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Inquiry based mathematics education and lesson study

Teachers' inquiry based learning about inquiry based teaching

PhD Thesis

Jacob Bahn

Supervisor: Carl Winsløw. Co-supervisor: Charlotte Krog Skott
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Inquiry based mathematics education and lesson study

Teachers' inquiry based learning about inquiry based teaching

ヤコブ　バーン
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Doctoral dissertation

University of Copenhagen
Department of Science Education

July, 2018

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Abstract

As a response to the somewhat discouraging circumstance that pupils in general are dissatisfied with mathematics teaching, research suggests some promising solutions for teachers' inquiry based learning about inquiry based teaching. Yet, the auspicious attempts in the West to develop and implement inquiry based education have generally had little impact. As opposed to Western efforts, Japanese efforts to both develop and implement inquiry based teaching have proven impactful and lasting. This is often attributed to lesson study, a research-like format for teachers to inquire into pupils' learning, with the aim of improving knowledge and skills to design and conduct lessons. Through lesson study, Japanese teachers and teacher educators devised *open approach method*, which provide guiding principles for inquiry based teaching. This thesis investigates the hypothesis, that lesson study and the guiding principles of open approach method can also help Danish teachers to develop knowledge and techniques of inquiry based teaching.

For the design and analysis of experiments, theoretical tools derived from the Theory of Didactical Situations are employed. The tools comprise models to analyse *didactical situations*: pupils' interplay with the didactical milieu (the mathematical problem and pertinent resources to solve it) and the teachers management of that interplay, and *paradidactical situations*: teachers' interplay with the paradidactical milieu (the teaching problem and the didactical situation hypothesised to solve it).

In three articles, I investigate research questions concerning:

- ① which conditions in the milieu are significant to realise principles of open approach method: From the analysis of one research lesson, four promoting conditions in the milieu and one hindering condition in the teachers' traditional management of situation are identified. This also suggests, that even when good problems and materials for inquiries are developed, managing pupils' inquiries is far from trivial.
- ② what teachers learn in lesson studies and how that learning emerges and evolves: From the analysis of teachers' *paradidactical* activities, specific teacher learnings are identified, and exemplary analyses of emergence and evolution are illustrated. The results suggest that learnings are strongly related to teachers' anticipation of didactical situations, and that interactions leading to learning are interdependent.
- ③ how teachers' anticipation of didactical situations evolve: From the analysis of pre-didactical situations (prior to the research lesson) of teachers' first and third lesson study it is illustrated how teachers' paradidactical practice evolved in step with a quantitative and qualitative increase in their anticipation of didactical situations.

In addition the thesis includes ④, a co-authored chapter proposing theoretical arguments for and tools to theorising lesson studies.

Abstract in Danish

Som respons på det noget nedslående forhold, at skoleelever generelt ikke er tilfredse med matematikundervisningen, peger forskning på nogle potentielle muligheder for læreres undersøgelsesbaserede læring om undersøgelsesbaseret undervisning. Det til trods har de ellers lovende forsøg på at udvikle og implementere undersøgelsesbaseret undervisning i Vesten generelt ikke haft den store gennemslagskraft. I modsætning til bestræbelser i Vesten har japanske bestræbelser, både på at udvikle og implementere undersøgelsesbaseret undervisning, vist sig at have både effektfuld og varig virkning. Det tilskrives ofte lektionsstudier, et forskningslignende format for professionsudvikling, hvor lærere, gennem undersøgelser af elevers læring, styrker deres professionelle viden om og færdigheder til at udvikle og forestå undervisning. Gennem lektionsstudier har japanske lærere og læreruddanere i fællesskab udviklet open approach method, der opstiller vejledende principper for undersøgelsesbaseret undervisning. Nærværende afhandling undersøger hypotesen der lyder, at lektionsstudier og de vejlednde principper fra open approach method også kan hjælpe danske lærere til at udvikle viden om og teknikker til at forestå undersøgelsesbaseret undervisning.

Eksperimenter er blevet designet og analyseret med teoretiske værktøjer fra Teorien om didaktiske Situationer. Værktøjerne inkluderer modeller til at analysere *didaktiske situationer*: elevers samspil med det didaktiske miljø (det matematiske problem og relevante ressourcer til at løse det) og lærerens regulering af samspillet, samt *paradidaktiske situationer*: læreres samspil med det didaktiske miljø (undervisningsproblemet og den didaktiske situation, der antages at kunne løse det).

Igennem tre artikler undersøger jeg spørgsmål angående

- ① hvilke forhold ved miljøet, der er væsentlige for at realisere principper fra open approach method: På baggrund af analysen af én studielektion identificeres fire fremmende forhold i selve miljøet og ét hæmmende forhold i lærerens traditionelle forvaltning af situationen. Det antyder også, at selv om man har udviklet gode problemer og materialer, er det langt fra trivielt at håndtere elevers undersøgelser.
- ② hvad lærere lærer i lektionsstudier og hvordan læring opstår og udvikler sig: På baggrund af læreres *paradidaktiske* aktiviteter identificeres specifikke læringer, og der gives eksempler på, hvordan læring opstår og udvikler sig. Resultaterne antyder, at læringerne er stærkt forbundne med lærernes anticipation af didaktiske situationer, og at interaktioner, der fører til læringer, er indbyrdes afhængige.
- ③ hvordan læreres anticipation af didaktiske situationer udvikler sig: På baggrund af analyser af pre-didaktiske situationer (forud for studielektionen) i læreres første og tredje lektionsstudie bliver det illustreret, hvordan læreres paradidaktiske praksis udviklede sig i takt med, at deres anticipation af didaktiske situationer udivklede sig både kvantitativt og kvalitativt.

Afhandlingen indeholder desuden ④, et kapitel, hvor to medforfattere og jeg argumenterer for og foreslår teoretiske redskaber til at teoretisere lektionsstuider.

List of all papers and abstracts

Papers marked with encircled numbers are included in the present thesis.

Bahn, J. (2015). “Lektionstudier i skolen”. In: *MONA* 4, pp. 95–96. URL: `https://tidsskrift.dk/mona/article/view/36343/37687`.

Bahn, J. (2016a). “Conditions and Potentials for Danish Mathematics Teachers to Design and Conduct Inquiry Based Teaching”. In: Yerme Summer School 8, August 13-20, 2016. Poděbrady, Czech. Submitted.

Bahn, J. (2016b). “Inspiration der rykker”. In: NOMUS XIII Conference, November 1-5, 2016. Gentofte, Copenhagen. Submitted.

Bahn, J. (2017a). “An Experiment with Open-Ended Approach in Grade Four Probability Teaching”. Under revision for: *Recherches en Didactique des Mathématiques* ①.

Bahn, J. (2017b). “Teachers’ Learning from Their First Lesson Study”. Under revision for: *Journal of Mathematics Teacher Education* ②.

Bahn, J. (2017c). “Evolution of Teachers’ Anticipation of Didactical Situations in the Course of Three Lesson Studies”. In review for: *Annales de Didactique et de Sciences Cognitives* ③.

Bahn, J. (2017d). “How Infrastructures of Lesson Studies Impact on Teachers’ Learning”. In: WALS International Conference, November 24-26, 2017. Nagoya, Japan. Submitted.

Bahn, J. (2017e). “Tsukuba Teachers’ Inspiration for and Impact on Our Lesson Study Endeavors”. In: 算数授業研究 *(Sansuu jugyou kenkyuu)* 114, pp. 42–43.

Clivaz, S., Takahashi, A., KimHong, T., Bahn, J., and Rasmussen, K. (2017). “Mathematics Lesson Study around the World: Theoretical and Methodological Issues”. In: WALS International Conference, November 24-26, 2017. Nagoya, Japan. Submitted.

Bahn, J. (2018). “Japanese Vocabulary - a Proposal for Standard Transcriptions”. In: *Mathematics Lesson Study Around the World: Theoretical and Methodological Issues*. Ed. by M. Quaresma, C. Winsløw, S. Clivaz, J. da Ponte, A. Ní Shúilleabháin, and A. Takahashi. Springer book series of ICME. Springer.

Winsløw, C., Bahn, J., and Rasmussen, K. (2018). “Theorizing Lesson Study: Two Related Frameworks and Two Danish Case-Studies”. In: *Mathematics Lesson Study Around the World: Theoretical and Methodological Issues*. Ed. by M. Quaresma, C. Winsløw, S. Clivaz, J. da Ponte, A. Ní Shúilleabháin, and A. Takahashi. Springer book series of ICME. Springer ④.

Bahn, J. and Winsløw, C. (2018). “Doing and Investigating Lesson Study with the Theory of Didactical Situations”. Under revision for: *Theory and Practices of Lesson Study in Mathematics: An International Perspective (Tentative)*. Ed. by R. Huang, A. Takahashi, and J. P. Ponte. Springer

Introduction

Personal motivation

The motivation for beginning the work on this dissertation was to a large extent linked to my frustrations as a teacher of mathematics. I cannot count the times I have left a classroom well aware that I did not succeed in my teaching project: though pupils would solve problems correctly and express signs of learning today's topic, I knew by myself - and was too often right - that the next time we would revisit the same topic, most pupils would not have really understood. I have aired these frustrations to colleagues at the same and other schools, as well as to supervising mathematics teachers (*matematikvejledere* - see later), school leaders and teacher educators during my time as a teacher. The well-intentioned but futile responses I received group into four perspectives, a mixture of consolation and advice: *pupils' conditions* (e.g. Peter suffers and poses so many difficulties), *teaching materials* (why not try an alternative textbook), *location* (authentic environments promote learning) and *theory about learning* (we must teach in a constructivist manner). While these perspectives are not irrelevant, the concrete responses I have met all lacked one crucial element: knowledge about how to relate directly to the pupils' learning of given knowledge under specific circumstances, and about what it would take to establish that relation.

Neither during teacher education nor when writing my master thesis (Bahn 2009), about teaching and learning in outdoor education (*udeskole*), did I find substantial answers to how teachers can develop good teaching, that is, how to establish the conditions which in practice allow pupils to construct a specific piece of knowledge. I have studied and been lectured about e.g. *didactical relations*, *constructivism*, *flow*, *learning by doing*, etc. numerous times, but I never came across concrete answers to how to design and enact 'good' teaching.

As will be discussed later, one interesting aspect regarding this is that while there is little dispute about inquiry based education holding a rich potential to raise pupils' motivation for learning mathematics and their actual results, there seems to be a lack of practical tools for teachers who wish to teach in an inquiry based fashion. As a result, in Denmark and many other countries, there is no solid tradition for teachers to teach inquiry based - except in theoretical and general terms.

Founding state of the problem

My personal experiences aligned only too well with research literature on Danish pupils' relationship with mathematics in school. My research project is further motivated by the well-documented beliefs and attitudes among teachers and pupils, such that *"Danish pupils are generally not very positive*" towards mathematics (Allerup 2012, p. 11, my translation), their *"self confidence is low"* (ibid., p. 11, my translation), and only a minority *"understand the purpose of the subject of mathematics"* (ibid., p. 12, my translation). This has been seconded by several reports on the state of Danish pupils mathematical competences (Niss, Andreasen, et al. 2006; Niss and Højgaard Jensen 2002; Sørensen 2014), and in these and other reports, authors have invariably pointed out the need to develop and support mathematics teachers' practice (e.g. Jespersen and Jon 2013).

Research suggests that inquiry based education holds a great potential to improve pupils' performance through increasing their motivation to learn, and a number of large-scale efforts have been launched and conducted to introduce teachers to and implement inquiry based teaching (Artigue and Blomhøj 2013; Q-model 2017; Rocard et al. 2007). However, in general, and outside protected environments of development projects, little development is seen in the teachers' practice (Artigue, Dillon, et al. 2012; Rocard et al. 2007), which is still dominated by pupils' work with routine tasks, led by the teacher (Mogensen 2011; Niss, Andreasen, et al. 2006). There is, so to say, a *gap between actual teaching and official ideals of teaching*.

There may be several reasons why teachers' practice does not develop towards inquiry based education. Research suggests that the barrier to develop and adapt new teaching methods comprises these aspects: the task is difficult and constraining and is thus unlikely to be performed by solitary teachers (Lindhart 2007; Rocard et al. 2007), yet teachers (of mathematics) in general do not take part in professional discussions with peers (Artigue and Blomhøj 2013; Rocard et al. 2007) or peer observation, live (Bruce and Ross 2008; Jackson and Bruegmann 2009) or recorded (Seidel et al. 2011; Tripp and Rich 2012).

In this respect, it is promising that research can point to some solutions even if many problems remain to be solved. In the preliminary studies for this dissertation, three leading perspectives emerged, which together offer three crucial perspectives: efficient guidelines for inquiry based teaching and collaborative development of teaching, teachers' reality as a starting point and focus on the connection between teaching and learning.

Leading perspectives

Since the 1990's, and particularly since the appearance of *The Teaching Gap* (Stigler and Hiebert 1999), it has been well-known in the West, that Japanese teachers have devised ways to implement inquiry based education throughout their school system, and that these ways have developed organically over a long period of time, beginning in the latter half of the 19th century (Isoda 2007; Makinae 2010). For the present thesis, two aspects of these "ways" offer particularly interesting perspectives:

Firstly, since the early 1970's Japanese teachers, teacher educators and researchers have employed substantial and collective efforts to develop and disseminate guiding principles for inquiry based teaching. These, first known as *open-ended approach* (Becker and Shimada 1997; Nohda 1995) and later *open approach method* (Nohda 2000), have developed into what is now known as Structured Problem Solving (Stigler and Hiebert 1999) or Problem Solving Approach (Isoda 2015).

Secondly, the development and diffusion of these principles have been done through certain structures that enable teachers' collaborative inquires into pupils' learning (Isoda 2007). While these structures, known as lesson study in the West, appear quite simple on the surface, they are complex and have been the object of decades of research (Fernandez and Yoshida 2004; Stigler and Hiebert 2016).

These two aspects are intertwined, as lesson study conveys teachers' inquiry based learning about inquiry based teaching, as modelled in e.g. open approach method. It also means that unlike the common Western "ideals" and "theories" about mathematics teaching, the

Japanese principles are grounded in experimental work that continuously involved and still involves a very large number of ”normal” teachers, collaborating with teacher educators and education researchers (roles which are often assumed by the same person). Accordingly, their compatible affordances and potentials are well-tested as well as well-documented.

Independently of the Japanese endeavours to develop knowledge about teaching and learning, and techniques of teaching through lesson study, the research field of *didactics* was developed in France over decades, starting in the late sixties. The core, the theory of didactical situations, offers advanced and well-developed tools and notions to model teaching and learning of mathematics, in alignment with the constructivist hypothesis of learning. The theory was originally developed to design and analyse experimental teaching-learning situations, but has proven powerful to analyse the dynamics of other learning situations as well. Hence, it offers efficient and versatile tools to analyse pupils' and teachers' learning activities in relation to teaching-learning experiments in lesson studies.

A note on vocabulary

As the subtitle suggests, the object of this thesis is the dynamical matter of teachers' learning about teaching, through inquiries. The matter of teachers' learning naturally put teachers in the role of learners. Moreover, the learning in question here is connected with teachers' inquiries, i.e. systematic studies, which puts the teachers in the role of students. Since the learning obtained through the teachers' inquiries regards teaching, it also *per se* regards pupils or, as they are also referred to, students. The terms *teacher* and *pupil* are considered not to cause confusion, but in their generality *learner* and *student* might. That generality though is pertinent as it clarifies how the principles and processes of learning and studying apply to any agent, regardless of age and status. Hence, in this thesis the generality of *learner* and *student* is maintained and may both refer to either teacher or pupil, depending on the context.

In addition, when teaching-learning situations are depicted in the literature, the reference to teachers and pupils as *she* or *he* often causes confusion. Again, this possible source of confusion is passed on in quotations. In the original text of the thesis this possible confusion is dealt with by generally using *teacher* and *pupil* and only occasionally using *she* to denote the former and *he* to denote the latter. The disposition of gender has no attached significance.

Given the Japanese origin, the literature on open approach method and especially lesson study comprise a number of notions and terms which do not easily translate into English. Since there does not exist a unified protocol for how to transcribe Japanese words, the Japanese vocabulary in the pertinent research literature can at times be confusing. As a response, I have proposed a standard for transcriptions (Bahn 2018), which is - naturally - followed in this thesis. See the Appendix of the thesis for further detail.

Lesson study

Lesson study refers to structures of teacher-led collaborative inquiries into pupils' learning (C. Lewis 2016, p. 571; Murata 2011, pp. 1-2). The purpose is to develop knowledge about and techniques of teaching specific subject matter knowledge, with the ultimate goal of enhancing pupils' learning outcome and motivation to learn (Miyakawa and Winsløw 2009).

The English *lesson study* is a translation of the Japanese *jugyou kenkyuu*. The latter part, kenkyuu, generally translates to *study*, *research* or *investigation*. Jugyou translates to such terms as *lesson*, *class (work)*, *teaching* and *instruction*. Hence, jugyou kenkyuu might as well be translated into *investigations of teaching* or *research on teaching*. The purpose of mentioning this is not to alter the English term we use, but to point our attention to the difference between a timespan in school filled with activity (lesson), and the activity of teaching (and learning) filling a timespan in school.

In Japan, lesson study is part of a county-wide system of generation, development and sharing of new knowledge and ideas of teaching, which in principle connects knowledge and practice across horizontal and vertical axes of the whole Japanese school system (Miyakawa and Winsløw 2017). In the narrowest sense, this is related to the individual lesson study in which a group of teachers share their ideas and develop new knowledge and techniques together. In the widest sense this means that teachers and other stakeholders from across the country and every layer of the school system exchange and develop ideas and knowledge. This includes structures for the re-thinking, testing, revision and implementation of the national course of study (curriculum). These wider structures and the manifold, often intangible, variations of lesson study are not elaborated further. Throughout this thesis, lesson study refer to the far less intangible structures described below.

The predecessor to lesson study, *open class* (Isoda 2007, p. 12) or *criticism lesson* (Makinae 2010, p. 7), is recorded as far back as the late 1890's (ibid., p. 7). Here, it was used as a means to educate student teachers in the elementary schools attached to the teacher education school (ibid., p. 7). Within a few years, it was adopted to the increasing use of what was known as *hihyou kai*, criticism conference, or *jugyou kenkyuu kai*, lesson study conference (ibid., p. 1) in connection with initial teacher education. Thus, lesson study dates back more than a century.

In the beginning of the 1960s, in-service teacher training was introduced in the form of *kounai kenshuu* (Fernandez and Yoshida 2004, p. 13), which literally means 'training within (the) school'. *Kounai kenshuu* was not in itself related to lesson study, but "the strategy of combining *[kounai kenshuu]* and lesson study was already well established by the middle of the 1960s" (ibid., p. 13). Lesson study has for long been the primary form of professional development for teachers in Japan (C. Lewis, Perry, and Hurd 2009, p. 285; Stigler and Hiebert 1999), with typical distinctions between "school-based, district-based, and national-level lesson study" (Fujii 2016a, p. 413).

There are no set rules of what constitutes a lesson study or fixed protocols of how a lesson study must be conducted. This may to some extent explain the variation in success for non-Japanese lesson studies (Fujii 2014; Takahashi and McDougal 2016) (and probably Japanese as well). Nevertheless, there is a widespread consensus, that a lesson study is revolving around a live *research lesson* in which observational data is collected from the test of a teaching-learning experiment (Murata 2011; Takahashi and McDougal 2016, pp. 514-515). Accordingly the teaching-learning experiment is prepared prior to the research lesson, and - in natural consequence of an experiment taking place - the collected data are reflected on, posterior to the research lesson. An illustration of the basic structure of a lesson study is presented in figure 1.

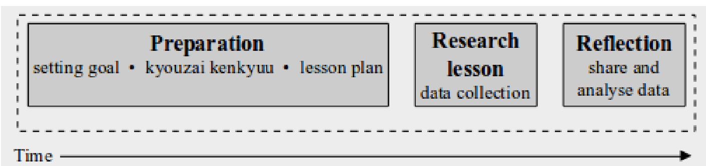

Figure 1: Basic model of a lesson study partitioned into three phases: the pre-lesson preparation, the research lesson and the post-lesson reflection session. Adapted from Winsløw, Bahn, and Rasmussen (2018).

Particularly in the mature lesson study culture of Japan, it is furthermore an apparent unanimity that preparation includes such activities as *goal setting, kyouzai kenkyuu* and *lesson planning*, that the research lesson is conducted by one teacher and observed by others, and that reflection includes sharing and analysis of the collected data (from the conducted experiment) in proportion to the planned experiment and in particular, its explicit goals.

As for lesson study as a whole, goal setting should reflect that knowledge is answers to questions. Hence, ”Lesson Study begins with a question” (Fujii 2016a, p. 412) and ”*identifying* this question ... is the first step in the [lesson study] process” (ibid., p. 412, my emphasis). Goal setting typically comprise two levels of goals: general and specific. The specific goal, the objective of the lesson, specifies the mathematical knowledge the pupil’s are intended to learn in the lesson in question. Prior to setting that (short-term) goal, a more general long-term goal for the schooling as such must be set, since this should govern the specific short-term goal. Long term goal may focus e.g. on how the the lesson can support the school’s general goal of ”preparing students to live in human society ... through interaction with others, pupils build a sense of mathematical values, and based on that they act in society by constructing and modifying their thinking” (Miyakawa and Winsløw 2013, p. 192).

The Japanese notion of *kyouzai kenkyuu* is a complex technical term which is not easily translated. It comprises an in-depth study of resources related to the knowledge to be taught and the constraints to teach it (Miyakawa and Winsløw 2009, p. 203; Watanabe, Takahashi, and Yoshida 2008). Takahashi (2011) reports that ”Japanese educators confirm that anticipating student responses, which includes not only several correct answers but also typical misunderstandings or wrong answers to the question, is one of the crucial processes while planning lessons” (ibid., p. 79). This is seconded by e.g. Fujii (2014), who furthermore points out that ”designing the task is the essential activity of [kyouzai kenkyuu]” (ibid., p. 4). In fact, kyouzai kenkyuu is often regarded the most crucial part of a lesson study (e.g. Isoda 2015, p. 85; Watanabe, Takahashi, and Yoshida 2008).

Hence, kyouzai kenkyuu functions as a preliminary study of knowledge about the mathematical topic, its connection with curricula, and pupils’ learning of it, including teaching materials’ conditioning of such learning. This forms the background for devising a hypothesis of a possible solution to the teaching problem and a proposal of how to test that hypothesis, i.e. the lesson plan.

In particular for the later analyses, it is essential to observe the different roles of the teacher teaching the research lesson, and the others observing it: The teacher teaching the lesson is part of the test, the observers are not. The test and the observers are, so to say, detached from each other, enabling the latter to observe the former at analytical distance. With the purpose of lesson study in mind - to develop knowledge about and techniques of teaching specific subject matter knowledge - this distinction, between the research lesson and the observation of it, helps to clarify that the research lesson plays the important role to test the underlying hypothesis about teaching and learning but is not important in itself: though indispensable, it is merely a tool to allow for the collection of data.

The number and institutional affiliation of participants of a lesson study may also vary. Some lesson studies are conducted by teachers at the same school (Fujii 2016b; Hino 2007), while others are conducted on the basis of study groups across institutions (Miyakawa and Winsløw 2017). Occasionally single teachers prepare a lesson study, but even so they will confer with colleagues, who will also participate in observation and reflection of the research lesson (ibid.) (This practice was confirmed by teachers and the vice-principal of the Attached Elementary School of the University of Tsukuba in personal discussions during 2015 through 2017. It was furthermore the practise presented in a video used to introduce the Danish teachers participating in the present research to Japanese lesson study. See later).

Not least due to the systematic (in the sense of focus, method and repetition) and systemic (in the sense of the Japanese elementary education system) use of lesson study (C. Lewis 2016; Miyakawa and Winsløw 2017), some teachers develop a relatively high expertise. In many lesson studies such experts, in English known as *knowledgeable others*, will give their comments at the end of the post-lesson discussion (Fujii 2016a, p. 412). There is no formal educations to become a knowledgeable other, and they may be affiliated with another school, a university (as an educator and/or researcher) or work in private (Takahashi 2014, p. 4). Takahashi (ibid.) states that, ”through their final comments, knowledgeable others play an important role in helping lesson study be effective” (ibid., p. 15), and elaborates that the role of a knowledgeable other may comprise such responsibilities as ”(1) bringing new knowledge from research and the curriculum [to the teachers]; (2) showing the connection between the theory and the practice; and (3) helping others learn how to reflect on teaching and learning.” (ibid., p. 10). Elsewhere, he also points out, that ”there is a lack of experienced lesson study practitioners outside Japan” (Takahashi 2011, p. 80) which makes ”beginning a lesson study [] extremely challenging” (ibid., p. 81).

Not all teachers achieve to become knowledgeable others, but many develop other forms of expertise which they use to facilitate or support lesson studies at their own school or by request from other schools. The role of such experts in Japan, often senior or head teachers, is underexposed in the English literature. There may be several reasons for that. One might be, that due to the integrated use of lesson study, many teachers can serve the - or a - role of facilitating less knowledgeable colleagues. At the same line, it also seems plausible that Japanese teachers are generally so skilled in lesson study work, that a facilitator as such is not necessary. Moreover, it might be that expert teachers facilitating a lesson study is as much like ’air’ to the Japanese teachers as the famous quote reveals lesson study is (Fujii 2014, p. 15). In the end, facilitation of lesson studies in Japan remains underexposed and calls for investigation.

J. M. Lewis (2016) seems to sum-up many non-Japanese lesson study practitioners’ view when she states that ”high-quality implementation of lesson study requires skillful facilitation” (ibid., p. 527), but that ”little is known about how teacher developers learn to facilitate professional development where teachers drive much of the process” (ibid., p. 527), as is the case in lesson study. While there are several accounts of how facilitators (typically researchers) facilitated in given lesson studies (e.g. Inoue 2011; C. Lewis and Hurd 2011), we do not have precise knowledge about what a ’good’ facilitator is and which knowledge and skills it take to be one.

Just as there are no set rules for doing lesson study, there is no ratified model of its organisation. Many models proposed by Western research describe lesson study as organised in four or five phases, but as described above, the model used here (see figure 1) regards the research lesson as the point of reference, with a pre-lesson preparation session and a post-lesson reflection session. Furthermore the differentiation between the research lesson and the observation of it is taken into account, and accordingly the model is extended to illustrates how preparation, observation and reflection, which all together take place outside of the research lesson, runs in parallel to the research lesson.

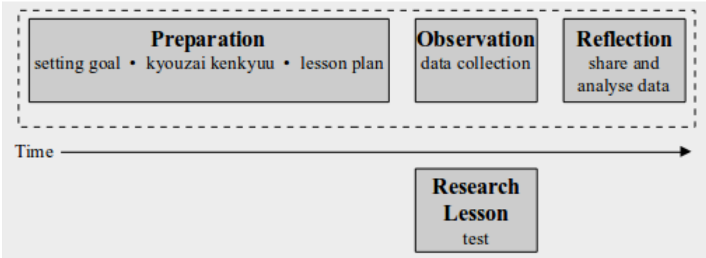

Figure 2: Basic model of a lesson study, illustrating the lesson study activities running in parallel to the research lesson. The indispensable research lesson enables data collection but is detached from the teaching-learning experiment (and other lesson study activities). Adapted from Winsløw, Bahn, and Rasmussen (2018).

It should be noted that the activities of goal setting, kyouzai kenkyuu and lesson planning during preparation are not strictly separated, and may influence each other back and forth. This is only natural since the perception of a given teaching problem and its possible solutions gets ”increasingly refined” (Murata 2011, p. 2), after expanding one’s insight into it, e.g. through kyouzai kenkyuu, and after kneading ideas to solve it (possibly orchestrated by a facilitator, see later), when devising a the lesson plan.

Lesson study is usually presented as a cyclic process (Murata 2011, p. 2; Takahashi and McDougal 2016, p. 521). There seems to be unanimity between Japanese practitioners and researchers that lesson study is an iterative process in the sense, that (some of) the knowledge applied in one lesson study was developed in an earlier one (Fujii 2014, p. 3; Murata 2011, p. 2). This is a crucial feature, since it ensures the sharing of knowledge across generations of teachers (Isoda 2015, pp. 87-88). In addition to safeguarding the sustainability of the ecological system of knowledge for teaching via generational overlap, cycles are also used within individual lesson studies. This gives the teachers the opportunity to strengthen their hypotheses through test-based revisions and the re-test of these. There is, though, an ongoing discussion related to the question of iteration.

According to Fujii (2014), re-teaching the research lesson is a ”misconception” (ibid., p. 11) which is in ”opposition to the core value of lesson study” (ibid., p. 12). Seemingly, the remarks primarily regard lesson study outside Japan, in which the hypothesis is often revised and re-tested in a sequence of research lessons (Hart, Alston, and Murata 2011; Quaresma et al. 2018) (this is also the case in the lesson studies in the present research). However, there are also several records of re-teaching in Japan (e.g. Ermeling and Graff-Ermeling 2014; Fernandez and Yoshida 2004, p. 28; Hino 2007, p. 511). Accordingly, there is no doubt that revision and re-teaching of the lesson (plan) also takes place in Japan, and, in light of the discrepancy, the question of cycles and re-teaching deserves further research (the results of this thesis support that revision and re-test can be beneficial, at least for teachers novice to lesson study).

In cyclic lesson studies the activities of the reflection session at some point change from analysing the tested hypothesis (based on the shared observation data) into revising the hypothesis and prepare a new test (of that revised hypothesis), and so forth. The pattern of such lesson studies are clearly cyclic (since similar types of processes are repeated in the same pattern) and typically represented as a circle or as an enclosed ’chain’ af phases (e.g. Takahashi and McDougal 2016, p. 521). Considering the progression with regard to temporality and (at least desired) evolution, it appears to be more suitable to stretch out the ’spiral spring’ - not least for the sake of analysing teachers’ learning in the course of lesson studies. Accordingly, the model developed to analyse the three-cycled lesson studies investigated in this thesis is founded on a tripled version of the basic structure of a lesson study illustrated in figure 1. While the analytical model will be presented later, its founding triple-version of the basic model is illustrated in figure 3.

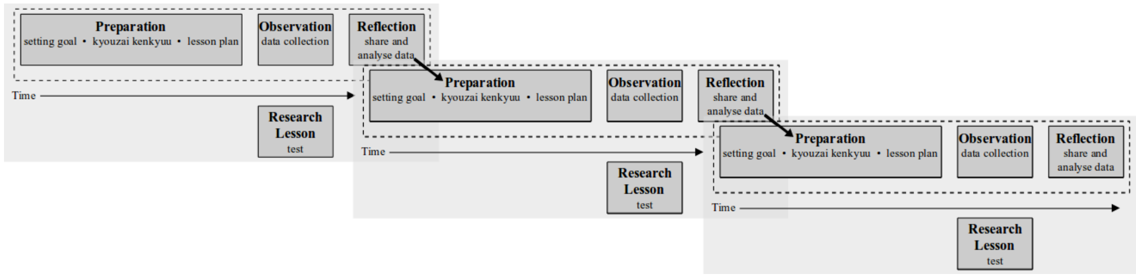

Figure 3: Basic model of a three-cycled lesson study. In the reflection session, analysis of the observed research lesson shift into preparation of the next. Based on figure 2.

Lesson study in a Danish context

In a Danish context, lesson study is still a fairly young concept. It was brought to Denmark by Carl Winsløw who studied for his Ph.D. thesis in Japan (1992-1994) (Winsløw 2004, p. 1). Following up on his studies in pure mathematics, he caught an interest in mathematics education, and in 2000 he was introduced to Japanese elementary school teaching (ibid., p. 1). The first presentation of lesson study in Danish (or by a Danish author) appears to be chapter 7 in *Didaktiske elementer* (Didactical Elements), a textbook for teacher education from 2006 (Winsløw 2006). Carl Winsløw has since authored and co-authored a series of papers on lesson study, e.g. (Miyakawa and Winsløw 2009, 2013, 2017; Winsløw 2012) and a number of papers in Danish.

Other contributions to Danish research involving lesson study comprise three Ph.D. dissertations: *Point-driven mathematics teaching studying and intervening in Danish classrooms* (Mogensen 2011), *Praxeologies and institutional interactions in the Advanced Science Teacher Education* (Rasmussen 2016), and *Teori-praksis-problematikken i matematiklæreruddannelsen - belyst gennem lektionsstudier* (The theory-practice problem in mathematics teacher education - illustrated by means of lesson studies, Østergaard 2016).

Open approach method

Open approach method is a more recent version of *open-ended approach*, which was originally developed with the aim of devising ”methods of evaluating higher-order-thinking skills in mathematics education” (Becker and Shimada 1997, p. vii). During the developmental research efforts, in the early 1970’s, to produce such methods, it became evident ”that lessons based on solving open-ended problems as a central theme have a rich potential for improving teaching and learning” (ibid., p. vii). These discoveries developed into a teaching method, the open-ended approach (Isoda 2007, p. 14; Nohda 2000). As will be elaborated in the following, the openness of open-ended approach refers to the use of a ”problem situations in which the solutions or answers are not necessarily determined in one way” (Becker and Shimada 1997, p. 23). Open approach method was later devised on the bearing principles of open-ended approach, but broadened the notion of openness to not only regard problems whose end is open. The wide-spread use of inquiry in present Japanese mathematics teaching (Hino 2007; Stigler and Hiebert 1999) is largely founded on the principles developed in connection with open-ended approach and open approach method (Hino 2007; Isoda 2015; Nohda 2000). Open approach method in its original ’pure’ form is rarely used in contemporary Japanese teaching, but it still thrives through the widespread practice of structured problem solving (Isoda 2015, pp. 96-97).

Given what is often reported to be the case in Western countries, the prevalence of inquiry oriented practice in Japanese mathematics teaching is remarkable. Open approach method has successfully been disseminated in the Japanese school system as a whole, and in that context has been developed further. There has been a widespread and concerted effort to adopt open approach method which includes adjustments to the national curriculum, development of appropriate teaching materials and textbooks, and a persistent effort to develop the practice of open approach teaching through pre-service and not least in-service teacher education (Isoda 2015; Miyakawa and Winsløw 2009).

Open approach lessons aim to enable pupils to approach the problem ”in response to their own mathematical power” (Nohda 2000, p. 41) and ”according to [their] own abilities, interests and emotions” (Nohda 1995, p. 58). As with lesson studies, there are no ’rules’ of how to conduct open approach lessons, but the method proposes some founding tenets that must be met. The lesson should be planned on three grounding principles (Nohda 2000, p. 42):

- The first principle relates to *students’ autonomous work* in the sense of appreciating the value of their activities.
- The second principle concerns the epistemological structures in mathematical knowledge, and considers that the acquisition of *essential knowledge* supports derivation of yet new knowledge, even later on.
- Related to the two first principles, the third principle regards the *teacher’s expedient decision-making* which should appreciate unexpected mathematical ideas and put them into play to enrich the learning situation all pupils.

Typically, open approach lessons are planned to progress in three phases (ibid., p. 42), in which pupils:

- (re-) formulate the mathematical problem.
- test methods to find solutions.
- advance new ideas and hypotheses.

It is the objective of the open approach method to ”recite, foster or develop students’ ideas through discussions between [the] teacher and [the] students, and among students under the teacher’s orientation” (Nohda 1995, p. 57). According to the method, this is achieved by establishing (preparing and managing) situations in which pupils can modify and develop further their ideas based on observation and investigations of other pupils’ discoveries and methods while comparing them to their own (Becker and Shimada 1997, p. 23). In this sense, one pupils’ knowledge and ideas can enrich other pupils’ resources. In order to enable (and force) pupils to develop their own mathematical thinking, it is crucial that the teacher does not assume mathematical authority but leaves that to the nature of the problems, the materials and the pupils’ shared ideas.

As both Becker and Shimada (ibid.) and Nohda (1995, 2000) note, designing problems and conducting lessons in accordance with the objective and the principles of open approach method is most challenging (Becker and Shimada 1997; Nohda 1995, p. 59, 2000, p. 50). In this respect lesson study as been invaluable as a means to not only develop the methods and principles but also to their implementation in the form of dissemination and teachers acquisition of pertinent knowledge and techniques to outlive them (Isoda 2015, p. 14).

A typical open approach lesson focusses on pupils' investigation of one open problem in the sense there are multiple answers to it (i.e. end is open), there are multiple ways to solve it (i.e. process is open) and/or there are potentially multiple ways the problem can be developed further (i.e. problems are open) (Isoda 2007, p. 14; Nohda 2000, pp. 43-44).

End is open can be exemplified with the "marble problem" (Nohda 2000, pp. 46-49): Figure 4 depicts how three players' marbles scattered when thrown on a piece of paper. Under the rules, that the winner is the one whose marbles scattered the most, the problem is to determine, "who do you think is the winner?" (ibid., p. 47).

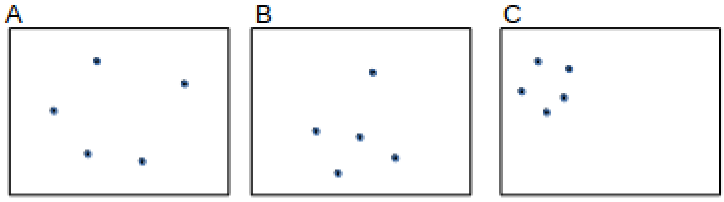

Figure 4: The marble problem: Each of three players have thrown five marbles on a piece of paper. The winner is the one whose marble are scattered the most. Who is the winner? (Nohda 2000, p. 47).

The openness in this problem lies in its several possible answers, which could be based on ideas related to e.g. distance between marbles, area within the polygon made by the marbles or radius of circles circumscribing all marbles (ibid., p. 44).

Process is open may be illustrated via the problem of "same form" (Miyakawa and Winsløw 2009): The problem here is to investigate if two quadrilaterals "are of the same form, or not?" (ibid., p. 206).

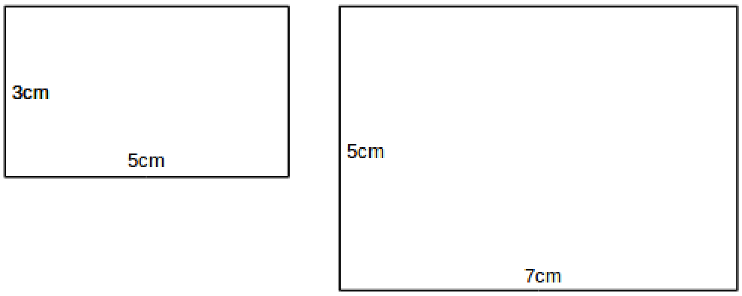

Figure 5: The problem of 'same form': are the two quadrilaterals of the same form, or not? (Miyakawa and Winsløw 2009, p. 206).

The initial question can be answered with a simple 'yes' or 'no', but here the openness regards the numerous ways in which one can (try to) solve the problem, e.g. by comparing the angels of the diagonals, by superimposing (with or without diagonals drawn), by proportion between heights or widths (ibid., p. 208). An important task in solving the problem is to recall or construct a working definition of what it means for two figures to be "of the same form"; this gives a slight "open-endedness" to this problem, as one might in principle consider rectangles to all have the same form.

Problems are open may be exemplified by the "matchstick problem" (Nohda 2000, pp. 44-45): Based on an illustration of the pattern shown in figure 6, the problem is to determine how many matchsticks are used if there are eight squares. In the exampled provided by Nohda (ibid.), the initial question is followed up by asking the pupils to write down their answer and how they found it, make up new variations of the initial problem and write down which of your own problems you like the best and why (ibid., pp. 44-45).

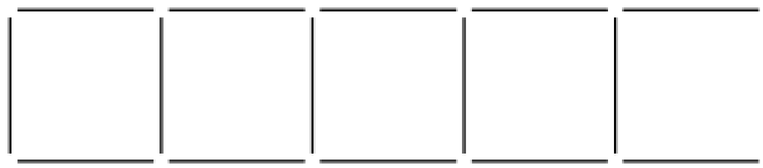

Figure 6: The matchstick problem: Each of three players have thrown five marbles on a piece of paper. The winner is the one whose marble are scattered the most. Who is the winner? (Nohda 2000, pp. 44-45).

This problem poses a double openness in the sense that the initial problem in it self can be solved in multiple ways (process is open), while the proceeding tasks potentially can develop in infinitely many ways (ways to develop are open). For example the initial (or a succeeding) problem can be developed by changing the number of squares or changing the shape from square to triangle or diamond, or an inverse problem can be developed: to determine how many squares can be made with a given number of matchstick (Nohda 2000, p. 45).

It is a recurring trait of open approach problems that they are easily adjusted in difficulty, either by altering the teaching material itself or the appertaining problem. Indeed, many problems designed in consideration of the principles and objectives of open approach method can be used at any grade level. This relates to one of the fundamental differences between traditional transmissive teaching of compartmentalised facts and procedures based on teachers' knowledge and modern Japanese teaching of fundamental concepts and their structural connections based on pupils' knowledge.

Despite crucial differences, there are some striking similarities between the conditions and objectives for pupils' work in open approach lessons and teachers' work in lesson studies. In both cases the learner faces an open approach problem to which there is not only one solution, and through collaborative processes of investigations of the participants' knowledge and ideas, possible solutions to the problem are developed, examined and developed further. In this sense, ”lesson study and teaching mathematics through problem solving are two wheels of a cart: one cannot succeed without the success of the other” (Fujii 2018, p. 2).

Yet, there are crucial differences, some which simultaneously pose reasons and constraints for teachers' use of lesson study to experiment with open approach method. For instance, the ”major difficulty [of] how to give a suitable problem to the students” (Nohda 1995, p. 59), i.e. how ”to make or prepare meaningful mathematical problem situations” (Becker and Shimada 1997, p. 24) and how ”to pose problems successfully” (ibid., p. 24), is highlighted. Nohda (2000) states that with the transition from open-ended approach to open approach method, the challenge of ”constructing the open problem [was] overcome” (ibid., p. 43).

Theoretical framework

One thing that differentiates the job of a teacher from other pedagogical jobs is the role of the teacher in the transposition of subject matter knowledge (Chevallard 1988), i.e. teaching. Teaching tasks occasionally occur in other pedagogical and non-pedagogical institutions or jobs, but in general teaching aims at promoting someone else's (usually a pupil's) acquisition of specific knowledge which is part of a more or less official curriculum. Accordingly, in order to address issues of teaching and learning, practical and theoretical knowledge about the transposition of knowledge not only deserves but requires intensive attention.

In this thesis the scientific pursuit of such knowledge - about the connection between teaching and learning - is referred to as *didactics*. For some, the notion of didactics regards an inexact variety of often intangible theories and methods about teaching. For others, the notion has the somewhat derogatory meaning of lecturing attached to it. In the following, the precise and sophisticated theoretical construct from which the meaning of didactics employed in this thesis is adopted, is presented.

In principle the notion of didactics can be used for any subject, e.g. *didactics of biology*, *didactics of history*, etc. For the sake of simplicity, and since we consider only the case of mathematics here, in this thesis *didactics* specifically refer to *didactics of mathematics*, i.e. knowledge about the connection between teaching and learning of mathematics.

Didactics refers to the knowledge about the connection between teaching and learning, and recognises that this always depend crucially on the specific knowledge intended for the student to learn, the *target knowledge*. Didactics, thus, is not a sub-domain of pedagogy but rather a super-domain of mathematics: One cannot teach what one does not know, and without a very deep knowledge about the mathematical topic one wants to teach, one cannot know and even less develop appropriate ways to teach it.

Didactics has been developed as an individual research field since the days of Felix Klein (1849-1925), but a new and more scientific understanding of the field emerged from the late 1960's with the theory of didactical situations (TDS - Brousseau 1997), a highly developed and powerful theory of the teaching and learning of mathematics. The theory and its notions have been very influential in design based research on mathematics education, and is a central paradigm within contemporary didactics of mathematics.

TDS was originally developed to design and analyse experimental teaching-learning situations of mathematics. The theory is closely connected with *didactical engineering* (Artigue 2015), which was adopted as the main methodology (Clivaz 2015; Margolinas and Drijvers 2015), and with which it shares its two interrelated purposes: research and production (Artigue 1994, p. 30). The methodological research tools of didactical engineering are structured into four (main) phases:

- Preliminary analysis (epistemological, institutional and didactical analyses).
- Conception (of research hypotheses) and a priori analysis (of potential didactical situation).
- Realisation (of the didactical situation), observation (of student-milieu interplays) and data collection (in accordance with the research goal)
- A posteriori analysis (of the realised didactical situation with reference to the a priori analysis) and validation (of the research hypotheses)

Didactical engineering thus rejects the "validation paradigm based on the comparison of control and experimental groups" (Artigue 2015, p. 471) but rather, "its validation is internal and based on the comparison between the *a priori* and *a posteriori analyses* of the didactic situations involved" (ibid., p. 471).

In addition to its original purposes, a number of studies "[confirm] that TDS is not bound to bear on the designs which enabled and formed it initially" (Miyakawa and Winsløw 2009). TDS has proven applicable to describe and analyse a variety of other activities regarding and surrounding teaching, including ordinary teaching-learning situations and teachers' learning. For instance, Hersant and Perrin-Glorian (2005) used TDS to analyse two long sequences of ordinary teaching practice, and Miyakawa (2015) employed it to identify what could be meant by a 'good' Japanese (open approach) lesson. In addition, Miyakawa and Winsløw (2009) used the theory to analyse and compare two different didactical designs, one resulting from the open approach method and the other from didactical engineering based on TDS. While these studies analysed practices in the teaching-learning situation, Margolinas, Coulange, and Bessot (2005) used TDS to analyse what teachers *can learn about* practices in and from the teaching-learning situation. Furthermore, Clivaz and Shúilleabháin (2016) have suggested to use notions of the theory to support the study of teachers' development of mathematical knowledge for teaching, as they participate in lesson studies. From a theoretical perspective, Clivaz (2015, 2018) further advanced the shared affordances of TDS and lesson study. Supported by empirical examples, including cases from the thesis at hand, Winsløw, Bahn, and Rasmussen (2018, included in the present thesis) further argues for *theorizing lesson study* by the use of TDS (as well as the anthropological theory of the didactic).

One of the strengths of TDS is its close relation to practice. Its instigator, founder and chief developer for many years, Guy Brousseau, started his carrier as a teacher and from early on he began conducting teaching-learning experiments. Only interrupted by military service, Brousseau spend the most of a decade as an 'investigative' teacher, before he in 1962 commenced his academic carrier, which would eventually lead to the development of a whole new research field: *didactique* or, as we call it by its English name, didactics. The main point here is that while a lot of desk work has been necessary on the way, TDS is not a desktop project: it is based on carefully designed and repeated experiments in real teaching-learning situations and has as such maintained a close relation to practice.

The following introduction to basic concepts and models of TDS serves only as a brief account of the theory. For more complete surveys, one can refer to e.g. Måsøval (2011), Warfield (2013), or Brousseau (1997) (ordered by increasing volume and abundance of details).

Indicated by the name of the theory, the pivotal concept is that of *didactical situation*. The notion of *situation* emphasises that learning is not a static event but a dynamic process. Specifically, it is a basic assumption in TDS that essential learning takes place in the process of a learners' adaptation to a milieu, with a reference to the work of Piaget. Unlike in Piagetian theory, the milieu that students are presented with is artificial; it is constructed by a teacher with a didactical intention to promote specific and explicit learning. In the promotion of that learning, the teacher moreover regulates the students' interplay with the milieu, and accordingly the conditions under which the pupil can - or cannot - adapt to it. The milieu presents the students with a problem and, if so intended, resources to solve the problem. Since learning of specific knowledge can not happen through adaptation to *any* milieu, it must be constructed in close consideration of the target knowledge. In fact, it is a basic assumption in TDS, that for any mathematical knowledge there is a (sequence of) milieu(s) which can "provoke the learning of some key mathematical knowledge" (Perrin-Glorian 2008, p. 3). The situation corresponding to such (a sequence of) milieus is referred to as *fundamental situation*. Fundamental situations are not so much for the direct use in teaching, but comprise a knowledge base of a "set of conditions defining all (or most) such possible situations ... to learn the target knowledge" (ibid., p. 3). Milieus appropriate for fundamental situations are "very difficult to find but the mere search of them is very productive from a didactic perspective" (ibid., p. 3), i.e. for the production of didactical knowledge.

Even when didactical design work does not result in a fundamental situation, the milieu is crucial, since interplays with a milieu is the only way to establish an adaptive learning situation: The problem of the milieu is the (possible) conveyor of the target knowledge in the sense the latter is the answer to the former. Accordingly, the milieu must be designed specifically to meet the potential and limitations of the students intended to interact with it. Though the purpose of pupils' interactions with the milieu is for them to obtain official knowledge (i.e. from the curriculum), the acquisition of it requires each pupil's personal adaptation to the milieu. Subsequently the teacher must ensure alignment between the personal knowledge and the official knowledge (see later).

The model of the didactical situation, presented in figure 7, helps to clarify the teacher's tasks and responsibility regarding the didactical situation: to devise and prepare the milieu (prior to the didactical situation), to hand over the milieu to the students, and to manage the student-milieu interplay. All with an eye to the target knowledge on the one hand, and the students' possible acceptance of the responsibility to (try to) solve the problem on the other hand.

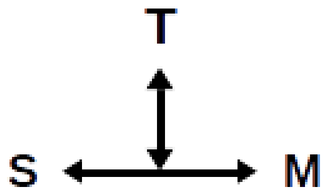

Figure 7: The didactical situation: The double interplay between the teacher (T) and the interplay between student (S) and milieu (M). Adapted from figure 4 in Brousseau (1997, p. 56)

TDS assumes learning to evolve through different levels of the situation. At the basic level 0 the situation is established. Through levels 1, 2, and 3 the pupils' dialectic interactions with the milieu serve for the pupils to approach the problem, hypothesise about the problem and to verify hypotheses about the problem. At level 4, the validated knowledge is aligned with curricular knowledge. Level 0 and 4 are didactical since they are sustained by the teachers' didactic intentions and actions. Level 1 through 3 are (ideally) adidactical since they are sustained by the pupils autonomous interactions with the milieu. Figure 8 illustrates the matryoshka-like interrelations between levels 1 through 4 (after the situation is established at level 0): As the situation progresses, pupils' interaction with the problem at one level becomes the milieu at the next higher level and so forth. Hence, "action at an upper level supposes reflection on the previous level" (Perrin-Glorian 2008, p. 4). Followed by the student-milieu matryoshka, each level is presented in slightly further detail here:

- Level 0, *situation of devolution*: The act of (successfully) getting pupils to accept the *responsibility* to solve the problem under the conditions of the milieu is denoted *devolution* and is crucial. If students do not assume responsibility, they cannot adapt to the problem, and (the intended) learning cannot take place. While the acceptance of responsibility lies with the pupils, it is the teacher's responsibility to have design an appropriate milieu, which poses a real problem to the pupils and at the same time indeed is possible for the pupils to solve, and then to devolve it (devolution can take such meanings as *transference of right* or *transfer of power*). The situation of devolution is didactical in the sense that the teacher directly leads the pupils to something (the milieu) which shall help them to obtain the target knowledge.

- Level 1, *situation of action*: If the milieu is appropriate and devolution is successful (which is assumed in the following), we say that the situation is *adidactical*. In an adidactical situation the students, by having accepted the responsibility for solving the problem, have freed themselves from from their dependency of the teacher (to tell them what to do). That is, even the milieu was constructed with a specific didactical intention, the milieu itself is without intention - it is just there to be dealt with - and the pupils now act oo this *adidactical* milieu regardless of the teachers intentions. With reference to the students' so far immediate actions on the milieu, this type the adidactical situation is denoted *situation if action*. The appropriate milieu retro-acts to the pupils' actions, providing them with feedback. At some point, the feedback enables the students to develop new ideas about how to solve problem.

- Level 2, *situation of formulation*: The adidactical situation and its milieu has now shifted dramatically in character. What was before the adidactical situation in itself has now, enriched with the knowledge conveyed by the feedback from the student-milieu interplay, become a new milieu for students to interact with. With reference to the knowledge obtained in the situation of action, students begin to formulate and share hypotheses about the problem to be solved. This type of adidactical situation os denoted *situation of formulation*.

- Level 3, *situation of validation*: Once again, the character of the adidactical situation changes significantly. In the new situations the previous situation of formulation is the milieu, and the students' interactions with it serves to verify formulated hypotheses or claims. This type of adidactical situation is denoted *situation of validation*. If the hypotheses are rejected, the students may 'return' to a situation of action or formulation, if the hypotheses are validated, the situation may progress into the next level.

- Level 4, *situation of institutionalisation*: If the formulations are confirmed in the situation of validation, the situation may once again transform, though not into a new adidactical situation. The validation of a hypothesis represents knowledge, but this knowledge is contextual and personal. One of the responsibilities of a teacher is to make sure that students obtain the official knowledge of the curriculum, which is void of connections to persons and contexts. In what is called *situation of institutionalisation*, the teacher seeks to de-personalise and de-contextualise the knowledge that the students have personally obtained in the context of their adidactical work with problem.

The interconnection between levels of situation and milieu, as described above and depicted in figure 8, bear some resemblance with the *vertical structure of the milieu in a didactic situation* presented by Perrin-Glorian (2008, p. 4) (which is itself modified from figure 2 in Brousseau (1997, p. 248)). But whereas, Perrin-Glorian (2008)'s model - with good reason - makes the didactical situation a point of reference (i.e. designated as level 0), the model presented here take takes as its starting point (i.e. level 0) the initiation of the (theoretical) sequence of situations: the situation of devolution. While this can be justified in itself, it is moreover consistent with Brousseau and Gibel (2005)'s levels of reasoning in adidactical situations, or, as it is formulated "the classification of forms of reasoning according to their function and type of situation" Brousseau and Gibel (ibid., p. 18).

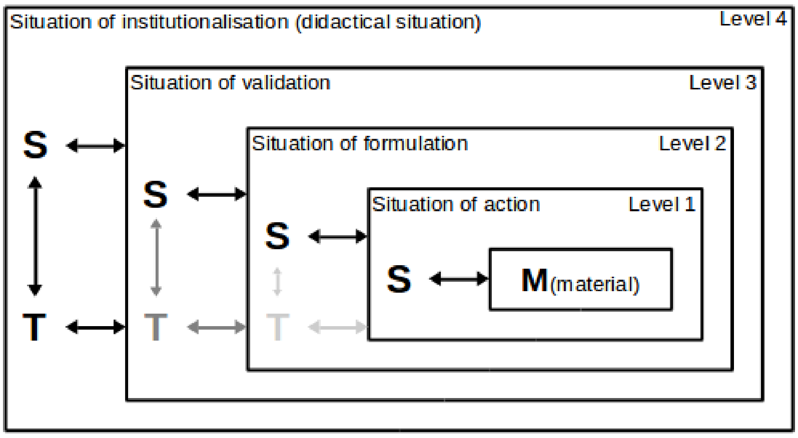

Figure 8: The situation-milieu matryoshka. Students' interaction with the problem (i.e. the situation) at one level becomes the milieu at the next higher level. Levels 1 through 3 are adidactical since they are sustained by the students' autonomous interaction with the milieu, i.e. independent from the teachers didactic intentions. The grey scaling of T at level 2 and 3 indicate typical levels of teacher intervention (which interrupt the adidactical situation). Level 4 is didactical since it is sustained by the teacher's didactic intentions and actions. Inspired by figure 2 in Brousseau (1997, p. 248) and figure 2 in Perrin-Glorian (2008, p. 4).

The situation of institutionalisation at level 4 is necessarily led by the teacher: only the teacher holds the (didactical) knowledge and power to institutionalise. Furthermore, the institutionalisation regards mathematical knowledge developed in pupils' previous situation of validation, and is conveyed through communication with these. In figure 8 this is illustrated by the (opaque) 'T' and its interactions at level 4. The level of opacity (i.e. the level of 'greyness') of 'T' and its interactions at levels 2 and 3 indicate that though the situations theoretically are adidactical, in reality, the teacher may feel a need to intervene, and these possible interventions can be qualitatively and quantitatively gradual. The possible 'need' for intervention is produced by tensions in the didactical contract (see later). Unless the student follow direct instructions (from the teacher or teaching materials), the interactions at level 1 are always adidactical.

It is important to observe that the (group of) students in different situations need not be the same (Brousseau 1997, p. 249). One student's interplay with the milieu at one level (e.g. one pupil's immediate action and the retroaction of the milieu at level 1) may function as a situation of reference, i.e. (part of) a milieu, for a different student at another level (e.g. formulations of hypotheses at level 2). In this sense, a situation involving one student may enrich the milieu of another.

The knowledge to be learned is at first concealed in the milieu, but emerges and evolves gradually through level 1 through 4. At level 1, the situation of action, the knowledge is still implicit. At level 2, during the situation of formulation, the knowledge emerges. At level 3, by way of the situation of validation, the knowledge has become explicit and is consolidated at level 4, in the situation of institutionalisation.

Indeed, learning and knowledge is not always explicit in the didactical situation. Even when a student is not aware of some new knowledge he or she has obtained, this knowledge can be revealed through a change of practice. At the same time, what appears to be an expression of learning may in reality be either a (wild) guess, or an effect of the *didactical contract* (e.g. Brousseau 1997; Hersant and Perrin-Glorian 2005).

Didactical contract is a central notion in TDS and allows us to analyse if knowledge and learning appearing in the didactical situation is the result of (adidactical) student-milieu interactions or of students' efforts to meet (what they conceive as) the teachers' expectations. The former is the desired effect of contracts for didactical situations planned to meet the principles of constructivist learning situations. The latter is an unfortunate effect of the predominant paragraph of the contract that *something must be learned* (Brousseau 1997, p. 208). If the pupils have not been able to take on the responsibility for solving the problem, pupils and teachers alike may be led to take shortcuts to evade the cumbersome and time-consuming efforts of whole-class inquiry based learning (but then, in fact, lose out on its benefits). This could be the case if the students do not understand the conditions of the milieu or do not think they have the knowledge to solve the problem or if the teacher does not have the knowledge to adjust the situation or to even realise its inappropriate inconsistency.

The usually implicit contract and its effects stem from the pressure between a task and some expectations of how to carry out that task, including how to perceive the task. Hence, the contract is "a way if regulating the mutual expectations of the teacher and the students with respect to the mathematical notion at stake" (Hersant and Perrin-Glorian 2005, p. 116). The contract is in principle negotiated from one didactical situation to the other, but are often based on earlier contracts. The negotiations generally take the form of indirect, unspoken, and even unconscious, exchange of perceptions and expectations, but the design and devolution of the milieu plays a crucial role.

Owing to the work of e.g. Hersant and Perrin-Glorian (ibid.), we consider three levels in the didactical contract: the *macro-contract* refer to the "mathematical field", the *meso-contract* regards the "didactical status of the knowledge and the characteristics of the situation, and the *micro-contract* concerns the apportionment of responsibility (ibid., p. 120). While the existence of a contract is an inevitable consequence of the social aspect of didactical situations, the analysis of it provides crucial insights into what is not so apparent at first sight.

During the research for this thesis, the notion of *didactical idea* emerged. A didactical idea is a hypothesis about the connection between specific teaching and learning under specific circumstances: that the devolution of a certain milieu to specific students will lead the students to specific potential student-milieu interplays which, under further specified regulations, will lead the students to a certain point (of knowledge, awareness etc.).

TDS and open approach method

The research for this thesis includes an analysis of a lesson based on the principles of open approach method by tools from TDS. This is out of the ordinary, both for the teachers involved and for TDS, which was developed to investigate and experiment classical teaching-learning situations. Yet, Miyakawa and Winsløw (2009) and Miyakawa (2015) has provided successful examples of such analyses, and indeed, as is also illustrated by e.g. Hersant and Perrin-Glorian (2005), TDS applies directly and immediately to describe and analyse a wide range of important elements of mathematics lessons not designed within the framework of TDS.

TDS and open approach method obviously share several traits, but by closer examination some differences are revealed. One of the differences regards how the mathematical (piece of) knowledge in question relates to the target knowledge.

- In TDS-based lessons, the acquisition of the mathematical knowledge in question is the *objective of study*, it is the *end* of the lesson in itself, it is often specific and constitute the target knowledge.

- In open approach lessons, on the other hand, the mathematical knowledge in question is the *object of inquiry*, the *means* to acquire the target knowledge, which is often less specific.

As the objective of open approach method is to ”recite, foster or develop students’ ideas through discussions” (Nohda 1995, p. 57), the target knowledge in lessons based on the open approach method is often less tangible than in TDS, but generally regards students’ ”mathematical problem solving ability” (Hino 2007, p. 507). In OEA, the target knowledge comprise not only mathematical knowledge but also ”mathematical ways of thinking, beliefs, and meta-knowledge of ”how to learn.”” (Nohda 2000, p. 47).

In both, pupils’ autonomous work is deemed essential, but whereas adidactical situations in TDS is the means for the pupils’ to develop the target knowledge, they are in open approach method a means to develop initial ideas and arguments to establish a mathematical discussion. Both TDS and open approach method highly value the social aspect of learning, but in the above sense, open approach method has put a relative larger emphasis on it. This requires for the teacher to not only design and devolve a rich material milieu, which can (more or less) in itself sustain a fruitful student-milieu interplay, but furthermore to prepare for *expedient decision-making* to establish and sustain the continuous enrichment of pupils’ milieu by other pupils’ situations or results thereof.

While this difference does challenge the *application* of TDS for the analysis of open approach lessons (since it is different from traditional TDS analyses), it does not so much challenge the *applicability* (i.e. whether or not it is useful for analysing open approach lessons). When investigating an open approach milieu or its corresponding situation, TDS offers effective tools and models to analyse possible student-milieu interactions, and hence to anticipate students’ ideas, and the possible ways in which they can be used to enrich the milieu of other students. For this purpose, the situation-milieu matryoshka of figure 8 is beneficial.

TDS and lesson study

Though Lesson study and TDS may appear somewhat distant from each other, there are at least two connections, which the research in this thesis takes advantage of: Firstly, the aim of lesson study is to develop and share knowledge about the complex dynamics of the teaching and learning of specific subject matter knowledge and its dependency on the design of the lesson. TDS provides a set of precise tools and vocabulary to analyse and describe such complex dynamics. Secondly, the methodologies of TDS, i.e. didactical engineering, and lesson study both profess internal validation through the realisation of a teaching-learning hypothesis in a live lesson. The research for this thesis assumes that with proper transposition these connections apply for both *research on lesson study* and *teachers development of knowledge in lesson study*, as well as for *facilitation of lesson study*.

When teachers engage in a lesson study, they become learners - students who interact with a milieu in order to develop knowledge which can solve a problem. In lessons, pupils’ milieu include a problem of mathematics and pertinent resources to solve it. In lesson study, teachers’ milieu comprise a teaching problem, along with a didactical situation, i.e. a pertinent resource to solve the problem. Thus, with reference to the learning situation of students (figure 7), the learning situation of teachers engaging in a lesson study can be modelled as in figure 9:

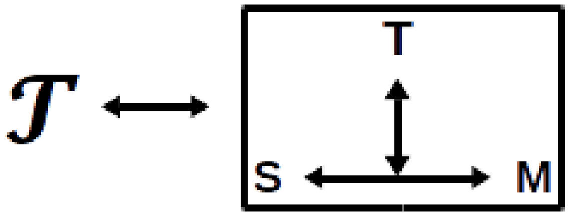

Figure 9: The interplay between a group of teachers ($\mathcal{T}$) and their milieu in paradidactical situations. Adopted from Winsløw, Bahn, and Rasmussen (2018).

Though there is in this sense a distinct similarity between pupils' learning situations in lessons and teachers' learning situations in lesson study, it is important to observe a defining difference: In the didactical situation, the mathematics lesson, the teacher is on the side of a mathematical situation, which unfolds between pupils and their milieu, and is in fact trying to enhance the mathematical practice in and by it. But the activity of the teacher is not mathematical. Likewise, in a lesson study, teachers place themselves on the side of a didactical situation, which unfolds between a teacher and the pupils and their milieu, and the group of teachers placed on the side are in fact trying to enhance this didactical practice. But their activity is not didactical. Since teachers' practice in lesson study run in parallel to their practice in the didactical situation, we refer to it as *paradidactical* (cf. Winsløw 2012). Accordingly the paradidactical practice takes place through teachers' *paradidactical interplays* with the *paradidactical milieu* in *paradidactical situations*, etc. Notice that the paradidactical milieu is the teachers' milieu mentioned above, comprising the teaching problem and the didactical situation which serves as a resources to solve it.

Moreover, since there is no teacher of the teachers in paradidactical situations, the milieu is not devolved and there is no pre-established knowledge to be institutionalised. Paradidactical milieus essentially take three forms:

1. Anticipated didactical situations, DS_A: The anticipated or theoretical didactical situation against which the hypothesis is developed and which may materialise into the lesson plan.

2. Real didactical situations, DS_R: The live research lesson in which one member, T, of the group of teachers, $\mathcal{T}$, is teaching.

3. Observed didactical situations, DS_O: The observed version of the realised didactical situation, as each and all members of $\mathcal{T}$ (except T) or other observers observed (and recorded) it.

Considering the methodological and time-wise conditions of the three types of paradidactical milieu, we define the corresponding paradidactical situations as below. The correlation between paradidactical situations and milieus is depicted in figure 10.

1. Pre-didactical situations, PrS, in which didactical ideas are developed. The milieu is of type DS_A: Didactical ideas and their corresponding milieu are developed through a *virtual* process similar to the sequence of adidactical situations. $\mathcal{T}$'s interplay with the milieu (DS_A) and resembles that of adidactical situations at level 1 and 2.
   The explicit hypotheses put into the lesson plan resemble *formulation* (level 2) whereas the details of the lesson plan which do not reflect a (part of a) hypothesis, resemble *action* (level 1). The (final) actions and formulations, represented in the

lesson plan, constitute the action on the subsequent milieu, DS_R.

2. Observational situations, ObS, in which data from tests of didactical ideas (in didactical situations) are collected. Here DS_R is the milieu: While the action on DS_R is launched from $\mathcal{T}$ in PrS, the retroaction from DS_R is observed in ObS.

3. Post-didactical situations, PoS, in which the collected data are analysed and the didactical ideas revised. Has DS_O as a milieu: $\mathcal{T}$ś interaction with DS_O corresponds to that of adidactical situations at level 3: Through the analysis of (elements of) DS_O, $\mathcal{T}$ confirm or rejects their hypotheses (at the present state). This resembles *validation* (level3).

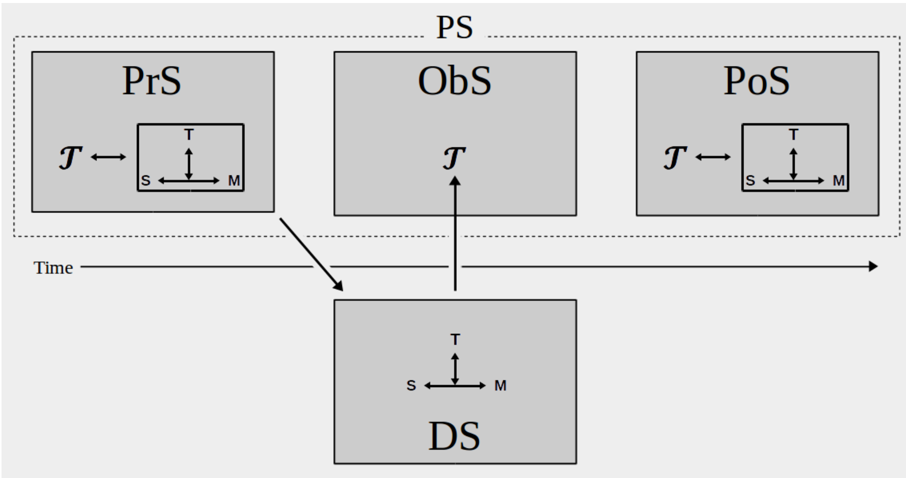

Figure 10: The correlation between paradidactical situations and their milieus in a lesson study. In the predidactical situation, PrS, the group of teachers, $\mathcal{T}$ interplay with anticipated didactical situations (DS_A). Eventuelly, the didactical situations hypothesised to be a solution to the teaching problem are put into the lesson plan, to be enacted in the didactical situation, DS, of the research lesson. In the Observation situation, ObS, the retroaction from the real didactical situation (DS_R), which is the didactical situation, DS, itself, is observed. The arrow from PrS to DS illustrates the $\mathcal{T}$'s action and formulation on DS, and the arrow from DS to ObS illustrate the retroaction from DS. Since not every single detail of the real didactical situation can bed observed, and some details even may be incorrectly perceived, the observed didactical situation is not identical with the real didactical situation. In the postdidactical situation, $\mathcal{T}$ interact with the observed version of the didactical situation. Adapted from figure 2

In the predidactic situation (PrS), the didactical idea to be tested in the research lesson is formulated. The research lesson is comparable to a laboratory in which the idea is tested and data are collected via observations (in ObS). In the post-didactic situation, the hypothesis of the didactical idea is validated, based upon the feedback from ObS. The realisation of the lesson plan, or even the lesson plan itself, may include some immediate attempts, i.e. actions of which the effects have not been hypothesised. Hence the lesson study process clearly resembles the adidactical learning process of levels 1 through 3, illustrated in the situation-milieu matryoshka (figure 8) (without a possible intervention of a teacher of the teachers).

When teachers, in the predidactic situation, devise the didactical ideas to be tested in the research lesson, they undergo a similar processes a number of times. When an activity or a milieu is proposed, this is similar to an immediate attempt to solve the teaching problem, i.e. it is a situation of action. Anticipations of how pupils would respond to that action produce some virtual feedback which allows the teachers to confirm or reject that action as a possible solution to the teaching problem. Whether based on the virtual feedback from a situation of action or on teachers prior experience and knowledge, didactical ideas, that are hypothesised to (partly) solve the teaching problem, are devised. Teachers' anticipation of the corresponding didactical situation resembles a virtual test of the didactical idea. The (virtual) feedback from the sequence of virtual tests, conducted throughout the pre-didactical situation, allows the teachers to gradually develop more and more coherent didactical ideas. Eventually, the didactical ideas that are hypothesised to (best) solve the teaching problem are put into the lesson plan.

The anticipatory interaction with the didactical situation is an important activity of lesson study (e.g. Fujii 2016a, p. 420; Takahashi 2011, p. 79), and the result of level-1-through-3 processes regarding, the DS_A put into the lesson plan, function itself as a situation of formulation regarding DS_R (which is observed as DS_O).

For the researcher, this distinction between didactical situations, in which teachers' didactical knowledge *is put into practice*, and paradidactical situations, in which teachers' didactical knowledge is *developed through (paradidactical) practice*, helps to discover and analyse specific phenomena and their possible interconnection. An expanded version of the model is particularly convenient in the case of multi-cycled lesson studies. Figure 11 illustrates a model of the paradidactic structure of a three-cycled lesson study, as those under scrutiny in this research:

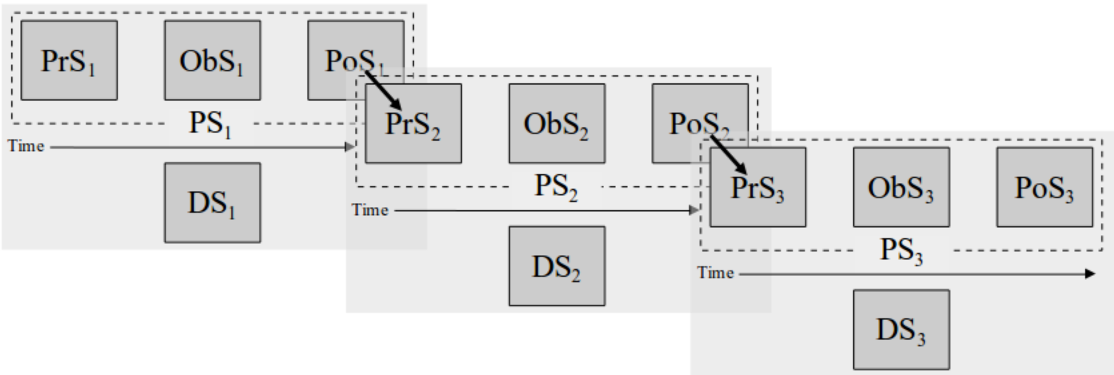

Figure 11: Paradidactic structures of a three-cycled lesson study. Here DS, PS, PrS, ObS and PoS refer to didactic, paradidactic, predidactic, observational, and postdidactic situations, indexed according to the cycle. Adapted from Winsløw, Bahn, and Rasmussen (2018).

The arrow from PoS_n to PrS_{n+1} illustrates that while PoS and PrS are distinct in nature, they may (in fact, in our experiments, did) take place in the same session: In PoS the observational data from the preceding research lesson is shared and analysed. When the feedback from this motivates a revision of the didactical ideas to be tested in the succeeding research lesson, the situation has turned into a PrS of the following cycle.

Together with the models of learning, the model of the paradidactical structure offers a wide range of affordances which all apply well with the functions and (various) configurations of lesson studies (Winsløw, Bahn, and Rasmussen 2018). Depending on the objective, the models are beneficial to analyse the connections (and possible gaps) between:

a PrS and PoS: reflecting change in teachers' didactical knowledge
b PrS and DS: reflecting the realisation of the hypothesised didactical situation
c PrS_n and PrS_{n+x}: reflecting change in teachers' paradidactical practice (and hence didactical knowledge) when planning a research lesson, e.g. in the form of their interac-

tion with DS_A)

(modified from Winsløw, Bahn, and Rasmussen 2018).

The research presented here especially takes advantage of (a) and (c), which are applicable for research to analyse the development and evolution of teachers' learning. (b) is important for researchers and teachers alike, for analysing of the possible realisation of hypothesised interplays in didactical situations. In principle, teachers always employ this pattern to some extent in PoS, which again provides material for the researchers' analysis of PoS.

Research questions

The overall purpose of this dissertation is to study *under what conditions lesson study can contribute to teachers' development of didactical knowledge conducive to devise and manage pupils' inquiry based learning*. The research questions below are indexed (encircled number) according to the article in which it is investigated:

① Which conditions in the didactical milieu are of significance to realise principles and potentials of open approach method?

② What did the teachers learn from participating in lesson study, and how did that learning emerge? Specifically, which concrete mechanisms led to that learning?

③ In which ways did teachers' anticipation of didactical situations evolve in the course of three lesson studies? To which extent did the teachers' attention to different aspects change from the first to the last lesson study? How did these aspects depend on each other, and were some consistently considered more important by the teachers?

Context, data and analysis

The empirical studies for this research took place at three public schools in Lyngby-Taarbæk Municipality, just North of Copenhagen, Denmark. The research project was initially designed independently of the municipality and the schools. The research data pool comprises 136 hours of video clips, representing the total amount of time spent on lesson study activities and seminars, all recorded in (almost) full accordance with the initial plan. The successful collection of these data is to a high degree due to two aspects: a close attention paid to the daily conditions and routines of schools and teachers during design, and the strategic efforts to inform and engage schools and teachers during recruitment.

Based on my five years of experience as a teacher (of mathematics, science and physical education), the empirical studies were carefully planned to obviate possible obstacles to participating teachers' attention. On a large scale these comprise such events as holidays and special feature weeks. On a smaller scale it regards e.g. teachers' day to day schedule and tasks, and the possible absence from and neglect of classes. Regardless of the scale, conflicts or unsettled issues concerning such issues potentially pose an obstacle to the wholehearted attention of the participating teachers. This further motivates the call for careful considerations about design and recruitment. Acknowledging the school leaders' leverage in this respect, it was from the beginning an explicit ambition to establish binding and supportive connections between the participating teachers and their school leaders.

Furthermore, the ambition comprised a corresponding connection between the school leaders and the administration of the municipality. Fortunately officials of the administration shared this ambition.

Context

At each of the three schools involved, one team of three or four experienced teachers from the same or neighbouring grades conducted three three-cycled lesson studies in the course of one school year (2015-2016). In each lesson study, the teachers developed and tested a lesson plan which was guided by the principles of the open approach method. A facilitator (the researcher) supported the teachers during all sessions. Besides basic introduction to lesson study and open approach method, concepts and models from the theory of didactical situations were introduced to support the teachers work. Selected sessions from across the nine lesson studies are analysed with reference to the research questions.

With the point of departure in one school year, the time-wise boundaries for the design were established by the number of weeks available and the budget for compensation of the time teachers would spend on the project during the year. In order to answer e.g. questions regarding the evolution of teachers' knowledge in the course of three lesson studies (research question 3), two weeks could be allocated to each of the nine lesson studies to be conducted. On that basis, the financial boundaries of the project allowed for compensation of twelve hours per lesson study per teacher, plus additional time for individual teacher's desk-top work and all teachers' participation in start-up and conclusive seminars. Based on a wish from the researcher, and as a result of the cooperation with the administration, the school leaders accepted to expand the time-usage to thirteen hours per lesson study plus two hours for *mutual visits* at the cost of the schools (the distribution of time and financial load will be elaborated on later, see e.g. figure 13).

The concept of 'mutual visits' refers to how the teams would visit each others' last (the third) research lesson and the following reflection session of each lesson study. Based on the Japanese tradition for sharing didactical ideas between teachers in and across schools via open lessons (whether for demonstration or study), mutual visits were believed to potentially amplify the teachers' outcome of the lesson studies, and hence the outcome of the research project. The school leaders' decision to accept the expansion and expense of the time each teacher was to spent on the project was made after their participation was agreed upon.

Originally two teams were considered to be enough, but three were chosen for two reasons. First of all, if one or more teams would have to abort the project, e.g. in case of a school's organisational change, participating teachers sickness, parental leave etc., the project as such would not be too vulnerable. Secondly, it was believed that the mutual visits would be more beneficial with three teams, though they would still function as intended, should one team leave the project. In reality, there were only minor changes to the schedule presented in figure 13: a few changes were made as to which team conducted lesson study in which weeks. Also, before one team's last lesson study, one teacher shifted workplace.

Recruitment and negotiation

At the core of the recruitment strategy was the ambition that participating teachers and school leaders should understand well the purpose, conditions and obligations of the project. This included how lesson study and open approach method relate to and is different from normal teaching and teacher development (TDS was not discussed before the start-up seminar; see later). In cooperation with the involved municipality officials, the following recruitment strategy was unfolded:

Under the impression that the administration considered the approaches in the project valuable, the teachers would be presented to the project from beneath, while at the same time the school leaders would be presented for it from above. The aim was to generate a shared interest and involvement in the project between teachers and leaders, in order to optimise the schools' willingness to join and to assume the necessary obligations to accomplish the three lesson studies.

One of the officials was *consultant of mathematics*, responsible for coordinating the network of *supervising mathematics teachers* (*matematikvejleder* in Danish) across the municipality. Supervising mathematics teachers are ordinary teachers who undertake a formal education and are appointed specific tasks related to the school's team of mathematics teachers. The specific tasks and the time allocated to perform them can vary from school to school, but usually three tasks are comprised: to organise the mathematics teachers' team meetings (typically of two hours, three or four times in a school year), to keep up-to date with and distribute information from official sources (ministry and municipality), and to support the other mathematics teachers when they have questions or needs regarding their teaching. These may vary from simple questions about supplementary (sheets of) tasks to complex and undefined challenges in the hot spot between social dynamics and learning (e.g. how to make pupils of different 'learning styles' engage in learning under the same conditions). The function as a supervising mathematics teacher is a supplementary task to ordinary teaching.

In contrast to its name, it is not the aim of the supervising mathematics teacher education to supply supervising mathematics teachers with neither broader nor deeper knowledge about mathematics let alone didactics (of mathematics). Furthermore there are no specific requirements to become a supervising mathematics teacher, except for the formal education (some bear the task of a supervising mathematics teacher without the education. These are often referred to as *coordinating mathematics teacher* (*matematiktovholder* in Danish), but as it plays no role in the present case, they are not distinguished here). In the context of this research, the supervising mathematics teachers involved in the initial phases played an important role due to their function as a link between the research project, the municipality, the school leaders and the mathematics teachers at each school. The role of the supervising mathematics teachers involved in the research project will be described later.

The consultant of mathematics was himself a supervising mathematics teacher at one of the schools, working two days a week for the municipality administration. Through centralised network meetings and meetings at schools, the consultant of mathematics informed the supervising mathematics teachers about the project and discussed the possibilities for participation, including how to engage the school leaders.

The other municipality official is *chief consultant* with responsibility of school development projects (among other things). In matters regarding school development (and other issues) she represents the municipality administration at the regular (and other) school leader meetings, where the leaders of the eleven schools in the municipality meet with the administration on a variety of matters. The project was brought up and presented in this forum at least three times. Both the chief consultant and the consultant of mathematics had prior superficial knowledge about lesson study and Japanese teaching.

In Denmark, where the research took place, the school year runs from August to June. Accordingly, a school year is planned in the preceding spring. For each school participating in the present research project, three or four teachers were to spend approximately 60 hours (in the final version; see later) in predefined patterns coordinated with two other schools during one school year. For most schools and teachers, that is a rather big commitment and hence it was crucial to negotiate all details of the conditions and the schools' obligations as early as possible. As the research project started January 1 (2015) there was a certain time constraint from the beginning.

Soon after the project was accepted for funding (in November 2014), the chief consultant presented the project at a school leader meeting. Concurrently, the consultant of mathematics began promoting it through the network of supervising mathematics teachers. Shortly after the official start of the project, I was invited to a school leader meeting at which I elaborated the purpose and design of the research and in particular the empirical studies.

I was afterwards invited to schools which were interested in participating in the project. Here I met with school leaders, supervising mathematics teachers and potentially participating mathematics teachers. I had had no influence on the choice of teachers other than a general advise to the leaders to consider teachers' they believed could participate 'wholeheartedly'. At these meetings at the schools, the background, purpose and content was elaborated further, giving the leaders and teachers the opportunity to pose questions to any part of the project. The basics of open approach method and lesson study were explained, and an example of the concrete plan was presented and discussed (see appendix a, slides 5-8: lesson study, slides 9-12: open approach method; in Danish). The purpose of these meetings at the schools was two-fold: to ensure the highest possible level of informed consent (or rejection) including the appertaining obligations and conditions, and to disclose and adapt the project to possible overlooked conditions. These meetings took place between January 21 and 30.

By the beginning of February, six schools had expressed their interest in participating (one withdrew due to other obligations), which put us in the dilemma of selecting and deselecting schools. The main criteria for selection was the degree to which the school displayed:

- concordance between the participating teachers, the supervising teacher and the school leader
- willingness to prioritise the project (among the schools' and the teachers' many other obligations)
- critical stance to the project (reflecting careful considerations)

At a third school leader meeting addressing the project on February 19, we presented the three selected schools (here referred to as school A, B and C), the selection criteria, the selection procedure and the subsequent procedures for negotiation. These included further negotiations of organisation, expectations and obligations to be summed up in a written and signed agreement. Another purpose of this meeting was to commend the schools which had most keenly engaged in the preceding process.

In the following two months, i.e. from mid-February to mid-April, I (and in some cases the consultant of mathematics) met with the teachers, supervising teachers and leaders at the selected schools a number of times to discuss and adjust the final details. In this period, the idea about mutual visits emerged during discussions with my supervisor, along with the wish for a little more time for preparing each research lesson. At a meeting with school leaders of the three participating schools on April 17, a draft for a written agreement was presented, to which all school leaders agreed. Subsequent to the original agreement the idea of mutual visits was presented along with the wish for one more hour of preparation in each lesson study. Both with the snag that the budget of the project could not bear such expenses. The school leaders agreed to these ideas and accepted to bear the extra expense (i.e. of nine hours per teacher). Subsequently, the final agreements were put forward and signed.

The above account for the procedures of recruitment and negotiation serve the purpose of illustrating the background of the successful conduction of the empiric studies (at least in the meaning of implementation and data collection). I strongly believe that my experience from and knowledge about the daily conditions and routines of teachers and schools, here utilised to lead a continuous and close dialogue with the municipality and especially the teachers and leaders, play a vital part in this. Likewise, I believe the successful engagement of schools and teachers owe to the recruitment strategy developed and implemented together with the two involved municipality officials.

Structure and organisation

Based on the final agreement with the schools, time for each lesson study was allotted as presented in figure 12.

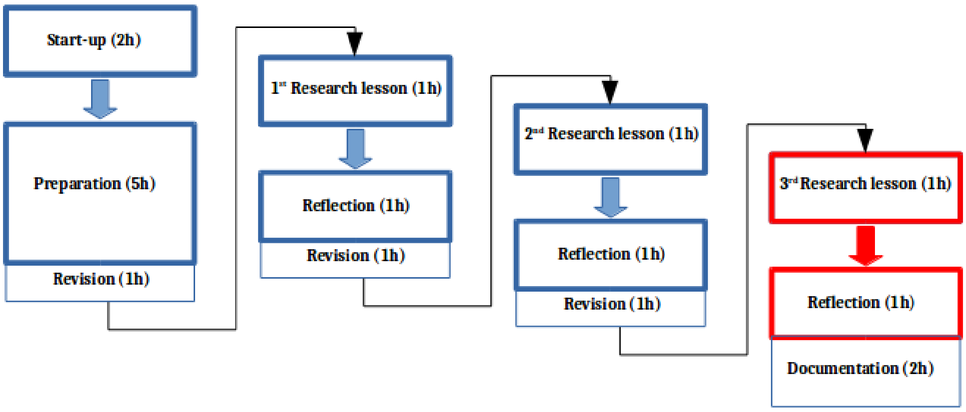

Figure 12: Distribution of time in each three-cycled lesson study. Thin-lined frames indicate individual desktop task, red-lined frames indicate *mutual visits*, i.e. participation of the other two teams. Please note the difference to figure 11 which illustrates the distribution of *types of situations* of each three-cycled lesson study.

On top of the 13 hours of team based activities, individual teachers would clerk in shift, such as to revise the writings of the lesson plan (one hour) and to write final documentation (two hours) in each lesson study. The schools and teachers were initially encouraged to divide the preparation into two, which could be conducted in different days. In practice though, preparation was planned and conducted on one day. One reason for this was to keep each lesson study within approximately two weeks, in order to be able to conduct them in a staggered pattern. This was necessary for sakes of facilitation (and observation) and my additional obligations (Ph.D. courses, study abroad etc.). Start-up meetings (not to be confused with start-up *seminar*) were scheduled on separate days, and for each research lesson, the reflection session followed immediately after.

In addition to the 14~15 hours spend on each of 3 three-cycled lesson studies, each teacher spend 6 + 4 hours at the start-up and conclusive seminars and another 3 x 2 hours for *mutual visits*. Hence, in total, each teacher spend approximately 60 hours, scheduled as illustrated in figure 13.

	start of school year		Start-up seminar							Study abroad					PhD course	Evaluation				Xmas holidays	Xmas holidays
Week number	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
School A			S	X	X												X	X			
School B			S										X	X							
School C			S			X	X														

Autumn 2015

			Conference			Winter holiday	Evaluation				Easter holiday										Conclusive seminar	school activities		End of school year
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
								X	X												C			
	X	X											X	X							C			
				X	X											X	X				C			

Spring 2016

Figure 13: Plan of the empirical studies during the school year (2015-2016). 'x' illustrates lesson study work activities.

Out of the approximately 60 hours each teacher would spend on the project throughout the school year, the schools took the expense of nine hours, equivalent of approximately one sixths. The residual hours (approximately 50 per teacher) were covered with funding from the project. While it would have been difficult for schools to engage in the project without compensation, their contribution illustrates commitment, which it may indeed have reinforced.

One of the major challenges was to ensure the teachers attention in the double sense of physical and mental presence during lesson study activities. These issues were addressed to the school leaders. The latter by calling the leaders' attention to it, and the former by presenting an overview of the overall plan and the time-wise obligations in-school and across schools (for mutual visits). See appendix c for further details.

Since all lesson study activities were to be recorded on video, there was an ethical and legal challenge regarding the research lessons in which pupils would naturally be recorded. In agreement with the school leaders we formulated and send a letter to the parents of the classes involved at each school. The letter informed about the purpose and procedures of project, including that recordings would take place in research lessons (for each pupil involved this would amount to a maximum of three lesson during the school year). Finally the letter, addressed from the school leader, requested that "If for reasons your child can not participate, please let me know" (Letter to parents, appendix d, my translation). No children could not participate.

Participants

From the three participating schools, a total of ten teachers participated. At two of the schools the supervising mathematics teacher (described above) also played a role. All ten teachers and the two supervising teachers had a minimum of ten years of experience teaching mathematics, for which they were all formally trained.

At school A, the team comprised three teachers in grade five. The school's supervising mathematics teacher, who was the only teacher involved in the project with some (i.e. a little) lesson study experience, was not formally a part of the team but attended most sessions. One of the teachers at school A was not deeply enthusiastic about the project (this did not shine through her participation, though) and had accepted to join for the sake of the colleagues. That is, if she had not joined, the team would only consist of two teachers which was considered too few to participate.

The team at school B consisted of four teachers of grades three and four. At this school, the supervising mathematics teacher was also involved, but to a lesser extent than at school A. As was the case of school A, one member of the team was initially not too exited about the project but had agreed to join in order for the colleagues to be able to participate (related to technical issues regarding the schedule at the school). This teachers' lower enthusiasm also did not seem to pose a negative effect, and in fact she turned out to be a crucial participant in the sense of instigating the team's deeper analyses of their ideas.

The three teachers constituting the team at school C taught in grades four and five. In the time span of the project, school C did not have the benefit of a supervising mathematics teacher. One of the teachers at school C was hired for a one-year position subsequent to the agreement for the school to join the project. She had apparently applied for the position because it involved participation in the project. As she could not prolong her position, she changed workplace before school C's last lesson study, which was then conducted with only two teachers (it is not part of the analyses of this thesis, but offhand this case appears to support the presumption, that two teachers (novice to lesson study) generate too little dynamics).

Except for the teacher hired for school B during the summer, all teachers had been introduced to lesson study and open approach method during negotiation meetings in the spring.

At the start-up seminar in the beginning of the school year, the teachers were introduced further to the (empirical part of the) project in general as well as to the concepts, models and roles of lesson study, open approach method and TDS. Furthermore it was restated that I would function as a facilitator, and would be present at all times. For the introduction to lesson study and open approach method, two videos were watched and discussed: one illustrating (elements of) open approach method (How many blocks? by Hiroshi Tanaka, http://www.criced.tsukuba.ac.jp/math/video/previous/index.html.en) and one introducing lesson study; its purpose, function and position in the professional development of teachers and school leaders in Japan, and an illustrative lesson study, including an exemplary research lesson by one of Japans most famous mathematics teachers (the video has been retracted due to copyright issues). Besides an oral presentation and the two videos, the participating teachers were handed out a handbook for the research project and to support their work during the lesson studies (see appendix f; in Danish). Since the handbook sums up the introduction to lesson study, open approach method and TDS well, I present in the following a translation of the corresponding passages (from handbook to teachers, appendix f, my translation):

> **Lesson study**
> Lesson study could rightly be translated to *studies of teaching* or *studies of teaching practices*. The fundamental idea is that one continuously, through studies of one's own and colleagues' teaching and didactical thinking, gain more and more knowledge about how to prepare and conduct good teaching for the target audience. By qualifying through punctuated studies, one develops knowledge of and skills for designing teaching, that is used to plan one's normal teaching which

thereby gets better.

In Japan a lesson study can take various shapes. Some lesson studies are mainly conducted by one teacher who seeks professional advice from colleagues. In others, a large group of teachers and researchers work together on experimenting to develop new methods (as in the case of e.g. open approach method). Some times it is just the team of teacher who participate in the research lesson s, at other times these are conducted at a conference with up to several thousands of participants! Lesson studies can also vary in length, as some proceed as long as for a year.

In this project there is a relatively tight frame for each lesson study, as you will conduct them in your team within two weeks. The basic structures are the same, though:

1) Preparation of a lesson plan based on the chosen topic, theme and goal.
2) A research lesson in which the lesson plan is tested and chosen focal points are observed.
3) A reflection session in which the lesson plan and observations are discussed - usually with a particular focus on the pupils' learning.

Phase 1-3 are often iterated in cycles in which phase 1 is typically shorter in the second and the third cycles. In our project vi talk about the iteration of phase 1 as a revision of the plan.

There is often a general theme not related to the topic (i.e. differentiation) and a subject specific topic within which a given fundamental knowledge is focussed on.

Open approach method

It is pivotal in open approach method to let the pupils work on one problem which has several correct solutions, let them work with and develop their own methods, and use the pupils' approaches and suggested solutions to discuss the problem and its possible solutions. In this sense, open approach method prepare the grounds for exercising reasoning, method(s) and known techniques on the pupils' own terms.

Open approach method builds on three principles of

1) The pupils' autonomy.
2) A fundamental knowledge.
3) The teacher's expedient guidance.

Open approach lessons usually consist of three phases

1) Mathematical (re-) formulation of the problem (which is often everyday-like).
2) Test of methods.
3) Promotion of advanced problems and hypotheses through classroom discussions.

It is not easy to design such lessons, and therefore lesson study has not only been used to develop open approach method but also to its implementation. The high level of Japanese pupils' mathematical knowledge and skills is to a large extent attributed to open approach method. This is grounded in the higher level of commitment which is prompted by well-planned and well-managed open approach lessons.

The theory of didactical situations - a didactical tool.
When planning and designing teaching, using a theoretical frame of reference can be useful. It can help to specify one's work and clarify communication within the team and outwards. In the following concepts and tools from the theory of didactical situations (TDS) are introduced.

TDS is a theory about teaching mathematics, based on practice. It is developed through studies of and experiments in practice in the course of several decades, primarily by Guy Brousseau, who was a teacher himself. The theory is complex but essential elements can be extracted for design and analysis fo teaching - which is at the core of our project. Furthermore there is a certain concordance between these tools and they we work in for instance lesson study and open approach method. Hence, TDS can function as an ancillary tool, which gives us some models and a shared vocabulary to discuss didactical design.

At the core and above all the theory states that learning (in school) occur in didactical situations. A didactical situation is constituted by to elements: a teacher with an intention to teach somebody something, and one or more pupil/s with the intention to learn something. This may appear as a banality, but it is crucial: if not the pupil/s has the intention to learn, there is no didactical situation, and learning of the targeted knowledge fail to happen.

The targeted knowledge er the aim of the teaching. Therefore the teachers strives to design a didactical milieu (tasks, materials, tolls, etc.) which inform and encourage the pupils to take the responsibility to work on finding a solution, which leads to the targeted knowledge. Fundamental learning (only) takes place through one's own realisation which is why the tasks /the didactical milieu) must result in an adidactical situation in which the teacher does not intervene: the individual pupil can work in his or her own way (method) on the basis on his or her own qualifications (knowledge). An adidactical situation can be seen as a parenthesis in the didactical situation, in which the adidactical aspect is constituted by the absence of the teacher's direct intervention.

If that was easy, it did not give rise to an independent theory!

For the purpose of design and analysis, TDS dissects the teaching into five phases (not all necessarily occur in every lesson): Open approach lessons usually consist of three phases

1) Phases of handing over, in which the responsibility for the task is handed over
2) Phases of action, in which the pupils feel their way
3) Phases of formulation, in which the pupils formulate ideas and hypotheses
4) Phases of validation, in which hypotheses are sought to be validated
5) Phases of certification, in which newly obtained knowledge is aligned

In the handbook's appendix on TDS, the model in figure 14 is featured:

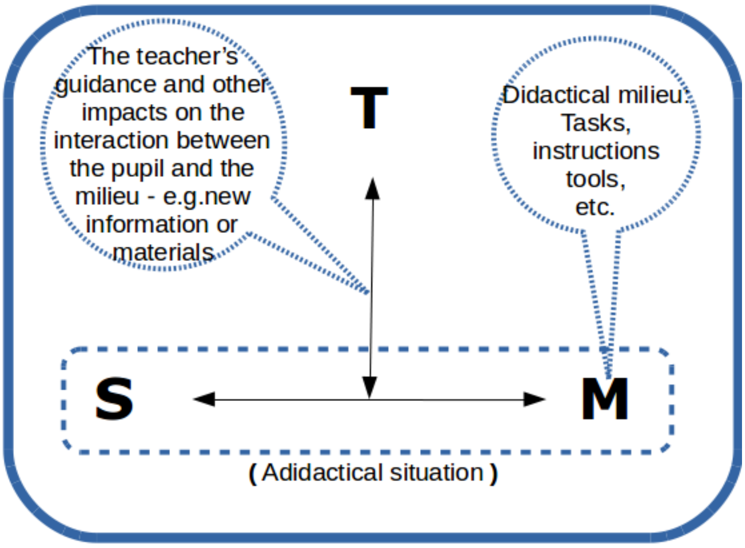

Figure 14: The didactical situation as illustrated in the teachers' handbook (Appendix f, my translation)

Though the presentation of lesson study, open approach method and TDS at the start-up seminar was more detailed than accounted for here, the above illustrates which elements were introduced and discussed.

As will leap out at the reader familiar with lesson study, open approach method and TDS, some adjustments were made in the communication of the basic concepts and models. These choices were intended to bring the concepts closer to the teachers' practice and vocabulary, ultimately with the purpose of minimising the gap between academia and practice. Whether this was necessary or had the desired effect (as opposed to keep the original vocabulary) is out of the scope of this thesis to answer.

The role of the researcher

The researcher was present during all teacher activities in the project, alternating between the role of researcher and that of facilitator. These two roles are different in nature, and at times even conflicting: the role of the researcher is to ensure that the research protocol is complied with, whereas the role of the facilitator is to support the teachers' process and outcome of that process. When the latter is the object of study of the former, such setup is not necessarily conflicting in itself, but it does require close attention and some conscious choices to be made.

This approach was chosen for two reasons: On the one side it was believed that the field notes the researcher could be able to collect when being present with the teachers could qualify the following analysis substantially. Both in the sense of providing additional data, and by supporting the interpretations and choices made during video analyses. On the other side, it was believed that it could be difficult for the teachers to adapt and stick to the new work forms and principles, confer the founding state of the problem. Hence, it was considered of great importance to the teachers' process and possible outcome of the (series of) lesson studies, that they would have the support of a facilitator at all times.

The dual role of researcher and facilitator seem to be not unusual outside Japan (Clivaz and Takahashi 2018, p. 157; Hart and Carriere 2011), but as Clivaz and Takahashi (2018) states, it is not typical in Japan (ibid., p. 157). One reason for this could be the "lack of experienced lesson study practitioners outside Japan" (Takahashi 2011, p. 80), applying to both teachers and facilitators.

Since the involved teachers had never worked with lesson study before, it was naturally a part of the facilitator's role to lead the way. A first 'target knowledge' for teachers novice to didactical studies, is that there is indeed something to be learned about the connection between teaching and learning. This includes to challenge the teachers' old knowledge, to create situations in which the teachers realise that their (old) knowledge is insufficient and often unrelated to their practice, and that their practice is often unrelated to theory about teaching and learning. Obviously, it is equally important to subsequently offer them opportunities to approach this 'missing link', to develop new knowledge which can (at least partly) fill out the gap. This includes situations in which the teachers' experience the didactical consequences of given didactical choices. While this is one of the purposes of the research lesson, it starts in the preparation of it, by means of teachers' interplay with anticipated didactical situations.

As is also demonstrated in article three of this thesis, teachers are generally not trained to anticipate didactical situations, and hence the role of the facilitator included to make teachers do such anticipations. This was done e.g. by presenting the facilitator's anticipations of a didactical consequence to proposed activities or by trying to provoke teachers' anticipation by repeatedly asking questions like ”what are you looking for?”, ”what answers would you like to see?”, ”what do you want [the pupils] to realise?” or ”you say *the methods*, but which methods actually exists?”.

The answers to such questions make explicit the teachers' (old) knowledge which can then be challenged by other team members or the facilitator. Once the insufficiency of that knowledge is realised, it becomes evident why *kyouzai kenkyuu* is considered crucial to lesson study. Yet, *kyouzai kenkyuu* is very complex (Watanabe, Takahashi, and Yoshida 2008) and it was not easy for the facilitator to convince teachers to engage in it. Nevertheless, it is an important role of the facilitator to drive the teachers into *kyouzai kenkyuu* for the double purpose of making teachers explicate their (old) knowledge so the team members and the facilitator can challenge it, and of obtaining new knowledge. This again changes the teachers' perception of the teaching problem and their hypothesised solutions to it.

Another way the facilitator tried to drive the teachers into *kyouzai kenkyuu* (with varying results), was to start examining all possible solutions or pupils' possible responses to a given problem. With a slowly increasing tendency, as is demonstrated in article two, one or more teachers would join these examinations in some of the latter lesson studies.

While the above examples illustrate techniques to instigate teachers' inquiries into knowledge regarding the subject matter and the teaching of it, the facilitator also took on the task of providing inspiration. This was done in several ways. One way was through the organisation and facilitation of mutual visit, in which teachers got inspiration from the other teams specific experiences and ideas. In addition the facilitator proposed specific didactical ideas, either of his own thinking or lent from other sources. For instance, in the lesson study examined in article one which examined ways to teach probability, the facilitator proposed the problem of determining which sack is was best to pick a black marble from: one with 3 black and 7 white or one with 2 black and 3 white. While the idea was later dismissed it led to further discussions about probability and appropriate materials to teach it (*kyouzai kenkyuu*) and the discussion related to it offered the facilitator further insight into the knowledge and the thinking of the teachers.

The chosen videos, shown at the start-up seminar, had been very inspirational to the teachers, who referred to them several times later. In some cases we re-viewed and analysed parts of the videos as a launchpad for discussions about e.g. teaching materials, milieu, devolution, adidacticity etc. (though primarily in other, non-technical, terms).

After school A and C's but before school B's first lesson study, we were so fortunate to have a visit form one of Japan's most esteemed mathematics teachers, Hiroshi Tanaka (Attached Elementary School of the University of Tsukuba). He conducted an open lesson which all participating teachers observed (together with many other observers), including the following paradidactical reflection. The techniques, ideas and knowledge displayed in that lesson, and discussed afterwards, were referred to multiple times in the subsequent lesson studies. Particularly techniques and materials related to devolution and adidacticity was discussed time and again.

In summary, facilitation was done by challenging teachers' ideas and knowledge, proposing didactical ideas, and providing opportunities to observe other teachers' didactical and paradidactical practices and experiences, including examples of expert teachers.

TDS played a central role in the facilitator's work. The concepts and models presented with the theoretical framework was used by the facilitator to analyse and interpret proposed activities into corresponding (anticipated) didactical situations. For instance, when the teachers proposed an activity and some materials for it, the facilitator interpreted these into a milieu and analysed the possible student-milieu interactions. This included an analysis of necessary knowledge of the pupils and the possible feedback from the milieu. Accordingly, the possible learning outcome and its connection with the expressed target knowledge was analysed. Furthermore, it was analysed how the student-milieu could be managed by the teacher, and what knowledge this would require. (an example regarding a proposed activity for pupils to realise ”the most expedient subtraction method” is given in the next section). When considered relevant, (parts of) those didactical situations - whether anticipated or observed - were again analysed together with the teachers, e.g. by asking questions like ”how could that help the pupils to learn?” or ”what options does that provide to the pupils?”.

During the initial presentation of the project, at the start-up seminar and when considered necessary over the course of the nine lesson studies, the facilitator repeatedly emphasised one crucial criteria of success: a good lesson study is not determined by a good research lesson but by wholehearted engagement in teacher learning with due attention to the purpose and the guidelines (which is the responsibility of the facilitator to provide and recall). Hence, teachers were continuously reassured, that responsibility for failure was on the facilitator, while credit for success was on the teacher.

General constraints for the teachers' work with lesson study

While the presence of a facilitator appears to be a conducive factor, other factors are more constraining. First and most of all, Danish teachers are not trained to analyse the connection between teaching and learning of given subject matter knowledge in a systematic way based on theoretical tools. In addition, Danish teachers are not educated to do, and there is no tradition for, didactical investigations in practice. Under these circumstances, introducing and realising lesson study pose considerable challenges.

In principle, teachers do generally not have the methodological prerequisites to conduct inquiries into their teaching and how it conditions the pupils' learning. First off, this constraint came out as a difficulty to realise a lack of connection between their intentions and their teaching. This difficulty seems to be related to a rather broad perception, that teaching is a subdomain of pedagogics and hence, teaching consists of general techniques which are independent of the knowledge to be taught. In accordance, the teachers also did not consider a need to examine and enhance their knowledge of the subject matter in question.

One example, taken from article two, is a teachers' proposal of how to teach "the most expedient method of subtraction" to pupils in grade three: If pupils are to sort cards of different colours which are imprinted with subtraction problems archetypical to different ways of subtraction, they *must* (somehow) realise which method is most expedient for which (type of) subtraction problem. The teachers did not consider, that this task requires the pupils to already know "the different" methods of subtraction in order to choose between them the most expedient one. They also did not consider the fact that such a milieu can offer no feedback, which would leave validation to the teacher's will. Furthermore, they did not observe that their own knowledge is imprecise (cf. the "different" methods), and how this imprecision would be passed on to the pupils.

Such observations are the foundation of inquiries into the connection between teaching and learning, and are related to the initial "target knowledge", that there is indeed something to be learned.

Moreover, the lack of methodological prerequisites include the lack of knowledge about research (like) methods. This challenge displayed itself as a lack of knowledge about how to design and follow a research protocol, i.e. a lesson plan. While each step before devising the lesson plan (i.e. goal setting and *kyouzai kenkyuu*) were difficult in themselves, it turned out to be an excessive challenge for the participating teachers to formulate and write a comprehensive and satisfactory lesson plan within the time available to the teachers. Accordingly, as will be elaborated later, the lesson plans provide little useful data.

From the facilitator's side, it was considered more important to work in-depth with *kyouzai kenkyuu* than to insist on the difficult editing of immaculate lesson plans. While this disposition is debatable, the rationale was that establishing a sufficient foundation for devising the hypotheses thoroughly would at any rate be a prerequisite to formulating them accurately in writing. From the researcher's point of view this is acceptable because the teachers' explicated knowledge and ideas about this were recorded on video. However, for the lesson studies as such, a more thorough preparation of the lesson plans might have led teachers to engage deeper into *kyouzai kenkyuu*, which again might have led to even more coherent didactical ideas, provide a better overview of the research protocol.

Even if the difficulties of devising a satisfactory lesson plan can be observed as a challenge related to the allocated time, it is probably foremost a question of training and knowledge.

It is the task of the facilitator to compensate for the teachers' lack of knowledge and experience, which led to the decisions and priorities described above.

From the beginning of the project, the basic concepts and models of TDS, as presented at the starts-up seminar, were intended to support the teachers' work. In practice though, TDS came to play a minor role as a tool for the teachers' work and in the communication between the facilitator and the teachers. Still, in combination with the concrete inspiration of Japanese teachers' teaching, some of the concepts came into play, though often designated with other terms. Especially devolution, material milieu and adidactical situation came into play. The notion of adidactical situation was usually denoted as such and devolution as *handing over*. The notion of milieu was used a few times, but was generally addressed in *teaching materials* and *problem*.

Relatively early, the facilitator decided to not put too much emphasis on using the vocabulary of TDS. This was chosen as a response to the observation, that by working under the new format of lesson study and with new principles of teaching, it appeared 'too much' for the teachers to also learn these new concepts.

Though using the vocabulary of TDS was not insisted on, the introduction of TDS appears to have supported the teachers in two ways: Firstly, the presentation of the concepts and models helped to illustrate some mechanisms and dynamics in teaching, not least under the principles of open approach method. Secondly, the teachers' adoption of some vocabulary, even after the facilitator dismissed emphasis, appears to have been useful for them in order to describe and work with central elements and details.

In hindsight it is not unlikely, that a soft and increasing persistence in using the TDS vocabulary could have led to more use and benefits of TDS by the teachers.

Data and analysis

Data was collected from the start-up seminar, the conclusive seminar and during all nine lesson studies. The data pool comprises lesson plans, teaching materials, teachers' notes, pupils' productions, pictures, field notes and video of 117 hours of lesson study activities from all of the nine three-cycled lesson studies conducted, as well as video recordings of the start-up seminar (6 hours) and the conclusive seminars (4 hours).

Each research lesson was recorded with two cameras (producing an additional nine hours of video), one focusing on the board (and the teacher) and one focussing on the pupils.

Originally it was considered to include interviews in the data pool. This was subsequently omitted due to recommendations proposed by researchers and teacher educators experienced with lesson study. Before the start-up seminar, I organised a one-day seminar for researchers and teacher educators who had worked with teachers' professional development through lesson study. One of the purposes of the seminar was to gather knowledge and experiences about conduction and facilitation of lesson study. The handbook presented to the teachers was based on recommendations and suggestions shared and discussed during that seminar.

The reason for recommending to not use interviews was partly that we cannot always fully rely on teachers' utterances in interviews. The main reason, though, was that the information we get from observing and analysing teachers actions and utterances while proposing, discussing and determining didactical aspects is already very rich and is in fact also a more reliable source to teachers knowledge, ideas and skills. With respect to the research questions, interviews were not considered to provide further insight.

Lesson plans were expected to offer a rich source to the ideas and intentions of the teachers, but in reality they offered little insight. This is hypothesised to be due to teachers' lack of training in developing and especially writing lesson plans, as discussed above. Transcribed videos have by far been the richest source for analysis, since these, especially from pre- and postdidactic situations, reveal in detail the teachers' knowledge and ideas about teaching and learning.

Table 1 depicts the various topics investigated in each school's first, second and third lesson studies.

School	First LS	Second LS	Third LS
A	Percentage	Variables and equations	Probability①
B	Subtraction②③	Area	Polyominos③
C	Whole or part	Area and volume	Combinatorics

Table 1: Topics in question in each school's first, second and third lesson studies. Encircled numbers indicate which lesson studies are included in which article(s).

Though data were collected in all sessions, only parts of the data have been analysed in the articles included in the thesis. With few exceptions, most sessions and lesson studies were considered to be relevant and rich sources of analysis, and hence equally useful. The last lesson study of School C was excluded from analysis, due to circumstance that the team had been reduced to two teachers (since one changed work place). The final decision of which lesson studies, sessions and episodes to include in the analyses presented in the papers, were to a large extent based on the specific research questions pursued in the individual articles, as well as the constraints imposed by journal articles. In standard journal papers, there is only space to present a few episodes and minimal contextual details, and so one has to choose those which can demonstrate the main answers to the questions without excessive use of space. After selection, the researcher made a verbatim translation of the video recordings of the selected sessions and sessions considered important for the analysis of those sessions. The latter comprise the research lessons of School B's first lesson study, which were relevant in order to analyse the surrounding pre- and postdidactical situations.

Table 2 illustrates which parts of which lesson studies were analysed in which article.

① School A	Third research lesson, first lesson study	
② School B	All pre- and postdidactical situations, first lesson study	
③ School B	First predidactical situation, first and third lesson study	

Table 2: Overview of which parts (marked red) of the lesson studies have been analysed in which article (encircled number). Note that ② and ③ target the same team of teachers, while school C is not represented.

The parts chosen for analysis do not reflect an elimination due to quality, and corresponding parts of other lesson studies could have been equally valuable for analysis. One specific reason for using School B's lesson study for ③ (with the consequence of omitting School C) was to investigate the connection between the teacher learnings analysed in-depth in ② and the long term evolution of teachers' anticipation of didactical situations. The second lesson study of School B was omitted since the analysis was aiming at evolution of teachers' knowledge from the initial state (in the first lesson study) to the final state (in the last lesson study). In this respect, this analysis would have been even more interesting, if the team had conducted more lesson studies.

Analysis

① In the first article (analysis of an open approach lesson) the unit of analysis was the didactical situation, specifically of the second activity in the third research lesson. The objective was to identify aspects of the milieu significant to realise the principles and potentials of open approach method.

In concordance with normal procedure in TDS based analysis of teaching episodes, an a priori analysis of the adidactical potential of the material milieu was conducted, which functioned as reference for an a posteriori analysis of the realised situations. In line with e.g. Miyakawa and Winsløw (2009), the a priori analysis was conducted after the lesson took place. The data for the a priori analysis included the transcribed video of the research lesson and the preceding predidactical situations, and the teaching materials. In addition, field notes were consulted, mainly for the purpose of clarification and verification. When the lesson plan added no value to the data, it was omitted.

In the a priori analysis, nine plausible methods for pupils to approach the problem were identified. These were divided into *comparison* and *measurement*, both with subdivisions. The a priori analysis displayed a rich adidactical potential of the material milieu (at level 1, cf. the situation-milieu matryoshka in figure 8). The possible realisation of (some of) this potential would correspond to the realisation of (some of) the *principles* of open approach method and would possibly enrich the milieus at higher levels, maintaining the possibility to realise the *potential* of open approach method.

Besides the transcribed video of the research lesson, pupils' productions supported the a posteriori analysis of the realised adidactical potential and hence the principles of open approach method. Here, the analysis took advantage of pupils' interactions with the teacher, in which they explained their mathematical ideas and their interactions with the milieu (*in situ* actions and formulations). Further, the videos were used to analyse the teachers' interplay with the pupils and the explanations of their interplay with the milieu.

The a posteriori analysis identified eight different approaches to solve the problem. Five of them were identified in the a priori analysis and one was a variant of one of these. Two were not fully completed. In this sense, a substantial part of the adidactical potential and open approach principles were realised. But the a posteriori analysis also pointed out that the teacher, in discordance with open approach method, did not establish a situation in which pupils could "share, compare and modify" the mathematical ideas developed in the classroom. Furthermore, the teacher returned to a more traditional way of teaching, and reclaimed responsibility for validation. In this sense, the potential that was released in pupils' situations of action and formulation, was not realised, and the full potential of open approach method was not realised.

② In the second article (analysis of teachers' learning) the unit of analysis was the pre- and postdidactical situations, specifically the teachers interplay with *anticipated* and *realised* (observed) didactical situation and their discussion of it. The objective was to teacher learnings, their emergence, and their evolution during a full lesson study. The data includes primarily transcribed videos from all the pre- and postdidactical situations, while field notes and to some extent teachers written materials, including the lesson plans, were consulted.

In this analysis, the model of teachers' learning as emerging and evolving through adaptation to a milieu is used. First, given teacher learnings are identified by an analysis of the teachers' utterances and actions. For each of the identified teacher learnings, the analysis moves backwards through the lesson study to identify situations in which the knowledge of each teacher learning evolved or initially emerged as a retroaction from the milieu of *anticipated* or *realised* (observed) didactical situations.

The teacher learnings identified divide into three: (specific) didactical knowledge (e.g. emphasising 'minus' means subtraction by take-away), generic didactical knowledge (e.g. examples can dictate pupils actions) and paradidactical knowledge (e.g. through iterative tests hypotheses and be confirmed and consolidated).

③ In the third article (analysis of the evolution of teachers' anticipation of didactical situations) the unit of analysis was the predidactical situation, specifically of the first and the last lesson studies. The objective was to elicit knowledge about how teachers' capacity to anticipate didactical situations (can) develop. The primary data are transcribed videos of the selected situations, which is supplemented by field notes, lesson plans, teaching materials and pupils' productions.

The overall analysis consists of a comparison of analyses of the predidactical situation of the first and the last lesson study respectively which focus on five aspects of *the target knowledge, the material milieu, pupils' anticipated response, adidacticity* and *the teachers' management of the student-milieu interplay*.

In the predidactical situation of the first lesson study, the teachers' interactions with the anticipated didactical milieu happens based on no former experience with or knowledge of didactical inquiries. In the last predidactical situation of the last lesson study, the teachers' interplay with the milieu can also draw on their experiences with interacting with didactical situations in two previous lesson studies, five mutual visits and the open lesson conducted by a Japanese expert teacher. By analysing the predidactical situation of each lesson study, the initial and the last state of their learning and knowledge can be compared.

The analysis assumes teachers' learning in paradidactical situations to take place as their adaption to their milieu: a teaching problem and a didactical solution hypothesised to solve it. The analysis furthermore takes advantage of teachers' utterances and actions in such situations. By means of TDS's model of dynamics of the didactical situation and mechanisms driving it, the teachers' utterances and actions are interpreted into what corresponds to an anticipated didactical situation. By comparing these corresponding anticipated didactical situation based on teachers' utterances and actions in their first and last lesson study, the evolution regarding the five aspects is identified.

In this sense, it was examined how the teachers' paradidactical practice, when designing and preparing a didactical situation for the research lesson, developed between the first predidactical situations of their first and third lesson studies.

With a special focus on *the target knowledge*, *the material milieu*, *pupils' anticipated response, adidacticity* and *the teachers' management of the student-milieu interplay*, teachers interactions' with didactical situations throughout the included predidactical situations are identified and analysed.

In practice, teachers' proposals of problems, teaching materials and feedback, and their peers reactions to these, are analysed to identify changes in the quality of the interplay possibly indicating an evolution. Similar data from other situations of the same lesson study have been consulted for clarity and verification.

Overview of presented works and their conclusions

I will now present an overview of my contributions to the research literature through the research papers contained in this dissertation. Texts ①, ② and ③ are empirical research articles, while text ④ is a theoretical synthesis written in collaboration with Carl Winsløw (professor and supervisor) and Klaus Rasmussen (postdoc and university college teacher). The latter text was chronologically produced between the first and the second articles, but is presented in the end here, in order to emphasise the chronology of the empirical articles and the common thread running through them in terms of the (time) span, the unit of analysis and application of the analytical tools:

① In *An experiment with open approach method in grade four probability teaching*, conditions in the teachers' design and management of the didactical situation which promoted or hindered pupils' inquiries are analysed. The analysis regards one research lesson, representing the outcome of teachers' predidactical efforts. The unit of analysis is the didactical situation, with a special focus on the milieu's significance for pupils' (possible) interactions. In order to analyse specific dynamics of open approach method, only minor adjustments were made to the well-established theoretical tools to analyse didactical situations.

② The analyses in *Teachers' learning from their first lesson study - analysed by the Theory of Didactical Situations*, embrace a full three-cycled lesson study, though the research lessons are not analysed in depth (as was done in the first article). The unit of analysis is the paradidactical situation, specifically the teachers' interplay with *anticipated* and *realised* didactical situations. The theoretical tools of TDS were extrapolated from analysing the double interplay of the didactical situation to analysing teachers' paradidactical interplay *with* didactical situations.

③ For the analysis in *Evolution of teachers' anticipation of didactical situations in the course of three lesson studies*, the predidactic situations of teachers' first and third lesson study are examined. The unit of analysis and the theoretical tools employed are the same as in the second article, but expanding from this, the analysis here include implicit teacher learning as observed as an evolution in paradidactical practice, and related to the paradidactical contract.

④ *Theorizing Lesson Study: Two related frameworks and two Danish case-studies* presents a general argumentation and specific suggestions for theorising lesson study. These suggestions include the Theory of Didactical Situations, which focus on the dynamics of learning situations, and the Anthropological Theory of the Didactic, which focus on institutional perspectives. The theories are highly compatible, and in unison they provide powerful tools to theorise lesson study as a whole.

I will now provide an outline of the analysed unit, results and conclusions for each article:

① An experiment with open approach method in grade four probability teaching (article under revision for *Recherches en Didactique des Mathématiques*)

In this article, I investigate a group of Danish teachers' experiment with open approach method. The purpose is to elucidate knowledge about conditions regarding the milieu, which are significant for the principles and potentials of open approach method to be realised.

In the analysed activity of the lesson, pupils were presented with a ”roulette type” wheel of fortune divided into six sectors, unequal in size and altering between two colours (red and yellow). The task for the pupils was to (explore ways to) determine which colour had the higher chance of success. Even so, the problem was devolved as to discover which of the two colours 'covers the most' by means of a ruler, a piece of thread, scissors and a protractor. Hence the original problem of probability was presented - and solved - as one pertaining, in fact, to geometry and more precisely measurement.

The analysis identified four conditions regarding the milieu which promoted the realisation of open approach principles and potentials: Firstly, the milieu was appropriately devolved, meaning that the pupils could and did undertake to solve the problem with the given resources. Secondly, the total size of sectors of each colour (approximately 185° and 175°) seemed to appropriately require and promote investigations, in the sense that pupils considered the answer to be not obvious but within reach. Thirdly, the sectors were appropriate to be measured by methods the students knew of, thus leading some pupils to rational hypotheses. Lastly, the tools of investigations provided the pupils with appropriate support and constraint for the pupils' autonomous investigations.

In accordance with the principles of open approach method, the rich student-milieu interplays, which unfolded, generated further potential, e.g. to ”recite, foster or develop students' ideas through discussions between a teacher and his/her students, and among students under the teacher's orientation" (Nohda 1995, p. 57). Yet, the analysis identified the teacher's habitual approach to teaching, in which validation is primarily done by the teacher, as a hindering condition.

In conclusion, this analysis has shed light on conditions regarding the milieu, i.e. devolution, design and the teacher's management of the student-milieu interplay, which proved significant to promote or hinder the realisation of potentials and principles of open approach method. While in this case, devolution and design of the milieu clearly promoted realisation, and hence developed further potentials, the teacher's habitual management of the student-milieu interplays hindered further realisation.

② Teachers' learning from their first lesson study (article under revision for *Journal of Mathematics Teacher Education*)

In order to adopt inquiry based teaching, teachers need to acquire knowledge about how to design and manage pupils' autonomous inquiries. Under fertile conditions, lesson study effectively promote teachers' development of knowledge about and skills of teaching. But how do teachers actually learn in lesson studies? Research on this is pivotal to help us fertilise the grounds of lesson study.

In this article, I study a team of teachers' reflexive interplay with *realised* and *anticipated* didactical situations. The paper serves two purposes: to present a fine grained analysis of how specific teacher learnings emerged and evolved in a three-cycled lesson study, and to demonstrate the analytical tools developed and employed to support such fine grained analyses.

In the lesson study under scrutiny here, teachers experimented with solutions to a teaching problem which included how to use pupils' sharing of methods of and knowledge about subtraction to teach the different methods of subtraction (on a number line) and their properties. Through the (a priori) development of hypotheses in predidactic situations, and the (a posteriori) validation of these in the postdidactic situation (based on the test conducted in the didactical situations of the research lesson), teachers gradually developed new knowledge about pupils' interactions with the milieu and their own possible management of these student-milieu interplays.

The analyses in the article illustrate the mechanisms leading to the emergence and evolution of the identified teacher learnings, and it is further demonstrated how the different types of phases (goal setting, *kyouzai kenkyuu*, lesson planning, reflection, etc.) in a lesson study constitute in themselves specific learning situations, each playing their part in how the teachers' learning evolved across them.

The identified teacher learnings fall in three parts: (specific) *didactical knowledge*, which is about the connection between teaching and learning related to the specific situations, *generic didactical knowledge*, which is about more general aspects of didactic practice such as organisation, time, management etc., and *paradidactical knowledge*, which regards learning about the connection between teaching and learning.

Examples of teacher learnings include e.g. *'minus' means take away* (use of 'minus' and '-' leads pupils to think of the take-away algorithm - didactical knowledge), *the force of examples* (examples presented by the teacher or pupils easily come to lead following actions and thinking - generic didactical knowledge) and *iteration* (how revision and re-teaching allows to see "how things can change" according to didactical decisions - paradidactical knowledge).

'Minus' means take away is used as one of the examples to illustrate how teacher learnings emerged and evolved: through paradidactical situations similar to those of *action, formulation* and *validation*, the teachers realise how the use of 'minus' and '-' seemingly force the pupils to think of 'take away', while moving the attention to 'difference' widens their minds to the various ways of subtraction, in an almost exemplary way.

Affirming points made by Clivaz (2015) and Miyakawa and Winsløw (2009), the analysis demonstrates striking consistence between structures and affordances of lesson studies and the learning model of TDS. This suggests the latter to be a powerful research paradigm for the analysis of why lesson study so effectively can help teachers to develop their didactical knowledge through a systematic interplay with teaching-learning situations.

③ Evolution of teachers' anticipation of didactical situations in the course of three lesson studies (article in review for *Annales de Didactique et de Sciences Cognitives*)

As an interesting parallel to the work forms promoted by inquiry based education, anticipation is at the heart of lesson study. Accordingly, it is plausible to hypothesise a self-perpetuating connection between teachers' capacity to anticipate didactical situations and what they learn from testing their hypotheses in practice, but how does this capacity of anticipation evolve?

In this paper I investigate and compare how teachers' develop didactical ideas in the first and last of three lesson studies. The purpose is to elicit knowledge about how teachers' capacity to anticipate didactical situations (can) develop, with a focus on five aspects regarding *the target knowledge, the material milieu, pupils' responses, adidacticity* and *the teacher's management of the student-milieu interplay*.

Since teachers' anticipation of didactical situations is a predidactic activity, which is hypothesised to develop over time, its evolution is investigated through the pre-didactical situations of teachers' first (from 'scratch') and third lesson studies (the last of this project). In the first lesson study (which is also under scrutiny in the second article) the teaching problem comprised how to use pupils' sharing of methods of and knowledge about subtraction to teach the different methods of subtracting integers (on a number line) and their properties. In the third lesson study the teaching problem revolved around pupils' development of strategies to determine the number of figures (polyominos) which can be made with a given number of matches (perimeter units).

Except for the aspect of adidacticity, examples presented in the article clearly illustrate both a quantitative and a qualitative progression in the teachers' anticipations of didactical situations. Roughly put, at the beginning of the first lesson study, the teachers did not appear to anticipate didactical situations which could correspond to their suggested activities at all. Gradually, teachers became more aware that little details in the milieu and the teachers' management of the student-milieu interplay can have crucial consequences to the didactical situation, and hence for the pupils' potential learning. Furthermore, in the last lesson study, the milieus and the teaching problem - including the target knowledge - were developed much more interactively, and in fact the teaching problem developed remarkably in accordance with the evolution of teachers' capacity for anticipation throughout the pre-didactical situation.

While the results may not document a long term and general shift in the teachers' *didactical* practice, they do illustrate a remarkable shift in their *paradidactical* practices as realised during these lesson studies. The study at hand appears to confirm the hypothesised self-perpetuating connection between teachers' capacity to anticipate didactical situations and what they learn from testing such.

④ Theorizing Lesson Study: Two related frameworks and two Danish case-studies (co-authored book chapter in: *Mathematics Lesson Study Around the World: Theoretical and methodological issues*, Springer book series of ICME)

In the wake of presentations in a Discussion Group on lesson study during the 13th International Congress on Mathematical Education (ICME) in 2016, it was proposed to compile a book presenting perspectives on *Mathematics Lesson Study Around the World: Theoretical and methodological issues*. My supervisor, Carl Winsløw, my colleague, Klaus Rasmussen, and I were asked to compose a chapter on *Theorizing Lesson Study* in which we would include the *two related frameworks and two Danish case-studies* from our research.

The chapter takes its starting point in the circumstance that scholars around the world observe both successes and difficulties in the numerous endeavours to implement lesson study. Hence, there is a rising need for finer methods to characterise and monitor the processes and objects which go into what is broadly referred to as lesson study.

In the chapter, we argue for the necessity of increased attention to theoretical precision in research related to lesson study, including more precise identifications of what practices the term itself refers to, and not least to describe and analyse essential parts of what happens in a lesson study. We also present two theoretical perspectives which we have found useful in this respect.

The two theoretical frameworks are compatible in the sense that they consider teaching and learning relative to explicit models of the knowledge to be taught, and for the analysis of the knowledge actually developed by students when implementing a research lesson. Unlike many commonly used frameworks, they do not only model students' learning but also the teaching and the knowledge that is taught. TDS and ATD have slightly different affordances, with TDS being more close to the perspective of teachers (constructing, devolving, and observing the effects of a milieu) and ATD focusing on the institutional perspective whose importance is undoubtedly more visible to researchers initiating and studying lesson study as a new element of paradidactic infrastructure.

In conclusion

In conclusion, the analyses presented here give us insight into some of the pivotal possibilities and constraints that comes with the challenge for teachers to develop knowledge and skills to act out inquiry based teaching. The specific conditions which are shed light on here regard the design and management of inquiry based teaching in experimental practice and teachers' inquiry based learning when developing ideas, techniques and materials to employ and test such. The investigations point to some difficulties but above of all, they also document a relative fast adaption to the principles and work forms presented by open approach method and lesson study. As is also evident, teachers' and the school system's adoption of and adaptation to inquiry based education in mathematics is not straightforward. Still, the positive results obtained by the teachers participating in this research strongly suggests that teachers' systematic and supported inquiry based learning of how pupils learn and how to enact the principles of e.g. open approach method through lesson studies is a promising path for teachers to learn to design and manage pupils' inquiry based learning.

Perspectives

The analyses presented in this thesis open up several new perspectives. First of all, additional analyses with the same purpose and focus is desirable in order to establish a subtle body of knowledge, which could support efforts to promote large scale realisation of inquiry based teaching. Further research from different angles and with different foci, could only broaden our knowledge base and hence provide further support for such efforts.

It is hard to see how teachers could adapt to radically new teaching paradigms, such as inquiry based teaching, without 'study and training facilities' like lesson study, which allow them to examine and test techniques and technologies to do so. In accordance with other research literature, the analyses of this thesis shows great potential for both the *initial* implementation of lesson study and teachers' subsequent benefits from it. This is in itself a good result, but it remains an open question what it takes for lesson study to not end on the "graveyard of [] educational reforms [as a] once-promising innovation that [was] poorly understood [and] superficially implemented" (C. Lewis 2002, pp. 20-21).

Indeed, the real power of lesson study lies in long term, structured efforts across teachers and institutions. But what does it take to ensure long lasting and solid effects? While the research of the sort mentioned above can help us understand possibilities and constraints regarding teachers in given lesson study situations, we need to study how the institutionally given conditions, i.e. the *paradidactical infrastructures*, support and constrain individual teachers' and the professions' adaptation of and to lesson studies. This includes the crucial study of facilitation: How does facilitation best support the teachers' inquiries in and outcomes of lesson study? It seems obvious that there are no simple answers to this but, as in the case of open approach method, in-depth research into the interplay between facilitation and teachers' interactions with their milieu (the didactical situation), might well bring efficient principles to the fore.

The theory of didactical situations has already proved effective in the analysis of pupils' and teachers' inquiries alike. In the analyses presented here, only basic notions of TDS have been put into play. It seems plausible that other notions of the theory could enhance such analyses further. Especially notions concerning *(para-) didactical obstacles* and *(para-) didactical contract* might prove powerful to examine interplays, constraints and learnings occurring in lesson studies.

As is noted in the article on theorising lesson study, it also seems as a viable hypothesis, that the Anthropological Theory of the Didactic can beneficially support TDS to analyse constraints and possibilities for implementing lesson study at an institutional level, for instance across schools and school districts.

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① An experiment with open approach method in grade four probability teaching

First article, under revision for *Recherches en Didactique des Mathématiques*

AN EXPERIMENT WITH OPEN APPROACH METHOD IN GRADE FOUR PROBABILITY TEACHING

JACOB BAHN

Abstract – Open approach method potentially helps pupils to develop their mathematical thinking through sharing and discussing their ideas in connection with mathematical inquiries. But how can teachers manage open approach lessons? In this article, a group of Danish teachers' experiment with open approach method in the setting of a lesson study is analysed. The focus is on aspects regarding the instigating problem and the resources available to the pupils significant for the principles and potentials of open approach method to be realised. Theoretical tools from the Theory of Didactical Situations are used to analyse the complex interplay between the teacher, the pupils, the problem and the resources. Specific conditions in the instigating problem and the resources are identified to promote pupils' initial inquiries while the experienced teacher's management of these hinders pupils' sharing of ideas.

1. INTRODUCTION

Pupils' inquiries are ideally at the core of teaching in what can loosely be described as constructivist paradigms, but how can teachers realise this ideal? In this article, a Danish experimental mathematics lesson based on the inquiry oriented principles of the open approach method method is analysed. Based on this some promoting and hindering conditions which teachers face in realising these principles and potentials are identified.

1.1 Open approach method

Open approach method (OAM for short) is a method, or rather a set of principles, for teaching mathematics through problem solving (e.g. Nohda, 2000). As a progression to *open-ended approach* (Becker & Shimada, 1997), the method was developed over decades by Japanese teachers, teacher educators and researchers through the research-like structures of lesson studies (Isoda, 2007; Nohda, 2000). The main difference between OAM and its predecessor open-ended approach is the expansion of *open* to equally embrace:

- problems with multiple answers (end is open),
- problems which can be solved in multiple ways (process is open), and
- problems which can be developed in multiple ways (problem is open)

(Isoda, 2007; Nohda, 2000).

OAM is the forerunner to what is known as problem solving approach (Isoda, 2015) or structured problem solving (Stigler & Hiebert, 1999), which is acknowledged as a pivotal source to Japanese pupils’ position as amongst the best performing in the world (Stigler & Hiebert, 1999), with a high ability to use mathematical knowledge in new situations and problems (Mullis, Martin, Gonzalez, & Chrostowski, 2004, pp. 190–191; 400–402).

The aim of OAM is to “foster both the creative activities of the students and mathematical thinking in problem solving simultaneously” (Nohda, 2000, p. 5), and the method advances principles and structures of how to activate pupils in mathematical investigations.

OAM-lessons usually emanate from one non-routine problem which is ”promotes varied [] approaches” (Nohda, 1995, p. 58) which students can apply ”in response to their own mathematical power” (Nohda, 2000, p. 41) and ”according to [their] own abilities, interests and emotions” (Nohda, 1995, p. 58). OAM builds on three principles of:

- The first principle relates to *students' autonomous work* in the sense of appreciating the value of their activities.
- The second principle concerns the epistemological structures in mathematical knowledge, and considers that the acquisition of *essential knowledge* supports derivation of yet new knowledge, even later on.
- Related to the two first principles, the third principle regards the *teacher's expedient decision-making* which should appreciate unexpected mathematical ideas and put them into play to enrich the learning situation all pupils.

And typically, the lesson advances in three phases, in which students:

- (re-) formulate the mathematical problem
- test methods to find solutions
- advance new ideas and hypotheses

(Nohda, 2000, p. 42). For further reading see e.g. Becker and Shimada (1997) and Nohda (2000).

A typical open approach lesson focusses on pupils' investigation of one open problem in the sense there are multiple answers to it (i.e. end is open), there are multiple ways to solve it (i.e. process is open) and/or there are potentially multiple ways the problem can be developed further (i.e. problems are open) (Isoda, 2007, p. 14; Nohda, 2000, pp. 43–44).

It is crucial in OAM, that students’ ideas are brought to the fore since in general ”a student has usually only one or two ideas of mathematics” (Nohda, 1995, p. 58) (regarding a given problem). Given this, one of the main objectives of OAM is to ”recite, foster or develop students’ ideas through discussions between a teacher and his/her students, and among students under the teacher’s orientation” (Nohda, 1995, p. 57). This interdependency calls for students to share, compare and modify, i.e.:

- seeing other students’ discoveries or methods
- comparing and examining the different ideas
- modifying and further developing their own ideas accordingly.

(Becker & Shimada, 1997, p. 23).

In this sense, one pupil's work, knowledge and experiences become a source, adding to and enriching the resources of other pupils.

The potentials of working with OAM comprise students' more active participation and extended frequency of communications, increased opportunity of comprehensive use of mathematical knowledge and skills, opportunity for all students to engage autonomously, intrinsic motivation to give proofs, rich experiences of discovery and approval from fellow students (Becker & Shimada, 1997, pp. 23–24).

Researchers of OAM highlight the "major difficulty [of] how to give suitable problem to the students (Nohda, 1995, p. 59). This regards both the difficulty "to make or prepare meaningful mathematical problem situations" (Becker & Shimada, 1997, p. 24) and "for teachers to pose problems successfully.." (Becker & Shimada, 1997, p. 24).

1.2 Lesson study

Lesson study is a research-like collaborative format for teachers' professional development which takes it starting point in the participating teachers' practice, knowledge and needs (Hart, Alston, & Murata, 2011). It is the major form of professional development for teachers in Japan, whose school children generally and steadily have been among the best performing in the world (Stigler & Hiebert, 1999). Lesson study take many forms and serve several purposes like in-service teacher education (the is the most famous form outside Japan), research on teaching and learning, testing and implementation of new curricular ideas etc. (Isoda, 2015).

Lesson studies are centred round a research lesson, in which hypotheses about the connection between teaching and learning is tested.

Prior to the research lesson the hypotheses are developed and the test of I prepared through phases of *goal setting*, *kyouzai kenkyuu* and *lesson planning*. *Kyouzai kenkyuu* comprise a thorough study of the subject matter knowledge in question: its position in the curriculum and its epistemological structure and how this conditions learning of it, teaching materials relevant to teach it etc. (Watanabe, Takahashi, & Yoshida, 2008). Based on *kyouzai kenkyuu* the hypothesis is devised and materialised in the lesson plan.

Posterior to the research lesson, data from observation of the research lesson are shared and analysed, and often a *knowledgeable other* links the specific results of the lesson study with general knowledge (Takahashi, 2017).

Lesson study is usually referred to as a cyclic process in two ways. One refers to how knowledge developed in one lesson study forms the basis of the next. The other refers to how the hypothesis of a given lesson study may be revised after one research lesson study and re-tested in another research lesson within the same lesson study. There is an ongoing discussion whether re-testing a revised (version of) the lesson plan is appropriate (e.g. Hino, 2007) or not (Fujii, 2014). The lesson under investigation in this paper was part of a three-cycled lesson study.

2. THEORETICAL FRAMEWORK

The Theory of Didactical Situations (TDS for short) has been developed over decades through studies of experimental teaching situations. It is mainly developed as a tool (box) for researchers to study didactic relations in teaching situations and has expounded precise knowledge of didactical mechanisms and accurate tools to reveal such (Brousseau, 1997; V. M. Warfield, 2006).

When using TDS to conduct didactical research, a priori analyses are developed to hypothesise didactical situations which are tested under observation. In the aftermath, the a priori analysis functions as a reference model for an a posteriori analysis. TDS has proven to be a powerful tool to design and analyse teaching of mathematics with respect to didactical situations (Brousseau, 1997; V. M. Warfield, 2006).

In 'traditional' TDS analyses, a didactical situation is designed and analysed a priori, that is, designed dialectically between intentions and hypothesised potentials. Interestingly, TDS has also proven to be powerful when analysing didactical situations not designed by means of TDS.

For instance, Hersant and Perrin-Glorian (2005) used TDS to analyse ordinary teaching on which they and the theoretical apparatus of TDS had had no influence (Hersant & Perrin-Glorian, 2005). Still, they applied the same strategy of conducting the a posteriori analysis with reference to an a priori analysis, even though the a priori analysis was actually conducted after the lesson. In this sense, while they could analyse the *potentials*, they disregarded the *intentions* of the lesson (as they had no access to it beforehand), and conducted the a posteriori analysis with reference to an a priori analysis of (hypothesised) potential situations. This approach is applied for the present article as well.

Miyakawa and Winsløw (2009) used TDS to analyse situations of two unrelated didactical designs, one (famous) situation designed within TDS and one designed within the the tradition of OAM, in order to compare the lessons and the didactical traditions in which they where each developed (Miyakawa & Winsløw, 2009). They did have access to the intentions of each design, and thus could make a priori analyses of the *intended* as well as *potential* situations of each lesson to form a reference model for the a posteriori analyses. In summary, they conducted a comparative analysis of didactical situations developed and performed in two unrelated settings even they had not themselves taken part in or observed the design or planing phases of either.

The name of the theory refers to the basic assumption of TDS, that (school) learning (of mathematics) is not a static event but a dynamic process which take place in (didactical) situations. A such is constituted by a teacher - with the intention to teach - and one or more students - with the intention to learn from the teacher. To fulfil a didactical situation, the teacher devolves - i.e. hands over - a didactical milieu, consisting of one or more tasks and if necessary, resources to solve these.

Situations, in which the students' autonomous interplay with the milieu, provides feedback which allows for them to construct new knowledge, are named adidactical situations. Students' such interplay takes place in situations of action (immediate attempts), situations of formulation (hypothesising correlations) and situations of validation (confirming or refuting formulated hypotheses). Adidactical situations are assumed to provide the most nutritious grounds for essential learning. As a consequence, the design of the milieu and the teachers management of the interplays are vital to pupils' learning. If so intended, the students' new knowledge may be institutionalised, i.e. aligned with and connected to the official knowledge of the curriculum, by the teacher. Though the description of the different types of situations appear linear, they are not necessarily in reality (Brousseau, 1997; V. M. Warfield, 2006).

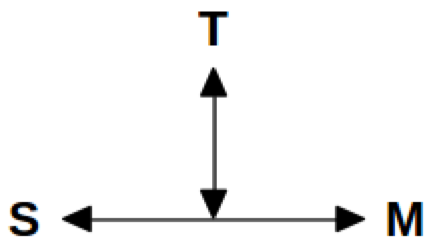

Figure 1: Didactical situation. (Adapted from figure 4 in Brousseau, 1997, p. 56)

Any situation has its own milieu. In an a situation of action (level 1 – see figure 2 below), the student interplays with the material milieu, i.e. the task and additional resources if supplied. When a situation of action provides the student with sufficient feedback, that situation itself becomes a milieu in the interplay with which, the students can formulate hypotheses about (mathematical) correlations (level 2). When testing, i.e. refuting or validating, these hypotheses, the student step out of a situation of formulation and into one of validation (level 3). Here, the situation of formulating a given hypothesis becomes a milieu in the interplay with which, the hypothesis is either refuted or confirmed. In the situation of institutionalisation (level 4), the situation of validation becomes the milieu of the student and teacher institutionalising communications. Hence, levels of situations are analogous to a matryoshka:

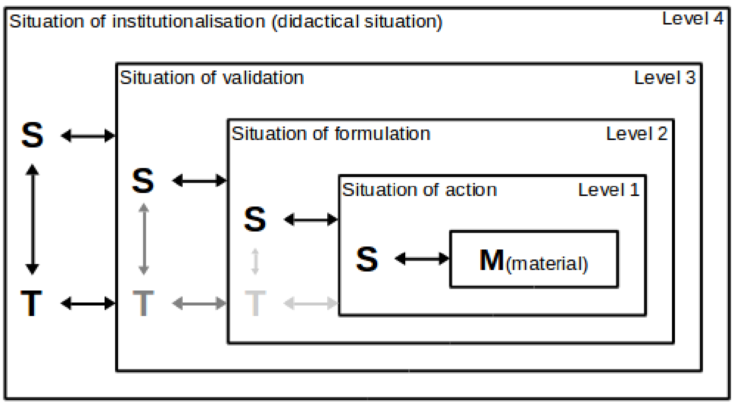

Figure 2: The situation-milieu matryoshka. Students' interaction with the problem (i.e. the situation) at one level becomes the milieu at the next higher level. Levels 1 through 3 are adidactical since they are sustained by the students' autonomous interaction with the milieu, i.e. independent from the teachers didactic intentions. The grey scaling of T at level 2 and 3 indicate typical levels of teacher intervention (which interrupt the adidactical situation). Level 4 is didactical since it is sustained by the teacher's didactic intentions and actions. (Adapted from figure 2 in Brousseau, 1997, p. 248) and (figure 2 in Hersant & Perrin-Glorian, 2005, p. 4)

It is important to observe that real in teaching-learning situations may fluctuate from the apparent linearity of the model, which is a theoretical illustration of interconnections between situations and milieus.

In figure 2 (as well as in figure 1), any S may at times be one pupil, at other times more or all pupils. Furthermore, pupils in different situations need not be the same (Brousseau, 1997, p. 249). One pupil's interplay with the milieu at one level (e.g. immediate attempts at level 1) may function as a situation of reference, i.e. (part of) a milieu, for a different pupil (e.g. formulations of coherences at level 2).

2.1 Analysing OAM with TDS

In OAM it is an aim to "...sophisticate [students'] ideas in mathematical activities by means of students' negotiations with others..." (Nohda, 2000, p. 41)n. Such negotiations can be expressed as sharing, comparing and modifying ideas (Becker & Shimada, 1997, p. 23) across levels of situations. Since milieus in higher level situations then are build on interplays and feedback from lower level situations, their richness depend on these. To realise the potentials of OAM, milieus at all levels must be rich and carefully aligned to match the abilities of the pupils, and design and analysis of milieus must take into account details that pose significant conditions to the student-milieu interplay. In this sense, TDS is well suited to analyse OAM-lessons.

Typically, pupils' mathematical negotiations are put into play in whole-class didactical situations of validation (level 3). Thus, the teacher orchestrates the pupils' sharing, comparing and modifying ideas in order to collaboratively advance new ideas and hypotheses.

When analysing a Japanese OAM-lesson using TDS, Miyakawa and Winsløw (2009) noticed that: "The description of the aims seems more focused on the process of students' thinking than the official program, which emphasizes the overall target knowledge for students." (Miyakawa & Winsløw, 2009, p. 207). This summarises the aim and focus of OAM (as described above)

3. PRESENTATION AND A PRIORI ANALYSIS OF THE LESSON

The lesson in question was chosen among a number of experimental OAM-lessons, none of which are 'genuine' OAM-lessons in the sense that none of them fully met the principles of OAM, and accordingly did not tap fully into its potential. One good question to this is, why the principles and potentials of OAM were not met? The lesson was chosen to illustrate which and how properties of the milieus, and the teacher's management of these, promote or hinder the principles, and thus potentials, of OAM to be realised.

The theme of the lesson is probability and the aim is for pupils to determine theoretical probability and to test empirical probability. The lesson consists of three activities. Only the second activity is based on OAM principles and is hence the object of analysis. Here I present a short and naïve outline of the lesson as a whole, followed by a more detailed a priori analysis of the OAM activity.

In activity one a spinning wheel (wheel no. 1), one half being red the other half being yellow, is hung on the blackboard. An arrow is pointing at the wheel from beneath. The class is asked to choose red or yellow and the wheel is spun. If the arrow points at the chosen colour the class gets a point and if it points at the opposite colour the teacher wins a point, and a game between the class and the teacher is established. After some spins a second wheel (wheel no. 2), divided into 2⁄3 of yellow and 1⁄3 of red, is placed on the blackboard. The game continues with the modification, that when a pupil chooses yellow, the teacher spins wheel 1 (pupils have a 1⁄2 chance for success) and when red is chosen, wheel 2 (pupils have 1⁄3 chance for success). Due to the uneven chance of red and yellow on wheel no. 1 and 2 respectively, and the unfair choice of wheel by the teacher, the pupils at some point complain about this. Then the teacher opens a short discussion about the probability of each colour on each wheel and why the teacher's choice of wheel is unfair. At the end of activity one, the teacher put one more wheel on the blackboard (wheel no. 3), which is divided into sectors of approximately 50° yellow, 101° red, 59° yellow, 15° red, 65° yellow and 69° red, which sums up to 174° yellow and 186° red. The teacher then asks the pupils to consider, which of the colours on this wheel has the higher chance of success, i.e. which colour "has the highest chance of winning". In this respect, activity one functions to devolve the milieu for activity two.

In activity two, the OAM-activity under scrutiny, the task is to (discover methods to) determine which of the two colours (red and yellow) on wheel no. 3 has the higher probability.

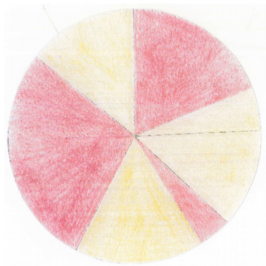

Figure 3: Material object of investigation, Wheel no. 3

After a short talk about which colour has the higher probability, the pupils are asked to determine it by investigation. Every pair of pupils (one group of three) are given two wheels identical to but half the size of wheel no. 3 and a set of investigation tools: scissors, ruler, thread and protractor. By the end of this activity, the pupils share their results, method and considerations through a teacher led presentation.

In activity three, a third copy of wheel no. 3 is distributed to each group, with a pin and sticky gum-tack to make the wheel spin-able on the table. An arrow is drawn on each table, and through 30 test spins pupils record the number of times each colour has success. By the end of this activity the results of each group are shared and written on the blackboard. The overall result is discussed in relation to the shared and not shared results of activity two, but no conclusion is made on the numerical results.

3.1 A priori analysis of the OAM activity (activity two)

Apparently the main problem to be solved in activity two is to determine which of red and yellow has higher (theoretical) probability for success. Yet, the real problem is to find ways to determine this by use of the presented tools. Hence the mathematical answer in question is not the *objective* of the lesson but serves as an *object* of investigation, and the objective is for pupils to develop their "mathematical problem solving ability" (Hino, 2007, p. 507).

Specifically, the lesson aims for pupils to use and/or develop further their existing knowledge about probability, proportionality and geometry. It is interesting, that though the lesson is on probability, from the pupils viewpoint the problem and solution of this task can in itself be seen as purely geometrical. Due to the design of the wheel, the relative probability can be determined by the size of the sectors, whether one does this by arc length, angle or area.

Depending on the strategy chosen, none or little computation is necessary but it is crucial to develop and/or apply a strategy. Useful computation includes adding or subtracting measures of sectors and the whole wheel, e.g. sum of angles or arc length. If someone uses a strategy of calculating the area, use of formula for the area of sector and circle will be necessary, but as indicated by the a priori analysis beneath, this strategy is not expected from this grade.

The material milieu is constituted by:

- Problem: Which colour has the higher probability of success?
- Cardboard wheel divided into six unequal sectors (adding up to 51.7% of red and 48,3% of yellow - see figure 3).
- Tools of investigation: Scissors, ruler, thread and protractor.

The problem that pupils face is presented as one of probability, but the problem is finally stated as: which colour covers the most? Hence it is thought to be resolved as one of geometry of plane figures, in this case a circle and its properties, e.g. sector angles and arc length. Proportion is pivotal since the probability and the size of sectors are proportional as well as is the arch length and the sector size.

Since the material object of their investigations is a circle, it is an advantage if pupils do not possess rooted knowledge of determining the size of sectors. This will deprive the milieu of its richness, since investigating by developing and testing methods is the core idea. Yet to be able to pursue investigations, they must poses some of the following knowledge:

- Geometry
  - that (sums of) angles, area and perimeter represents the size of a sector of a circle.
  - that the sum of area, angles or perimeter of sections of one colour determines if it's larger or smaller than the opposite colour.
  - The angle sum of a circle is 360° and hence the half is 180°.
  - Use of protractor: That angles in a circle is measured from the centre along radii.

- Proportion
  - that the size of area, centre angel and perimeter of one sector are proportional to each other.
  - That the size of a sector measured by means of angle, area or perimeter is proportional to its share of the whole circle, e.g. that a sector of 90° = 25% of the area.

- Probability
  - That the theoretical probability is independent, i.e. one outcome does not affect later outcomes.

OAM situations are contingent upon a rich milieu which allows for a variety of possible approaches to solving the problem, including appropriate feedback for pupils to evaluate the quality of their actions, formulations and validations. The material milieu of activity two allows for (at least) the following approaches:

C. By comparison

d. Direct

1. cut, assemble by colour, compare side by side: The larger has higher probability

Figure 4: Approach C.d.1

2. cut, assemble by colour, compare by overlap: The larger has higher probability

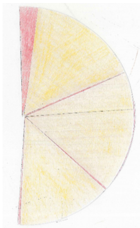

Figure 5: Approach C.d.2

3. cut, piece by piece, compare by overlap: The remainder has higher probability (not accurate
4. cut, piece by piece, compare by overlap, adjust (cut pieces to match): Surplus colour has the higher probability

i. Indirect

- cut, assemble by colour, compare with straight line, e.g. ruler or table edge: Crossing straight line means higher probability, not crossing means lower probability.

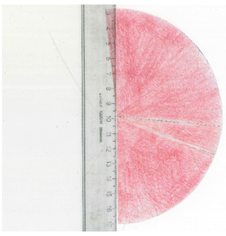

Figure 6: Approach C.i

M. By measurement

a. Angles

1. (cut,) measure each piece by protractor, add up per colour: Colour with larger total angle has the higher probability
2. (cut,) measure each piece of one colour by protractor, add up: More than 180° total angle = higher probability, less than 180° = lower probability

p. Arc length (perimeter)

- (cut,) measure arc length by use of thread or ruler: Colour with longest total arc length has the higher probability

c. Chord

- (cut,) measure distance between each sector's intersections with perimeter: False since in the case where the sum of angles of sector a and b are equal to the angle of sector c, the sum of chords of sectors a and b is larger than the chord of sector c.

Though some approaches seem similar, each approach reflects different ways of thinking and hence call for varied management of the student-milieu interplays. In the same way, different kinds of feedback have different value to each pupil depending on their knowledge and ideas. One kind of feedback that might bring one pupil further, might not enlighten another pupil.

Learning situations, i.e. situations of validation (level 3), under OAM are based on pupils sharing, comparing and modifying mathematical ideas. In OAM-lessons the *potential* student-milieu interplays in higher level situations strongly depend on the *realised* student-milieu interplays in situations at lower levels. Hypothetically, each of the above listed possible approaches offer in themselves a description of potential (elements of) milieus of higher level situations, i.e. levels 2 and 3.

4. METHODOLOGY

The object of analysis for this article, is an experimental lesson designed and conducted as part of a lesson study (Clivaz, 2015; Hart et al., 2011; Takahashi, 2017), itself being a part of a larger research project. The lesson was the last of three iterative lessons of the last of three lesson studies conducted through one school year by one team of four teachers of mathematics at an ordinary public elementary school in Denmark.

Neither the lesson study, in which the lesson in question was developed in, nor the series of lesson studies it was a part of, is an item of investigation here. The lesson studies, including the teachers' introduction to OAM, obviously had an impact on their design of the lesson, but this too is not taken into account. This article solely seeks to investigate which elements of a lesson can promote or hinder the realisation of principles and potentials of OAM. For this purpose investigating an ordinary lesson would probably not be fruitful, whereas one guided by OAM principles might.

The lesson took place in a grade four class. The teacher teaching the lesson was not the usual teacher of the class. She had 13 years of experience and in addition to her (main) role as teacher, she served as a supervising mathematics teacher (*matematikvejleder* in Danish). Supervising mathematics teachers do not necessarily (as was the case here) have more or better theoretical or practical knowledge of mathematics or didactics than other teachers, but has taken a formal education which includes theoretical knowledge of supervision, pupils' special needs, digital tools and the like.

In the research project, teachers experimented with teaching based on the principles of open approach method. Prior to the series of lesson studies, the teachers were introduced to the principles of OAM, essentially as presented in the introduction to this article supported by some examples of OAM problems and one video of a lesson including OAM principles (Tanaka, 2005). Furthermore, the teachers were briefly introduced to basic notions and models of TDS, i.e. the model of didactical situations as depicted in figure 1, including the notions of adidactical situation, and the five phases: devolution, situation of action, situation of formulation, situation of validation and institutionalisation (by and large TDS was not considered by the teachers).

As a part of the overall lesson study research project, the lesson in question was observed and recorded. The recordings were conducted with two stationary video cameras and one portable microphone attached to the teacher.

In combination with observational notes and pupil productions (notes, results and wheel 3), these recordings have been analysed to reveal qualitative information about pupils' potential and realised interplay with the milieus across levels of situations. That is, to which extent OAM potential was established and realised in milieus across levels of situations. Furthermore, conditions that appear to be significant for potentials and principles to be realised or not were identified

Specifically, an a priori analysis outlines the OAM potential of the milieus, while an a posteriori analysis examines to which extent the potential was realised and which conditions seem significant to this.

Following the purpose of the article, the aim is to provide answers to the research question of: "*Which conditions regarding the milieu are of significance to realise principles and potentials of open approach method?*".

5. A POSTERIORI ANALYSIS OF THE OAM ACTIVITY (ACTIVITY TWO)

The devolution of the material milieu as described above apparently caught the pupil's interest. Furthermore their attention was led to half a circle being equal to a fifty-fifty chance and hence, that sizes larger than half means more than 50% percent chance while less than half means less than 50% chance for success.

During devolution the pupils were instructed to "...investigate by means of those tools and discover whether it is red or yellow that has the highest chance to win" which was repeated and rephrased to "...discover, is it the red or is it the yellow which covers the most?". Though the answer to this question is closed (red or yellow), the answer can be found in multiple ways, and hence this is a *process is open* type of problem. Then the pupils were asked to guess the answer beforehand, and were finally asked: "What do your eyes tell you? Is it yellow or is it red which actually covers the most?".

The two expressions "which covers the most" and "What do your eyes tell you" seemed to be pivotal and were repeated throughout the activity. In conjunction with the apparent geometric fashion of the material milieu, it appears plausible to claim that "which covers the most" supported understanding the problem at hand as a geometric problem. "What do your eyes tell you" seems to function as a way to overcome reluctance to pose a guess and to lead pupils to hypothesise and reason about the problem and possible solutions/strategies to resolve the problem.

Hence finding the right answer becomes a means to reach the aim: To hypothesise and test approaches. In the following, examples of situations representing the variety of applied approaches are presented.

5.1 Example no. 1: Guess and immediate reasoning (which is not included in the a priori analysis)

This episode illustrates, that in addition to the variety of potential approaches to solve the problem at hand, the milieu inspired to immediate reasoning and could provide some feedback; just enough to keep the reasoning alive, strengthen the pupils' first guess and to infuse further investigations.

Robert	I think it will be red because if there is, if there is the lowest change it is that one (pointing at 15° red). And if it should be higher but nearer it must be this one (points at 69° red). And the largest is that one (points at 101° red). These are almost the same size (points at the three yellow sectors). I also think that if it was, if this was a pie, one could say for example.. cut some off here and put it over there.. Then they are almost equally big (indicates apparently, to separately assemble the sectors of the two colours).

When Robert considers how to actually substantiate that red is larger, he reasons further:

Robert	How, how can I show, that I think... Should I say, that I think... Should I say, that I think that there is the least chance (points at 15° red) and a fifty-fifty chance (points at 65° yellow and 69° red) and (points at 101° red)_ (interrupted by the teacher)

From the video and audio track it seems plausible to assume, that the next step was to compare 50° yellow and 59° yellow to 101° red and declare them more or less equal in size. Since 65° yellow and 69° red too seem to be similar in size, red must be in surplus by 15° red and hence hold a higher probability. The feedback provided for this immediate reasoning (action) is dubious; though the 'answer' cannot be refuted, it cannot be relied on either. In order to attain certainty, Robert has to develop another approach and test it.

Since Robert was cut off, this reasoning is partly imaginary. Whether or not it holds to the end is not crucial, though. The point is that properties of the milieu led Robert to use this sort of reasoning which again provided him with feedback that led to investigate further (encouraged by the teacher).

The teacher makes no attempt to have Robert share his idea with other pupils.

5.2 Example no. 2: Direct comparison by assembled overlap (C.d.2)

It's not easy to determine which of the overlapping approaches is put into play, but from what can be watched in the video, it seems to be that of direct comparison by assembled overlap.

Peter	We put them on top of each other
Karen	to see if they.. see if they.. they are.. it's because we think it's difficult to see..
Teacher	Yes, yes. Good thinking
Karen	It looks as if the red is bigger.
Teacher	Because, Elsa
Elsa	Because it's.. The yellow covers..
Karen	There is more of it (the red)
Teacher	Yes. Well, but that is actually a way to prove it. Yes.

In this group they formulated the (hypothesis about) approach of direct comparison by assembled overlap. The milieu provided feedback, that this approach is valid, as it led to a plausible result.

Ignoring to led the pupils share and discuss their ideas and hypotheses, the teacher validated the finding immediately.

5.3 Example no. 3: Indirect comparison with straight line (C.i)

Anna	Well, I think.. because before it was put together and then we put a ruler. And I thought they were equally big, and Paul thought they were erhm, that red was just a tiny bit bigger..

Though this sequence does not reveal whether Anna and Paul received appropriate feedback, it demonstrates that the milieu did indeed incite this approach to the problem.

5.4 Example no. 4: Measuring angles by protractor and compare with opposite colour or 180° (M.a.1 or 2)

It is unclear from the video, if this is solved by measuring all sectors or solely the red ones. The conversation indicates that only measures of red angles were added, while knowing the sum of degrees of a circle led the pupils to realise that red was larger.

1.

Mike	185 on the red and 175 on the yellow
Teacher	And how did you find that out? By means of…
Mike	I measured the angle with the protractor
Teacher	Yes, yes. Really good. So, that means that, by use of the protractor (Mike: the protractor)? Yes. Well done
Mike	And because a circle is 360 degrees, then.. (teacher talks, so Mike's explanation cannot be heard).

The last line is interrupted, but it seems that Mike was about to express, how they summed up angle measures of red and subtracted that from the total 360° to find the angle sum of yellow. It seems that the milieu was not appropriate to Mike (i.e. it posed no challenge).

5.5 Example no. 5: Measure angles and compare to opposite colour - variation of (M.a.1)

In this episode the pupils measure the angle of sectors of one colour each. Since they are not aware of the sum of angles in a circle, and they don't question the group mate's result, they do not realise that they are wrong. Ignoring these aspects, the milieu cannot give appropriate feedback that their approach has a flaw.

Clara	We have tried to measure the angles of this one
Teacher	Yes
John	Both of us have tried to measure it. We get exactly the same
Clara	We get exactly the same. We both get 185. But…
Teacher	Oops then. OK.
John	(unclear and low-voiced) ..10 degrees too much for a circle..
Clara	It's because..
Teacher	Well, yes..

Clara	But it’s because, when we try to.. When we try to.. add up the red ones, you know like we put the red ones here, and like this, then red becomes larger..
Teacher	Then red becomes the larger. But I am thinking, you haven’t cut it up. It is just when your eyes sh.. say
Clara	Yes

The sequence does not reveal whether Clara and John measured sectors of the same colour on each their wheel, or if one of them simply measured imprecisely.

They don’t notice anything at first, but when explaining to the teacher, John notes that “10 degrees [is] too much for a circle”, hence he must know that something is wrong with their work (and not so much with their approach). Considering OAM this consideration should have been shared among pupils.

5.6 Example no. 6: Measure angles and compare to 180° (M.a.2)

This group first tried to solve the problem by direct comparison by assembled overlap (see example no. 2). Here they express and use knowledge about the sum of degrees in a circle to determine which colour is largest.

Peter	We’ve proven that ehm.. it is larger, just by measuring the red at the angles, because it was more than 180

Peter and his group used approach M.a.2, recalling that the sum of angles in a circle is 360°. It is unknown whether that approach was fully formulated from the beginning or it evolved after measuring the red sectors, but since the result seems plausible to the pupils, their approach was obviously found to be valid (”we have proven..”).

5.7 Example no. 7: Measure chord (M.c)

As seen on the picture of figure 7, at least one group tried to solve the problem by measuring the chords. It is unknown for how long they pursued this approach and which feedback (or lack of feedback) made them cease this pursuit.

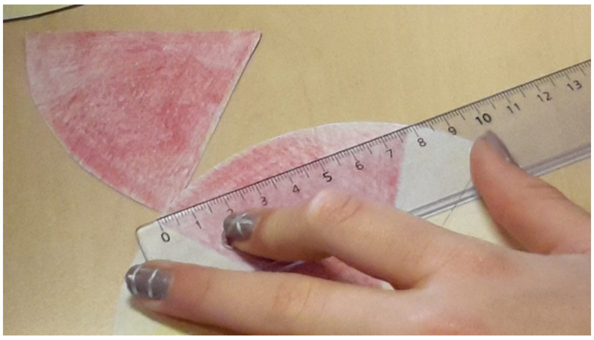

Figure 7: Attempt to solve by chord

It would have been interesting to see how other pupils would respond to this approach, especially since it will lead to no certain answer (whether right or wrong).

5.8 Example no. 8: Calculate area.

By the end of the OAM activity, the teacher led a presentation of results. Against the principles of OAM the teacher did not facilitate an exchange and discussion of approaches and ideas, but rather a simple presentation (to the teacher) of results.

Though inappropriate in regard of OAM, the summation revealed attempts to solve the problem by calculating area. A few pupils actually knew how to calculate the area of the whole circle, but had no knowledge of how to calculate area of a sector. Still it is noteworthy, that the milieu led them to try this approach.

Peter	We also tried to solve it in a different way. We multiplied the length with the width and got.. it was the red. We got the red to be 20.7 cm and the.. or was it.. yes, and the yellow to be 20.5 cm.

Another group succeeded to calculate the area to be ”145.7.. or something like that .. or was it 151.7”. Considering the slight unevenness of the (perimeter of the) circle, the actual area is approximately 154 cm^2.

During summation, the main focus seems to be on the actual answer, i.e. which colour actually is largest. The summation is set off by the teacher getting everyone’s attention and asking the whole class: ”What [conclusions] did you reach?”. The following typify the sorts of answers from the pupils:

5.9 Typical answer 1: Simple answer

Minna	Well, [] we got 185 and 175
Teacher	Regarding which colour by what means
Minna	Well, er, yellow 175
Bob	And red was 185
Minna	Yes
Teacher	Okay

The teacher did not revisit the question of means.

As seen below, some answers included ideas and progression that could have based the foundation for a potential discussion across groups.

5.10 Typical answer 2: Elaboration of answer

Teacher	OK. And at one point I heard you explain, what do you know about it, when one is 185 and 175. Then what?
Karen	Well, if.. if one of them is over 180 then, er, then the other is under. Because the sum of angles must be 360 degrees, you see.. (shortly after followed by) Er, first we measured red because we were quite sure that red was biggest. Because that was also what we had concluded from cutting it out
Teacher	Yes. And when you saw it was more than 180 you thought, then you had…
Karen	Yes
Teacher	...proved that..

Though a few ideas were presented to the class, they were not used to establish a discussion and as such do not qualify to be shared in the sense of OAM.

6. RESULTS AND DISCUSSION

From the a posteriori analysis conducted above, the results divide into conditions promoting or hindering the realisation of principles and potentials of OAM. Interestingly the promoting conditions are related to situations of action and formulation at levels 1 and 2, while the hindering to realise the principles and potentials primarily takes effect in situations of validation at level 3.

Firstly, the material milieu was appropriately devolved in the sense that pupils were not uncertain about the initial problem. *Secondly*, the difference between red and yellow appear to be most appropriate. With a too large difference between the total size of red and yellow sectors respectively, the answer would have been to obvious and investigations not incited. On the other hand, had the difference been too small, the answer could only be found by precise measurement, obviating investigations. *Thirdly*, the size of each sector were significant too. For instance, Robert's hypothesis in example no. 1 is driven by the fact, that a (rough) comparison of sectors convinced him that 15° red, was in surplus. *Lastly*, the tools of investigation provided pupils with an appropriate palette of possibilities to approach the problem, though the piece of thread played no big role.

While the three latter conditions regard the milieu itself, cf. how "to make and prepare meaningful mathematical problem situations" (Becker & Shimada, 1997, p. 24), the first condition regards the devolution of it, cf. how the teacher "pose[s the] problem successfully" (Becker & Shimada, 1997, p. 24).

The identified hindering condition too does not regard the milieu itself but lies with the teacher's management of the student-milieu interplays. Sticking to a more traditional approach of teaching, the teacher assumed (more or less) total responsibility of validation, which furthermore took place with only one group at the time. In this sense, milieus at level 3 were never enriched with promoting conditions, but were rather left with conditions hindering OAM principles and potentials to be realised further.

Hence, principles and potentials of OAM were successfully realised through actions and formulations at level 1 and 2. This, in itself generated a rich potential for whole-class didactical validation at level 3. However, this potential for students to advance new ideas and hypotheses through sharing, comparing and modifying ideas was never realised.

7. CONCLUSION

In this article, a Danish experimental lesson on probability in grade four has been analysed in order to investigate conditions significant to realise principles and potentials of OAM. While the analysis focusses on conditions regarding the milieu, i.e. the mathematical problem and the resources available to the pupils, the significant conditions identified fall into two: promoting and hindering conditions.

In the lesson, pupils were to investigate on a spinning wheel divided into six unequally sized sectors, which of two colours had the highest probability of success. Though a problem of probability, it was solved as one of geometry, i.e. which colour covered the most.

Four specific promoting conditions were found, all directly or indirectly regarding the initial material milieu: appropriate devolution, appropriate difference between total size of sectors, appropriate difference in size between each sector and appropriateness of tools for investigation (ruler, protractor, thread and scissors).

Owing to these promoting conditions, principles and potentials of OAM were initially realised to a great extent, in itself developing potential to enrich subsequent milieus. Identified as a hindering condition, the teacher stuck to a traditional approach to teaching in which validation was done primarily by the teacher and only with small groups. Hence, the pivotal objective of OAM to ”recite, foster or develop students' ideas through discussions between a teacher and his/her students, and among students under the teacher's orientation” (Nohda, 1995, p. 57) was omitted and the potential developed (by the pupils) in situ was not realised,

The analysis presented in this article took advantage of advanced tools derived from TDS. The theory was not originally developed to analyse lessons not designed with TDS, but especially Hersant and Perrin-Glorian (2005) and Miyakawa and Winsløw (2009) have proven it to be efficient in this respect. The present analysis both benefited from and supports this.

Though inquiry-based teaching is not straightforward, by aiming at the principles of OAM during a lesson study the teachers of this research did to a certain extent manage to establish an inquiry based situation. Still, outliving the potential to the full is not trivial, and we need to explore further how teachers can learn to better design and manage complex teaching-learning situations. Due to the intricate dynamics of teaching-learning situations, the advanced tools of TDS could play a crucial role in this quest. Perhaps with a special view to teachers' learning through lesson studies?

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② Teachers' learning from their first lesson study

second article, under revision for *Journal of Mathematics Teacher Education*

Teachers' learning from their first lesson study
- analysed by the Theory of Didactical Situations

Jacob Bahn

Abstract. How do teachers actually learn in lesson studies? Though lesson studies are reportedly effective means to develop teachers' knowledge about the connection between teaching and learning, we still lack precise knowledge about what and especially how participating teachers can learn. In particular, we need to develop powerful analytical tools for this research.

In this article it is investigated what and how teachers of mathematics learned about the teaching and learning of mathematics in the classroom during a three-cycled lesson study. The tools to make fine grained analyses stem from the theory of didactical situation which is a highly developed theory developed specifically for design and research of teaching and learning situations in mathematics. The tools are based on a model of the teachers' learning as emerging from their reflexive interplay with teaching-learning situations, whether *realised* in the research lesson or *anticipated* during preparation and post-lesson reflection.

In addition to specific teacher learnings and their evolution, detailed examples of the latter is illustrated while demonstrating how the analytical tools allow for fine grained analyses of how learning evolved throughout a lesson study.

1 Introduction

Lesson studies are reportedly effective means to develop teachers' knowledge about connections between teaching and learning (Stigler and Hiebert 1999). Since the end of the nineties, a rising number of initiatives to introduce lesson studies to schools and teachers have seen the light of day (Murata 2011). Not all of these have been equally successful (Takahashi and McDougal 2016). To better investigate how one can realise the potential of lesson studies, i.e. how teachers can learn what, we need precise scientific models of the possible learning processes and hence precise scientific tools to examine it (Lewis, Perry, and Hurd 2009; Shuilleabhain 2017; Tepylo and Moss 2011).

The aim of this article is to present a fine grained analysis of teachers' learning from a lesson study, and how that learning emerged. What triggered the learning process? What made the learning possible? And, what shaped the learning under way?

An important part of lesson studies is that teachers are 'forced' to speak out their knowledge and ideas, in the process of collaboratively developing and testing hypotheses about solutions to a teaching problem (Fernandez and Yoshida 2004; Stigler and Hiebert 1999; Takahashi 2011). These processes are to a large extent in line with the learning model of the Theory of Didactical Situations (TDS), from which the analytical tools derive. In TDS, pupils' essential learning is assumed to take place through similar processes of hypothesising and testing solutions to a mathematical problem (the theory is specific to mathematics). Hence, we will take advantage of specific structures and properties shared between TDS and lesson study.

The purpose of this article is to expand our knowledge about teacher learning in lesson studies, to provide insight into mechanisms conditioning this learning and propose tools to investigate the mechanisms further.

2 Lesson study

Lesson studies have been characterised as collaborative teacher-led action research (Miyakawa and Winsløw 2009, p. 203), due to its research-like structure and nature of developing and testing hypotheses. The collaborative and experimental approach is at the base of outcome of lesson studies, well described in literature (e.g. Hart, Alston, and Murata 2011; Lewis, Perry, and Hurd 2009; Stigler and Hiebert 1999). Though teacher-led, the support of a facilitator, external to the participating teachers, is regarded as crucial to promote teacher learning (**hart2011b**Takahashi 2011, p. 79).

While varying a lot, lesson studies are often described to be structured as illustrated in figure 1 (**english2009**Murata 2010, p. 576**winslow2011**):

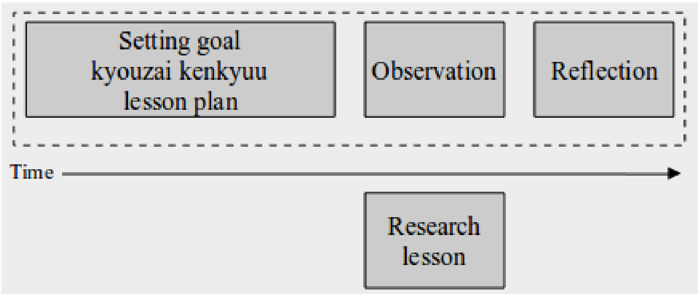

Figure 1: General structure of a lesson study. Modified from **winslow2017**

The research lesson is pivotal to a lesson study in the same way a test is pivotal to a scientific experiment. Before the research lesson can be carried out, it is carefully prepared through goal setting, *kyouzai kenkyuu* (study of the subject matter and related materials, e.g. Doig, Groves, and Fujii 2011, pp. 182-183; Watanabe, Takahashi, and Yoshida 2008) and developing a lesson plan. Concurrent with the research lesson, the teaching-learning situation is observed. After the research lesson, data from the observation are shared and analysed through joint reflections.

Outside Japan (Fujii 2014, pp. 11-12), lesson studies often have a cyclic structure, in which the lesson plan is revised based on the reflections conducted after the research lesson, and tested anew (e.g. Dudley 2015, p. 8; Fernández 2010, p. 352). There are no set rules, and hence the number of cycles vary. In the lesson study case of this article, three cycles were conducted.

Due to the specific formats of lesson studies, participating teachers' knowledge is explicated through posing and discussing ideas and hypotheses (Fernandez and Yoshida 2004; Stigler and Hiebert 1999; Takahashi 2011). This provides us as researchers with a unique access to the epistemological process which teachers undergo. As will be elaborated on later, the teachers are being 'forced' to express, discuss and put into action their knowledge when putting forward and discussing objectives and proposals for the study and this is a rich source of information to the analysis.

3 Theoretical framework

3.1 Theory of Didactical Situations

Didactics has different meanings. In this paper, the term is adopted from the French theories of didactics of mathematics (e..g Artigue 2009) and refer to knowledge about the connection between teaching and learning of a given subject. Here, *didactics* relate only to mathematics, but in general, it can be used regarding any subject, e.g. *didactics of biology, didactics of music* etc.

In the Theory of Didactical Situations (Brousseau 1997; Warfield 2006) learning is assumed to take place in *situations* and to emerge from the students' interplay with a *milieu*, i.e. the problem(s) and the pertinent resources available (e.g. Måsøval 2011, pp. 32-65).

In the *didactical situation* the milieu is *devolved* - passed on - from the teacher to the students, who may then interplay with it, through three types of situations:

- *situation of action*: Spontaneous actions on the milieu
- *situation of formulation*: Hypothesising mathematical coherences
- *situation of validation*: Refuting or confirming hypotheses

Since essential learning emerges from the student-milieu interplay, these situations ideally involve *adidactical situations*, which means that the teacher does not intervene directly. Hypotheses made in situations of formulation are based on feedback from situations of action, while the hypotheses are tested and refuted or confirmed in situations of validation. The newly gained knowledge may be *institutionalised* by the teacher (e.g. Clivaz 2015, pp. 248-251).

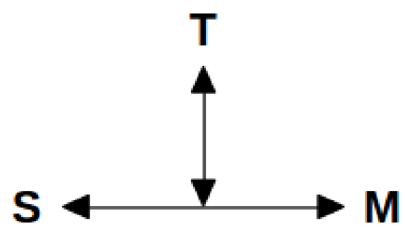

Figure 2: Didactical situation: The double interplay between the teacher (T) and the interplay between students (S) and milieu (M). When T refrains from direct interference in the student-milieu interplay, the situation is adidactical.

TDS was developed to design and analyse experimental teaching situations (of mathematics). Hence, in the situations normally under scrutiny, the learners are expected to develop specific mathematical knowledge, determined by the teacher.

In lesson studies, where the learners are the participating teachers, the target knowledge is didactical and there is no specific target knowledge, known in advance.

Still, there is a clear resemblance between the modelled learning process and the working structure of lesson study. Since in lesson studies there is no 'teacher' of the teachers, the student-milieu model suits the purpose, as a kind of 'adidactical learning situation'.

In didactical situations, the pupils' milieu holds a mathematical problem which can be solved (only) by mathematical practice. The teachers' milieu on the other hand, holds a didactical problem. The possible solution to the problem is of a didactical nature, but working to find its possible solution is not. Since lesson study activities, are not didactical practice in themselves, but are *about* didactical practice, we refer to them as *paradidactical*.

Didactical practice applied as a possible solution to a teaching problem comprise not only the mathematical problem which pupils have to face, but at least *some* idea about how to present this problem. Such a *didactical* idea can be more or less thought out, and when the teachers reflect on the possible effect of it, they are *anticipating student responses* to that idea, i.e. they apply the didactical idea to an *anticipated* didactical situation (student-milieu interplay). Didactical ideas adopted to the lesson plan are applied to a *realised* didactical situation in the subsequent research lesson.

The model of teachers' learning in lesson studies, depicted in figure 3, illustrates not only how the paradidactical learning situation *is about* didactical situations, but also the very learning process itself: When teachers apply an idea to an *anticipated* or *realised* didactical situation, they receive feedback from that didactical situation. This will allow the teachers to either refute or confirm the idea (in its present state) to be a possible solution to the teaching problem. Hence, teachers' learning from a lesson study evolves in concordance with the evolution of didactical ideas, and is formed by the feedback.

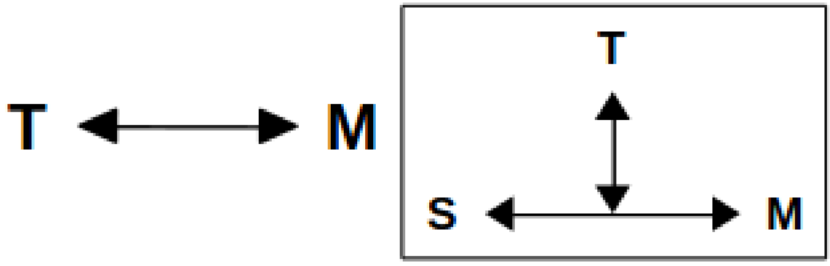

Figure 3: Teachers' learning situations in lesson studies. Teachers' actions on a didactical situation, i.e. a learning situation for pupils, produces feedback which inform the teachers. The didactical situation may be anticipated or realised.

The model illustrates some similarities between the student-milieu interplays in adidactical situations and the paradidactical interplay between teachers and didactical situations. But the model also reveals a crucial difference. Contrary to didactical situations, paradidactical situations (in lesson studies) do not involve a teacher to design the milieu and to regulate the learner-milieu interplay. This relates to lesson studies being teacher-led. Depending on the case, the facilitator's role may adopt similar traits as those of a teacher with a didactical intention. The facilitator may also adopt functions belonging to the milieu, or act as a catalyst for the teacher-milieu interplay.

3.2 Modelling structures of lesson studies

The core activities of a lesson study are to develop and test hypotheses about teaching-learning connections, i.e. didactical ideas. Lesson study activities are not in themselves didactical situations, they are *about* such. Hence, we can distinguish between the didactical situation and the *paradidactical situation*, the latter being *about* the former. Figure 4 models lesson studies as paradidactic structures, the didactical situation being the point of reference:

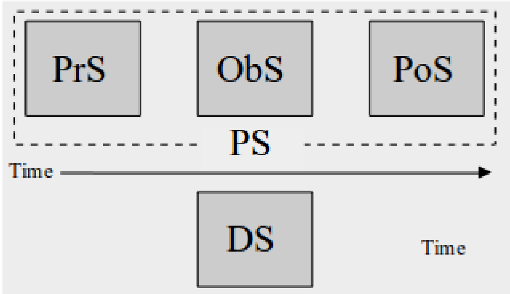

Figure 4: Paradidactic structures of a lesson study. Modified from **winslow2017**. DS, PS, PrS, ObS, PoS refer to didactic, paradidactic, predidactic, observational, and postdidactic situations.

From an analytical point of view paradidactic situations differ in nature whether unfolding before, concurrent with or after the research lesson. Accordingly, the model helps us analyse the interplay between the different types of paradidactical situations and the didactical situation, and the possible outcome of this interplay. This is especially convenient when analysing multi-cycled lesson studies, where certain relations between PoS and PrS are crucial.

Figure 5 illustrates the paradidactical structures of a three-cycled study (as deployed in the case of his article):

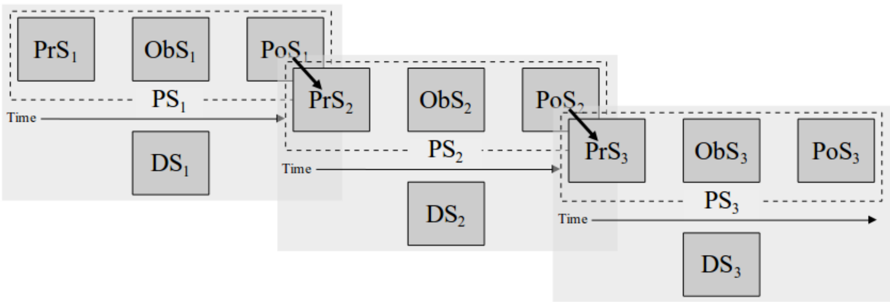

Figure 5: Paradidactical structures of a lesson study with three cycles. Here DS, PrS, ObS, PoS refer to didactic, predidactic, observational, and postdidactic situations, indexed according to the cycle. Adopted from **winslow2017**

4 Methodology and context

4.1 Context of the lesson study

The lesson study under scrutiny here was part of a larger research project in which teachers used lesson study as a format to experiment with inquiry based teaching, based on the Open-ended Approach (OEA, Becker and Shimada 1997; Nohda 2000). OEA was developed to allow for pupils to learn in a process of mathematising a rich problem and test methods to its solution, in order to advance new ideas and hypotheses. It is essential that this process involves sharing and discussing ideas among classmates. In OEA-lesson, the problem is designed to allow for pupils to develop a wide variety of solutions. The teacher is expected to have prepared to make expedient decisions to support the pupils' learning (Nohda 2000, p. 42).

The research project took place in three schools, in each of which a team of three or four mathematics teachers conducted three lesson studies in one school year. The lesson study we consider here is the first for one of those teams, and followed a general pattern of 2 + 5 hours for the initial preparation and 3 times 2 hour for research lesson and reflection. Prior to the first lesson study of each team, a joint start-up seminar was held. Here the teachers were introduced to the lesson study format and the principles of OEA. At the start-up seminar, the full schedule for the (practical part of the) research project was presented, including a concluding seminar. In each lesson study, the two other teams observed the last research lesson and joined the reflections (ObS_3 and PoS_3). For all research lessons, the reflection session was conducted immediately after.

All schools and teachers volunteered, and the schools were financially compensated for a part of the time teachers spent on the project. All participating teachers had at least ten years of experience. For anonymity, all real names are replaced. Pupils are anonymised by replacing their names with variants of 'pupil'.

The author played the roles of researcher and facilitator. Facilitation was characterised by (at least) two parallel approaches: Ensuring a focus on didactics in general and OEA in particular, and keeping the process in motion. In general, the facilitator took active part in the investigations, asked clarifying and motivating questions and pointed to general and specific principles (e.g. of LS or OEA) and theory (e.g. TDS or teaching-learning in general) when necessary.

4.2 Data

The selected sequence of lesson studies is chosen among three. Any of them could be useful for this analysis, and the chosen one is not particularly more representative than the two others.[[]]

The data for the analysis consists of video recordings, field notes and written lesson plans. The primary videos (approximately nine hours), comprising recordings of all pre- and postdidactic situations, have been transcribed (verbatim). Field notes and recordings of the lessons have been consulted to verify teachers' observations. The lesson plans play a minor role, since their quality is marked by the fact that this is the teachers' first lesson study: formulating research questions, goals, reasoning, anticipated student responses etc. is a discipline to be learned over time, as is writing the proposed plan for the lesson.

In accordance with the analytical tools, the pre- and postdidactical situations have been examined to expound the evolution of knowledge from the teachers' interplay with anticipated and realised didactical situations. Selected, exemplary learning evolutions will be analysed in full using the theoretical tools, while other learnings will be described in outline.

Specifically, for each learning observed, the analysis searches to elicit the mechanisms leading to that knowledge. In practice, the analysis starts at the final session (PoS_3) and proceeds backwards, to PrS_3, then PoS_2 etc.

To support the overview of roles in a given situation, the teacher teaching the lesson is identified by underlining his or her alias when quoted.

4.3 Research Questions

While observing the aims and the purpose of this analysis, the research question we sought to answer is: *What did the teachers learn from participating in the lesson study, and how did that learning emerge? Specifically, which concrete mechanisms led to that learning?*

5 The case

5.1 Starting point - the teaching problem

The following paragraphs serve to introduce the reader to the concrete the lesson study. First the teachers' initial thoughts about the teaching problem to be addressed are depicted. Subsequently, since the teaching problem regards methods of subtraction, basic theory about these is presented.

Before reporting on the results of the teachers' learning, we provide an outline of the lesson developed in PrS_1 and revised in PrS_2 and PrS_3 respectively.

When initially discussing the teaching problem in PrS_1, the teachers stated that:

Pierre	We sometimes experience in mathematics that they have learned a method, and then they use that method no matter what. Hence, some times they don't realise that it could be expedient for instance to take away or count [upwards]. It is that connection we would like them to [see].
Robert	Well, it is basically number sense, right. Because we work with their number sense and method choice .. look at a subtraction problem and then say, which method should I use."
Pierre	we have written: develop expedient methods to solve subtraction problems
Robert	... we are looking for pupils beginning to see that there is a connection ... that it pays off to use the 'count-up method' rather than exchange [e.g. tens to ones].
Henry	The goal is that the pupils understand that (..) [different] methods can be used on different subtractions.
Joan	The pupils should learn to understand to use different methods on individual problems.

Hence, the initial teaching problem was that of how to make pupils realise that there are several methods, and that one may be more expedient to solve a given problem.

5.2 Subtraction

The mathematical topic addressed in the lesson study in question is that of subtraction of positive integers by counting, using a number line. In van den Heuvel-Panhuizen and Treffers (2009) two models of subtraction are outlined, taking away (TA) and determining the difference (DD) (ibid., p. 108).

TA implies the use of a counting down strategy, i.e. starting at the minuend (the larger number). DD may be performed either by counting up from the subtrahend to the minuend or by counting down from the minuend to the subtrahend.

In the following, 'TA' refers to a problem (as recognised by the student) of the TA model solved by counting down strategy, 'DD_{up}' a problem of the DD model solved by counting up strategy and 'DD_{down}' a problem of the DD model solved by counting down strategy. Hence there are three ways of performing subtraction when counting on a number line.

This knowledge was not explicit among the teachers prior to the lesson study in question. There is in fact little consensus on terminology, at the teachers use expressions such as (direct translations):

TA	Withdraw, withdraw backwards, count down, count downwards

DD_{up}	Count down, count downwards, find difference, withdraw
DD_{down}	Count up, count down, withdraw, find difference

So, several expressions were used for each method. This changed during the lesson study.

5.3 Outline of the lesson

In the next section, the progression of the teachers' learning throughout the lesson study is illustrated. The purpose of this article is not to present all details of the experimental lesson designed, but to give an overview of the teachers' learning. We now outline the intended didactical situation as eventually designed.

It consisting of three activities:

1 Sentences: Talking about different given expressions related to 'minus', aiming to draw the pupils' attention to 'difference'.
2 Maths stories: Producing maths stories, to increase pupils' awareness of the meaning of the sentences and its relation to 'difference'.
3 Number line: Pupils present their own methods of choice, in order for each pupil to observe the various methods in action.

The main role of the research lesson is to provide feedback the teachers' didactical hypothesis. Seeing how the pupils responded to the (experimental) teaching, allow for teachers to analyse the function of the hypothesised teaching-learning mechanisms. In this regard, activity two turned out to be not very informative with respect to the teachers' learning. Hence, activity two plays a minor role in the following analysis, while activity one and three play major roles.

In activity one, the class was talking about the following sentences (all grammatically correct in the original language, Danish), presented on a signboard and discussed one at the time:

a How far is it between 16 and 11?
b How much is 18 larger than 15?
c How much is 4 smaller than 19?
d How far is it from 5 up to 7?
e What is the difference between 10 and 3?
f How far is it from 12 down to 2?

The didactical idea of activity one was that: talking about how all problems could be represented by phrases of type e, and that this phrasing even works when swapping the order of the 'smaller number' and the 'larger number', would lead pupils to begin to think more of 'difference' than of 'subtraction'.

For activity two, the didactical idea builds on the tradition of working with maths stories. The intention was for pupils to use the six sentences. This was assumed to help clarify the meaning of 'difference' and to illuminate how all sentences could be said to be about finding difference.(As a side note it is interesting that throughout the lesson study, teachers keep discussing the difficulties for pupils to work meaningfully with maths stories. Still they chose to let the pupils works with them).

In activity three, pupils made their own choice of method while solving a subtraction problem on a number line. Subsequently, they would each explain their method to a classmate. Lastly, volunteering pupils illustrated their method to the class by drawing on the board where they start, which direction they go, and where they end. The didactical idea is that when pupils can see for themselves each method and their affordances in concrete examples, as performed by their peers, each pupil is offered anew opportunity to adopt one(s) they consider useful.

6 Results

The results comprise two inseparable elements: The teacher learning itself and how that learning emerged. First, a set of *types of learning situations* is presented. Following that, the teacher learnings observed and their generation are presented.

6.1 Learning situations

Different paradidactical situations constitute different types of learning situations. During predidactical situations, teachers anticipate didactical situations, and hence the learning here is based upon reflecting new ideas against teachers' private and joint knowledge, possibly consulting the literature, teaching materials, the facilitator or other sources.

The didactical ideas retained for the lesson plan are tested against a realised didactical situation. Observational situations do not so much constitute learning situations in themselves, as they serve the function of providing feedback from the realised didactical situation. In the postdidactical situation, a main source of learning lies with the analysis of data collected during observation. The analysed data bring new knowledge to the teachers or confirm the knowledge they already possessed. When the new data is used to re-consider the teaching problem and the didactical ideas, a new predidactical situation is established.

The first predidactic situation, PrS_1, is unique in that it is timewise much longer, and forms the foundation for the whole lesson study: here the pivotal teaching problem is defined and the subject matter in question is studied. Likewise, in PrS_1 the fundamental didactical ideas are devised. These elements are repeatedly revisited, in the following postdidactic situations, but never as extensively. Still, the same type of paradidactical situation is considered to provide the same *type* of learning situations.

At least five types of learning situations are identified:

- Developing the teaching problem (PrS)
- Studying the subject matter in question and related resources - *kyouzai kenkyuu* (PrS)
- Developing didactical ideas and testing these against an anticipated didactical situation (PrS)
- Analysing the feedback provided from testing didactical ideas in realised didactical situations (PoS)
- Throughout a lesson study, knowledge about how to conduct lesson studies (paradidactical knowledge) is developed (PrS and PoS).

Most teacher learning appeared as the outcome of a complex process. Before presenting fine grained analyses of some of the identified teacher learning, we present an overview of what the teachers learned through the lesson study.

6.2 Teacher learnings

The teacher learnings fall into three types relating to *didactical knowledge*, which is about the connection between teaching and learning, *paradidactical knowledge*, which regards learning about the connection between teaching and learning, and *generic knowledge*, which is about e.g. pedagogical practice, organisation, management etc. Here, the different examples of teacher learnings are grouped by the type of knowledge it relates to. Detailed analysis of the two teacher learnings presented under didactical knowledge is given later.

6.2.1 Didactical knowledge

Methods of subtraction

In PrS_1, while discussing the teaching problem and possible solutions to it, it became clear, that the teachers (and the facilitator), did not have precise knowledge about which subtraction methods exist. Through teachers' collaborative reflections and small investigation into it, the three methods of TA, DD_{up} and DD_{down} were discovered and verified. The facilitator pushed a number of times for teachers to engage in the investigation of methods, but they did not commence it before one of the teachers anticipated the student-milieu interplay of a given didactical idea.

'Minus' means take away:

One of the key didactical assumptions, fit into the lesson plan, was that pointing pupils to minus being about finding the difference would open their minds to various ways of subtracting. In ObS_1 though, the teachers observed that pupils only employed TA. In PoS_1, the teachers concluded that this was related to the teaching teacher putting too much emphasis on minus when speaking and on the board: when pupils were to illustrate how they solve a problem, this was presented as e.g. "7-5", putting emphasis to 'minus'.

The teachers hypothesised that putting more emphasis on difference, and less on minus, would lead the pupils to employ all three methods. In the revised lesson plan (for DS_2), the teacher was only to use the word minus when responding to pupils using the word, and only in 'neutral' ways, i.e. in ways that the teacher would not expect to let the pupils think of minus as an essential part of the lesson. Likewise, the problems for activity three were to be presented as to *show how you would find the difference* between e.g "7 and 5", "19 and 4" etc.

During ObS_2, the teachers observed all three methods being used and eventually illustrated on the board. In PoS_2, they concluded their hypothesis to be confirmed.

The emergence of this teacher learning is illustrated by a fine grained analysis later.

6.2.2 Generic knowledge

The force of examples

Though it probably is not in its essence new knowledge, it is evident that from the interplay with realised didactical situations, the teachers learned about the force of examples. This is evidently true concerning the teachers' choice of examples to enrich the pupils' milieu, and it also holds for pupils' examples.

Robert	... it's a funny thing, that herd mentality ... one pupil start by saying that [the numbers] represent centimetres and then a number [of pupils] say centimetres ... when one has said it, well then that's the way it is ... (regarding maths stories) ... the majority, they start by making stories, where ... "how many remain?"... I think there were two examples of "how many remain?". And then, that is what they lean against.

Robert	(Encouraged to discuss the secondary research problem: "how can differently formulated problems of subtraction influence the pupils' choice of method?") I would rather grab hold of another issue I am left with, the crucial point of how important it is to choose the right examples .. the interesting part as a teacher is you have to walk around and have an eye on what they are working on. ... the pupil you choose to pose an example, it is insanely crucial... the choice of who get to explain what a maths story is at all, they presented a maths story which actually set it in motion in the whole group, so that [the stories] by and large all became uniform... it is like the gregarious [mentality], like the pupil who said "centimetre" and then the other pupils say "centimetre" as well. I think it's sort of the same, if you use a pupil as an example, and they say like "ahh, I understand that", and then that's what they do. And that is quite interesting, when you as a teacher have to go around and se their work and decide, this is a good example, and this an opposite example, it is those two example I want to be brought to the fore.
Pierre	... regarding what Robert said about the importance of choosing the right example ... it is really interesting to see ... how significant the examples you give or the explanations you give have for the rest of the lesson or the way you trigger the children's way of thinking, you know kick starting it, in a lesson. If you use some example, then one example can lead them to think like this for the rest of the lesson, and a different one (lead them to) make them think in a different way...
Facilitator	Robert talked about examples in the milieu, but think as well of the things we bring up as teachers?
Pierre	Well. I think that what we have learned from these three lessons, just how we present things, even just how we present a signboard, which word are we emphasising, are we saying "now we work with minus" or don't we say at all what we are working with.. how significant an influence that has proven to have on the progress of the entire lesson, on how they used a number line, how did they choose to compose their maths story. You know, how much significance it has as to where we end up or can end up. That is perhaps the most interesting from this [lesson study].

It is interesting to consider that these examples are not in themselves unique to a teaching situation. Still, as with the realisations about how specific words and formulations can impose unwanted response, they lead to distinct situations of learning.

6.2.3 Paradidactical knowledge

Iteration

In Japan, the origin of lesson studies, research lessons are usually not repeated. This is however often recommended outside of Japan. As is already demonstrated above, the iteration of lesson study cycles appear to have a positive effect on teachers' learning possibilities. In terms of paradidactical knowledge, teachers also reflect on working with multiple instances of the same lesson. In the first two examples, Henry and Pierre compare didactical situations (namely DS_1 and DS_2) with a different outcome. Hence, feedback from their actions on two different but similar didactical situations enrich each other and the teachers' learning:

Henry ...I also think, that issue of language, it shows how it simply supports a wider variation of approaches. It brings about a broader palette. So, I think our reflections from yesterday are really well supported by the today's lesson.

Pierre ... but really, that ... the formulation ... influences their choice of method is obviously evident in this lesson in comparison to Yesterday.

Pierre It was interesting, what we discussed .. but that is because we have tried now, three mornings in a row, to do these things, and to see how things can change. First we inadvertently said 'minus' very early and then they more or less all used take-away through out the [lesson]. In the second [lesson] we more or less didn't use the word 'minus', and various [methods] appeared. And this time as well, I think that the word 'minus' relatively early comes into play. But I don't know if the concept of difference too comes so well into play that they don't really choose the traditional take-away [method]. As really many did the first day, where we experienced that they quickly thought that minus was like, you have something and then you take something away. It has been interesting to see how different it has been from [lesson] to [lesson].

Here, Pierre compares feedback from all three didactical situations, which in its combined form provides him with more learning that they do individually. Pierre continues to elaborate, repeating the power of iteration:

Pierre	... I think that what we have learned from these three lessons, just how we present things, even just how we present a signboard, which word are we emphasising, are we saying ”now we work with minus” or don’t we say at all what we are working with.. how significant an influence that has proven to have on the progress of the entire lesson, on how they used a number line, how did they choose to compose their maths story. You know, how much significance it has as to where we end up or can end up. That is perhaps the most interesting from this [lesson study].

Apparently, being able to try out either similar or alternative ways to present a given problem (devolution and milieu) in comparable situations is considered beneficial, allowing for repeated explorations regarding the specific teaching-learning mechanisms.

6.3 Evolution of teacher learning

In this section, the tools of analysis are put to use to identify the evolution of teacher knowledge. The two examples presented are chosen because they represent important findings, but also because their evolutions illustrate how lesson studies (can) provide teachers with rich learning situations.

In the first example, it is shown how interplays with didactical situations across the lesson study affect and support each other. The analysis is started from the final postdidactical situation (PoS_3), in which the participating teachers account for some of there experiences and learning to their colleagues from the two counter-teams. Yet, the origin of the knowledge lies earlier, and its emergence is the outcome om reflexive interplays between the teachers and didactical situations, as will be illustrated here..

The second example takes place during PrS_1, and regards the mathematics in question: methods of subtraction. From the outset, none of the teachers, all with at least ten years of experience, can account for which methods exist. The facilitator tried to get the teachers to engage in investigations of the methods, but with no success. But when one of the teachers started to anticipate the pupils situation when working with the methods, she and the team became aware, that knowing the methods in detail was crucial. Hence, the emergence of this learning stems from the feedback of teachers’ interplay with an anticipated didactical situation.

6.3.1 The evolution of *’minus’ means take away*

By the end of the lesson study, one teacher learning stood out: That of how pointing out *minus* led pupils to think of only TA, while focussing on *difference* led pupils to think of the two variants of DD *in addition* to TA. Early in PoS_3, Henry, the teacher teaching the lesson, stated his observation that:

Henry	... all over the classroom there were pupils trying out all three subtraction methods ... All three methods came about, all over

Robert agreed to this and added, that:

Robert In [the first research lesson] it was made clear that it was about minus, and instead of 'and' between [the numbers] we had written 'minus'. This led many to solve it as a problem of minus (i.e. TA) ... In [the second research lesson] it wasn't made much of [minus], and there was written 'and'... and so we had many working with 'difference'.

These observations were seconded by Pierre. To inform the colleagues from the two other teams of the research project, he elaborated:

Pierre First we inadvertently said 'minus' very early and then they more or less all used take-away throughout the [lesson]. In the second [lesson] we more or less didn't use the word 'minus', and the various [methods] appeared. And this time as well. I think that the word 'minus' relatively early comes into play, but I don't know, perhaps the concept of difference too comes so well into play that they don't really choose to employ the traditional take-away [method]. like so many did the first day, where we saw that they quickly thought of minus as you have something and then you take something away.

The observations and reflections presented in PoS_3 are - as indicated - more or less a repetition of the corresponding ones of PoS_2. While the analysis in PoS_2 bore resemblance to a confirmation of the pivotal hypothesis of the didactical idea, the teachers seems to consider the re-confirmation in PoS_3 as a re-enforcement of the hypothesis.

The crucial feedback regarding the emphasis on *minus* came in PoS_1 though. Shortly into PoS_1, the teacher who taught the lesson, Robert, shared his observations and considerations:

<u>Robert</u> It was a shame ... when it was about minus, they write it as a problem of subtraction (i.e, TA). They seemed to quickly forget it was about finding the difference. Because, when they go to illustrate on the board, they.. And actually, you can say that the board (with problems and the number line) that I had made myself ... there I say that these are the subtraction problems that you must give an explanation to. And I can see, well they have already... I had put it as minus. In other words, that's how they put it too, as minus (i.e. TA).

Robert had no doubt, that presenting the problems with en emphasis on minus rather than difference, led pupils to think of only TA. Keeping in mind, that this was the very first time (ever) for the teachers to conduct a post-didactical reflection, the observing teachers did not in the first place engage in the issues Robert raised. This was later re-raised by the facilitator:

Facilitator .. I was thinking, instead of putting the subtraction problems in the form of seven minus three, [how about] *show on the number line how you find the difference.* That's what we discussed (in PrS_1), to make a point of difference. We actually haven't brought that into the last activity.

Henry That's right, we talked about putting it on the board, like it would be just the numbers...

The above is an excerpt of a longer dialogue regarding the question of how to present (devolution and milieu) the problems, meant for the pupils to solve. For the sake of space and overview, a number of items considered are omitted, e.g. how many numbers, the physical distribution of numbers on the board, the order of the higher and the lower number(s) respectively, separation by commas, space, columns, *and* etc (the size of the numbers and the choice of using a number line were discussed and decided in PrS_1). The point is to illustrate and analyse considerations taken into account when re-designing the milieu and the didactical idea. This reflects a modification of the teaching problem: Whereas the teaching problem forming the hypotheses in the lesson plan is still valid, a focus on how to avoid pupils from thinking on TA has been added.

Meanwhile, the teachers had discussed how to promote better the focus on difference, including how to make pupils consider and realise, that phrases of e.g. type e, *what is the difference between 7 and 5*, can be used independently of the order of the higher and the lower number while others, e.g. phrases of type b, *How much is 18 larger than 15?* can not.

Regarding the part of the teaching problem discussed here, i.e. which related to activity three, it was hypothesised that if the teacher would successfully direct the pupils' awareness to difference, presenting the subtraction problems on the board as *Use the number line to explain how you find the difference between* e.g. [7 and 5], would lead pupils to think of and present all three methods. This didactical idea was tested in DS_2, and in the following PoS_2, the teachers shared and discussed the feedback of the realised didactical situation:

Pierre ... there is something about the whole plan, the way it was rolled out, because a lot of them, they actually ended up, contrary to Yesterday, they ended up using the method of filling up. Because that concept, difference, came to be a strong presence in some way. So, I think that was a success. I think, that they somehow succeeded in putting those concepts into play, at least the concept of difference... there really was a lot of filling-up [DD_{up}].. the methods employed were either filling-up or ... some of them showed [going] backwards on the number line ... not all the way back but backwards to the difference [DD_{down}], or how to say it. So, I think that was a success... a minority of them used the method of going all the steps backwards [TA].

Facilitator ... it is evident that yesterday, one of the first things we discussed (in PoS_1) was how our expectations were not satisfied. We don't talk about that today. [The lesson] has to a larger extent stuck to what we anticipated to emerge. It has taken us by surprise that it is not apparent to us and the pupils that it's about minus. But from our observations today it is evident that they use the formulations [of the sentences] a lot more and they were much more versatile. Both when it comes to how they used the formulations, and how they worked with the number line. The difference in how the problem[s] were presented today, were clearly significant.

Robert	At any rate, I find it really interesting that since it is not apparent to them that this is about minus, it becomes clear for a larger number of pupils, that it is about difference, or at least one of the other formulations (of the sentences). It is just an assumption. As a result, many more, when they look at the number line, realise it's about difference. That makes me think - it's and assumption - since they see it as a difference, they count upwards and count what is in between [the two numbers]. They count the difference by counting upwards.
Joan	Yes, or down
Robert	But it is still the difference.
Joan	Yes. Yes.
Robert	and it was much more evident today that it was about difference, and basically not about minus. Whereas yesterday, it was all about minus and therefore hey start at the larger number and go backwards [TA]. I just assume, that because Yesterday the problem was presented like it was about minus, it's about minus. Today that wasn't obvious, but it was about difference, or some of the other notions, and so that's what came... You know, it's really crucial how you... bring focus on... whether you bring focus to minus or bring focus to difference.

The teacher learning regarded here was explicitly stated by the teachers in PoS_3, but was first expressed in PoS_2. Whereas the third cycle of the lesson study then functioned to cement the hypothesised solution to the teaching problem, the learning evolved via interplays with didactical situations over the course of PrS_1 through PoS_2.

The feedback of ObS_2 confirmed the hypothesis, that downplaying minus and emphasising difference from the teachers side would lead the pupils to present the three methods of subtraction. This hypothesis, on the other hand, emerged from the feedback of ObS_1, in which the teachers realised that speaking of minus and presenting problems of subtraction as e.g. 7-5, led pupils to think of taking away. But this interpretation again is based on the teaching problem and the didactical ideas developed in PrS_1, which involves for pupils to work with difference in a broader sense, not only narrowed down to that one method, each pupil is believed to usually stick to. Hence, the teacher learning does not just happen at an instance but is the outcome of a longer reflexive interplay between the teachers and didactical situations.

6.3.2 The evolution of *methods of subtraction*

Even before the lesson study began, there was little doubt that it would concern a teaching problem regarding subtraction. Early in PrS_1, Pierre read aloud the focus discussed as early as at the start-up seminar: [for pupils to] develop expedient methods to solve subtraction problems. Throughout the lesson study, other topics were never discussed.

In the beginning of PrS_1, the teachers talked about methods of subtraction, the learning of them and their expediency without accuracy. When talking about *methods of subtraction* the teachers used such expressions as: go kind of backwards, use the filling-up method or some take-away method, algorithms (Pierre), writing down on paper, look up on a hundred-table, fill-up, go forward, go backward (Robert), number line, place above each other, filling-up (Henry).

While most of these expressions are not necessarily problematic in themselves, the use of them here is symptomatic of what Pierre observes:

Pierre	... we happen to have mixed up methods, algorithms, number table, counting table, write down, filling-up method, take-away method...

Even though Pierre made this comment, the teachers did not engage in clarifying the notions even if they were pivotal to the lesson study from the very beginning. Likewise, the facilitator at a number of times tried to push for teachers to start investigating the mathematical topic, their ideas about the teaching problem and the possible solutions centred about.

Facilitator	What are you looking for? ... Which answers would like to see. What di you want [the pupils] to realise?

At a later point, when teachers are discussing the maths-supervisor's comments about (given at an earlier time) about tools and strategies to solve problems of subtraction, the facilitator asks, "what do you do (when solving problems of subtraction)?", to which one teacher jokingly answers "use a calculator", and follows up by saying that he doesn't really know (Pierre).

A little later, the facilitator tries again to make the teachers aware of there imprecise and apparently insufficient knowledge, hoping to instigate an investigation.

Facilitator	What is the problem, you want [them to] solve? Which mathematical subject knowledge , do you think is difficult [for the pupils]? What do you want them to work with?
Henry	They should work with finding out when - and I guess that is the mathematical problem - to find out when do I use the most expedient method to solve these particular problems.
Facilitator	But, how many different methods do they know?
Henry	I don't know...
Facilitator	It is much more difficult to consider.. to design a problem which aims at when they use which method, when is it most expedient to us one over the other method, if one for instance knows only one method, or you actually isn't aware of a method.

This triggers Robert to speak at length about his general experiences from teaching, but at no point does he or any other of the teachers think of initiating an investigation about methods to subtract. The point here is not so much to illustrate that teachers don't know the methods or that they don't initially engage in investigations, but rather to show what eventually made them investigate and figure out the three methods described earlier.

At one point, Henry develops a didactical idea in which pupils should categorise subtraction problems, and proposed that pupils should sort problems, according to which method they think would be the easiest to solve them (Henry). Anticipating the corresponding situation, Joan reflects on its feedback:

Joan But doesn't that require that you have tried every single method on every single problem? To be able to figure out which one is most suitable.

Henry Well, then we take four...

Joan To be able to answer that I think you would have to test all the methods. Like when you have to find out how to do the best long jump, then you have to try different things, too...

Facilitator But we don't have four methods, do we?

Henry Well, three are those I just... number line, place above each other, filling-up and..

Pierre Filling-up and take-away are actually a number line as well...

Facilitator But isn't that..I am thinking, how many methods are there, Because when you say number line, I think you can use it to take away as well as to fill up.

Joan Yes

Henry Well, yes that right.

Facilitator And so I am thinking, how many different methods are there actually?

Robert For minus? I start at the lowest number and count upwards. That's one. And then there is the opposite, you start at the highest nuber and count downwards.

Facilitator So, there are two methods?

Robert Well, yes I would say so

This new knowledge is not challenged at any time, but on a few occasions, some of the teachers need to reaffirm it:

Robert — I am just... We are actually talking about d... well, if you are going one way or the other way on the number line it doesn't matter, because you are finding the difference. So, you have a method called *find the difference*, and you have one method, called take away. You don't find the difference when you know you've got 148 cubes, and you have to remove 20. That's not finding the difference. I would say that's what is called take away and then there is one called find the difference.

This particular formulation is consistent with (van den Heuvel-Panhuizen and Treffers 2009).

It is interesting how, in this example, what apparently instigated the teachers to engage in an investigation of the methods of subtraction was one teacher, (Joan), beginning to anticipate how pupils would and could respond to the problem. At a glance the emergence and evolution of this teacher learning may seem different from the previous example of *minus leads to tale away.* But along with others, not presented in detail here, the two examples share decisive traits in that the teacher learnings involved are the outcomes of teachers' interplay with didactical situations, realised or anticipated.

7 Discussion

The results illustrate that analysing teachers' learning as and through interplay with didactical situations is beneficial, and they suggest that the various forms of interplay itself is a triggering factor for teachers' learning in lesson studies.

Furthermore, the results indicate that repeating cycles of the lesson study is beneficial to the learning outcome, at least to teachers not experienced with lesson studies. Iteration is in contradiction to typical Japanese lesson studies (Fujii 2014), but the analysis of this article, as well as others (**bahn2017**) suggests that iterations can be profitable, apparently because the repetition offers a chance to adjust or confirm teaching-learning hypotheses.

In an interesting way, the tools and methods of analysis used here have profited from the feature of iteration while concurrently suggesting iteration to be profitable (to teachers and researchers alike).

It is interesting, that lesson studies and TDS, developed in two different and detached context, share so many affordances. As Miyakawa and Winsløw (2009) demonstrates, there are differences as well, but generally, TDS is most appropriate to analyse lessons, as is also pointed out by Clivaz (2015).

8 Conclusion

This article serves two purposes: To present specific teacher learnings from a lesson study, including examples of how that learning emerged and evolved, and to demonstrate how the analytical tools put into use allow for such fine grained analysis.

The analytical tools, derived from the TDS, take advantage of lesson studies being structured as paradidactical situations in which the teachers develop knowledge about didactical situations. Based on the learning model of didactical situations, the model developed for this analysis depicts teacher learning as taking place through an interplay between the participating teachers and their milieu, comprising the didactical situation for which they have to develop a solution. Throughout the lesson study teachers interplay with realised or anticipated didactical situations, and it is illustrated in this article, how such interplays (can) lead to the emergence and evolution of teacher learning.

This is perhaps not so surprising when it comes to the realised didactical situations of the research lessons, but as shown here, it is the case for anticipated didactical situations as well. Perhaps this is the 'truth' behind "anticipating student responses … [being] one of the crucial processes while planing lessons" (Takahashi 2011, p. 79).

The teacher learnings discovered here divide into three: didactical knowledge, generic knowledge and paradidactical knowledge. It is also illustrated, how the different types of phases (e.g. goal setting, *kyouzai kenkyuu* etc.) in a lesson study constitute in themselves specific learning situations and how the teachers' learning evolved across these, each playing their part.

The 'key example' of this article (minus means take away) illustrates how teachers, through interplay with their milieu, which comprised the various stages of the didactical situation of the research lesson, developed knowledge about the teaching problem: to promote pupils to share different methods of subtraction. The design of the initial milieu led pupils to think of only one method (take away) in the realised didactical situation. Based on the feedback from that situation, the teachers developed and tested hypotheses about how to not narrowing down pupils options inadvertently, but to support their own choice of method. The feedback from the subsequent didactical situation confirmed the hypothesis which was re-confirmed in the final didactical situation.

In accordance with e.g. Clivaz (2015) and Miyakawa and Winsløw (2009), the present analysis points out, in a more concrete way, that there are apparent similarities between structures and affordances of lesson studies and the learning model of TDS, making the latter appropriate to analyse the former.

Teachers interact with didactical situation daily, so why is lesson studies so much more efficient to develop teacher knowledge? It seems plausible to say, that it is due to the infrastructures (see Miyakawa and Winsløw 2013**winslow2011**) of lesson study. Yet, this needs further scrutiny. It is hypothesised, that TDS can also be used to analyse infrastructures' impact on teacher learning, hence providing us with crucial knowledge about e.g how to optimise possible learning outcome by facilitation.

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③ Evolution of teachers' anticipation of didactical situations in the course of three lesson studies

Third article, in review for *Annales de Didactique et de Sciences Cognitives*

Evolution of teachers' anticipation of didactical situations in the course of three lesson studies

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Abstract. Teachers' anticipation of pupils' possible responses in teaching situations is at the heart of lesson study. In this article, it is investigated how teachers' anticipation about teaching-learning interplays evolved in the course of three lesson studies. Taking advantage of teachers being 'forced' to communicate and display their knowledge when preparing and planning a research lesson collaboratively, episodes from one team of teachers' first and third lesson studies are analysed and compared. Tools derived from the Theory of Didactical Situations were employed for the analysis, and the results illustrate a clear evolution - quantitatively and qualitatively - of teachers' anticipation regarding five pivotal aspects of teaching situations: *the target knowledge*, *the material milieu*, *pupils' response*, *adidacticity* and *the teacher's management of the student-milieu interplay*.

Introduction

Anticipation is at the core of scholarly research (Artigue, Dillon, et al. 2012), which is the role model of IBE, inquiry based education (Artigue and Blomhøj 2013). With a strong reference to e.g. the works of Dewey, IBE is widely endorsed as holding the potential to raise pupils' motivation to learn (Rocard et al. 2007). Yet, most teachers are not well-trained in the structured use of anticipation seen in research, and hence have little means to pass on the research-like work forms of IBE (Artigue and Blomhøj 2013; Rocard et al. 2007). Accordingly, endeavours to implement IBE fail to take root in most countries (Stigler and Hiebert 1999). Interestingly - and most promising - IBE has become an inherent element of Japanese teachers' practice, inside as well as outside the classroom (ibid.).

Since the late nineties - and in particular since *The Teaching Gap* appeared (ibid.) - it has been well known and widely acknowledged that Japanese teachers' widespread use of IBE is attributable to an educational culture founded on *lesson study*, a professional development paradigm involving practices similar to those of research (Miyakawa and Winsløw 2009, p. 203). Japanese teachers are educated and work in an educational environment in which anticipation is actively practised (used and trained) with the purpose of developing their personal and collective knowledge about the interaction between teaching and learning (Stigler and Hiebert 1999). It is particularly interesting, that the Japanese(') ideal and practice of e.g. IBE is the direct result of decades of systematic investigations of how to give life to ideals and principles of classical educational philophers such as Dewey (Becker and Shimada 1997; Nohda 2000).

The case of Japan suggests that there is a powerful self-perpetuating connection between studying teaching-learning relations through inquiry based investigations and managing pupils' autonomous investigations through inquiry based teaching. It seems plausible that Japanese teachers' experiences with and knowledge from such structured research-like processes develop their capacity to anticipate global teaching-learning dynamics not only prior to a lesson (planning) but also to instantaneously anticipate local teaching-learning mechanisms within episodes of the lesson (improvising).

The learning model used for analyses in this paper also includes a central role for anticipation, in the form of hypotheses to be tested, as a crucial step towards learning. It is adapted from the Theory of Didactical Situations (Brousseau 1997), which was originally developed to design and analyse experimental teaching (in mathematics) (Måsøval 2011; Warfield 2006).

Here, the learning model is extrapolated to the analysis of teachers' learning through inquiries. In practice it is investigated, how teachers' anticipation of teaching-learning situations evolved from their inquiries in the course of three lesson studies.

The research presented here is based on the hypothesis that the effects of lesson studies will apply to Danish teachers as well, and it is hypothesised that teachers' inquiries into the connection between teaching and learning will develop their capacities for anticipating crucial events during teaching-learning situations.

Lesson study

Lesson study comprises manifold, often intangible, structures of inquiry into pupils' learning (e.g. Takahashi and McDougal 2016, p. 515). It is one of the most important forms of professional development for teachers in Japan (e.g. ibid., p. 515). In general though, lesson study refers to less intangible structures of research-like processes performed at schools (Murata 2011; Miyakawa and Winsløw 2009).

As the purpose of lesson study is to develop knowledge about and techniques of teaching pupils specific curricular knowledge, the central activity in lesson studies is to study pupils' learning (e.g. Miyakawa and Winsløw 2009, p. 207). This is done through activities similar to those of research: Once a teaching problem is defined, a hypothesised solution to it is devised through thorough *kyouzai kenkyuu*: studies of the subject matter in question and related teaching materials, pupils' general learning, the specific pupils' prior learning of the topic, and the sequence of learning it is or should be at part of. The thoroughly devised hypothesis of a solution to the teaching problem is then tested under observation in a live research lesson. Subsequently the observations are shared and analysed (e.g. Fernandez and Yoshida 2004).

It is not unusual for some Japanese teachers to devise the lesson plan by themselves, although they will typically consult colleagues sometimes during the work. Outside Japan, it is most common for teachers to undertake lesson studies as a team, collaboratively defining the teaching problem and devising the hypothesis (Lewis and Hurd 2011). In both cases, colleagues are observing the teacher teaching the research lesson.

Particularly in the case of a team-based lesson study, researchers can take advantage of the fact that teachers explicitly share and apply their knowledge and ideas when devising the lesson plan and when they analyse observation data (Fernandez and Yoshida 2004; Stigler and Hiebert 1999; Takahashi 2011).

There are no set rules of what constitutes a lesson study. But there is a broad agreement that the aim is to investigate teaching-learning processes linked to a specific target knowledge and a specific group of children, and that a single lesson study is structured as illustrated in figure 1 (Winsløw 2011; English 2009; Murata 2010):

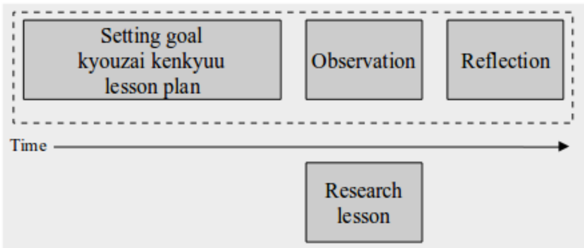

Figure 1: Generally agreed structure of a lesson study as an activity with a detached research lesson as the point of reference. Adapted from Winsløw, Bahn, and Rasmussen (2018)

It is a common misconception that the research lesson is the most important part of a lesson study (Fujii 2014). Rather, it is a means to collect data and can be considered as detached from the core lesson study activities of anticipating and analysing the students' learning in the lesson, as figure 1 illustrates. Japanese researchers and practitioners consider *kyouzai kenkyuu* and particularly *anticipating pupils responses* as the most important element of a lesson study (Doig, Groves, and Fujii 2011; Watanabe, Takahashi, and Yoshida 2008). This viewpoint is consistent with the research presented here.

Theory

In the theoretical framework for this analysis, the Theory of Didactical Situation (TDS - see e.g. Brousseau 1997), *didactics* refer to knowledge about the connection between teaching and learning of specific subject matter (target) knowledge.

The pivotal notion in the TDS is that of *didactical situation* (ibid.), in which students' learning evolves through their interplay with a *didactical milieu*. The milieu comprises the problem to be solved and pertinent resources to solve it, and the student-milieu interplay unfolds in *situations of action* (immediate attempts), *situations of formulation* (hypothesising) and *situations of validation* (Måsøval 2011). Essential learning depends on rich student-milieu interplays, which again requires the milieu to hold a rich potential for feedback, on the basis of which the knowledge to be learned can emerge as the solution to the given problem. Situations in which student-milieu interplays unfold without the direct intervention of the teacher are *adidactical*.

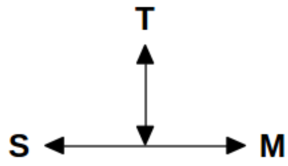

Figure 2: The double interplay between the teacher (T) and the interplay between students (S) and the didactical milieu (M): *the didactical situation*

The teachers' didactical practice involves to prepare and devolve - hand over - the milieu and to manage the student-milieu interplay.

The analysis in this paper is concerned with teachers' learning *about* didactical practice, and the model of learning in the didactical situation is extrapolated to model teachers' learning as it unfolds in *para-didactical situations* (Winsløw, Bahn, and Rasmussen 2018; Winsløw 2011). Due to the absence of a "teacher of teachers" with a didactical intention, paradidactical situations are comparable to adidactical situations. The milieu is constituted by the teaching problem together with a didactical situation, along with pertinent resources such as textbooks, curriculum, teaching materials, a lesson study facilitator, etc.

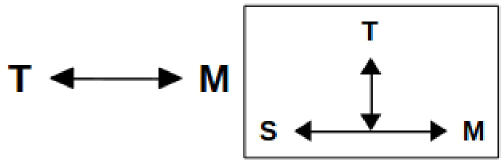

Figure 3: The interplay between teachers and the teaching problem of their milieu in *the paradidactical situation* which may be either *realised* or *anticipated*. Adopted from Winsløw, Bahn, and Rasmussen (2018)

Notice that in the chronological structure of lesson studies, depicted in figure 1, the interplay between the group of teachers and the didactical situation, $\mathbb{T} \leftrightarrow \mathrm{DS}$, is apparent while it concerns, in each of the three phases, different didactical situations: the one which is being planned, the one which is observed in real time, and the observations from the past lesson which is being reflected on. The purpose of the realised didacitcal situation, $\mathrm{DS_R}$, is to test hypotheses about possible teaching-learning connections, referred to as *didactical ideas*, which are developed and tested through processes of:

situations of action	in which spontaneous proposals and materials for teaching-learning processes are put to the test
situations of formulation	in which specific teaching-learning connections are hypothesised
situations of validation	in which hypotheses of teaching-learning connections are rejected or confirmed to be a possible (partly) solution to the teaching problem

In lesson studies, it is a key feature to develop a lesson plan, based on one or more didactical ideas, intended to be tested in DS_R. In the *pre-didactical situation*, didactical ideas are developed, tested and validated virtually, i.e. against corresponding *anticipated didactical situations*, DS_A. Those didactical ideas which the teachers consider to be viable teaching-learning hypotheses, may be put into the lesson plan to be tested in the DS_R of the research lesson. We distinguish between *anticipated didactical situations* (DS_A) tested virtually in the predidactic situation, and *realised didactical situations* (DS_R) tested in the reality of a research lesson.

As illustrated in figure 3, DS_A provides feedback to the group T of teachers, which allow the teachers to test and evaluate a given didactical idea. In this paper, we consider the teachers' anticipation regarding the following aspects: *the target knowledge, the material milieu (problem and resources), pupils' responses, adidacticity (feedback potential)* and *the teacher's management of the student-milieu interplay*.

Context and research question

The lesson studies under investigated here were organised as part of a research project on teachers' development of inquiry based teaching through lesson studies. On each of three schools, a team of three or four teachers conducted three lesson studies, each spanning approximately two weeks, within one school-year. Before conducting the lesson studies, all teachers participated in a start-up seminar in which they were introduced to lesson study, principles of IBE as devised in the Open-ended Approach (e.g. Becker and Shimada 1997; Nohda 2000) and to basic notions and models of TDS. All lesson studies were three-cycled, in the sense that the phases illustrated in 1 were carried out three times for a research lesson which, naturally, evolved in the process. The paradidactical structure of a three cycled lesson study is shown in figure 4:

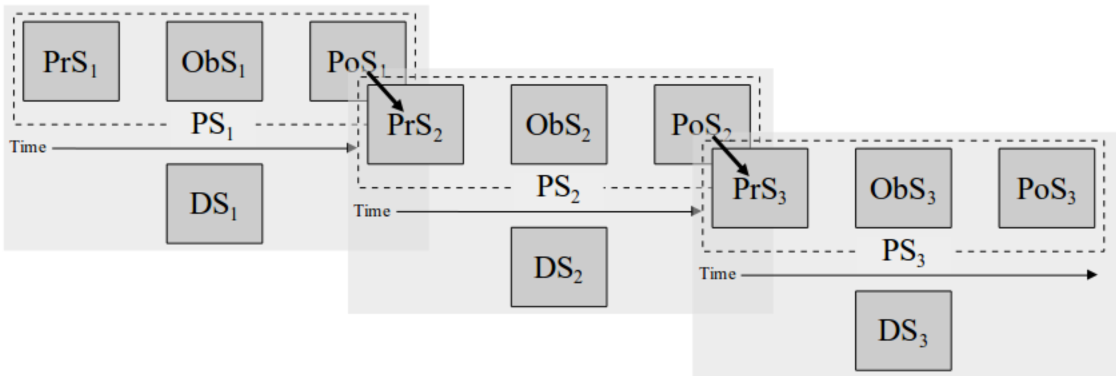

Figure 4: Paradidactical structure of a three-cycled lesson study. PS, PrS, ObS, PoS and DS refer to paradidactical, predidactical, observational, postdidactical and didactical situations, indexed according to the cycle. Adopted from Winsløw, Bahn, and Rasmussen (2018).

Teachers spent seven or eight hours on the predidactical situation of the first cycle, PrS_1, which is when the basic development and planning of the research lesson was carried out. During PrS_2 and PrS_3 the teachers resume the planning based on the analysis of data, which was carried out in the preceding postdidactic situation, PoS. A PoS and the following PrS took place within the same session of 1 hour, while being clearly separated in time.

All nine lesson studies were scheduled in different weeks, and for each lesson study the teachers from the two other teams took part in ObS_3, PoS_3. These exchanges were referred to as *mutual visits* and enabled the teachers of all three teams to take part in observation and reflection related to all nine research lessons (in their third version).

Just as pupils' knowledge becomes explicit and can be observed when they interact with a milieu and its certain mathematical challenges, teachers' knowledge can be observed when they interplay with *their* milieu, the didactical situation, thanks to the collaborative 'enforcement' of explicitation. In this paper, we focus on, PrS_1 of the first and the last lesson study of the team of teachers from one school, and investigate the following research questions::

RQ: *In which ways did teachers' anticipation of didactical situations evolve in the course of three lesson studies? To which extent did the teachers' attention to different aspects change from the first to the last lesson study? How did these aspects depend on each other, and were some consistently considered more important by the teachers?*

Data and method

The author of this article participated in all sessions of the lesson studies conducted, including the two PrS_1 of one team's first and last lesson study, which are being investigated here. Video and transcripts of the two sessions are the primary sources of data, which also include field notes, lesson plans, teaching materials and pupils' productions. When relevant, similar data from other situations of the lesson studies are consulted.

It was chosen to focus on PrS since this is where $\mathbb{T} \leftrightarrow DS_A$ comes into play, and LS_2 is omitted from the main analysis because the purpose is to analyse the overall development of teacher knowledge during a lesson study. Two other teams conducted a similar sequence of lesson studies and their processes could have been chosen just as well. While analysing the evolution of anticipation across all three teams would have been fruitful and have put further weight to the study, it was chosen to focus on one for reasons of time, resources and simplicity.

Based on the hypothesis presented above, that *teachers' inquiries into the connection between teaching and learning will develop their anticipation of such teaching-learning situations*, the research question regarding *in which ways did teachers' anticipation of didactical situations evolve in the course of three lesson studies* is sought to be answered by identifying situations in the relevant PrS where they discuss *the target knowledge*, *the material milieu*, *pupils' anticipated response*, *adidacticity* or *the teacher's management of the student-milieu interplay* in connection with anticipating a didactical situation or situations comparable to that (e.g. an activity).

Results

We now present and analyse some carefully selected episodes in which teachers anticipated aspects of didactical situations in the first predidactical situations of their first and third lesson study. Each example is analysed in accordance with its relevance to the research questions and when pertinent, the analysis draws on other examples or other situations of the lesson study, to illustrate a possible evolution or constant assumption in teachers' anticipation of didactical situations. The episodes are presented in chronological order.

Lesson study 1

The first lesson study, LS_1, took its starting point from teachers' experience that their grade three pupils tended to consider 'minus' as a problem to be solved only by the take-away method (TA, see e.g. van den Heuvel-Panhuizen and Treffers 2009). Initially it was discussed how to teach pupils that there are several methods, and that for a given problem one method may be more expedient than others. Through *kyouzai kenkyuu*, the teachers determined subtraction methods to comprise take-away and determine difference (DD, see e.g. ibid.), which can be done by counting down from the higher number (DD_{down}) or by counting up from the lower number (DD_{up}). Eventually, the question of expediency was dismissed, since teachers in the end considered it to depend on personal preferences, and focus was put on difference and how to find it.

In the first recurring activity, after discussing properties and solutions to various formulations of difference, e.g. *How far is it from 5 up to 7?, How much is 4 smaller than 19?* and *What is the difference between 10 and 3?*, pupils were to consider how they would solve given problems and how they would use a number line to illustrate the applied method to classmates. In the final activity, volunteering pupils demonstrated and explained how they would solve given subtraction problems to the class, using a number line on the board.

In the first research lesson, almost only TA was employed. After specific changes in devolution and milieu, all three methods were more or less evenly applied in the second and the third research lesson: In he first lesson, the teacher had explicitly made sure, that pupils were aware that difference is about minus, and problems, which solution were to be presented on the board, were given as e.g. *show how you solve 7-5*. In the second and third lessons, the teacher only used 'minus' in connection to students' usage of the term, and on the board problems were presented as *show how you find the difference between 7 and 5*, with the opposite order of the larger and smaller number for some problems.

Hence, a key learning from this lesson study was how important small details, for instanse the use or non-use of the term 'minus', can be for the outcome of a lesson. Furthermore, the teachers realised how powerful examples can be, and that a posed example may not only enable students to solve similar problems, but also - inadvertently - eliminate relevant forms of reasoning that are not present in the example. At the same time, the teachers relationship to methods of subtracting integers became much more explicit and precise during the process.

Episode 1.1: Typical proposal of activity

One of the first didactical ideas proposed during PrS_1 was to present the pupils with cards imprinted with subtraction problems, and coloured in accordance to which method of subtraction was most expedient to solve the problem. The task for the pupils would be to investigate "*why.. the yellow [cards] are yellow, and the blue [are] blue and why the red [are] red*" (Pierre). Then, there “*might be some [pupils] who figure out to actually look at the subtraction problem and then they say, well, that’s because...*” (Pierre).

One teacher responded, “*I think it’s a fine idea, your colour codes. At least when it comes to ask [the pupils] the question of … why are some yellow, why are some blue*” (Robert).

The teachers did not mention what could make the pupils *figure out* the target knowledge (namely, that certain methods of subtract are more expedient than others, for a given problem). In fact there seems to be no feedback potential from such a milieu, and the task appears more as a test or practice: If you master beforehand (all) the methods you might be able to categorise which problem is best solved with which method. But in the activity, as proposed here, the pupils were to categorise the cards with no further information. To solve the problem, one would have to know by which characteristics the cards should be sorted, and if one knows those characteristics, there is no new knowledge to be learned.

The facilitator tried to ’force’ the teachers to anticipate the possible connection between the activity and pupils’ learning:

Facilitator	What are you looking for? When you design the problem in this way, what is it .. you are looking for? … What kinds of answers do you want to extract? What do you want [the pupils] to realise?
Pierre	Well, I think we want them to realise.. if for instance we say, well it is very basic, that you had, kind of have.. [if we] then just wrote nine minus eight, ten minus nine, eleven minus ten (..) and then [they] figured out that those were the yellow [ones]. And then figured out that there was a distance of just one [between the numbers].. a very small distance. So, if you had some blue ones, where there were ten minus one, eleven minus two or something, and then figure out, well there is long.. large distance between those two numbers. So when I get the result I have to go kind of backwards. You know, it is that idea between they can see if you have to use the filling-up method or some take-away method...
Robert	And we would also like them to .. categorise these subtraction problems. You know, I am just thinking, it would be clever if we could see that a pupil thinks, hey, this one has to be in the blue category...

Rather than anticipating a didactical situation the teachers *imagine* a situation in which pupils by some sort of inherent logic or automatic mechanism realise underlying mathematical principles by analysing different kinds of exemplary problems. At this point, the solution to the teaching problem, i.e. the (imagined) didactical situation, does not take into account pupils’ possible responses, other than them *figuring out* what is obvious to the teachers (though their possible agreement on the matter is not tested). The pupils' interplay with the milieu is not considered, and in fact the milieu does not seem to contain any ressources for developing the desired explanations, or for validating them.

Teachers also did not realise, that the target knowledge of this activity is not objective, but subjective in the sense that the 'correct' answer is determined by the teachers' opinion of which method is most expedient to solve a given problem. In the lesson study it was later discussed, whether such a target knowledge makes sense at all: If a pupil is unaware or not confident that a subtraction problem can be solved be determining the difference between two integers (e.g. count the distance between the two numbers) , it *is* most expedient for that pupil to take away, i.e. count the number of steps equivalent to the smeller number from the larger number.

Episode 1.2: Questioning pupils options

After discussing the difference between the teachers and the pupils defining which method is most expedient for which problem, and about teaching different approaches, one teacher asked rhetorically: "*So what is out? Colour codes are out*" (Pierre).

Still, the idea that pupils *figure out* mathematical connections through pairing problems and methods to solve them, continued to surface in the discussion, and activities similar to that of example one were proposed a number of times.

This reemergence of a rejected idea seems to reflect the teachers' lack of experience with common and explicit anticipation of student activities in an open or inquiry oriented adidactic situation.

The following episode contains the proposal and discussion of an activity similar to that of example one, the main differences being to use plastic pockets, instead of colours. Pupils' task woud be to put notes with given subtraction problems into the plastic pocket representing a given method which the pupil considered to be appropriate to solve the problem. In opposition to the above episode, one teacher responded to the proposal by anticipating how pupils' could respond to the milieu:

Henry	[So]..they get a pile of problems ... we make some kind of plastic pockets, which the pupils are to put [the problems] into.. they get the card which says 10 minus 9. Which of the plastic pockets will I use? Will I use the one that is called filling-up? Will I use the number line? Will I use the one [where the numbers are] above each other?.. And then I think, they must begin to become aware ... which techniques.. So we go through the techniques first and then they get the problems, and then they have to place them according to what they think is the best technique for each problem ... The task is to systematise the group (of problems) … say 12 problems … and then they have to divide them into [the pocket] they think [represent the] easiest [method]. Why did I choose [that or that]? Because then it is definitely.. [about] getting to understand which method is best in a given situation..

Joan	Doesn't that require that you have tested every single method on every single problem then, to figure out which one is most expedient? ... For me, as a child, to be able to answer that, I think I would have to test the [all the] methods.

Joan's comment is the first clear record of $\mathbb{T} \leftrightarrow DS_A$ in the sense of anticipating of students' activity, based on explicitly attending to the feedback potential of the proposed milieu. Her anticipation of pupils' possible responses led her to realise that to determine which method is most expedient for any of the given problems, pupils would have to test each combination of problems and methods. This would require for the pupils to master each method, in which case the activity would function more as a test or training. None of the other teachers opposed her analysis.

Suggestions of the same type were still proposed after this episode, but Joan's anticipation appeared to be a turning point in the sense that from this point on, the teachers (reluctantly) started to examine the target knowledge more systematically: Which methods exist and what are their properties?

Episode 1.3: Considering the milieu's role regarding pupils' acquisition of the target knowledge

One teacher returned to an earlier comment of the facilitator, that when (some) Japanese teachers introduce new concepts, they tend to use relatively simple examples (e.g., small numerical values) in order to reduce the difficulty to only the essential features of the target knowledge.

Pierre	Perhaps they shouldn't use paper and pencil.. If the [problem] is so easy, the challenge will be to formulate.. instead of giving them 443 minus something couldn't it just be 20 minus 2? Or 20 minus 18?..
Robert	..it is their explanation to it..
Pierre	...make it so obvious to them what the answer is.. some of them will not even be able to say why it makes 2 or how they figured it out. And then that is what we dig into. What have you actually done ... a lot of things they can make without paper and pencil.. perhaps that is the way to go. Simplify it a lot for them, but then demand explanation or formulation from them to a higher extent.

At this time, the teaching problem has turned from a didactical situation in which pupils realise which method is most appropriate to solve given subtraction problems to one in which pupils consider difference and various ways to find it by working consciously with their own method(s) and observe others employ different methods for different and same problems. There has, then, been a shift from students *figuring out* the teacher's preferences for methods, applying, examining and explaining their own methods. Then, teachers also anticipated that too high numbers may be an obstacle for pupils' interplay with the real problem: To understand and explain different methods of subtraction.

Episode 1.4: Anticipating pupils' responses to specific problems

In immediate continuation of the episode of example three, the following episode took place. The facilitator asked directly to the teachers' anticipation of pupils responses.

Facilitator	Let's say .. 20 minus 18 .. What would you expect the pupils to do? In which different ways would the pupils use e.g. centicubes? ... if we gave the pupils 20 centicubes and the problem of 20 minus 18, wouldn't most [pupils] .. remove the 18? Would anyone take 18 and then count up to 20?..
Robert	Wouldn't it be interesting if we put 20 centicubes in front of them and say 20 minus 18.. it could be quite interesting to see how many would just take (away) 2 and say that's the result..
Joan	I don't think anyone will..
Robert	You don't think so
Joan	No, I actually don't think so
Robert	Well, I would hope there would be some [who would]. If they look at the problem and they know it is 2. Then you say, show me the result with centicubes. And if you start to take away 18 …

Robert feels rather confident, that if the pupils have 20 centicubes in front of them, they - or at least some pupils - would take two to represent the difference. When Joan questions this, Robert - almost unnoticeable - alters his scenario to include pupils being aware that the difference is two - i.e. to represent the result instead of the problem. To this, Joan alters her anticipation accordingly:

Joan	Yes, if you say it like that, if you say show me the result, but if you say the problem is 20 minus 18…

Here, Joan changes the focus of the discussion, to how the pupils would respond to different milieus (problems), and it seems that Henry is in line with that:

Henry How big is the difference from 18 to 20.. then possibly someone would start at 18 and go upwards.. but again, that's about formulation.

Though the teachers do not elaborate much on their ideas and anticipations, it seems that here they are trying to anticipate how pupils would respond to different (variations of) the milieu. From here on, a longer discussion about different formulations and the size of numbers to be included in the problems succeeded.

Summary of episodes from PrS_1 of LS_1

During PrS_1 of their first lesson study, the teachers displayed both a lack of habit to anticipate didactical situations but also to some extent a progression. It has been exemplified how the teachers readily proposed activities, but primarily focused on aspects regarding the target knowledge and the problem (material milieu), and only in few cases the pupils' potential responses. The aspects of adidacticity and the teacher's management of the situations were not regarded.

Intermezzo: Teachers' experiences and learnings between PrS_1 of LS_1 and LS_3

We now turn to provide a short overview of paradidactical activities related to the lesson studies in between, and in particular some central teacher learnings.

Firstly, the didactical ideas developed in PrS_1 of LS_1 were tested, analysed and revised three times. From this, three particular teacher learnings were identified (cf. also Bahn 2017):

- There are essentially three ways to subtract integers on a number line: Take away (TA) and determine difference by counting up (DD_{up}) and down (DD_{down}) between the minuend and the subtrahend.
- For (these) pupils, the word and symbol of 'minus' indicate a task to be solved by TA; more generally, that small details (e.g. minus vs difference) may have crucial consequences to the didactical situation.

In LS_2, the teachers designed a lesson in which pupils were presented with different two-dimensional figures and were asked to 'order' them. After discussing the possibilities of doing so according to both area and perimeter, the teacher devolved the following task which focuses only on area. The pupils were offered some tools (thread, transparent greaseproof paper, scissors and pencil) to examine the area of the figures in order to confirm or revise their proposed order. As a final activity, pupils were asked to order four three-dimensional shapes by size - a pyramid, a sphere, a cube and a drinking glass. The first teaching problem aimed at pupils learning to distinguish between area and perimeter by becoming confident with one of them - area. The last activity also comprised volume, and was intended to help clarifying the concept of area. Especially in the first research lesson there was some confusion about the area and perimeter and volume, but as the devolution and tasks were clarified during revisions, the last research lesson made it possible for pupils to readily engage more or less adidactically in solving the problems.

The teacher learnings from this lesson study comprise, again, how critical little details of devolution and milieu can be to the adidactic potential of the situation and that their own use of words related to area and perimeter (e.g. *what is inside* and *surrounding*) is often imprecise and confusing. Following experiences from bot LS_1 and LS_2, the teachers also discussed how important it is to reduce the teacher's speaking time and to increase the time pupils have to engage adidactically with carefully prepared problems.

In addition to the two lesson studies (LS_1 and LS_2) which the teachers undertook themselves, the teachers experienced six *mutual visits* (described above). Two of these unfolded as the other teams' visit to the final research lesson and reflection session of LS_1 and LS_2 of this team, while the in other four, the teachers in question visited the final research lesson and reflection session of LS_1 and LS_2 of the the other two teams. Furthermore, between LS_1 and LS_2, the teachers observed an open lesson by a Japanese expert teacher and the reflection meeting that followed it.

These activities in and outside of their own lesson studies also had a great impact on the evolution of their paradidactical knowledge and practice, but will not be further elaborated here.

Lesson study 3

The third lesson study, LS_3, started from the teachers' wish to focus om competencies (Niss and Højgaard 2011) rather than a particular skill. Through discussions, the teachers agreed to focus on the *problem solving competence* (ibid., p. 55), which the teachers imprecisely considered as any situation where there is a problem for the pupils to solve.

Whereas the teaching problem was determined relatively early in PrS_1 of LS_1, it gradually developed throughout PrS_1 in LS_3. Inspired by Pehkonen (1999), the teachers agreed to use problems with polyominos, in which pupils were to find (the) possible polyominos with a given number of perimeter units, *pu*. The teachers did not use (or know) the notion of *polyomino*, but referred to them as *figures*.

For a long time the considered teaching problem aimed at pupils' strategies to discover a system of determining how many polyominos you can make with the perimeter of a given number of units. For example, as illustrated in 9, with four and six pu you can make one polyomino, and with eight pu you can make three.

4 pu 6 pu 8pu

Table 1: Possible polyominos with a given number of perimeter units, *pu*

One of the teachers in the team had for a long time pursued the idea that there must be a definite number of polyominos with a given pu, and hence the problem of finding an expression for that number in terms of the given pu (formally, a functional relationship, but the teachers did not seem to think of it that way). He had also discussed the problem with colleagues including some from the team. It should be noted right away that this remains an open problem even in scholarly mathematics, while a number of special cases are known, cf. e.g. Leroux and Rassart (1999).

The teachers failed to discover an overall system, but it was long believed, that it would be easier to find a system if you look only at how many different polyominos can be made within a given rectangle which shortest side is 2 pu. Figure 5 illustrates the possible polyominos with 12 pu within its 2x4 rectangle. The teachers discovered that under these rules the 2x2 rectangle of 8 pu, 2^1 polyominos are possible while the 10 pu 2x3 rectangle allows for 2^2 polyominos and the 12 pu 2x4 rectangle holds 2^3 polyominos. When finally discovering, that only 14 polyominos ($2^4 - 2$) can be made within the 2x5 rectangle of 14 pu, the teachers considered they could not find a system.

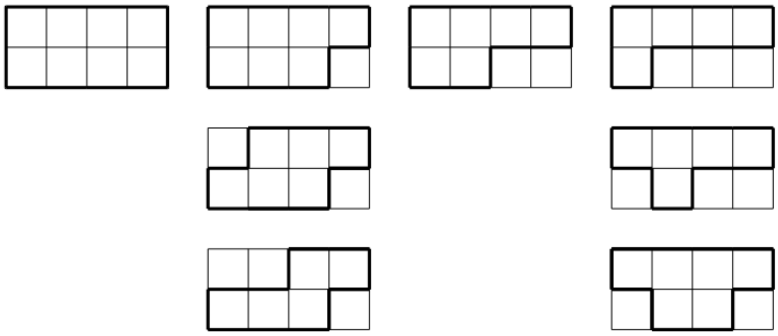

Figure 5: Possible polyominos with 12 matches within a 2x4 rectangle

Until this was realised, quite late in PrS_1, the target knowledge had been for pupils to explore ways to discover the system. From then on, the target knowledge comprised the observation that the connection between the number of pu and of possible polyominos is not linear. Still, it was an aim for the teachers to investigate if pupils would examine the problems systematically and if so, how.

Like in most of the lesson studies conducted in this research, the research lesson was constructed as three consecutive activities. After devolving the rules for polyominos (still referred to as figures), pupils were to be presented with geoboard papers and ten headless matches (as pu). The problem was to find as many figures as possible. When summing up, all figures should be illustrated on the board while a table of the connection between the number of matches and the number of polyominos would be drawn:

Matches	Figures
1	0
2	0
3	0
4	1
5	0
6	1
7	0
8	3
9	0
10	6

Table 10: *

Table 2: Connection between perimeter units (matches) and number of polyominos (figures)

Discussion about the connection and especially about odd numbers were to prepare for the next task: to guess how many polyominos can be made with eleven and twelve matches. The pupils were anticipated to realise that you cannot make polyominos with an odd number of matches, and the teachers anticipated them to guess that with twelve matches between seven and ten figures could be made (the actual number is 11). Hereafter, the pupils were to find as many polyominos as possible, while the teacher would continuously put figures pupils had found on the board. In the end all polyominos should be visible. The teachers had prepared all the 25 possible polyominos with 12 pu, in case some were not found by pupils. After examining if all were found, the corresponding number of matches and polyominos would be added to the table on the board (table 10), and the final question of the lesson coukd be posed: How many figures can be made with 14 matches. Pupils' guesses should be shared, but they would not pursue the problem any further in the lesson. In fact, the teachers did not know how many polyominos can be made with 14 pu (the number is 86), but this was not considered a crucial problem, since it was not part of the target knowledge.

Episode 2.1: Inquiring target knowledge and the material milieu

In contrast to the two former lesson studies, teachers soon began to investigate the problem themselves (how many polyominos can be made with a given number of perimeter units = matches) and the material milieu (if matches fit on the line between intersections of a grid paper). After agreeing to use a problem with polyominos to focus on the problem solving competence, the facilitator started examining which polyominos can be made from different numbers of pu. One teacher actively entered this investigation as soon as he realised what the facilitator was doing, and kept working on it through most of the rest of PrS_1.

Another teacher had brought grid paper, and a third teacher went to fetch some matches. One of the teachers started to examine how well they fit the grid. Soon after all teachers (one was absent) examined and discussed how much the paper would need to be resized in order for the matches to fit. One of the teachers immediately went to resize the grid paper using a xerox machine.

While this episode did not involve direct reference to the DS_A, it reflects the teachers' experiences from the two first lesson studies of how important the material milieu is to realise a desired DS in which the target knowledge can be conveyed through a fruitful student-milieu interplay. Moreover, possibly due to the challenging problem, it is also clear that the teachers were to a larger extent motivated to engage with mathematics themselves, to anticipate what students might be able to do.

Episode 2.2: Teacher's management of the situation

Most of the teachers' discussions of the didactical situation (in all the lesson studies) regarded devolution and milieu. This episode illustrates one of the not so many anticipations regarding feedback to the teacher for managing the situation.

A pivotal didactical idea involved to have all possible polyominos with ten matches presented on the board while at the same time constructing a situation in which less productive pupils could contribute to the total production of the class.

It had been proposed that the pupils should have a grid paper and 12 matches. The task would then be to find as many different polyominos as possible with a perimeter of 12 matches.

Joan	Shouldn't they draw the figures? Because, I think we are not able to get around to see what they [all] do because bang, bang and then (indicating that figures are made quickly after each other, leaving little time to observe one figure before the matches are used to make the next)

Again, a teacher anticipating the corresponding didactical situation ignites a discussion in which ideas are tested and validated. Joan's comment initiated the following discussion of teachers' possible feedback from the student-milieu interplay:

Robert	..they'll get one of these (a large grid paper) to make the figures on [and] they get a notebook..
Joan	o.k... to draw on..
Pierre	..or we print out geoboard paper .. it's the same .... Then there would like 12 geoboards on one sheet of paper
Robert	That depends on.. well, no matter what, it'll be more untidy on a geoboard paper
Joan	Because there are no (indicating lines with her finger)
Pierre	In a notebook? ... [No]
Robert	You don't think so?
Pierre	No

Robert	Well, first of all..
Pierre	Untidiness doesn't matter anyway
Robert	Well, that's the question.. it's kind of.. ..you also have to think that you have to go around and keep an eye on which [figures] they have made
Joan	Yes, that's [right]
Robert	because, if the idea is for them to draw the figure before they [remove the matches to make a new figure], then the [note] paper is what you have to keep an eye on.

This discussion illustrates how teachers consider little details that might be crucial for the teacher to get the feedback necessary to manage the situations of formulation and validation. Especially Joan and Robert appear to have realised that the feedback in this particular stage of the lesson is crucial, and the they all agreed manage the situations of formulation and validation the didactical idea to make use of grid paper as a part of the materials included in the lesson plan. Hence, this episode regarded not so much the target knowledge or adidacticity, but rather how the material milieu could support the teachers' (possible) management of the situation.

Episode 2.3: A precise milieu

At one point, a pivotal idea was to lead pupils to examine the possible figures in a systematic way by 'forcing' them to consider first only the possible rectangles. For instance, with 12 matches you can make one rectangle that has the height of one and is 5 matches long. If you make a rectangle which is 2 matches high, it will be 4 matches long, hence referred to as a 2x4 rectangle. With twelve matches polyominos can also be made within a 3x3 rectangle, which would be a following task. The idea was to make pupils consider the possible figures within a given rectangle, hoping that they would develop a systematic approach, e.g. as illustrated in figure 5 on page .

Until now, the teachers had been working with grid paper of 5x7 squares on an A3 sheet. In continuation of a discussion about how to devolve the sub-problem of finding polyominos within a rectangle - or how to devolve the rules of it - one teachers holds the paper in front of himself and wonders:

Robert	So, I am thinking, how can we set the task in which [the figure] may only have the height of 2? We could do that by cutting [the grid paper] to be only 2 high ... You know, if the idea is that they should start by finding.. rectangles. That would work, asking them to find a rectangle with 1, 2 etc. matches. After that, I just thought it's really difficult to phrase [that task] ... Because that is not a rectangle (pointing at a non-rectangular figure similar to one of those in 5). And afterwards, they get one [paper with the height of 3]. How many can you make now?

Though the sub-problem of finding polyominos within a given rectangle was later dismissed, the other teachers went along with this idea immediately. This episode illustrates the increasing attention of the teachers to how the material milieu can support pupils to understand the problem and to solve it autonomously, while the previous episode show the teachers' attention to how it may enhance the teacher's monitoring of the adidactic situation. Hence, while not explicitly, adidacticity is in play here. In a later episode, the teachers decided that the activity might not be productive towards the overall target knowledge.

Episode 2.4: Teachers' knowledge vs the milieu (problem)

Throughout PrS_1 of LS_3, the teachers have been searching for a system of determining all possible polyominos with a given number of pu whivh they consider the target knowledge. It is late in PrS_1, one teacher considers their efforts in relation to the - so far - intended target knowledge:

Pierre	Now, we [keep] saying there is a pattern, but we are struggling to find it. And if we can't find it, can we demand that the children can see some sort of pattern in it? Isn't it just arbitrary numbers [to them]?

The teachers soon realised, that they could not find a system, and in fact they became convinced that there isn't one. While this reflection came late, it is the first time during the preparation of three lesson studies, that the teachers were explicitly concerned about their lack of knowledge as a real problem. Once again, it is when anticipating how the pupils might receive the problem that the teachers realise the flaw and its significance.

Summary of episodes from PrS_1 of LS_3

While there are still some signs of teachers' reluctance or lack of capacity to anticipate didactical situations and what consequence given choices may have on the didactical situation, PrS_1 of LS_3 also illustrates a specific evolution. As soon as the teachers had agreed on a teaching problem, they started a thorough investigation of the target knowledge, related materials and didactical ideas of how to teach. In fact, the teachers' continuous examination of polyominos and a possible system to predict the number of polyominos that can be made with a given number of pu, led them to realise that the initial target knowledge was unclear to them, and they changed the teaching problem accordingly. This is in opposition to LS_1, in which the teachers readily proposed activities and materials, but without examining their possible advantages and disadvantages for the student-milieu interplay and hence for the pupils' learning. In LS_3 we also see direct anticipations of the teachers' possibilities to obtain feedback for the management of the situation, and there are indirect anticipations regarding the pupils' possible adidactic interplay with the milieu.

Discussion

As is evident from the above episodes, it is not always clear if what we observe in predidactic situations is an expression of reasoned anticipation or if it is just a (wild) guess. As long as teachers' do not express clearly if they are in fact anticipating, we can not know for sure. On the other hand, even if a teacher would express clearly that he or she is anticipating a didactical situation, we cannot know for sure whether he really aims to do so, or it is an effect of the *paradidactical contract*, i.e. the mostly unspoken rules of the paradidactical situation (see e.g. Brousseau 1997). The notion of paradidactical contract is an extrapolation of Brousseau (ibid.)'s notion of *didactical contract*, which regards the rules of the didactical situation. As Brousseau (ibid.) points out, the contract has some crucial effects on the situations (see e.g. Brousseau 1997; Måsøval 2011, pp. 37-46).

An example of an effect of the paradidactical contract is the teachers' rapid proposals of activities lacking a connection with a learning potential in the first lesson study: the contract states that a lesson plan involving activities related to a teaching problem must be designed, and the teachers do their best to fulfil the contract. The notion of paradidactical contract can indeed shed more light on the above episodes. For instance, some of the examples are recorded not because of an explicit anticipation but because teachers' change in paradidactical practice implicitly indicate an anticipation. The change in practice is connected to a change in contract which may be based on specific experiences and learnings, or on other external factors, e.g. the facilitator, observation of colleagues' paradidactical practice or the like.

It seems apparent that anticipation is not only an *aim* in lesson studies, it is a *means* as well. While not always explicit, the representative examples presented above indicate that before crucial examinations took place, one or more teachers anticipated a didactical situation corresponding to a proposed milieu or desired student-milieu interplay. This is similar to a *powerful self-perpetuating connection between studying teaching-learning relations through inquiry based investigations and managing pupils' such relations through inquiry based teaching*. If this inquiry can be maintained over longer periods, as in the process set up for this project, one could indeed expect that teachers improve their capacity to anticipate didactical situations.

The analysis presented above only regards teachers' practice in paradidactical situations, but the main purpose of lesson studies lies in improving teachers' practise in didactical situations, and one pressing question occurs: Which effect does evolution in paradidactical practice have on their didactical practice both in the research lesson and in their "normal" teaching? What we have seen here has not reflected the possible changes in how the teachers manage the didactical situation, but it seems reasonable to hypothesise, that a changed capacity for anticipating student-milieu interplay beforehand, with practice may entail a changed capacity for anticipating such interplays immediately before they take place, thus enhancing the teachers management of the situation.

Conclusion

In this paper, examples of teachers' paradidactical practice are used to analyse the evolution of their anticipation of didactical situations with a focus on five aspects regarding *the target knowledge*, *the material milieu*, *pupils' response*, *adidacticity* or *the teacher's management of the student-milieu interplay*.

The examples illustrate both a quantitative and a qualitative shift in how the teachers anticipate the didactical situation. Roughly put, in the beginning the teachers did not seem to anticipate a didactical situation which could correspond to their suggested activities. Later, they reluctantly began to anticipate how pupils might respond to a given activity. Later again, the anticipation of pupils responses became more present, and in the latter part of the third lesson study, teachers seemed to be well aware that little details in the milieu could have crucial consequences to the didactical situation.

The mathematical subtleties of the target knowledge and the material milieu were more or less present at all times, but their roles changed under way. In the first lesson study, the teachers did not seem to recognise their lack of knowledge about the target knowledge as a crucial problem. In fact they did not start to examine it before one teacher anticipated a didactical situation, claiming that the proposed activity might obstruct pupils' learning. In the third lesson study, teachers started examining the target knowledge, as soon as a teaching problem was agreed upon. A similar evolution is evident regarding the milieu. Early on, milieus - in the form of activities - were readily proposed but their corresponding effect on the didactical situation was not analysed. Later on, milieus were not only examined but they were to some extent the outcome of considerations of a desired student-milieu interplay. In the last lesson study, milieus and the teaching problem - including the target knowledge - were developed much more interactively.

Anticipation of pupils' responses play a vital role in anticipating didactical situations. Still, the explicit anticipation of specific pupil responses were scarce, especially in the first lesson study. Even in the latter lesson study, teachers did not explicitly convey anticipations of pupils' responses much but the character of their reflections and practice indicate a higher degree of awareness about possible student-milieu interplays.

Throughout the lesson studies, the teachers make very little direct reference to the adidacticity in the sense of pupils' autonomous interaction with the milieu. While we have detected a qualitative and quantitative increase in the teachers' anticipation of pupils' response to a milieu, this mainly regards the pupils' first encounter with it, i.e. in situations of action and (much less) formulation. Because of a lack of attention to the potentials of adidactic validation situations, the teacher continued to play a vital role in validation.

Though the teachers did talk about the observation of pupils' interplay with their milieu as a source of data to manage the situation, several times through the lesson studies, this was only explicitly discussed very few times. The example presented in this paper is the only instance, where the role of feedback from the student-milieu interplay was explicitly anticipated and had direct impact on a didactical idea, and even had significant effects on the lesson plan.

While these results may not indicate a shift in the teachers *didactical practice* they do illustrate remarkable shifts in their paradidactical practices as conducted in these lesson studies. Whether that shift has lasting effects or not is out of the scope of this paper. It is also out of the scope of this paper to say anything about what led to this evolution. Both of these questions are important and deserve further scrutiny. One approach could be to examine how the infrastructures of lesson studies impact on teachers learning and change in paradidactical practice, e.g. by analysing the interdependency between the infrastructures and the *paradidactical contract*.

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④ Theorizing Lesson Study: Two related frameworks and two Danish case-studies

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Mathematics Lesson Study Around the World: Theoretical and methodological issues

Springer book series of ICME

Chapter 7
Theorizing Lesson Study: Two Related Frameworks and two Danish Case Studies

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Abstract Lesson study refers to certain well-established professional development practices for teachers in Japan. Over the past 30 years, the phenomenon drew the attention of scholars in other countries, and their writings have inspired several 'movements' of lesson study implementation. As scholars observe both successes and difficulties in these endeavors, the need arises for finer methods to characterize and monitor the processes and objects which go into what is broadly referred to as lesson study. This brief presents an overall characterization of lesson study in terms of the notion of paradidactic infrastructure, in relation to specific adaptations of two related theoretical frameworks. We argue that the use of these frameworks can help sharpen researchers' understanding of lesson study as a phenomenon. We exemplify the use of these tools with cases from our own work on pre- and in-service teacher development in Denmark.

Keywords Lesson Study; Theory of Didactic Situations; Anthropological Theory of the Didactic

7.1. Why and how should researchers theorize lesson study?

In this section, we argue for the necessity of increased attention to theoretical precision in research related to 'lesson study', including more precise identifications of what practices the term itself refers to, and not least to describe and analyze essential parts of what happens in a lesson study. We also present two theoretical perspectives which we have found useful in this respect.

7.1.1. Why theorize lesson study?

'Lesson study' was introduced into the Anglophone educational literature in the second half of the 1990s as a translation of the Japanese notion *jugyou kenkyuu*. Stigler and Hiebert's (1999) bestseller *The Teaching Gap* hugely increased the impact of an idea which had otherwise mainly appeared in scientific papers just a few years before. Among these, Lewis & Tsuchida (1997) do not use the term 'lesson study' (corresponding to the Japanese *jugyou kenkyuu*) but instead report on the phenomenon of 'research lesson', a translation of the Japanese term *kenkyuu jugyou*. They underline and exemplify that "research lessons take several distinct forms" and subsequent scholarly work – beginning with the volume by Fernandez and Yoshida (2004) – have presented case studies of some of these. Lewis (2016) further discusses various types of lesson study in Japan.

Soon after the publication of *The Teaching Gap*, efforts to introduce lesson study in the United States and other (mainly) English speaking countries began to emerge, and with these efforts more researchers (outside of Japan) became involved in the development and research associated to lesson study. Widely used handbooks on 'how to do lesson study', such as Stepanek, Appel, Leong, Mangan & Mitchell (2007), present detailed instructions for the different 'phases' of lesson study, usually as elements of a four or five step cyclic process of preparatory study and planning, observing, discussing and revising a lesson. Such publications evidently helped support the increasing number of lesson study groups in many countries, and it also made lesson study accessible as a phenomenon and experimental device to a much larger number of researchers. At the same time, the efficient marketing and dissemination of a phenomenon as foreign as *jugyou kenkyuu*, together with the further adaptations which are often made by novice practitioners, inevitably lead to a certain distance between the resulting practices and the Japanese original. At the same time, lesson study was certainly not promoted as something which could take several distinct forms, but rather as a definite type of teacher-led activity which should follow the steps indicated and is explained in the guidelines.

It did not take more than a few years of American experiments with lesson study before Lewis (2004, p. 132) noted that "many emerging examples of lesson study in the US diverge substantially from lesson study as it is practiced by Japanese" and pointed out that:

> The graveyards of U.S. educational reform are littered with once-promising innovations that were poorly understood, superficially implemented, and consequently pronounced ineffective. If lesson study is to be any different, it will require a deep understanding of what it is and why it has been useful to Japanese teachers, and how it can be adapted to the very different setting of the US (Lewis, 2004, p. 134).

For researchers and lesson study promoters alike, this evidently raises the questions of what this deep understanding would consist of, what it would be an understanding of, and who should have it. Given the claim of a manifest distance between lesson study in the US and in Japan, it seems natural to interpret the quote in such a way that the deep understanding would concern lesson study as it is practiced in Japan; however, most teachers and researchers abroad could only read about this in the sources already outlined, not observe and participate in the original (model) activity themselves. Indeed Lewis (2004, p. 134) considered that "the likelihood of success [*of lesson study*] would be increased by the participation of Japanese educators who could help to figure out the essential qualities of lesson study". There is no doubt that the involvement of US based Japanese born experts has had an impact on parts of the implementation of lesson study in that country (Fernandez 2003), and possibly also elsewhere. But in general, promoters of lesson study are not likely to be satisfied with this solution, though in most countries outside of Japan lesson study would remain a very marginal activity if it required the frequent or even occasional participation of Japanese experts; and even if available, the needs for adaptations and explicitness of what lesson study is or could be would remain.

Additionally from a research point of view, there is something fundamentally unsatisfactory about knowledge remaining simply individual 'understanding' linked to people and their personal experience (or even national background), rather than being explicitly and accessibly described. It also seems to be a bit paradoxical, given that one of the often cited features of lesson study is to create shared and documented knowledge, rather than (just) private experience and wisdom. It has even been claimed that:

> Lesson study also provides mechanisms for teachers to move squarely into Popper's World 3 -developing knowledge that is intended for public discussion and examination (Hiebert, Gallimore & Stigler, 2002, p. 10).

If teachers' 'understanding' of teaching should be amenable to move from individual skills and beliefs to public and accumulating knowledge, and lesson study supplies important avenues to facilitate that move, it seems hard to accept that these 'mechanisms' could not also be subject to public and transparent analysis and description.

The various activities called lesson study (or, depending on the country, *jugyou kenkyuu*, or *lektionsstudier*, etc. …) are clearly phenomena which are contingent upon cultural and institutional conventions. There is obviously no straightforward 'model' which applies to them, but on the other hand, without explicit models we are left with practices that are at best understood by the participants themselves, but remain more or less opaque to outsiders. Simplistic models, such as the cycle or step-by-step descriptions, clearly miss many essential aspects of lesson study including its purposes in terms of promoting teacher and student learning at various stages of the study. More importantly, a theoretical framework (with explicitly defined categories and terms) is needed to move the analysis of mechanisms and principles of lesson study away from the culturally contingent narratives about lesson study, with which the literature abounds. Also, research on lesson study needs to strive towards a more international and explicit stance (or, if you want, "move squarely into Popper's world 3"). In scientific terms, this requires more precise models of what lesson study is and is about – based on theoretical frameworks which are shared and developed by researchers. As in all modeling, the point is not to achieve complete or non-reductive descriptions and analyses, but to make assumptions and *foci* explicit and open to scrutiny. Thus, we need to get beyond 'lesson study' as if it is one well-defined package of activity, and as if we (or at least someone) understand what it consists of without having to describe it further than the few overall stages it involves.

In the sequel, we refer to lesson study as the (broad class of) teacher-led activities, which are referred to by this name in the Anglophone literature, or by practitioners themselves. Certainly this literature contains a number of 'local' (*ad hoc*) models of what lesson study is and how it works (e.g. Lewis, 2016). But we consider it an important and largely outstanding task for researchers to capture its elements, boundaries, typology and prerequisites within well-established theoretical frameworks for research in mathematics education, which could also help relate the phenomena to other forms of teacher professional development. This task should not be confused with the (equally important) efforts to make use of general theoretical frameworks, such as variation theory, as a tool *within* lesson study (e.g. Huang, Gong & Hang, 2016).

7.1.2. Types of research related to lesson study

A quick search on the database MathEduc (https://www.zentralblatt-math.org/matheduc/) shows that in the mathematics education journals referenced by that database, about 165 papers (March 2017) have been published with 'lesson study' as part of the title (while broader searches suggest a total number close to 300). Many of the most recent papers on the topic are from a special issue of the German journal ZDM (vol. 48 no. 4). Browsing through these papers, we can identify three types of research:

1. Papers describing and analyzing what lesson study is in Japan;
2. Papers describing and analyzing what lesson study is or could be in other countries;
3. Papers reporting on experimental research using lesson study as a method to investigate specific questions related to mathematics education.

Some papers can be hard to classify, but it is clear that the second type is by far the most common. The first appears mainly before 2010, while English language papers of the last type are rare and appear only very recently (this could of course be due to 'lesson study' not appearing in the title of papers where lesson study is just the method, not the object of research). Notice that teachers' research (*modulo* all the debates there could be about what constitutes 'research' in the academic sense) would fall into the third point as above, and is more commonly published in Japan than in other countries. Looking at the literature more broadly, we propose a rough classification of the research field as shown in Table 7.1.

Table 7.1 Four kinds of research related to lesson study, with sample papers reporting on it

Examples of work	OBJECT: Lesson study in Japan	OBJECT: Lesson study (and the like) outside Japan
Descriptive research: lesson study as a research object	Lewis & Tsuchida (1997) Fernandez & Yoshida (2004) Lewis (2016)	Lewis (2004) Rasmussen (2016a)
Intervention research: using lesson study as a method	(Mostly in Japanese; some translated papers were presented in JSME, 2000)	Bradshaw & Hazell (2017)

In the descriptive research strand, the researchers are not necessarily initiators or participants of the lesson study activity. Such studies are basically about what lesson study *is* and *how it happens* – for instance, what lesson study consists of, how it works in particular contexts, and what it takes to implement (the latter, of course, mostly in studies focusing on non-Japanese contexts). From a scientific viewpoint this clearly leads to a need for developing precise *models* of lesson study; this necessitates a theoretical framework to answer questions not only about lesson study as an activity, but also about the *objects* of this activity. Intervention research using lesson study as method may draw on such a model in order to explain and justify the methods used, but will otherwise focus on theoretical models related to the didactical research questions investigated (e.g. the study by Bradshaw & Hazell (2017) of students' different approaches to solving mathematical problems). Nevertheless, in contexts where lesson study is not a well-known working method for teachers, some kind of theoretical description will be needed in this case too.

7.2. Two theoretical frameworks for research on lesson study

We now describe how two related theoretical frameworks (both originating in French didactics of mathematics) can be used to model lesson study: the theory of didactic situations, which allows researchers to analyze the fine mechanics of a lesson and (in our adaptation) also those of the teachers' learning from studying it; and the anthropological theory of the didactic, which proposes a more global viewpoint on the institutional conditions and constraints which affect the realization of lesson study as a 'paradidactic' activity.

7.2.1. A global theoretical perspective: the Anthropological Theory of the Didactic (ATD)

An ATD approach to lesson study takes as a point of departure two key notions of the theory, originally formulated by Chevallard (1999) and outlined as far as needed for this chapter below:

- *praxeologies* used to model human practice and knowledge
- *institutions* as ecologies (habitats) of praxeologies, formally structures of positions which human may occupy relatively to the exercise and development of praxeologies.

Here, a praxeology consists of *praxis* (the types of tasks, needs or questions which the practice aims to deal with, along with the techniques or actions used to do so), and of *logos* (the discourse about the practice and similar practices – roughly, 'theory'). The most relevant categories for mathematics education are *mathematical praxeologies* (where the tasks are 'mathematical') and *didactic praxeologies* (where the tasks are about teaching specific mathematical praxeologies). Focusing on the latter, the *institutions* in which didactical praxeologies are carried out are basically what we call schools (with the well-known variety), and it is also clear how different positions – most notably *teacher* and *pupil* – exist in these institutions, relative to didactic praxeologies, and are fundamental to their workings.

Lesson study is not simply didactic practice, but it is a *study* of didactic praxeologies – including a specific didactic *praxis*, the lesson which the participants plan, observe and reflect on. But these three fundamental *tasks* – planning the research lesson, observing it, and reflecting on it, are *not* didactic practice – they are *about it*. Similarly, any teacher can engage in praxeologies which are very closely related to specific teaching tasks (and more broadly didactic praxeologies), but which are not themselves didactic (the didactic ones being teaching tasks, techniques etc.); for instance, planning lessons, evaluating students' results, etc. We call these praxeologies of the teacher *paradidactic praxeologies*. Following Miyakawa and Winsløw (2013), we distinguish the different paradidactic praxeologies which teachers may engage in and develop in relation to a specific didactic practice, according to whether they occur *before* or *after* the didactic practice, or *at the same time*. These are called, respectively: *predidactic praxeologies*, *observational praxeologies* and *postdidactic praxeologies*. Notice that these notions are very general and do not only concern lesson study or mathematics teaching.

In lesson study, the predidactic praxeologies sometimes involve a group of teachers, and possibly an adviser. It is characteristic for lesson study that predidactic praxeologies involve the study of a number of materials, formulating precise didactic tasks, techniques and hypotheses about how they will affect the students' development of mathematical praxeologies in the lesson. It is also characteristic of lesson study that observation praxeologies involve other teachers rather than the person who is engaged as teacher in the didactic praxis itself, and that these teachers participate in a specific postdidactic praxis where observations are shared in order to build the participants' didactic knowledge about the didactic tasks which the lesson has tried to solve, as well as more broad forms of didactic knowledge. All of these paradidactic praxeologies involve different positions and have tasks which are *relative* to the didactic praxis observed in the lesson (along with students' and teachers' mathematical praxeologies). Therefore, we talk of a paradidactic *system* (participants and praxeologies considered together) and of a pre- and post didactic system, when needed (Winsløw, 2011). The institutionally given conditions and constraints for the function of paradidactic systems are jointly called the *paradidactic infrastructure*. Therefore, lesson study furnishes a specific kind of paradidactic infrastructure, which itself may depend on other conditions and constraints on teacher's paradidactic praxeologies (e.g. time resources for their work to prepare and evaluate teaching, material conditions for teachers' work at the school, etc.).

The overall goal of lesson study as such is to improve the participants' knowledge about didactic praxis (both in terms of explicit knowledge about techniques, and in terms of didactical theory). A central focus is a specific set of didactic tasks, which in turn are more or less specific to a mathematical praxeology to be taught; this mathematical praxeology is indeed a very central part of what the participants work with in lesson study (both in the preparation and reflection parts), and what they learn about (in relation to the didactic praxis). At the same time, broader *theoretical* didactic knowledge may be drawn on and developed from this concrete basis (see e.g. Miyakawa & Winsløw, 2013).

7.2.2. The dynamics of the lesson studied, and of lesson study: the Theory of Didactic Situations

The Theory of Didactic Situations in Mathematics (TDS), whose foundations were established by Brousseau (1997), models teachers' knowledge about mathematics and its teaching in terms of *didactic situations* (DS), which is a dynamic triad unfolding in time consisting of an interaction between teachers, students and a didactic *milieu* (see Fig. 7.1). Here, the didactic milieu typically contains a problem and the resources related to solving it; the students interact with the milieu and their attempts to adapt their knowledge to the milieu (solving the problem using the resources it offers) is the main mechanism of learning in situations. This interaction is initiated and, in various ways, regulated by the teacher.

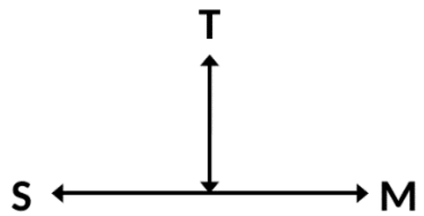

Fig. 7.1 Didactic situations as interplay between a teacher (T), students (S) and milieu (M)

In order for the students to be able to interplay and accept to interplay with the milieu, this must not only be appropriate in itself but the teacher has to *devolve* it – hand it over – to the students appropriately. The teacher may also institutionalize the knowledge that the students' have obtained through this interaction.

In lesson study, teachers plan, observe and reflect on a didactic situation. Lesson study may be viewed as a sequence of *paradidactic situations* where the teachers interact with the didactic situation as a kind of milieu. Here they formulate, experiment and validate certain hypotheses about students' learning in the didactic situation (Fig. 7.2). *The paradidactic system* thus consists of the teachers participating in the lesson study, together with the various stages of the didactic situation itself. Considering the dynamic interaction of the system in the phases considered above, we may talk of *predidactic situations* (PrS), *observational situations* (ObS) and *postdidactic situations* (PoS).

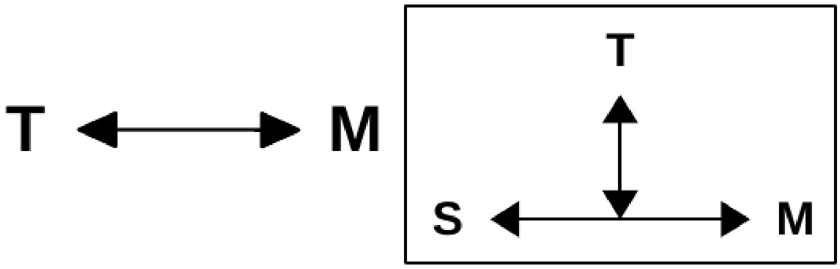

Fig. 7.2 The learning situation of lesson studies: The interplay between the teachers and their milieu, the didactical situation

During our observation and analysis of nine instances of lesson study, the term *didactical idea* emerged. A didactical idea is the hypothesis that the devolution (i.e. the 'handing over') of a certain milieu with further specified regulations will lead the students to a certain point (of knowledge, awareness etc.) through corresponding didactic situations. Didactical ideas may be(come) more or less explicit.

Analysis of realized *didactic situations* provides us with insight into students' actual interplay with the milieu and the ways in which it supports their learning. This is useful not only to a researcher but also to the teachers, and results from a feedback from the milieu of the teachers (i.e. the *didactical situation*). Furthermore, the teachers' explicit analysis of the didactical situation, in pre- and postdidactic situations, gives researchers insight into the development of teachers' knowledge as it develops from interaction with their milieu. This means that the researcher can investigate three levels of knowledge: the students' mathematical knowledge, the teachers' didactic knowledge, and the teachers' paradidactic knowledge (e.g. on how to generate knowledge through lesson study, how and what to hypothesize, and what to observe and analyze). Following the teachers' work with lesson study over time (multi-cycled lesson study and/or several lesson studies) allows us to analyze the evolution of such knowledge. This is particularly interesting when teachers in PrS are anticipating situations, which is one of the core ideas of lesson study: "Anticipating student responses … is one of the crucial processes while planning a lesson" (Takahashi, 2011, p. 79).

The collaborative approach in lesson study 'forces' teachers to express, discuss and operationalize their didactical knowledge when discussing objectives and proposals for the research lesson. The teachers put their didactical knowledge to the test when proposing milieus and anticipating situations in PrS, and when subsequently analyzing their observations in PoS. As demonstrated by Miyakawa & Winsløw (2009), TDS provides a strong tool to compare situations. With a special focus on the properties and roles of the milieus proposed by the teachers, we can then qualitatively identify the teacher knowledge that is put into these milieus; how it develops, and what drives this development.

When we talk about didactical situations, we assume a teacher is present, with the intention to teach specific target knowledge. In paradidactic situations such as those involved in lesson study where the teachers are the learners, there is no teacher with the intention to teach a specific knowledge. In this case we can use TDS to analyze how teachers adopt their didactic knowledge to the milieu (the didactical situation, cf. Fig. 7.2). This model allows us to observe the relationships between the teachers' actions on the milieu, the didactic situation, and the milieu's feedback to the teachers. It is especially interesting to study the relationships (and possible gaps) between:

i. PrS and PoS (reflecting teachers' didactical learning)
ii. PrS and DS (reflecting potential and realized didactic situation)
iii. PrS and PrS' (reflecting the planning of different versions of a lesson)

Notice that (i) and (iii) are particularly interesting for the study of how teachers' didactic knowledge evolves, and hence the effects of lesson study. When using lesson study as a research method rather than as an object of study, (ii) is the essential mechanism offered by a TDS-based analysis.

7.3. Case 1: lesson study with experienced Danish teachers

This case study is based on the situational approach offered by TDS. Outside of Japan, the consecutive situations of research lesson, post-lesson reflection and revision of the lesson plan are often iterated, leading to 'cycles' of paradidactic situations. This was also the case in the intervention which we consider in this section. In a cycle following a previous one, the learning resulting from the PoS of the previous cycle feeds into the PrS of a following cycle. The model shown in Fig. 7.3 illustrates a lesson study with three cycles.

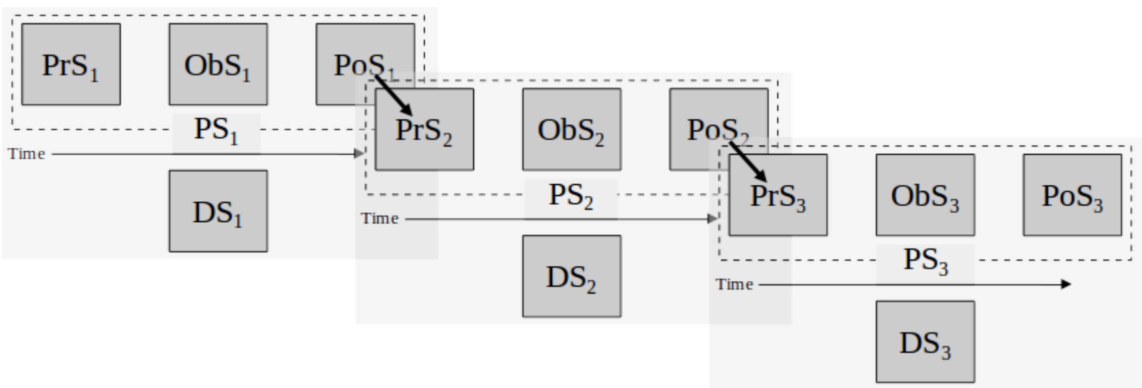

Fig. 7.3 A lesson study with three cycles. Here DS, PrS, ObS and PoS refer to didactic, predidactic, observational, and postdidactic situations, indexed according to the cycle

We now analyze a segment of observations and results from one lesson study (Bahn, 2017), which was part of a research project where a team of three or four teachers at three different Danish schools conducted three lesson studies within one school year. That is, each team of teachers conducted three lesson studies each, with a total of nine research lessons. The lesson studies were structured as illustrated in Fig. 7.3, i.e. with three cycles comprising research lessons in three different classes. PrS_1 consisted of a start-up meeting of two hours and lesson (plan) preparation for five hours, a total of seven hours over two different days. PoS_1/PrS_2, PoS_2/PrS_3 and PoS_3 lasted approximately one hour each. The participants all had ten or more years of experience as teachers. During lesson study the facilitator (a researcher) was present at all times and guided the teachers using questions and suggestions to notice and analyze didactical mechanisms so that the teachers would be able to construct didactical ideas. The analyses are based on observational notes and video recordings.

The lesson study addressed here was the first of the three, and for the illustrative purpose only elements of the two first cycles are included. The lesson study showed that from the teachers' experience their grade three students tended to choose inappropriate strategies of subtraction. The research question therefore addressed how students can be expressly taught the different strategies and the properties of each, i.e. their advantages and disadvantages. The mathematical topic addressed in the lesson study in question was that of subtraction by counting, using a number line. Van den Heuvel-Panhuizen & Treffers (2009, p. 108) outline two models of subtraction, *taking away* (TA) and *determining the difference* (DD). TA always implies the use of *counting down* strategy, i.e. starting at the minuend (in this case being the larger number). DD may be performed by either *counting up* from the subtrahend to the minuend or *counting down* from the minuend to the subtrahend.

There are essentially three ways of performing subtraction when counting on a number line. In Table 7.2, 'TA' refers to a problem (as recognized by the student) of the TA model solved by counting down strategy, 'DD_{up}' a problem of the DD model solved by counting up strategy, and 'DD_{down}' a problem of the DD model solved by counting down strategy.

Table 7.2 Paradidactic situations in two lesson study cycles, as viewed by teachers and researchers

Paradidactic situation	Teacher perspective	Researcher perspective
PrS_1	The discussion of proposals for the lesson design leads to new knowledge of three	Without knowing which strategies for subtraction that ex-

	ways of subtraction by counting on a number line (epistemological condition). The *a priori* analysis also convinced the teachers that each of the strategies would appear in the lesson, since each strategy would be known or found by at least some students (cognitive condition). Furthermore, the analysis led the teachers to think that rather than hearing the 'truth' from the teacher, that seeing actual application by and hearing explications from peer students in a controlled setting (e.g. not too large numbers) would hold a larger potential for students to actually watch, listen and understand (didactic assumption). Didactical Idea: If various students present and explain their strategies, all students will then know the existence and properties of all three strategies.	ist, teachers first proposed a milieu in which problems of subtraction were written on colored cards, the color representing which of the (unknown) strategies is 'easiest' to use when solving the problem. In addition to discovering the three strategies and their properties, the a priori didactical analysis and discussions led the teachers to realize that the proposed milieu may cause students to try to guess what the teacher expects, rather than further their mathematical thinking (since the 'easiest' to use strategy depends, among other things, on prior knowledge). These discoveries (and other reflections) led the teachers to propose milieus in which students were to consider and explain their strategy when solving a problem of subtraction, and for students to show their strategy on a number line on the black board (for everybody to see), while explaining it.
ObS_1	When students explained their strategies and when they presented it on a number line on the black board, only TA came into play.	
PoS_1	From the *a posteriori* didactical analysis, teachers suggest that the teacher uttering 'minus' and problems presented as '7-5', for example, leads students to think only of TA.	A crucial feedback from the ObS_1 was that no student mentioned DD_{up} or DD_{down}. Previously, the teachers were convinced that all would come into play and they immediately started to discuss aspects in the milieus that could be possible reasons for this. From analyzing the observations it was suggested that 1) due to verbalizing 'minus' when devolving the milieu, students were led to think that the tasks were of

		TA type since this represents the 'standard' algorithm of subtraction, and 2) the way in which problems were written on the blackboard only represents TA and therefore reminds students of this strategy, leading them to think that the problems should be solved as such.
PrS_2	The above leads teachers to suggest not using the word 'minus' but instead to only talk about 'difference'. On the blackboard, problems should be presented, for example, as 'find the difference between 5 and 7'.	The teachers suggested altered milieus, where students were to consider 'finding the difference' instead of 'minus', and where problems on the blackboard were also formulated as finding the difference between two numbers, even changing the order of the minuend and subtrahend for some of them to further dissociate the problem from 'prescribing an operation'.
ObS_2	All three strategies were presented, illustrated and explained.	
PoS_2	The analysis of DoS_2 strengthen the teachers' hypothesis that verbalizing 'minus' and presenting problems of subtraction as 'minuend-subtrahend' (e.g. '7-5') lead students to solve the problem as they read or hear it.	Regarding the verbal and written formulations representing subtraction problems the teachers formed the hypothesis, based on the a *posteriori* analysis of ObS_2, that particular words and formulations lead the pupils to understand the problem differently from what was intended by the teachers.

In summary, the infrastructure of lesson study made it possible for teachers to realize two key points:

1. First, they discovered crucial correlations between *how students respond to small details in the milieu and the form of the devolution*, as observed in ObS_1 and analyzed in PoS_1. In PrS_2 the milieu and devolution were adjusted in accordance with the analysis in PoS_1, and in ObS_2 the desired student activity was observed. This led to a conclusion – a strengthened hypothesis – about the correlations in PoS_2.

2. With the aim of proposing a lesson plan for DS, the didactical analysis of PrS_1 led teachers to discover not only the mathematical knowledge needed to design the lesson, but also their hypothesizing didactical situations made them reject one inappropriate milieu and develop one potentially more appropriate. In ObS_1 the teachers observed unwanted and counterproductive student activity, which in PoS_2 they analyzed to be related to details in milieu and devolution. Aiming at a new DS, these details were altered in PrS_2 in accordance with the analysis of PoS_1. In ObS_2 students responded as hypothesized in PoS_1, and in PoS_2 the teachers consolidated their new hypothesis.

It is interesting to observe how the analysis of dynamics within a paradidactic situation, between didactical situations, and between DS and paradidactic situations, can generate insight into how teachers dynamically develop didactical knowledge

. This provides us with tools to examine the role of each paradidactic situation. For instance, there are different opinions as to whether lesson studies should comprise more than one cycle (e.g. Huang & Shimizu, 2016). In this respect, the analysis in question informs that discussion as it appears – at least with lesson study novices – that working with more than one cycle may be beneficial.

As has been illustrated above, the model of paradidactic situations supports the analysis of the details and interdependence of the various phases of lesson study. The notions of TDS provide tools to analyze how the details of one situation are conditioned by and determine other situations. TDS also allows us to analyze hypothetical or potential situations; looking at lesson study in terms of paradidactic situations enables us to analyze the evolution of teachers' knowledge in interaction with the didactic situation.

7.4. Case 2: lesson study in Danish teacher education

This section is about the endeavor to use lesson study in Danish pre-service teacher education. This case study is based on the institutional approach offered by ATD. As explained in the theory section above, we view lesson study as a paradidactic infrastructure, and as such it has similarities and notable dissimilarities to existing paradidactic infrastructures in the Danish institutional context. A number of conditions work against an easy incorporation of lesson study in Denmark.

When lesson study is to be conducted within the auspices of pre-service education in Denmark, it can take place in connection to either 1) practicum or 2) coursework.

Regarding 1), practicum takes place at schools that have a contract with one of the University Colleges (UCs) where the pre-service teachers do their coursework. The institutional documents and agreements between schools and the UCs do not hold any (official) information or requirement about lesson study as part of practicum. Furthermore, practicum is defined at the UCs as a discipline in its own right with its own curriculum, textbooks etc., so it is not subsumed under any subject matter course. When pre-service teachers are in practicum, they are guided by a regular teacher employed at the school, who – except for very rare cases – has no knowledge of lesson study. Therefore, for lesson study to take place it has to happen in some 'experimental setting' where special arrangements are made. Otherwise practicum is primarily controlled by the school with rather tenuous connections to the school subjects. It should be noted here that Danish primary and lower secondary schools teachers are educated as a generalist, having to teach at least three different subjects. The coursework related to specific school subjects and their didactics occupies only half of the total educational load, while the other half is first and foremost general pedagogics. To our knowledge, there have been only three Danish projects using lesson study in connection with practicum for pre-service teachers (Jørgensen, Rostgaard, & Mogensen, 2016; Rasmussen, 2015; Østergaard, 2016).

Regarding 2), lesson study has been tried out as a part of the coursework specific to the school subjects, under the guidelines for so-called '*associated practice*' (Danish: *praksistilknytning*) which can be many forms of cooperation between teachers from the schools, and educators or students at the UCs. The particular framework for associated practice is unique for each UC, and there is a lot of leeway for individual initiatives, hence also for lesson study (e.g. Nyboe & Rasmussen, 2015). So far the initiative to do lesson study in the context of *associated practice* has come from educators at the UCs, having their students prepare elaborate lesson plans, and then 'borrow' a class of pupils, e.g. from their previous years practicum school. The first example of such an initiative was reported at ICME13 in poster form: 'Consecutive cycles of "whole class" lesson study: A format for development of shared teacher knowledge in pre-service teacher education' (Rasmussen, 2016). In that reported initiative, it is uncovered that the immediate challenge of incorporating lesson study in associated practice is that coursework traditionally takes place during activities where whole classes of pre-service teachers attend to the same things, without necessarily being split into smaller working groups. Thus, if no particular changes are made to the paradidactic infrastructure of lesson study for small groups of in-service teachers, it becomes obvious that only a marginal fraction of pre-service teachers truly engage in the lesson study process; the rest opting for a much more peripheral involvement. This has led us to develop a much more explicit paradidactic structure for pre-service lesson study which involves parceling out *kyouzai kenkyuu* tasks, conferences (sessions where ideas are shared among groups), rotation of groups responsible for each iteration (cycle), and a lottery to decide on the pre-service teacher who will teach the research lesson.

We take our case from Rasmussen (2016), where the notion of praxeology is used to precisely pinpoint the changes in mathematical tasks proposed to the students, as well as the techniques that students were anticipated or even suggested to employ. And most significantly, we mapped the reasons (the didactic logos) which pre-service teachers gave for choosing to employ and forecast these tasks and techniques. The analysis is straightforward but quickly becomes rather extensive as it mirrors the great complexity of what goes on in just one (essentially repeated) lesson.

The lesson study analyzed in this case initially centered on the theme "Who is going to pick up school milk for the class?" (See Table 7.3) This alludes to a context which is familiar to students in Danish lower secondary school. Denmark is a dairy producing country, and school milk is made available to all children at a low cost. At school, the box with the milk for each class has to be picked up from a central refrigerated storage. The student teachers' didactic idea (in the sense of Sec. 2) is to have the students play a game of flipping coins to determine who has to go to fetch the box, and investigate if someone is most likely to 'lose'. The main *task* facing the students in the first research lesson was to play a game using two coins in groups of three students. The first technique to be deployed was to take count of who won and lost in a number of consecutive games. But to increase the number of experiments, which was to be realized within the relatively short time span of one lesson, the pupils would also simulate the game using a mobile phone app. The pre-service teachers employed a self-paced video instruction to help the students do the simulation on the app. This we characterize as associated *didactic* techniques. From a didactical point of view, the main justification of these techniques is to make pupils aware of statistical probability as a means to decide the long-term outcome of a stochastic situation. However, what is most explicit in the justifying discourse among the pre-service teachers is that using the familiar context of the theme (picking up school milk) could somehow engage the pupils.

Table 7.3 Praxeological analysis of initial lesson study cycle

Initial task type (T) put to the pupils, anticipated techniques (τ)
T_1: Who in the class is most likely to be selected to pick up milk?
T_2: Of three pupils, investigate who is most likely to be the one to pick up milk, if they use two coins.
T_3: Make a combinatorial argument to answer T_2.
τ_1: Peer/class discussion based on prior experiences.
$\tau_{2.1}$: Perform physical simulation, take count.
$\tau_{2.2}$: Perform large number of simulations using a mobile phone app.
τ_3: Draw a schematic of sample space.
Reasons (θ) for tasks and techniques
θ_1: Pupils should become aware of subjective beliefs about probability.
$\theta_{2.1}$: Pupils become aware of statistical probability.
$\theta_{2.2}$: Pupils become aware that statistic probability variation decreases.
θ_3: Pupils become aware of combinatorial probability.
Associated didactic techniques ($\tau^\dagger$)
$\tau_1\dagger$: Use of familiar context engages the pupils.
$\tau_{2.2}\dagger$: Watch video instruction about how to do simulation.

The following conclusions emerged from the post-lesson reflection: The simulation using the app was difficult for the pupils, leading to the suggestion to use a computer spreadsheet which the students might be more familiar with. Moreover, in the research lesson, the work with the app simulation was carried out by students individually; the post-lesson discussion led to suggesting it to be done in groups.

Going on to the next (revised) research lesson (see Table 7.4), we see that nearly all tasks and techniques are the same.

Table 7.4 Praxeological analysis after first revision of lesson plan (Note the slight change to τ2.2)

Task type (T) put to the pupils, anticipated techniques (τ)
T_1: Who in the class is most likely to be selected to pick up milk?
T_2: Of three pupils, investigate who is most likely to be the one to pick up milk, if they use two coins.
T_3: Make a combinatorial arguments to answer T_2.
τ_1: Peer/class discussion based on prior experiences.
$\tau_{2.1}$: Perform physical simulation, take count.
$\tau_{2.2}$: Perform large number of simulations using ICT (Excel spreadsheet).
τ_3: Draw a schematic of sample space.

No new **reasons for tasks and techniques (θ)**
Associated didactic techniques ($\tau^{\dagger}$) $\tau_{2.1}$† : Provide a table in which to record the results of physical simulation. $\tau_{2.2}$†: Step-by-step video and written instruction of how to do simulation.

The reasons are identical, and only the didactic techniques are altered (in accordance with the conclusions from the post-lesson reflection). This second research lesson progressed to a very high degree as the first; the students apparently still engaged poorly and reluctantly with the simulation task. At the second reflection session, the talk is ripe with concerns for more fundamental challenges in the lesson design, particularly it is now questioned if the students realized the purpose at all – the *raison d'ętre* – of the simulation task. It is suggested that the lesson is revised into a lesson with a problem situation that calls more urgently for the statistical view of chance: the investigation of a game using asymmetrical dice whose probabilities cannot be determined by combinatorics. (See Table 7.5)

The third revision of the research lesson exhibits mathematical tasks where coins are exchanged with the asymmetric dice but are otherwise quite similar in surface structure, whereas the didactic techniques and the logos are much different, and naturally the didactic technique of using the asymmetrical dice is prominent.

Table 7.5 Praxeological analysis after second revision of lesson plan

Task type (T) put to the pupils, anticipated techniques (τ) T_1: Given one crooked dice, who should do the dishes? What are the rules to make a fair game? T_2: What happens if two or more crooked dice are used? What are the rules to make a fair game in this situation? T_3: Make combinatorial arguments to answer T_2. $\tau_{1.1}$: Peer/class discussion based on 'intuition' about how an irregular dice will perform. $\tau_{1.2}$: Perform physical simulation, sample frequency as probability. τ_2: Perform physical simulation with two dice. τ_3: Draw sample space (with unequal probabilities).
Reasons (θ) for tasks and techniques. $\theta_{1.1}$: Pupils should be able to make subjective inferences about probability on the basis of prior experience. $\theta_{1.2}$: Pupils realise the value of statistical probability to determine probability. θ_2: Investigate a more complex situation using statistics. θ_3: Make a connection to combinatorial probability.

Associated didactic techniques (τ†)

τ1.1† : Use of crooked dice to make lesson more oriented towards problem solving.

τ1.2† : Use of crooked dice generates a true need for statistical probability.

The third revision of the research lesson exhibits mathematical tasks where coins are exchanged with the asymmetric dice, but otherwise quite similar in surface structure. However, the didactic techniques and the logos are radically different, and naturally the didactic technique of using the asymmetrical dice are prominent. Furthermore, after the third iteration, it was concluded that the lesson now worked well for the use of one dice (T_1), but difficulties were encountered at the introduction of the second dice (T_2). It was also realized that every pupil should have dice of the same crookedness, in order to make the pooling of individual results for greater frequency stability of the sought probability possible. ICT should not be used at all.

The analysis in terms of praxeologies allows us to follow in detail the incremental changes in pre-service teacher knowledge. It also suggests to us that in the case of pre-service teacher education, in a culture unaccustomed to lesson study, the paradidactic structure of lesson study needs to include several iterations of the same lesson, in order for fundamental changes to be undertaken.

In this case, we used ATD to analyze paradidactic praxeologies and associated didactic praxeologies, looking for changes appearing with each cycle of the research lesson. The overall conclusion is that pre-service teachers are hesitant to draw conclusions from the research lesson and post-lesson reflection. They are reluctant to change their initial lesson plan based on just one experience. They believe the outcome of the lesson is much too contingent on the students, the performance of the pre-service teacher who teaches the lesson, and specific conditions at the particular school. Pre-service teachers are initially unable to identify from observations in the research lesson what could be more fundamental consequences of the didactic choices made in the lesson plan in relation to how students learn mathematics. After the second run, what is contingent and what is not stands out more clearly for them. One might even be pleased that pre-service teachers want the stronger evidence provided by a second cycle, which could be seen as a sound research principle. Our experiences therefore suggest that teachers need to have considerable systematic experience of lesson study (and more generally, a knowledge of teaching and learning) in order to benefit from standalone research lessons. Such experience and knowledge is naturally not available to pre-service teachers; and thus repeated cycles could be of particular significance in pre-service education, particularly when the practicum teachers are also more or less new to lesson study.

7.5. Conclusion

We have presented and showcased two related theoretical perspectives for research on lesson study. As illustrated by Case 1, TDS focuses on the dynamic interplay between a milieu and (individual) learners adaptation to the milieu as it unfolds in time. This allows for a detailed analysis of how both students (in research lessons) and teachers (carrying out lesson study, observing students' learning) develop their knowledge in various phases of lesson study. On the other hand, ATD offers an institutional perspective on mathematical and didactical knowledge, which can be modelled in any desired detail using the notion of praxeology, and used to further analyze the viability of lesson study in the presence of other paradidactic infrastructures. We outlined how this works in Case 2 on practicum systems within initial teacher education. The ATD perspective appears particularly useful for research on the extent to which lesson study as a paradidactic infrastructure 'fits' into institutional frameworks already in place, such as teacher education. The TDS approach was historically closely linked to the researchers' infrastructure of didactical engineering (cf. Miyakawa et al., 2009) but when viewed as a series of paradidactic situations, this approach can also be adapted to research employing lesson study as a methodology, whether initiated by teachers themselves or by a researcher as the first case study (Sec. 2) exemplifies. For such research, we hypothesize that the TDS approach also has the advantage that the classical elements of the theory – such as didactic milieu, and the typology of didactic situations (devolution, action, formulation, validation and institutionalization) – have the advantage of being quite close to teachers' perspectives, while the analysis of students' and teachers' knowledge in terms of mathematical and didactical praxeologies seems to be most appropriate to a researcher perspective. But it should be noted that ongoing research by Carlsen (2017) also experiments with the operationalization of certain elements of the ATD framework for teachers.

The two theoretical frameworks are compatible in the sense that they consider teaching and learning relative to explicit models of the knowledge to be taught, and for the analysis the knowledge actually developed by students when implementing a research lesson. Unlike many commonly used frameworks they do not only model students' learning but also the teaching and the knowledge that is taught. TDS and ATD have slightly different affordances, with TDS being more close to the perspective of teachers (constructing, devolving, and observing the effects of a milieu) and ATD focusing on the institutional perspective which are so visible to researchers initiating and studying lesson study as a new element of paradidactic infrastructure.

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Appendix 1: Japanese vocabulary - a proposal for standard transcriptions

Note in:

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Japanese vocabulary - a proposal for standard transcriptions

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A proposal for a standard use of Japanese lesson study vocabulary

In the wake of an increasing interest in Japanese lesson study, the amount of research literature covering it has grown significantly during the last twenty years. Since most authors and readers are not familiar with Japanese, one challenge is how to transcribe Japanese vocabulary into Western languages. There is no agreement to one of the various transcription systems, and in some cases, a mix of these is used. In this brief comment on the use of Japanese vocabulary in research literature, the author suggests the use of one common system of transcription, namely modified Hepburn *without* contraction of vowels and splitting words.

Written Japanese comprises four types of characters: *Kanji*, *hiragana*, *katakana* and Roman (Latin) letters. Kanji bear meaning and are often complex. Hiragana and katakana are sets of syllables and carry no meaning, comparable to an alphabet. The two sets are phonetically identical, i.e. they represent identical sounds, but have different purposes. Roman letters are used as well to a minor degree. Numbers are usually written with Arabic numerals (also known as European digits) but are also written with kanji. The process of transcription is often referred to as *romanizing*.

Most words we adopt from Japanese are originally written in kanji, but – a complicated matter made simple – it suffice here to say, that any word in kanji can be meaningfully written with hiragana syllables. Since we cannot transpose the meaning of a kanji into Latin script, we can (in a simplified way) consider transcription as an act of 'translating' a Japanese word's 'spelling' with hiragana into Roman letters.

Transcriptions take compromises and for the sake of communication a common standard is necessary (as for any language). There are a number of systems for transcribing Japanese in use. In many cases, one person will consistently use one of these systems to transcribe, but in other cases the systems are mixed. For readers and writers not familiar with Japanese, the different transcriptions may be confusing (and occasionally that goes for Japanese writers and readers too).

There are a number of historical and linguistic reasons for the several systems, and hence good reasons for each of them. Still, especially for readers and writers not familiar with Japanese, it is inconvenient with different transcriptions. This is the sole motivation to suggest a standard of transcription.

Commonplace cases

Hiragana (and katakana) comprise 46 *monographs*, i.e. single characters, of which some can be combined into *digraphs*, two characters combined to function as one. Most of the monographs cause no confusion, but some do. For instance, the hiragana つ can be romanized as *tsu* or *tu*. Likewise し is written as *shi* or *si*. To distinguish digraphs from two monographs, the second character is smaller, e.g. きや, *kiya*, (monographs of equal sizes) and きゃ, *kya*, (digraph, one normal and one small). Confusing cases of transcribing digraphs include じゅ (*ju* and *jyu*) and しょ (*sho* and *syo*).

One of the differences is whether the transcription differentiates between short and long vowels. Written with hiragana, the length of a vowel is directly represented with the number of characters. For examples, the sound きょ has a short vowel, which can be represented as *kyo*, whereas きょう has a long vowel, often represented as *kyou* or *kyō* (with a macron) but often as *kyo* as well.

Contrary to Western languages, space between words is not used in Japanese. When applying Japanese vocabulary to Western writing, this calls form some choices regrading readability. As a result, 授業研究 (lesson study) romanizes as *jugyoukenkyuu, jugyou kenkyuu* or *jugyou-kenkyuu*. Combined with the above illustrated multiple variations regarding romanization of each character, this allows for a myriad of ways to transcribe.

There are other not so frequent cases of confusion, but for the moment the main difficulties summarize to

- General romanization
- Long vowel spelled out or contracted (with or without a macron)
- Combined or split words

The following examples – taken from recent literature – illustrate the problem. These are not examples of an author using a wrong transcription, but solely examples of the use of different transcription systems, resulting in different 'spellings'. In the headline of each example, the word in question is romanized by the suggested system, spelling out vowel and splitting words. Emphasis is added to the Japanese word.

***Jugyou kenkyuu* - じゅぎょうけんきゅう - 授業研究**

*" Lesson study (*jugyou kenkyuu*) is a form of professional development that has been credited for supporting profound changes in teaching in Japan..."* (Takahashi & McDougal, 2016, p. 513)

Comments: Split words, long vowel spelled out.

"LESSON study (jugyoukenkyuu) is a Japanese professional development process..." (Fernandez & Chokshi, 2002, p. 1)

***Kyouzai kenkyuu* - きょうざいけんきゅう - 教材研究**

"*In Japan, teachers consider* kyozaikenkyu *inherent in a teacher's life so they are actively involved in this endeavour in the hope of improving their level of teaching*" (Fujii, 2014, p. 7)

Comments: One word, contracted vowels [o] and [u] (unidentifiable)

"Kyouzai kenkyuu, *the careful study of academic content and teaching materials, is integral to lesson study...*". (Takahashi & McDougal, 2016, p. 520)

Comment: Split words, long vowel spelled out.

"*...addressed and the material developed for it – that is, the result of* kyōzaikenkyū *(study of teaching materials)...*" (Miyakawa & Winsløw, 2017, p. 7)

Comment: One word, contracted vowels (identifiable).

Kenkyuu in other contexts

"*... in the setting of teacher study meetings (in Japanese,* kenkyu-kai*).*" (Miyakawa & Winsløw, 2013, p. 186)

Comment: Hyphened word, contracted long vowel (unidentifiable)

”Here, “practice research”(jissenkenkyū) is a broader term that denotes the study and research on teaching practices, carried out mainly by an individual teacher or a group of teachers...” (Miyakawa & Winsløw, 2017, p. 2)

Comments: One word, contracted long vowel (identifiable).

The examples presented here illustrate various ways of spelling the same word, different approaches to splitting words or not and a variety of writing specific hiragana.

Suggested system for transcription

There are already a number of systems in use, but – without going into further detail – they each hold some difficulties regarding ease of use and readability for readers and writers not familiar with Japanese. The suggestion presented here is, based on the author’s experience with the initial encounter with Japanese and – years later – as a teacher of basic Japanese, believed to be a good balance between simplicity and precision. One of the major systems of transcription of Japanese is called Modified Hepburn System. From its widespread use and from the author’s experience, this system to a high degree meet represents the sounds of each syllable well. In Hepburn, long vowels are contracted though, represented with a macron (e.g. ō). For the sake of simplicity and consistence, it is suggested to omit this and instead spell out each hiragana (which too will dissolve the issue of *oo* vs *ou*).

In addition, it is suggested to split but not hyphen words. This is in accordance with English and other Western languages.

The list of words presented below serves as an example of how given words are romanized, following the suggested system.

Table 1. Examples of Japanese lesson study vocabulary transcribed with the suggested system.

Meaning	Transcription	Japanese
Lesson study	Jugyou kenkyuu	じゅぎょうけんきゅう（授業研究）
Research lesson	Kenkyuu jugyou	けんきゅうじゅぎょう（研究授業）

Research meeting/conference	Kenkyuu kai	けんきゅうかい （研究会）
Open lesson	Koukai jugyou	こうかいじゅぎょう （公開授業）
Study of topic, curriculum, learning, learning progression and related teaching materials	Kyouzai kenkyuu	きょうざいけんきゅう （教材研究）
Practice research	Jissen kenkyuu	じっせんけんきゅう （実践研究）
Instruction and/or guidance including formative assessment at the pupils' desks.	Kikan shidou	きかんしどう （机間指導）
Lesson plan	Gakushuu shidou an	がくしゅうしどうあん （学習指導案）
Knowledgeable other	Koushi	こうし （講師）
Attached school	Fuzoku gakkou	ふぞくがっこう （付属学校）
Elementary school Mathematics	Sansuu	さんすう （算数）
Mathematics	Suugaku	すうがく （数学）
Calculate/calculation	Keisan	けいさん （計算）
Education	Kyouiku	きょういく （教育）

The list is not exhaustive and should be expanded.

References

Fernandez, C., & Chokshi, S. (2002). A practical guide to translating lesson study for a US setting. *Phi Delta Kappan*, *84*(2), 128–134.

Fujii, T. (2014). Implementing Japanese Lesson Study in Foreign Countries: Misconceptions Revealed. *Mathematics Teacher Education and Development*, *16*(1), 2-18.

Miyakawa, T., & Winsløw, C. (2013). Developing mathematics teacher knowledge: the paradidactic infrastructure of "open lesson" in Japan. *Journal of Mathematics Teacher Education*, *16*(3), 185–209. doi:10.1007/s10857-013-9236-5

Miyakawa, T., & Winsløw, C. (2017). Paradidactic infrastructure for sharing and documenting mathematics teacher knowledge: a case study of "practice research" in Japan. To appear in JMTE.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: maximizing the impact of lesson study. *ZDM Mathematics Education*, *48*(4), 513–526. doi:10.1007/s11858-015-0752-x

Appendix 2: Appendices of the introduction (a-f)

Appendix a: Slides from the initial presentation of the project

Undersøgelsesbaseret matematikundervisning og lektionsstudier

- Baggrund
- Problemstilling
- Løsningsforslag

1

Undersøgelsesbaseret matematikundervisning og lektionsstudier

Forskning viser, at

- Ca. fra femte klasse falder danske skoleelevers motivationen for matematik.
- Mange forstår ikke faget og formålet med undervisningen
- I test præsterer de lavere end dem, vi ønsker at sammenligne os med.

- Undersøgelsesbaseret matematikundervisning øger motivation og læring.
- Det er krævende at eksperimentere med at anvende nye didaktiske metoder.

- Faglig kollegial sparring er et stærkt middel til at styrke læreres faglige og didaktiske udvikling, herunder at eksperimentere med nye didaktikker.
- Vi ved det, men bruger det sjældent og dårligt.

2

Undersøgelsesbaseret matematikundervisning og lektionsstudier

Problemer:

1) Hvordan gør vi vores elever dygtigere til matematik?
2) Hvordan støtter vi lærerne til at løse problem nr. 1?

Løsningsforslag:

Ad 1) Ved at anvende matematiske undersøgelser i undervisningen.

Ad 2) Ved at lade lærerne udvikle og designe undervisningen i fællesskab.

3

Undersøgelsesbaseret matematikundervisning og lektionsstudier

Matematiske undersøgelser → *Open-ended Approach* (OEA)

Teambaseret udvikling af undervisningen → *Lektionsstudier* (LS)

Begge er fuldt udviklede metoder, der er integrerede i det japanske skolesystem, herunder læreruddannelse.

Danske, europæiske og amerikanske pendanter er på forsøgs- og implementeringsniveau.

4

Lektionsstudier

5

Lektionsstudier – overordnet beskrivelse

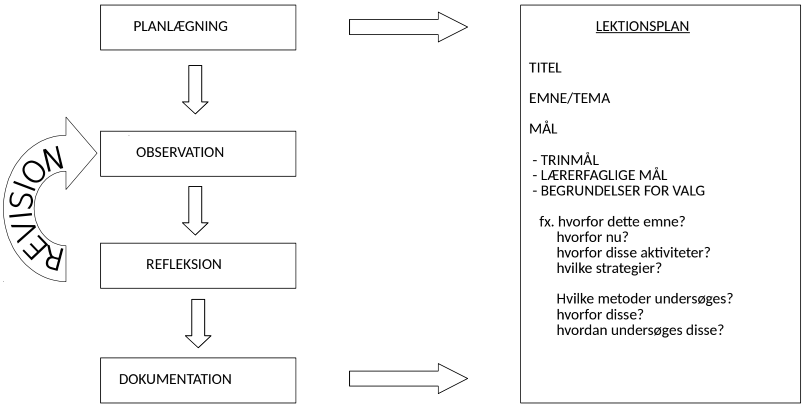
- Lærerstyret aktions*forskning*
- Teambaseret udvikling og design af undervisning (vi-form).
- Fokus på én lektion, men det primære udbytte = didaktiske kompetencer.
- Fokus er på elevernes læring (modsat lærernes undervisning).

Procedure:

- Beslutte fagligt mål (tilsigtet viden)
- Undersøge det matematiske emne, curriculum, undervisningsmaterialer osv.
- Planlægge og designe detaljeret og nøje tilrettelagt lektionsplan (viden, erfaringer og forestillinger)
- Gentagen observation af og refleksion over lektionsplanen ift. tilsigtet viden.
- Afsluttende refleksion og dokumentation.

6

Lektionsstudier - forløbet



7

Lektionsstudier – faglige didaktiske overvejelser

LEKTIONSBESKRIVELSE

LÆRINGSAKTIVITETER OG

LÆRER-SPØRGSMÅL

FORVENTET TID (pr. aktivitet)

FORVENTEDE ELEVREAKTIONER

LÆRERSTØTTE

EVALUERINGSTEGN OG -DATA

= FAGTEAMETS SAMARBEJDE OM

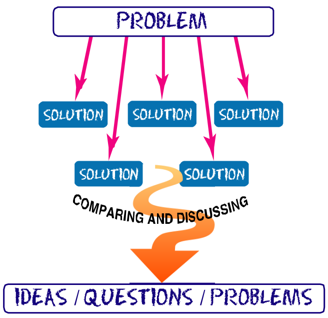
- Didaktisk planlægning
- Didaktisk udvikling
- Forenklede fælles mål
- Synlig læring
- Differentiering

8

Open-ended Approach

9

Open-ended Approach



Ét velformuleret problem med flere korrekte løsninger

- Høj grad af autonomi
- Hypotese og afprøvning
- Frit metodevalg og -udvikling
- Analyse og diskussion

10

Open-ended Approach – principper og struktur

I modsætning til almindelige åbne opgaver præsenteres eleverne i Open-ended Approach (OEA) *altid* for et problem, der har flere *rigtige* løsninger.

OEA bygger på 3 principper om:

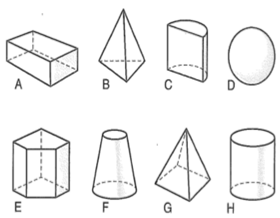
1) **Elevernes autonomi**
2) **Essentiel viden**
3) **Lærerens hensigtsmæssige vejledning**

OEA-undervisningen består at 3 faser:

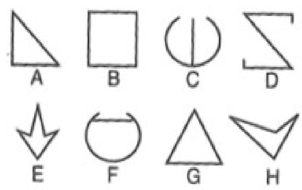
1) **Matematisk formulering af problemet**
2) **Afprøvning af metoder**
3) **Fremsætning af avancerede problemer**

11

Open-ended Approach - eksempler



Hvilke rumlige figurer har noget til fælles med figur B? Hvad har de til fælles?



Find så mange karakteristika ved figurerne som muligt. Grupper figurerne ud fra disse karakteristika.

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Organisering

13

Organisering - årsplan

Team	År / uger														
1	X		X				X				X				X
2	X			X				X				X			X
3	X				X				X				X		X
	1)							2)							3)

1) Fælles opstartsseminar for alle involverede lærere

2) 3 lektionsstudieforløb i løbet af skoleåret

3) Fælles afslutningsseminar for alle involverede lærere

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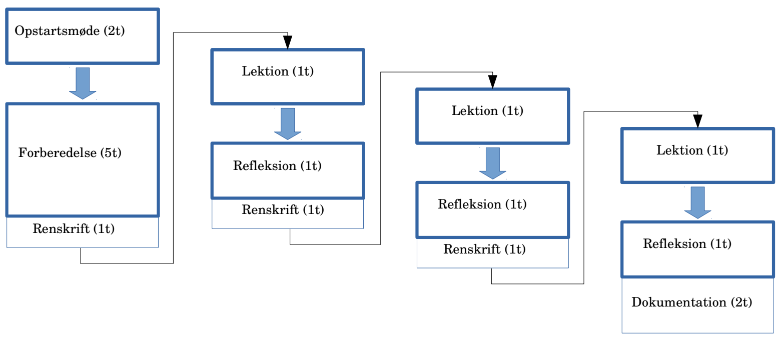
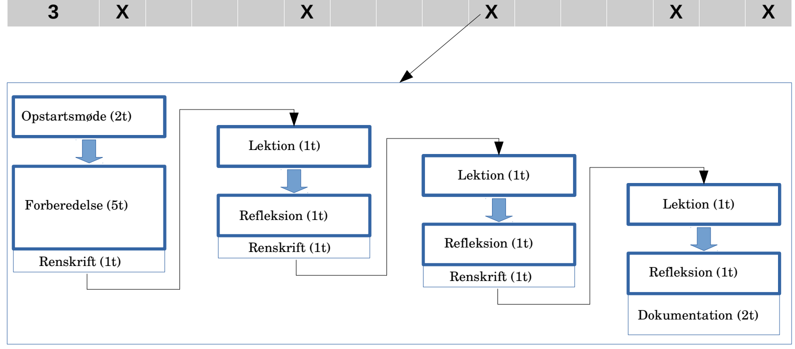
Organisering – LS-forløb



15

Organisering – forløb over året

Team	År / uger														
1	X		X				X				X				X
2	X			X				X				X			X
3	X				X				X				X		X



16

Udvikling og design af undersøgelsesbaseret matematikundervisning

Desuden...

- Interview med alle lærere før og efter
- Interview med udvalgte elever før og efter
- Audio/videooptagelser
- Indsamling af materialer: Fx. elevbesvarelser, undervisningsmaterialer
- Anvendelsesrettigheder
- Eventuel anonymisering (lærere og elever)
- Eventuel tilladelse til fremvisning
- Eventuel **gensidig deltagelse i hinandens (sidste) lektion+refleksion hver gang?**

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Appendix b: Letter of agreement

Lektionsstudier og undersøgelsesbaseret matematikundervisnining.

Kære lærere og ledere.

Tak fordi I deltager i projektet. Projektets baggrund er et ønske om at afprøve nye rammer for læreres udvikling af effektiv undervisning. Disse rammer er: 1) En konkret model for samarbejde, refleksion og vidensudvikling i lærerteams, og 2) en konkret didaktisk model for, hvordan man kan bygge sin undervisning op om matematiske undersøgelser.

Det drejer sig naturligvis i sidste ende om, at vi som lærere bliver endnu dygtigere til at lave god matematikundervisning for vores elever. Grundet projektets fokus på praksis, ligger der en stærk forventning om, at selve forløbet bliver udbytterigt for alle parter. I vedlagte projektbeskrivelse uddybes både baggrunden for projektet og projektets udformning, herunder en overordnet plan for året. Planen kan tilpasses efter aftale, men det prioriteres højt, at planen ikke ændres markant med mindre det er nødvendigt - både for at sikre et godt forløb på den enkelte skole, og for at sikre den samlede plan på tværs af de tre skoler.

At der er tale om et forskningsprojekt betyder, at vi får rig lejlighed til at eksperimentere, og at der ikke ligger en forventning om et givent udbytte. Hverken I eller jeg bliver bedømt på, om vi opnår noget bestemt, heller ikke om det vi eventuelt når frem til er et positivt resultat. Personligt tror jeg, at vi både kommer til at opleve nogle små fiaskoer og nogle store succeser. Men det væsentlige er alt det, vi lærer om samarbejde, undervisningsudvikling og undersøgelsesbaseret matematikundervisning, mens projektet står på.

Projektet drejer sig om lektionsstudier og undersøgelsesbaseret matematikundervisning, men foregår i jeres hverdag, så vi kommer til at arbejde tæt sammen om at få det hele til at gå op i en højere enhed. Både for at kunne støtte mest muligt i den sammenhæng, og for at kunne observere mest muligt til mit videre analysearbejde, vil jeg være til stede både som forsker og vejleder i alle faser undervejs.

Jeg ser frem til at arbejde sammen med jer.

Venlig hilsen,

Jacob Bahn

Beskrivelse af forskningsprojektet: Lektionsstudier og undersøgelsesbaseret matematikundervisning.

Formål

Formålet med projektet er at undersøge, om givne metoder og redskaber kan understøtte matematiklærere i at designe og gennemføre undersøgelsesbaseret undervisning.

Fokus

Projektet har to overordnede og sammenhængende fokuser:

1. *Lektionsstudier* (LS), der er en konkret model for samarbejde, refleksion og vidensudvikling i lærerteams, og
2. *Open-ended Approach Method* (OEA), der er en konkret didaktisk model for, hvordan man kan bygge sin undervisning op om matematiske undersøgelser.

Jeg har først og fremmest valgt disse modeller fordi forskningen viser, at der er et potentiale for at skabe en mere engagerende og motiverende undervisning ved at anvende dem. Desuden er de begge fuldt udviklede og anvendes på daglig basis i Japan. Til sammenligning er danske og europæiske varianter af lignende metoder stadig under udvikling.

Opbygning og overordnet plan

I projektet undersøges bl.a., hvilke betingelser der er af betydning for at realisere de faglige potentialer i OEA-metoden, og hvordan lektionsstudier kan bidrage til lærerteamets vidensudvikling.

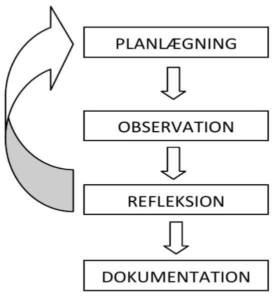
I praksis vil matematiklærerteams på xxxxxx-skolen, xxxxxx-skolen og xxxxxx Skole i Lyngby-Taarbæk Kommune afprøve metoderne i tre omgange over et skoleår. Hvert team vil i løbet af skoleåret 2015-2016 gennemføre tre lektionsstudier centreret om én lektion, der er bygget på principperne for OEA. Et lektionsstudie strækker sig over ca. to uger. Teamene består af matematiklærere på mellemtrinnet på én eller på hinanden følgende årgang(e). I hvert lektionsstudie gennemføres tre lektioner – *studielektioner* - uanset antallet af lærere i teamet.

I starten af skoleåret skyder vi projektet i gang med et opstartsseminar (*6 timer*), hvor alle lærere deltager. På seminaret introduceres metoderne og de lærerværktøjer, der er udviklet til projektet grundigt. Inden skoleåret slutter, holder vi et afslutningsseminar (*4 timer*), hvor vi i fællesskab reflekterer over forløbet og udbyttet.

I løbet af skoleåret kommer hver lærer til at bruge ca. *17 timer* pr. lektionsstudie, i alt 6+4+(3 x ca. 17) ≈ 60 timer. Jeg skelner mellem *timer* a 60 minutter og *lektioner* a 45 minutter. Desuden vil lærerne blive bedt om at stille sig til rådighed til et interview (ca. 30 minutter) før og efter projektets gennemførelse.

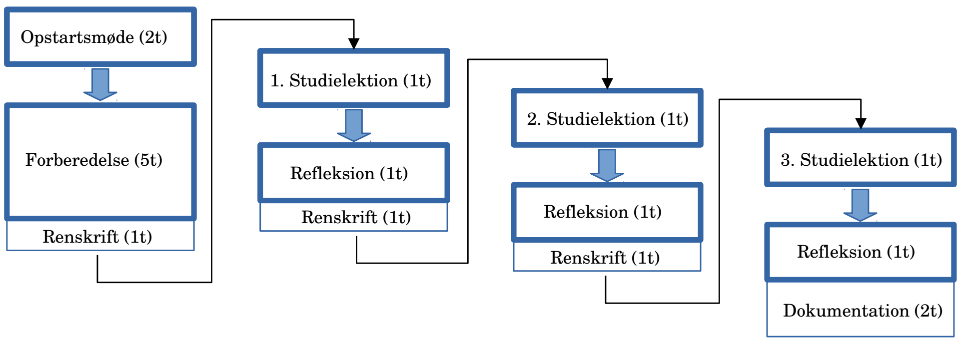
Så vidt muligt er jeg til stede under samtlige fælles aktiviteter, men også gerne i forbindelse med individuelle aktiviteter (se nedenstående).

Detaljeret plan



I projektet flettes lektionsstudier og Open-ended Approach sammen, så planlægning af de givne lektioner foretages på baggrund af teamets detaljerede overvejelser og diskussioner. Denne reflekterede planlægning munder ud i en velbeskrevet lektionsplan, som lærerne på skift afprøver i sin undervisning (dvs. en studielektion) under observation af teamets øvrige medlemmer. Efter studielektionen følger en fælles refleksion over, i hvor høj grad eleverne fik det udbytte, man havde tiltænkt, og om lektionsplanen eventuelt skal revideres. Efter sidste studielektion dokumenteres lektionsstudiet med den endeligt reviderede lektionsplan og notater om væsentlige overvejelser, udfordringer og resultater.

Et lektionsstudie forløber over ca. to uger og kommer til at se nogenlunde sådan her ud:



Til hvert teams sidste studielektion + refleksion (2 timer) i hvert lektionsstudie deltager de øvrige teams. Under opstartsmødet besluttes emne og fokus for lektionsplanen. Det meste arbejde foregår i teamet, men i tiden mellem opstartsmødet og den egentlige forberedelse overvejer hver lærer, hvad vedkommende ved om emnet, hvilke ressourcer og materialer vedkommende vil tage med til forberedelsen osv. Jo mere tid, der er til dette, jo bedre. Desuden udføres arbejdet med revision og dokumentation af de enkelte lærerne på skift (markeret med en tynd ramme). Refleksion af studielektionen bør foregå i umiddelbar forlængelse af denne.

Hvert team deltager således i to seminarer (fælles for alle) og udfører tre lektionsstudier. Planen ser i udgangspunktet således ud, men kan ændres i henhold til aftalen:

Opstartsseminar

Alle	Uge 35 – torsdag d. 27/8 2015 kl. 9:00-15:00	
Lektionsstudier		
Team A	Ugerne 45-46, 2-3 og 14-15	(xxxxxx-skolen)
Team B	Ugerne 36-37, 49-50 og 9-10	(xxxxxx-skolen)
Team C	Ugerne 38-39, 5-6 og 17-18	(xxxxxx Skole)
Afslutningsseminar		
Alle	Uge 22 – tirsdag d. 31/5 2016 kl. 09:00-13:00	

Aftale om deltagelse i forskningsprojektet

Lektionsstudier og undersøgelsesbaseret matematikundervisning.

Deltagerne i projektet udgøres på hver af skolerne af skoleledelsen, de deltagende lærere og Institut for Naturfagenes Didaktik (IND), repræsenteret ved phd-stipendiat Jacob Bahn. Projektet udføres i øvrigt i samarbejde med Lyngby-Taarbæk Kommunes Center for Uddannelse og Pædagogik (CUP) og Professionshøjskolen UCC. De officielle aftaleparter er skolernes ledelse og institutledelsen.

Projektet er finansieret af en bevilling fra Phd-rådet for uddannelsesforskning.

1. Nærværende aftale bygger på ovenstående projektbeskrivelse inklusiv plan.
2. Projektet forløber i hele skoleåret 2015-2016 på xxxxxx-skolen, xxxxxx-skolen og xxxxxx Skole, alle i Lyngby-Taarbæk Kommune.
3. Efter hhv. første og andet lektionsstudie foretages en evaluering af projektet. Her aftales eventuelle ændringer i udformning og indhold. Sådanne aftaler kan om nødvendigt også foretages på andre tidspunkter, hvis det findes hensigtsmæssigt. Erfaringer og beslutninger fra evalueringsmøder deles med de øvrige skoler/team.
4. Afholdelse af lønudgifter i forbindelse med lærernes deltagelse sker i et samarbejde mellem IND og skolernes ledelser. Projektet bidrager med en pulje på kr. 138.000 til kompensation af lærernes tid. Puljen dækker et fuldstændigt frikøb af lærerne, inkl. forberedelsestid m.m. i samlet 529 klokketimer, beregnet med kr. xxx pr. time pr. lærer. Hver lærer kommer ca. til at bruge 60 klokketimer på projektet i løbet af året, hvoraf projektets pulje dækker godt 50 timer pr. lærer, mens den enkelte skole dækker 9 timer pr. lærer. Ifølge planen vil der være et lille overskud af timer i puljen. Disse bruges evt. på timer til skolernes matematikvejledere eller andre, for hvem det kan være relevant at følge projektet.
5. For xxxxxx Skole deltager 3 lærere på x klassetrin, og det er budgetteret, at kompensation for lærernes deltagelse over året udgør kr. xxxxx.
6. Afregning forudsætter fremsendelse af faktura og sker efter ovennævnte evalueringer, efter hhv. første og andet lektionsstudieforløb, samt efter afslutningsseminaret med 1/3 af det samlede beløb pr. afregning.
7. Hvis en lærer ikke kan fuldføre sit engagement i projekt (fx pga. sygdom, barsel eller jobskifte) prioriterer ledelsen det højt at finde en afløser, der kan træde ind i projektet med fuldt engagement.
8. Skolerne har frie hænder ift. medieomtale af projektet, og Jacob deltager gerne efter forudgående aftale. I det omfang medieomtale foregår uden Jacobs deltagelse tilstræber ledelse og lærere at nævne samarbejdet med IND som projektets værtsinstitution.
9. Skolerne og Lyngby-Taarbæk Kommune har ret til at anvende de af lærerne udviklede materialer ifm. projektet. Materialer der helt eller delvist er udviklede af Jacob/IND kan anvendes efter aftale. Jacob/IND har ret til at anvende ethvert materiale og alle data indsamlet i projektet under hensyn til almindelige principper for god videnskabelig praksis, herunder eventuel anonymisering.
10. Jacob Bahn deltager som konsulent og observatør under alle skemalagte aktiviteter i projektet. I nogle tilfælde deltager Jacobs vejledere eller andre interessenter. Hvis ledelsen eller lærerne ønsker andre deltagere kan dette aftales med Jacob.

11. Til videre bearbejdelse (analyse) optages alle aktiviteter i projektet på video og til tider kan der suppleres med still-billedoptagelser. Desuden indsamles alle lærermaterialer (fx opgaver og evalueringer) og relevante eksempler på elevmateriale (fx besvarelser). Alle optagelser og materialer samt det efterfølgende analysearbejde opbevares og behandles som fortroligt materiale, og kun relevante personer i tilknytning til projektet får almindelig adgang til materialerne.
12. Anvendelse af optagelsesmateriale, der er relevant for Jacob/IND ifm. formidling eller andet, kan kun ske efter samtykke fra ledelsen, evt. efter anonymisering. Ledelse og lærere vil være behjælpelige med de nødvendige tilladelser.

Underskriver:	Dato:
Skoleleder xxxxxx xxxxxx: ________________________	___________
Institutleder Hanne Andersen: ___________________________	___________
Phd-stipendiat Jacob Bahn: _____________________________	___________

Appendix c: Note to timetable coordinators

Oplysninger til skemalæggere

Kære skemalæggere.

Jeg er klar over, at I står over for en stor opgave med at få det hele til at hænge sammen. Jeg har her prøvet at skabe et overblik over de ting, I skal tage højde for. Tjek gerne for eventuelle fejl eller detaljer, jeg har overset!

For det førte ser årsplanen sådan ud:

	skolestart		OPSTARTSSEM							efterårsferie					jb kursus	EVALUERING				juleferie	juleferie
uge	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
B			O	X	X												X	X			
A			O					-		jb japan			X	X							
C			O			X	X														

				jb konference			vinterferie	EVALUERING				påskeferie						kr.himmelfart!				AFSLUTNINGSSEM			
uge	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B									X	X												A			
A		X	X											X	X							A			
C		-			X	X											X	X				A			

Opstartsseminar

Alle Uge 35 – torsdag d. 27/8 2015 kl. 9:00-15:00

Lektionsstudier

Team B Ugerne 36-37, 49-50 og 9-10 (xxxxxx-skolen)

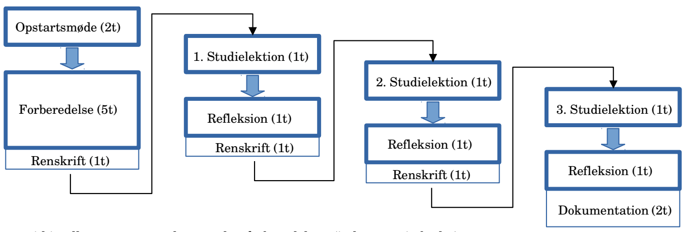
Team A Ugerne 45-46, 2-3 og 14-15 (xxxxxx-skolen)

Team C Ugerne 38-39, 5-6 og 17-18 (xxxxxx Skole)

Afslutningsseminar

Alle Uge 22 – tirsdag d. 31/5 2016 kl. 09:00-13:00

For hver af de to-ugersmoduler lektionsstudierne forløber i, skal er det planen, at programmet skal have nogenlunde dette forløb:



OBS!

- Der må gerne gå noget tid imellem opstartsmødet og selve forberedelsen (jo længere jo bedre)
- Forberedelsesfasen må gerne deles over i to, evt. tre.
- Det er en fordel, at refleksionsseancen ligger umiddelbart efter studielektionen (den lektion, hvor lektionsplanen afprøves)
- Hvis det bliver nødvendigt at lægge to studielektioner samme dag, er det ok.

Som en lille **ekstraudfordring** vil jeg bede jer *prøve* at tage højde for, at i jeg uge 36-41 skal på skolebænken mandag og onsdag eftermiddag. Hvis det bliver for besværligt at planlægge udenom, har projektet absolut førsteprioritet – så får jeg bare lidt mere hjemmearbejde:-)

Jeg er ikke klar over, hvordan det skal lade sig gøre at planlægge besøgene hos de andre teams under deres 3. studielektion+refleksion – måske bliver I 'bare' vikardækkede? Den foreløbige plan for gensidige besøg ser således ud →

Uge	besøgende teams			afholdende team
37	A	C	→	B
39	B	A	→	C
46	B	A	→	C
50	B	C	→	A
3	A	C	→	B
6	B	C	→	A
10	A	B	→	C
15	C	A	→	B
18	B	A	→	C

Hvis jeg på nogen måde kan hjælpe til, må I sige til. Jeg rejser til Japan 3. juni, men er på mail dagligt.

Rigtig god fornøjelse og endnu bedre sommer! Jeg glæder mig meget til at arbejde sammen med jer...

Appendix d: Letter to parents about video recording

Kære forældre.

X Skole har fået mulighed for at deltage i et forskningsprojekt med i alt tre skoler næste skoleår. Forløbet fokuserer på hvordan vores lærere kan blive dygtigere til at undervise i matematik. Mere konkret drejer projektet sig om, hvordan man som lærer kan bruge teamet til at udvikle undervisningen, og hvordan man kan arbejde med matematiske undersøgelser. Projektet udføres af phd-stipendiat Jacob Bahn i samarbejde med læreruddannelsen på UCC og Institut for Naturfagenes Didaktik på Københavns Universitet.

Forløbet er afgrænset til mellemtrinnet. De lærere, der deltager, skal eksperimentere med nye arbejds- og undervisningsmetoder. Tre gange i løbet af skoleåret skal deres arbejde efterprøves i nogle undervisningssituationer, som lærerne efterfølgende analyserer og reflekterer over i fællesskab.

For at kunne analysere forløbet, udviklingen og udbyttet, vil vi optage lærernes arbejde med deres undervisning sammen med eleverne på video. Det gælder også efterprøvningen af deres arbejde, og over skoleåret optager vi video i tre lektioner i hver klasse, der deltager. Optagelserne har stor betydning for forskningen og lærernes udvikling, og erfaringen viser, at de ikke får negativ indflydelse på elevernes læring. Eleverne kommer med på optagelserne, men det er lærernes undervisningsarbejde, der er i fokus. Optagelserne vil blive opbevaret og behandlet som fortroligt materiale og er til forskningsbrug. Ingen videoklip videreformidles uden forudgående aftale med jer forældre.

Inden selve forskningsprojektet går i gang efter sommerferien, har forskeren brug for at teste nogle af de redskaber, han skal bruge. Derfor vil han besøge jeres børns klasser enkelte gange i den kommende tid, hvor han bl.a. optager video. Disse optagelser er alene til træningsbrug og behandles i øvrigt som beskrevet ovenfor.

Hvis der er grunde til, at jeres barn ikke kan deltage, vil jeg bede jer gøre mig opmærksom på det.

Jeg håber, I vil følge projektet med interesse, og Jacob Bahn stiller sig gerne til rådighed ved et forældrearrangement eller lignende, hvor han kan uddybe eventuelle spørgsmål.

Med venlig hilsen

Skoleleder

Appendix e: Slides from the start-up seminar

Opstartsseminar

Undersøgelsesbaseret matematikundervisning og lektionsstudier

- Hummeltofteskolen
- Kongevejens Skole
- Trongårdsskolen

1

Kort fra Japan

Mine erfaringer med japansk undervisning:

- **Undervisningen er rolig: Typisk ét problem**
- **Mindst én lang fase med selvstændigt arbejde**
- **Eleverne kommer med svarene (og ofte også spørgsmålene)**
- **Aktiviteter bygger på fagligt indhold, ikke på pædagogisk ramme**
- **Struktureret brug af tavlen (og magneter!)**

2

Spørgsmål til japanske lærere

- **Skriv løbende de spørgsmål ned, der opstår ift. japansk undervisning, elevers præstationer og læreres arbejdsmetoder og -forhold.**
- **Så samler jeg sammen senere, og spørger i Japan!**

(man må også gerne skrive andre spørgsmål ned)

3

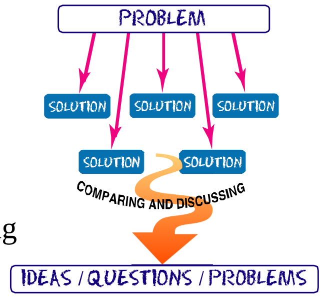
Lektionsstudier (LS)

- **Lærerstyret, teambaseret aktions*forskning***
- **Kompetenceudvikling og vidensdeling i direkte relation til lærerens praksis**
- **Fokus på elevernes læring**
- **Mål: At udvikle didaktiske kompetencer**

(film)

4

Open-ended Approach (OEA)



- **Ét problem med flere svar**
- **Tre principper:**
  - Elevers autonomi
  - fundamental viden
  - hensigtsmæssig vejledning
- **Tre faser:**
  - Matematisk problemformulering
  - afprøvning af metoder
  - udviklede problemer/ideer

(film)

5

Et didaktisk hjælperedskab

Som hjælp til at designe og analysere undervisning, låner vi begreber og modeller fra

Teorien om didaktiske situationer (TDS)

- **Præcision i vores arbejde**
- **Modeller for didaktiske forhold**
- **Fælles sprog**

6

Teorien om didaktiske situationer

TDS

- **Praksisbaseret teori om undervisning i matematik**
- **Udviklet gennem praksisstudier og eksperimenter**
- **God overensstemmelse med principper og strukturer i lektionsstudier og OEA**

7

TDS 1: didaktisk situation

- **Læring foregår i *didaktiske situationer***
- **De konstitueres af to forhold:**

1) En lærer med intention om at lære fra sig

2) En elev med intention om at lære

8

TDS 2: didaktisk miljø

- **Målet for en didaktisk situation er elevens tilegnelse af *den tilsigtede viden* (=undervisning)**
- **Til det formål forbereder læreren nogle opgaver og instruktioner og evt. nogle materialer og redskaber.**
- **Disse er fysiske og betegnes det *didaktiske miljø***
- **I en didaktisk situation indgår således**

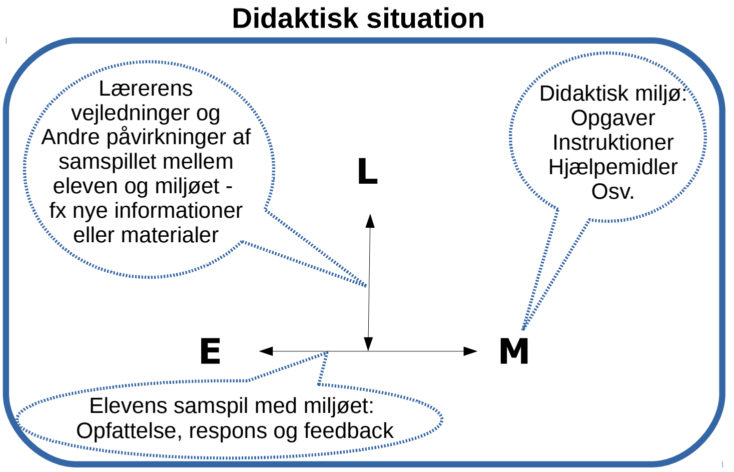
  1) En lærer (L)

  2) En eller flere elever (E)

  3) Et didaktisk miljø (M)

9

TDS 3: Lærer, elev og miljø



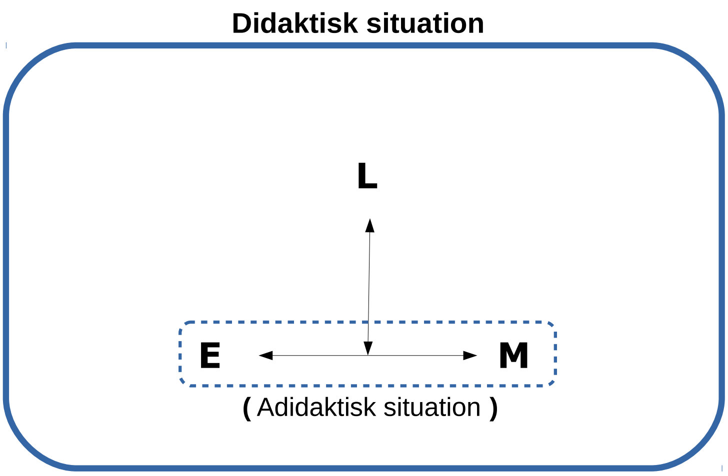
10

TDS 4: *adidaktisk* situation

- ***Didaktisk*** **refererer til** ***direkte*** **undervisning**
- ***Adidaktisk*** **refererer til** ***indirekte*** **undervisning**
- **En** ***adidaktisk situation*** **er, når eleverne arbejder selvstændigt (autonomt), uden lærerens indblanding.**
- **Det kræver, at det didaktisk miljø er tilstrækkelig tydeligt og yder passende feedback.**

11

TDS 5: adidaktisk situation



12

TDS 6: faser

Overdragelse
- Ansvaret for opgaven overdrages

Handling
- Eleven prøver sig frem

Formulering
- Eleven formulerer ideer og hypoteser

Validering
- Ideer og hypoteser søges valideret

Bekræftelse
- Nyerhvervet viden afstemmes

13

Teorien om didaktiske situationer

Læs side 4 i håndbogen (og kig evt. på side 15)

14

Workshop: At komme i gang

- Diskuter jer frem til et matematisk emne, I er enige om, er svært for eleverne at lære.
- Udpeg emnets fundamentale viden – der kan være flere, vælg én som tilsigtet viden.
- Diskuter, hvad udfordringerne er ved hhv. at lære og at undervise netop dette.
- Formuler et problem, hvis løsning I forventer kan lede til forståelse af den tilsigtede viden.
- Afprøv problemformuleringen og forestil jer elevernes reaktioner.
- Prøv at lave en drejebog (husk OEA og TDS)

15

Appendix f: Handbook for teachers

Håndbog

i forbindelse med forskningsprojektet

Undersøgelsesbaseret matematikundervisning og lektionsstudier

1

Kære lærere.

Tak fordi I er med i projektet, som jeg håber – og tror – vil give os alle et stort udbytte. Jeg har været meget omhyggelig med at prøve at udforme projektet, så både forskning og praksis kan få gavn af det – og af hinanden. Vi kommer så at sige til at lægge forskningsfronten i (jeres) praksis og (jeres) praksis i forskningsfronten. Hvis vi kommer i mål med det – og det er jeg overbevist om, at vi gør – har vi virkelig gjort det godt!

Mål og baggrund

Målet med projektet er at eksperimentere og skabe erfaringer med undersøgelsesbaseret matematikundervisning og lektionsstudier. Det sidste bl.a. som en vej til det første. Projektet tager udgangspunkt i Open-ended Approach (OEA), der er en model til – eller rettere nogle principper for – at lade elever arbejde undersøgende i matematikundervisningen. Modellen kommer, ligesom lektionsstudier, fra Japan.

Selv om det af uransagelige årsager ikke er alment kendt, har man siden slutningen af 90'erne vidst, at Japan ligger i front, hvad angår udvikling af undervisning og læreres kompetencer. En af de betydeligste årsager til dette er brugen af lektionsstudier, igennem hvilke lærere og andre væsentlige aktører fra hele uddannelsessystemet på forskellig vis kan bidrage til denne udvikling. Det er bl.a. vha. lektionsstudier, man har udviklet principperne bag OEA, der tilskrives stor betydning for de japanske skoleelevers høje niveauer af engagement og læring.

Lektionsstudier

Lektionsstudier kunne med rette være oversat til til undervisningsstudier eller studier af undervisningspraksis. Den grundlæggende ide er nemlig, at man løbende, gennem studier af sin egen og kollegers undervisning og (fag)didaktiske tænkning, bliver klogere på, hvordan man laver god undervisning for målgruppen. Ved at dygtiggøre sig gennem punktvise studier udvikler man kompetencer til at designe undervisning, som man bruger til at planlægge sin normale undervisning, der dermed bliver bedre.

I Japan kan et lektionsstudie tage mange former. Nogle studier udføres primært af én lærer, der søger sparring hos nogle kolleger, i andre arbejder en stor gruppe af lærere og forskere sammen om at eksperimentere med at udvikle nye metoder (som fx OEA). Nogle gange er det kun lærerteamet, der deltager i studelektionerne, andre gange udføres de til en konference med flere tusinde deltagere! Også længden kan variere, idet nogle lektionsstudier varer op til et år.

I dette projekt er der sat en relativt stram ramme op om hvert lektionsstudie, idet I skal gennemføre det i jeres team inden for to uger. De grundlæggende strukturer er dog de samme:

1) Udarbejdelse af lektionsplan, pba. det valgte emne, tema og mål.
2) En studielektion, hvor lektionsplanen efterprøves og udvalgte fokuspunkter observeres

2

3) En refleksionsseance, hvor studielektionen diskuteres ift. observationer og lektionsplanen – som regel med særligt fokus på elevernes læring.

Fase 1-3 gentages ofte i en cyklus, hvor fase 1 typisk er kortere i anden og senere omgange. I vores projekt taler vi om gentagelsen af fase 1 som en revision af planen.

Ofte arbejder man med et overordnet ikke-fagspecifikt tema (fx differentiering) og et fagligt emne, inden for hvilket man fokuserer på en given fundamental viden

Open-ended Approach

Det centrale i OEA er, at lade eleverne arbejde med ét problem, der reelt har flere rigtige løsninger, at lade dem arbejde med og udvikle deres egne metoder samt at bruge elevernes tilgange og løsningsforslag til en diskussion af problemet og dets mulige løsninger. OEA lægger således til rette for at træne bl.a. ræsonnement, metode(r) og kendte teknikker på elevernes præmisser.

OEA bygger på tre *principper* om

1) Elevernes autonomi
2) En fundamental viden
3) Lærerens hensigtsmæssige vejledning

og består typisk af tre *faser*

1) Matematisk (om)formulering af problemet (der typisk er dagligdags)
2) Afprøvning af metoder
3) Fremsætning af avancerede problemer og hypoteser gennem klassediskussion

Det er ikke let at designe sådanne lektioner, og derfor har lektionsstudier ikke bare været brugt til at udvikle OEA, men også til at implementere det. OEA tillægges stor betydning for japanske elevers høje faglige niveau. Begrundelsen er det øgede engagement, som veltilrettelagte OEA-lektioner medfører.

Kommentarer om japansk matematikundervisning

Det er interessant at observere og analysere japansk matematikundervisning, og der er et par egenskaber, der går igen. For det første er både undervisningen og læreren meget rolig; en lektion fokuserer typisk på ét problem/emne. For det andet er der typisk mindst én lang fase, hvor eleverne arbejde selv (individuelt eller i grupper). For det tredje er det næsten altid *eleverne*, der kommer med svarene – og ofte også spørgsmålene! For det fjerde bygger aktiviteter på matematikkens faglige indhold og ikke så meget på en pædagogisk ramme. For det femte har japanske lærere en tilsyneladende simpel men nøje gennemtænkt og struktureret brug af tavlen og andre materialer (de er meget glade for magneter).

3

Et didaktisk hjælperedskab
I arbejdet med at planlægge og designe undervisning, kan der være hjælp at hente i at bruge en teoretisk referenceramme. Det kan være med til at præcisere ens arbejde og tydeliggøre kommunikationen i teamet og udadtil. I det følgende introduceres begreber og redskaber hentet i Teorien om didaktiske situationer (TDS).

TDS er en praksisbaseret teori om undervisning i matematik. Den er udviklet gennem praksisstudier og eksperimenter over flere årtier, primært af Guy Brousseau, der selv var lærer. Teorien er kompleks, men man kan trække essentielle elementer ud, der egner sig som redskaber til design og analyse af undervisning – hvilket jo er kernen i vores projekt. Ydermere er der en vis overensstemmelse mellem disse redskaber og den måde man arbejder i hhv. lektionsstudier og OEA. TDS kan således fungere som et hjælperedskab, der giver os nogle modeller og et fælles sprog til at snakke om didaktisk design med.

Helt centralt og overordnet står, at (skole)læring sker i *didaktiske situationer*. En didaktisk situation konstitueres af to elementer: En lærer, der har intentioner om at lære noget bestemt fra sig, og (mindst) en elev, der er indstillet på at lære noget. Det kan lyde banalt, men det er centralt: Hvis ikke eleven/erne er indstillet på at lære noget, er der ikke en didaktisk situation, og læringen af den *tilsigtede viden* udebliver.

Den tilsigtede viden er målet med undervisningen. Derfor tilstræber læreren at designe et *didaktisk miljø* (opgave, materialer, redskaber osv.), der oplyser og udfordrer eleverne, så de tager ansvaret for at arbejde med at finde en løsning, der leder til den tilsigtede viden. Essentiel læring sker (kun) gennem egen erkendelse, hvorfor opgaven (det didaktisk miljø) må føre til en *adidaktisk situation*, hvor læreren ikke blander sig; den enkelte elev kan arbejde på sin egen måde (metode) ud fra egne forudsætninger (viden). En adidaktisk situation kan ses som en parentes i den didaktiske situation, hvor det adidaktiske består i fraværet af lærerens direkte indblanding.

Hvis dét var nemt, affødte det ikke en selvstændig teori!

I design og analyseøjemed deler TDS undervisningen op i fem faser (ikke alle indgår nødvendigvis i alle lektioner):
1. *Overdragelsesfaser*, hvor ansvaret for opgaven overdrages
2. *Handlingsfaser*, hvor elever prøver sig frem
3. *Formuleringsfaser*, hvor eleverne formulerer ideer og hypoteser
4. *Valideringsfaser*, hvor hypoteser søges valideret
5. *Bekræftelsesfaser*, hvor nyerhvervet viden afstemmes

Se i øvrigt bilag om TDS

4

GUIDE

5

Introduktion

Lektionsstudier kan sammenlignes med et forskningsprojekt. Grundlæggende er det forskning i egen praksis; en form for aktionsforskning, hvor man undersøger et centralt spørgsmål fra flere vinkler.

Først udvælger og præciserer man, hvad man vil undersøge og hvordan man vil undersøge det – man opstiller nogle forskningsspørgsmål og hypoteser, der er styrende for studierne. I et lektionsstudie svarer dette til *opstartsmødet*.

Dernæst forbereder man den eller de undersøgelser, man mener kan give svar på på forskningsspørgsmålene. Ofte må man undersøge væsentlige problemstillinger først, så man øger chancen for faktisk at få svar på sine spørgsmål. Det svarer til *forberedelsesfasen* i et lektionsstudier.

Derefter gennemfører man sine grundigt tilrettelagte hovedundersøgelser og observerer dem ud fra udvalgte kriterier – man skal vide, hvad man leder efter, og have gjort sig forestillinger om, hvilke observationer, der kan belyse det, som undersøges. I lektionsstudier sker dette i *studielektionen*.

Til sidst analyserer man de informationer, man har fået ud af sine observationer og fortolker dem – hvilke svar har man fået, og hvilke nye spørgsmål leder de til? Det er, hvad der foregår i *refleksionsfasen*.

Med sin trinvise opbygning har kvaliteten af hvert trin i et studie stor betydning for kvaliteten af det følgende trin.

Husk: I forskning er der ingen garantier for, at man finder, det man troede. Til gengæld er alle resultater vigtige, idet de som minimum kvalificerer følgende undersøgelser. Det gælder også i lektionsstudier og er vigtigt.

De følgende sider er tænk som en praksisguide og er vejledende. Guiden bygger på praksiserfaringer fra bl.a. nogle af jeres kolleger her i Lyngby-Taarbæk. Den er produktet af et seminar med det selv samme formål at samle de vigtigste erfaringer sammen til jer, hvilket har ledt til formen: Korte, punktvise råd og anbefalinger samt spørgsmål, man med fordel kan stille sig selv undervejs. Desuden indgår elementer fra TDS som støtteredskaber ift. design og analyse.

En overordnet kraftig anbefaling er at tage mødetider alvorligt og at sikre, at man ikke skal andre ting i den tid, der er sat af til aktiviteter i lektionsstudiet. Vi-aspektet fremhæves som afgørende for, at alle føler sig trygge og yder sit bedste for teamet og dermed sig selv. Det er et fælles projekt, og lektionsplanen er et fælles produkt.

1) Opstartsmøde

Opstartsmødet er det forum, hvor lektionsstudiet bliver sat i gang. Det er her, de grundlæggende og styrende rammer, fokuser og spørgsmål bliver bestemt. Det er vigtigt senere at holde sig stringent til disse, så vær grundig og pas på ikke at blive for ambitiøs – især de første gange.

Anbefalinger

- Udpeg/inviter en facilitator, dvs. en, der hjælper teamet med at holde fokus.
- Bestem et overordnet tema for lektionsstudiet (fx differentiering, metode eller selvstændig tænkning)
- Diskuter jer frem til et matematikfagligt emne, som I er enige om er vigtigt og indeholder udfordringer for elevernes læring. Udpeg – i henhold til principperne for OEA – hvad, der er fundamental viden inden for emnet (der kan være flere).
- Anvend lærerbøger, Forenklede Fælles Mål, lærervejledninger m.m.
- Afdæk, hvad I ved om emnet og vurder, om I har brug for at vide mere inden planlægningsfasen.
- Definer den tilsigtede viden og udform ét undersøgelsesspørgsmål og fokuser på det – husk at det skal rette sig mod OEA.
- Læg en klar plan for resten af studiet.
- Vælg en tovholder.

Spørgsmål

- Hvad er afstanden mellem det, vores elever kan nu, og det vi vil give dem?
- Hvad vil vi gerne udforske ift. børns læring af et givent emne?
- Hvordan skaffer vi information om det igennem et lektionsstudie?
- Hvad har vi brug for at vide, før vi kan designe en brugbar studielektion?

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2) Forberedelse

Det synlige mål med forberedelsesfasen er en grundigt udtænkt lektionsplan. Lektionsplanen er fælles og skal kunne bruges af alle lærere i teamet. Da lektionsplanen skal bygges op, så eleverne *selv* tilegner sig den tilsigtede viden, er det vigtigt gøre sig forestillinger om, hvordan eleverne reagerer; hvad der kan lede eleverne derhen.

Generelle anbefalinger

- Brug jeres forskelligheder konstruktivt
- Vær dedikeret ift. teamet og jeres plan
- Brug én og samme skabelon til planlægningen
  - Beskriv baggrund for og intention med lektionsplanen
  - Udform en detaljeret drejebog
- Skriv detaljeret, så der ikke opstår misforståelser, og så observatørerne ikke bliver i tvivl om fokus
- Sæt jer godt ind i emnet

Anbefalinger og spørgsmål ift. at udfærdige lektionsplanen

- Undersøg og diskuter i fællesskab, hvad der er det vanskelige ved hhv. at lære og at undervise i den tilsigtede viden
- Overvej og diskuter, hvordan I tror, eleverne kan opnå den tilsigtede viden
  - Hvilke problemer med tilhørende løsninger kan lede eleverne hen til en forståelse af den tilsigtede viden?
  - Hvilke strategier vil vi bruge?
  - Hvilke problemer vil vi støde ind i?
  - Hvilke 'misforståelser' vil eleverne etablere?
  - Hvordan kan evt. misforståelser bruges?

Anbefalinger ift. efterfølgende observation

- Lektionsplanen skal kunne bruges som værktøj til observation og skal indeholde eksplicitte hypoteser og spørgsmål
- Definer tydelige mål for observatørerne
- Definer kriterier og parametre for observatørerne
  - Hvad skal de observere
  - Hvem skal de observere?
  - Højst to fokuspunkter – nedskrives i lektionsplanen!

Uddyb og studer eventuelt mere i fx fag- og tekstbøger, artikler (fagtidsskrifter, web-artikler, forskningsartikler), FFM, jeres årsplan, andre lektionsplaner, anden litteratur osv.

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Ud over ovenstående anbefalinger og hjælpespørgsmål, er her en række fokuspunkter, der kan hjælpe jer til at udforme lektionen med OEA. Denne del af guiden flettes med begreber fra TDS.

Anbefalinger ift. design af det didaktiske miljø

- Fokuser på ét problem, hvor *den tilsigtede viden* ligger i løsning af opgaverne (der kan godt være flere opgaver)
- Problemet skal være åbent i den forstand, at der skal være flere brugbare svar
- Opbyg det *didaktiske miljø* (opgaver, instruktioner, redskaber osv.), så I tror, at det er muligt reelt at *overdrage* ansvaret for at løse opgaven. Ideelt foregår læring i en *adidaktisk situation*, dvs. uden lærerens indblanding.
- Forbered hensigtsmæssig vejledning pba. forventninger om, hvordan elever reagerer, og hvordan de (hver elev!) bedst støttes til maksimalt udbytte.

Anbefalinger ift. design af den didaktiske situation (dvs. undervisningssituationen)

- Det er centralt, at eleverne i en *adidaktisk situation*, får lejlighed til at arbejde selvstændigt ud fra egne forudsætninger og med egne faglige interesser og metoder.
  - Overvej, hvordan opgaverne skal *overdrages*, så eleverne forstår dem fuldt ud og er helt indstillet på hvad de skal – så de *kan* tage ansvaret.
  - Efterprøv jeres problemformulering: Lad én 'overdrage' problemet og lad de øvrige prøve at gøre det, de forestiller sig eleverne ville gøre.
  - I OEA er det typisk, at eleverne skal omformulere et dagligdags problem til et matematisk problem.
- Når den *adidaktiske situation* er etableret, kan elevernes *handlinger* og evt. *formuleringer* observeres. Brug disse til at vurdere elevernes metoder, fremskridt og behovet for hensigtsmæssige vejledninger.
  - Overvej, hvilke opgaver, der minimerer behovet for lærerens indblanding.
  - Overvej, hvilke *handlinger* og *formuleringer* eleverne kan frembringe og forbered, hvordan de kan bruges til at diskutere tænkemåder og forståelse.
- Når elever arbejde selvstændigt og tilvejebringer forskelligartede svar, er det vigtigt at disse svar valideres. I henhold til OEA foregår dette (bl.a.) i en lærerstyret klassediskussion pba. elevernes *formuleringer (hypoteser).*
  - Overvej, hvordan en sådan diskussion ledes hen imod alles forståelse af den tilsigtede viden, og i hvor høj grad læreren skal blande sig i selve diskussionen.
- Sørg for, at eleverne får *bekræftet* deres nytilegnede viden af læreren.
  - Overvej, hvordan læreren kan afslutte timen, så elevernes nyerhvervede viden bekræftes.

Husk at tænke ovenstående ind i mål for observationen.

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3) Studielektion og observation

Studielektionen er helt central i lektionsstudier. Studielektionen er så at sige der, hvor empirien skabes.

Helt overordnet er det vigtigt, at man allerede inden studielektionen har besluttet, hvad der skal observeres, og hvem der observerer hvad eller hvem. Dette skal stå nedfældet i lektionsplanen.

Fokuser på elevernes læring. Kommer man til at evaluere læreren i stedet, misser man pointen – at undersøge elevernes læring – og man skaber utryghed i teamet. Observationer af elevernes *handlinger og formuleringer* er vejen til forståelse af, hvad og hvordan de tænker.

Anbefalinger ift. proceduren.

- Vær helt stille – forstyrrelser forstyrrer!
- Lad være med at interagere med lærer, elever *eller andre observatører*!
  - Mind dine kolleger om det, hvis de glemmer det!
- Undgå 'eksterne' forstyrrelser
  - Gør andre opmærksomme på, at I ikke vil forstyrres.
  - Hæng evt. et skilt på døren, så andre lærere eller skoletandlægen ikke afbryder.
- Cirkuler frit, når eleverne arbejder selvstændigt (dvs. når eleverne arbejder *adidaktisk)*, men stå ellers ude ved siden.
- Pas på med at blokere elevers udsyn.

Anbefalinger ift. indsamling af data

- Tag noter under hele lektionen i henhold til planen.
- Noter løbende elevernes respons på det didaktiske miljø (*handlinger og formuleringer*). Også dem, der er udtryk for misforståelser
- Dokumenter elevernes interaktion med læreren.
- Illustrer, hvordan elever konstruerer deres viden gennem aktiviteter og diskussioner (*formulering* og *validering*).
- Nedskriv variationen i de løsningsforslag, eleverne kommer med.

Suppler eventuelt med foto og video, hvis der er observatører nok (mindst tre).

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4) Refleksion og revision

Refleksionsfasen er kronen på værket. Det er her, vores nye viden om og forståelse af elevernes læring virkelig kan tage form. Lektionsstudiet skal ikke bare munde ud i en konstatering af om eleverne lærte eller ej, men er en undersøgelse af, hvad der gjorde eller ikke gjorde at de lærte – hvordan de lærte. Det er at undersøge, hvad der (tilsyneladende) *virkede*, for at kunne opstille en hypotese om, hvad der *virker*.

Refleksionen er bygget op om en simpel struktur. Det væsentlige – og vanskelige – består i, at analysere lektionen kort og præcist med det rette fokus.

Struktur

1. Facilitatoren indleder med referat af undersøgelsesspørgsmål og hypoteser (facilitatoren tager også referat – i stikordsform).
2. Læreren fremlægger sin oplevelse af undervisningen (iht. lektionsplanen)
3. Teamdeltagere på skift (under ledelse af facilitatoren)
4. Evt. deltagende ekstern vidensperson (fx matematikvejleder)
5. Åben diskussion
6. Facilitatoren afrunder, med resume af evt. ændringer til lektionsplanen (referat overdrages til næste lærer/revisor)

Facilitatoren minder om refleksionsseancens kodeks:

- Det er et fælles lektionsstudie
  - Vi- og ikke jeg-sprog
- Struktur og mødekultur (ingen afbrydelser – tag noter)
- Fokus på elevernes læring ift. lektionsplanen (intentionen)
  - Det gælder også under lærerens fremlæggelse
- Vi er her for at blive klogere og dygtigere – vær kritisk
  - På en konstruktiv måde!

Husk, det kan være meget følsomt at tale om, hvordan eleverne opbygger misforståelser – det kan let opfattes som kritik af læreren – men forståelse af netop dette er essentiel. Træd varsomt!

På baggrund af refleksionen bestemmes eventuelle ændringer i lektionsplanen. Disse tilføjes efterfølgende af den næste lærer, der er på, eller den endelige revisor.

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5) Dokumentation

Dokumentationen har to anvendelsesmuligheder: Internt og eksternt.

Internt tjener dokumentationen det formål at fastholde jeres viden om, hvad I undersøgte, hvordan I gjorde det, og hvad I fandt ud af. I første omgang tjener fastholdelsen det formål at bevidstgøre jer om jeres arbejde – når I i fællesskab vælger ud, hvad der er vigtigt at dokumentere, repeterer I samtidig processen og resultaterne for jer selv. I anden omgang tjener fastholdelsen af essensen af jeres arbejde til på længere sigt at kunne huske, hvad I arbejdede med og hvordan.

Eksternt kan dokumentationen bruges til vidensdeling med nære – og fjerne! - kolleger eller ligestillede. I forhold til vores projekt er det oplagt at dele viden og erfaringer med de øvrige teams, men det kunne også være relevant at dele med øvrige kolleger, eller andre interesserede personer, der ikke er med i projektet.

I lektionsplanen står, hvad I nåede frem til, men der står intet om, hvordan I nåede frem til det og intet om, hvilke udfordringer i stod over for undervejs, og hvordan I kom omkring dem. Notatet skal dokumentere det væsentlige i det af jeres arbejde, der ikke er synligt i lektionsplanen.

Den dokumentation, vi lægger op til her, består af

- Jeres lektionsplan
- Referater fra refleksionsseancerne
- Et afsluttende notat over jeres arbejde.
- Personlige refleksioner.
- Bilag: De to første lektionsplaner.

Notatet bør beskrive

1. Processen
    - Hvordan startede i ud?
    - Hvilke gode beslutninger tog I?
    - Hvilke beslutninger var uhensigtsmæssige?
    - Hvorfor tog I de beslutninger?
    - Hvilke udfordringer løb I ind i?
    - Hvordan løste I dem?
    - Hvilke overraskelser fik I?
2. En konklusion
    - Et sammenfattende svar på jeres undersøgelsesspørgsmål.

Personlige refleksioner kan være et kort notat om generelle eller specifikke tanker eller opdagelser.

6) Skabelon

Team

Skole: Klasse(r): Team:

Lektionsstudie

Titel: Emne: Tilsigtet viden: Undersøgelses- spørgsmål:

Studielektion

Underviser: Mål:

Fokuspunkter for observation

Baggrund, hypoteser og spørgsmål

Hvorfor har vi valgt dette emne for et lektionsstudie?

Hvorfor giver det mening ift. elevernes læring at have denne lektion nu?

Hvorfor har vi valgt denne opbygning af lektionen?

Hvorfor har vi valgt de beskrevne aktiviteter?

Drejebog

Tid	Hvad skal der ske	Forventninger til, hvordan eleverne reagerer på det didaktiske miljø (opgaver, lærers instruktion osv.)	Hvad læreren skal gøre og sige	Udtryk for hvordan eleverne tænker matematik (handlinger, formuleringer)

Refleksionsnoter

Bilag

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Bilag

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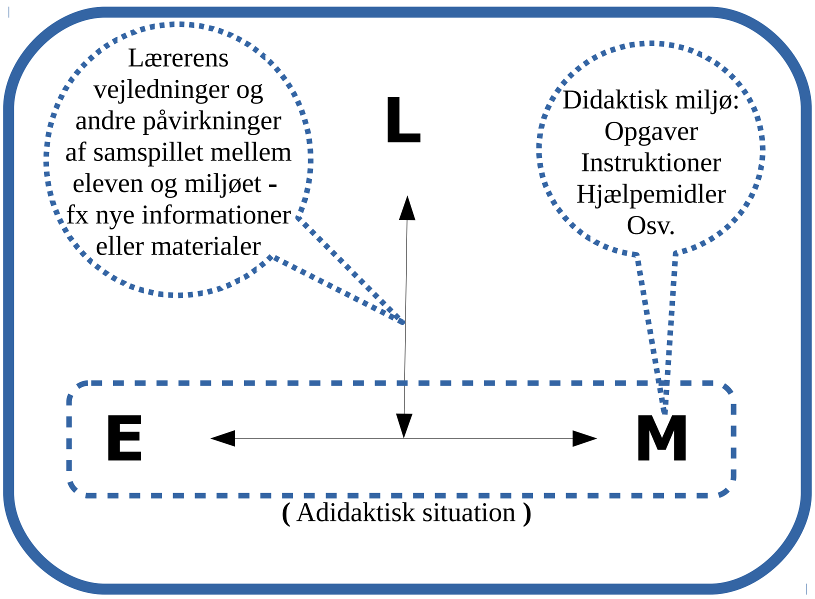
TDS – Teorien om didaktiske situation

	Lærerens rolle	**Elevernes rolle**	**Didaktiske miljø**	**Situation**
Overdragelse	Igangsætte afklare	Modtage og forstå opgave	Etableres	Didaktisk
Handling	Observere Reflektere	Handle Reflektere	Problemfelt Udforskningsfelt	Adidaktisk
Formulering	Organisere Spørge	Formulere Præcisere	Åben diskussion	Adidaktisk eller didaktisk
Validering	Lytte Evaluere	Argumentere Reflektere	Styret diskussion Bedømmelse	Normalt didaktisk
Bekræftelse	Præsentere Forklare	Lytte Reflektere	Officiel viden	Didaktisk

Modificeret efter figur 11 i Didaktiske elementer af Carl Winsløw (2006)

Didaktisk situation



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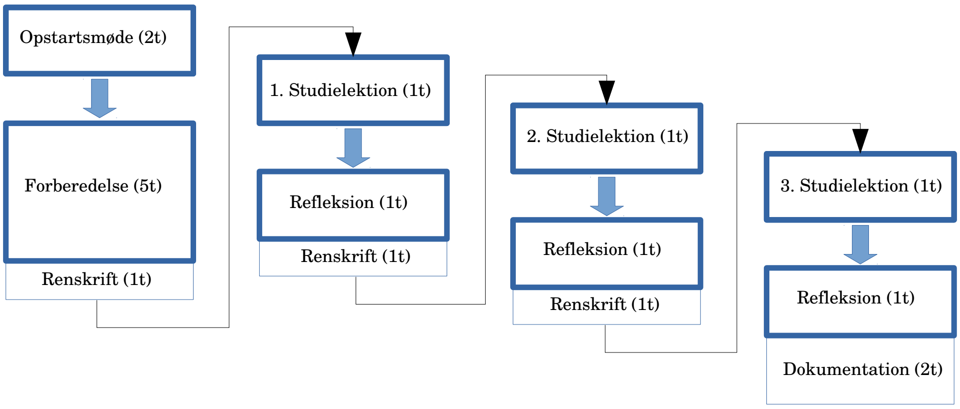
Årsplan 2015

	skolestart		OPSTARTSSEM							efterårsferie					jb kursus	EVALUERING				juleferie	juleferie
uge	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
B			O	X	X												X	X			
A			O							jb japan			X	X							
C			O			X	X														

Årsplan 2016

			jb konference			vinterferie	EVALUERING				påskeferie						kr.himmelfart!				AFSLUTNINGSSEM			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
								X	X												A			
	X	X											X	X							A			
	-			X	X											X	X				A			

Model for lektionsstudieforløb



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Dedication

To the teacher
Courage is contagious