After the introduction, the pupils receive a right-angled triangle cut of cardboard and the teacher asks the key question: "You shall pretend that you are a photocopier and draw an enlarged and a reduced copy of the triangle". This is the main mathematical task of the *knowledge actually taught*. When the pupils have drawn the two triangles they must contact the teacher. The teacher then asks them two questions: "How did you construct the triangles?" and "How can you convince me that the two triangles are similar to the one cut of cardboard?" The two questions encompass the transcending of the horizontal theory-practice axis from the practice block to the theory block in knowledge meant to be taught. The teacher's didactic praxeology in connection to this mathematical praxeology is therefore a key issue of the lesson. The task is chosen in order to pursue three different learning goals from the Danish curriculum: *Similarity of right-angled triangles, Reasoning competency and Aids and tools competency* (for further details about *competencies* in Danish curriculum see Niss & Højgaard (2011)).

The crucial mathematical praxeology to be developed in the lesson study is based on the type of task: *Given a right-angled triangle, how can you reduce/enlarge the size without changing the form*? A possible, predictable – and desirable – technique is to copy two or three angles from the cardboard triangle for instance by putting it on top of the paper and draw the angles and then reduce/enlarge the length of the sides. The technology to be realized by the pupils is firstly, that equiangular triangles are similar and secondly; the ratios between the lengths of equivalent sides are constant. Theory – in this case the mathematical definition of similarity – is framing and justifying technology. The lesson plan also points out some pivotal didactic praxeologies. The first one is concerning the inquiry based education mentioned in the category *mathematical working methods*: "*working in pairs – inquiry based education*". Inquiry Based Education (IBE) is a significant trend in mathematical and scientific education in the last decade. The Danish word for inquiry "*undersøge*" is mentioned 206 times in the 89 pages mathematical curriculum for primary and lower secondary school of the time (*Undervisningsministeriet* 2009*b*). The origin of IBE is in John Dewey's philosophy of education (for further details see Artigue & Blomhøj (2013)). IBE is concerned with the teaching-learning relation in the model – a theoretical idea about how pupils learn and from this how to teach. The example shows the delicate interplay between the pupil's and the teacher's didactic praxeology. The appertaining type of task in the teacher's didactic praxeology is how to set up a learning environment that makes the pupils investigate the mathematical task. The task is not explicitly mentioned in the lesson plan but two different techniques to solve the task appear in the following quotes: "*The pupils work inquiring with concrete materials and get the opportunity to reason on their own*" and "*Tips for the teacher: Be careful not to unveil the points*". So, the two main didactical techniques are to use concrete materials and to give the pupils opportunity to work out their own solutions (in pairs) without a standard procedure presented by the teacher.

The second didactic praxeology is connected to the mathematical goal of the lesson: "*The pupils shall reason that angles and ratios between the lengths of equivalent sides are constant in similar triangles*". The goal articulates the technology of the mathematical praxeology in knowledge meant to be taught. The crucial didactical task is to give the pupils opportunity to extend their understanding of the everyday word *similar* (in Danish: *ligedannede*) to a more exact mathematical interpretation. What do we mean when we claim that e.g. two polygons are mathematically similar? The pupils have an intuitive idea of the word/concept and the aim is to build up their mathematical knowledge on the basis of this comprehension, in particular two pivotal properties of similar triangles. To attain this goal the teachers and student teachers use the technique to take a well-known situation from the pupils' everyday life about reducing and enlarging two-dimensional figures (photocopying) and ask them to study what is happening when you enlarge/reduce a right-angled triangle. The selected mathematical tasks cause two problems. Firstly, the most obvious and expected technique – copying the angles – is only connected to the first part of the goal concerning the angles. The pupils are not encouraged to make any further mathematical investigations of the similar triangles and none of the tasks involves the ratios between the lengths of the sides. Secondly, the lack of a mathematical investigation prevents the pupils from *reasoning* referring to the theory block – answering *why*. Because the pupils' only knowledge about similar at this stage is the everyday word, they are only able to answer the question "*How can you convince me that the two triangles are similar to the one cut of cardboard?*" referring to this – answering *how* – the practice block.

This example shows how the model captures underlying mathematical and didactical considerations and the relations between these. In this case, the model is primarily used descriptively to analyze the lesson plan but it can as well be used normatively for instance to improve the design of the lesson plan template in the example about the ratios between the lengths of the sides by stressing the connections between mathematical and didactical praxeologies or type of task, technique and technology.

Video recordings of the lessons

The video recordings show that the student teachers to a great extent conduct the lesson as it is described in the lesson plan. They have experience with lesson study and know that this is important to focus the attention on the teaching instead of the teacher. During the section of the lesson where the pupils work with the problem in pairs they stick to the manuscript of the lesson, e.g. "*Be careful not to unveil the points*", and pose the planned question. For instance in the following situation in grade 6:

Pupil 1:	This one is double size
ST:	How can you convince me, that it is the same triangle? Can you argue that they are similar?
Pupil 1:	It has the same shape – and it has three sides
Pupil 2:	And it is right-angled
ST:	Yes. But so is this triangle (the teacher shows a triangle from another group). And your triangles are not similar to this one

The student teacher finds a new triangle which is definitely not similar to the group's triangle

ST:	Look at this one. Is it similar to your triangle?
Pupil 1:	No
ST:	No, but they both have a right angle and three sides. Try to find out what the similar triangles have in common but these have not. Think about it...

The student teacher leaves (my translation).

The student teacher's first question is almost verbatim from the lesson plan. This question is difficult to answer for the pupils. Nevertheless, Pupil 1 refers to "same shape" as a colloquialism but unfortunately, the teacher do not respond to the suggestion and so the pupil does not get the opportunity to create a link to the mathematical concept – equal angles. As stated in the previous section, this is the task of the didactic praxeology – to extend their understanding of the everyday word similar to a more exact mathematical interpretation. The example (and others alike) shows that the question does not encourage the pupils to investigate mathematical properties about the similar triangles and thereby get an opportunity to become acquainted with the theory block of the mathematical praxeology. The technique to solve the didactical task seems to fail. Maybe as a consequence of this, the student teacher improvises and reformulates the question: "*Try to find out what the similar triangles have in common but these have not*." This question is not mentioned in the lesson plan but it leads the pupils to examine mathematical properties because the question is posed in mathematics. An obvious answer to the question is that similar triangles have angles in common but ratios of the length of sides are not in the same way immediate obvious for pupils at this age. A new didactical task is therefore how the teacher can pose questions to lead the pupils to examine the ratios of the length of sides without “unveiling the point”? Analyzing the situation by means of the model could for example lead to a question like “What will happen if you multiply the length of the three sides with the same number – 2 for example?” The example shows that a problem concerning the didactic praxeology requires an analysis in details of the appertaining mathematical praxeology.

The recapitulation in the end of the lesson is very different in the two completions of the lesson. In the first completion (grade 6) only few of the pupils participate – hesitating and insecure. In the second completion (grade 7) several of the pupils contribute to the conversation, and it seems as though most of the pupils have a growing understanding of similarity. The teacher’s first question to initiate the class discussion is concerning *how* the pupils reduced and enlarged the triangle. In both classes two different kinds of correct solutions – “measuring and copying the angles” and “measuring and multiplying the length of all sides by e.g. 2” – are suggested, however, in grade 6 multiplying the length of the sides was explained very unclearly. During the dialogue, the teacher tries to generalize the pupils’ techniques by saying: “similar triangles are equiangular” but the pupils do not go into this discussion – they stick to answering the “how question”.

After a brief exposition the teacher concludes by framing a precise theorem about angles and the ratio between corresponding sides in similar triangles. It does not seem immediately obvious that the pupils transcend the axis between the praxis block and the theory block and it is difficult to tell if the teacher realizes this on the basis of the video recordings. The teachers are both determined to complete the planned lesson although it is very clear – especially in grade 6 – that the pupils have not reached this point.

The video recordings show that the student teachers are very determined to follow the lesson plan as it is planned by the participants. The comprehensive preparation of the lesson and the very close connection to the theoretical education at the university college gives the student teachers an opportunity to try out their theoretical knowledge – both didactical and mathematical – in practice. Because they stick very carefully to the lesson plan there is a close connection between knowledge meant to be taught and knowledge actually taught and between the mathematical and the didactical praxeology – this is a crucial challenge in teacher education. Obviously, this challenge should be taken up in teaching practice but student teachers often find this very isolated from the theoretical education at the university college. In teaching practice the student teachers are “forced to act” – they have to teach a fixed number of lessons each week. Therefore, they often experience teaching practice as complex and stressful and fall into short-lived performance without coherence to their learning outcome from the theoretical education.

Interviews

Dialogue and working relationship between teachers and student teachers are – of course – important learning resources about school practice for student teachers. The interviews show that both teachers and student teachers experience significant differences between the dialogue involving teachers and student teachers in the lesson study compared to the usual teaching practice situation:

> Teacher (about teaching practice): "Usual, when you have student teachers, it is vulnerable. Very often, you tell them what they did wrong or what they shall be aware of next time in the class instead of sticking to the point, the lesson, the content.(...)"
> Teacher (about lesson study): "Focus is on the lesson and not on the student teachers. We are not supposed to supervise them. We discuss what is working and what is not working about the lesson. We share a common responsibility to make the lesson work. We don't evaluate the student teachers but the lesson."
> Student teacher: "In teaching practice, the teacher watches you when you teach, whereas in lesson study we are equal. We should all participate in the preparation of the lesson and we could all contribute to the lesson."

In teaching practice the student teacher usually prepare the teaching and teach a single lesson, while 1-3 of his or her fellow students and the teacher observe the lesson. Afterwards, the teacher supervises the student teacher in very close connection to the student teacher's presentations and interactions with the pupils in the lesson. The student teacher then tries to "*correct the mistakes*" before the next performance – some descriptions by student teachers and teachers indicates an inappropriate "trial and error" method. The strong focus on the student teacher's performance emphasizes the practice block of the teacher's didactic praxeology and – to a lesser extend – the teaching-learning relation in the ATD-model. According to both teachers and student teachers, it is unusual to discuss didactical and mathematical theory, curriculum and other topics in connection with the teaching on a general level – the two columns to the left in the model are almost absent in the dialogue. The interviews show that the teacher's didactical praxeology in knowledge actually taught is most often disconnected from both the appertaining mathematical praxeology and the mathematical and didactical praxeologies in the other columns. This is evident in many of the dialogs between teachers and student teachers in connection with teaching practice. Such practice off course implies a risk of widening the gap between student teacher's experience of theory and practice during their teacher education.

The interviews show two main differences between the dialogue between teachers and student teachers in connection with usual teaching practice and lesson study. Firstly, the dialogue in lesson study take place both before and after the teaching and especially the very long time spent preparing the lesson was emphasized as fruitful. The common preparing together with the lesson plan template makes the participants discuss and consider knowledge meant to be taught, the internal didactical transposition and the interplay between mathematical and didactical praxeologies. Secondly – as stressed by the teacher in the quote above – focus is on *teaching* and not the *teacher* in lesson study. This implicates for instance that knowledge actually learnt to a much higher degree is included in the dialogue in connection with lesson study than it is in the dialogue in connection with teaching practice and thus, the interplay between knowledge actually learnt and knowledge actually taught can be examined, discussed and related to knowledge meant to be taught.

There is a very clear consensus between teachers and student teachers that the dialogue in connection with lesson study to a much higher degree than the dialogue in connection with teaching practice includes a broader range of pivotal problems in teaching and learning mathematics. As a consequence, theory-practice axis in the model are treated and transcended more often.

Conclusion

The ATD model points out three different theory-practice problems in mathematics teacher education. It is crucial to put focus on all three axes and give student teachers opportunities to establish coherence between theory and practice in connection to the three axes.

Through different examples from a lesson study it is shown, that the model can be a fruitful tool to analyze teaching and learning contexts in teacher education. Firstly, the model can be used as a descriptive tool to analyze and criticize planned teaching (lesson plan), actually completed teaching and the participants' experiences of the teaching with a special focus on theory-practice problems. Secondly, the model can be used as a normative, prescriptive tool for making adjustments to or changes in didactical designs for teacher education.

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