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INSTITUT FOR NATURFAGENES DIDAKTIK

The Mathematics of Map Projections

An Analysis and Design of Teaching Material for an Interdisciplinary Lesson

Josefine Dixen Zwisler
Speciale i Matematik & Didaktik

Vejledere:
Carl Winsløw

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Abstract

This is a theoretical thesis in Mathematics Education which examines how map projections, as an interdisciplinary topic between mathematics and geography, can be communicated to Danish upper secondary students. Given the limited literature on teaching map projections in Danish upper secondary schools, the thesis focuses on transposing scholarly knowledge on the subject into actual teaching material. To do so, Didactical Engineering is used as the methodology, with emphasis on its preliminary analyses as the foundation of the design. The analyses include an epistemological analysis of the mathematics at stake, an institutional analysis identifying potential conditions and constrains affecting the development of the teaching material, as well as an didactical analysis about existsing didactic literature, which relates to the thesis. Based on these analyses, the Theory of Didactical Situations is used as a framework for the design process.

The thesis demonstrates, through the example of the teaching material, how and what aspects of the theory of map projections can be communicated. It suggests that as an interdisciplinary topic, the geographical aspects of map projection and the use of maps, can serve as a meaningful way to present the theory of map projections, as a transition to the mathematics of map projections. In addition to this, the design suggests that most of the scholarly knowledge about map projections, especially the mathematics, has the potential to be transposed into concepts mentioned in the curricula and textbooks, and provides an example hereof. However, the analyses also suggest that there might be some limitations to how much that can be transposed. The designed teaching material then provides an example on how these limitations can be addressed differently.

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1 Introduction

Interdisciplinarity between the natural science subjects in upper secondary schools in Denmark has been part of the school's organization since the 2005 reform. At that time, it was introduced into the curriculum as specific interdisciplinary courses, particularly the course *Almen Studieforberedelse* (abbreviated AT), but also through the course *Naturvidenskabelig Grundforløb* (Hansen, 2007). However, with the 2017 reform, AT was abolished. Nevertheless, interdisciplinarity remains a requirement and is explicitly stated in the curricula, though it is now up to the individual upper secondary school to incorporate this and how the collaboration is carried out (Rasmussen, 2021). According to Rasmussen (2021), the removal of AT may have led to increased uncertainty among teachers regarding interdisciplinary teaching, as AT provided a structured framework. However, it may also have granted greater flexibility. This thesis does not assess the merits of AT, or of interdisciplinarity in upper secondary schools, but instead explores how an interdisciplinary lesson could be designed and take form.

The interdisciplinarity will be between my major, mathematics, and my minor, geography. Such as other natural sciences, geography is not isolated from mathematics. Here, mathematics functions mainly as a practice-oriented tool, naturally integrated into methods for processing, analyzing, and calculating geoscientific data. In my view, there are numerous opportunities for interdisciplinarity between mathematics and geography — the only thing I was missing was a topic where it was evident that mathematics serves not only as a tool for calculations but also as a foundation for reasoning.

I would like to thank my supervisor, Professor Carl Winsløw, for steering me in the direction of **map projections**. Map projections are the method for producing maps, for which maps are one of the most important tools in geography, used to organize and analyse spatial data and to visualize geographical phenomena. More interesting, especially in regards to world maps, map projections will always distort the Earth's surface. As we will see, mathematics forms the foundation of map projections, making it a strong topic for interdisciplinarity between mathematics and geography. To guide the work of this thesis, I have formulated the following research question:

”*How and to what extent can map projections, as an interdisciplinary topic between mathematics and geography, be communicated to Danish upper secondary students?*”

The assumption is that teaching map projections, whether it is interdisciplinary or not, is not particularly widespread in Danish upper secondary schools (if even existing). Thus, in order to investigate the research question, I have designed my own version of a teaching material on map projections, which could frame a potential interdisciplinary lesson. To do so, I have made use of the research methodology Didactical Engineering, especially its preliminary analyses and a priori analysis, and the Theory of Didactical Situations to guide the design and shape the teaching material.

As a preparation to the design, I have studied the theory of map projections, for which I have selected and included in this thesis. Thus, this thesis is an *external didactic transposition* process. The *external didactic transposition* (abbreviated EDT) is part of the *didactic transposition*, a key notion from the Anthropological Theory of Didactics, or just ATD. Didactic transposition refers to the process of transforming *scholarly knowledge* (scientific knowledge, knowledge produced by research communities, like at universities) into *knowledge to be taught* (proposed knowledge to be taught at schools). The *knowledge to be taught* is then transposed into *knowledge actually taught* by teachers in a teaching situation, which is then assimilated by students as *learned/available knowledge* (Chevallard and Bosch, 2020). The EDT concerns the selection, modification, and organisation of *knowledge to be taught*, starting from *scholarly knowledge* and culminating in the development of curricula and teaching materials that will be used in a specific course (Bosch et al., 2021). Even though ATD provides the notation of external didactic transposition to describe the work of and structure of this thesis, ATD will not be part of the didactical framework that surround the thesis.

The thesis is structured as follows: first, a brief description of both the Theory of Didactical Situations as well as the method of Didactical Engineering. Then a presentation of the selected scholarly knowledge on map projections. Afterwards I will preform the preliminary analyses, posed by Didactical Engineering, followed by a conception and a priori analysis, where I describe the design and choices I have made, regarding the designed teaching material. Since I will not test the teaching material through an actual lesson, there will be no empirical data to analyse or discuss, hence the discussion that follows will focus on the choices made in the design-process. The thesis will end with a conclusion, in which the research question is answered.

2 Didactical framework

2.1 Theory of Didactical Situations

The Theory of Didactical Situations (TDS in the following) is a well known and used theory in mathematics educations, developed in the 1960's by Guy Brousseau (González-Martín et al., 2014). There exists many texts about TDS and much research has been done, based on TDS. However, this section will be a short introduction to TDS, presenting only some of the concepts and aspects from TDS, which will be used designing of the the teaching material. For a more detailed description of TDS, I would recommend reading the translated works of Brousseau (1997).

According to González-Martín et al. (2014, p. 118), "*TDS is a theory that analyses variables of teaching practice and explores their relationship with the production of mathematical knowledge*". In addition to this, TDS provide some conceptual tools to construct and analyse *Situation(s)*, that is "*the ideal model of the system of relationships between students, a teacher, and a mathematical milieu*" (González-Martín et al., 2014, p. 117). *Situations* can be modelled according to two levels; an *adidactical* level and a *didactical* level (González-Martín et al., 2014).

At the *adidactical* level, the focus is on the students interactions with the *milieu*, and the feedbacks from the *milieu*, which helps the students to form strategies for solving and producing new knowledge (González-Martín et al., 2014). In line with Brousseau (1997), in an *adidactical situation*, the teacher leave the students to work on the problems independently, without much interference from the teacher. One can identify *adidactical situations* based on the activity, which lead to knowledge production; Brousseau (1997) calls these situation of *action*, of *formulation* and of *validation*.

In the situation of *action*, the students forms strategies to solve problems, or in general, gaining new knowledge through physically interacting with the milieu. In the situation of *formulation*, the students formulate and discuss answers or strategies for solving, progressively adapting an appropriate language. However, the students reasoning are often insufficient or incorrect, hence situation of *validation* must lead the students to discuss their implicit validations, get them to evolve, redefine or replace their theory with the right one (Brousseau, 1997). The targeted knowledge can be understood through one or more *adidactical situations* which preserves the true meaning (the epistemology) of the mathematical knowlegde at stake, what Brousseau (1997) calls *fundamental situation*(*s*). Only when the student can apply and relate the mathematical knowledge to the real world or different contexts, outside the classroom, the student has truly acquired the knowledge (Brousseau, 1997). Furthermore, as González-Martín et al. (2014) notes, the feedback provided by the *milieu* may sometimes be insufficient to ensure full adidactivity, hence the teacher can not disappear completely from the situation and must step in and provide additional tools and/or rules.

The teacher's role become more explicit at the *didactical* level, which include *devolution* and *institutionalisation* (González-Martín et al., 2014). In *devolution*, the teacher places the students in a *adidactical* situation by devolving (think of it as delegating or "turn over") the broader situation, involving the mathematical knowledge at stake, to a problem or situation that enable the students to work independently and which provides meaningful interaction. Through the *institutionalisation*, the teacher connects the students works with the scientific knowledge and the didactical project (Brousseau, 1997).

2.2 Didactical Engineering

According to Artigue (2014), Didactical Engineering (abbriviated DE) is a research method for designing and analysing classroom realizations, developed in the 1980's in close connection to TDS, hence "*this theory became [...] the natural support of DE*" (Artigue, 2014, p. 468) or as González-Martín et al. (2014) notes, "*DE relies on TDS to implement Situations which aim to to give students maximal responsibility in producing new mathematical object and techniques which appear as optimal mathematical tools to the problems they are given*" (González-Martín et al., 2014, p. 121). The methodology consists of four main phases: preliminary analyses; conception and *a priori* analysis; realization, observation and data collection; an *a posteriori* analysis including validation. In DE, the validation is internal and based on the comparison between *a priori* and *a posteriori* analyses, compared to other more traditional theories of didactics in mathematics education (Artigue, 2014).

The preliminary analyses consists of three dimensions: an epistemological analysis, an institutional analysis and a didactical analysis (Artigue, 2014). The epistemological analysis examines the mathematical content at stake, often including its historical development, which helps the researcher to define what the students should learn. This include identifying *epistemological obstacles*, that might lead to misconceptions about the mathematical content among the students. Epistemological obstacles arise when prior knowledge interferes with learning new concepts. Artigue (2014) notes, this is important when later identifying fundamental situations. The institutional analysis takes into account the institutional conditions and constraints that surround the DE. According to Artigue (2014), these conditions and constraints may be situated at different levels, from the curricular choices regarding the teaching, available resources, evaluation, etc. to the general curricular choices regarding the chosen content. In relation to this, Artigue (2014) states that it is important to understand that the (current) curricular choices and organizations also has undergone a historical development, which also needs to considered when identifying the conditions and constrains. Lastly, the didactical analysis is the research for other studies involving the (mathematical) content at stake. This can be used as an inspiration or guide for the following design (Artigue, 2014).

Conception and the *a priori* analysis is about the choices made in the design, how they relate to the preliminary analysis, identifying main didactic variables for each situation and posing conjectures about how students might react with the milieu. Artigue (2014) distinguishes between choices of different levels; *macro-choices*, which guide the overall design, and *micro-choices*, which affect a specific situation. These choices lead to identification of the *macro-didactic* and *micro-didactic variables*. The conception and the *a priori* analysis is an important part of DE, as this is where research hypotheses are formulated and held up against theoretical didactical situations (Artigue, 2014).

The last two phases in DE is about the realization, data collection and an *a posteriori* analysis. Since this is a theoretical thesis, I will not conduct a lesson, hence the these phases will not be of much relevance in this thesis. Nonetheless, the realization and data collection phase is about putting the researchers work made in the preliminary and *a priori* to a test in a classroom and observing and collecting the students' work (Artigue, 2014). During this, the researcher is able to make changes to and adapt the design, which is important for the following *a posteriori* analysis. In the *a posteriori* analysis, data and observations are interpreted and held up against the hypotheses posed in the *a priori* analysis, from which the hypotheses are evaluated (Artigue, 2014).

What I have presented are the main characteristics of DE as a research method, which I find relevant for my thesis. As Artigue (2014) notes, these principles are not as rigid as described here; the methodology is very versatile, and can be used to study other aspects of mathematics education beyond just lesson design.

3 A brief history of map production and map projections

The purpose of this section is to provide a brief introduction to the historical development of map production, called *cartography*, and map projections [1]. This section is (primarily) based on Snyder's *Flattening the Earth* (1993), and you are interested in more details of this development, I recommend looking into *Flattening the Earth*, which also contains detailed descriptions of all known map projections throughout history.

Map projections is the method for producing maps, that is transforming the round Earth so it can be displayed on a flat surface, and can be be traced back to ancient Greece. Among the ancient Greeks, there was a broad consensus that the Earth was round, but also the sky that surround the Earth was round, and for a long time, mathematics, astronomy and map projections were closely linked. Hence, the development of projections was primarily to construct star maps. However, there are examples of maps of the Earth's surface dating back to this period, and the early map makers were aware that there were some limitations when trying to display the round Earth on a flat surface.

Nevertheless, they still tried. For this, and much like we do today, there was a need to develop an artificial grid of lines of latitude and longitude, which could be used to define locations on the Earth. Here, the Greek astronomer and mathematician Hipparchus (c. 190-126 BCE) formalized a system of longitude and latitude, which inspired Ptolemy's works on maps of the Earth's surface. Ptolemy (c. 100-178 CE) was a highly influential figure, which made a significant impact on (among other things) cartography, especially with his *Geography*, in which he standardized the system of *meridians* and *parallels* (see p. 10 for more information of these) - Figure 1 shows a reconstruction one of Ptolemy's world maps. But the use of mathematics in constructing maps were very limited.

[1] To study the history of the content at stake can be proved useful in an epistemological analysis.

Figure 1: A reconstruction of Ptolemy's world map, c. 1500 (Wikipedia, 2025).

Although Ptolemy and a handful of others had provided fairly detailed descriptions for constructing maps of the Earth's surface, it seems that maps produced in the period before the Renaissance, i.e., during the Middle Ages, were more rooted in philosophy. With the Renaissance (1470–1669), a new era of map production and projections emerged. The Renaissance was a period of great geographical discoveries, leading to an increasing demand for more accurate maps of the Earth's surface. The relatively simple method of constructing maps resulted in distortions of shapes and, more importantly, failed to represent lines of constant bearing, called *rhumbs*, as straight lines. This made existing maps rather impractical for navigation. This issue led Gerard Mercator (1512–1594) to develop his projection, now known as the Mercator projection, which was used by sailors for many years to come. However, how he actually accomplished this remains a subject of debate.

Whilst the Renaissance brought many mathematical advancements and new discoveries[2], mathematics still played a minor role in map projections. It was not until after the Renaissance that we see a greater focus on mathematics in map production. Specifically, J.H. Lambert (1728-1777) revolutionized map production by using the newly developed calculus in his map projections, making it possible to create projections that preserved certain properties of the Earth's surface. Furthermore, the Earth itself became subjected to the mathematical analysis, as well as the development of tables of logarithms made computations easier. In addition to this, more accurate measurements of the Earth resulted in the conclusion that the Earth's shape approximates closer to a rotating, flattened ellipsoid (also known as a *spheroid*) rather than a perfect sphere. All of this resulted in the mathematics behind map projections becoming more advanced as well as more precision when mapping the Earth.

The works of Lambert showed there was room for advancement in the field of cartography, and with the 19th century brought one of the greatest contributions: the foundation of map projections on firm mathematical principles. How and why this happened, Snyder (1987) does not explicitly states, but one could infer that the introduction of analytical geometry in the 17th century, including coordinate systems, made it possible to describe spherical coordinates and transformations algebraically (Katz, 2014). And the development continue to today. The modern day cartography deserves a whole section itself, which I will not provide. Today, computer programs like GIS (Geographic Information Systems) have made it possible to solve geometric problems directly from large databases, containing spatial and geographic information. However, even though much has become digitized, Lapaine (2017) argues that maps and map projections are still very important for every GIS, used for presenting the output data; it is important to be familiar with the map projection, their formulas and their origin, and thus Lapaine (2017) concludes, and which I will end this section with: "*Hence, the computer aided method in the map production and first of all GISs have not reduced, but increased the importance of map projections*" (2017, p. 254).

[2] This includes the further development of plane and spherical geometry, but also a greater focus on algebra for calculations and problem-solving, the beginnings of analytical geometry, and the invention of calculus (Katz, 2014)

4 The theory of map projections

The following section contains two main parts; a more ”geographical” aspect on map projection, which include the classes and characteristics of map projections, and the mathematical construction of map projections, including geometry of the sphere and conversion formulas. But first, a short (also rather ”geographical”) presentation of geographic coordinates, which is important for understanding map projection classes and characteristics.

4.1 The geographic coordinate system

The *geographic coordinate system* is a spherical, 3-dimensional coordinate system used to define locations on the Earth's surface (Esri, 2021). The locations are expressed by *latitude* and *longitude*, both angles measured in degrees from the center of the Earth to the location on the surface (Figure 2). Then these can be visualised on the *reference model* (a model of the Earth - see p. 11 for more); the latitudes are visualised by lines, or in fact circles, parallel to the equatorial line (or circle, often referred to as the Equator), called the *parallels* of latitude, and the longitudes visualised by curved lines going from north-south, called the *meridians* of longitude. The zero latitude is along the equatorial line and the zero longitude is defined as the Greenwich meridian (or Prime meridian), the meridian passing through Greenwich in England (Esri, 2021). Locations on the Earth can then be expressed using the latitude and longitude of that location. The longitude is measured from 0° (the equatorial line) to 90° followed by either N or S to note whether the location is north or south of the equatorial line, respectively, and the longitude is measured from 0° (the Greenwich meridian) to 180° followed by either E or W to note whether it is east or west from the Greenwich meridian, respectively (Bolstad, 2012). As we will see later, conversion from geographic coordinates to planar Cartesian coordinates is an important part when doing map projections.

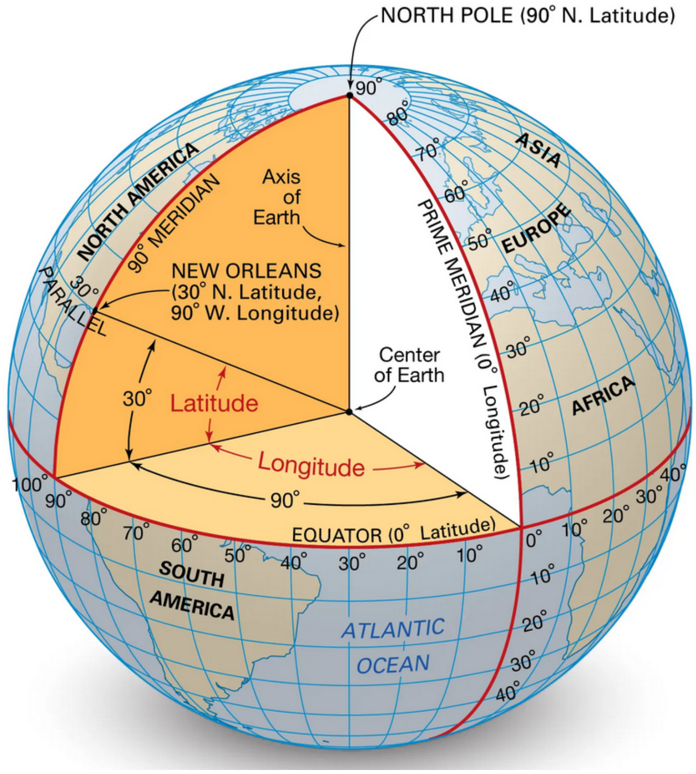

Figure 2: The latitude and longitude, as well as the *parallels* and *meridians* (The Editors of Encyclopedia Britannica, 2025).

4.2 What kind of map projections are there?

4.2.1 The classes of map projections

Map projections can be understood geometrically as projections from a *reference surface* onto the surface of a geometrical shape, which is tangent to the *reference model* (Small Farm Link, 2024). Here, the *reference surface* is the surface of a *reference model*, which I define to be the mathematical model, used to represent the Earth. The shape of the Earth is complex, the surface is not smooth and in reality there does not exists a perfect model of Earth. Therefore, when doing projections, one must decide on which model to use, which can be expressed mathematically. Some may use the terms *reference globe*, *reference sphere* or *reference ellipsoid* to refer to the spherical shape of Earth (Robinson and The Committee on Map Projections, 2017; Bolstad, 2012). I will stick to *reference model* for a more general term. Later on, I will specify which model I will use.

The geometrical shape must be able to be "cut open" and made into a flat, 2-dimensional surface (a map), without additional distortion[3]. One way of classifying map projections is based on the geometrical shape, the reference surface is projected onto (Small Farm Link, 2024). The three main geometrical shapes are a plane, a cylinder and a cone, which define the three main classes of map projections: *azimuthal (perspective) projections, cylindrical projections* and *conic projections* (Small Farm Link, 2024; Robinson and The Committee on Map Projections, 2017). There exists map projections, which does not fall under these classes. However, I will only present these three classes of map projections.

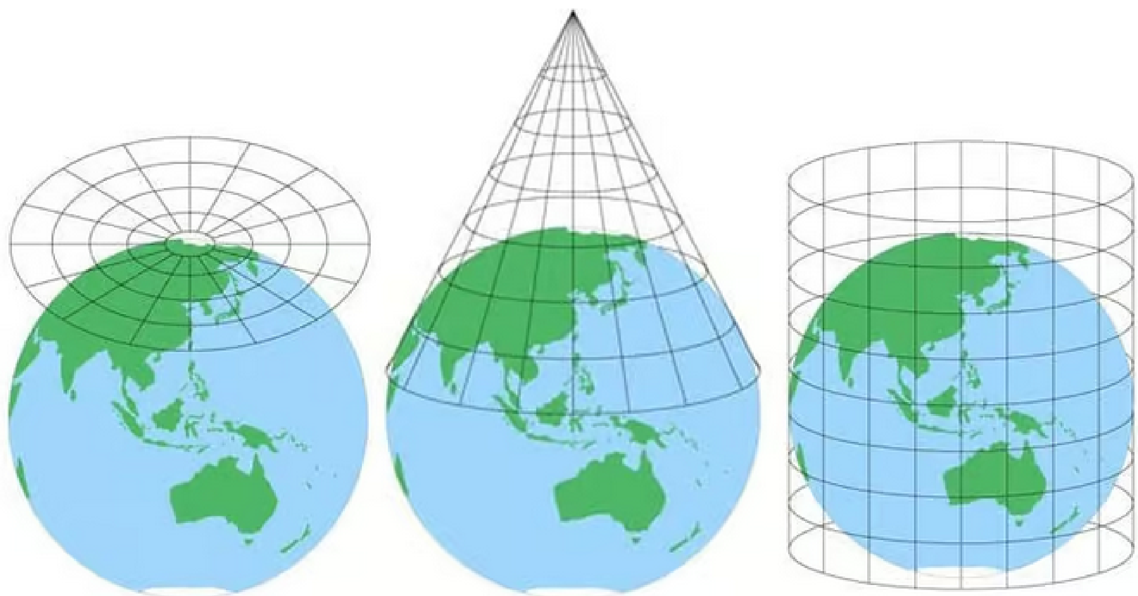

Figure 3: The three different classes of map projections. Starting from left: azimuthal, conical, cylindrical (Anderson and Kessler, nd).

[3]This is why it is sometimes mentioned as a *developable surface* within the theory of map projections.

The class of *azimuthal (perspective) projections* is when the reference surface is projected onto a simple, 2-dimensional plane (Figure 3). A plane is made tangent to some point, *standard point* (or *point of tangency* (Small Farm Link, 2024)), and the point of perspective is either center of the reference model, the antipodal point[4] of the standard point on the reference surface or placed at infinity (Small Farm Link, 2024). The projections are named *azimuthal* since they preserve the *azimuth* angle (Figure 4), that is "*the angle, in degrees from north, between the great circle arc and the meridian*" (Robinson and The Committee on Map Projections, 2017, p. 27) - see p. 18 for definition of great circle. The outline of the map will become circular with great distortion, the further away from the standard point one get (Robinson and The Committee on Map Projections, 2017).

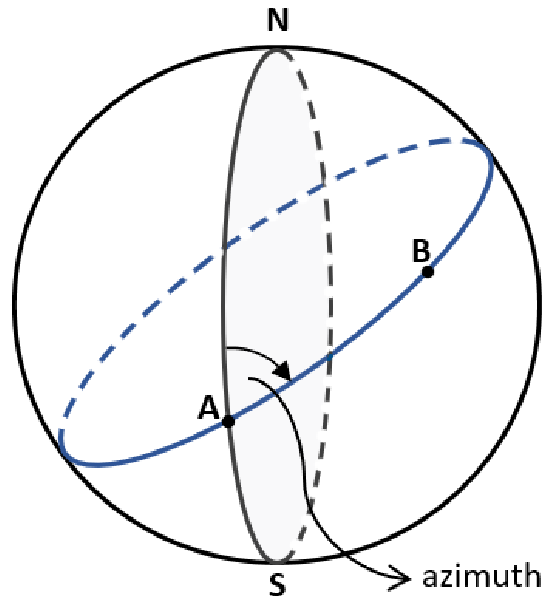

Figure 4: Visual representation of the azimuth angle. Own illustration.

In the class of *cylindrical projections*, the reference surface is projected onto a cylinder that is "wrapped around" the reference model (Figure 3). The cylinder is tangent to the reference model at a *standard line*, hence the diameter of the cylinder can be thought of as the same as the reference model's (Small Farm Link, 2024). The outline of the map will become rectangular, and can be made to preserve different kind of characteristics which result in different kind of distortions (Robinson and The Committee on Map Projections, 2017).

With the class of *conic projections*, the reference model is projected onto a cone, which is placed over the reference model and made tangent along a standard line (Small Farm Link, 2024) (Figure 3). Conic projections are quite complex, hence not a class of map projections I will use more time on in this thesis, but I still think it is important to mention the existence. For more details about conic projections, I

[4]Antipodal points: two points that are diametrically opposite.

would recommend Snyder's *Map Projections: A Working Manual* (1987), pp. 97-140.

Lastly, I would like to note that the different kind of classes of map projections can further be divided by the types of *aspects*. Firstly, I have mentioned the plane, the cylinder or the cone is tangent to the reference model. There are also a secant case, where they cut the reference model at two places (Small Farm Link, 2024). Then there are the *normal, transverse* and *oblique* aspects, which does not have an distinct definition but depends on the class (Small Farm Link, 2024). However, I will not focus on other aspects than the tangent and normal. For the azimuthal (perspective) projections, this means that the polar aspect is the normal aspect, i.e. the plane is tangent to the North Pole (or South Pole) and the meridians is projected to straight lines that "ray" from the North Pole (Small Farm Link, 2024). For the cylindrical projections, the normal aspect is the equatorial aspect, i.e. the cylinder is tangent to the Equator, and the parallels and meridians are projected to straight lines, intersecting each other perpendicular (Small Farm Link, 2024).

4.2.2 The characteristics of map projections

Map projections can further be classified based on what they preserve from the reference surface when transforming it onto a flat surface, so called characteristics of map projections. Since no projection can preserve all the characteristics, each type introduces distortions in different ways.

Firstly, map projections can be *equal-area* (or *equivalent*), meaning that regions on the map maintain their correct relative sizes. However, shapes gets distorted, especially near the edges of the map. In many cases, angles and shapes of landmasses are significantly altered. Despite these distortions, equal-area projections are valuable for applications where accurate representation of spatial distributions and relative sizes is essential, such as in thematic maps displaying population density or land use (Small Farm Link, 2024; Robinson and The Committee on Map Projections, 2017).

Then there are map projections which preserve local angles; these are called *conformal* map projections. Preserving local angles and shapes means that small (infinitesimal) features retain their correct form. However, when maintaining local angles, these projections significantly distort area; regions far from the standard lines or points appear much larger than they are in reality. This characteristic makes conformal projections especially useful for navigation and meteorology, where preserving direction and local shape is more important than preserving size. Notably, no map projection can be both conformal and equal-area (Small Farm Link, 2024; Robinson and The Committee on Map Projections, 2017).

Some map projections may preserve some distances/lengths and these are called *equidistant* map projections. Equidistant projections preserve distances along specific lines or from specific points, but they do not maintain area or shape across the entire map. In most cases, distances are preserved either from a standard point to all other points or along specific lines, such as meridians or parallels. These projections are commonly used in applications where accurate measurement of distances from a particular location is important, such as airline route maps (Small Farm Link, 2024; Robinson and The Committee on Map Projections, 2017).

Then there exists map projections which preserve the azimuth angle, as already mentioned in the previous about the class of azimuthal projections. Azimuthal projections maintain accurate directions (azimuths) from a central point to all other points on the map (why also called *true-direction* projections). This makes them particularly useful for navigation, especially in aviation and maritime contexts. Depending on how they are constructed, azimuthal projections can also be equal-area, conformal, or equidistant (Small Farm Link, 2024; Robinson and The Committee on Map Projections, 2017).

In this thesis, I will present Fajstrup's (2006) selection of map projections: the three azimuthal (perspective) map projections *gnomonic, stereographic* and *orthographic*, as well as the two cylindrical map projections, *Lambert's equal-area* and *central cylindrical.* I will also include the *Mercator projection*, another cylindrical map projection. In the following table, I have presented the six map projections of focus, based on Snyder's (1987) presentation of the map projections[5]. However, Snyder does not include the central cylindrical projection in *Map Projections: A Working Manual* (1987), but he does so in this *Flattening the Earth* (1993), which I then have used.

[5]Only some of the characteristics and features, noted in Snyder (1987), are included in the table, that is the characteristics and features I find relevant to my presentation of the map projections.

Name	**Class**	**Characteristics and features**	**Distortion**	**Usage**
Gnomonic.	Azimuthal.	Preserves azimuth angle. No distortion at standard point. All great circles are shown as straight lines.	Distortion of angles (not conformal), areas and lengths (except the great circles).	Star maps or to show great circle paths.
Stereo-graphic.	Azimuthal.	Preserves azimuth angle and is conformal. No distortion at standard point.	Distortion of areas and lengths.	To show only one hemisphere. Most used for polar maps.
Ortho-graphic.	Azimuthal.	Preserves azimuth angle. No distortion at standard point.	Distortions of angles (not conformal), areas and lengths.	To show only one hemisphere.
Lambert's equal-area.	Cylindrical.	Equal-area. Meridians and parallels are shown as straight lines at normal aspect.	Distortion of angles (not conformal), directions and lengths.	Rarely used.
Central cylindrical.	Cylindrical.	Meridians shown as equally spaced straight lines. Parallels shown as unequally spaced straight lines.	Distortion of angles (not conformal), azimuth angle, areas and lengths.	Not used for much, only a textbook example.

Mercator.	Cylindrical.	Conformal. Meridians shown as equally spaced straight lines. Parallels shown as unequally spaced straight lines.	Distortion of areas and lengths.	For navigation. Often used for world maps.

Table 1: Some map projections and their class, characteristics and features, distortions and usage.

I will later present some examples of maps constructed using these map projections. Having now presented some concepts to describe distortions on maps, we can now move on to presenting the mathematics of map projections and why map projections can not preserve all the characteristics.

4.3 The mathematics behind map projections

The primary source I have used for the mathematical theory is Lisbeth Fajstrup's lecture notes *Kortprojektioner og forvanskinger* (2006) for the land surveying program at Aalborg University. This is clearly seen in the structure of the theory. To make the following sections more readable, without constantly referring to the same source, I want to clarify from the beginning that this is not a theory I ”came up with myself”, but one attributed to Fajstrup's notes[6]. However, there will be instances where I felt Fajstrup's notes were insufficient in some way, and I have used other sources for further exploration. I will make this clear by explicitly referencing these sources in the text.

4.3.1 Spherical geometry

Because Earth has a spherical shape, spherical geometry is crucial in the study of map projections. No matter what (spherical) reference model one chooses, spherical geometry lies as the foundation when doing map projections. Moving forward, I will use the sphere with radius r as the reference model of the Earth. The sphere is the most used reference model of the Earth, beside the ellipsoid. Fajstrup (2006) does present the theory for both the ellipsoid and the sphere. However, I prefer the sphere for simplicity. Then spherical geometry provides the foundation to calculating distances and angles on the sphere, and when transforming points on the sphere to the plane.

Just like planar, Euclidean geometry is the study of point, lines and angles (among others) in the plane, spherical geometry revolves around points, angles and lines on the sphere. But lines on the sphere are not like lines in the plane, it is curves. A curve on the sphere can be seen as a segment of a *great circle* on the sphere (called *great circle arcs*), and are the most important circles on the sphere, since they are used to define many features on the sphere's surface:

Definition 1. *A great circle is a circle on the sphere that appears as the intersection curve between a plane passing through the sphere's center O and the surface of the sphere. Its radius r is the same as the sphere's.*

[6] Fajstrup's (2006) lecture notes are in Danish, hence everything I have included in this thesis from the notes have been translated to English by me.

Figure 5 shows a great circle on a sphere. From this definition, it is clear that:

- Great circles are the largest circles on the sphere.
- Infinitely many great circles pass through two antipodal points. Conversely, only one great circle passes through two non-antipodal points.
- The equatorial line make up a great circle and meridians are great circle arcs.
- The shortest path between two points on a sphere is along the great circle arc, connecting the two points.

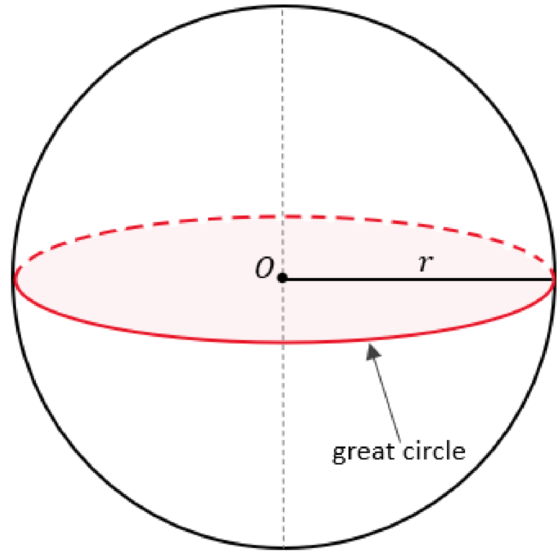

Figure 5: An example of a great circle (red) on a sphere. Own illustration.

The second and third point can easily be argued:

The equator is a great circle: Place the sphere of radius r in a 3-dimensional Cartesian coordinate system, then the sphere can be expressed as $x^2+y^2+z^2=r^2$. The equator can be defined as the set of points where $z=0$, or the intersection of the sphere with the plane given by $z=0$. Since the center of the sphere is $(0,0,0)$, which also lies on the plane $z=0$, it is clear from the definition that the equator is indeed a great circle.

Infinitely many great circles passes through two antipodal points...: Let A and B be to antipodal points, then there is a straight line from A to B which passes through the center of the sphere. The plane containing the line, also intersects the center, and hence the intersection of the plane with the sphere must be a great circle by definition. Since there are infinitely many planes containing this line, there must be infinitely many great circles that passes through the two antipodal points A and B. However, if A and B is not anitpodal, then A and B cannot be be connected with a straight line that passes the center. Hence the plane that intersects A and B and cointain the line, and which passes through the center, must be unique.

The last point requires a fair amount of mathematics, which I will omit in this project (like Fajstrup (2006) also notes). However, I encourage to read Andersen's proof for *Geodesics on a spherical surface* (Andersen, nd), in which he proves that the geodesic[7] on a sphere is indeed a great circle arc.

Then there is the angles on the sphere. In this thesis, angles will be measured in radians. By definition 1, great circles are obtained by planes that passes through the center of the sphere and intersects with the surface of the sphere. Two great circles intersect in some point (or in reality two antipodal points) and will create an angle on the sphere [8]:

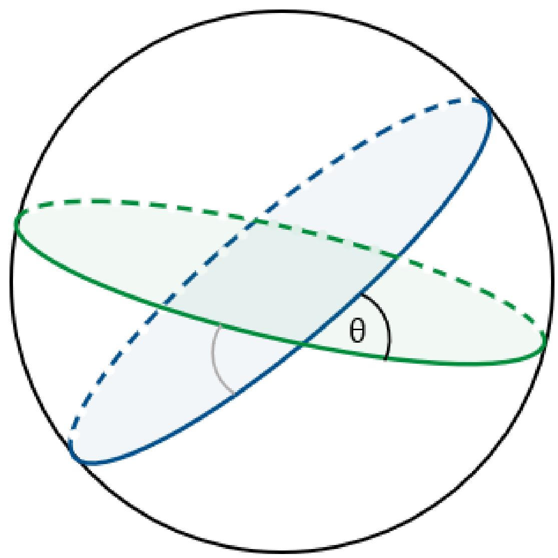

Figure 6: Visual representation of an angle θ sphere. Own illustration.

Definition 2. *An angle θ on the sphere is measured to be the acute angle between the two planes, posed by the great circles. The area spanned by the acute angle is called a lune*[9].

As we see in Figure 6, the intersection of two great circles produce not only one but two identical acute angles (as well as two identical obtuse angles, but we are not interested in these), hence creating another lune on the other side of the sphere.

Determining the acute angle between two great circles is the same as determining the acute angle between the two planes, induced by the great circles. This is similar to determining the angle between two vectors in 2-dimensions, but with the normal

[7]A local length-minimizing curve.

[8]Fajstrup (2006) does not go into much detail about angles on the sphere; this is my own elaboration.

[9]Or sometimes a *digon* (Encyclopedia of Mathematics, 2014).

vectors, $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, of the two planes:

$$\cos\theta = \frac{\overrightarrow{n_1}\cdot\overrightarrow{n_1}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$$

Now imagine three great circles on the sphere, pairwise intersecting, creating a *spherical triangle* (Figure 7):

Definition 3. *A spherical triangle is a triangle on the sphere, consisting of three points, pairwise connected by three great circle arcs.*

The spherical triangles differs from from triangles in the plane by having curved sides (great circle arcs) and a sum of angles greater than π:

Proposition 1. *Given a spherical triangle ABC with angles α, β, γ, on the sphere of radius r, the sum of the angles is given by:*

$$\alpha + \beta + \gamma = \pi + \frac{A}{r^2}$$

where A is the area of the spherical triangle.

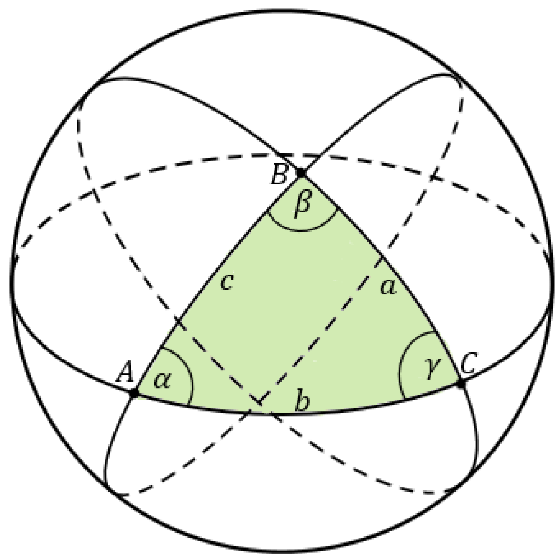

Figure 7: Visual representation of a spherical triangle ABC on the sphere, consiting of three great circle arcs a, b, c and the three angles α, β, γ.

Proof. Recall that the area of a sphere with radius r is $4\pi r^2$.
Let there be given a spherical triangle on a sphere, consisting of three great circles arcs, a, b and c, with A, B, C being the intersection points of these great circles. Let α, β, γ be the three angles in points A, B, C respectively, generated by the pairwise intersection of the curve segments (see Figure 7).
The three angles in the spherical triangles generates 3 lunes (one for each angles) as well as 3 lunes on the backside of the sphere. Recall the area of a sphere and that a lune with angle π will cover the whole sphere, then the area of a lune with angle θ

must be: $\frac{\theta}{\pi}4\pi r^2 = 4\theta r^2$.
Overall, the triangle is covered 3 times by the three lunes, which is 2 too much, as well as the triangle on the backside of the sphere is covered 2 times too much. This means that the entire area of the sphere is covered up, including twice the area of the spherical triangles, hence one get the area to be:

$$4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 = 4\pi r^2 + 4A$$

$$\implies \alpha + \beta + \gamma = \pi + \frac{A}{r^2}$$

□

The fraction $\frac{A}{r^2}$ is called *spherical excess* and denotes the amount for which the sum of the angles in a spherical triangles exceeds π radians.

If one wants to work with degrees instead of radians, the formula for the angle-sum is a bit different, because one needs to "correct" for the angles in radians. Hence, if the angles α, β, γ is in degrees, then the formula for the angle-sum is given by:

$$\alpha + \beta + \gamma = 180° + \frac{A \cdot 180°}{\pi \cdot r^2}$$

4.3.2 Geographic coordinates

Geographic coordinates is an example of spherical coordinates, highlighting the relationship between spherical geometry and map projections. In the work with map projections, it is essential to know how to transform the geographic coordinates to Cartesian coordinates. These play a central role when expressing some map projections mathematically.

I have already given a short presentation of the geographic coordinate system, as parallels of latitude and meridians of longitude on the Earth's surface, which will also apply to the reference model of the Earth - in our case, the sphere. Hence *geographic coordinates* are used when wanting to express a location on the Earth or the sphere, and consists of *latitude* and *longitude*, expressed in degrees from their respective zero point. However as noted, we will use radians. Mathematically, we can define the latitude and longitude by placing the sphere in a 3-dimensional Cartesian coordinate system, with center of the sphere O colliding with the center of the coordinate system $(0,0,0)$. Let the x-axis go through the Greenwich Meridian, and the z-axis point through the North Pole. Then for a point P on the sphere:

Definition 4. *Latitude φ is the angle between the xy-plane and the position vector $\overrightarrow{OP}$. We have $\varphi \in [-\pi/2, \pi/2]$, with 0 being the equatorial line, negative angles being south of the equatorial line and positive angles being north of the equatorial line.*
Longitude λ is the angle between the Greenwich Meridian and the meridian passing through the point P. We have $\lambda \in [-\pi, \pi]$, with 0 being the Greenwich Meridian, negative angles is west of Greenwich and positive is east of Greenwich.

See Figure 8 for a visualisation of geographic coordinates on the sphere. Mathematically, we denote a geographic coordinates by (φ, λ)[10].

Hence determining the 3-dimensional Cartesian coordinates for a geographic coordinate is the same as calculating the coordinates for the position vector $\overrightarrow{OP}$ (Figure 8). Given the geographic coordinate (φ, λ), we use the following transformation formulas:

$$x = r\cos\varphi\cos\lambda$$
$$y = r\cos\varphi\sin\lambda$$
$$z = r\sin\varphi$$

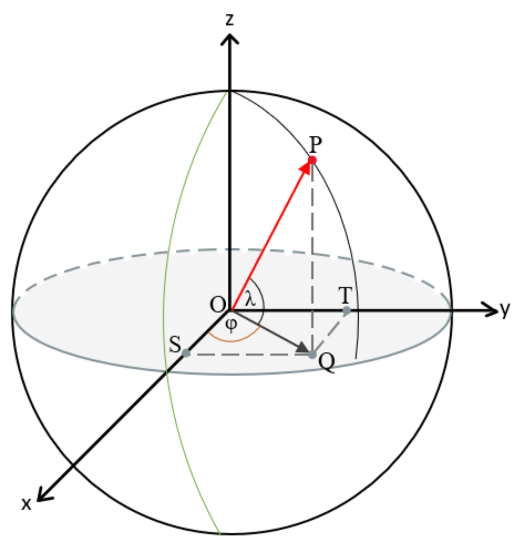

Figure 8: A geographic coordinate P on the sphere. Own illustration.

which can be derived by using sine for right triangles on the triangles seen in Figure 8. See for example Kro (2003, p. 25-31).

A more practical use of geographic coordinates is calculating the shortest distance between two locations on the Earth, that is the *spherical distance*. Recall that the

[10]Note that Fajstrup (2006) writes geographic coordinate as (λ, φ). The other sources I have consulted, such as Bolstad (2012), writes (φ, λ). I will stick to this notation.

shortest distance between two points on the sphere is the length of shortest great circle arc between the two points.

Let d denote the spherical distance between two points, $P_1 = (\varphi_1, \lambda_1)$ and $P_2 = (\varphi_2, \lambda_2)$, on the sphere with radius r. Then d is given by:

$$d = r \cdot \sigma$$

with $\sigma = \arccos\left[\sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cdot \cos(\lambda_1 - \lambda_2)\right]$ being the *spherical angle*, that is, the angle between the two positions vectors for P_1 and P_2 (Weisstein, 2024). Hence it comes as no surprise, that the derivation of the spherical angle σ follows from the formula for determining the angle v between two vectors. Letting the sphere have radius 1 and express the two position vectors by 3-dimensional Cartesian coordinates, we get:

$$\begin{aligned}
\cos\sigma &= \overrightarrow{OP_1} \cdot \overrightarrow{OP_2} \\
&= \cos\varphi_1 \cos\lambda_1 \cos\varphi_2 \cos\lambda_2 + \cos\varphi_1 \sin\lambda_1 \cos\varphi_2 \sin\lambda_2 + \sin\varphi_1 \sin\varphi_2 \\
&= \cos\varphi_1 \cos\varphi_2 (\cos\lambda_1 \cos\lambda_2 + \sin\lambda_1 \sin\lambda_2) + \sin\varphi_1 \sin\varphi_2 \\
&= \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos(\lambda_1 - \lambda_2)
\end{aligned}$$

(Weisstein, 2024).

4.3.3 Mathematical expressions of map projections

Given the conversion of geographic coordinates, one can express the different map projections mathematically. As noted earlier, one can see map projections as projections from one surface to another, or from one set of points to another. As Fajstrup (2006) notes, a map projection can be described analytically as a map

$$\mathbf{f} : S^2 \to \mathbb{R}^2$$

$$\mathbf{f}(\varphi, \lambda) = (f_1(\varphi, \lambda), f_2(\varphi, \lambda))$$

where $S^2 \subseteq \{(\varphi, \lambda) \in \mathbb{R}^2 | -\pi\backslash 2 \leq \varphi \leq \pi\backslash 2, -\pi \leq \lambda \leq \pi\}$ is the surface of the sphere. It is pretty clear that we would need the functions f_1, f_2 to be injective (and differentiable, we will see why later).

Using Fajstrup (2006), I will now present three different kinds of azimuthal (perspective) projections: gnomonic, stereographic and orthographic (see Figure 9).

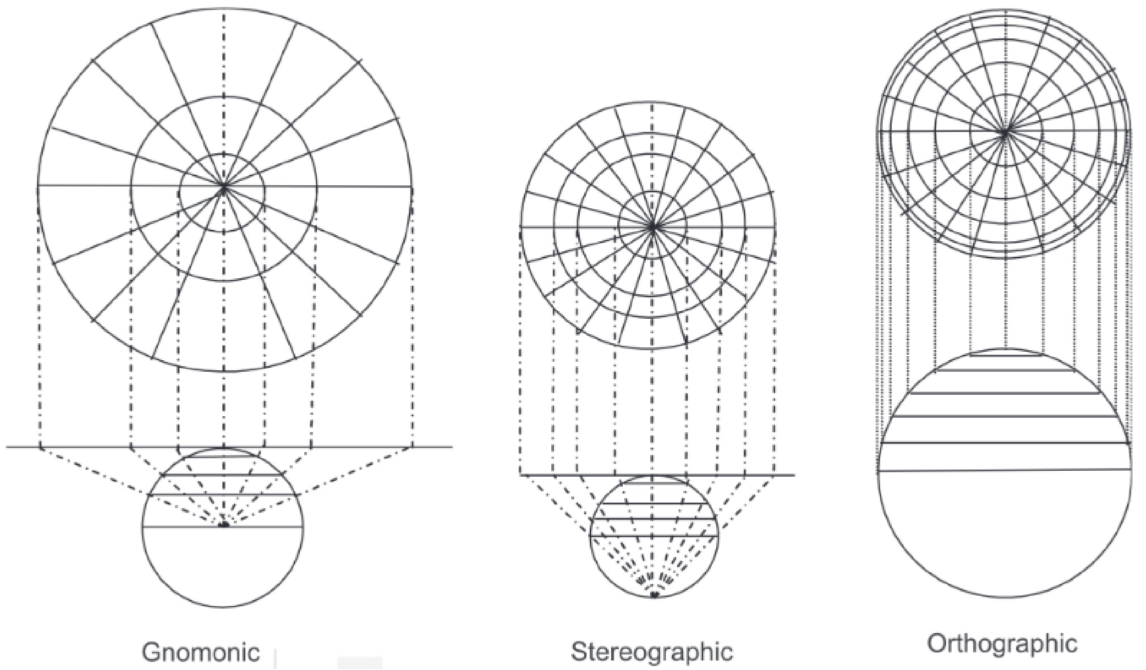

Figure 9: Construction of the azimuthal projections: gnomonic, stereographic and orthographic (Raj, 2020, p. 102).

For simplicity, let the sphere have radius 1 in the following.

Gnomonic projection. Let the projection plane be given by $z = 1$, with the perspective point being the center of the sphere (see Figure 9). Then one can write the parametric equation of a line through a point $P = (\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi)$ and center of the sphere as:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} \cos\varphi\cos\lambda - 0 \\ \cos\varphi\sin\lambda - 0 \\ \sin\varphi - 0 \end{pmatrix}$$

Since $z = 1$, the sphere intersect with the projection plan when $1 = t\sin\varphi \iff t =$

$1\backslash\sin\varphi$, hence the analytical description of the gnomonic projection is given by:

$$(\varphi, \lambda) \to \left(\frac{\cos\varphi}{\sin\varphi}\cos\lambda,\ \frac{\cos\varphi}{\sin\varphi}\sin\lambda\right) = \cot\varphi(\cos\lambda,\ \sin\lambda).$$

In this description, the perspective point is going through the North Pole. The following figure (Figure 10) is how a map with gnomonic projection with perspective point at the North Pole would look like:

Figure 10: A map where the gnomonic projection has been applied, with North Pole as perspective point (Wikipedia, 2024c).

Stereographic projection. Again, let the projection plan be given by $z = 1$. Let the perspective point be the South Pole, $(0, 0, -1)$. Hence the parametric equation for the line passing through a point $P = (\cos\varphi\cos\lambda,\ \cos\varphi\sin\lambda,\ \sin\varphi)$ on the sphere and the perspective point is:

$$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} \cos\varphi \cos\lambda - 0 \\ \cos\varphi \sin\lambda - 0 \\ \sin\varphi + 1 \end{pmatrix}$$

so $1 = -1 + t(\sin\varphi + 1) \iff t = 2/(1 + \sin\varphi)$. Then the analytical description of the stereographic projection is given by:

$$(\varphi, \lambda) \to \left(2 \cdot \frac{\cos\varphi}{1 + \sin\varphi} \cos\lambda,\ 2 \cdot \frac{\cos\varphi}{1 + \sin\varphi} \sin\lambda\right) = 2 \cdot \frac{\cos\varphi}{1 + \sin\varphi} (\cos\lambda,\ \sin\lambda).$$

Recall that the stereographic projection is conformal. Figure 11 shows how a map with the stereographic projection would look like:

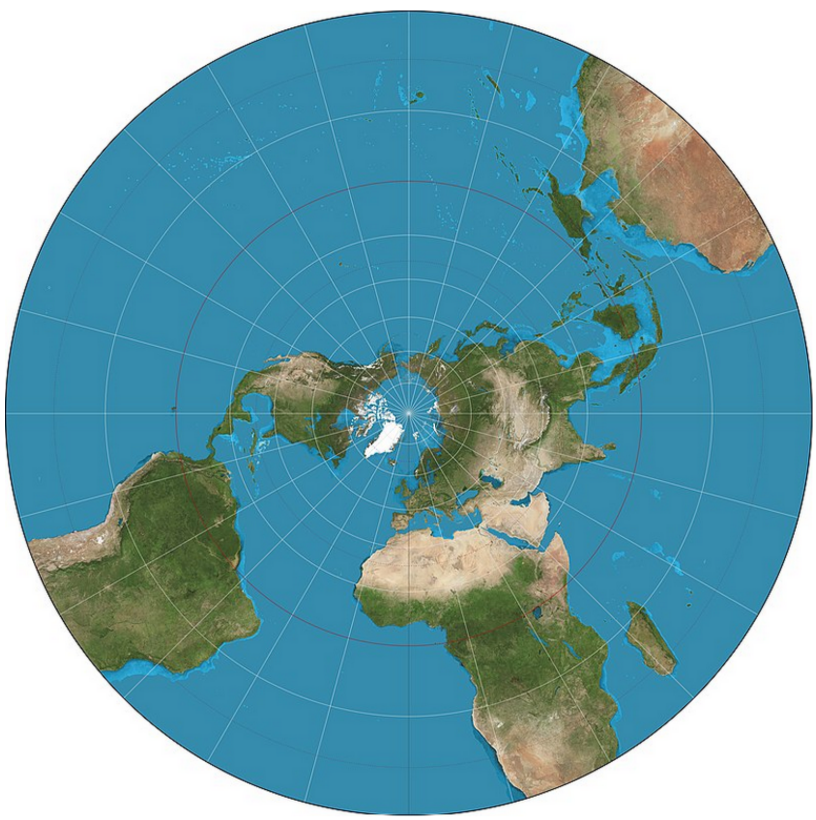

Figure 11: A map where the stereographic projection has been applied, with South Pole as perspective point (Wikipedia, 2024f).

Orthographic projections. The perspective point is placed at infinite distance, or in other words, projecting perpendicular on the projection plane $z = 1$. Then the analytical description is given by:

$$(\varphi, \lambda) \to (\cos\varphi\cos\lambda,\ \cos\varphi\sin\varphi) = \cos\varphi(\cos\lambda,\ \sin\lambda).$$

Figure 12 provides an example of a map with the orthographic projection.

Figure 12: A map where the orthographic projection has been applied, with an equatorial aspect (Wikipedia, 2024e).

The second class of projections are the cylindrical projections. Based on Fajstrup (2006), I'll describe the Lamberts cylindrical projection[11] and the central cylindrical projection (see Figure 13 & 14).

[11]In Fajstrups lecture notes she calls it Archimedes' equal-area projection, but all other sources ascribe it to Lambert (see for example Snyder, 1987).

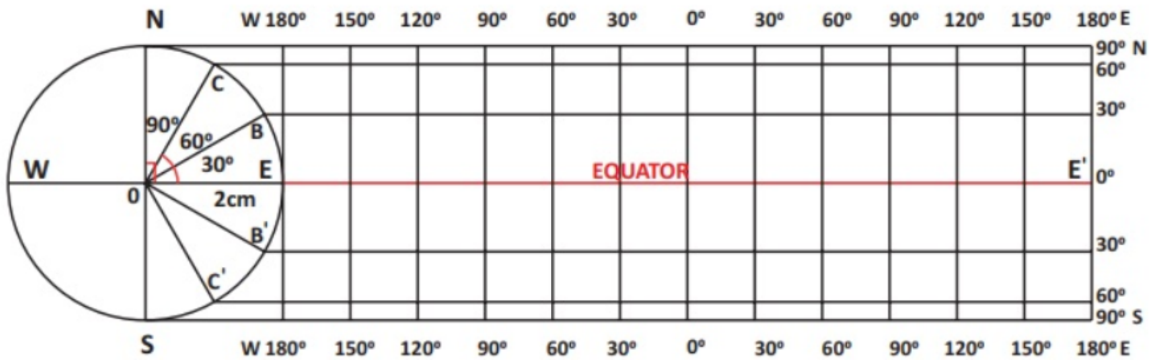

Figure 13: Construction of Lambert's cylindrical equal-area. Note that the lines of latitude and longitude are in degrees (BrainKart.com, 2023).

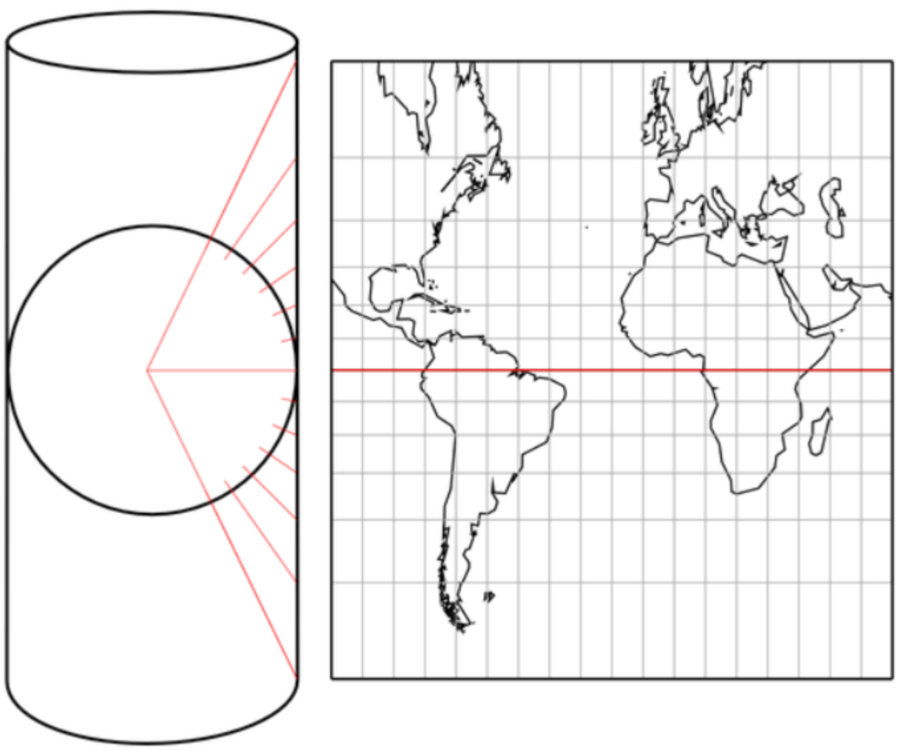

Figure 14: Construction of central cylindrical projection (Jung, 2019).

Again, let the sphere have radius $r = 1$.

Lambert's cylindrical equal-area projection. A point on the sphere is projected perpendicularly onto the tangent plane, that is the cylinder, with the equatorial line as standard line (see Figure 13). From the sine of right triangles:

$$(\varphi, \lambda) \to (\lambda, \sin\varphi).$$

As the name for this projection suggest, the Lambert cylindrical equal-area projection is equal-area. Figure 15 shows a map with the use of Lambert's cylindrical equal-area projection.

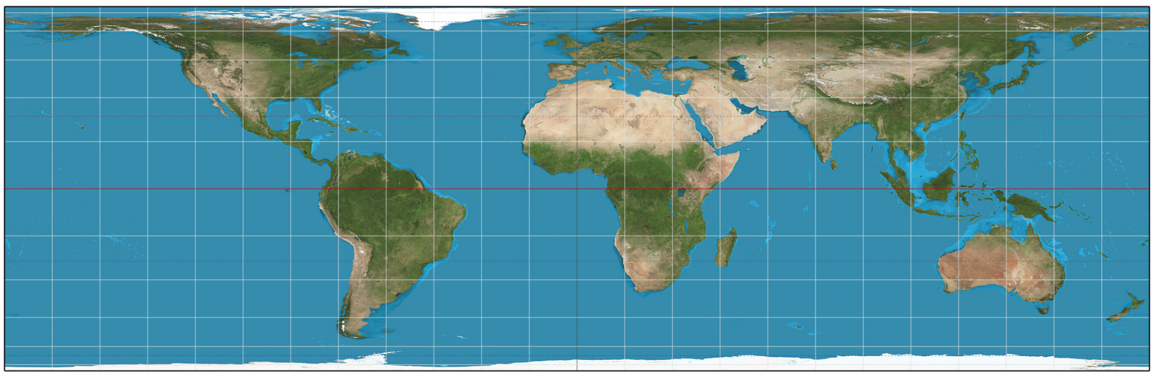

Figure 15: A map where Lambert's cylindrical equal-area projection has been applied (Wikipedia, 2024b).

Central cylindrical projection. The central cylindrical projection is given by letting perspective point be the center of the sphere, the standard line is again the equatorial line. Then consider a straight line from the perspective point through a point on the sphere and further onto the cylinder (see Figure 14). This projection can be expressed analytically by:

$$(\varphi, \lambda) \to (\lambda, \tan\varphi)$$

Figure 16 is a map constructed by the central cylindrical projection.

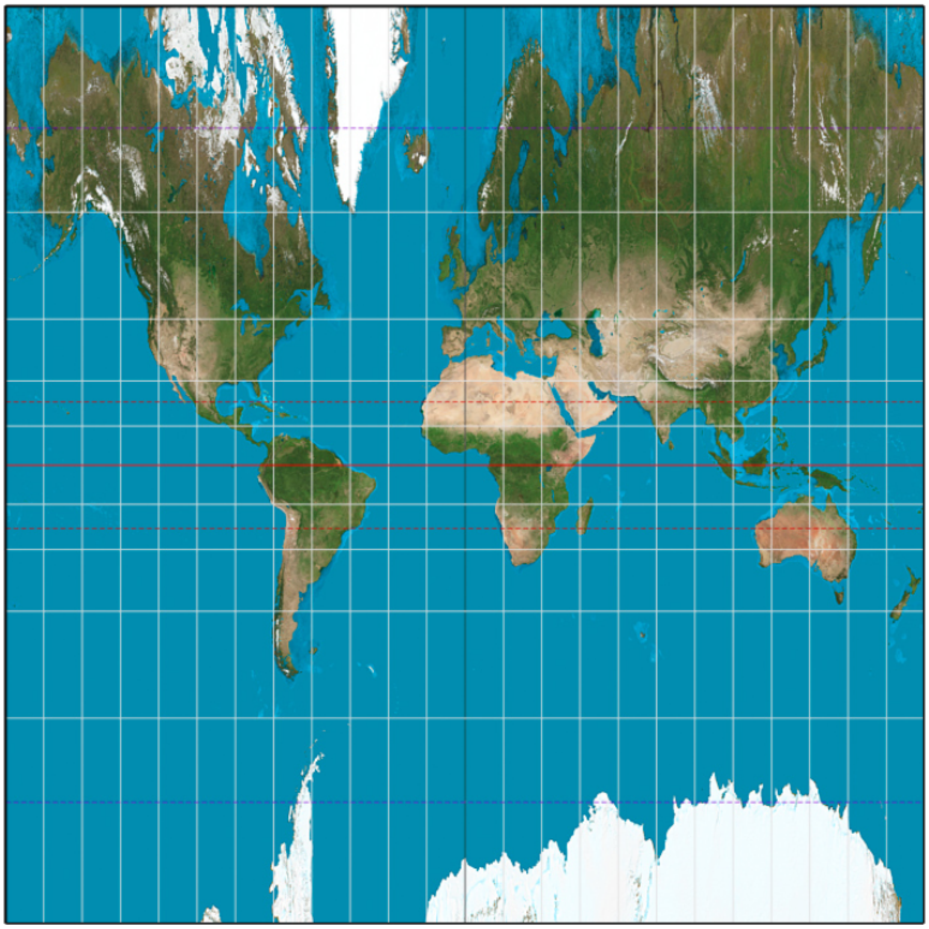

Figure 16: A map where the central cylindrical projection has been applied (Wikipedia, 2024a).

Mercator projection. As already mentioned, the Mercator projection is one of the most known and used map projections (Snyder, 1987). Compared to the other analytical expression of the map projection, the expression for Mercator is a bit more complex and not so intuitive. I will still use Fajstrup's (2006) notation, but the expressions is from Snyder (1987):

$$(\varphi, \lambda) \to \left(\lambda,\, \ln\left[\tan\left(\frac{1}{4}\pi + \frac{1}{2}\varphi\right)\right]\right)$$

or as McClure (2018) writes it (still using Fajstrup's notation):

$$(\varphi, \lambda) \to (\lambda,\, \ln(|\sec\varphi + \tan\varphi|))$$

I will go into more details about how one can derive to this expression in the following section. A map constructed by the Mercator projection is shown in Figure 17, as well in Figure 17.

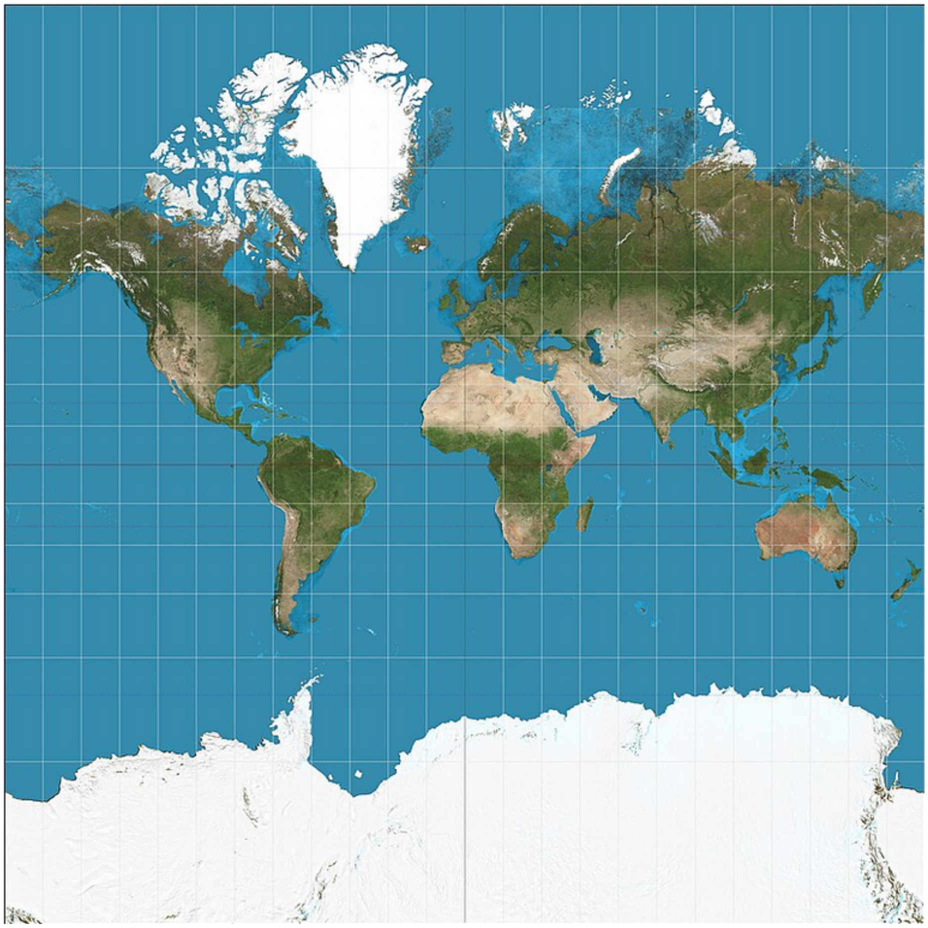

Figure 17: A map where the Mercator projection has been applied (Wikipedia, 2024d).

Note that the cylindrical projections has the general expression $(\varphi, \lambda) \to (\lambda, h(\varphi))$, with h some function of the latitude φ (McClure, 2018).

4.4 Why doesn't there exists a perfect map?

If there exists a perfect map, Fajstrup (2006) notes that the map must be conformal, equal-area and preserve all distances on the reference model. This also includes that great circle arcs on the sphere are mapped to straight lines on the map, ensuring accurate measurement at all times - however, this is not possible. Fajstrup (2006) proves the following theorem in the following way (see p. 11 in (Fajstrup, 2006)):

Theorem 1. *There does not exists a map from an open set U from the surface sphere S^2, which maps great circle arcs to line segments in $\mathbb{R}^2$ and is conformal.*

Proof. Assume for a contradiction that such map in Theorem 1 exists. Let a spherical triangle from U be mapped to $\mathbb{R}^2$, then the great circle arcs that make up the spherical triangle gets mapped to line segments. The map is also conformal, hence the angles in the spherical triangle, α, β and γ, gets preserved. From Proposition 1, we have that $\alpha + \beta + \gamma = \pi + \frac{A}{r^2}$. However, in $\mathbb{R}^2$, it is only possible that $\alpha + \beta + \gamma = \pi$. Hence one reach a contradiction, so there does not exists a map $U \subseteq S^2 \to \mathbb{R}^2$ which is conformal and maps great circle arcs to line segments[12]. $\square$

Given the criteria posed for a perfect map, one can conclude from Theorem 1 that there does not exists a map projection that preserves the Earth's surface perfectly. Actually, Fajstrup (2006) notes furthermore it is possible to prove that there exists no map projection which 1) preserves distances, and 2) preserves both angles and areas. The proofs for these statements are found in the field of differential geometry, and follows from the theorem *Teorema Egregium* (Kerkovits, 2023), proved by Carl Friedrich Gauss in 1827 (Lapaine and Divjak, 2017). I will not go into more details about the proofs, as it require the theory of differential geometry of curves and surfaces which is not presented in this thesis, but I would recommend to look at either Schlichtkrull's treatment of Teorema Egregium (2018, pp. 93-105) or even Gauss himself in his *Investigations of Curved Surfaces*[13] from 1827[14]. In brief, using the *Gaussian curvature*, one can conclude that there does not exists a length-preserving projection (called an isometry) between the sphere and the plane, because they do not have the same curvature (Schlichtkrull, 2018). Another proof of the non-existence of a map projections which preserve both angles and areas is found in Conrad's *Math 396: Map Making* (2006), which also uses differential geometry but with some different concepts than used in Teorema Egregium.

All above states there does not exists a map projection without distortion, and will always distort either angles, areas or distances. It is possible to measure the distor-

[12]In this thesis (and other places), it is presented that Proposition 1 (or something like Proposition 1) is necessary for the proof of Theorem 1; this may not be the case, as a simple counterexample is enough.

[13]Original title: Disquisitiones generales circa superficies curves (Lapaine and Divjak, 2017).

[14]Translated work can be found in The Project Gutenberg's *General Investigations of Curved Surfaces of 1827 and 1825* (2011).

tion on a map, that is measuring the map scale distortion. In the theory of map projections, a distinction is made between two types of map scale: the *principal scale* and the *scale factor* (Small Farm Link, 2024). The *principal scale* is measured as the ratio of distance on the sphere to the distance on Earth, and is constant (Small Farm Link, 2024). There is always a downscaling of the sphere when doing map projections, hence the principal scale describes this downscaling. This is expressed as the representative faction $1 : n$, which indicates that 1 unit on the sphere corresponds to n units on the Earth (Small Farm Link, 2024). How much 1 unit on the sphere correspond to on the Earth can be found by dividing the circumference of the sphere by the circumference of Earth. Hence the principal scale does not refer to the scale of the map, but the scale of the sphere (or any kind of reference model) (Small Farm Link, 2024).

The *scale factor* on the other hand, relates directly to the scale of the map, i.e. is a local scale. This scale varies throughout the map, and is measured as the ratio of the distance on the map to the corresponding distance on the sphere (Small Farm Link, 2024). Hence one can say, that the scale factor quantifies the distortion on a map. The distortion pattern on a map can be visualised by *Tissot's indicatrix* (Figur 18), distortion ellipses which change size, shape and orientation based on the map projection (Small Farm Link, 2024).

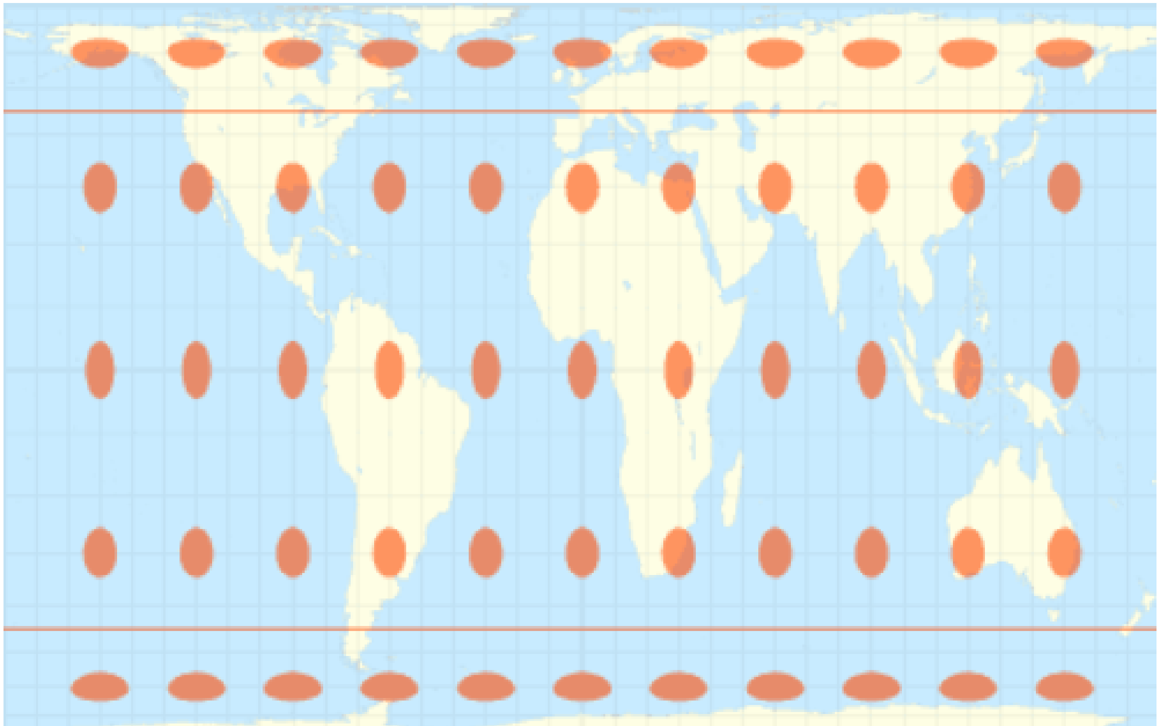

Figure 18: The Gall-Peters projection with Tissot's indicatrix (Small Farm Link, 2024).

The scale factor varies from one map projection to another since each projection distorts in its own way. Instead of calculating the scale factor for every distance or point on the map, a general expression for the scale factor can be derived for each projection. Fajstrup (2006) does so, using the *first fundamental form*, a notion from differential geometry used for determining metric properties on a surface, such as curve lengths, areas and curvature (Gaussian curvature) (Schlichtkrull, 2018). For determining the scale factor at a point P and for every direction $\gamma'(t)$ for a curve γ on the sphere, Fajstrup (2006) derives the following (using the expression on how to calculate the length of a curve):

$$\lim_{t \to t_0} \frac{|(\mathbf{f} \circ \gamma)'(t)|}{|\gamma'(t)|} = \frac{|(\mathbf{f} \circ \gamma)'(t_0)|}{|\gamma'(t_0)|}$$

where $\mathbf{f} : S^2 \to \mathbb{R}^2$ is the map projection. Then the fundamental form provides the tools to determine $(\mathbf{f} \circ \gamma)'(t_0)$ and $\gamma'(t_0)$ for given a given map projection (Fajstrup, 2006). How to do so, I leave with a reference to Fajstrup (2006), pp. 25-33.

McClure (2018) gives a rather simple example of how to derive the scale factors for cylindrical projections. As we know, cylindrical projections can be conform (like the Mercator) or equal-area (but not both!). In brief, if a cylindrical projection is to be conformal, it must be scaled by equal factor in both the horizontal and vertical directions. If it is to be equal-area, then if one direction is scaled by a certain factor, the other direction must be scaled by the reciprocal of that factor (see McClure (2018), p. 6). We know that the cylindrical projection maps the parallels from the sphere to equally long lines on the map, so the scale factor along the parallels will be $\frac{2\pi}{2\pi \cos\varphi} = \sec\varphi$, given the sphere is of radius 1 (McClure, 2018). The scale factor for the meridians is given by:

$$\frac{h(\varphi + t) - h(\varphi)}{t}$$

with t the change of distance on the globe. Since it is the local scaling factor, let $t \to 0$, and hence the scaling factor along the meridian is $h'(\varphi)$ (McClure, 2018) - this is why we would need f_1 and f_2 to be differential. Recall, the cylindrical projections can generally be expressed as $(\varphi, \lambda) \to (\lambda, h(\varphi))$. This means, for a cylindrical projection to be conformal, then one need $h'(\varphi) = \sec(\varphi)$. For this to happen, then:

$$h(\varphi) = \int_0^{\varphi} \sec(\phi) d\phi = \ln(|\sec\varphi) + \tan(\varphi)|)$$

and we see that this corresponds with the Mercator projection being conformal (McClure, 2018). Based on this, McClure (2018) proceeds with a general analysis to determine the scale factors that ensure conformality or equal-area properties for map projections, using partial derivatives and the Jacobian matrix (McClure, 2018).

This ends the section of the selected scholarly knowledge of map projections. The following will concern the preliminary analysis of the content at stake, preparing for the design, conception and a priori analysis.

5 The preliminary analyses of DE

The following sections deal with the preliminary analyses, as described in the section about DE, used to support, and in some way justify the *knowledge to be taught*. This involve an epistemological analysis, a institutional analysis and a didactial analysis. These will be influenced by my chosen target group, which I already determined at the beginning of the work of this thesis. The intended target group for the teaching material will be students, preferably in their final year of the general upper secondary education program (stx) in Denmark, typically aged 16-18, who are taking Mathematics and Physical Geography[15] at the highest level possible.

5.1 An epistemological analysis

As noted, the epistemological analysis is an analysis of the mathematical content at stake, in this case the presented scholarly knowledge about map projections. In line with DE, I will analyse the scholarly knowledge, identifying some overall aspects on what should be taught, when teaching map projections to upper secondary students, as well as identifying some possible epistemological obstacles and related misconceptions.

The assumption, thus expectation, is that upper secondary students have never been introduced to map projections (neither in Physical Geography or Mathematics) and that, in a potential interdisciplinary lesson on this topic, they would need to acquire a significant amount of new knowledge without having many prior prerequisites to work on. In reality, the students may not be able to be taught *everything* presented in the previous sections (depends on the time allocated to the teaching, as well as the findings in the institutional and didactical analysis), however, there are some key aspects which is important to include when teaching map projections.

First, maps, hence map projections, will always distort the surface of the sphere. There might be misconceptions about the accuracy of maps in general, that is maps depicts the Earth's surface perfectly. There are two facets in this, that is important to address through a lesson on map projections: map projections are not projections

[15] In Danish upper secondary education, Physical Geography refers to the subject of geography. See more about this in the institutional analysis.

from the Earth's surface to a flat surface, but from a model of the Earth to a flat surface, and there are different kinds of map projections, which imply different kinds of distortions. Hence, the teaching material must include situations which address these rather common misconception. This can be supported by including the impossibility theorem, in which mathematics appears as the argument and reasoning to this problem.

In relation to this, to understand why map projections distort (and also to understand the impossibility theorem), differences between the geometry on the sphere and the plane must be made clear. It might be enough to just acknowledge, that it is impossible to make something round, flat. However, for a deeper epistemological understanding, it is important to introduce spherical geometry, particularly the angle sum of spherical triangles. Especially in relation to teaching upper secondary students, because there might be some epistemological obstacles regarding this; through their studies of planar geometry, students know that the triangles angle sum is 180°, and might think this also holds for spherical triangles. In addition to this, it is likely that students also have a misconception about shortest distances on maps, since they have been taught that the shortest distances in planar geometry is a straight line. Including a situation, in which the students studies spherical triangles and distances on the sphere can help correct these misconceptions.

To fully understand how map projections connects mathematics and geography, the analytical expressions for map projections can be introduced. Assuming students prerequisites about map projections are limited, they may think that computers and satellites are used to produce maps (which is not wrong seen from a contemporary perspective) but without giving much thought to how the computers then construct maps (using the analytical expressions). Hence, for students to gain a deeper knowledge about how mathematics is used in map projections, presenting the analytical expressions can be deemed valuable. However, to do so, students must learn about conversions of geographical coordinates.

However, it is notable that the quantification and generalisation of distortions on maps is somewhat detached from the rest of the scholarly knowledge, meaning that there is a need for a whole different theory of geometry, differential geometry, when calculating general expression of the scale factor for each map projection. The reason for can be found in the historical perspective of the theory, since differential geometry came much later than spherical geometry. In relation to the teaching of upper secondary school students, it is reasonable to think that this might be too much to expect for students to learn, since it might require to much to learn - beside calculating principal scales and scale factors for specific distances. With this perspective, and this hold for all previous mentioned, it might seem more fitting to support the learning of map projections and their distortions by letting students interact with maps. Maps are inherently graphical, and students might trust the appearance of distortions on maps more that the numbers and mathematical explanations. However, there might occur some obstacles only showing statical maps, since students might not know how the real world looks like, the accurate shapes and sizes of the landmasses. Letting students interact with 3D models of the Earth or interactive maps can help support the understanding of the different kind of distortions.

How and to what extend all of these aspects of the theory of map projections can be communicated will be further examined through the following institutional analysis.

5.2 An institutional analysis

In the institutional analysis, I will examine the institutional conditions and constraints that surround the DE, in this case, the designing of teaching material about map projections. This will include an analysis of curricula and textbooks, since these somewhat describe the institutional expectations about what should be taught and how. In Denmark, the Ministry of Children and Education (Danish: Børne- og Undervisningsministeriet) is the public authority responsible for the children and education sector, covering day care, primary and lower secondary education, upper secondary education, and higher education. The ministry oversees, among other things, the development of curricula (Danish: læreplaner) for upper secondary education, including stx. More specificity, curricula for each subject offered are developed by a ”subject consultant” (Danish: fagkonsulent) with experience of teaching the subject.

5.2.1 The curricula

The curricula serve as guidelines that dictate a subject's purpose, identity, learning objectives (Danish: læringsmål), core content (Danish: kernestof), evaluation and feedback, and examination formats, depending on the level. In short, they define what students should be taught, what they should gain from the instruction, and how they should be assessed. As mentioned earlier, my target group consists of students taking Mathematics and Physical Geography at the highest possible level. In Denmark, the levels are divided into A, B, and C, with A being the highest and C the lowest[16]. Mathematics at A-level is quite common in Danish upper secondary schools, whereas the availability of Physical Geography, which is only offered up to B-level, has been decreasing.

With the 2005 reform for upper secondary schools, the subject *Geography* was renamed *Physical Geography*, and the number of teaching hours was reduced (Malm and Madsen, 2015). According to Conradsen et al. (2024), there has been a political de-prioritization of Geography, affecting primary schools, upper secondary education, and teacher training programs in Denmark. In relation to this, fewer students now choose study programs that include Physical Geography, which could indicate that the subject is gradually disappearing from both upper secondary education and in general. Conradsen et al. (2024) even describe this de-prioritization as paradoxical (which I strongly agree with) since the subject's core areas address some of the most pressing global challenges today, such as climate change, migration, inequality, and sustainability.

This decline in priority may explain why Physical Geography is only offered up to B-level, thereby limiting both the expectations for students and the depth of content covered. At the same time, this can restricts the possibility of teaching map projections as an interdisciplinary topic between Mathematics A and Physical Geography B. While such interdisciplinary teaching could be valuable, it may be difficult to implement if there are few, if any, study programs that include both subjects at these levels. However, in 2013, Geoscience A was introduced as an experimental interdisci-

[16]The level determines the duration of the subject: A-level lasts 3 years, C-level lasts 1 year. For B-level, it is taught for 1 year if students have previously taken the subject at the C-level, or for 2 years if the B-level subject is part of the study program.

plinary subject between Physics B and Physical Geography B, which has since become a permanent subject offered at some upper secondary schools (Malm and Madsen, 2015), and requiring Mathematics A as part of the study program (Uddannelses-guiden, 2021). This suggests a potential institutional space for an interdisciplinary approach to map projections, though my focus remains on the curricula for Mathematics A and Physical Geography B.

The curriculum for Geography B (Børne- og Undervisningsministeriet, 2024b) provides a few conditions for teaching map projections. Maps or map projections are not mentioned in relation to the core content, which primarily covers landscape processes, climate and weather, as well as sustainability and resource distribution. The only place where maps are explicitly mentioned is in the learning objectives, which state that students should be able to "*seek out, assess the quality of, interpret, and apply a range of geoscientific representation forms such as texts, data, maps, [...]*"[17] (Børne- og Undervisningsministeriet, 2024b, p. 1). Thus, maps are regarded as a geographical tool among others, used to illuminate topics within the core content rather than as a subject that can be analysed and understood in its own right. An argument for the curriculum supporting a topic on map projections is that students must be able to "*assess the quality*" of maps; does this mean that students should be familiar with different types of distortions? Since curricula does not propose actual teaching situations, this question can be explored further by examining how maps are presented in textbooks for Physical Geography B.

However, the curriculum pose conditions for the use of mathematics in the teaching of Geography B. In the learning objectives, it is stated that students must be able to "*understand and critically apply complex geoscientific models and simple mathematical models as representations of reality*" (Børne- og Undervisningsministeriet, 2024b, p. 1). As mentioned, map projections are a mathematical construction and not, in themselves, a model but rather a method for creating a model, namely a map of Earth's surface. However, in order to understand and critically apply, let's say, maps, one could argue that knowledge of map projections and their distortions is essential for comprehension and application.

[17]This quote and the following in this section has been translated from Danish to English by me.

The requirement that mathematics should also be applied within other subject areas is also evident in the learning objectives of the Mathematics A curriculum. Among other things, it states that students must be able to "*use mathematics as a means to formulate, analyse, and solve problems within the subject itself or in other subject areas and in relation to the world around them*" (Børne- og Undervisningsministeriet, 2024a, p. 1). In the topic of maps and map projections, mathematics is used as a means to express map projections and, not least, to solve problems within this field, such as the fundamental issue that it is impossible to create a projection that preserves all properties of Earth's surface simultaneously. The mathematics used (such as spherical geometry, trigonometry, and to some extent analytical geometry) has the potential of covering some of the core content stated in the curriculum. About geometry, trigonometry and vectors, the curriculum states (highlighting some of the most important concepts that is used in the mathematics of map projections):

"*Vectors in the plane and space:* ***Coordinate sets****, vector arithmetic, length,* ***angle between vectors****,* ***dot product****, projection. In the plane: Determinant, area of a parallelogram, line equation determined by a point and a normal vector, angle between lines, parametric representation of a line and a circle.* ***In space: Cross product, parametric representation of a line in space, plane equation and parametric representation, the sphere as well as intersections, distances, and angles in space.***" (Børne- og Undervisningsministeriet, 2024a, p. 2).

In this quote, we also see *projection* mentioned. However, this relates to projection of vectors on vectors[18], and is not about projections from one space to another.

One potential limitation is that functions of multiple variables are no longer part of the core content[19], which could make it challenging to present map projections using analytical expressions, since this require an understanding and use of functions of two variables.

[18]See for example *Matemat10k* (2014), p. 38.

[19]Looking at the 2017 Mathematics A curriculum, functions of two variables were included, but unfortunately (for my case) they have been removed in the 2024 curriculum. You can find the 2017 curriculum on the Danish Ministry of Children and Education's website: `https://www.uvm.dk/gymnasiale-uddannelser/fag-og-laereplaner/stx-laereplaner`.

5.2.2 Textbooks

Although curricula represent the official institutional requirements and expectations for each subject, it is also relevant to look beyond the curricula, and examine textbooks for each subjects. The role of textbooks in relation to institutional constraints and constrains is significant, as they bridge the gap between the intended curriculum and actual classroom practice. I have examined two textbooks: *Naturgeografi - vores verden* (2023), a textbook designed for teaching Physical Geography C and B, as well as *Matema10k* (2014) for teaching Mathematics A.

Like the curriculum for Physical Geography B, the textbook *Naturgoegrafi - vores verden* (2023) does not place much emphasis on maps or map projections, hence does not pose many conditions for teaching of map projections, which in itself can be a constrain. However, the book does introduce potential issues related to the use of maps, but provides no explanation whatsoever, stating that "*the map [...] is one of the most important tools in geography. [...] One of the major challenges in creating maps is transferring the round globe onto a flat map. This creates certain problems and inaccuracies in the map that are important to be aware of*" (Kristiansen et al., 2023, p. 11). The quote ends with a reference a figure[20], Figure 19, which illustrates the three classes of map projections, similar to Figure 3. Furthermore, the book contains no exercises, only pure text accompanied by figures. This, of course, provides some freedom to develop and formulate exercises related to the book's content. From this perspective, only imagination sets the limits for what kinds of exercises can be created in connection with this otherwise very brief section on maps.

[20]The figure is somewhat misleading. At first glance, it appears to show the three classes of map projections (cylindrical, conic, and planar) and below them, maps constructed by these map projections. However, under the conic projection, a globe model of the Earth is depicted, which can be misinterpreted as suggesting that using the conic projection would allow for the construction of a spherical map, which is not correct.

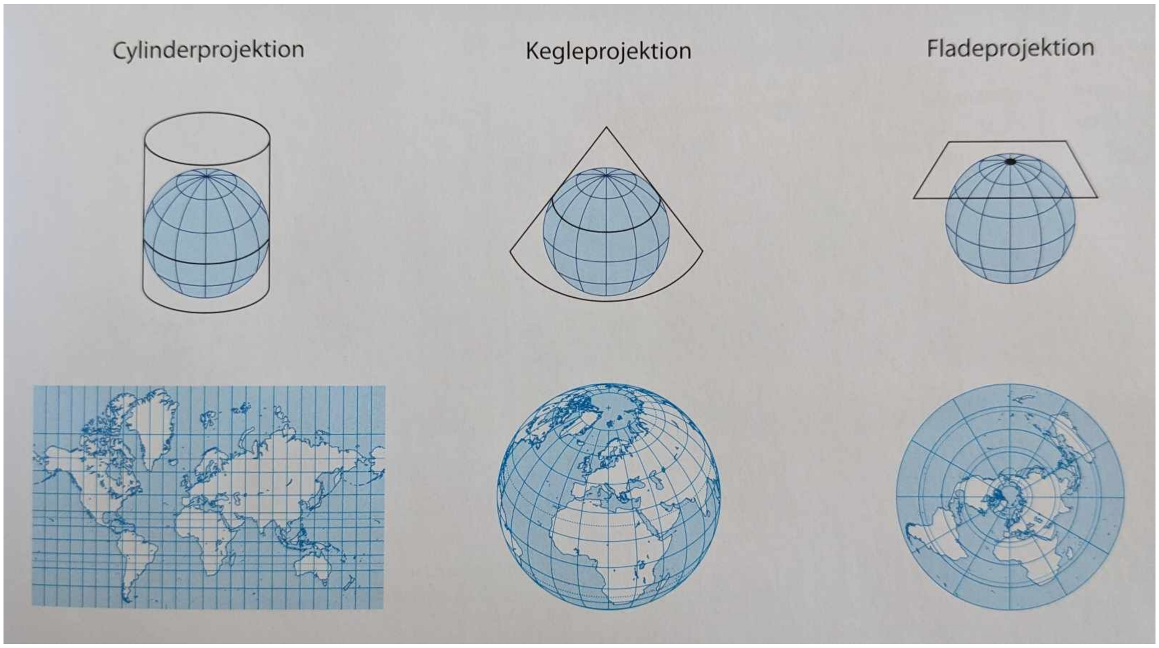

Figure 19: The figure from the Physical Geography textbook (Kristiansen et al., 2023, p. 11).

The Mathematics A textbook *Matema10k* (2014) extensively covers the core material related to geometry, trigonometry, and vectors (see for example pp. 77-121 in the textbook). Many of the mathematical formulas presented in this thesis can be recognized in the textbook, including those for the angle between two vectors, the angle between two intersecting planes, as well as the parametric representation of lines and planes. Numerous exercises are provided to practice these concepts. However, the book does not relate these topics to map projections, despite presenting other examples of interdisciplinary connections with different subjects, but does provide the mathematical foundation for discussing map projections.

It is also worth noting that the curricula for both Geography B and Mathematics A were revised in 2024. However, both textbooks were published before this revision, meaning they may no longer fully align with the current institutional expectations for these subjects and instead reflect older curricula. The Geography B textbook was published in 2023, which suggests that it could still be relevant today. However, the Mathematics A textbook was published in 2014. Since then, multiple versions of the curriculum have been released, making it likely that this textbook is no longer entirely representative of the current requirements. Nevertheless, it still covers the same mathematical topics outlined in the core content, which is why it is referenced in this thesis (with reservations).

5.3 A didactical analysis

A natural part of working on what I assume applies to all theses is the search for literature that, in some way, can be used in and support the purpose of the thesis. One of the first searches I conducted as part of my thesis was "*kort projektioner gymnasium*" (English: "*map projections upper secondary schools*"), which led me to the following website: `https://people.math.aau.dk/~fajstrup/UNDERVISNING/GYMNASIE/KORTPROJEKTIONER/`. Here, one can find Fajstrup's lecture notes on map projections and distortions (Fajstrup, 2006), as well as a Word document containing notes on spherical geometry aimed at upper secondary schools: *Sfærisk Geometri* (2005). These notes were developed by Fajstrup in collaboration with Dorthe Nielsen from Vesthimmerlands Gymnasium, and based on the document information, the last edit was made in 2005 — meaning the notes are now about 20 years old. Nevertheless, these teaching notes can be a great source of inspiration for my own work in designing educational materials, in relation to the structure and organisation of the teaching materials.

The content of the notes seem rather technical and the exercises are often accompanied by hints, such that students will be able to solve the exercises themselves without help from the teacher. The hints might also assist the teacher, as it is also possible that the teacher might have limited knowledge about the content. Comparing the notes with the presented scholarly knowledge, one see a greater emphasis on spherical trigonometry in the notes, which is not included in the scholarly knowledge. Additionally, the notes do not offer much from a geographical aspect; there is no mention of the classification of map projection, the purpose and use of the different map projections and so on. This indicates that the notes were only developed with only mathematics in mind, hence not meant to address map projections as an interdisciplinary topic between mathematics and geography. From this perspective, it seems reasonable why spherical trigonometry is also treated. In the preface to the teaching notes, Fajstrup mentions that the material was tested at Espergærde Gymnasium in the spring of 2004, where the teacher and students provided feedback with suggestions for modifications. Unfortunately, there is no available empirical data on this specific teaching experience.

Fajstrup's and Nielsen's work appears to be the only Danish example of literature on map projections and spherical geometry in an educational context[21]. However, there is also very little English-language literature on this topic. To search for other didactic literature, I have used Google Scholar, where I employed search words such as *map projections, didactics, mathematics, geography, upper secondary schools, high school* — as well as various combinations of these. A few research articles on teaching map projections appear, but none that I feel would contribute meaningfully to my thesis.

But I came across the article *Maps as Representation: Expert-Novice Comparison of Projection Understanding* (2002) by Anderson and Leinhardt. In this, they examine how varying levels of expertise influence the perception of maps as representations of the Earth's surface, with a focus on shortest distances on maps (created using the Mercator projection) versus the Earth's surface. In short, Anderson and Leinhardt (2002) conclude that the more experience one has in the field (which correlates with the length of time spent working in the field), the easier it is to solve geometric problems, such as determining the shortest distance on a map; "experts" knew that the shortest route on the map between two locations followed a curved path, whereas participants with less experience had difficulties realising this and often assumed it was a straight line. This is probably not surprising, but the most interesting aspect of the article is their discussion of educational implications for K-12 students[22]. Here, it is also evident, and not surprising, that students who had received the most instruction (i.e., the older the students) had a deeper understanding of maps, map projections, and distortions. Anderson and Leinhardt (2002) then suggest instructional strategies that support students' learning about distortions:

[21]Afterwards, my advisor made me aware that there also exists the following, *Projekt 5.5: Sfærisk Geometry og Introduktion til Korprojektioner* (2014), provided by L&R Uddannelse (Egmont): `https://lru.praxis.dk/Lru/microsites/hem/fra_gymportal/docs/Projekt_5-5_Sfaerisk_geometri_og_introduktion_til_kortprojektioner.pdf`.

[22]In Denmark, we are not accustomed to this notation, but a quick internet search shows that the term covers students from kindergarten to 12th grade (i.e., up to upper secondary school).

"*One way students could learn to account for distortion in the map due to projection is by reading and interpreting maps with different projections and connecting them back to the globe through a specific task. [...] In our study, we found evidence that using lines of latitude and longitude was a successful strategy. However, no matter how the concept of projection is taught, students need to have access to explicit experiences documenting that the curve of the earth's surface results in a great circle being the shortest distance between locations on the earth's surface. These activities would help students understand the relation between the flat map and the curved surface of the earth, while learning that this relation will change (i.e., the distortion) depending on the type of map projection used.*" (Anderson and Leinhardt, 2002, p. 316).

The findings from Anderson and Leinhardt (2002) suggest that incorporating inquiry-based tasks, in which students study distortions on maps and relates these back to the sphere, could foster a deeper understanding of map distortions. Likewise, this can be supported by letting students work explicitly with shortest distances on the sphere, all of which is already stated in the epistemological analysis. This reinforces the necessity of structured instructions that leverage both theoretical and practical aspects to help students grasp the complexities and limitations of representing a 3-dimensional Earth on a 2-dimensional surface.

6 Conception and a priori analysis

This section presents the conception and *a priori* analysis of the designed teaching material. The teaching material (in Danish) can be found in Appendix A, as well as the materials used in some of the exercises can be found in Appendix B, C and D[23]. Many considerations and choices have gone into the development of the teaching material, some more explicit, others more implicit. For obvious reasons, I will not go into every detail about the development of the teaching material, but will here present the most important choices that have had a significant impact on the final result.

[23]Please note that the format of the materials in Appendix D is misleading. In reality, the material are meant to be printed in paper-size A3. Also, the map in which Lamberts cylindrical projection has been used, has been rotated the wrong way.

The first macro choices concern the content and the overall structure of the teaching material. Many considerations lie behind the selection of the content; firstly, the main goal is to teach students about map projections, as the method for producing maps, and that there exists no map projection that depicts the Earth's surface perfectly - as mentioned in the epistemological analysis, the students might think that there exists a map that depicts the Earth perfectly. In order for students to understand this, they must learn about the distortions and why map projections distorts. To support this knowledge development, classes and characteristics about map projections was chosen to be included.

Of course, this can be done without ever presenting the mathematics behind map projections, but the teaching material is developed with regards to an interdisciplinary collaboration between mathematics and geography, hence some mathematics must be included, also in relation to preserve some epistemology. The selection of which mathematics the teaching material should contain, was based both on the institutional analysis and the epistemological analysis. As we have seen, students much undergo a course about geometry, trigonometry and vectors in 2- and 3-dimensions, hence it was assessed that some of the scholarly mathematical knowledge can be transposed to fit into a high school context, as much of it involves circles, planes, and vectors in 3-dimensions. Therefore, it was decided to include topics such as spherical geometry - especially spherical triangles and their angle sum - and the analytical geometry, the conversion formulas. As mentioned in the epistemological analysis, this can support the introduction of the analytical expressions.

This leads to identification of some prerequisites, for which it is expected that students have learned about, to be able to follow content of the teaching material:

- Basic arithmetic (including unit conversions).
- The circle, its radius, diameter, circumference, and particularly the calculation of circular segments.
- The two-dimensional coordinate system and coordinate sets.
- Triangles in the plane, including area and the sum of interior angles.

- Trigonometry for right-angled triangles, as well as the sine and cosine rules and conversion to radians.
- Vectors in 2-dimensions, especially the angle between two vectors, position vectors, normal vectors, and vector calculations.
- An introduction to vectors in 3-dimensions.
- An introduction to planes in space, including the plane equation and normal vector.
- An introduction to the sphere (and unit sphere) in space, including its equation, radius, diameter, and circumference.

These choices and considerations further lead to the macro choice of the overall structure of the teaching materials. An example was presented in the didactic analysis, namely the notes by Fajstrup and Nielsen (2005), which serves as inspiration for structuring the teaching materials; these notes have influenced the overall mathematical structure but not how mathematics is connected to map projections. In order to strengthen the interdisciplinarity between geography and mathematics (as mathematics is used to describe geographical tools), highlight the relevance of the subsequent mathematics, and provide an accessible introduction to the topic, it was decided that students need to be introduced first to the classification of the map projections (the classes and characteristics).

Decisions like these lead to macro-didactic variables concerning aspects such as the amount of time allocated and the organization and the number and nature of the exercises included. For this teaching material, it is estimated that 3.5–4 hours (including breaks) should be sufficient. Additionally, the exercises are designed so that students can work on them in pairs or small groups, fostering discussion, strategy development, and knowledge-sharing among students. These exercises themselves constitute micro-didactic variables, as they can be adjusted in terms of difficulty level, length, and the amount of information provided to support students in solving them. It should be noted, student does not need any other types of CAS-tool, other than a calculator which include the trigonometric functions.

As mentioned earlier, TDS is used to guide the design of the teaching materials. Although TDS is most commonly applied to orchestrate and implement an actual lesson, the theory is here used to support the design of teaching materials. This is because TDS compels one to consider the progression of the materials and the development of exercises that are meaningful and encourage knowledge construction. One could argue that this is also a type of macro choice, as it influences both the overall structure of the teaching materials and how they evolve. At the same time, it constitutes a micro choice for each teaching situation that the teaching material encourage. For instance, considerable thought was given to how didactical situations could be incorporated into the teaching materials — specifically, devolution and institutionalisation — either as a lead-in to the exercises or as a means to connect the knowledge gained from the exercises to the scholarly knowledge.

The situations of devolution and institutionalisation thus function as didactic variables that the teacher can adjust, determining how much knowledge should be ”broken down” and provided to students before they attempt to solve an exercise, as well as to what extend the teacher connects this to the scholarly knowledge. Wherever possible and meaningful, efforts have been made to incorporate devolution and institutionalisation, particularly in cases where the exercises exhibit a more adidactic nature.

The following conception and a priori analysis will be divided into similar sections as in the teaching material:

6.1 First section: The introduction

A choice was made to begin the teaching material with an open-ended question (Exercise 1, p. 2 in Appendix A), which does not necessarily fall into the category of macro- or micro-didactic variables — yet in a way, it does. By posing an open-ended question, the aim is to encourage students to reflect while also partially assessing their prior knowledge about maps. Students are not expected to develop a specific strategy for answering the question, as there is no definitive solution. However, they are expected to provide examples of what maps can be used for and how they are created; for instance, through the use of computers and satellites.

6.2 Second section: What are map projections?

The second section (pp. 2–4 in Appendix A) also begins by assessing students' prior knowledge about how maps can be created without the use of computer programs and satellites. This represents a micro choice. Exercise 2 (p. 2 in Appendix A) is designed to place students in a situation of action, where they can use Google Earth to sketch their own version of a world map. It is expected that many students will resort to freehand drawing, though some may recognize the potential of using the graticule displayed on the Earth model in Google Earth. However, it is likely that these students will not fully understand how to utilize the graticule effectively and will instead adopt the strategy of freehand drawing. The main goal of the exercise is to lead students to realize that constructing a perfect map of Earth's surface is a highly challenging — if not impossible — task. In Exercise 2, the use of IT also appears as a didactic variable, allowing for adjustments based on which digital tools students are already familiar with that might serve a similar function to Google Earth.

After Exercise 2, a devolution follows in which map projections are introduced as the method for producing maps, along with an explanation of the overall work behind map projections. This devolution is intended to prepare students for Exercise 3 (p. 3 in Appendix A). A micro choice was made to present an alternative model of the Earth during this devolution, but it is possible to introduce more. A macro choice was made to use the sphere as the model of the Earth for analytical purposes in the subsequent work, which includes macro-didactic variables such as how much and in what way the sphere is presented in the teaching material. The devolution also serves to correct students' potential misconception regarding map projections by explaining what a projection is and clarifying that map projections are not simply a direct projection from Earth's surface onto a flat map.

Exercise 3 is a situation of action in which students must calculate the circumference of both the Earth and the sphere, then (implicitly) determining the principal scale. It was decided to include the numerical calculations for the map scales, since it is rather simple and the principal scale is often mistaken as the actual scale of a given map. In this exercise, students are expected to apply their prior knowledge of the sphere, though fundamentally, an understanding of circles is sufficient. The key challenge for students is realizing that calculating the circumference of a sphere is the same as calculating the circumference of a circle. If students have already been introduced to the formula for the circumference of a sphere, the expected strategy is that they will apply this knowledge directly. Additionally, if students know about unit conversion and orders of magnitude (maybe this is more of a question about whether they remember or not), they should be able to determine how much 1 cm on the sphere corresponds to on Earth by dividing the two circumferences. However, some students may struggle to identify the appropriate strategy, in which case the teacher may need to guide them toward this realization.

A decision was made to include an exercise, Exercise 4 (s. 4 in Appendix A), which aim is to get the students to reflect and try to explain why map projections distort the surface of a sphere. It was considered, if this exercise should include a hint (for example: ”Can you make a ball completely flat?”), but the intention with this exercise is not necessarily for students to give a ”correct” answer. The intention is for students to discuss possible reasons, and to provide possible answers, and maybe in collaboration, derive the correct answer.

6.3 Third section: What kind of map projections does there exists?

The following section (pp. 4–7 in Appendix A) treat the different kind of map projections. This allows students to use the concepts later on and gives them insight into how map projections distort the surface of the sphere before providing a mathematical explanation for this, thus can be seen as a fundamental situation. Consideration was given to how to make this presentation as adidactic as possible, so that students would have a sense of autonomy in their learning and better opportunities for institutionalisation. Several micro choices were made in this context. The first was to allow students to derive the classes of map projections themselves, which led to Exercise 5 (p. 4 in Appendix A). In this exercise, students are placed in a situation of action, where they are given three maps that represent the three classes of map projections (Appendix B), and they must identify the geometric shapes that form the surface that is being projected onto. It is expected that students will follow the instructions given in the exercise, cutting out the maps and attempting to manipulate them into the desired shapes (cylinder, cone, plane). The plane projection here serves somewhat as a wildcard, as students do not need to "manipulate" the map in this case; instead, they can position it relative to a sphere (this can be either a fictional or physical sphere, depending on the teacher's choice).

Institutionalisation of the classes of map projections follows, where latitude and longitude are also explained through a fact box, as students need this knowledge for the institutionalisation of the classes. The geographical coordinates will be further studied later in the teaching material. Here, a macro choice was made to focus on the cylindrical projections, as they are the easiest to understand and describe mathematically (aside from the Mercator projection). It was determined that students would not be able to derive the characteristics of map projections themselves, because of the uncertainty to how this should unfold, and whether students could in fact derive the characteristics without any prior knowledge. Instead, the characteristics are presented as part of the devolution leading to Exercise 6 (p. 7 in Appendix A), except that of being azimuthal. This was based on the decision about only focusing on cylindrical projections (which are not azimuthal anyway), as well as students might find it difficult to distinguish between the characteristics of angle-preserving and azimuthal, as they are both angles.

Exercise 6 aims to train students in how to visually assess the properties of a map projection (Appendix C) and apply geographic reasoning to evaluate and explain what the map can be used for based on its characteristics — a type of situation of formulation, but also a fundamental situations according to the epistemological analysis. Once again, students are allowed to use Google Earth, and it is expected that they will take advantage of this tool. The idea is that they will employ a process of elimination: if they observe that sizes on the map are distorted (e.g., Greenland appears much larger on the map than on the model in Google Earth), they can conclude that the map is not area-preserving and must therefore be conformal. Determining whether a map is area-preserving can be more challenging since this property distorts shapes, which can be misleading.

The exercise also includes a task where students must draw what they believe to be the shortest path on the map. It was deliberately decided not to explain shortest distances to the students at this point in order to keep all possibilities open regarding how they approach the task. The expected strategy is, of course, that they will draw a straight line between the points, as they are familiar with this as the shortest distance in planar geometry. However, some students may again use Google Earth and notice that the shortest path on the model behaves differently from what they expect. After the students are done with the exercise, the teacher can show the actual shortest distances between the location, or wait until later when the concept of shortest distances is connected to mathematics.

6.4 Fourth section: The mathematical prerequisites for map projections

Many choices and considerations were made in the section on the mathematics behind map projections (pp. 7-11 in Appendix A). The content was weighed against what could be expected given students' prior knowledge. The first macro choice was to introduce some formal definitions essential for spherical geometry and, more specifically, for the theory of map projections. Here, macro-didactic variables include the definitions of great circles and angles on the sphere's surface. Another macro choice was to include key facts about great circles on the sphere, each of which can also function as a didactic variable. It was decided not to have students derive these results themselves, as is done in Fajstrup and Nielsen's notes (2005). The reason for this decision was that students are only expected to have been introduced to the sphere and planes in 3-dimensions but are not yet fully comfortable working with these objects.

All this serves as a devolution leading to Exercise 7 (p. 8 in Appendix A), where students must determine the angle on the sphere's surface — a situation of action that forces them to apply both new and prior knowledge. Some students may struggle to derive the normal vectors from the equations, in which case the teacher will need to step in and assist. From this, it is expected that students will be able to use the formula for determining the angle between two vectors.

Great circles are then used to define spherical triangles, where a key result is presented regarding the angle-sum in a spherical triangle (p. 9 in Appendix A). Here, one must be aware of students' possible epistemological obstacles regarding the angle-sum, which is why a micro choice was made for students to work with the proof of the theorem, creating a fundamental situation. Exercise 8 (p. 9 in Appendix A) serves multiple roles: in addition to being a micro-didactic variable, it also includes a devolution and functions as both a situation of action and formulation. In the first subtask, students are asked to explain the formula: $4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 = 4\pi r^2 + 4A$. A hint (which is also a didactic variable) is provided, which students can use to understand and explain the formula. This also aims to test and develop students' mathematical reasoning. It is expected that students will try to use the hint, and some might also use the formulas provided in the devolution (the intended strategy), however some might get stuck and need help to realise, that they also need to use the formulas presented in the devolution. In the second subtask, students are asked to explain how one arrives at the formula for the angle sum, as presented in the theorem. If students are comfortable with basic arithmetic, they should recognize that it simply requires dividing all terms by $4r^2$. Some students may not immediately see this, so the teacher can guide them by asking some leading question, for example about how to "remove" $4r^2$.

After this, the geographic coordinates are presented from an analytical perspective. As mentioned in the epistemological analysis, this is important to include, if one decide to present the analytical expressions. Exercise 9 (p. 10 in Appendix A) is again a situation of action, where students must derive the conversion formulas using a figure with right-angled triangles - inspirered by Fajstrup and Nielsen (2005). The expectation is that students will follow the suggested strategy, i.e., using sine for right-angled triangles. However, some students may be uncertain about which angle they should apply sine to, even though it is relatively clear from the figure. The second part of the exercise may appear more challenging, where students must derive the conversion formulas for φ and λ, as well as the radius of the sphere. A hint is also provided to help students with the formula for the radius. Additionally, students may need some extra assistance recalling the inverse of cosine and sine functions, but after this, they should be able to apply the expected strategy to determine the conversion formulas. The subsequent Exercise 11 (p. 11 in Appendix A) requires students to apply the formulas they have just derived. There may be a need for an institutionalisation of the knowledge from Exercise 10 before students proceed to Exercise 11.

In continuation of the section on geographic coordinates, it was deemed appropriate to include a fundamental situation, which address students' possible epistemological obstacle regarding the shortest distances on a map by providing a deeper explanation of distances on the sphere. A micro choice was made to have students work with distances on the sphere first and then later connect this to distances on the map and distortions (which, in essence, is also then a macro choice). A devolution takes place, where the formulas for calculating distance and the internal angle are introduced. It was considered whether students should derive the formula for determining the internal angle themselves, as it simply requires using the formula for the angle between two vectors or even presenting the cosine and sine relations for spherical triangles, as done in Fajstrup and Nielsen's notes (2005). However, another approach was chosen, as it was assessed that time and space needed to be allocated to other aspects (crudely put).

Again, students are placed in a situation of action in Exercise 11 (p. 11 in Appendix A), where they must use the results from Exercise 10. Students are expected to convert the coordinates from Exercise 10 to radians before applying the distance formula. Although an attempt is made to recall students' knowledge of converting to radians, some students may still use the geographic coordinates in degrees. Here, a choice may arise in the teaching situation; whether to point out the mistake to these students or allow the situation to remain adidactic.

6.5 Fifth section: The mathematical expressions for map projections

It was decided to include the mathematical expressions for map projections to show students why map projections are essentially a mathematical construction (pp. 12-13 in Appendix A). Several considerations were made regarding whether this section should be included at all.

First, there is an institutional constrain in that students are not familiar with functions of multiple variables. This imposes limitations on how the mathematical expressions can be communicated. Given that students have knowledge of conversion formulas for geographic coordinates to Cartesian 3-dimensional coordinates, a macro choice was made to present the mathematical expressions using the same approach, even if this may come at the expense of the epistemology.

Second, the question arose: what purpose would this serve for the students beyond simply being informed about it? It was considered whether it would be possible to design an exercise in which students themselves derive these expressions, for example, using figures. While it is likely that students could manage this in the teaching situation with the right guidance from the teacher, it was ultimately decided that they would instead engage with this material through an assignment. Consequently, this section takes on a more explanatory role, essentially didactic in nature, without direct student involvement.

6.6 Sixth section: Why does map projections distort the Earth's surface?

The final section (pp. 14-16 in Appendix A) serves the dual purpose of linking spherical geometry to distortions and introducing scale factor as a quantitative method for measuring distortions on maps. Similar to Fajstrup and Nielsen's notes (2005), the impossibility theorem is presented as one of the main results concerning map projections. The impossibility theorem was actually one of the first macro-didactic variables identified when I was working on explaining the mathematics behind map projections. However, unlike in Fajstrup and Nielsen's notes (2005), a micro choice was made: students should not only be presented with the proof of the impossibility theorem but also contribute to the main argument of the theorem. This led to the inclusion of Exercise 12 (p. 14 in Appendix A), where students must provide a counterargument to the claim that a map projection can exist in which great circle segments are depicted as straight lines while also preserving angles. The exercise thus functions as a situation of formulation. It aims to highlight the difference between planar and spherical geometry (if that distinction was not already clear to students) while also training their mathematical reasoning through proof by contradiction. The assumption is that Danish upper secondary students work with proofs in their mathematics education but are still not entirely familiar with proof by contradiction. This is also the reason why students are not expected to present the full proof of the impossibility theorem themselves.

This exercise is designed to challenge even the most advanced students. Naturally, students who are less comfortable with mathematics will struggle to formulate the argument and will therefore require significant assistance from the teacher in reaching this insight. The expectation is that students will particularly rely on their partners for sparring and discussion as part of their knowledge development. Some students will use the information provided in the exercise and apply their acquired understanding of the differences between spherical triangles and planar triangles (differences in side lengths and angle sums) to formulate the counterargument. Since the full proof is not included in the teaching material, the institutionalisation must come from the teacher, who will go through the proof with the whole class.

It was decided to spend the last part of this section on scale factors on maps, for the same reason as presented earlier in relation to Exercise 3. A devolution takes place, including an example. Exercise 13 (p. 15 in Appendix A) aims to have students apply their acquired knowledge (thus creating a situation of action) using a physical map. Several micro-didactic variables are incorporated, including the choice of maps, the amount of information provided to students, and the selected geographic points. The exercise consists of seven subtasks, all of an adidactic nature. In the first two subtasks, students are expected to use the same strategy as in Exercise 3. Some students may once again calculate the Earth's circumference, but most are expected to reuse this from Exercise 3. In subtasks 3 and 4, students must apply their knowledge of geographic coordinates and distances on a sphere, which they are expected to manage since this content should still be fresh in their memory. In particular, subtask 3 serves as a situation of formulation. Subtasks 5 and 6 focus on scale factors, where students engage in a situation of action by measuring distances on the map (Appendix C), calculating the scale factor, and then evaluating how this aligns with distortions on the map. The students are expected to measure the distances on the map using a ruler, and then use the example as inspiration for solving. A key consideration was how these location pairs should be positioned relative to each other, in order to use the result in relation to distortions. If the locations were paired along the same latitude, developing the exercise would require significantly more effort, as the shortest distances on the map would also be curved lines. Therefore, it was decided that the locations should be paired along the same longitude instead.

The last subtasks (4,5,6 and 7) in Exercise 13 was inspired by Anderson's and Leinhardt's suggestion, mentioned in the didactical analysis. The goal is to get the students to realise, that one can easily determine whether or not a map is distorted, by studying the lines of the parallels and meridians as well as the shortest distances on maps.

After further consideration, it was decided that students should conclude a potential lesson on map projections with a written assignment. An assignment provides students with an opportunity to apply and reinforce the knowledge they have acquired through the teaching material, as well as to reflect on what they have actually learned. The purpose of this assignment is to fill in any gaps that have arisen or have not been addressed in the teaching materials (or the lesson itself). In this way, certain didactic variables can be identified that should be incorporated into the assignment: maps that have utilized the map projections presented on pp. 12-13 in Appendix A, along with the mathematical expressions for these projections.

Students are first asked to find maps (using the internet, which is also a didactic variable) where the map projections presented have been applied, describe (again) the visible distortions, and assess what these maps are used for. The students have already seen maps with cylindrical projections in previous exercises but have not encountered a map using the orthographic plane projection. Additionally, the students have only worked with one map at a time, and in this assignment, they must gather all the maps together, refresh their memory on the properties, and assess the distortions. They will also be tested on their (geographical) knowledge of what these maps can be used for. Furthermore, students are asked to explain the mathematical expressions for the map projections. Here, they must use their knowledge of the conversion formulas for geographic coordinates to Cartesian coordinates, as well as their knowledge of trigonometry for right-angled triangles. The assignment can be done in small groups or pairs, but it could also be interesting if it is completed individually. Since this is a home assignment, students might just use the internet as a strategy for solving this assignment. However, it is intended that the students use the teaching material and the exercises for reference.

7 Discussion

The design of the teaching material for map projections was guided by both theoretical (like the preliminary analyses and TDS) and practical considerations. The goal was to introduce upper secondary students to map projections an an interdisciplinary topic, supporting both the development of geographical and mathematical knowledge. Although it can be difficult to discuss one's own product without empirical data, as it risks becoming too speculative, this discussion seeks to critically reflect on some of the choices made in the design, considering their effectiveness, possible limitations, and areas for possible improvement.

The first critique is regarding some of the macro choices made in the beginning of the design. A key decision was to focus on spherical geometry and analytical geometry due to their relevance to map projections. While this aligns well with the Mathematics A curriculum and the findings in the epistemological analysis, one could question whether the mathematical depth is appropriate for all students. The inclusion of spherical triangles and coordinate conversions, which will be entirely new knowledge for students to assimilate, provides a strong foundation but may challenge students with weaker mathematical backgrounds. More exercises in which the students assert distortions on maps, before engaging with formal mathematical concepts, could make the topic more accessible for students which are less comfortable with mathematics. Likewise, the structure of the material, where the classification and characteristics of map projections were introduced before the mathematical formulation, is aimed to strengthen the connection between geography and mathematics. This approach helps students understand the necessity of mathematical tools but also risks creating a disconnection between theory and application, even though not intended. A more integrated method, intertwining mathematical concepts with the characteristics of map projections, could reinforce understanding more effectively. For example, a map projection being area-preserving might seem obvious, but what does it really mean to be angle-preserving? When introducing angles on the sphere, it could be effective to recall the characteristic of angle-preserving. However, this was not done, because one would need the understanding of infinitesimal quantities, which is not a prerequisite required by students.

TDS was employed progress the exercises and guide student knowledge construction. The incorporation of devolution and institutionalisation phases (even though the institutionalisation phases might not be explicitly stated in the teaching material) aimed to create meaningful learning situations. However, a critical reflection on the exercises, particularly those related to mathematical content, suggests that some may resemble more traditional mathematics exercises where students apply given formulas rather than developing their own solving strategies. While this approach ensures efficiency, it may not fully align with the principles of TDS, which emphasize the need for students to develop knowledge through adidactic situations. This raises the question of whether more open-ended exercises could have been incorporated to encourage multiple solution strategies and deeper engagement with the concepts. Likewise, the exercises does not include a situation of validation, which again may limit the students knowledge production and the feeling of autonomy. In relation to this, another important consideration is that many exercises allow for only one method of solving, limiting students' ability to explore alternative strategies. This could restrict their ability to develop flexible problem-solving skills and their mathematical reasoning. In some cases, providing multiple ways to approach a problem or encouraging students to devise their own methods before introducing formulas might have supported more meaningful learning experiences. The potential trade-off between efficiency and conceptual exploration is an important factor in assessing the overall effectiveness of the teaching material.

It could also be discussed whether the choice of only focusing on cylindrical might not be as appropriate as intended. At first seem like a way to ensure that students were able to work with mathematical concepts in relation to map projections. However, not including other map projections at all, other than the orthographic map projection, could result in students not gaining a broader understanding of map projections and distortions in regards to other classes of map projections. This also lead to the omission of the azimuthal characteristic, not providing students a rather important characteristic. However, this relate to a more general discussion whether or not it is better to gain a deeper understanding of one or few aspect of some content, or a broader understanding.

The teaching material incorporates didactic variables, such as group work and structured exercises. While group work fosters collaboration and knowledge-sharing, it also introduces the risk of passive participation. More structured individual accountability, such as requiring students to present findings, might help this issue. Additionally, the decision to present some mathematical results (such as the results about great circles, or the formula for the spherical angle in relation to distances on the sphere) rather than having students derive them aimed to streamline learning but may have reduced opportunities for deeper conceptual engagement. Encouraging students to actively construct knowledge, rather than passively receiving it could enhance their learning experience. The assignment serves as opportunity to assess the students individual learning, as well as an opportunity for students to synthesize their learning by analysing real-world maps. However, there is a risk that it becomes a superficial exercise if students rely too heavily on internet searches (or the use of large language models) rather than engaging deeply with the teaching material.

8 Conclusion

The goal of this thesis was to examine the following:

"How and to what extend can map projections, as an interdisciplinary topic between mathematics and geography, be communicated to Danish upper secondary students?"

Even though the purpose of the use of Didactical Engineering was to develop teaching material on map projections, which should serve as an example to answer the research question, its preliminary analyses has also proved useful to investigate the research question itself.

The assumption was that students are not taught about map projections in Danish upper secondary schools, which is supported by the results of the institutional analysis. The institutional analysis also suggests a possible limitation in the implementation of such teaching (as it is not prioritized in curricula or textbooks, and because Mathematics A and Natural Geography B are not necessarily part of the same study program). However, there are opportunities for meaningful transposition of the theory of map projections. According to the curriculum, and as supported by textbooks, students must go through a course on geometry, trigonometry, and vectors in 2- and 3-dimensions. This indicates that as long as the theory of map projections, especially the mathematics, can be linked to concepts within this core content, it should be possible for students to understand a significant portion of the scholarly knowledge, while also preserving some epistemology. The didactical analysis identified an example of how and which aspects of the mathematics behind map projections can be communicated to upper secondary students, and also how students, through inquiry-based tasks, can develop an understanding of distortions in maps — an approach that has inspired the designing of the teaching material. The teaching material thus presents an example of how and to what extent map projections can be communicated to upper secondary students.

It demonstrates that a more geographical approach to map projections and their classification is possible and can serve as a meaningful transition to the mathematical aspects. It highlights the interdisciplinarity between mathematics and geography, showing how mathematics functions both as a tool and an explanation, while providing students with the means to describe the distortions they observe on maps. Similarly, incorporating elements such as circles, planes, and vectors in the mathematical presentation ensures that students can relate to previously learned concepts, making the material more accessible and easier to grasp. However, challenges may arise when presenting the analytical expressions for map projections, as students lack the necessary background to understand them as functions of multiple variables, or to comprehend how distortions on maps can be quantified. Nevertheless, the teaching material offers alternative approaches to addressing these challenges.

The thesis contributes to a broader understanding of how complex mathematical concepts can be adapted and communicated in upper secondary education. While the theoretical development of the teaching material provides a possible approach to teaching map projections, the discussion shows there might still be room for improvement and further clarification. Further studies, such as practical implementation and testing, would be valuable in assessing its educational impact, providing more insight to how and what extend map projections can be communicated, such that it ensure students development of knowledge also. A future expansion could focus on student responses, teacher experiences, and the development of alternative teaching methods that further enhance students' understanding of both the mathematical and geographical aspects of map projections.

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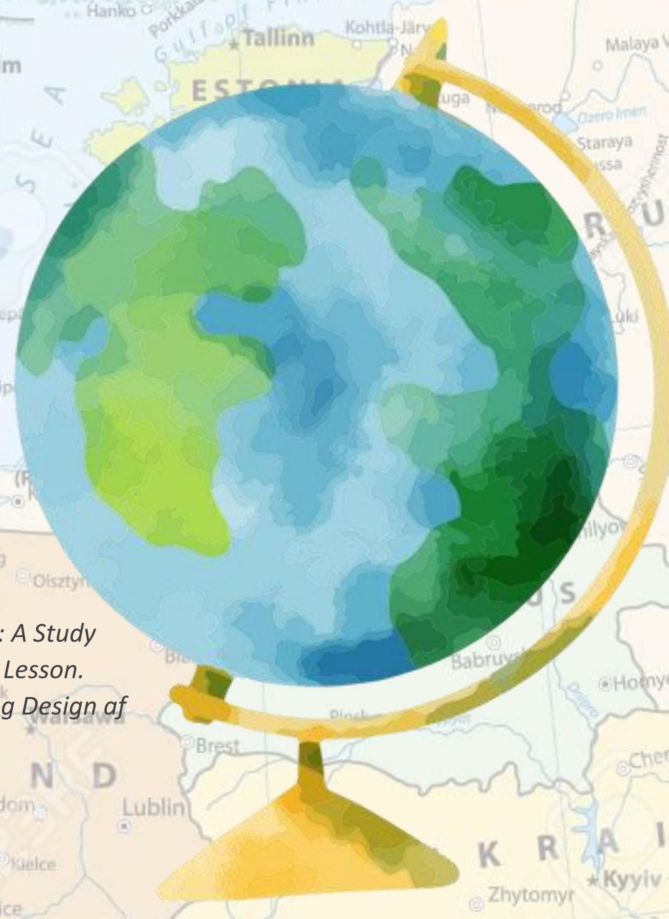
9 Appendix A

The teaching material in Danish is found on the next pages.

KORTPROJEKTIONER

Et undervisningsforløb om kortprojektioner og forvrængninger.



Noter skrevet af Josefine Dixen Zwisler.

En del af specialet *The Mathematics of Map Projections: A Study and Design of Teaching Material for An Interdisciplinary Lesson. (Dansk: Matematikken bag Kortprojektioner: Et Studie og Design af Undervisningsmateriale til et Tværfagligt Forløb).*

1. Introduktion

Kort; hvad er et kort egentlig? I kender måske mest til kort gennem jeres brug af Google Maps eller Kort-appen på iPhone. I har helt sikkert også stødt på nogle kort i jeres geografibog, eller måske set et kort i ny og næ i tv'et. I geografi er kort et meget benyttet værktøj, hvor det bruges til at beskrive rumlige mønstre og sammenhænge på Jordens overflade.

Øvelse 1:

Giv nogle eksempler på, hvad et kort kan bruges til. Hvordan tror I, man laver et kort?

Produktionen af kort har fundet sted i mange tusinde år. Vi tilskriver de gamle grækere som de første til "rigtigt" at arbejde med kort produktion, hvor de kortlagde stjernehimlen, men også landjorden. Kortene dengang var faktisk meget nøjagtige, med tanke på, at de ikke havde samme teknikker og værktøjer som vi har i dag[1]. I dag bruger vi satellit-data og GIS til at producere (digitale) kort. Produktionen og læren om kort er en gren af geografien, kaldet *kartografi*. Som I kommer til at se, danner matematikken de vigtigste grundsten i kartografien.

Som I nok ved, så findes der kort der viser byer, regioner, lande og endda hele verden. Det er specielt sidstnævnte, såkaldte *verdenskort*, der er fokusset i dette forløb. Derudover kan kort også have nogle forskellige temaer, som de forsøger at vise, f.eks. højder (topografiske kort), socioøkonomiske forhold, grænser, arealanvendelser osv. Verdenskort er specielt interessante, fordi der opstår nogle problemer, når man forsøger at vise Jorden på én gang, hvilket vi kommer til at se nærmere på i dette forløb.

2. Hvad er kortprojektioner?

Øvelse 2:

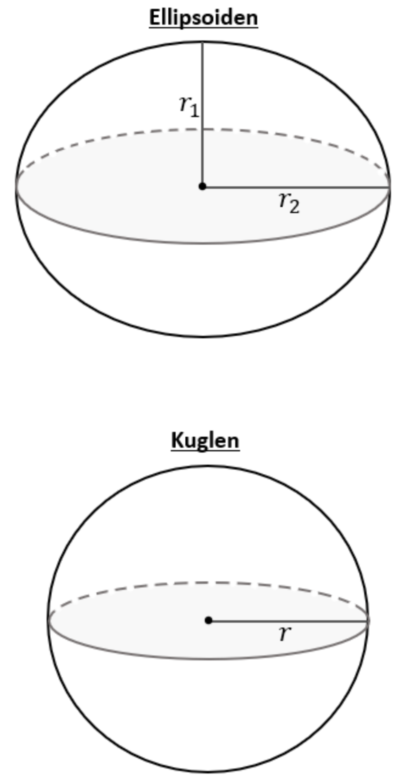
Hvordan forestiller I jer, at et verdenskort ser ud? Lav en hurtig skitse af et verdenskort. I kan evt. benytte jer af Google Earth[2], hvis I er i tvivl om hvordan Jordens overflade nu ser ud.

Det I lige har forsøgt jer med, er en slags *kortprojektion*, dog uden brug af de rigtige teknikker. En kortprojektion er metoden man bruger til at producere kort, og er i sin natur (i dag) en matematisk konstruktion. Vi kommer her til at gennemgå den overordnede tankegang bag kortprojektioner, før vi dykker ned i den matematiske del.

[1] Hvis I er interesseret i mere viden om kort produktion før computere og satellitter, så anbefales det at se YouTube-videoen *How Our Earth was Mapped before Satellites* af Interloop (https://www.youtube.com/watch?v=yoO5O_QJ3N8).

[2] Det er værd at notere, at Google Earth selvfølgelig kun er en model af Jorden. Men den er baseret på en model og en masse data, der gør, at dens beregninger er stemmer overens med virkeligheden, ned til få meters afstande. Derfor bruger vi den som reference til hvordan Jorden nogenlunde ser ud i virkeligheden.

En *projektion* er en afbildning, en funktion, der sender et objekt fra et fler-dimensionelt rum hen på en flade. I tilfældet med *kortprojektioner*, *projicerer*[3] man altså ikke bare Jordens overflade hen på et fladt kort. Først skal vi bestemme, hvilken (matematisk, geometrisk) model vi vil repræsentere Jorden med. Jordens form er kompleks, der er svær at gengive, og derfor ser man sig nødsaget til at vælge en model, man kan beskrive matematisk. Den skal være rumlig og 3-dimensionel, ligesom Jorden. Det er af generel opfattelse, at Jorden har form som en kugle. Dette er til dels korrekt, men Jorden er ikke perfekt kugleformet. Jorden bliver lidt sammentrykt ved polerne grundet rotation, og har derfor mere form som en såkaldt *ellipsoide*[4]. En kugle har, som I nok ved, samme radius til hvilket som helst punkt på kuglens overflade, hvorimod en ellipsoide har forskellig radius til de forskellige punkter på dens overflade (se Figur 1). I tilfældet med Jorden, så bliver Jorden trykket sammen ved polerne, dvs. Jordens radius mod polerne er mindre end mod Ækvator. Vi vil dog gå med kuglen som den simplere model af Jorden, da det er lettere at antage, at Jorden har én radius i stedet for flere forskellige.



Figur 1: Ellipsoiden og kuglen.

Før vi kan begynde arbejdet med kortprojektioner, så skal vi gøre denne kugle mindre. Det er for at gøre størrelsesforholdet mere håndgribelige. Dette kommer også til at have betydning for, hvor stort vores kort ender ud med at være. Hvis vi lod kuglen have ramme radius som Jorden (rettere sagt, Jordens radius langs Ækvator), vil vi ende med et enormt stort kort, der vil have samme størrelsesorden som Jorden. Det går selvfølgelig ikke, så vi nedskalerer kuglen til en væsentlig mindre radius.

Øvelse 3:

Antag, at Jorden er en perfekt kugle med konstant radius på 6378 km. Hvad er Jordens omkreds i km og i cm? Ydermere, så skalerer vi kuglen ned til en radius på 60 cm. Hvad er denne kugles omkreds i cm? Udregn også, hvad 1 cm på kuglen svarer til på Jorden.

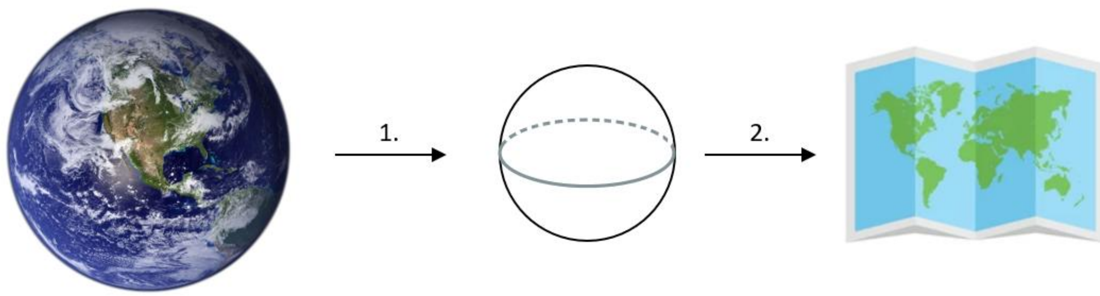
Udgangspunktet for kortprojektioner er denne mindre kugle (som fremadrettet bare bliver kaldt for *kuglen*), som repræsenterer Jorden. Kortprojektioner vil altid forvrænge enten former, arealer, længder, ja endda vinkler, fra kuglens overflade, så de misvises på kortet.

[3] Det hedder *projicere*, og ikke *projektere*. At *projektere* har noget at gøre med projektarbejde.
[4] Også kaldet *omdrejningsellipsoiden.* Selv dette er en idealisering af Jorden.

Øvelse 4:

Hvorfor tror I, at kortprojektioner forvrænger kuglens overflade?

Arbejdet med kortprojektioner indebærer altså at man udvælger en model af Jorden (kuglen), nedskalaer den og så projicere kuglens overflade (som egentlig bare består af en masse punkter) hen på en flad overflade (Figur 2).



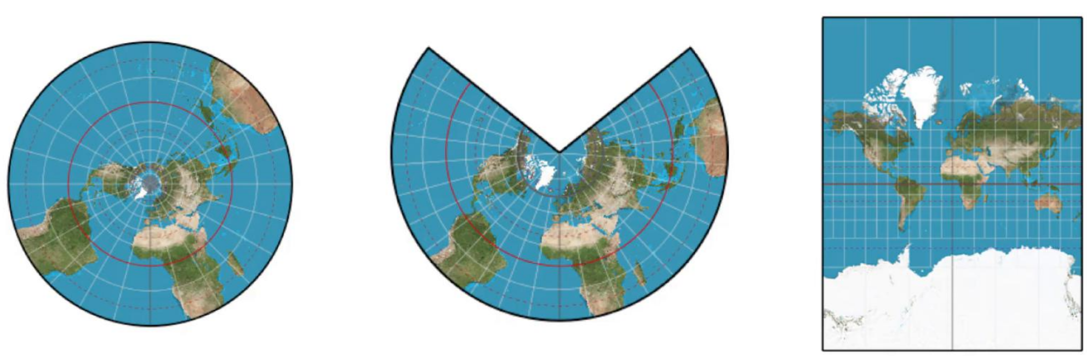
Figur 2: Groft set arbejdet med kortprojektioner. Trin 1: udvælgelse og nedskalering af model af Jorden. Trin 2: projicering af kuglens overflade hen på en flade, så man får et kort.

3. Hvilke kortprojektioner findes der?

Der findes forskellige slags kortprojektioner, grundet kortprojektioner forvrænger. Her skelnes der mellem deres *typer* og deres *egenskaber*. Kortprojektionens *type* baseres på følgende: da Jordens og heraf kuglens runde form er umulig at gøre flad, har man behov for nogle andre geometriske former, som man kan projicere hen på og som derefter kan gøres flade som et kort.

Øvelse 5:

I er givet tre kort (som i Figur 3), hvor der er benyttet tre forskellige *typer* kortprojektioner. Klip disse ud og prøv at finde ud af, hvilken figur der er tale om. I er givet en kugle (miniudgave af Jorden), hvordan vil I placere formen på kuglen?



Figur 3: Lignede kort, som I har fået udleveret med Øvelse 4.

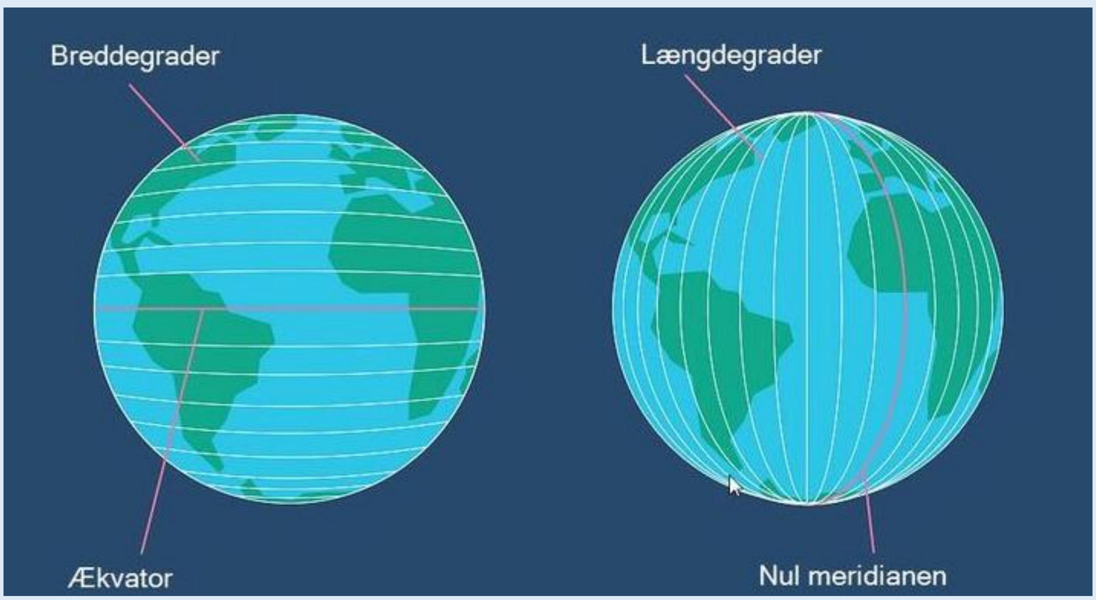
Faktaboks: Breddegrader og længdegrader.

Til lokationsbestemmelse på Jordens overflade bruger vi *breddegrader* og *længdegrader*. Som begreberne antyder, måles begge i grader, og altså vinkler set fra Jordens centrum. Ved visualisering af bredde- og længdegraderne danner man et slags geografisk koordinatsystem på Jordens overflade. Dette koordinatsystem kan også bruges på kuglen, da omtales det som sfærisk koordinatsystem.

Breddegrader måler, hvor nord eller syd en lokation er, i forhold til Ækvator. Graderne varierer fra 0° ved Ækvator til 90° ved polerne, og man skelner mellem nord og syd for Ækvator ved at notere med hhv. N eller S. På modeller af Jorden visualiseres breddegrader som *parallelle* linjer eller cirkler til Ækvator.

Længdegrader, også kaldet *meridianer*, måler hvor øst eller vest en lokation er, i forhold til Greenwich meridianen (nulmeridianen). Her varierer graderne fra 0° ved Greenwich til 180°, og noteres med enten Ø (øst for Greenwich) eller V (vest for Greenwich). Længdegrader visualiseres som buede linjer, langs overfladen, der går fra Nordpolen til Sydpolen.

Tilsammen udgør de *geografiske koordinater*, hvor breddegrad noteres først og længdegrad bagefter, f.eks. 55.68°N, 12.57°Ø. Bemærk, her skrives kommatal med punktum.

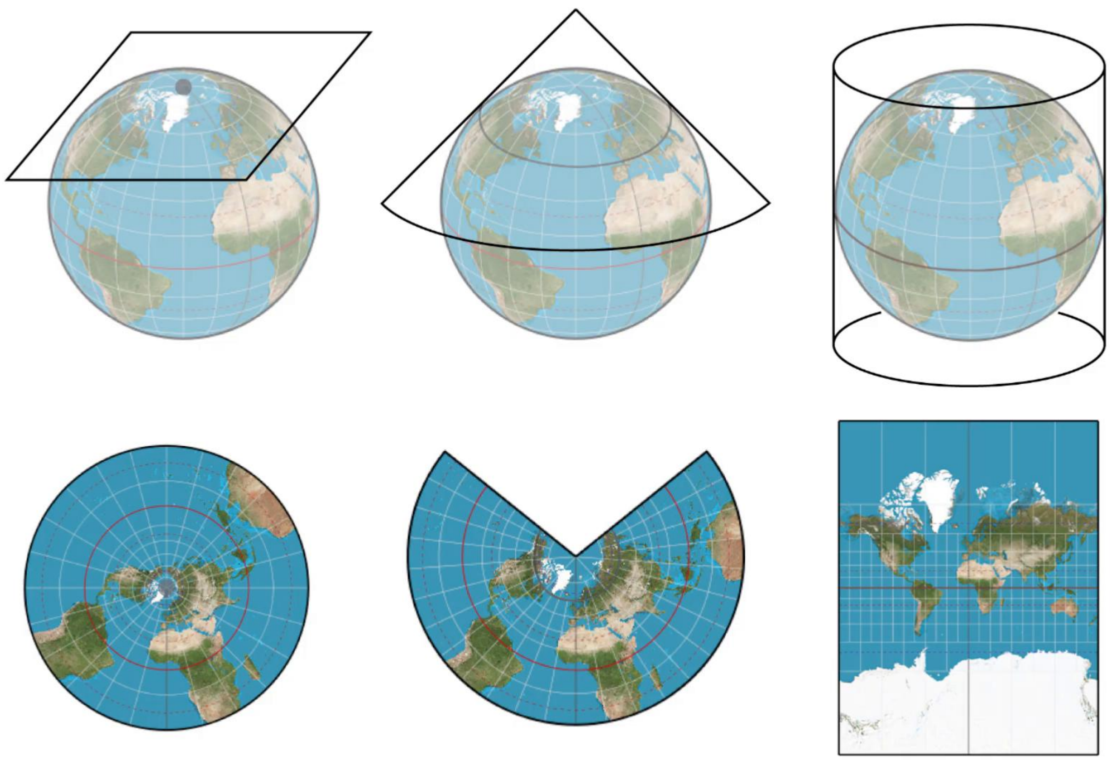


De tre forskellige slags kort, I har fået udleveret, er resultatet af de tre typer kortprojektioner: *planprojektionen*, *kegleprojektionen* og *cylinderprojektionen* (Figur 5). Der findes mange forskellige typer kortprojektioner, men disse tre er de mest kendte og benyttede.

Ved *planprojektionen* bliver der projiceret direkte hen på en flade, også kaldet *en plan* i matematikken, der tangerer (rører) kuglen i ét bestemt punkt. Derfor får kortet en cirkulær form. Denne type kortprojektion bliver typisk brugt til fremstilling af polare kort, og egner sig mest til visning af halvkuglen. I disse tilfælde fremstår meridianerne som rette linjer, der alle stråler ud fra Nordpolen eller Sydpolen (Figur 5).

Næste er *kegleprojektionen*. Her kan I forestille jer, at man placerer en kegle hen over kuglen, så den tangerer kuglens overflade langs en linje (eller cirkel, breddegrad cirkel) på kuglens overflade, og projicerer hen på keglen. Kortets form bliver, som hvis man klipper keglen op og flader ud (Figur 5). Ved denne kortprojektion vil meridianerne også fremstå som rette linjer, der stråler ud fra keglens top.

Den sidste type der nævnes, er *cylinderprojektionen*. Kuglen placeres "inde i" en cylinder med samme radius, der også tangerer cylinderen langs en linje, og der projiceres hen på cylinderen. Ved denne type projektion får kortet en rektangulær form. Her vil både breddegraderne og meridianerne fremstå som rette linjer på kortet.



Figur 5: Fra højre mod venstre: planprojektion, kegleprojektion, cylinderprojektion. Fra: https://mapscaping.com/understanding_map_projections/.

Vi kommer ikke til at gå i flere detaljer om kegleprojektionerne, og heller ikke så meget med planprojektionerne. Som sagt, så egner planprojektionerne sig ikke til verdenskort, og kegleprojektioner er meget komplekse at fremstille og forstå. Så fremadrettet vil vi arbejde cylinderprojektioner. I vil dog se et eksempel på, hvordan man kan udtrykke en bestemt planprojektion.

Ud over typer af kortprojektioner, kan kortprojektioner (uanset type) have forskellige *egenskaber*. Nogle kortprojektioner er *arealbevarende*, dvs. bevarer arealer fra kuglens overflade til fladen, kortet. Nogle kortprojektioner er *vinkelbevarende*, dvs. bevarer lokale (i et punkt eller mellem to linjer) vinkler på kuglens overflade til kortet. Det betyder, at der bevares retninger. Kortprojektioner med denne egenskab kaldes også for *konform*. Så er er kortprojektioner, der kan bevare nogle afstande, ikke alle! Disse egenskaber er noget et kort arver.

Hvis der fandtes en kortprojektionen, som bevarede Jordens overflade perfekt, eller rettere sagt kuglens overflade, så skulle kortprojektionen bevarer alle afstande, samtidig med arealer og vinkler. Men det er matematisk umuligt, og kortprojektioner vil altid forvrænge kuglens (Jordens) overflade i en eller anden grad. Derfor må man gøre sig nogle overvejelser i forhold til hvad formålet med kortet er: hvad skal kortet bruges til? Dette bestemmer hvad der er vigtigt for kortprojektionen at bevare.

Øvelse 6:

I får nu udleveret et kort med en cylinderprojektion og skal nu til at overveje følgende:

a) Tror I den er arealbevarende eller vinkelbevarende?
b) Hvad vil I mene dette kort kan bruges til?
c) Der er markeret nogle lokationer på kortet: tegn de korteste afstande mellem lokationerne, som I tror den er i virkeligheden.

I kan til fordel igen benytte jer af Google Earth til sammenligning.

Vi vil senere se nærmere på, hvorfor det er umuligt for kortprojektioner at bevare alle egenskaberne. Indtil da skal vi se nærmere på den grundlæggende matematik bag kortprojektioner, så vi nemlig kan forsvare, hvorfor der ikke findes en kortprojektion, der bevarer alle egenskaberne.

4. Matematiske forudsætninger for kortprojektioner

Arealer, vinkler, linjer, afstande er alle væsentlige begreber fra geometrien. I har arbejdet med geometri i 2-dimensioner, i planen. Der eksisterer også geometri på kuglen, som kaldes for *sfærisk geometri*. Kortprojektioners opgave er at bevare noget af sfæriske geometri, når vi projicerer fra kuglen til en flade. Det viser sig at være lidt af opgave, da den sfæriske geometri er anderledes end geometrien i planen.

Kuglen er en 3-dimensionel figur, som vi anskuer i det 3-dimensionelle rum med x-, y-, og z-akser. Lad kuglens centrum være i (0,0,0), noteret med O.

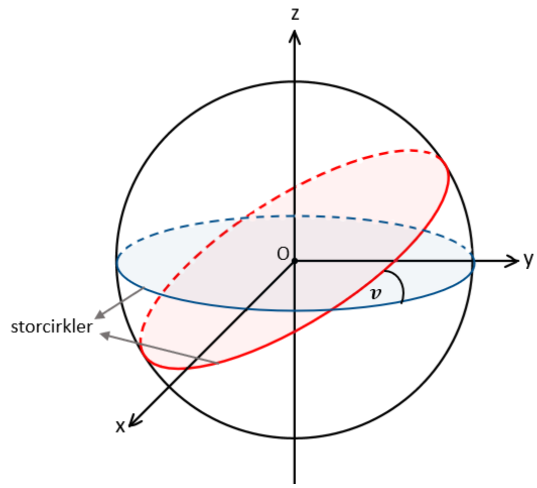
Linjer på kuglens overflade er ikke linjer, som I kender dem fra planen, men er linjestykker af *storcirkler*:

Definition 1. En storcirkel er den cirkel på kuglens overflade, der fremkommer ved skæringen mellem kuglens overflade og en plan, der går gennem kuglens centrum.

Linjestykker af en storcirkel kalder vi *storcirkelstykker*.

Faktum:

- Storcirkler er de største cirkler på kuglen.
- Ækvator er en storcirkel. Meridianer er storcirkelstykker.
- Den korteste afstand mellem to punkter på kuglens overflade er langs den korteste storcirkelstykke, der forbinder de to punkter.
- To storcirkler vil altid skære hinanden i to punkter, der er diamental modsatte (dvs. antipoder). Specielt går der uendelig mange storcirkler gennem to punkter, der er diamental modsatte.



Figur 6: Storcirkler på kuglen, inklusiv vinkler på kuglen.

Vi bruger storcirkler til at definere vinkler på kuglen. Pr. definition 1 opstår en storcirkel ved at lade en plan, der går gennem centrum, skære kuglens overflade. Har man to storcirkler, har man altså to planer, som skærer hinanden i to punkter på kuglen (diamentale punkter).

Definition 2. En vinkel v på kuglens overflade, er den stumme vinkel, dannet af skæringen mellem to storcirkler. Denne er den samme som den stumme vinkel mellem de to skærende planer, der danner storcirklerne.

At udregne vinkler på kuglens overflade er altså det samme som at udregne vinklen mellem to planer. Bemærk, her taler vi om den spidse vinkel v, som kan ses i Figur 6. Vinklen v danner en såkaldt *tokant*. Faktisk dannes en lignende tokant på ”bagsiden” af kuglen.

Er vi givet ligningerne for planerne kan vi nemt finde vinklen mellem de to planer. Vi husker fra vektorer i 2D, hvordan man finder vinklen mellem to vektorer. Lignende tanke gør sig gældende for udregning af vinklen mellem to planer, men her benytter man sig af planernes normalvektorer. Så lad a og b være to planer, der går gennem kuglens centrum og skærer med kuglens overflade. Lad $\overrightarrow{n_a}$ og $\overrightarrow{n_b}$ være normalvektorerne til planerne a og b, henholdsvis. Da kan man bestemme vinklen mellem de to planer ved:

$$\cos(v) = \frac{\overrightarrow{n_a} \cdot \overrightarrow{n_b}}{|\overrightarrow{n_a}| \cdot |\overrightarrow{n_b}|}$$

Øvelse 7:

Lad to planer a og b, der skærer enhedskuglen i centrum, være givet ved ligningerne: a: $x + 2y + 2z = 0$ og b: $2x - y + z = 0$. Udled normalvektorerne for planerne og bestem vinklen mellem planerne, dvs. vinklen på kugleoverfladen.

Er man givet tre storcirkler på kuglens overflade, vil man få dannet en trekant på kuglen – en *sfærisk trekant* - hvor storcirklerne parvis skærer hinanden i punkterne A, B, C, og derved danner vinklerne α, β, γ (Figur 7). For at gøre det fremadrettet arbejde "lettere", skal vi minde os selv om, hvordan vi regner en vinkel v målt i grader om til radianer:

$$\text{radianer } = v \cdot \frac{\pi}{180°}$$

Dette gør det lettere at arbejde med vinkler fremadrettet. Her er et vigtigt resultat om sfæriske trekanter:

Sætning 1. Givet en sfærisk trekant ABC, med vinkler α, β, γ, da er vinkelsummen givet ved:

$$\alpha + \beta + \gamma = \pi + \frac{A}{r^2}$$

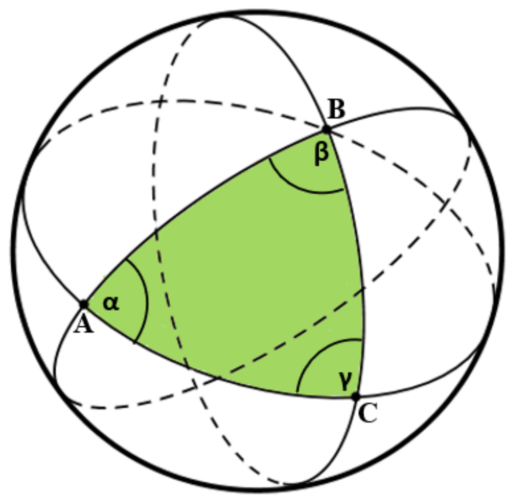
med A arealet af den sfæriske trekant og r kuglens radius.

Øvelse 8:

Formålet med denne øvelse, er at forstå argumenterne bag beviset for Sætning 1 ovenfor.

I bund og grund handler det om at anskue arealer af de forskellige (sfæriske) geometriske overflade figurer, man får dannet, når man har en sfærisk trekant på kuglen.

Se på den sfæriske trekant ABC i Figur 7. De tre vinkler α, β, γ danner tre tokanter. En tokant med en vinkel på π radianer vil dække hele kuglen, så arealet for en tokant med vinkel v er: $\frac{v}{\pi} \cdot 4 \cdot \pi \cdot r^2 = 4vr^2$.



Figur 7: En sfærisk trekant på kuglens overflade.

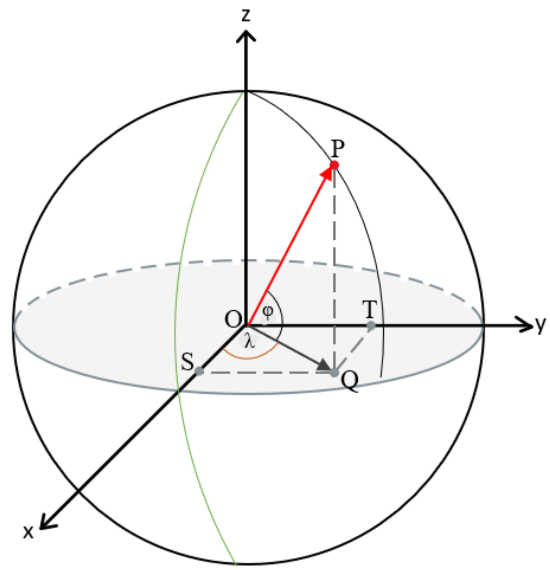
Dette følger af, at arealet af en kugles overflade er $4\pi r^2$. Vi lader A betegne arealet af den sfæriske trekant ABC. I beviset for sætningen, udleder man følgende formel om arealerne for de områder, der bliver dannet af tokanterne og den sfæriske trekant:

$$4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 = 4\pi r^2 + 4A$$

1) Forklar formlen. Hvorfor ser den ud som den gør? Hint: tage et led ad gangen og forklar hvad det er for et areal, der er tale om. Brug gerne Figur 7 til forståelse.
2) Hvordan ender vi med formlen for vinkelsummen: $\alpha + \beta + \gamma = \pi + \frac{A}{r^2}$

4.1. Geografiske koordinater

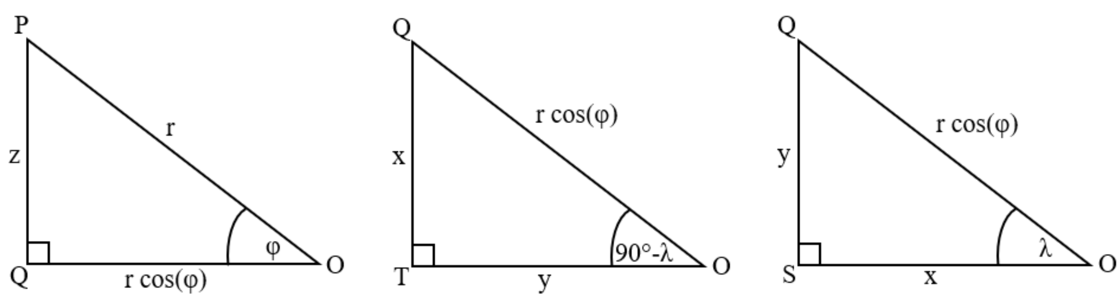
I er blevet introduceret til de geografiske koordinater (se Faktaboks på s. 5). Når vi snakker om kortprojektioner, vil vi gerne kunne udtrykke de geografiske koordinater som et punkt P på kuglen i rummet. Det gør vi ved at bestemme koordinaterne til stedvektoren $\overrightarrow{OP}$ i det 3-dimensionelle rum (Figur 8).



Figur 8: Et punkt P på kuglen i rummet. Den grønne meridian er Greenwich meridianen.

Lad z-aksen gå lodret op gennem Nordpolen. Vi ser nu *breddegrader* og *længdegrader* som decideret vinkler. Lad P være et vilkårligt punkt på kuglen. Breddegraden φ er vinklen mellem stedvektoren $\overrightarrow{OP}$ og xy-planen. Vi lader $\varphi \in [-90°, 90°]$, hvor negative grader er vinkler syd for Ækvatorlinjen og positive grader er nord for Ækvatorlinjen. Længdegraden λ er vinklen mellem Greenwich meridianen (vi lader x-aksen gå gennem denne) og meridianen der går gennem P. Vi lader $\lambda \in [-180°, 180°]$, hvor negative grader er vest for Greenwich meridianen og positive grader er øst for Greenwich meridianen.

På Figur 7 ser vi, at hvis vi lader punktet P projicere vinkelret ned på xy-planen, kan vi danne punktet Q og dermed stedvektoren $\overrightarrow{OQ}$. Da stedvektoren $\overrightarrow{OP}$ har længde r, må stedvektoren $\overrightarrow{OQ}$ have længde $r \cdot \cos(\varphi)$. Punktet Q kan projiceres vinkelret hen på hhv. x- og y-aksen, hvor vi finder punkt S og punkt T, hhv. Vi får følgende retvinklede trekanter:



Figur 9: Retvinklet trekanter udledt af Figur 7.

Øvelse 9:

Brug Figur 9 til at udlede konverteringsformlerne for koordinaterne x, y og z til det 3-dimensionelle koordinatsæt (x, y, z). Hint: brug sinus for retvinklede trekanter og $\sin(90° - v) = \cos(v)$.

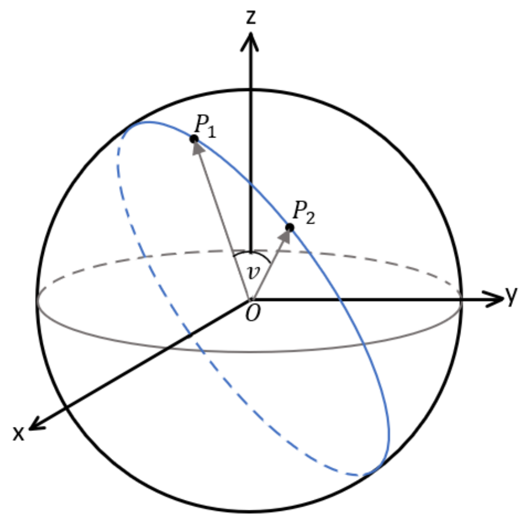
Givet koordinatsættet (x, y, z), udled formlerne til udregning af de geografiske koordinater (φ, λ) og radius r. Hint: kuglens ligning er givet ved: $r^2 = x^2 + y^2 + z^2$, for en kugle med centrum i (0,0,0).

Øvelse 10:

Vi er givet de geografiske koordinater på Jorden for København ved $(55.7°, 12.6°)$. Omregn dette til 3-dimensionelle koordinater (husk, Jorden har radius 6378 km). Derudover, så er vi givet de 3-dimensionelle koordinater på Jorden for Tanga, Tanzania: $(374.16\,, 2180.95\,, 5981.83)$. Omregn disse til geografiske koordinater.

4.2. Afstande på kuglen

Givet geografiske koordinater (φ, λ) kan vi altså omregne til 3-dimensionelle kartesiske koordinater, men er vi givet to sæt geografiske koordinater (φ_1, λ_1) og (φ_2, λ_2) kan vi f.eks. bestemme den korteste afstand mellem disse på kuglens overflade. Husk, at den korteste afstand mellem to punkter på kuglen er langs den korteste storcirkelstykke mellem punkterne (Figur 10). Dette er årsagen til, at korteste afstande på kort ikke nødvendigvis er en ret linje. Lad $P_1 = (\varphi_1, \lambda_1)$ og $P_2 = (\varphi_2, \lambda_2)$, da kan vi igen danne stedvektorerne $\overrightarrow{OP_1}$ og $\overrightarrow{OP_2}$. Afstanden bliver udregnet ved hjælp af vinklen v, målt i radianer, mellem disse stedvektorer (Figur 10). Denne kalder vi *indre vinkel* mellem punkterne.



Figur 10: To punkter på kuglens overflade og vinklen imellem.

Den korteste afstand d på kuglen udregnes som følgende:

$$d = r \cdot v$$

hvor r er radius af kuglen. Er man kun givet de geografiske koordinater, skal vi først have udregnet den indre vinkel v. Husk, at de geografiske koordinater er i grader, så de skal også omregnes til radianer, før vi kan bruge formlen:

$$\cos(v) = \sin(\varphi_1)\sin(\varphi_2) + \cos(\varphi_1)\cos(\varphi_2)\cos(\lambda_1 - \lambda_2)$$

Denne formel er et eksempel på brugen af cosinus-relationerne i en sfærisk trekant, og kan udledes ved at bruge formlen for vinkler mellem to vektorer (her de to stedvektorer).

Øvelse 11:

Givet de geografiske koordinater fra Øvelse 10, udregn den korteste afstand d på Jordens overflade mellem København og Tanga.

5. Matematiske udtryk for udvalgte kortprojektioner

Kortprojektioner handler som nævnt om at projicere kuglens overflade over på en flade, en geometrisk figur, der omkranser kuglen og som kan ”klippes op” og flades ud. Denne projicering kan vi beskrive med matematik. Vi har allerede snakket om konverteringsformlerne for geografiske koordinater til 3-dimensionelle koordinater. Det er sådan set samme tankegang vi bruger, når vi vil beskrive kortprojektioner matematisk; at konvertere punkter på kuglen til punkter hen på en flade, så man får et kort. Kortet er dog ikke 3-dimensionel, men kun 2-dimensionel. Så vi er kun interesseret i at konvertere til x- og y-koordinater.

For simpelhedens skyld, lader vi kuglen, vi projicerer fra, have radius 1.

Vi ser først på et eksempel med en planprojektion:

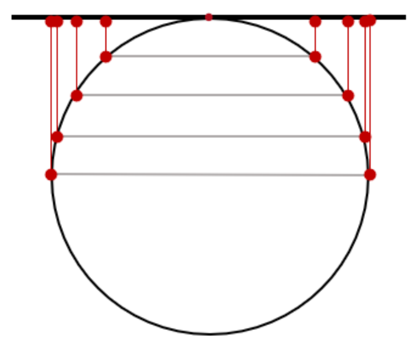
Ortografisk planprojektion:

Her lader vi projektionsplanen tangere kuglen i et punkt, og projiceret *vinkelret* hen på planen (Figur 11). Fra konverteringsformlerne for geografiske koordinater til 3-dimensionelle koordinater ser vi:

$$x = \cos(\varphi)\cos(\lambda)$$

$$y = \cos(\varphi)\sin(\lambda)$$

Den viser meridianerne som rette linjer på kortet.



Figur 11: Ortografisk planprojektion.

Vi kommer nu til nogle eksempler på cylinderprojektioner:

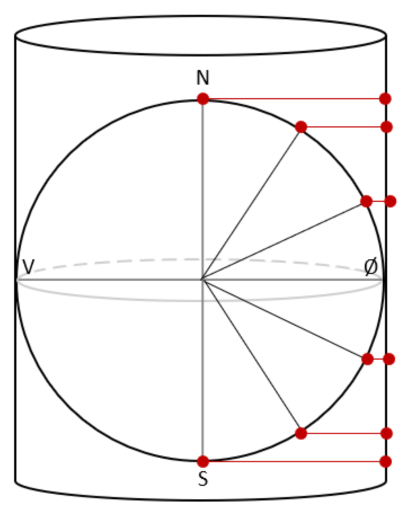
Lamberts arealbevarende cylinderprojektion:

Ved Lamberts arealbevarende cylinderprojektion, projiceres punkter på kuglen *vandret* hen på cylinderen (Figur 12), hvor cylinderen tangerer langs Ækvatorlinjen. Vi får:

$$x = \lambda$$

$$y = \sin(\varphi)$$

Som I nok kan regne ud fra navnet, så er denne cylinderprojektion arealbevarende. Den viser breddegrader og meridianer som rette linjer.



Figur 12: Lamberts arealbevarende cylinderprojektion.

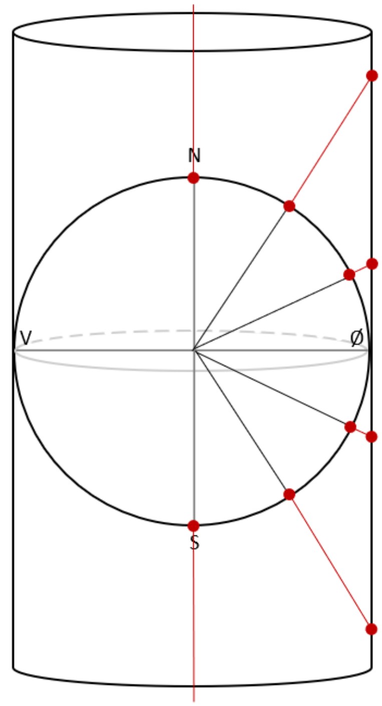
Central cylindrisk projektion:

Her projiceres punkterne på kuglen langs en ret linje, der går fra centrum af kuglen, gennem punktet, og videre ud på cylinderen (Figur 13). Formlerne er:

$$x = \lambda$$

$$y = \tan(\varphi)$$

Den central cylindrisk projektion har ikke rigtig nogle bevarende egenskaber, andet end den som alle andre cylindriske kortprojektioner viser bredde- og længdegrader som rette linjer. Men den er let at forstå, set fra et trigonometrisk synspunkt.



Figur 13: Central cylindrisk projektion.

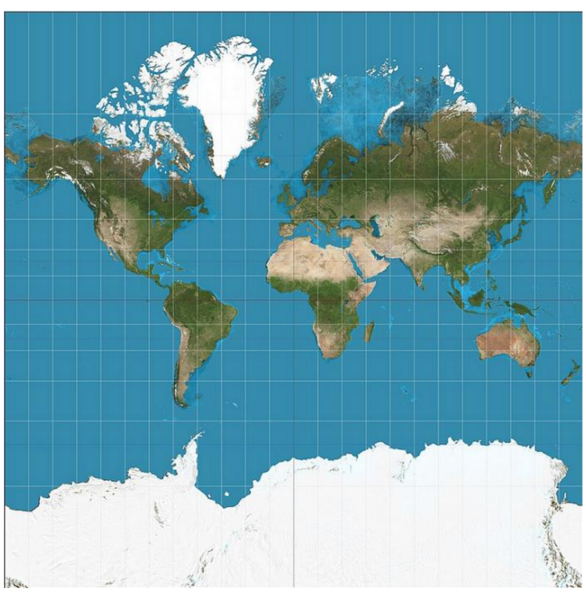
Mercators cylinderprojektion:

En af de mere kendte projektioner, men også mere indviklet. Hvor de andre kortprojektioner måske virker indlysende, er det svært at gennemskue hvordan man kommer frem til formlerne:

$$x = \lambda$$

$$y = \ln(|\sec(\varphi) + \tan(\lambda)|)$$

Mercators cylinderprojektion er såkaldt konform, og dette er et eksempel på, hvor komplekst det kan blive, hvis man vil have ens kortprojektion til at bevare visse egenskaber. Figur 14 er et eksempel på et kort med Mercators cylinderprojektion.


Figur 14: Mercators cylinderprojektion. Fra: https://en.wikipedia.org/wiki/Mercator_projection.

6. Hvorfor er det, at kortprojektioner forvrænger Jordens overflade?

Som nævnt tidligere, så vil kortprojektioner altid forvrænge, kuglens, heraf Jordens overflade. Hvis der fandtes en ideel kortprojektion, som perfekt bevarede Jordens overflade, skulle den bevare både alle arealer, alle vinkler og ikke mindst alle afstande, herunder de korteste afstande mellem vilkårlige lokationer. Vi har set på cylinderprojektioner som projicerer breddegrader (paralleller) og længdegrader (meridianer) til rette linjer, men ikke de korteste afstande. Faktisk kan vi generelt vise at:

> **Sætning 2:** Der findes ikke en kortprojektion, som projicerer alle storcirkelstykker til rette linjer på kortet, og samtidig bevarer vinkler.

Øvelse 12:

Hovedargumentet i beviset for Sætning 2 tager udgangspunkt i sfæriske trekanter[5] og hvad der sker, når man projicerer en sfærisk trekant en på en flade (en plan for eksempel). Kan I formulere et modargument til, at der skulle findes en kortprojektion, der projicerer storcirkelstykker til rette linjer på kortet og samtidig bevarer vinkler?

Det er også muligt at vise, at der ikke findes en kortprojektion, der bevarer både vinkler og arealer. Det er dog ikke noget vi vil gøre her. Det er nok med denne sætning, at vi kan konkludere, at der ikke findes en kortprojektion, som ikke forvrænger Jordens overflade på en eller anden måde.

Målforhold viser sig at være helt central i forklaringen om, hvorfor kortprojektioner forvrænger Jordens overflade. Man skelner mellem to målforhold: et *globalt målforhold*, der måler forholdet mellem Jorden og kuglen, og så det *lokale målforhold*, som fortæller noget om forholdet mellem kuglen og kortet.

I regnede et eksempel på et globalt målforhold i Øvelse 3. Den bliver som regel udtrykt som forholdet $1\!:n$, som siger noget om, at 1 cm på kortet svarer til n cm i virkeligheden (på Jorden). Denne er konstant, og det er den vi ser udtrykt på fysiske kort.

Det lokale målforhold et i sin forstand det sande målforhold på kortet. Den udregnes som forholdet mellem længder på kortet og så længde på kuglen, dvs.

$$\text{lokalt målforhold} = \frac{\text{længde på kortet}}{\text{længde på kuglen}}$$

Et hurtigt eksempel:

Vi er givet et kort, hvor der er benyttet en kortprojektion, og et globalt målforhold $1\!:52.000.000$, dvs. 1 cm på kuglen svarer til 520.000.000 cm eller 520 km i virkeligheden. På kuglen udvælges der to punkter der ligger 2 cm fra hinanden, på samme længdegrad

[5] Faktisk burde det være nok med et modeksempel.

(meridian). Dvs. der er 1040 km mellem disse punkter, jf. det globale målforhold. De samme punkter finder vi på kortet. Vi måler til gengæld at der er 2,2 cm mellem punkterne på kortet, eller $2{,}2 \cdot 520 = 1044$ km, oversat til virkeligheden. Det lokale målforhold er derved: $\text{lokalt målforhold} = \frac{1044}{1030} = 1{,}00385$, og det fortæller os faktisk, at der sket et "stræk" (nord-syd) af kuglens overflade ved brug af kortprojektionen.

Øvelse 13:

I har fået udleveret et kort med en af de præsenteret cylinderprojektioner. Noteret er også det globale målforhold. Radius af kuglen, som projiceres fra, er noteret på kortet og Jorden har en radius på 6378 km.

1) Beregn kuglens og Jordens omkreds.
2) Hvordan er man kommet frem til det globale målforhold?

På kortet ser I nogle lokationer markeret, der parvis ligger på samme længdegrad. Deres geografiske koordinater er følgende:

① (76.02°, -65.11°)

② (51.92°, -65.11°).

③ (37.56°, 104.14°).

④ (-3.31°, 104.14°).

3) Ligger lokationerne ①, ②, ③ og ④ på den nordlige eller sydlige halvkugle, og hvad med vest eller øst?
4) Beregn de korteste afstande d (på Jorden) mellem lokationspar ① og ②, samt ③ og ④.
5) Brug en den udleveret lineal til at finde afstanden mellem punktpar ① og ② samt ③ og ④ på kortet. Hvad er det lokale målforhold for disse punktpar?
6) Sammenhold de udregnede lokale målforhold med forvrængningen på kortet. Kan I finde en sammenhæng mellem det lokale målforhold og forvrængningen?
7) Se på bredde- og længdegrad linjerne der er visualiseret på kortet. Hvad fortæller om forvrængningen der sker på kortet? Sammenlign evt. med modellen i Google Earth.

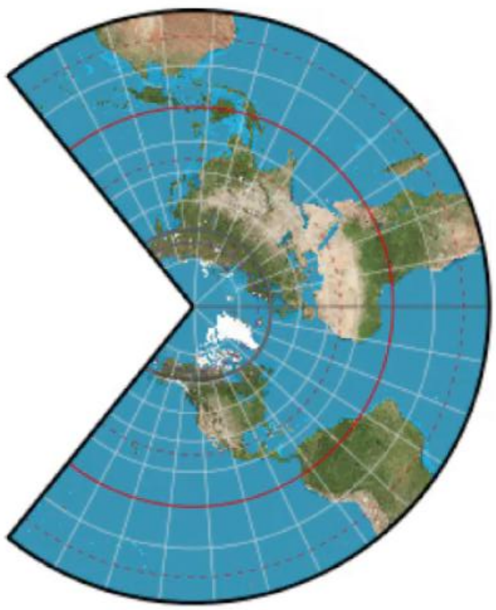
De lokale målforhold kan faktisk generaliseres for hver kortprojektion, for hvert punkt. Det kræver dog noget matematik, som er meget over niveau af hvad der forventes af jer. Det matematiske udtrykt for Mercators projektion er specifikt baseret på dets lokalt målforhold, der sikrer, at projektionen bevarer vinkler.

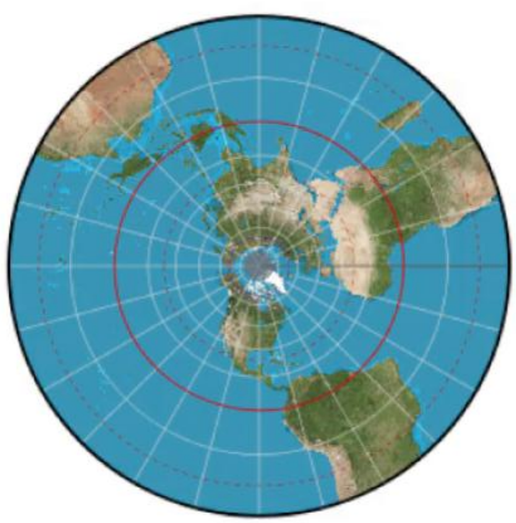
Arbejdet med kort produktion i dag (kartografien) laves over computeren, vha. satellit-data og GIS. Vi kan nu lokationsbestemme, udregne størrelser, finde de hurtigste ruter på få sekunder og med få centimeters unøjagtighed. Men selvom alt dette er til rådighed, er det endnu ikke lykkedes at få skabt et perfekt kort af hele Jorden (vi har netop set hvorfor det er umuligt). Jorden er og vil altid være en kompleks størrelse, men det har ikke fået det videnskabelig samfund til at miste interessen for kort. Faktisk har teknologien åbnet for en hel ny måde at arbejde med kortprojektioner på, og vi ser stadig en udvikling inden for kartografien, selvom man ikke længere gør det i hånden.

Som aflevering: Se på de fire kortprojektioner (s. 12-13). Brug nettet til at finde kort, hvor disse kortprojektioner er benyttet. Beskriv kortenes egenskaber (herunder hvad det betyder for forvrængningerne) og kom med flere eksempler på, hvad disse kort kan bruges til. Derudover ønskes der også en forklaring på, hvorfor deres formler ser ud som de gør (brug evt. Figur 11, 12, 13) – det er ikke et krav, at I skal kunne forklare formlerne for Mercator projektionen, men I er velkommen til at prøve!

10 Appendix B

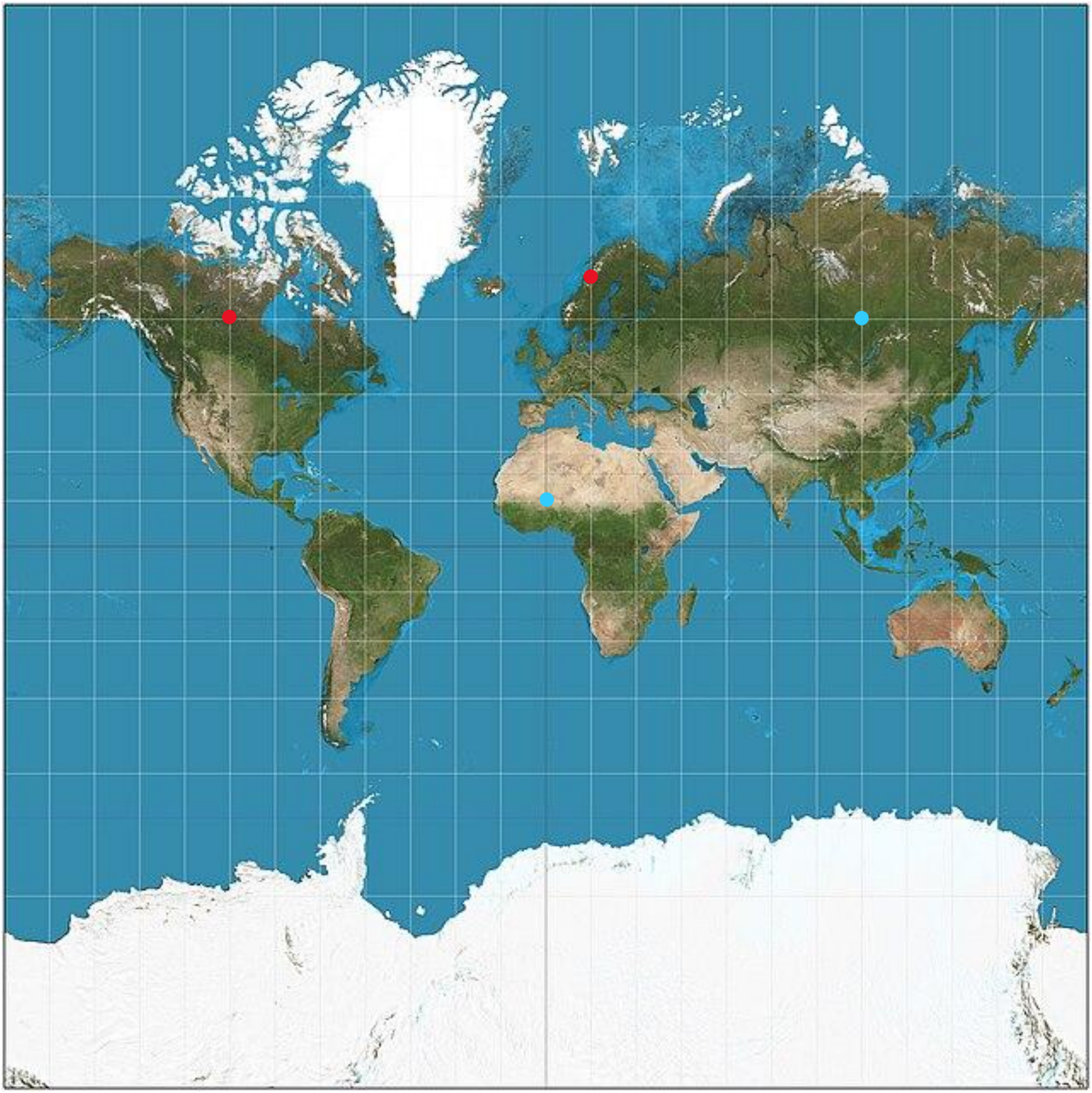
The material for Exercise 5 in the teaching material is found on the next page.

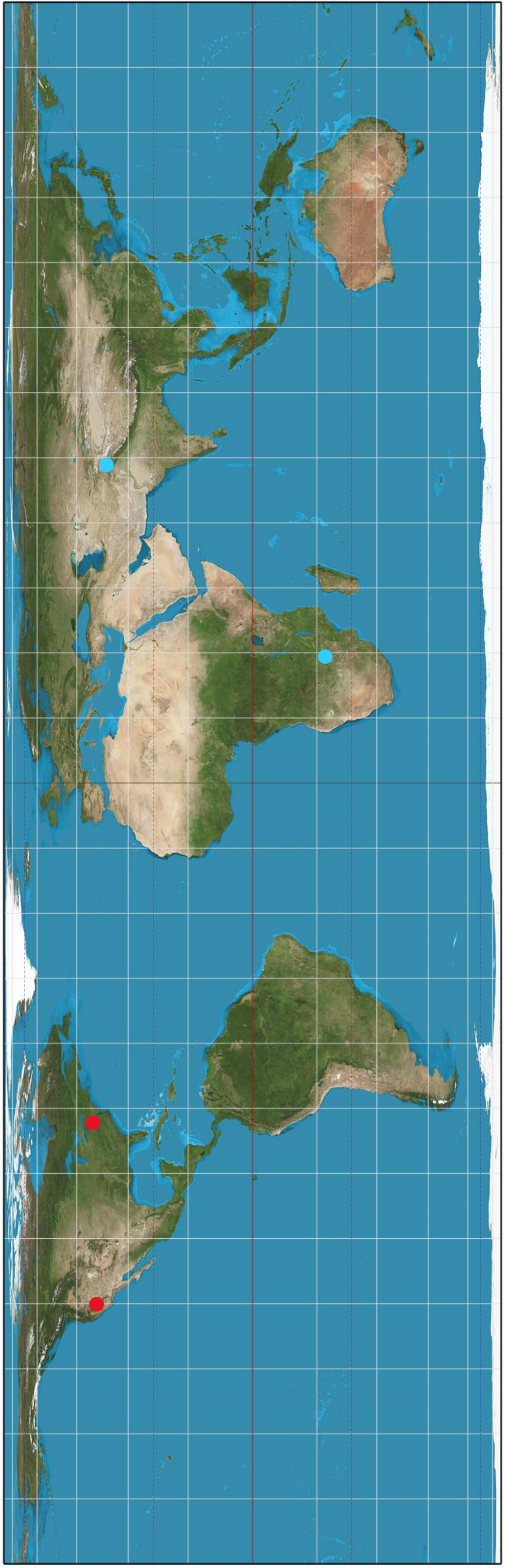




11 Appendix C

The material for Exercise 6 is found on the next pages.



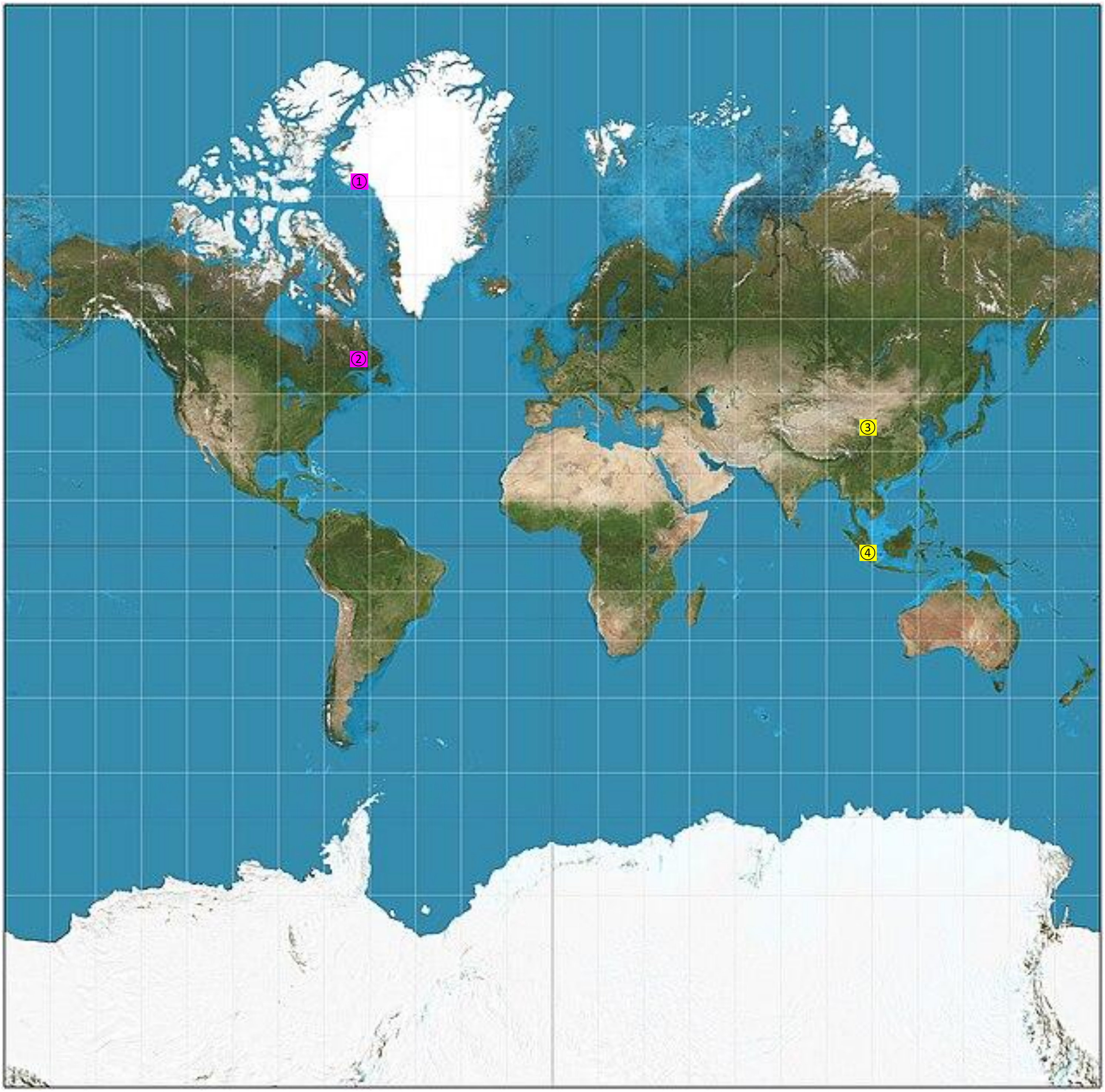


12 Appendix D

The material for Exercise 14 is found on the next pages.

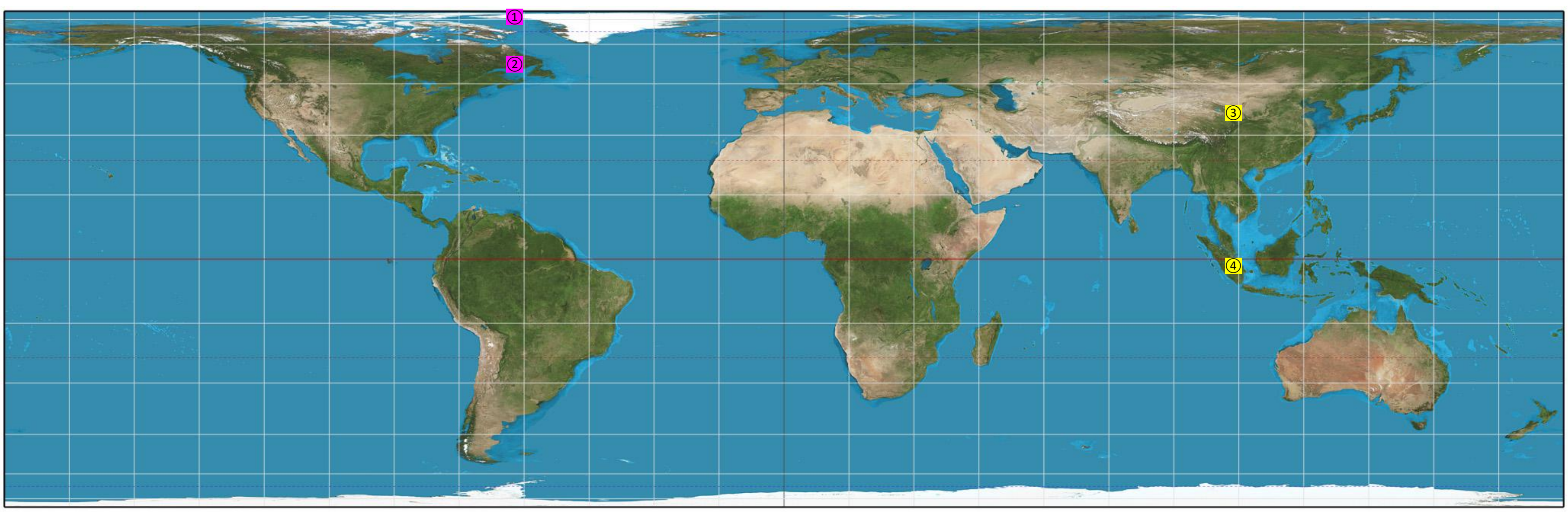
Kort med Mercator projektionen. Målforhold: 1:134.932.660

Radius af kuglen: 29,7 cm.



Kort med Lambert’s areal-bevarende projektion. Målforhold: 1:95.416.667

Kuglens radius: 42 cm



Kort med central cylindrisk projektion. Målforhold: 1:134.932.660

Radius af kuglen: 29,7 cm.