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Important prerequisites to understanding the definition of limit

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UNIVERSITY OF COPENHAGEN

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Master of Science in Mathematics

Important prerequisites to understanding the definition of limit

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ABSTRACT

The transition from high school to university level mathematics has been reported to cause difficulties for many students. The transition has been described as a move from describing to defining and from arguing to constructing proofs based on definitions. The formal definition of limit is often one of the first concepts which is taught to students in a way that requires post-transitional thinking. For this reason (and many others), student understanding of limits has attracted a lot of research interest. This study aims to compare the importance of five competencies, which have previously been shown or hypothesised to be important to learning the formal definition of limits or transitioning to university level mathematics in general:

- Being able to distinguish between valid and invalid proofs
- Having an active meaning orientation to studying
- Understanding the role and nature of definitions in mathematics
- Not being coloured by common misconceptions of limits
- Being able to solve inequalities involving absolute values and connect the result to an image of a subset of the number line.

154 students took part in our study. They were first tested in each of the competencies above. Then they were taught the formal definition of limits, and finally their understanding of the concept was measured. We found that all of the competencies on the list above which are specific to mathematics (that is all of them except orientation) had moderate to strong correlations with understanding of the formal definition of limit, and that this correlation was significant, even when we controlled for general mathematical competency (as measured by a grade in an unconnected subject within mathematics). We did not find any strong indications of which competencies out of the four were most important. Orientation to studying was found to correlate only very weakly to understanding of the formal definition of limits. Hence this research indicates that if teachers want to improve student understanding of the definition, it is more important that they focus on the mathematical prerequisites rather than trying to change the study orientation of the students. Further research is needed to determine the order in which the different competencies should be taught.

CONTENTS

Part I

INTRODUCTION

1

INTRODUCTION

Understanding the formal definition of limit is an important step in the mathematical career for many students. First, because the concept is useful in its own right [Nagle 2013]. For example when making sense of continuous functions close to points, where they are not defined. Second, because understanding the concept of limit is fundamental to further studies in analysis. Many topics such as continuity, differentiability, and integrals are either founded on limits or on definitions with a very similar structure [Williams 1991, Swinyard and Larsen 2012, Cornu 1991]. Third because limits "often serves as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally quantified mathematical statements, and transitioning to abstract thinking" [Swinyard and Larsen 2012]. Therefore it is unfortunate that many students are reported to struggle with understanding limits [Cottrill et al. 1996, Nagle 2013, Oehrtman 2009]

The mathematics students at the University of Copenhagen seem no different in this respect. They learn to understand the formal definition of limit in a course called Analysis 0 (An0) about half a year after they start their university education. They have been told by the older students, that this is the moment, when they will finally be taught "real mathematics". In the beginning of the course, they are understandably excited, and when the formal definition of limit appeared on the screen in the lecture we observed, an audible whoosh went through the auditorium. However, the course responsible report that the enthusiasm of the students is quick to dim, and halfway through the nine weeks of the course he usually has to calm down the teaching assistants, who are very concerned about how little the students understand. It is only towards the end of the course, that the students gradually start to see the pattern, and a large majority of them are able to pass the exam [personal communication with course responsible].

Our motivation for writing this thesis is to improve this situation. We wonder if it is possible to prepare the students better for learning the formal definition of limits, e.g. by teaching important prerequisites in the six months they are at university prior to taking An0.

Going through the literature, we found eight different explanations to why students have problems understanding the formal definition of limits or acclimatising to university level mathematics in general. Most of these explanations came with recommendations to the teachers on how to improve the situation. However, it is unfeasible to implement all of these suggestions, and we were not able to find any investigations of which proposed solutions were the most effective in improving student understanding of limits. Therefore, we chose five of the proposed explanations and decided to test empirically which of them was likely to have the greatest effect on the learning of the formal definition of limits for our students. The five proposed explanations we chose were

- **Proof Validation:** The students are not able to distinguish between valid and invalid proofs [Alcock and Weber 2005]
- **Active meaning orientation:** The students' orientation to learning is not suited for studying at university level [Entwistle and Ramsden 1982]
- **Use of definitions:** The students do not understand the role of definitions in mathematics [Vinner 2002, Edwards and Ward 2004]
- **Misconceptions about limits:** The students have misconceptions about limits, which will coexist with their developing knowledge of the formal definition and create confusion [R. B. Davis and Vinner 1986]
- **Inequalities involving absolute values:** The students do not connect inequalities involving absolute values to an image of a subset of the number line [**fernandez˙studentstake˙2004**]

And our research question hence becomes: Which of the following items are the best predictors of the student's ability to learn the formal definition of limit during the course An0?

- Proof validation
- Active meaning orientation
- Use of definitions
- Misconceptions about limits
- Inequalities involving absolute values

See figure 1 for an illustration of the conceptual framework of the thesis. To ease communication, we will call the items on the list above "the proposed prerequisites to learning the formal definition of limits" or simply "the prerequisites".

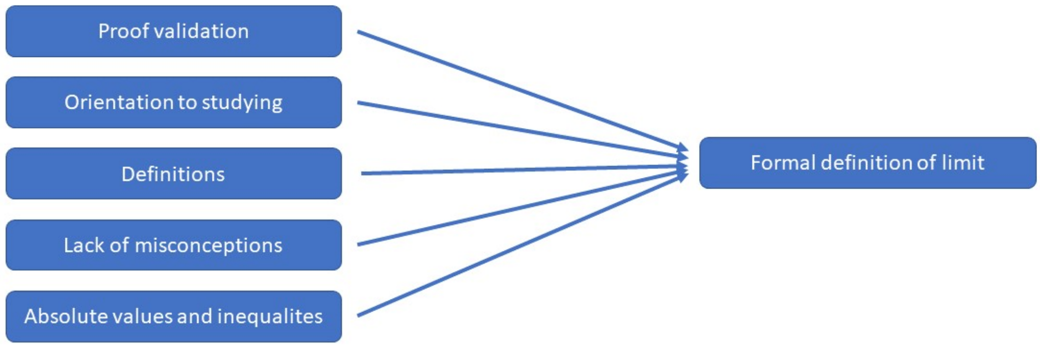


Figure 1: Conceptual framework of the thesis.

In order to answer the research question, we will test the students' prerequisites before they start learning to understand proofs involving the formal definition of limits. After they have received the normal teaching in An0, we will test to what extent they understand the formal definition of limits.

We will be interested in the correlations between the prerequisites and understanding of the formal definition of limits. Our research setup does not allow us to establish whether any of the prerequisites *cause* increased learning in An0, but we will attempt to eliminate some of the alternative explanations.

In the following chapter, we will define the key terms of this thesis such as "limit" and "understanding of limits". In chapter 3, we will expand on what we meant by "transitioning to abstract thinking" in the beginning of the introductions, and in chapter 4, we will use the terminology of chapter 3 to describe the institutional context. In chapter 5 we will describe each of the prerequisites in greater detail. In chapters 6 and 7, we will describe how we carried out the investigation. Chapter 6 is dedicated to the overall structure of the investigation, including how we went about trying to establish cause and effect, and chapter 7 deals with how we created the many different tests. In chapter 8, we present the results of the investigation, and in chapter 9 we pick out three results, which we discuss in greater detail. The thesis is rounded of by a conclusion in chapter 10.

Part II

BACKGROUND

2

UNDERSTANDING THE FORMAL DEFINITION OF LIMIT

In this chapter, we will expand on the introduction by defining what we mean by "limit" and "understanding the formal definition of limit". We aim to describe in detail what exactly we want the students to learn and how to measure it.

2.1 LIMITS AND UNDERSTANDING OF LIMITS

In this thesis, our examples will mostly involve limits of functions $f : \mathbb{R} \to \mathbb{R}$, when x tends to some $a \in \mathbb{R}$. However, this is just a special case of the definition of limits given in An0 (our translation):

Definition 1. Let A be a subset of $\mathbb{R}^k$ and $f : A \to \mathbb{R}^m$ be a function. Let further $a \in \overline{A} \backslash A$ and $b \in \mathbb{R}^m$. Then, we will say that f has *limit* b when $x \to a$ if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; ||f(x) - b|| < \varepsilon \text{ for all } x \in A \text{ with } ||x - a|| < \delta$$

If the reader is not familiar with this definition, we recommend Lindstrøm 1995, which has an excellent explanation.

We note that, at times, we will also use the word "limit" to describe the behaviour of sequences when $n \to \infty$ or real functions, when $x \to \infty$. The definitions of limit in these two cases are very similar to the definition given above.

Having settled the question of what we mean by the word limit, we turn to the question of what we mean by "understanding the formal definition of limits". Certainly it is not enough to be able to state the definition. We should require that students can also bring it into play somehow. For the context of this thesis, we chose to define understanding of limits as the ability to *understand simple proofs which apply the definition*. Examples of the types of proof we have in mind are "showing that the sum of two functions converge to the sum of their limits" and "showing that a particular affine function converges". We chose this definition because it aligns with the exam requirements of An0 (to be described in section 7.6.1) and because we judged it to be one of the simplest ways to "bring a definition into play" in formal mathematics.

2.2 PROOF UNDERSTANDING

Having defined "understanding the definition of limit" as "understanding simple proofs which apply the definition of limit", we are left with the task of defining what we mean by understanding a proof. Before we can do that, we need the following definition

Definition 2. By *practising mathematician* we shall mean a person who either teaches mathematics at university level or is research active within the field of mathematics. The current collection of all practising mathematicians in the world will be denoted by "the current mathematical community".

Now we are ready to define understanding of a proof:

Definition 3. An individual's *understanding of a proof* is the mental model that the individual has of the proof. An individual's proof understanding is *judged* by how closely the individual's responses match those accepted by the current mathematical community whenever the individual brings his understanding into play. If the individuals responses always align with the accepted knowledge in the mathematical community, we say that the individual *understands* the proof.

Note that we are skirting the issue of absolute truth. We are defining a good response as a response which is accepted by the current mathematical community. To use this definition, we have to assume consensus in the mathematical community. We will get back to this issue in section 5.1

Note also that this definition skirts the issue of what the individual has in his mind. As we can never measure this, we will not be judging what is in the mind of the individual directly. One individual may see a proof as a flower with a beautiful surface and deep connection to other parts of mathematics, others may see it as a house, which is build layer by layer from the foundation and up. We are only interested in the outcome, when the individual tries to apply his understanding to something.

2.2.1 *How to measure proof understanding*

Having defined proof understanding as the consistent ability to give acceptable responses in mathematical situations involving the proof, we are left with the task of finding good situations to use, when assessing the proof understanding of students.

One situation, which is often used is oral examinations in which the students present a proof and answer questions about it without referring to their notes. This sort of test certainly requires that the student remembers the proof and the questions ensure that it is not just rote memorisation. However, oral examinations have been shown to lack reliability as the outcome for example depends on what questions the students were asked [M. H. Davis and Karunathilake 2005].

To avoid this issue, one can for example use the guideline for questioning, which was developed by Mejia-Ramos, Fuller, et al. 2012. The guideline is divided into two parts: one evaluating the local proof understanding of the student, and one evaluating the global. On the local level, the article explains how to ask questions which measure whether the students can

1. understand the meaning of words and statements in the proof.
2. understand the role of an individual statement in the proof structure (for example in proofs by contradiction or induction).
3. explain why one statement follows from the previous.

On the global level, the article gives guidelines to questions measuring whether the student can

4. split the proof into relevant parts, and explain the relationship between the different parts.
5. illustrate part of the proof with an example.
6. summarise the proof via high-level ideas.
7. transfer the general idea or methods to another context.

According to our definition of proof understanding, there is no limit to how many different ways we should ask the student to interact with the proof. In practice we of course have to make a selection. Mejia-Ramos, Fuller, et al. 2012 made the above selection by demanding that

1. At least one piece of previous literature should treat the particular aspect of proof understanding as important.

2. At least two out of nine practising mathematicians should mention the aspect in a semi-structured interview the authors conducted.

Giving credence to the selection, is the fact that the guidelines in the article have already been used as an operational definition of proof understanding in several investigations [Hodds et al. 2014, Weber 2015, D. Zazkis and R. Zazkis 2016].

For these reason, we choose to adopt the model of Mejia-Ramos, Fuller, et al. 2012 as one operational definition of proof understanding. However, we will also rely on oral examinations which are not based on this guideline to measure proof understanding. We will get back to this in section 7.6.1.

3

THE TRANSITION TO UNIVERSITY LEVEL MATHEMATICS

In the introduction, we briefly mentioned that limits "often serve as a starting point for [...] transitioning to abstract thinking" [Swinyard and Larsen 2012]. In this chapter we will explain what we mean by "a transition to abstract thinking". We will see that several researchers have been interested in this transition and that their theories illuminate different aspects of how university level mathematics is different from high school mathematics (see Gueudet 2008 for a review).

3.1 PRAXEOLOGIES

One way to understand transitions is through praxeologies. A praxeology is a combination of

- types of tasks
- techniques for solving the tasks
- technology or discourse to describe the techniques
- theory to justify the techniques

The types of tasks and techniques are referred to as the *practical block* of the praxeology, and the technology and theory are together called the *logos block*. [Barbé et al. 2005]

Praxeologies may have different sizes. The smallest praxeologies are the ones which contain only a single type of task. These praxeologies are called punctual. Examples of punctual praxeologies could be the one explaining how to 'differentiate products of functions with known derivatives' or the one explaining how to 'calculate the limit of rational functions when both numerator and denominator tends to zero in the limit'. Punctual praxeologies can be integrated into local praxeologies, which contain several types of tasks. So for example, the second praxeology above could be integrated into a local praxeology concerning the calculation of limits

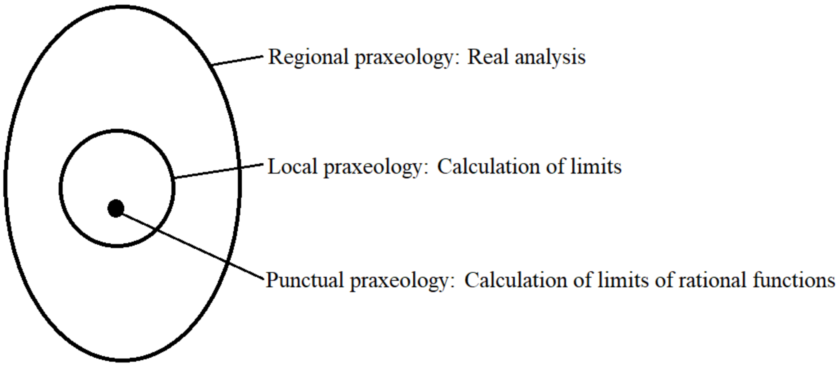


Figure 2: An illustration of nested praxeologies

[Barbé et al. 2005]. The theory block of this praxeology would contain different rules for calculations of limits such as l'Hôpital or 'the limit of a sum is the sum of the limits'. Going one step ahead, local praxeologies which depend on the same theory can be integrated into a regional praxeology. So for example, the praxeologies we have mentioned above are all part of the regional praxeology: real analysis. We view the logos block of real analysis as applications of the axioms of the real numbers structured as a metric space to certain real valued functions. See figure 2 for an illustration of the idea.

Winsløw et al. 2014 use praxeologies to describe a transition which is often observed between high school and university level mathematics. In high school the assessment is often focused on the practical block of the praxeologies. Students can learn to recognise types of tasks and apply suitable techniques to solve them. This can be done without reference to the theoretical block of the praxeology, which the students are hence free to ignore [Winsløw et al. 2014]. In a situation like this, the curriculum can become atomised into punctual or local praxeologies. Only the teacher, who has access to the theoretical block of bigger praxeologies, can see how the smaller praxeologies are connected [Barbé et al. 2005].

At university on the other hand, the assessment system is more likely to force the students to work with the theoretical block of the praxeologies, and the theoretical block is often in the forefront at lectures, where the practical block is often only visible in introductory examples [Winsløw et al. 2014]. Hence the move from high school to university can for some be seen as a move from a sole focus on the practical block, to an incorporation of the two with focus on the theoretical block.

3.2 THEORETICAL THINKING VS. PRACTICAL THINKING

Sierpinska 2000 describes the transition in another way. She focuses on the ability of mathematicians to change between two different thinking modes:

- Theoretical thinking is characterised by organised systems of concepts and conscious reflection on the meaning of words and symbols. For example by explicitly stating definitions, which completely determine the object.
- Practical thinking is the opposite: it uses prototypical examples and reasoning based on the logic of action.

According to her, mathematicians can use both thinking modes. Most of the time, in a familiar context mathematicians think in practical ways. They switch to theoretical thinking, when they are confronted with a contradiction, want to answer a confusing question or have to defend their theory against criticism.

Sierpinska 2000 interprets some of the difficulties encountered by first year students as consequences of practical thinking. As an example, she mentions students who think that "linear transformations are rotations, dilations, projections, shears, etc. and their combinations with constant parameters" (p. 225). "These students focus on the prototypical examples they know, and do not develop a theoretical understanding of the concept of linear transformations." [Gueudet 2008]

3.3 ADVANCED MATHEMATICAL THINKING

In the book Advanced Mathematical Thinking [Tall 1991] a group of authors set out to describe a way of thinking which is characteristic of expert mathematicians. They are not able to present a coherent theory, but they present a collection of smaller theories, which describe different aspects of advanced mathematical thinking such as:

- Mathematical creativity
- Mathematical proof
- The role of definitions
- Generalisations and abstractions

They do not claim that the above theories apply only to mathematics as it is taught at universities. But they do describe a transition in the required mathematical thinking, which many undergraduate students experience in their first year at university. This transition is described as a move from "describing to defining" and from "convincing to proving in a logical manner based on definitions".

3.4 SUMMARY

To sum up, the transition from high school to university has been described in several different ways. From the three descriptions above, we gather that the transition can include

- A move from a practical focus to a more theoretical focus, through which the connections between different tasks become visible.
- A move from descriptions to precise definitions, which completely determine the object and are consciously reflected upon.
- A move from arguing to presenting formal proofs, which rely on the definitions.

This transition need not take place between the last day of high school and the first day of university. Depending on the high school, some students may have started the transition before they enter university, and at university, some parts of the transition may be postponed to later courses. Furthermore, the transition is never complete. As Sierpinska 2000 points out, even expert mathematicians use practical thinking most of the time. The transition consists in the ability to use the theoretical way of thinking *as well*.

4

INSTITUTIONAL CONTEXT

At UCPH the academic year is split into four blocks of nine weeks. In each block the students have two courses. This means that students take four courses before Christmas in their first year and four courses after.

In this chapter, we will describe the courses which are relevant for this thesis. We will find that part of the transition described in the previous chapter takes place in the course called An0, which is taught in block three of the first year (the first block after the Christmas vacation). Incidentally An0 is also where students are first expected to understand proofs involving limits.

4.1 PRE-TRANSITIONAL COURSES

We view most of the courses, which the students take in their first year before Christmas, as pre-transitional in the sense that they do not require

- That the students work with the theoretical block of the praxeologies
- That the students understand the roles of the definitions of the courses
- That the students construct formal proofs based on the definitions

The theoretical block of the praxeologies are presented in the lectures, but for the most part the students are only asked to apply the theory to concrete functions. So for example, the students are asked to argue that the function

$$f(x,y) = 3x + 2y^2$$

has a maximum on the set

$$\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, x-2 \leq y \leq x\}$$

and to calculate the limit of

$$\lim_{x\to 1} \frac{\sin(\pi x)}{e^x - e}$$

In both of these exercises, the students have to refer to theorems, which were proven in the lectures (the extreme value theorem and l'Hôpital), but they do not need to know how the exercises are connected in the praxeology real analysis. Neither are they asked to write formal proofs which rely on definitions. Hence it may come as no surprise that some of the students do not learn to understand the role of definitions in mathematics. In fact, we find that 28% of the students in our sample, most of whom have passed all courses in the first half year, mistake a theorem for a definition in one of the questions of our quiz.[1]

4.2 DISRUS

One course called DisRus is an exception. It is placed before Christmas, but acts as a stepping stone towards the transition. In this course the students are introduced to mathematical definitions, and how to use them in simple proofs. For example, the students see the proof of $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, which uses the definition of pre-image, and are asked to prove $A \subseteq f^{-1}(f(A))$ in a hand-in.

Thus in some ways, DisRus marks a significant step in the direction of advanced mathematical thinking: It requires that students work with definitions and produce proofs. However, we will argue that DisRus is only part of the way. In DisRus, the students are introduced to many different topics : Relations, functions, exists- and forall quantors, permutations and groups. Some of the topics are linked theoretically (functions are examples of relations, permutations are examples of groups), and discussions about proofs and definitions apply in all the topics, but compared to later courses, the students are likely to consider the praxeologies of DisRus to be relatively small and disconnected. Hence the students are not trained in seeing how theory is connected, which is part of theoretical thinking as defined by Sierpinska 2000. Also we would argue, that the students of DisRus do not have to take responsibility of the logos block of the praxeologies in DisRus. It is true, that they have to construct proofs, but we regard proof construction in DisRus as part of the practical block of the praxeology. Take for example induction. The lecturer argues why induction is a valid proof technique, and presents example proofs. Then the students construct a number of induction proofs following the example in the lecture. The proofs of the students are not used to tie pieces of theory together in any of the

1 Here we are referring to question 3 of the test in definitions. The test is described in section 7.3

topics of DisRus. Finally, the proofs of DisRus are very simple in comparison to the proofs of An0. Take the proof of $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$$
\begin{aligned}
x \in f^{-1}(A \cup B) &\Leftrightarrow f(x) \in A \cup B &(1)\\
&\Leftrightarrow f(x) \in A \vee f(x) \in B &(2)\\
&\Leftrightarrow x \in f^{-1}(A) \vee x \in f^{-1}(B) &(3)\\
&\Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B) &(4)
\end{aligned}
$$

This proof is a straight forward application of the definitions of equality of sets, pre-images and unions of sets. If we look back to the definition of proof understanding of Mejia-Ramos and Weber 2014 in section 2.2, we see that some of the measures of global proof understanding do not apply as the proof cannot meaningfully be split into smaller parts, and the only high-level idea in the proof is to use the definitions.

4.3 POST-TRANSITIONAL COURSES

After Christmas, the students take An0, which is post-transitional in the sense that it requires all of the elements on the list above. Incidentally this course is also where students are first expected to understand proofs involving limits.

To pass the exam in An0, the students have to pass two assignments and an oral examination. In the assignments they are for example asked to prove that if a function is increasing and bounded, it must have a limit as $x \to \infty$ or to show that a continuous function which converges both for $x \to \infty$ and $x \to -\infty$ is uniformly continuous. In the oral exam, the students are asked to present a part of the theory including relevant definitions and at least one proof. The students just get a headline like “the Riemann integral”, and then they have to create their own presentation. This means that the student must also show understanding of how the different elements of the theory are connected.

4.4 SUMMARY

In this chapter, we have described how most of the courses that the students take before Christmas in their first year would be considered as pre-transitional in the language of the previous chapter. DisRus is an exception. In DisRus, the students are taught about definitions and required to construct proofs which are based on them. However, DisRus is only a stepping stone towards the transition. A significant part of the transition is postponed to An0, which is taught after Christmas in the first year. An0 is the first course in which the students are required understand proofs which are based on formal definitions in the context of a large and interconnected praxeology and the first time the students are asked to understand proofs with a complicated global structure. The students are well aware that this transition exists. They have been told by the older students that An0 is where you finally learn “real mathematics”, and their excitement is tangible in the first lectures of An0.

5

BARRIERS TO UNDERSTANDING THE CONCEPT OF LIMIT

The literature and the lecturers at UCPH agree: The students find it difficult to learn to understand proofs involving limits [Nagle 2013, Swinyard and Larsen 2012]. However, once you start to ask "why", the agreement is no longer evident. In this chapter we will first briefly describe all the different explanations, that we have found either in the literature or among the teachers of first year students. Then we will go into more detail with the five explanations we chose to investigate in this thesis.

5.0.1 *Possible barriers to understanding the concept of limit*

By reviewing the literature and talking to teachers of first year students, we found the following hypothesised prerequisites, with direct relevance for understanding the concept of limits:

- Inequalities involving absolute values: How fast and reliable is the student in solving inequalities involving absolute values, and connect the answer to the image of interval(s) on the real number line? [**fernandez˙studentstake˙2004**]
- Misconceptions about limits: To what extent is the student influenced by some of the common misconceptions about limits such as "the limit can never be reached"? [R. B. Davis and Vinner 1986]

And the following hypothesised prerequisites for understanding many different topics in advanced mathematics, for example limits:

- Implications: How well does the student understand that $A \Rightarrow B$ *only* means "if A then B" and that one cannot use it to conclude A given B? [Hazzan and Leron 1996]
- Definitions: How well does the student understand the role of definitions in mathematics and how to work with them? [Vinner 2002,Edwards and Ward 2004]

- Proof validation: How reliable is the student in making judgements about the validity of mathematical arguments? [Alcock and Weber 2005]
- Language: Do the students have difficulties in picking up on important subtleties in the language (such as grammatical differences between theorems and definitions) because they do not speak or read Danish well enough? [Barton et al. 2005]
- Quantification: How well is the student able to understand and work with complicated statements involving the quantifiers $\forall$ and $\exists$ [Dubinsky et al. 1988, Cornu 1983]

And the following prerequisites which have been shown to be important for university students of all subjects

- Self-efficacy: How well do the students believe they will do academically? [Richardson et al. 2012]
- Active meaning orientation: In what ways does the student actively seek to understand the meaning behind what is presented? [Entwistle and Ramsden 1982]

Note that the list is not meant to be exhaustive. It is simply a summary of the interesting prerequisites we were able to find given the time restrictions.

As we only had an hour in total to test the students, we decided to choose five of the above proposed prerequisites for further investigation. Our selection was based on

1. Measurability: How good a measure we expected to be able to get by using only 10 minutes of the students' time
2. Institutional value: Whether it would be institutionally feasible to train the competency, if it turned out to be important
3. Diversity: A desire to test very different prerequisites
4. Influence: How influential did we judge the idea to be in the literature (based on citations and number of articles on the topic)
5. Experience: How important the course responsible judged the prerequisite to be

After much debate, we landed on the five possible prerequisites

- Inequalities involving absolute values, because it was very different from the other proposed prerequisites and it is comparatively easy to measure. [diversity and measurability].

- Misconceptions, because it is widely discussed in the literature and we judged it to be comparatively quick to improve in teaching [influence and institutional value].
- Definition, because the idea is influential in the literature and the course responsible judged it to be likely to be important [influence and experience]
- Proof validation, for the same reasons as definition [influence and experience]
- Active meaning orientation, because the course responsible judged it to be interesting and we were hoping to find "quick fixes". If we were able to point to a specific thing the successful students did that the unsuccessful students did not (like take notes or independently test the theory on examples), it would be very institutionally feasible to help the unsuccessful students improve [experience and institutional value]

In the following sections we will expand on what is already known about the proposed prerequisites. We will change the order around as some of them are connected, and we want to improve readability of this thesis.

5.1 PROOF VALIDATION

We will start by describing the proposed prerequisite proof validation. First we will define what we mean by proof validation, then discuss why this might be difficult for students and finally we will present an arguments to why proof validation is considered an important ability to have when learning mathematics at university level.

5.1.1 *What is proof validation?*

Proof validation is the process of judging whether a proof fulfils the requirements of the current mathematical community. To expand on what types of requirements, the mathematical community might have, we need a definition of proof. For the sake of this thesis, we will use a slight modification of the definition in the Meriam-Webster dictionary:

Definition 4. A *proof* is an argument which establishes the validity of a statement by derivation from other statements (called the premise) in accordance with principles of reasoning.

According to this definition, there are only two possible reasons to reject a proof

1. We do not accept the validity of the premise in this context. For example because the premises are based on axioms, we do not accept, because it has not been proven in this context yet, or simply because it is false.

2. The proof relies on principles of reasoning, which cannot be applied in this setting, which are applied incorrectly, or which are invalid in general.

If a mathematician finds both the premise and the principles of reasoning to be sound, he should also accept the conclusion.

However, a proof is an abstract entity, and in order to critique it, we must have a physical representation of it (usually an oral description, chalk on a black board or letters on paper). The physical representation can be critiqued for:

3. The sender does not give sufficient amount of detail to why the different steps in the proof are valid.

To give an example of different reasons to reject a proof, consider the following theorem and invalid proof:

Theorem 1. If f is a continuous, real function and $a, b \in \mathbb{R}$ then

$$\int_a^b f(x)\ dx \leq \int_a^b |f(x)|\ dx$$

Proof. We prove the theorem by splitting into cases: $f(x) \geq 0$ or $f(x) < 0$.

If $f(x) \geq 0$, then $f(x) = |f(x)|$ and thus $\int_a^b f(x)\ dx = \int_a^b |f(x)|\ dx$.

If $f(x) < 0$, then $\int_a^b f(x)\ dx \leq 0$. In contrast to this $|f(x)| \geq 0$ so $\int_a^b |f(x)|\ dx \geq 0$.

Thus we see that in both cases $\int_a^b f(x)\ dx \leq \int_a^b |f(x)|\ dx$ □

We tested this proof on a first year student of mathematics in the pilot phase of this study. The first year student (rightly) rejected the proof. We had expected him to argue along the lines of principles of reasoning: The proof is a proof by cases, but the cases are not exhaustive as there exists functions which change sign within the interval. Hence proof by cases cannot be applied here, and the proof is invalid. The first year student had another reason for rejecting the proof, though. He argued that the premise was not certain, since he did not know if the integral of a positive/negative function is always positive/negative. He was in the stage of his studies, where he was very careful with what could be assumed. He likely did not doubt in the truth of such a theorem, he had just internalised that he could only use theorems that he had seen proven.

His objection raises a valid point. When we judge the validity of a proof, we are not just judging the proof in itself. We are also judging how it fits into the wider network of theorems and proofs. We need to make certain how it fits into the mathematical world, we are constructing, and in particular, which other theorems we can assume and when.

However, this sort of judgement is certainly not a prerequisite to learning to understand proofs involving the definition of limits. The prerequisite we will be interested in, is whether the students notice logical mistakes in an argument. That is, we are only interested in testing whether the students can validate proofs according to number 2 on the list above. Even with this restriction in place, we still have to ask whether there is consensus in the mathematical community on which rules of reasoning we are allowed to use, when and how.

Unfortunately, we have seen historically that this is not the case. Famously, the intuitionists surrounding L. E. J. Brouwer in the beginning of the 20th century did not accept proofs by contradiction [Hesseling and Avigad 2006] and a large part of the mathematical community did not accept the use of computers in the proof of the four-colour problem [Bueno and Azzouni 2005]. Inglis, Mejia-Ramos, et al. 2013 investigate whether there is any ongoing disagreement in the mathematical community, which affects even the well-understood, established and non-controversial area: elementary calculus. They do this by giving the following theorem and proof to 109 research active mathematicians.[1]

Theorem 2.

$$\int x^{-1}\,dx = \ln(x) + c$$

Proof. We know that $\int x^k\,dx = \frac{x^{k+1}}{k+1} + c$ for $k \neq -1$.

Rearranging the constant of integration gives $\int x^k\,dx = \frac{x^{k+1}-1}{k+1} + c'$ for $k \neq -1$.

Set $y = \frac{x^{k+1}-1}{k+1}$, and take the limit as $k \to -1$ as follows.

Let $m = k+1$, and rearrange $y = \frac{x^{k+1}-1}{k+1}$ to give $x^m = 1 + ym$ or $x = (1+ym)^{1/m}$.

Set $n = \frac{1}{m}$. Then $x = (1+ym)^{1/m} = (1+\frac{y}{n})^n \to e^y$ as $n \to \infty$, by properties of e.

As $n \to \infty$ we have $m \to 0$, so $k \to -1$.

In other words, $x \to e^y$ as $k \to -1$, so $y \to \ln(x)$ as $k \to -1$.

So $\int x^k\,dx = \frac{x^{k+1}-1}{k+1} + c' = y + c' \to \ln(x) + c'$ as $k \to -1$.

So $\int x^{-1}\,dx = \ln(x) + c'$. □

And alas. The mathematicians do not agree: 73% of the mathematicians decree that the proof is invalid and 27% decree that it is valid. The mathematicians report three different reasons for rejecting the proof

1 56 PhD students and 53 academic staff associated with Australian and Canadian universities in the fields: applied mathematics, pure mathematics, or statistics.

1. c' cannot be treated as a constant, since it depends on k.
2. The proof relies of switching the order of integration and limit. As this is not in general a valid move, the author must argue why we can do so in this case.
3. It is not clear how e and ln are defined. A standard way of defining ln is as the integral mentioned. Since the author is obviously not using that one, we need to know which definition of ln the author is using to validate the proof.

Only the first objection falls within the scope of this thesis. We consider it to be a very valid objection about the rules of reasoning concerned with taking limits of expressions involving constants. In fact c' tends to infinity as $k \to -1$, so treating it is a constant is wildly inappropriate. The second objection is not about rules of reasoning as the interchange of limit and integration is valid in this case. So the mathematicians are not rejecting the proof because logic is unsound, they are rejecting it because they want more detail. The third reason is about the premise of the proof, and is of the same type as the student's in our earlier example.

The fact that some of the mathematicians discarded the proof because of reason number 1 and some did not is a source of worry for this thesis. If they agree on the rules of reasoning, they should agree about whether reason number one is significant enough to discard the proof.

However, Inglis, Mejia-Ramos, et al. 2013 note that this may be due to a "performance error" on the part of the mathematicians who accept the proof: it may be that they simply do not notice the mistake, and if the mistake is pointed out to them, they would change opinion and reject the proof. If this is the case, we would assume that the mathematicians would reach a consensus if given time and space to discuss the proof, and then we can use this consensus as the aim of proof validation.

Inglis, Mejia-Ramos, et al. 2013 test whether "performance error" may be the cause of disagreement between the mathematicians in their study. Unfortunately for our project, they only discus reason number 2 in this setting, so we cannot learn from this second part of their investigation, and we do not have the time to carry out our own investigation.

Hence for this thesis we have to make the (unsupported but not implausible) assumption, that within the relatively well understood, established and non-controversial topic elementary calculus, the mathematical community agrees on which principles of reasoning are valid and when they can be applied.

5.1.2 *The process of proof validation*

What do mathematicians do, when trying to validate a proof? To answer this question, we follow Inglis and Alcock 2012 and A. Selden and J. Selden 2015 in distinguishing between a local and a global level.

Local level

On the local level, we find that the term "warrant" as described in Alcock and Weber 2005 is useful for understanding. A warrant is an argument to why one statement in a proof follows logically from the previous. Sometimes a warrant is not required and sometimes is it explicitly stated. As an example of this, take the following proof

> **Theorem:** There exists a real number x such that $x^4 = x + 1$
>
> **Proof:** Set $f(x) = x^4 - x - 1$
>
> $f(1) = -1$ and $f(2) = 13$
>
> As f is continuous, the intermediate value theorem gives that there exists a c between 1 and 2 such that $f(c) = 0$
>
> c has the desired property.

This proof contains a number of statements:

- S_1: $f(x) = x^4 - x - 1$
- S_2: $f(1) = -1$ and $f(2) = 13$
- S_3: f is continuous
- S_4: There exists a c between 1 and 2 such that $f(c) = 0$
- S_5: c has the desired property.

Logically S_4 follows from S_2 and S_3. The argument to why the implication is true, is the (explicit) warrant "by the intermediate value theorem". S_1 is a definition, so it does not require a warrant.

S_3 is a good example of a statement in a proof, which is not supported by an explicit warrant. It is, however supported by the implicit warrant "f is a polynomial and all polynomials are continuous".

According to Inglis and Alcock 2012, when a reader is validating a proof, the reader has to

1. accurately determine when a warrant is required

2. correctly infer an implicit warrant

3. evaluate the warrant's mathematical validity

Alcock and Weber 2005 show that students find this difficult if both the premise and the conclusion of an argument are obviously true. They present the students with the following invalid proof:

Theorem: The sequence $a_n = \sqrt{n}$ tends to infinity as $n \to \infty$
Proof: We know that $a < b \Rightarrow \sqrt{a} < \sqrt{b}$
$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n
This shows that $\sqrt{n} \to \infty$ for $n \to \infty$

which is invalid, since the last statement "$\sqrt{n} \to \infty$ for $n \to \infty$" does not follow from the previous. Only three out of thirteen students noticed this mistake, which Alcock and Weber 2005 takes to indicate, that students do not check warrants if both the premise and the conclusion are true.

Global level

One could argue, that if a reader has checked how each statement in the proof follows from the previous, then the reader can be sure the proof is valid. However, in practise, this is not sufficient for mathematicians. Weber and Mejia-Ramos 2011 asked nine research active mathematicians how they read proofs. All nine participants indicated that understanding a proof was more than just understanding how each statement followed from the previous. It is also a matter of seeing the key ideas of the proof. Some of the students interviewed by Weber 2009 feel the same way. Weber 2009 describes 13 instances in which the students appear to accept all the steps in a proof, but does still not find the proof convincing. A student, who is talking about the proof in question 6 of our quiz, is quoted to say

> "Each of the steps made sense though. I think it's right, but I'm not sure."

In the language, we have just developed, we would say that the students have achieved local proof validation, but that he feels he is still missing something.

Inglis and Alcock 2012 call this "missing something" global proof justification. They describe it as "decomposing the proof into methodological moves and evaluating whether these moves fit together to imply the theorem". They describe "methodological moves" as "encapsulated strings of logical derivations that together form coherent chunks of the whole argument".

To understand what they mean, we look at an example from Rav 1999, who coined the term "methodological moves". The example concerns the Theorem of Lagrange. The theorem states that for any finite group G and any subgroup $H \leq G$, the number of elements in H divides the number of elements in G. The proof of this theorem can be divided into the following methodological moves:

1. Define the notion of cosets.
2. Show that any two cosets are disjoint.
3. Show that all cosets have the same number of elements as H.
4. Show that any element of G must be in one of the cosets.

Each move is an encapsulated chunk of the whole argument, and together the moves imply the theorem.

According to Thurston 1994, ones ability to understand a proof depends on the mental infrastructure one has available. Each mathematical subfield has its own standard terms, standard ways of proving and standard metaphors. Within a subfield, communication is easy, as the basic mental constructs are already in place, but between subfields, communication is hard. Thurston 1994 gives as example of a proof which takes 2 minutes to communicate to mathematicians in one subfield, but an hour to communicate to mathematicians in another subfield.

We hypothesise that one reason why students have trouble understanding proofs is that they do not have the necessary mental infrastructure in place. A trained mathematician who looks at the proof of the Theorem of Lagrange will already be familiar with

- forming new sets from old sets by applying a transformation to each element of the old set
- proving that two sets are disjoint by assuming an element lies in both sets and achieving a contradiction
- proving that two sets have the same size by constructing a function from one to the other, and proving the function is a bijection

So the appropriately schooled mathematician is able to encapsulate all the detail of the proof into just four methodological moves. Each move only takes up a small part of the working memory of the mathematician, so he is able to hold all four moves in mind at the same time. Hence the mathematician is able to zoom out and see how the methodological moves fit together to imply the theorem.

In the language of our hypothesis, a student, who looks at the same proof, might not have the mental infrastructure in place, so he may be unable to encapsulate each of the chunks into a methodological move. Hence he cannot hold all the moves in his working memory at the same time, so he cannot perform global proof validation.

Our hypothesis is inspired by the APOS theory, in which mathematical processes can become encapsulated into object, which can then be used in higher order processes [Cottrill et al. 1996]. In our hypothesis, chunks of argument become encapsulated into methodological moves, which can then be used in higher order arguments. In long proofs, the higher order arguments may then be collected into chunks, which are encapsulated as methodological moves and so on.

5.1.3 *Why is proof validation necessary for learning to understand proofs involving limits?*

Several mathematical textbooks start their introduction by encouraging the reader to be critical of everything they read in the book, because the author believes this will help them understand the material better [Allenby 1961, Thorup 2007]. Despite this, several of the practising mathematicians we talked to as well as some didactic researchers [Sierpinska 2007, Anthony 1996] note that many students rely on the teachers to provide validation of mathematical arguments instead of trying to validate them independently. Alcock and Weber 2005 point out that much of the teaching at universities is through proofs presented at lectures, so it is important that students are able to learn from this sort of teaching. They question how much students learn, if the students cannot independently judge the validity of the arguments presented at lectures.

5.1.4 *Summary*

We started this section by defining proof validation to be “the process of judging whether a proof fulfils the requirements of the current mathematical community”. We found three different reasons to why a mathematician might reject a proof, but we decided to limit our investigation to one of them: “The proof relies of principles of reasoning, which cannot be applied in this setting, which are applied incorrectly, or which are invalid in general”.

We then went on the describe the process of proof justification. We found that it splits up in two parts: local and global proof validation. We will need these terms, when we analyse the questions in the test of proof validation.

Finally we presented an argument to why proof justification can be seen as a prerequisite to learning the formal definition of limits.

We now turn our attention to the next proposed prerequisite. We will return to how proof validation can be measured in the chapter on methods.

5.2 ACTIVE MEANING ORIENTATION

In this section we develop the hypothesis that the students' behaviour in teaching situations and their beliefs about what is expected from them influence how much they learn. Several authors have described behaviours and beliefs among students, which are likely to have an influence on learning mathematics [Weber and Mejia-Ramos 2014, Sierpinska 2007, Anthony 1996, Alcock 2012], however we have not been able to find any theory within didactics of mathematics which unify the observations of these authors. Therefore, we will start this section by describing more general theories of learning approaches and orientations, which apply to students of any discipline at university. We will then use the above articles to specialise the general theory to the learning of mathematics.

5.2.1 *Deep vs. surface approach*

The distinction between deep and surface approaches to learning was first described by Marton and Säljö in 1976 and later developed by Entwistle in the 1980's [Marton 1997]. In the original experiment, 30 Swedish university students were asked to read an article. Marton and Säljö found that the students showed very different levels of understanding of the text, but also that the students varied in their account of how they went about the task. The main division line was between what students using something they called a deep approach and those using something they called a surface approach. We base our descriptions of the terms on Entwistle and Ramsden 1982:

- Surface approach: Intention to memorise. Constrained by specific task requirements and anxious about that constraint. Interpreting the text as an aim in itself instead of seeing the text as a means through which one could learn about the world. A quote from a student using surface approach: *In reading the article, I was looking out mainly for facts and examples... I thought the questions would be about the facts in the article... This did influence the way I read; I tried to memorise names and figures quoted , etc.*

- Deep approach: Intention to understand. Interacting actively with arguments by relating them to previous knowledge and their own experience and by trying to see to what extend the author's conclusions were justified by the evidence presented. A quote from a student using the deep approach: *I was looking for the argument and whatever points were used to illustrate it... My feelings about the issues raised made me hope he would present a more convincing argument than he did, so that I could formulate and adapt my ideas more closely, according to the reaction I felt to his argument*

Marton and Säljö found that a deep approach to learning was connected to a higher level of understanding, and the result has been confirmed several times in later studies [Marton 1997].

It has later been argued that deep approach should be subdivided into an active and a passive variant, as it is possible to intend to understand a text without actively engaging with the arguments. [Entwistle and Ramsden 1982]. Take for example the following quote from a student in one of the investigations of Entwistle and Ramsden 1982:

> "I did not expect the questionnaire to ask for any details from the article. Consequently I read it with impartial interest – extracting the underlying meaning but letting facts and examples go unheeded"

Entwistle and Ramsden 1982 calls this approach deep passive approach, to distinguish it from the deep active approach described above. Students, who use a deep passive approach, do intend to understand the article, but do not interact actively with the details of the argument by for example judging if the conclusion follows from the evidence [Entwistle and Ramsden 1982].

5.2.2 *Study orientations*

The distinction between deep and surface above describe how the student approaches a single task, and it may vary from task to task. Entwistle and Ramsden 1982 expands the concept to describe more general study orientations, by which they mean the tendency of a student to approach tasks in a specific way. The context and the task is still assumed to influence the approach of the student, but it is assumed that each student has a tendency to approach tasks in a certain manner, and this tendency is fairly consistent over time [Entwistle and Ramsden 1982].

An important part of their research consists in investigating how different dimensions of student behaviours and approaches are related. Among the dimensions they investigate are

- Intrinsic motivation: Interest in learning for learning's sake. Representative item: "I find academic topics so interesting, I should like to continue with them after I finish this course"
- Syllabus-boundness: Relying on staff to define learning tasks. Representative item: "I like to be told precisely what to do in essays or other assignments"
- Strategic approach: Awareness of implications of academic demands made by staff. Representative item: "Lecturers sometimes give indication of what is likely to come up in exams, so I look out for what may be hints"
- Organised study methods: Ability to work regularly and effectively. Representative item (reverse scoring): "My habit of putting off work leaves me with far too much to do at the end of term"
- Internality: Sees truth as coming from within, not from external sources. No representative items given in Entwistle and Ramsden 1982 nor in the article he cites.

Through factor analysis they find that a deep approach to studying is often related to intrinsic motivation and internality, surface approach is often related to syllabus-boundness while strategic approach is often related to organised study methods. From this and other investigations they create their descriptions of different study orientations:

- Meaning orientation: The name of the factor which associated deep approach, intrinsic motivation, and internality (and others). It was later split into an active and a passive variant to reflect the difference between active and passive deep approach. Both types of students look for meaning, but students with an active meaning orientation "use evidence critically, argues logically and interprets imaginately" whereas students with a passive meaning orientations thrive "on personal interpretation and integrative overview", but neglects evidence.
- Reproducing orientation: The name of the factor which associates surface approach, syllabus-boundness and others. Described as students who are motivated by the job-perspectives of the education, can accurately describe facts and components of arguments, but not relate this to any clear overview.
- Strategic orientation: The name of the factor which associates strategic approach, organised study methods and others. Student with this orientation are described as employing approaches focusing on meaning or detail according to what they perceive the teacher to require. They often get good grades (with or without understanding).

- Non-academic orientation: Will not be used further in this thesis, but for the sake of completeness, we will mention it. The stereotypical personality of students with this orientation is social extroverts with few academic interests or vocational aspirations. They often show little attention to detail and an over-readiness to generalise. They often get low grades.

A major question is whether the orientations above are stable enough over time or even across tasks for it to make sense to describe students according to them. Entwistle and Ramsden 1982 do not deal with this question. The students are only asked to describe their learning approach once, so we expect the measure, the above connections are build on, to be a snapshot of how the student view their own orientation at a particular point in time. We expect that this view is probably influenced by the students' recent experiences in learning. However, it was never the aim of Entwistle and Ramsden 1982 to find boxes to put all students into. Rather they were interested in how the orientation of the students are influenced by the learning environment, so in a very real way, they expect the orientations to be changeable.

5.2.3 *What is active meaning orientation in mathematics?*

The descriptions above are meant to describe students of any subject. However, Entwistle and Ramsden 1982 find that deep approach in science looks different to deep approach in humanities. If they are measured on the same scale, science students are for example less inclined towards taking a deep approach and are more syllabus bound. However, this may be because "in science, it seems that it may be necessary to use procedures which are empirically inseparable from surface approaches as a stage prior to taking a deep level appoach" [Entwistle and Ramsden 1982].

We do not know of any sources which describe active meaning orientation or active deep approach in mathematics, but a lot of authors have investigated student behaviours and beliefs about learning mathematics. Even though they are not explicitly referring to the concept "active meaning orientation", it seems to us that what they describe often carry the same distinctions between meaning and reproducing and active and passive.

Responsibilities when reading a mathematical proof

Weber and Mejia-Ramos 2014 investigate "what students believe is their responsibility when reading a mathematical proof and what mathematicians believe the students' responsibilities to be, highlighting discrepancies between the two".

He illustrates how mathematicians might read a proof by the following quote from a practising mathematician:

> there is a very elegant, short proof on the classification of wallpaper groups written by an English mathematician. So this is one where he's deliberately not drawing pictures because he wants the reader to draw pictures. [...] Each assertion in the proof basically requires writing in the margin, or doing an extra verification, especially when an assertion is made that is not so obviously a direct consequence of a previous assertion. [...] And also I try to check examples, especially if it's a field I'm not that familiar with, I try to check it against examples that I might know.

And contrasts this to the answer of two students who were asked what they thought made a good mathematical argument:

> P9: It's got to be really detailed. You have to tell every detail. Every step, it is very clear. I like doing things step by step.
>
> Interviewer: So you like having every detail spelled out as much as possible?
>
> P9: Yeah, yeah, yeah.
>
> P10: Well, it has to cover all the bases so that it is in fact a complete rigorous proof. For me, as a student, what else I would like to see are all the intermediate sorts of steps, things to help along, graphs, and visual things. Things that recalled facts that perhaps I should know but you know, maybe not immediately at the tip of my tongue. That's to me what makes a good mathematical argument.

In order to find out if this discrepancy between the expectations of the teacher and the students is a wide spread phenomenon, Weber and Mejia-Ramos 2014 a survey to 50 large state universities (one in each state). In this survey the mathematics students of the university were asked which of the following statements they preferred

1. In a good proof, every step is spelled out for the reader. The reader should not be left wondering where the new step in the proof came from.

2. When reading a good proof, I expect I will have to do some of the work to verify the steps in the proof myself

And the mathematicians at the university were asked a similar question, but instead of asking about their own experience of reading proofs, the questions were phrased about mathematics majors. They received replies from 175 undergraduate mathematics majors who had taken a mathematics class in which they were expected to write proofs regularly and 83 mathematicians who had taught such a course. They found that 75% of the mathematics majors preferred 1 and 14% preferred 2, while 27% of the mathematicians preferred 1 and 52% preferred 2.

They found a similar pattern, when the participants were asked whether they preferred 1 or 2 below:

1. When reading a good proof, if a diagram can help my understanding, it should be included. I should not be expected to draw a diagram myself.

2. When reading a good proof, sometimes diagrams are not included. I expect I sometimes have to draw these diagrams myself.

Where again the mathematicians were asked about the students and the students were asked about themselves. In both cases the difference between the group of mathematicians and students was significant, and Weber and Mejia-Ramos 2014 concludes that this difference in perception may be one of the causes why students learn little from proofs in advanced courses.

The question raised by Weber is not explicitly mentioned in the theory above, as it is specific to mathematics, but we find that it corresponds well to "interacting actively with the arguments" and "seeing if the argumentation justifies the conclusion", which are some of the examples of an active deep approach.

I need the teacher to tell me if I'm right or wrong

Sierpinska 2007 studied adult students of a pre-university course required by a university for admission into academic programs such as psychology, engineering or commerce. She tries to understand why the students depend on teachers for the validity of their solutions and lack sensitivity to contradictions. To illustrate the problem, she quotes the results from one question in the end of course evaluation, where 96 students were asked if they agreed with the statement: "I need the teacher to tell me if I am right or wrong". 67% of the students agreed.

She also quotes the PhD instructors of the course, who all agree that the students disliked theory and proofs. They preferred worked out examples of typical examination questions. Furthermore, they said that the students preferred to memorise more rules and formulas than to understand how some of them can be logically deduced from others and memorise fewer of them. As one instructor puts it

> [Students don't want to reason from definitions about] those rules, [although] all the rules come from the definition (...) It's especially true when we learn (...),

> the seven rules of exponentia[tion]. Sometimes, I just try to let them know that [it is enough to just] know four [rules], or even three, if one knows the definition well. You don't need to put so much time on recalling all those rules in your mind. But when I try to explain those things, they don't like it. They ask me 'Why, why you do this?'

Sierpinska 2007 is related to the above theory as she deals with the question of whether the student sees the truth as coming from within or external sources, which was shown to be related to a meaning orientation.

Different kinds of active learning

Anthony 1996 followed a class of 6th form (year 12) students in their mathematics classes for a year in order to study their use and awareness of learning strategies in their authentic learning environment. In addition to class room observations, some of the students were also interviewed between classes. Anthony 1996 concludes that "having students involved in activities such as discussions, question answering, and seatwork problems does not automatically guarantee successful knowledge construction. The nature of students' metacognitive knowledge and the quality of their learning strategies are seen to be critical factors in successful learning outcomes."

To illustrate her point, she describes two different students Gareth and Adam, who are both active and appear to be on task in class, but who in fact have very different learning strategies. Where Adam made sense of incoming information by adding details, explanations, examples, and mental images that relate the information at hand to prior knowledge, Gareth focused solely on the current work without attempting to look for connections with what was done previously. Where Adam selectively attended to the conceptual material rather than the calculations, Gareth's focus on the procedural and arithmetic steps of a problem was often at the expense of attention to conceptual material. Where Adam critically evaluated methods on the criteria of efficiency, ease, and completeness, Gareth accepted the teacher's answers uncritically, even in cases where they were incorrect. Where Adam was conscious of his learning goal, planned his learning, monitored his progress towards the learning goal, and was able to take remedial action where necessary, Gareth equated problem completion with success, and as a consequence was blissfully ignorant of the extent of his difficulties. Where Adam felt that he did not need to make an extra effort when it comes to revising for tests because he has done a thorough job on his homework, Gareth uses revision strategies involving memory strategies such as rereading and rehearsal: “If I have to memorise the formula that they won’t give me, or a graph, I just write it our a few hundred times.”

To illustrate the problem with Gareth’s learning strategy, we describe an example where Gareth’s strategy of copying step-by-step procedures from the worked examples resulted in a very telling mistake. He was completing the exercise: “Find the turning point of the function $y = x^2 - 6x + 11$, and the values for which the function is increasing or decreasing”. Gareth follows the explanation steps in a similar worked example: “differentiate and solve for 0”, to successfully find the turning point (3,2), but then mistakenly continues by applying the next explanation steps: “substitute x = -2” and “substitute x = 0” which comes from another example.

This article is interesting in that it describes the same distinction between students as Entwistle and Ramsden 1982 (Adam is the deep learner, who connects new knowledge to previous knowledge and is critical of the “truth” the teacher presents and Gareth is the surface learner who tries to learn by rote memorisation), but if this is true, then the article adds a possible mathematical interpretation of the distinction between deep and surface approaches. Namely that surface learners learn the procedures by heart without understanding them while the deep learners are more interested in the concepts. It is not that deep learners of mathematics ignore the procedures, it is just that they also want to understand why they work. The article is also interesting as it adds a new concept, which is not covered by the theory of Entwistle and Ramsden 1982, namely the metacognitive ability of the student. That is the ability to monitor ones own learning, and actively seek out ways to fill the holes in ones understanding. This is mentioned in both this article and the next.

Advice to new students from experienced teacher

The book “How to study for a mathematics degree” [Alcock 2012] contains an extensive guide to new students of mathematics from Lara Alcock, who is both a teacher of first year students and a researcher of the didactics of mathematics.

In all types of situations, where the student is being presented new knowledge either through a book or a lecture, Alcock 2012 for example advise that students

- look up definitions of words they do not know (right away or whenever they have the chance)
- take control of what they do not know and find ways to seek out answers

- when being presented with an example: check that the definition/theorem applies in this case.
- when reading in the book, have a pen and paper ready, and do the calculations themselves

When reading or being presented a theorem, Alcock 2012 for example advises that, the students ask themselves questions such as

- Does the theorem sound intuitively plausible?
- What happens if you relax the premise? Is the conclusion still true, or can you find a counterexample?

When solving exercises, Alcock 2012 for example advice that students

- Before solving an exercise, take a minute to think about which procedure to apply instead of just rushing headlong into the first thing that comes to mind.
- When solving an exercise, stop and reevaluate from time to time to check they are not wasting time on something, which is clearly not working.
- After solving an exercise, ask themselves
  - Why did the procedure work?
  - What could have changed in the question, so it would still work?
  - What could have changed in the question, so the procedure would no longer work?
  - Can the procedure be modified to work here as well?

Alcock 2012 is extremely specific about what students can do to improve at mathematics. Theoretically it seems to be well connected to the active deep approach in that it seems to be an elaboration on what it means in mathematics to "interact actively with arguments by relating them to previous knowledge" and to judge "to what extend the author's conclusions are justified by the evidence presented". Also as mentioned previously it also places emphasis on the importance of monitoring ones own learning.

5.2.4 *Summary*

In this section, we described the theory of Entwistle and Ramsden 1982, which is concerned with different approaches to learning. The main distinction is between the deep and the surface approach. However, Entwistle and Ramsden 1982 further subdivided the deep approach into an active and a passive variant. Students of both variants seek to understand the meaning, but only students with an active deep approach interact actively with the material by

- Relating it to previous knowledge
- Relating it to own experience
- Judging if the evidence justifies the conclusion

Entwistle and Ramsden 1982 showed that an active deep approach is linked to internality (seeing the truth as coming from within) and intrinsic motivation (interest in learning for leaning's sake). They coined the term active meaning orientation to describe the orientation of a student who is inclined to use an active deep approach, is intrinsically motivated and sees truth as coming from within.

Mathematics educators have also been interested in describing different student behaviours and beliefs which are likely to influence learning. We found that the descriptions of Weber and Mejia-Ramos 2014, Sierpinska 2007, Anthony 1996 and Alcock 2012 corresponded well to the description of active meaning orientation above. We see their descriptions as adding mathematical interpretations of the more general theory. For example, what Entwistle and Ramsden 1982 calls

- internality can be interpreted in mathematics as sensitivity to contradictions and non-reliance on the teacher for checking the validity of solutions
- "interacting actively with the material", can be instantiated in mathematics by examples such as filling in small arguments in proofs and verifying the calculations on a piece of paper next to the book
- "intention to understand" can for example be interpreted in mathematics as attention to both concepts and calculations.

However, not everything that the mathematicians described was covered by the theory of Entwistle and Ramsden 1982. The mathematicians added the extra aspect of being able to metacognitively monitor one's own learning, and actively seeking to fill gaps in one's understanding.

In this thesis, we wanted to test the hypothesis that the student's orientation to learning is important in connection with learning the formal definition of limit. The specific orientation to learning, we are interested in, is the active meaning orientation combined with metacognitive monitoring of own learning, where these concepts have been interpreted in terms of how they show up in mathematics education.

We will return to how this can be measured in the chapter on methods. For now, we turn to the next proposed prerequisite.

5.3 USE OF DEFINITIONS

In this section we discuss how mathematical definitions are thought to associate words to meaning and contrast this to psychological theories for how the human brain functions in every day life. We hypothesise that students, who have not understood the special nature of mathematical definitions, will find it difficult to understand the formal definition of limits.

5.3.1 *Definitions in mathematics*

In mathematics, definitions have a very important role

> the mathematicians, ever since Euclid at latest, have been making their own meaning for words. 'By a *denumerable* series', they say, for example, 'we shall mean a series which you can put into one-to-one correspondence with the positive integers without changing its order.' This is not a historical description of what has been meant by 'denumerable' in the past or is commonly meant by it now. It is an announcement of what is going to be meant by it in the present work, or a request to the reader to take it in that sense. [Robinson 1963]

Edwards and Ward 2004 call this sort of definition a stipulated definition. Stipulated definitions do not report usage – they create usage. Therefore, stipulated definitions have to create clear boundaries between what is and what is not meant by the word and once a word has been defined through a stipulated definition, the word is expected to be free from associations connected to previous use of the word. Mathematicians may deduce new facts about the concept from the definition, but their previous conceptions about the word have to be treated with care [Edwards and Ward 2004]. For example, if we define increasing functions, to be functions f which satisfy

$$x_1 \geq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

Then we have to expand our image of increasing functions to also contain the constant functions [Alcock 2012].

5.3.2 *Definitions in everyday life*

Outside mathematics this sort of process is rarely needed, as most words are not connected to meaning through stipulated definitions. To see this, consider the word “bird”. If it had a stipulated definition, we should be able to come up with a list of necessary and sufficient conditions, which clearly delineates between instances and non-instances of birds. But this is surprisingly difficult! Some obvious conditions could be [have feathers], [lays eggs], and [can fly], however, not all birds have feathers, but some clothing accessories do, not all birds lay eggs, but some turtles do, not all birds fly, but most air planes do. [Gleitman et al. 2011]

Problems like the above have led psychologists to suggest other theories for how word meanings are represented in our brains. One of these theories is the prototype theory. According to this theory, we all have a mental prototype for every word we know – a prototypical bird, a prototypical piece of furniture and so on. These prototypes are constructed during our lives as a sort of mental average of all the instances of the word we have seen. For example, a person living in Denmark has seen more sparrows than ostriches, so our prototypical bird is likely to resemble a sparrow more than an ostrich. According to this theory, when we are asked to judge if something is a bird, what we do is we mentally compare it to our prototypical bird. So for example, a lark is obviously a bird, a penguin is a border case bird, and a row boat is not a bird at all [Gleitman et al. 2011].

This theory would help explain why some instances of a category appear to exemplify the category better than others. For example a Labrador seems like a more typical dog than a Pekingese, an armchair is somehow more furniture-ish than a lamp and a study showed that Americans spend longer time before answering “yes” when asked about the truth of the statement “an ostrich is a bird” than the statement “a robin is a bird” [Gleitman et al. 2011].

5.3.3 *Navigating definitions in mathematics*

Vinner 2002 has investigated the clash between the demands of formal mathematics and the way in which humans normally deal with the meaning of words. He distinguishes between

- the “formal definition” of a word, which is the definition as it appears in the relevant textbook,
- a persons “concept definition”, which is whatever the person answers when you ask them what the definition of the concept is,

- and a persons "concept image" which is whatever pictures, impressions or experiences appears in the mind of the person, when you present them with the word.

He demonstrates that many students do not have concept definitions which are in accordance with the formal definition. For example, he shows that only 57% out of 147 students, who studied higher level mathematics in grades 10 and 11 and had been presented with the formal definition of functions, had a concept definition which was in line with the formal definition. 14% of the students thought that a function must be given by an algebraic formula. We take this as evidence that the students assigned meaning to the word "function" according to the prototypical theory of word meaning. They probably have seen very few or no functions which were not defined by a formula, so their mental average function is defined by a formula.

Interestingly, he also shows that some students give answers to exercises, which are not in accordance with their concept definition. For example, he shows that only one third of the students, who correctly defined "function", answered three questions of the type "is this a function?" correctly. This he interprets as the students relying on their concept image instead of their concept definition, when replying to certain questions. Tall and Vinner 1981 argue that these students may find it difficult to see the value in formal proofs. They give an example of students, who do not find it meaningful to prove that if $s_n \to s$ for $n \to \infty$ and $s_n, s \neq 0$ for all n, then $1/s_n \to 1/s$ for $n \to \infty$. The students claim that this fact is obvious (based on their concept image), and the proof is an unnecessary complication.

5.3.4 *Summary*

We have found two different reasons why students may have trouble understanding the use of definitions in mathematics. First, definitions in mathematics are stipulated and create clear boundaries between what is and what is not meant by a word. The prototypical theory of word meaning indicates that the human brain does not naturally work that way. Rather, it assigns meaning to a word based on a mental average of all the instances we have seen and uses this average to judge *to what extent* something is covered by a definition. To succeed in mathematics requires that we train our brains to understand the nature of stipulated definitions.

Second, Vinner 2002 indicate that even if an individual has formed the correct concept definition, he or she may not refer to it. The concept definition may become a separate piece of information which is at times opposed to the concept image. Such an individual may find proofs which involve the formal definition bewildering.

This concludes our section on the proposed prerequisite, “use of definitions”. The fourth proposed prerequisite, that we want to describe is “misconceptions of limits”.

5.4 MISCONCEPTIONS OF LIMITS

Many students enter higher education with concept images of limits which are not in accordance with the formal definition. For example, some students express the opinion that sequences can never reach their limit, or that limits act as an upper bound [Williams 1991, Nagle 2013]. These misconceptions may not always be visible to the teacher, as many of the students *also* have correct conceptions. When confronted with a situation which requires a notion of limit, the students may retrieve a conception which is in line with the formal definition, or they may retrieve one which is not [R. B. Davis and Vinner 1986]. For example Tall and Vinner 1981 show that 14 out of a group of 36 mathematics students, who just arrived at university in one instance reply that

$$\lim_{n\to\infty}\left(1+\frac{9}{10}+\frac{9}{100}+...+\frac{9}{10^n}\right)=2$$

and in another that $0.\overline{9} < 1$. It is apparently possible to hold both conceptions at the same time without it causing any cognitive conflict. It is only when the two conceptions are retrieved simultaneously that conflict emerges [Tall and Vinner 1981].

It is thus possible to go through an entire university course in elementary calculus with several misconceptions about limits intact [Przenioslo 2004]. The question we attempt to answer in this thesis is whether these misconceptions are a hindrance in using the formal definition of limits. We could imagine that the formal approach at university is sufficiently different from what the students associate with limits, that their misconceptions are never evoked. Tall and Vinner 1981 are not so optimistic, though. They argue that “Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous” and this may “seriously impede the learning of a formal theory”.

5.4.1 *Different misconceptions of limit*

Reviewing the literature, we found a number of different misconceptions, which some students are reported to have:

- Dynamic conceptions of limit (to be explained below) [Núñez 2006, Monaghan 1991, Nagle 2013, Williams 1991, Cornu 1991]
- Limits can never be reached [Nagle 2013, Williams 1991, Cornu 1991, Tall and Vinner 1981]
- Limits cannot be overstepped [Cornu 1991, Monaghan 1991]
- Limits are determined by plugging in numbers closer and closer to a given number [Williams 1991]
- A limit is an approximation of the function value [Nagle 2013]
- The limit is equal to the function value [Nagle 2013]
- Sequences increase/decrease monotonously towards their limit [Cornu 1991, R. B. Davis and Vinner 1986]

Due to our limited time, we had to make a choice of which misconceptions to focus on. From our own experience, we judged that the first three misconceptions were likely to be most widespread among the first year students in mathematics at UCPH, so we chose to focus on those. Below, we will describe each misconception more in detail:

Dynamic notion of limit

The formal definition of $\lim_{x\to a} f(x) = L$ is:

For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

According to this definition, nothing is moving! Every epsilon, delta and x is a fixed number on the numberline. The definition gives a condition which must be fulfilled for every x which *is* close to a. It does not somehow make an x move closer to a.

Nevertheless, 80% of the 341 students in a second semester calculus course who took part in the investigation by Williams 1991 marked the following statement as true:

A limit describes how a function moves as x moves toward a certain point

According to Monaghan 1991 and Cornu 1991, part of the reason for this conception to exist among students is the language used around limits in English and Danish as well. Words like "tends to" and "approaches" (in Danish "går mod" and "nærmer sig") already have meanings in our everyday lives and these meanings involve motion. Hence, even if a teacher introduces limits only by way of a static definition, the language in itself may persuade the students to develop a dynamic understanding of limits.

However, it is unlikely that many students have met a purely static introduction to limits. Núñez 2006 uses analysis of gestures to demonstrate that even some practising mathematicians have dynamic images of limits. When explaining limits, they voluntarily or involuntarily move their hands to imply movement. To Núñes this is not a misconception of limits. Rather, it indicates that not every aspect of mathematics is captured by formalisms and axiomatic systems. Rather, mathematics relies on conceptual metaphors which allows mathematicians to make sense of concepts which cannot be experienced in real life. To Núñes, it is not difficult to understand why students have trouble understanding the epsilon-delta definition of limit.

> The reason is (cognitively) simple. Static epsilon-delta formalisms neither formalize nor generalize the rich human dynamic concepts underlying continuity and the "approaching" of locations.

Limits can never be reached

70% of the students in the investigation of Williams 1991 marked the following statement as true:

> A limit is a number or point the function gets close to but never reaches.

Williams 1991 is curious as to whether he can make the students change their minds by presenting them with a cognitive conflict. Therefore, he gives them a constant function and asks them to find the limit. After this intervention 8 out of 10 students now agreed that functions may reach their limits. The two last students were still not convinced: one made an exception for limits of constant functions and the other decided that constant functions weren't functions at all.

Again the language we use in England and Denmark around limits may be partially to blame. Words like "approaches" and "gets close to" in English or "nærmer sig" and "går mod" in Danish do carry a sense of never reaching the goal [Cornu 1991, Tall and Vinner 1981].

Limits can not be overstepped

33% of the students in the investigation of Williams 1991 marked the following statement as true:

> A limit is a number or point past which a function cannot go

Again part of the reason for this misconception probably lies in the language. According to Cornu 1991 and Monaghan 1991, the most common association connected to the word limit is an impassable limit, such as a speed limit.

It was interesting to see that even the students of R. B. Davis and Vinner 1986, who we know to have been exposed to sequences which oscillate around their limit were able to develop this misconception. To us this indicates the truth of the opening remarks of this chapter: students (and humans in general, we suspect) really are able to hold opposing ideas in mind at the same time. In any given situation, either a correct conception or an incorrect can be retrieved. Neither answer guarantees that the other conception does not exist as well.

5.4.2 *No concept definition*

In general, the research we have just described indicate that many students do not have a stable conception of limits which is consistent with the formal definition. Nagle 2013 argue that this may be because in many high schools, when limits are taught, only a very short time is spent on the definition after which focus shifts to calculating limits. Tall and Vinner 1981 gives an example of students who can calculate

$$\lim_{x \to 1} \left(\frac{x^3 - 1}{x - 1} \right)$$

but cannot define limits. This focus on calculation lead students to develop prototypical concept definitions based on what they have experienced in class, so for example, it is a widespread belief that sequences must be given by a single formula, and

$$s_n = \begin{cases} 0 & \text{, if n is odd} \\ \frac{1}{2n} & \text{, if n is even} \end{cases}$$

is thus not a sequence [Tall and Vinner 1981].

Another interesting explanation of the students willingness to accept contradictory ideas as true is found in Williams 1991. He found that “students tended to accept as true many different statements of limits”. To them, the truth of a statement depended on what function they were applied to and “mathematical truth, then, was truth for particular cases”. As one student put it

> And I thought about all the definitions that we dealt with, and I think they’re all right – they’re all correct in a way and they’re all incorrect in a way because they can only apply to a certain number of functions, while others apply to other functions, but it’s like talking about infinity or God, you know. Our mind is only so limited that you don’t know the real answer, but part of it

5.4.3 *Summary*

From our discussion above, we see that misconceptions of limits may be caused by a lack of understanding of mathematical definitions. Hence this topic is intimately connected to the previous section. Our research question is: Which of the following is most important in learning to understand proofs involving limits?

- Understanding the nature of mathematical definitions.
- Having a concept definition of limit, which is in line with the formal definition of limits and only using elements of the concept image in situations, where they provide intuitions which are in line with the formal definition.
- Not being influenced by any of the misconceptions: "functions never reach their limit", "limits cannot be overstepped" or "numbers move when taking limits" in cases where they are at odds with the formal definition.

We see that the three proposed prerequisites above are not independent. Rather they are hierarchically ordered with the most general competency at the top and competencies which apply to fewer situations below. Nevertheless, from this analysis we cannot conclude which level the teachers should focus on, if they want to prepare the students for learning to use the definition of limits in proofs.

We turn now to the final proposed prerequisite of our investigation.

5.5 INEQUALITIES INVOLVING ABSOLUTE VALUES

In chapter 3 we described a transition that many students have to overcome in the first year of university: From describing to defining, from arguing to proving and from smaller to larger praxeologies, where connections between the theoretical elements of the smaller praxeologies become visible. We argued that teaching pre-transition was different to teaching in An0, and that learning how to use the definition of limits in proofs was a post-transition task.

In section 5.1 and 5.3 we hypothesised that certain aspects of advanced mathematical thinking (ability to validate proofs and understand definitions) were important to have before attempting to learn to use the definition of limits in proofs in An0. In this section, we question this hypothesis. The teacher of An0 know that the course marks a transition to the students [personal communication], so we assume he has designed the course with this in mind. That is, advanced mathematical thinking is the aim of the course, not a prerequisite. If this is the case, then what may be important prerequisites to the course? I.e. which pre-transitional abilities are important to learning the formal definition of limits? The only item on our list of proposed mathematical prerequisites which we judged to be purely pre-transitional was inequalities involving absolute values. Hence we chose to include that one in our battery of proposed prerequisites.

We are interested in answering the question: What is more important in learning the formal definition of limits as it is taught at UCPH: to have learned certain aspects of advanced mathematical thinking before An0 starts or to have mastered the pre-transitional competency of being able to solve inequalities involving absolute values and understand what they represent. An answer to this question could have important implications for teaching at UCPH as it would allow teachers to know whether the first half year prior to An0 should be used perfecting elementary mathematical thinking or laying out stepping stones for advanced mathematical thinking.

Part III

METHOD

6

DESIGN OF THE STUDY

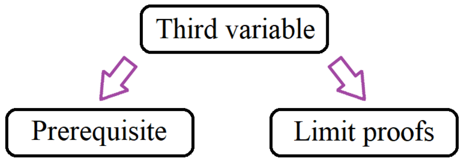
To answer the research question, we tested all the students of An0 twice during the course. In the beginning of the course, they were tested in all the proposed prerequisites, and in the end of the course they were tested in how well they understood proofs involving limits. From this experimental setup we hoped to learn which of the prerequisites (if any) were most important for student success.

6.1 CORRELATION DOES NOT IMPLY CAUSATION

From the setup above, we can learn about correlations between the prerequisites and the learning goal. However, we wanted to be able to advise teachers about which prerequisites are important to have *before* one attempts to learn to understand limits. For that, we needed to know if differences in prerequisites *causes* differences in learning outcomes.

However, there are at least two alternative explanations to a high observed correlation. One alternative hypothesis is that students who understand proofs involving limits at the end of the course also did so at the beginning of the course, and that this skill developed first. That is, they are good at the proposed prerequisites, *because* they practised them in the context of proofs involving limits

Another alternative hypothesis is that some underlying third variable is the cause of both how well the student performs in the test of the prerequisite and in the test of proofs involving limits.



From our study we will not be able to conclude anything rock solid about the direction of causation. However, we can take certain steps to minimise the possibility of the two alternative hypotheses above. The first alternative hypothesis, Prerequisite ⇦ Limit proofs can be eliminated by testing the students in their understanding of proofs involving limits in the beginning of the course, and excluding students from the study who already understand proofs involving limits, when they are tested in the prerequisites. Then we can use "the simple logic that the causes must be in place before the effects" [Gleitman et al. 2011]. The second alternative hypothesis cannot be ruled out completely as we can never be sure we have thought of all the possible third variables. We will only consider two possible third variables in this thesis. They are described in the following two subsections.

6.1.1 *Intelligence*

Performance in intelligence test and tests of mathematics have been shown to have correlation coefficients of 0.55-0.7 [Krutetskiĭ 1976]. Therefore, we expect that most measures of mathematical abilities will correlate significantly with each other for the simple reason that intelligent students do well in all of them. However, one has to be careful about which group is being investigated. Jensen 1998 shows that the correlation between intelligence and academic performance decreases as you move up through the educational ladder. In elementary school the correlations are often between 0.6 and 0.7, in high school the correlations are often 0.5-0.6, in college they are 0.4-0.5 and in graduate school we are down to something like 0.3-0.4. Jensen 1998 explains this by pointing to the decrease in variability of intelligence as you move up through the rungs of education. Each move from one educational level to the next involves a choice, and the group of students with low IQ is less likely to continue in the educational system. Hence the group who stays have lower variance in intelligence than the group of the previous educational step, so other factors than intelligence will begin to play a more prominent role, hence lowering the correlation between academic performance and intelligence.

As we are investigating students at university, we expect the correlation between general intelligence and academic performance to be in the low end of the correlations mentioned above, so we chose to also consider another background variable, which may play a bigger role in our experiment.

6.1.2 *General mathematical competency*

Among some of the students and lecturers at UCPH there exists a narrative about a general mathematical competency. They claim that some students are good at both algebra, analysis and the calculation heavy tests in MatIntro and that other students would probably be better of by studying a less mathematically challenging subject [personal communication]. However, we have not been able to find any theory concerning a general mathematical competency in the literature. In fact Krutetskiĭ 1976 review the literature with the aim of answering

1. "The question of the specificity of mathematical abilities. Do mathematical abilities proper exist as a specific formation, distinct from the category of general intelligence?"
2. "The question of the structuredness of mathematical abilities. Is mathematical giftedness a unitary (single, undecomposable) or an integral (complex) property?"

They reach no final conclusion on the questions, but do cite some interesting results along the way. For example, they cite Oldham 1937 who studied 410 schoolchildren between the ages of 9 and 15 through tests of intelligence, arithmetic, algebra, and geometry. The author found the following correlations: "0.60 between algebra and arithmetic, 0.47 between arithmetic and geometry, 0.59 between algebra and geometry. The correlations of the tests for arithmetic, algebra, and geometry with the intelligence tests were significantly lower (0.40, 0.27, and 0.31, respectively)." [Krutetskiĭ 1976].

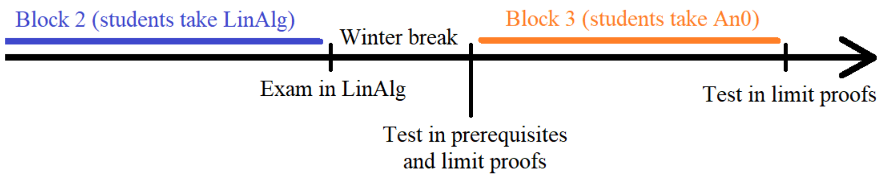
Our takeaway from Krutetskiĭ 1976 is that the term "general mathematical competency" has yet to be defined, and perhaps does not even make sense. Nevertheless there seems to be some connection between different skills within mathematics which are stronger than what they would be if based on intelligence alone.

Hence we chose to include an extra mathematical variable in our experiment rather than a measure of general intelligence. The mathematical variable that we chose was the grade of the student in the course Linear Algebra (LinAlg). For the purpose of this thesis, it is not necessary to know all the different aspects of LinAlg. We are simply interested in the factors which systematically influence LinAlg *and* one or more of our variables. Factors which systematically influence LinAlg, but none of our other variables will just be noise in our experiment. We chose to use LinAlg as background variable because, to the best of our judgement, it is not necessary to understand definitions or validate proofs to pass the exam in LinAlg and there are no inequalities with absolute values or limits in LinAlg either. Hence we assume that any overlap in the grade of LinAlg and one of the above prerequisites is due to a more general shared competency in mathematics and not the specific prerequisite. The only prerequisite, which has a strong connection to LinAlg is orientation, as all the questions are about orientation to studying in LinAlg. Hence for that variable we have to be careful about which conclusions we draw from models in which both LinAlg and orientation are included as predicting variables.

7

HOW THE COMPETENCIES WERE MEASURED

In the previous chapter, we presented the general design of our investigation. As a kind of summary of the previous chapter, we now present a time line of the tests and exams, that the students were asked to complete:



In this chapter, we will go into detail with each of the tests: how they were constructed and what they were intended to measure. We have already talked about LinAlg in the previous chapter, so this chapter will focus on the tests of the five proposed prerequisites:

- Proof validation
- Active meaning orientation
- Use of definitions
- Misconceptions of limits
- Inequalities involving absolute values

and the tests of understanding of proofs involving limits.

7.1 PROOF VALIDATION

As we constructed the test in proof validation ourselves, we want to analyse the questions to understand exactly what is measured by the test. In this section we therefore conduct an a priori description of the proof validation test. The test can be found in the appendix.

7.1.1 *Which types of mistakes do we anticipate?*

Before looking at the questions individually, we want to examine known reasons why students make mistakes in proof validation tasks. Weber (2009, 2010) discuss an experiment in which twenty-eight mathematics majors, who had recently completed a transition-to-proof course, were asked to validate proofs. He found several different types of mistakes. We will now present each of these, and consider how they are likely to influence our experiment.

Required knowledge

Is defined to be mistakes which happened because the student lacked the relevant mathematical knowledge or is unable to bring it to mind in the situation.

All proofs in our quiz are within the topic of elementary calculus, which all the students were taught at least half a year prior to taking the test. Hence all students should have the necessary prerequisites. Nevertheless, not all students are equally good at elementary calculus, so this test will inevitable measure both the student's ability to validate proofs and their familiarity with the topic.

Performance error

Is defined to be errors caused by the student overlooking that a logical rule of reasoning is either false or applied incorrectly.

As P. J. Davis 1972 points out, even trained mathematicians make performance errors, so it should come as no surprise that students do as well.

Adding to this general difficulty, the students in our experiment were also under time pressure. The students in the pilot study were highly sensitive to this pressure, and many of them rushed through the questions in an attempt to go for quantity over quality. To overcome this problem, we told the students, that they would be awarded negative points for wrong answers. In the pilot phase of the study, this had a positive effect on how long time the students spent with each question. However, we still found that many students made mistakes, which they could easily correct after the test, when they were less stressed.

We expect performance errors (both those due to time pressure and those which would also have been made in more relaxed conditions) to occur frequently in this experiment. We expect performance errors to cause students to accept invalid proofs.

Local validation failure

Is a specific type of performance error, in which the student does not check a false warrant whose premise and conclusion are both true. See section 5.1 for further explanation of what we mean by local proof validation.

This type of mistake can help explain why students accept invalid proofs, though only one of our proofs requires warrant checking, so the effect of this mistake should only be visible for that question.

Global validation failure

Is a mistake where the student is able to understand all the steps in the proof, but is still not able to understand the proof, and therefore has to guess whether it is correct or not. See section 5.1 for further explanation of what we mean by global proof validation.

This type of mistakes can explain both why some students reject valid proofs.

Perception of what it means to validate a proof

Weber 2009 showed that some students seemed to have a majority approach to proof validation: if most of the proof was correct, they judge it to be a valid proof even if they find a mistake in the proof. This was for example the case in the proof we have used in question 5. In Weber's study, the proof was longer, and only the last line was incorrect. Some students seemed to ignore the mistake in the last line of the proof as everything else was correct. In total, Weber reports that 10 out of the 28 participants judged one of the arguments to be a proof after they located a significant flaw in the argument.

Also Weber showed that 12 out of 28 of his participants judged at least one argument they did did not fully understand to be a proof. He contrasts this with his observations of mathematicians in another study, who only accept proofs they fully understand and who discard a proof as soon as they find one mistake.

This can help explain why some of the students may accept invalid proofs.

Requirements for valid proofs

As we discussed in section 5.1, we are only interested in whether the students can follow the logical reasoning in the proof. We are not interested in whether the judge the presentation to be suitable, the level of detail to be sufficient or the premise to be acceptable in this setting. In order to indicate this to the students, the quiz is prefaced with an example question and explanation, so they know which types of judgements they will be asked to make. Furthermore each question is presented in the same style, and the possible answer options are designed to avoid students rejecting proofs for reasons connected to the style of the proof or the amount of detail given.

Hence, we expect this sort of mistake to play a very small role in the experiment. If it plays a role, we expect it to cause students to reject valid proofs.

The student is guessing

Unfortunately, since our test is multiple choice, we cannot rule out the possibility that the students are simply guessing, which could cause them to reject some valid proofs and accept some invalid proofs.

7.1.2 *A priori analysis of the test items*

We now turn our attention to the specific proofs, in order to analyse which types of mistakes the students may make and why.

Question 1

In this question, the students are given two proofs which attempt to show that certain second degree polynomials are positive for all $x \in \mathbb{R}$, and asked to judge which of them is valid. The question is a modification of a proof in Weber 2010 and is meant to test whether the students can infer and check warrants. The first proof relies on the implicit warrant "the sum of positive numbers is positive", the second on the warrant "the difference of positive numbers is positive". The first is valid, the second is not.

However, warrant checking is not necessary in this case, because the conclusion in the second proof is false (the polynomial is negative for $x = -3$), so the students only need to test different values of x to validate this proof. However, the polynomial is positive for all non-negative x, and the first polynomial $f(x) = x^2 - 4x + 7$ has a minus in its definition, whereas the second polynomial $g(x) = x^2 + 6x + 5$ is defined using only plusses between terms. Hence students who are inclined to check only positive x's may incorrectly answer that the second polynomial is positive for all x and the first is not.

The topic specific knowledge required is

- an understanding of what it means for a function to be positive

- the identity: $(a-b)^2 = a^2 + b^2 - 2ab$
- the identity: $(a+b)^2 = a^2 + b^2 + 2ab$

In Weber's study, where the students were not under time pressure, they all said that they understood the proof fully. Hence, the most likely sources of mistakes are time pressure and a bias towards positive x's, not the lack of required knowledge.

Question 2

This question is created by the author and is inspired by a mistake which some of the students made in their first course at university. In the proof, the Intermediate Value Theorem is mistakenly applied to show that $x \mapsto x^2$ is positive for all $x \in [1, 10]$.

The Intermediate Value Theorem is a statement on the form $A \Rightarrow B$, so in a proof one can deduce B if one has already proven A. In question 2, however, the theorem is used to deduce $\neg B$ from $\neg A$, which is equivalent to the opposite implication $B \Rightarrow A$. Hence the proof is invalid.

The question tests whether the students notice that this theorem has a direction (from the premise you can deduce the conclusion, but not opposite).

The topic specific knowledge required is

- previous experience with the intermediate value theorem may not be required, as the theorem is stated in the question, but it is certainly helpful. The students are pressed for time, so anyone who has to spend time during the test to understand the theorem will be disadvantaged.
- a non-zero continuous function, which is positive for one value of x, will be positive for all values of x.

As the prerequisites for the question are either high school material or explicitly given in the question, we expect that the most likely mistake will be that the students deem the proof valid because they do not notice the direction of the intermediate value theorem.

Question 3

This question is created by the author of the thesis. It gives a proof and two theorems. Written with mathematical symbols, the theorems are as follows

$$\text{Theorem 1} : \forall x \in \mathbb{R}^+ \ \exists n \in \mathbb{N} : \frac{1}{n} < x$$

$$\text{Theorem 2} : \exists n \in \mathbb{N} \ \forall x \in \mathbb{R}^+ : \frac{1}{n} < x$$

The first of which is true and the second of which is false. In the test, the quantifiers are written out in words ("for all ... there exists..."). The students are asked to tell which theorem is proven.

The question tests whether the students notice that the order of quantifiers is important, and in particular how it affects a proof. It is significant that x is introduced before n in the proof, and the formulation "let x be" is significant too, as it is a standard way to prove something for all x.

The topic specific knowledge connected to the question is

- the natural numbers are not upwardly bounded
- when solving an inequality involving variables in $\mathbb{R}^+$, you can multiply or divide both sides by the variable without changing solutions to the inequality.

In this question there are 3 wrong answers: "Theorem 2", "None of the theorems" and "Both of the theorems". From the pilot phase of the study, we expect that most of the students, who answer incorrectly, will pick "both theorems", because they cannot see the difference between the order of the quantifiers. However, some may also pick "none of the theorems", because they cannot perform global proof validation. And some may pick "Theorem 2", since it is formulated to sound more plausible. $\forall x$ is stronger than $\exists x$, so formulations involving the former sound less likely to be true. Both theorems have one $\exists$ and one $\forall$, but more words are devoted to explain the $\forall$ in the first theorem than in the second.

Question 4

The proof is borrowed from Weber 2015. It is a straight forward application of the intermediate value theorem to show that there exists an $x \in \mathbb{R}$ such that $x^4 = x + 1$. In this thesis, the proof is meant to test how confident the students are in applying a theorem.

The required topic specific knowledge and abilities are

- previous exposure to the intermediate value theorem is again an advantage.
- remembering to also consider non-integers as examples. When asked to think about a random number, most people probably think about a whole number. But the theorem is false, if $x \in \mathbb{Z}$, so the students need to overcome this bias.

In this question, there are two wrong answers. We expect that the students who choose "the theorem is false" have been affected by the whole number bias. The students who choose "the proof is invalid" may be unable to perform global proof justification under time pressure.

Question 5

This proof is a modified version of the proof from Alcock and Weber 2005 and Weber 2010. It attempts to prove that $a_n = \sqrt{n}$ diverges to infinity, but only argues that a_n is increasing.

This question tests whether the students can infer and check warrants. All statements in the proof are true. The only issue is that the last statement does not follow from the rest of the proof.

The required topic specific knowledge is

- the meaning of the symbols $\Rightarrow$, $\sqrt{}$ and $<$
- the definition of what it means for a sequence to diverge to infinity
- $x \mapsto \sqrt{x}$ is an increasing function which tends to infinity for $x \to \infty$.
- not all increasing sequences diverge to infinity.

Alcock and Weber 2005 studied first year student responses to this proof extensively. They asked 13 first year students if the proof was correct. They found that

1. Three students inferred and falsified an implicit warrant having the last statement of the proof as its conclusion.
2. Three students distrusted the proof because it did not use the definition of diverging to infinity (for all $M > 0$ exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$).
3. Seven students accepted the proof initially. When their attention was directed to the missing warrant, five of them changed opinion and rejected the proof. The two others persisted in their opinion.

As our students have not yet been taught the formal definition of diverging to infinity, we expect that none of them will reject the proof for reason number 2 above. The five people in group number 3 who change opinion after having their attention directed towards the missing warrant seem to have made a performance error. We expect this to be common among our students too. The last two students in the study above seem to not find any flaws in the proof. Perhaps this is because in this situation, they are led to believe that all increasing function diverge to infinity. Weber 2010 found that 6 out of the 28 participants in his study reported this belief in connection to a similar proof, so we should note this as a likely reason for mistakes among our students, too.

Question 6

The proof is a slight modification of a proof in Weber 2015 and Weber 2010. The proof is a valid application of differentiability to find the minimal value of a polynomial and see that it is bigger than -30. An important move in the proof is to realise it is enough to investigate the stationary points.

The necessary topic specific knowledge is

- that even degree polynomials with a positive highest order coefficient tend to infinity for $x \to \pm\infty$.
- how to use differentiation to find maxima and minima

After having done the experiment, we realised that the theorem is not necessarily true as it stands. The students have recently learned about complex numbers, so when we compute the students' marks in the test, we have to award points both to students who accept the proof and to students, who reject the theorem because all non-constant complex polynomials have roots.

In Weber 2010 only 14 out of 28 students judged that that they fully understood the proof, which was by far the lowest for all the proofs (the second to least understood proof was fully understood by 20 out of 28 students). Weber 2010 does not go into detail about this, but he does give one example of a student who does not know the relevant fact about fourth degree polynomials and another student who understands all steps in the proof, but cannot wrap his head around the proof in its entirety. Despite this, 23 out of 28 said that the proof was valid in Weber's study.

We expect that most students will answer this question correctly. Our best guess at why some students would judge the proof to be invalid is that they do not know the fact about polynomials of even degree or that they fail in global proof validation.

Question 7

This proof is borrowed from Weber 2010. It proves a statement about integrals by analysing positive and negative functions separately. Logically, it is a proof by cases, which fails because the cases are not exhaustive: there exists functions which change sign within the interval.

The necessary topic specific knowledge is

- some functions change sign.
- the integral of a positive/negative function is positive/negative.

Weber 2010 found that 22 out of the 28 students in his study said they understood the proof fully. Despite this and the mistake in the proof discussed above, 15 out of the 28 believed that it was a valid proof. Only 9 students discarded the proof as invalid.

Based on our pilot study, we expect our students to perform similarly. We expect that a common error will be that the students do not notice that the cases are not exhaustive, but we also expect some students to answer incorrectly because they are not familiar enough with integrals. One student said that she only thought of integrals as the opposite of differentiation, and had forgotten that they were related to areas, so she did not believe that the integral of a negative function was necessarily negative.

7.1.3 *Summary*

To sum up, we find the following possible reasons for mistakes in our test

- Lack of required knowledge
- Performance error (causing students to accept invalid proofs)
- Failure in global proof validation (causing students to reject valid proofs)
- Mistaken idea of what it means to validate a proof (causing students to accept invalid proofs)
- Guesses

The required knowledge of the test is either high school material, or material from elementary calculus, but still we have to expect that, in the context of this test, some students will have trouble recalling knowledge such as

- Not all increasing functions diverge to infinity
- Polynomials of even degree with a positive highest order coefficient tend to infinity for $x \to \pm\infty$
- The integral of a positive/negative function is positive/negative

The likely performance errors, that the students may make during the test are to miss

- That they have a positive and whole number bias causing them to overlook other numbers (question 1 and 4)

- The direction of theorems (from A you can deduce B, but not opposite) (question 2)
- The importance of the order of quantifiers (question 3)
- False warrants when both premise and conclusion are true (question 5)
- To check if the cases are exhaustive in a proof by cases (question 7)

In the pilot phase of the study, the students who made these mistakes, corrected their reply once the problem was pointed out to them. Hence they were genuine performance errors.

7.2 STUDY ORIENTATION

In this section we present the questions in the questionnaire on study orientation, and connect them to the literature in section 5.2.

The questionnaire is designed with the aim of measuring active meaning orientation to learning mathematics in all the settings it may show up:

- During lectures
- When reading the book
- When solving exercises
- In exercise classes
- Outside the classes

In order to create questions which the students could meaningfully answer, we chose to ask questions about specific behaviour. The problem with this approach is that not every student with an active meaning orientation will exhibit the exact same behaviours. It may be that the next Ramanujan is out there and studying in a manner, which will lead him to great success, but may not earn him many points in the quiz, as his methods were not included in our rather narrow list of reflected active behaviour.

7.2.1 *Question 1: Behaviour during lectures*

The students were asked how often they did the following things during lectures in LinAlg:

- Write notes

- Attempt to fill small gaps in the argument, which the lecturer did not expand on.
- Attempt to predict what the lecturer would say next (for example how a definition should be, what the premise of a theorem must be or how a proof must progress).
- Independently check the theory on an example or find a counter example.
- Notice things they did not understand/would like to know more about and attempt to find an answer (either alone, with a friend, on the internet or by asking a teacher)

For each question the student was asked to choose between never, rarely, at about half the lectures, often and always. The question on note taking will not be included in the scoring as it does not necessarily indicate a meaning orientation. It was included in the questionnaire to lower the chance that some students incorrectly would answer that they did one of the other things often, because they did not like to answer they did nothing at lectures.

The questions are created by the author of this thesis, but they are all backed up by either Alcock 2012 or Anthony 1996.

7.2.2 *Question 2: Behaviour while reading textbooks*

The students were asked how often they did the following things while reading the textbook in LinAlg:

- Attempt to fill small gaps in the argument, which the book did not expand on.
- Look ahead in the book to see where the author was going or back to check up on something which was written earlier.
- Check the theory on an example or find a counter example.
- Draw a picture/diagram
- Notice things they did not understand/would like to know more about and attempt to find an answer (either alone, with a friend, on the internet or by asking a teacher)
- Attempt to explain a main message to themselves without looking in the book.

For each question the student was asked to choose between never, rarely, sometimes (about 50%), often and always.

The research of Weber and Mejia-Ramos 2014 is the inspiration for question 1 and 4. We tested Weber's question in the pilot study, but decided to change the wording of it as too many of the successful students picked the option, which indicated that they did not believe it was their job to fill in arguments when reading a proof (just like many of the mathematicians in Weber and Mejia-Ramos 2014's study did). The other questions on the list are created by the author, but most are backed up by Alcock 2012.

7.2.3 *Question 3: Why the methods work*

In question 3 the students were asked whether they were interested in learning why the methods of LinAlg worked, even though it was not important to their exam.

This questions is a reformulation of a question asked by Sierpinska 2007, and it is backed up by Anthony 1996.

7.2.4 *Question 4: The ideal assignment*

In this question, the students were asked how they thought an ideal assignment should be. They were asked how big a percentage of the assignment should be exercises of each of the following types:

- Exercises, they had seen presented on the black board, and where they now had to apply the same method to another example
- Exercises, where they had been presented to all the methods and ideas in class, but where they had to combine the elements in a new way.
- Exercises, where they had to contribute with something. For example by reading in the book or getting a good idea.

This question is created by the author. The aim is to capture whether students want to think for themselves when solving an exercise or whether they would rather just be asked to reproduce what the teacher has shown. It is related to advice given by Alcock 2012 and some of the remarks of Anthony 1996.

7.2.5 *Question 5: Is feedback always necessary*

In two of the previous courses the students had taken, they did not get feedback on all exercises they handed in. Only one out of two or three exercises in each assignment was corrected. In this question, the students were asked if this was a problem for them personally.

The question is meant to test if the students trust themselves in establishing mathematical truth or whether they rely on the teacher to do so. It is inspired by the main question in Sierpinska 2007. We chose to change the questions, as the original question "I need the teacher to tell me if I am right or wrong" was understood in very different ways by the students in our pilot study.

7.2.6 *Question 6: Others*

In this questions, the students were asked both how often they did the following things and how often they thought they ought to do the following things:

- Read about topics within their subject which are not part of the curriculum
- Make precise what they do not understand, so they can ask good questions.
- Contribute to the conversation in class, so it is not just one long monologue from the teacher.

The questions are made by the author of this thesis. The first question is most strongly connected to intrinsic motivation, the second to metacognitive monitoring of own learning and the third to activity (not necessarily meaning oriented). Especially the second question is deemed to be very important by both Anthony 1996 and Alcock 2012.

7.2.7 *What does this questionnaire measure?*

From our analysis, it should be clear that our questions are theoretically scattered. They are all related to active meaning orientation or metacognitive monitoring own learning, but the connection is not strong, as the themes have been reinterpreted to concrete behaviour exhibited by mathematics students in the first year of university at UCPH. The advantage of this is that if we find that some specific behaviour is very important to student success, it will be easy to communicate exactly what students should do. The disadvantage is that we will have trouble interpreting the scores of the students theoretically.

We hence see this part of our study as a pilot study. We are throwing out a wide net of questions related to active meaning orientation and metacognitive monitoring of own learning in mathematics, and we are hoping that the results will point us in a more specific direction.

7.3 USE OF DEFINITIONS

We constructed a test with the aim to measure to what extent the students understand the special nature of definitions in mathematics. We were interested in whether they had a concept definition which included boundary cases, or whether their concept definition only allowed for examples which were similar to the students prototypical example, and we were interested in whether they relied on their concept definition, or whether they could be brought to rely exclusively on a faulty concept image.

In order to test this, we asked 5 different questions, which were of three different types

1. Two questions asked students to decide if certain examples were covered by a definition, we expected them to know well (the definitions of function and tangent).
2. One question asked the students to recall a definition (of determinant) from a previous course, which they had all taken.
3. Two questions asked the students to work with definitions they had never seen before, but where the name of the concept being defined sounded like something they knew (unbounded and even).

The first two questions are taken from Vinner 2002, and the last three questions are created by the author of this thesis. The questions can all be found in the appendix.

We chose to ask three different kinds of questions, as they all had strengths and weaknesses. The first two questions are meant to test if the students can apply well known definitions to boundary cases, or whether they are influenced by their prototypical examples or faulty concept images. The questions were used in Vinner 2002. The weakness of the questions lies in the fact that our students come from many different high schools, so they have had many different teachers. One teacher may have gone through great effort to clear out possible misunderstandings of the particular definitions or he may have presented the students to the particular examples we use, while other teachers may not have done so. As we are interested in a general competency in understanding the nature of mathematical definitions, and not the familiarity of the student with a particular definition, these issues add noise to our data.

The second type of question has the advantage that we know all students received the same instruction, so it actually measures differences between the students and not between their teachers. We assume that students who understood the purpose of mathematical definitions were more likely to notice when a definition was presented and to remember it afterwards, as the definition is meaningful to them. Students who are used to rely exclusively on their concept image, or who create their definitions according to the prototypical theory of word meaning will not find the definition as meaningful, so they are more likely to forget it. The question is multiple choice, where some of the wrong choice could have been a definition of determinant, whereas others could not. Hence this question also tests if students are aware of the role of definitions in mathematics in bringing concepts to life, or whether they may have a similar misconception as Andre in Edwards and Ward 2004, who believed that a definition in mathematics is simply something which is true, so there is no difference between definitions and theorems.

The weakness of this question is that it relies on the memory of the students. Previous research has indicated that students who look for meaning in the material are more likely to remember facts, than students who do not [Entwistle and Ramsden 1982], so the assumption about the connection between students who find the definition meaningful and students who can remember it is not unjustified. But it is still a measurement by proxy, so it is bound to introduce some noise.

The last two questions are not influenced by the specific knowledge of a student or the student's memory. Mistakes in these questions indicate either that students find it difficult to extract meaning from formal mathematical definitions (perhaps because they are used to form concept definitions by way of examples and prototypes) or that the students are able to form a concept definition, but the wordings of the questions induce them to use a concept image which is based on previous use of the same words instead. The weakness of the last two questions is that students who are used to working with $\forall$ and $\exists$-quantors and elementary number theory have an advantage. Some of the students have taken a course in which these concepts were introduced and others have not. Hence these questions measure not only the ability of the students to work with definitions, but also their familiarity with specific mathematical concepts. Unfortunately, we were not able to come up with quiz items which relied only on knowledge we could expect all the students to have.

7.4 MISCONCEPTIONS

In this quiz, we tested if the students had the following misconceptions:

1. A dynamic notion of limit
2. Limits can never be reached
3. Limits cannot be overstepped

For this, we used a revised version of the test in Williams 1991. For each of the following statements:

1. A limit is the number that $f(x)$ moves towards when x moves towards a certain number
2. A limit is a number, which $f(x)$ cannot move past
3. A limit is a number which $f(x)$ is arbitrarily close to, if x is sufficiently close to a certain number
4. A limit is a number $f(x)$ gets close to, but never reaches.

the students were asked to indicate if they thought the statement was true or false. Afterwards the students were asked which statement was closest in meaning to their interpretation of limits.

The changes between our version and the one in Williams 1991 are in part due to translation. Our test was in Danish, and often a direct translation to Danish was not appropriate. Also we had to revise statement 3 several times, because we found that some older and successful students did not think it was close enough to the formal definition, so they marked it as false.

The students were also presented with the functions in table 1 and 2. For some of the functions they were just given a picture and asked to make their best judgement based on the picture. For some functions they were given both a picture and an algebraic definition of the function. For all but one function, the students were asked what they would say about the limit of the function as $x \to 0$:

- The limit in zero does not exist/the concept of limit does not make sense in this setting.
- The limit in zero exists
- I do not know

For the last function, the students were instead asked whether they would say that the function had limit 1 as $x \to \infty$. They were given the options

- No
- I cannot say anything for sure based on the behaviour of the function a bounded interval, but it looks like the function has limit 1
- I do not know

We chose to change this question as some older students had been indoctrinated thoroughly never to conclude anything based on the graph of the function on a bounded interval. Even though the original question asked them specifically just to make their best judgement based on the picture, they refused to commit to anything but "I do not know".

In table 1 and 2 we have listed all the functions, whether a definition was given for that function and which misconceptions a wrong answer might indicate[1].

These questions are meant to serve two purposes. First, they can be used to determine to what extend the student has a concept definition of limits which is in line with the formal definition and only use elements of their concept image when they align with the formal definition. Second, some of them can shed further light on which particular misconceptions the student may have. As we noted in the theory chapter, the student's misconception may not be visible in all questions which are designed to detect it, as some times the student may retrieve other conceptions. Thus we can see this test as providing five measures:

1. How likely is the student in retrieving conceptions which are in line with the the formal definition of limits?
2. Does the student have a dynamic conception of limit, and if yes, then how likely is it that the student retrieves it in cases where it is unsuitable?
3. Does the student have the misconception that limits can never be reached, and if yes, then how likely is it that the student retrieves it in cases where it is unsuitable?
4. Does the student have the misconception that limits act as bounds, and if yes, then how likely is it that the student retrieves it in cases where it is unsuitable?

How the measures were constructed from the replies will be discussed in the results section.

1 These tables each take up a whole page, so it is likely that Latex has placed them at the end of this chapter

7.5 INEQUALITIES INVOLVING ABSOLUTE VALUES

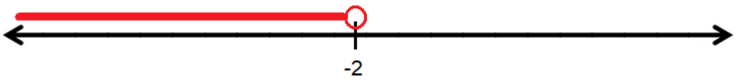
In the test of inequalities involving absolute values, the students were asked questions like the following:

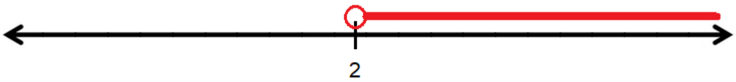
Four students have attempted to solve the inequality

$$|x| + 3 > 5$$

They all drew their answers on a number line.

Which solution is correct?

○ 

○ 

○ 

○ 

(The set of solutions is empty)

○ None of the students have solved the equation correctly

○ I don't know

The questions all require that students solve in inequality and connect the answer to an image of a subset on the number line. We included questions like the above, where none of the options are correct, to avoid students taking advantage of the multiple choice format in different ways to avoid doing the calculations.

Many of the inequalities, we asked the students to solve were of the form

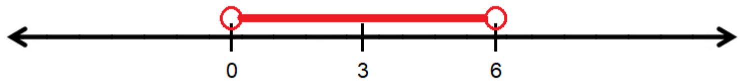
$$|x - a| < b$$

like the following:

Your teacher has told you, that the set of solutions to

$$|x - 3| < 3$$

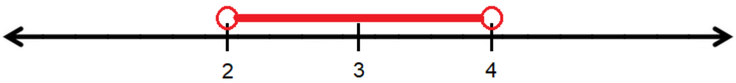
is

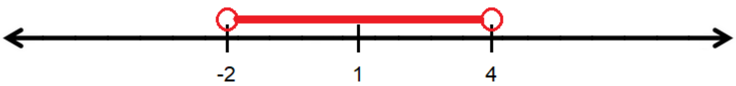


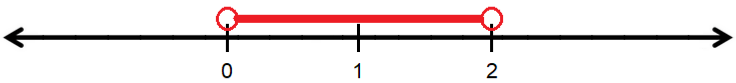
Three of your fellow students have attempted to solve the equation

$$|x - 3| < 1$$

Their replies are below. Who has solved the equation correctly?

○ 

○ 

○ 

○ None of the students have solved the equation correctly

○ I don't know

We included many of these questions, as they are directly related to many proofs involving the formal definition of limit.

As with the other tests, the students were given ten minutes to solve as many inequalities as they could.

7.6 UNDERSTANDING OF PROOFS INVOLVING LIMITS

This was tested twice: in the beginning and end of An0. In the beginning of the course, the course responsible was able to set aside a whole hour for our test of the students. Out of this hour, we used ten minutes on testing how well the students understood proofs involving limits (the remaining time was used on testing the prerequisites). Our test on understanding of proofs involving limits is a direct application of the guide in Mejia-Ramos, Fuller, et al. 2012, and is included in the appendix, so we will not discuss it further.

However, at the end of the course, the course responsible was unfortunately not able to set aside any time for testing, so we have to rely on the students grade in An0 as an indirect measure of their understanding of proofs involving limits. In the following subsection we analyse what this grade measures and whether it is an acceptable proxy for the ability we are interested in.

7.6.1 *An0 as a measure for understanding of proofs involving limits*

The grade in An0 is based 100% on an oral examination at the end of the course. At the exam, the students pick one of the following headlines at random (our translations)

- Main theorems about continuous functions
- Taylors formula in one variable
- Riemann integral
- The problem of antiderivatives in one variable
- Differentiability in $\mathbb{R}^k$
- The generalised chain rule
- Independence of order of differentiation
- Taylors formula in $n \geq 2$ dimensions
- Max and min for real functions of several variables
- Length of curves
- Integral over curves
- The problem of antiderivatives in $\mathbb{R}^k$
- Green's theorem

Then they have 30 minutes to prepare a presentation, after which they have 20 minutes to present without referring to their notes. Their presentation must contain all relevant definitions and at least one theorem and proof from the book. During the presentation, the examiners ask questions to clarify details.

All the major definitions in the course either directly build on the definition of limit:

Definition 5. A function $f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^k$ is said to be an *epsilon-function* in the limit $x \to a$, $x \in A$ if

$$f(x) \to 0 \text{ for } x \to a, x \in A$$

or they have the same structure as the definition of limit: For all epsilon there must exist a delta, such that if some things are closer to each other than delta then other things are closer to each other than epsilon:

Definition 6. A function $f : [a, b] \to \mathbb{R}^m$ is said to be an *Riemann integrable* if there exists a number $I \in \mathbb{R}$ with the following property: For all $\varepsilon > 0$ there exists a $\delta > 0$ such that any partition D of $[a, b]$ with $\text{norm}(D) < \delta$ and any choice of Riemann sum $M = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ with respect to that partition satisfies that

$$\left| I - \sum_{i=1}^{n} f(\xi_i)\Delta x_i \right| < \varepsilon$$

We will call this structure for epsilon-delta structure. Some of the proofs in An0 require that students use a definition with epsilon-delta structure directly in the proof and some do not. The proofs, which do not, instead relies on other theorems which were proven using the definition. For example, the proof of Taylor's formula relies on differentiating a polynomial and the Mean Value Theorem. The students do not need to understand the epsilon-delta structure of the underlying definitions to understand this proof.

Colloquially, we will refer to proofs in An0 which unpack the epsilon-delta structure of the definitions as epsilon-delta proofs. By going through the exam questions, we see that only one topic (Green's theorem) does not contain an epsilon-delta proof. We also judge that for some other topics, it is possible to create a good presentation without epsilon-delta proofs (for example Taylors formula in one variable). But for the absolute majority of the topics, the obvious choice is to include at least one epsilon-delta proof in the presentation.

Hence we judge that the majority of students will be assessed in understanding of an epsilon-delta proof. We will assume that students who are able to understand an epsilon-delta proof, would also be able to understand a proof which only relies on the definition of limits. Hence we will see the proving part of the exam in An0 as an indirect measure of understanding how to apply the definition of limits in proofs for almost all of the students (about 90%). The remaining 10% of the students who are not tested specifically in epsilon-delta proofs will have to be regarded as adding noise to our experiment.

Other aspects of the An0 exam

The exam in An0 is not just a presentation of a proof. The students are also required to present definitions and show understanding of the concepts.

Also we have seen in section 5.2 that success in examinations is linked to strategic study orientation, so students who are aware of the requirements of the course responsibles, and change their study approach to match this will do better. At this particular exam, the strategic students may take various actions, which will lead to a higher grade, but will not show increased understanding. For example, they may get the outlines of presentations from successful older students, or they may learn the definitions or parts of the proofs by heart.

Hence we see that the grade in An0 is only an indirect measure of the ability we are interested in. First because the proofs in An0 may not be based on the definition of limit but rather on other definitions of epsilon-delta structure. Second because the examination also tests understanding the definitions and coherent presentation of theory, and third because strategic study orientation is likely to have an effect on the mark in An0.

We will have to take this into account when we make judgements based on our results.

This is only mentioning the aspects of An0 which may have a systematic effect on our results. Previous research has shown that marks in oral exams are not very reliable, as they depend on the examiner, and also may be influenced by other factors as gender, style of clothes, ethnicity and accent [M. H. Davis and Karunathilake 2005]. However, we cannot take this into account, but will have to accept it as noise in our experiment.

7.7 SCORING OF THE TESTS

Most of the tests, we constructed, were multiple choice tests and had a time limit of 10 minutes. The students were instructed that they were not expected to finish all tasks and that wrong answers would be penalised, so they should only guess if they were more than 50% certain. The questions differed in how many wrong options they had. When possible, we created four to six wrong options, but sometimes we were only able to come up with one or two. In the final scoring, we decided to deduct points for wrong answers in such a way that

A student who guessed randomly would score zero points on average.

So for example if a question had one correct answer and three wrong answers, the correct answer would be rewarded with one point and a wrong answer would give the student -1/3 point. If the student answered this question with a complete guess, their expected score would be

$$\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \left(-\frac{1}{3}\right) = 0$$

More generally, if a question had n wrong answers, the correct answer would still be awarded with one point but every wrong answer would give $-1/n$ point. If students skipped a question or answered "I don't know", the student would score zero points for that question. Unless otherwise noted, every question in a test weighs equally towards the total score for that test.

7.7.1 *Exceptions to the rule*

In some questions we chose to award more negative points for some wrong answers than others, as we interpreted some of the wrong answers to indicate less understanding than others. For example, in the last question of the test of definitions, the students had to read the definition

Definition 7. $n \in \mathbb{N}$ is called *super-even* if an even number of digits in n are even.

And then they had to say which number was the smallest super even number. The correct answer is 1, as it is the smallest natural number, and it is also super-even (it has zero even digits, and zero is even). Many students answered 20, which is the smallest super-even number with more than zero even digits. We decided this was a markedly better answer than 0, 2, 4 or 10, which are not super-even. Hence we chose to award the answer 20 with zero points and all other wrong answers with $-1/5$ points.

7.7.2 *Scoring in test of orientation*

This test did not have wrong answers, but instead it had ordinal answers such as "always", "often", "rarely" and so on. All questions were marked on a scale from zero to one, where zero indicated no reflected activity and one was the maximal possible in this question. The options between the two ends were assumed to have equal distance between them. So if there were 5 options, they would be graded by 0, 0.25, 0.5, 0.75 and 1. Theoretically, trying to assign numbers to answer such as "often" and "rarely" makes little sense, but we had to get some way of obtaining one score from the replies of the students.

In one question, the students had to prioritise between three types of exercises, which showed increasing levels of reflections, by assigning each of them a percentage. For this question the percentages that the students entered for the two higher categories of exercises entered directly into the score, weighed so that they in total counted for about one and a half question.

As there were disproportionately many questions about reading the book compared to the other activities involved in being a student, we chose add an additional weight to the questions, so that activity during lectures accounted for 25% of the score, reading the book accounted for another 25% and the rest of the questions made up the remaining 50%.

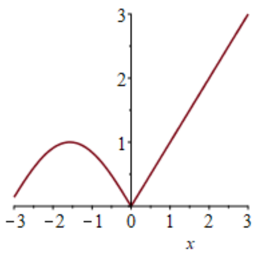
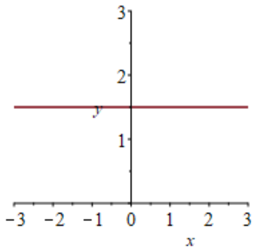
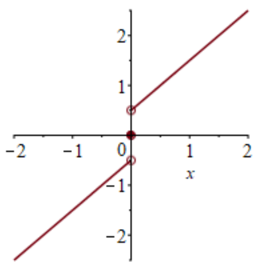
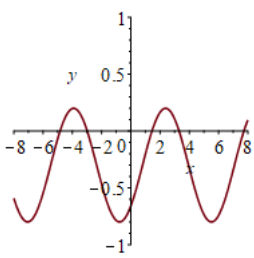
Function	**Is a definition given? Which misconception is tested for?**
	No definition given The question does not test for a specific misconception
	No definition given Misconceptions tested for • Dynamic conception • the limit is never reached
	No definition given Misconceptions tested for • Dynamic conception
	No definition given Misconceptions tested for • Limits act as bounds

Table 1: Four first functions used in the test of misconceptions and which misconception is tested for

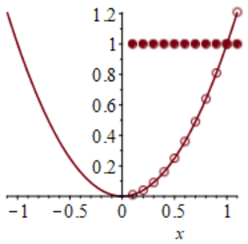
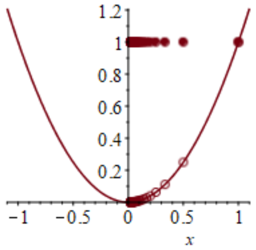
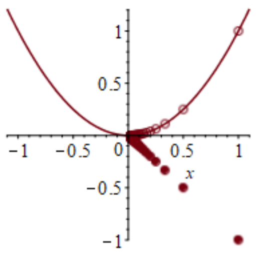
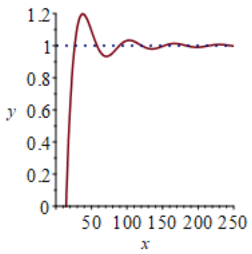
Function	**Is a definition given? Which misconception is tested for?**
	Definition given: $f(x) = \begin{cases} 1 & \text{, if } x = \frac{n}{10} \text{ for } n = 1,2,3,... \\ x^2 & \text{, otherwise} \end{cases}$ The question does not test for a specific misconception
	Definition given: $f(x) = \begin{cases} 1 & \text{, if } x = \frac{1}{n} \text{ for } n = 1,2,3,... \\ x^2 & \text{, otherwise} \end{cases}$ The question does not test for a specific misconception
	Definition given: $f(x) = \begin{cases} -x & \text{, if } x = \frac{1}{n} \text{ for } n = 1,2,3,... \\ x^2 & \text{, otherwise} \end{cases}$ The question does not test for a specific misconception
	No definition given Misconceptions tested for • Limits act as bounds

Table 2: Four last functions used in the test of misconceptions and which misconception is tested for

Part IV

RESULTS AND DISCUSSION

8

RESULTS

In this chapter we will present the results of our experiments. First we will make a couple of remarks about a statistical model, which we will be using many times. Then we will describe each of the variables in our experiment in light of the data we have obtained, and finally we present the results which will be used to answer our research question. The main conclusions and limitations will be discussed in the next chapter.

8.1 ORDINAL REGRESSION MODELS

We are interested in how different factors affect how well the student performs in An0. So we will need to run statistical models with An0 as the response variable. As we shall see, the grades in An0 are unlikely to be normally distributed, so we cannot run standard linear models. In stead we will use an ordinal regression model. In this model Y_i will be the grade student i obtained in An0. The model assumes that

$$\log\left(\frac{P(Y_i \leq j)}{1 - P(Y_i \leq j)}\right) = \theta_j - (\beta_1 \cdot x_{i1} + ... + \beta_n \cdot x_{in})$$

Where j runs through the different grades $\{-3, 00, ..., 12\}$, the θ's and β's are parameters of the model and x_{ik} is the score of student i in test k. We will not be interested in the sizes of the θ's, but we will be interested in whether β_k is significantly different from zero, as $\beta_k = 0$ means a change in x_k has no effect on the probability of getting a high grade. Note this is not necessarily because the variables are not connected, it may be because variable k is closely connected to another variable, j, so if x_k increases then x_j will also increase, and if $\beta_j > 0$, this will cause the probability of getting a high mark to increase. In this case we will say that the explanatory power of variable k is absorbed by variable j. This relationship does not say anything about cause and effect between variable k and variable j, and every time we observe a situation like this, we will have to use our knowledge of the variables to interpret the results.

8.2 DESCRIPTION OF THE VARIABLES

In this section, we give a description and initial analysis of each of the variables in our data set and give them a nickname, so we can more easily refer to them throughout the rest of the analysis.

Note that many of the nicknames we introduce in this section will have two meanings. “An0” and “LinAlg” will be used to describe both the courses An0 and LinAlg, but also the grades that the students obtain at the exams. Similarly, “limit proofs” will mean both simple proofs based directly on the definition of limits, but also the amount of points the students obtained in our test on the topic. And

- “Definitions”
- “Absolute values”
- “Orientation”
- “Proof validation”
- “Misconceptions”

will be used to denote both the proposed prerequisites and the score of the students in our tests on the topics.

8.2.1 *An0*

An0 is the mark the student received in the course An0. As we discussed in section 7.6.1, we take this mark as a proxy measure the student’s ability to understand proofs involving the formal definition of limit. But as we also discussed in section 7.6.1, An0 also contains other aspects such as the student’s understanding of the definitions and strategic study orientation.

The student scores are shown in figure 3. We note that the scores are not normally distributed, and that we are likely to have difficulties distinguishing between the good students, as the marks are capped at 12, so very good students get the same mark as good students. Also note that the category of students who got -3 on their exam is large as it consists both of students, who did not hand in all mandatory assignments, students who did not show up at the exam, students who showed up, drew a question and decided not to attempt to answer it, and students who attempted to answer the question they drew, but showed no understanding.

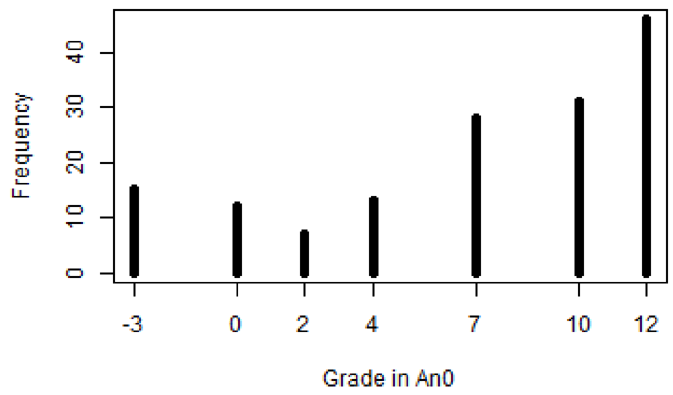


Figure 3: Histogram of grades in An0

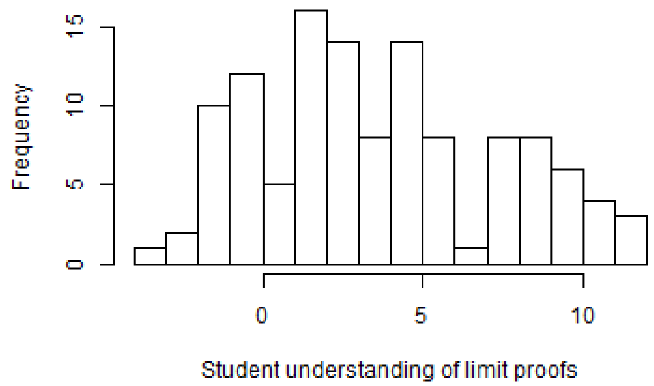


Figure 4: Histogram of student understanding of proofs involving limits after one lecture. The points are scaled, so 12 is the theoretical maximum.

8.2.2 *Limit proofs*

"Limit proofs" is the number of points the student scored in the quiz on understanding of simple proofs involving limits in the beginning of the course An0. We found that this score correlated more strongly with An0 than any other variable in our investigation (correlation coefficient 0.45), which gives some credence to the claim that they measure the same thing. The testing conditions were very different however and the two tests are separated by time, which helps explain why the correlation between our tests and the exams is not stronger.

The histogram of student scores can be seen in figure 4. We note that the test was apparently appropriately difficult to get a large variation in our student sample, but the scores do not look like they come from a normal distribution. To test this, we did a Shapiro-Wilk test, which rejected the hypothesis that the data is normally distributed with $p = 0.003$.

We also note that a surprisingly large group of the students show significant understanding of proofs involving limits already this early in the course. The test was constructed to pick out the few students who had learned about proofs involving limits through other sources, but judging from the number of students who did well, many students apparently were able to pick up a lot from the three hour long lecture on the formal definition of limits, which they had just attended. This interpretation is backed up by the text answers the students gave in two open questions of the test in limit proofs. Many students seemed to consider epsilon delta proofs a game like what was described in the lecture and they used the word “to parry”, which is not common in Danish, but was very common in lecture they just attended.

8.2.3 *Proposed prerequisites*

The five possible prerequisites were all tested at the same time as the limit proofs.

- “Definitions”: The extent to which the student understands the role of definitions in mathematics, is able to use stipulated definitions and to not be coloured by concept images which are at odds with the formal definition.
- “Absolute values”: The extent to which the student can solve inequalities involving absolute values and connect the result to an image of a subset of the number line.
- “Orientation‘”: The extent to which the student has an active meaning orientation to learning and metacognitively monitors their own learning.
- “Proof validation”: The extent to which the student is able to validate proofs within the topic elementary calculus.
- “Misconceptions”: The extent to which the student has a concept image of limits, which is is line with the formal definition and is able to use this without being coloured by a faulty concept image. The name of the variable may be misleading, as a high score in misconceptions indicates that the student did well in the test, and *does not have* many misconceptions about limits.

Common to the results for all the prerequisites is that we normalised the scores, so a hypothetical max was awarded twelve points, and no answers or complete guesses was awarded zero points. In figure 5 we have plotted histograms for each variable and scatter plots and

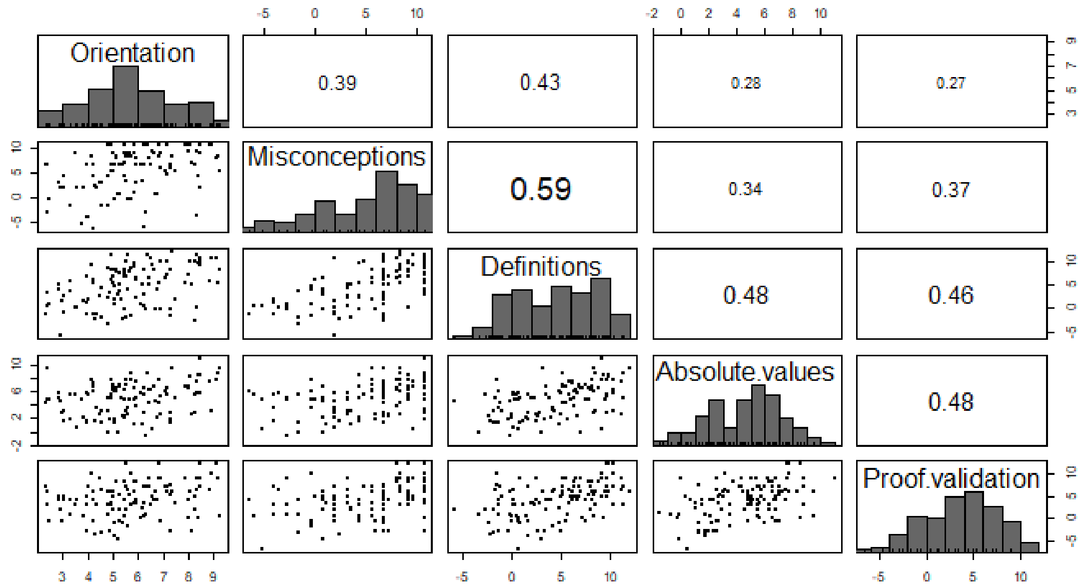


Figure 5: Histograms of each prerequisite and scatter plot plus Pearson correlation coefficient for each pair of prerequisites.

Pearson correlation for each pair of variables. We notice that the histograms do not all look like they come from normal distributions. And indeed, the Shapiro-Wilk test rejects the hypothesis that misconceptions, definitions and absolute values are normally distributed ($p = 4 \cdot 10^{-6}, 0.005, 0.044$ respectively).

In general it seems that all prerequisites are correlated. Orientation stands out as being the least correlated to the others. This makes sense for several reasons. First orientation is self reported, so we expect a lot of noise. Second, all the others are measures of the outcome of learning, whereas orientation is connected to precursors to learning. The effect of this variable is probably not immediate. Rather learning may be an effect of studying effectively over a long period of time.

In the other end of the scale, it seems like definitions and misconceptions are the most closely connected. This also makes sense, as definitions is a general competency in dealing with mathematical definitions and misconceptions is the same competency applied to the definition of limits. The apparent connection between these two variables becomes even more interesting, when we notice that the test in definitions did not contain the definition of limits. However, three out of five questions were connected to functions, so this may also explain the connection.

Lastly we are surprised to see that absolute values is just as correlated to definitions and proof validation as the two are to each other. We had expected absolute values to measure a different aspect of mathematical competency, as it does not require advanced mathematical

Variable	Correlation with LinAlg
Definitions	0.16
Orientation	0.17
Absolute values	0.17
Misconceptions	0.27
Proof validation	0.15
Limit proofs	0.29
An0	0.44

Table 3: Pearson correlation coefficients between the grade in LinAlg and the other variables of our experiment

thinking, but definitions and proof validation do. We suspect that either the idea that some students are good at abstract ideas and others are good at concrete ideas is wrong, or we have not been able to capture the difference between pre- and post-transitional mathematics in our tests. We will return to this issue in the discussion.

8.2.4 *LinAlg*

LinAlg is the mark the student received in the course Linear Algebra, which is taught just before An0 at UCPH. We only have grades for students who took LinAlg in 2018/19, so students, who did not progress directly from LinAlg to An0 in the school year 2018/19[1], will not enter into models which have LinAlg as a parameter. Theoretically we take any overlap between LinAlg and our proposed prerequisites to be a measure of general mathematical competency. In table 3 we have calculated the correlation between LinAlg and the other variables.

We note that as expected, An0 is the variable which is most correlated with LinAlg. We suspect this is because these two variables more closely measure ability to succeed in a university setting than our other measures. LinAlg and An0 differ from our tests in that the students knew what to expect, so they were able to prepare. Hence we expect strategic study orientation to play a larger role in LinAlg and An0 than in our tests. For both LinAlg and An0 there are certain things you can do, which will give you a higher grade, but which do not reflect better understanding. In LinAlg, you can get help with the hand-ins from your friends, in An0 you can learn from older students that you must present all relevant definitions when you are at the

1 for example because they passed LinAlg in their first go, but failed An0

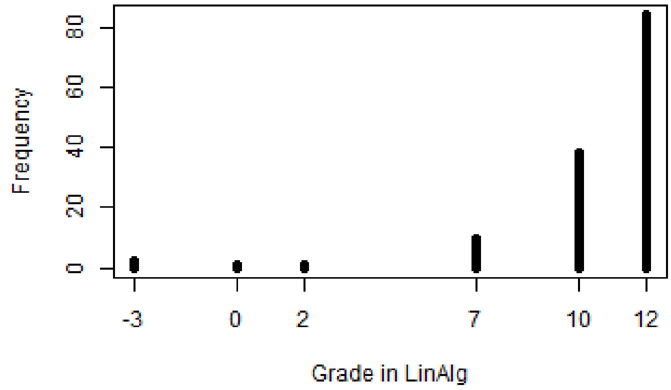


Figure 6: Histogram of grades in LinAlg

exam, and at both exams, the questions are predictable enough, that some rote learning can be helpful.

In general we note that the correlations are not very strong[2]. However, the lack of correlation may be because the LinAlg variable does not distinguish very well between the students in our sample. As can be seen from figure 6, more than half of the students in our sample scored 12 in LinAlg

8.2.5 *Different study programmes*

The course An0 is taken by students of many different study programmes. It is obligatory for students of pure mathematics, actuarial mathematics, mathematics-economics, and for students who take mathematics as a minor. This final group of students major in some other subject (often within humanities, physical education or music), but take courses in mathematics to obtain a double degree (often with the aim of becoming high school teacher). Furthermore, An0 is taken by a few students who are not enrolled at university for a full degree, but pay the university money to follow a single course and students who are enrolled in a study programme such as physics or computer science, but have an interest in mathematics, and therefore spice up their education with a few mathematical subjects here and there.

We found that some of these groups of students did perform significantly worse than others in our tests. In particular the students, who take mathematics as a minor, scored fewer points on most of the tests compared to the students who were enrolled in one of the mathematical

2 we cannot say if they are significant, as our variables are not normally distributed

bachelor degrees. However, we also found that once we controlled for LinAlg, this difference disappeared. Hence the important predictor seems to be the general mathematical competency and not the study programme of the student.

8.2.6 *DisRus*

Is a binary variable which indicates whether the student has passed the course DisRus in 2018. Note that not all students have DisRus a part of their study programme. None of the students of mathematics-economics or actuarial mathematics have it and only 18 out of the 65 students of pure mathematics took DisRus. The only group of students who all have DisRus are the students who take mathematics as a minor.

We included DisRus in our investigation, as we worried whether some of the questions favoured the DisRus students, because these students have learned about elementary number theory, the definition of functions and the quantifiers $\forall$ and $\exists$. This could influence their ability to answer the following questions:

- Question 3 in proof validation (quantifiers)
- Question 1 in definitions (function)
- Question 4 in definitions (quantifiers)
- Question 5 in definitions (number theory)

We performed χ^2 tests or t-tests on the questions, to learn whether DisRus had an influence on the students answers in the questions of definitions and proof validation. The results are in table 4. We found that in general, the DisRus students did at least as well as the other students in all questions, and especially in the four questions we had pointed to as possibly problematic.

We next investigated whether DisRus had statistically significant influence on the total variables. By comparing the mean of the sample of students who had passed DisRus with the mean of the sample of students, who have not passed DisRus in 2018, we found that DisRus only had significant influence on proof validation. That is, DisRus did not have significant influence on definitions, LinAlg or An0.

Our best interpretation is that the students who took DisRus have a small advantage in definitions, but it is so small, that statistics cannot pick it out. We will therefore ignore it. However in proof validation, the connection is stronger, so we have to take it into account. The problem is that we judge the connection to come both from the students being slightly

Effect of DisRus on definitions

	p-value
Functions*	0.2
Tangent	0.9
Determinant	0.7
Totally (un)limited*	0.1
Super-even*	0.2

Effect of DisRus on proof validation

	p-value
Q1	0.6
Q2	0.1
Q3*	0.001
Q4	0.4
Q5	0.003
Q6	0.4
Q7	0.8

Table 4: Results from testing the hypothesis: the student's ability to answer the question is independent of whether the student took DisRus or not. The questions which draw on topics from DisRus are marked with an asterisk.

better at proof validation but also from knowing about quantification. To remove the effect of quantification but keep the effect of general competency in proof validation, we chose to remove the one problematic question from our set. In that way we can exclude DisRus as an explanatory variable in the future statistical analysis.

Before we leave the variable, we want to point to another interesting possible conclusion. If our analysis above is correct, it seems like the group of students who took DisRus are on average as good students as the ones who did not (as judged by the mark in LinAlg). It also seems as if the students of DisRus did learn something which is related to proofs involving limits (quantification), however it also seems like the other students were able to pick up this competency when it became necessary.

8.3 ATTEMPTING TO ANSWER THE RESEARCH QUESTION

In this section, we first argue that we cannot use our data in the way we originally envisioned to answer the research question. We then look at the data from different perspectives to see if we may learn something from it anyway.

Originally we had imagined that most of the students would score close to 0 points in our test on understanding of limit proofs. Then we could remove the few students from the sample, who for some reason did understand limit proofs, and run the analysis on the remaining students. The results from the analysis could then be interpreted as showing to what degree

	Absolute values	Orientation	Definitions	Misconceptions	Proof validation
LinAlg	0.17	0.17	0.16	0.27	0.15
Limit proofs	0.54	0.42	0.58	0.58	0.40
An0	0.25	0.11	0.34	0.37	0.32

Table 5: Pearson correlation coefficients between the proposed prerequisites and LinAlg, Limit proofs and An0

the different prerequisites were important to have *before* one attempts to learn to understand proofs involving limits.

However, as we have already discussed, many of the students showed considerable understanding of limit proofs, when we tested them. Our data set is not big enough to remove all the students who show significant understanding. We therefore had to come up with different ways to look at the data to try to answer our research question. We note that, we will have problems establishing cause and effect, as the abilities, we call prerequisites, may not have developed first. It may be the other way around, so practising proofs involving limits, has caused the students to develop their other skills. To keep our word use consistent, however, we will continue calling the abilities and the orientation from the research question prerequisites.

8.3.1 *Simple correlations*

In this subsection, we will create scatter plots and calculate correlations between the different variables to see if this may help us answer the research question. The Pearson correlation coefficients are in the table 5 and the scatter plots are in figure 7.

Based on the scatter plots and the correlation coefficients, we will now make a number of observations, which will be discussed in later sections.

Orientation

It seems like orientation is the least connected to An0. It is however, moderately connected to limit proofs. Looking through the different subsections of orientation, we found that the main connection to limit proofs was found in the questions describing behaviour in lectures. We see two different explanations for why this moderate correlation arose: It may be because limit proofs is partly a measure of how much the student learn in a lecture, and students with an active meaning orientation learn more. It may also be because limit proofs is a measure of how

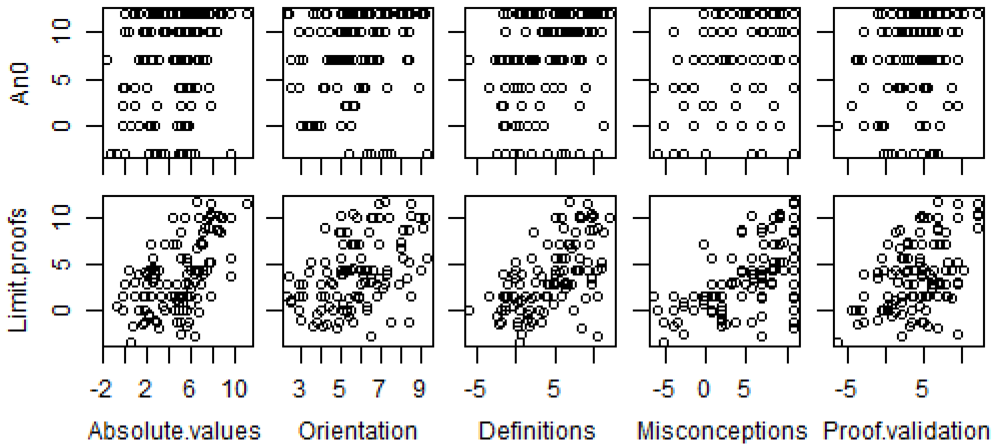


Figure 7: Scatter plots displaying the connection between the prerequisites and the ability of the students to understand proofs involving limits in the beginning (limit proofs) and end of the course An0 (An0)

well the students understood of limit proofs before entering the lecture, and the students who already knew about proofs involving limits are better able to think ahead and come up with good questions during the lecture. Thus they score higher on a questionnaire which rewards this sort of behaviour.

In any case, our measure of orientation does not seem to have any effect on the outcome of entire courses such as An0 or LinAlg. The questions in orientation are centred around behaviour in LinAlg, and we could expect behaviour to change between the two courses, so we would expect orientation to correlate stronger with LinAlg than An0. This is not the case, however. As we mentioned earlier, our questions in orientation are theoretically scattered, so we did go through all the different questions to see if any of them had moderate or strong correlations with LinAlg or An0. The only correlations we found were weak or very weak, which indicates to us, that perhaps our measure of orientation is not an important factor in doing well at university.

Necessary conditions

Judging from the scatter plots, none of the prerequisites are necessary for obtaining a high mark in An0. For all prerequisites, it is possible to give a complete guess in the test of prerequisites, but still get the mark 12 in An0. However, it does seem like the prerequisites are to some degree necessary for limit proofs. If this is true, it must mean that students can pick up any prerequisites they lack during An0, which should be encouraging to the course teacher, but slightly discouraging to us, as it means we cannot point to an obvious solution.

	Significant influence on An0?	Still significant influence, when LinAlg is taken into account?
Absolute values	p = 0.0004	p = 0.006
Definitions	p = 3e-5	p = 0.0007
Proof validation	p = 0.0002	p = 0.002
Misconceptions	p = 5e-5	p = 0.02

Table 6: In the first column we test if each of the prerequisites have significant explanatory power over An0. In the second column we test if this significance is maintained when LinAlg is added to the model as a second predictor.

Which is more important?

The aim of this thesis was to get an indication of which prerequisites would be most interesting to study further. The correlations indicate that orientation is the least interesting. The part of orientation concerning activity in lectures seems to be connected to level of understanding after the lecture, but this was not what we tried to understand here. Hence we will drop orientation from the rest of the statistical analysis.

Unfortunately the correlations do not provide a clear winner. Judging only from the correlations, the other prerequisites seem equally suited for further study.

8.3.2 *Linear models of An0 given LinAlg*

In this section we first investigate whether the connections we found above are statistically significant. Then we check if they can be explained away by referring to the background variable general mathematical competency (as measured by LinAlg), and finally we test whether each of them add new information or they are all measures of the same background variable.

To answer the first question, we ran the ordinal regression model with An0 as the response variable and each of the prerequisites as predictor. The results are in table 6. We see that all prerequisites have significant effect on An0. However, this may just be because both prerequisite and An0 are measures of the same background variable "general mathematical competency". To test if this was the case, we ran the models again, but with two predictors: The prerequisite and LinAlg. The results are also in table 6. We see that all prerequisites maintain their significance. To us it indicates that An0 is not just a continuation of LinAlg. Our experiments indicate that An0 requires some mathematical competencies which LinAlg does not require, and that our prerequisites may help us understand the difference between the subjects. However, as yet, we have not looked at whether these competencies are overlapping.

To test this, we ran the model with An0 as response variable a number of times. We found that the variable definitions absorbed the explanatory power of all other variables except LinAlg. This indicates to us that if we are trying to explain the difference between LinAlg and An0, then definitions is an interesting variable. Note that this conclusion is about the particular subjects not necessarily about proofs involving limits. A large part of what is tested at the exam in An0 is understanding of definitions, so the connection can simply be caused by this.

8.3.3 *Linear models predicting limit proofs*

As we discussed in section 7.6.1, the variable An0 is only a proxy measure of the ability of the students to understand proofs involving the formal definition of limits. Therefore we will use this section to repeat the analysis of the previous section with the variable limit proofs as our response variable in stead of An0 to test if the same conclusions hold.

Like in the previous section, we find that all of our proposed prerequisites have significant explanatory power on limit proofs and that this explanatory power is not absorbed by LinAlg. However, when we include several of our proposed prerequisites in the model, we get new results. We find that definitions still absorb all the explanatory power of proof validation, but not of misconceptions or absolute values. All of the other variables can coexist in a model without absorbing the explanatory power of each other. To us, this could indicate that definitions and proof validation are both a measure of the student's familiarity with advanced mathematical thinking as defined in section 3.2.

Hence the analysis of this section confirms the results from the previous section:

- all of the proposed prerequisites are more strongly connected to understanding of limit proofs than what can be explained by general mathematical competency
- definitions and proof validation seem to coincide in what variance they explain

8.3.4 *A model of the increase in understanding during An0*

Despite the fact that the students do not start from zero in An0, we could still be interested in knowing which factors influences how much they learn during An0. We assume that the course is difficult enough, that all students will have room for improvement.

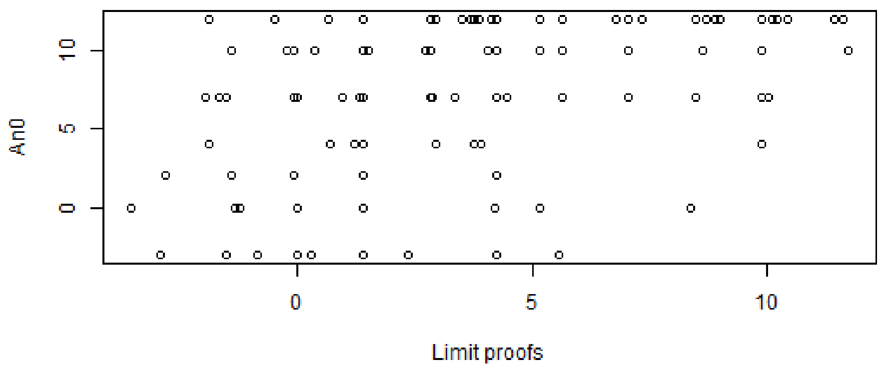


Figure 8: Scatter plot of limit proofs vs. An0

To investigate this, we fitted the data to an ordinal regression model. We ran the model several times in R. Each time with An0 as the response variable and with limit proofs as one of the predicting variables. We found that in this model, none of the prerequisites had significant additional explanatory power. That is, adding one or more of the prerequisites to the model, did not significantly increase its likelihood. From this, we conclude that whatever explanatory power the prerequisites did have, has been completely absorbed by the variable limit proofs.

One possible explanation to this could be that the students who scored high in our test of limit proofs already mastered the essential parts of An0 after the three hour lecture, so they did not improve much during the rest of the course. To test if this was the problem, we ran the model again but only with students who scored six or less on limit proofs. We chose the cutting point six as figure 4 shows not many students lie on the cutting point and figure 8 shows it is possible to get any mark in An0, if one scores less than 6 in limit proofs. We found the same result however. Whatever explanatory power the prerequisites had, was completely absorbed by limit proofs.

In a way, this should not be surprising. We hypothesised that the prerequisites should play an important role in learning to understand proofs involving limits in the context of university teaching, but our measurement limit proofs already contains this information. It is taken after the first lecture, so it measures both what the students already know, but also how much they are able to pick up from three hours of lectures. Hence if our prerequisites had additional explanatory power, it would mean that some later element in the course depended more heavily on them than the initial lecture. But this first lecture is exactly where proofs involving limits were first introduced, and the prerequisites were created with this moment in mind.

8.3.5 *Misconceptions*

Having found that the lack of misconceptions is important to learning proofs involving limits, we were interested in whether any particular misconceptions were more harmful to student success than others. To investigate this, we created a variable for each misconception:

1. Dynamic: A limit is the number that $f(x)$ moves towards when x moves towards a certain number
2. Bound: A limit is a number, which $f(x)$ cannot move past
3. Never reach: A limit is a number $f(x)$ gets close to, but never reaches.

The variables are created by summing the score of the students in all the questions which indicate the particular misconception. We scored each question negatively, so that a higher score indicates a greater tendency to be coloured by a misconception. Note that each variable is based on only three to four questions, so the work in this section is only exploratory.

Not surprisingly, we found that all three misconceptions correlated negatively with performance in An0 and limit proofs. I.e. the more the student was influenced by the misconception, the worse they did in An0 and limit proofs. We found no big difference in the size of the correlations:

	Correlation with An0	**Correlation with limit proofs**
Dynamic	-0.32	-0.46
Bound	-0.31	-0.46
Never reach	-0.22	-0.47

We wanted to investigate whether the misconceptions were important in themselves and not just as indicators of the fact that the student has little understanding of limits. Therefore, we created three other variables: non-dynamic, non-bound and non-reach. Each of these variables are created by summing the remaining questions of misconceptions. That is, by adding the scores of non-dynamic to dynamic, you get the score in misconceptions.

We ran a number of models in which each of the misconceptions were entered into models together with their non-version. We were interested in whether the different variables had significant explanatory power once the other variable was taken into account. The p-values of the analysis are presented below:

	Significance of variable		**Significance of non-variable**	
	An0	Limit proofs	An0	Limit proofs
Dynamic	p = 0.08	p = 0.006	p = 0.02	$p = 7 \cdot 10^{-5}$
Bound	p = 0.15	p = 0.17	p = 0.03	$p = 9 \cdot 10^{-6}$
Never reach	p = 0.66	0.08	p = 0.004	$p = 4 \cdot 10^{-5}$

We see that the non-variables consistently add explanatory power to the model, and for all but one model, this power eclipses the explanatory power of the misconception variables. This result indicates that for the most part, the particular misconceptions are not important in themselves. They are only important in that they indicate poor overall understanding.

The exception to this picture is the misconception dynamic. Our results on this variable are mixed, but it seems like this misconception may be important in itself. I.e. it is not just important that the student overall has a good understanding of limits, it is also important that the student does not have the misconception dynamic.

Knowledge in theory or in use

During the investigation above, we ran into a different result, which is interesting in its own right. The test was split into to parts. In the first part, the students had to say whether they thought different statements about limits were true, and in the second part, they had to apply their definition of limits to different example functions to say whether the function converged. We found that the important predictor of success was the second part of the test. The sum of scores from the first part had correlations 0.2 and 0.39 with An0 and limit proofs respectively, whereas the sum of the scores from the last part had correlations 0.38 and 0.56 respectively. Furthermore, the last part absorbed the explanatory power of the first part in both models predicting An0 and models predicting limit proofs.

To us, this indicates that the important predictor of student success is not how well they can put their knowledge into words, but how well they can apply their knowledge to problems.

9

DISCUSSION

The aim of this chapter is to discuss the results, which were presented in the previous chapter. We will proceed by first listing the main limitations of our study, and then discussing some of the conclusions reached in the previous chapter in light of these limitations. We chose to focus our discussion on the following three conclusions:

- Orientation to studying is only weakly connected to study success.
- An0 marks a significant transitional step towards advanced mathematical thinking for the students.
- Students, who did not take DisRus, perform as well in An0 and limit proofs as students who did.

9.1 MAIN LIMITATIONS OF THE STUDY

The main limitation of our study is that it cannot establish cause and effect between the prerequisites and understanding the formal definition of limit. Hence we cannot answer the motivating question of this thesis: "What should teachers focus on, if they want to prepare the students to learn the formal definition of limits?".

One way to overcome this limitation is through an effect study. In effect studies the students are randomly assigned to different groups, which are taught different material. Then the groups are compared according to how well they do on some response variable [Gleitman et al. 2011]. However, as we had many different hypothesises we wanted to test and not enough time to teach that number of classes, we decided on a less time consuming strategy. We planned to measure the proposed prerequisites *before* the students were taught the formal definition of limits, and then look at correlations between their mastery of the prerequisites and of limits. This separation in time cannot be used to establish cause and effect in no uncertain terms, but it would allow us to rule out the alternative hypothesis that the students improved in the prerequisites by practising the formal definition of limits.

However, as we already mentioned, by the time we measured the proposed prerequisites, the students were already well on their way to understand the formal definition of limits. Once we took their initial understanding of limits into account, none of our prerequisites added additional explanatory power. Hence our results will be even weaker than anticipated. We can only make conclusions about connections between different mathematical competencies, and not about the order in which they should be taught.

Another main limitation of our study is that we are using the grade in An0 as a measure of understanding of the formal definition of limits. As we discussed in section 7.6.1, this grade measures many other aspects of student understanding than the desired. We did find that the strongest correlation between any of our tests and An0 was between limit proofs and An0. However, we also found that definitions absorbed the explanatory power of the other variables in models with An0 as the response variable, which could be explained by the heavy weight examiners are reported to put definitions at the exams. Hence we have to be careful about which conclusions we can draw based on An0 alone. For this reason, we repeated many of the statistical analyses with both An0 and limit proofs as the response variable. We will now discuss three of the most important conclusions reached this way.

9.2 CONNECTION BETWEEN STUDY ORIENTATION AND STUDY SUCCESS

Our data clearly indicates that an active meaning orientation to studying is not important as a prerequisite to An0. Even though the questions in the questionnaire on orientation span a range of different ways student may show an active meaning orientation to studying, and they were specifically aimed at mathematics students after inspiration from several sources, we only found weak correlations with the grades in An0 and LinAlg. To us, this indicates that if we are narrowly trying to improve the grades of students in their courses over a short time frame, then the differences between students pointed to by Sierpinska 2007, Weber and Mejia-Ramos 2014, Anthony 1996 and the advice given by Alcock 2012 are not very important dimension.

Richardson et al. 2012 reviews 13 year of research into which factors influence the academic success of students (as measured by their grade point average). They find roughly the same correlation as we do: a correlation of 0.14 for meaning orientation, -0.18 for surface orientation and 0.23 for strategic orientation. Coffield et al. 2004 reviews different models of learning strategies, approaches, orientations etc., and in their review of meaning vs. reproducing orientation, they wonder out loud why it seems to be an implicit assumption in the literature that meaning orientations is superior, when the data shows strategic orientation to be more effective[1].

One possible answer is that meaning orientation originally comes from the concept deep approach, which describes how a student approaches a single learning task. According to Marton 1997 several studies have shown qualitatively different outcomes from students who were using a deep vs. surface approach to reading an article. Perhaps the assumption that students have a relatively stable study orientation is wrong? We did find moderate correlations between our measure of active meaning orientation in lectures and student understanding of limits after the first lecture in An0. It is very weak evidence as we were asking about their approach in lectures in the previous course not the one they had just attended, and the causation seems more likely to go the other way (students who already understood limits to a great extent had more time to think ahead, come up with questions etc), but it is striking that this was the only moderate correlation we found with any of our measures of study orientation.

We wonder whether the students are able to reflect on their own learning orientation over long time periods. As mentioned, it is interesting that the only moderate correlation we found between approach and any of the other measures was between understanding of a lecture they had just attended and their own experience of being active and reflected during lectures. To us this indicates that the students answers were very influenced by the lecture they had just attended, when trying to judge their average level of meaning oriented activity in lectures.

To avoid the uncertainties we just mentioned, we suggest that future studies, which try to quantify the effect of study orientations, focus on shorter time frames like a single lecture or a day of independent study. Over a shorter time period and with less varied tasks we expect the approach to change less. Hence its effect will be easier to quantify.

If we take our results at face value, they indicate that when teaching, it is more important to focus on teaching the mathematical content than on teaching active meaning orientation. We suspect this is especially true, if the tasks the students work on do not encourage it or the assessment system does not reward it.

Taking the limitations of our research setup into account, it is hard to make general conclusions about the importance of study orientations. However, our data do not support the hypothesis each student has a semi-stable approach to learning which affects academic success as measured by grades in a first year course.

1 The terminology in the two articles mentioned here is different from ours. They use the words deep approach, surface approach and strategic approach instead. But by looking at how they define the terms, we see that they are referring to what we call orientations

9.3 AN0 MARKS A TRANSITION WITHIN FIRST YEAR

In chapter 4 we claim that the transition in how limits are taught at UCPH coincides with a more general transition towards advanced mathematical thinking. Our study was not designed to test this claim, but we did obtain some support for it anyway. We found that the grade in An0 was not only determined by the grade in LinAlg in our models. Our mathematical prerequisites added significant additional explanatory power. We conclude that there is a difference between what is required of the students mathematically in LinAlg and in An0, and our prerequisites can help shed light on what the difference is. We found that our four prerequisites were unidimensional in the sense that in the models predicting success in An0, only one of them had significant explanatory power. In particular, we found that our variable definitions was consistently picked by the statistical models as the best predictor among our set of prerequisites. We repeated the statistical experiment, but this time trying to predict limit proofs. In this case our prerequisites were no longer uni-dimensional, but the explanatory power of proof validation was still absorbed by that of definitions. We conclude that the students' ability to work with mathematical definitions is an important difference between LinAlg and An0, but also that definitions and proof validation seem to both be measures of the same aspect of this difference.

This corresponds well with the theories of practical vs. theoretical thinking [Sierpinska 2000] and advanced mathematical thinking [Tall 1991] both of which lump understanding of definitions into the same group as understanding formal proofs. We hence see our results as confirming the above theories, that some transition in the way mathematics is taught does take place, and in this transition understanding definitions and proofs go together.

A big limitation of our study, when used to make this conclusion, is the fact that LinAlg is exceptionally bad at distinguishing between the top students, as more than half of the students in our sample scored the top mark. An0 has the same limitation, but to a much smaller degree. Hence the additional explanatory power of our prerequisites may simply be that they distinguish between the high achieving students of LinAlg. If we were to avoid this limitation in future studies, we should have more variables measuring pre-transition mathematics.

We thus see our data as providing support for the theories of different types of mathematics as described by the practical vs. theoretical thinking [Sierpinska 2000] or advanced mathematical thinking [Tall 1991]. Furthermore, our discussion points to an interesting way to test their theories empirically. If there are different types of mathematics, it seems likely that some students will be good at one type and others at the other type, so we should be able to see this reflected in data.

9.4 DOES DISRUS PREPARE THE STUDENTS FOR AN0?

One of our motivations for this thesis was to find out if any of the abilities that we called prerequisites actually were prerequisites for An0. I.e. if it is important that they are taught *before* the students attempt An0. We were not able to carry out the investigation we had planned, but we did get some data which support the hypothesis that the prerequisites can be taught as they are needed. First, we found that none of our prerequisites were necessary to have *before* the course started: it was possible to score zero points in a prerequisite and still get top marks in An0. Second, we found that the students who had taken DisRus were significantly better at quantifiers, but were not better at An0.

The implications for teaching are not clear cut. It seems clear that not every prerequisite can be taught when it is needed. Just imagine teaching operator algebra to a student who has never heard of topology, function spaces or algebraic objects like groups, rings, or vector spaces. But at least in this limited case, our data supports the hypothesis that the prerequisites we have focused on can be picked up during An0, and when it comes to DisRus and quantifiers it even seems like the students who took DisRus have no discernible advantage over the students who did not. Our best advice to teachers based on this data is to provide opportunities for students to pick up important prerequisites during their courses.

A big limitation to our conclusion considering the effect of DisRus is that the experiment is not random, so there are differences between the student populations who took DisRus and those who did not. The students who did not take DisRus are primarily students of mathematics-economics or actuarial mathematics. They all have high marks from high school as the competition to get into those study programmes is intense. The students who take DisRus are primarily students of pure mathematics and students who have another major, but take mathematics as a minor with the aim of teaching it in high school. The pure mathematics study programme is not very difficult to get into, so the students in this programme have very mixed abilities. We saw in section 8.2.5 that the students who take mathematics as a minor in general seem to perform worse than the other students on most of our measures. This makes sense. These students have another major than mathematics, so mathematics is only their second choice of subject. Hence it could be argued, that the students who take DisRus are on average likely to be less good at mathematics than the students who do not. The lack of difference between their performance in An0 can thus be explained as the pure mathematics students having the benefit of DisRus but the actuarial mathematics and mathematics-economics students having the benefit of better prerequisites from high school. To test this hypothesis, we looked at the grades in LinAlg, which we assumed to be less dependent on DisRus, as the exam does not test material from DisRus and the hand-ins only do so to a very small extent. Among our sample of students, the ones who had DisRus did as well as the students who did not, which indicates to us that the above hypothesis is false for the group of students in our sample. To investigate the second hypothesis further, we would advise that other measures of general mathematical competency, which do not rely on DisRus were used as well. For example the average grade from high school could be used as well.

Another limitation of our study is that we do not have measurements of the prerequisites closer to the exam date, so we do not know if the prerequisites are necessary or not. It could be that the students who do not have them in the beginning of the course pick them up quickly. If we wanted to investigate the question of timing the teaching of prerequisites, then these data points should be added to the set.

In conclusion, our data supports the hypothesis that the problem for students in An0 should not be solved by introducing more introductory courses before An0. Rather we should look at what can be done within An0. However the evidence is not very strong.

Part V

CONCLUSION AND FUTURE WORK

10

CONCLUSION

The aim of this thesis was to provide advice to teachers on how best to prepare the students for learning to understand proofs involving the formal definition of limits. It has previously been found that the formal definition of limits is taught at a pivotal point in time in a students university education. Up until this point in time, students are only required to perform tasks such as describing, explaining and calculating. After this time, they are also required to understand definitions and proofs which are based on these definitions. Even though our study was not intended for this purpose, we did find evidence that at the University of Copenhagen, part of this transition takes place during the course An0, which the students take about half a year after they start their university education. Incidentally this is also the course, where they are first asked to work with the formal definition of limits.

The transition to advanced mathematical thinking and to understanding limits has been shown several times to be very difficult for students, and hence much research has gone into investigating how to ease the transition. The aim of this thesis was to compare five different suggestions from previous research on what may be causes of the difficulties faced by students:

- They are not able to distinguish between valid and invalid proofs
- Their orientation to studying is not as suited for studying university level mathematics
- They do not understand the role of definitions in mathematics
- They have misconceptions about limits, which will coexist with their knowledge of the formal definition and create confusion
- They lack pre-transitional skills such as how to solve inequalities involving absolute values.

We hypothesised that the students in our sample would vary on the degree to which they were faced with the difficulties above, when they entered the course An0, and hence we could learn about which of the difficulties were most detrimental to student success by comparing the entry levels of the students on each of the hypothesised prerequisites and their final marks in the exam.

We found that all of the four mathematical competencies above (that is everything besides orientation) had moderate to high correlations with learning outcomes in An0, which indicates that previous research was correct in pointing to them as important aspects of student learning. However, we were not able to find clear evidence that any of the competencies were more important than others.

We were not able to establish cause and effect between the prerequisites and the final mark in the course, but we did try to eliminate some of the alternative hypotheses. We focused on the two alternative hypotheses:

- The competencies developed in the opposite order. That is, the students learned the prerequisites by practising them in the context of proofs involving limits (for example during the course MatIntro, where they were briefly introduced to the formal definition of limits).
- Students who are good at the prerequisites are also good at proofs involving limits, because both are mathematical competencies, and some students learn mathematics faster than others due to factors such as IQ and facility with mathematics in general.

We wanted to eliminate the first alternative hypothesis by removing students from the sample who already understood proofs involving limits when we tested them in the prerequisites. However, we found that most of the students showed moderate to good understanding already at the time of testing, and we did not have a big enough sample to work only with the students who showed little or no understanding. Hence we were not able to dismiss the first alternative hypothesis.

We tried to eliminate the second alternative hypothesis by controlling for the grade in LinAlg, when we ran the statistical analysis. The idea was that LinAlg is sufficiently different from any of the mathematical prerequisites we investigated, that any overlap would be due to a general mathematical competency. We found that all the mathematical prerequisites above maintained their significant influence on the grade of An0 even when we controlled for LinAlg. We concluded that the link between the proposed prerequisites and proofs involving limits is stronger than what is predicted by general mathematical competency alone. However, a big limitation here is that the grade in LinAlg is not a very good measure, as more than half of the students received the top mark.

To sum up, we found that the following four proposed prerequisites

- Proof validation
- Use of definitions
- Lack of misconceptions
- Ability to solve inequalities involving absolute values

had moderate to strong correlations with the ability to understand proofs involving limits. These correlations were stronger than what would be predicted by general mathematical competency alone, but we cannot establish what came first: the prerequisite or the understanding of proofs involving limits. On the other hand, orientation to studying was not found to be an important predictor of study success as it only correlated weakly with measures of learning outcomes.

10.1 IMPLICATIONS FOR TEACHERS

Based on this research, we recommend that teachers spend time on teaching all of the four mathematical prerequisites above, and that they spend comparatively little if any time on trying to change the study orientation of the students.

We were not able to find any clear answers on the question of whether the prerequisites should be taught before or simultaneously with proofs involving limits. However, we did find some indication that some of the prerequisites to learning about proofs involving limits (such as knowledge of proof techniques and quantifiers) can be picked up when they are needed. Students who were taught ahead of time showed no significant benefit.

10.2 FUTURE WORK

We recommend future research into the cause and effect relationship between our proposed prerequisites and understanding of proofs involving limits. For example by measuring the hypothesised prerequisites before the students start to learn about proofs involving limits, or by conducting an experiment in which the students are randomly assigned to groups and taught one of the prerequisites. After the intervention, the students could be taught proofs involving limits as usual, and the understanding of the different groups could be compared. In a study like this, the effect of the different prerequisites on each other would become more clear. For example, the relationship between definitions and proof validation could be explored by testing if the students improved in one by being taught the other. Results from this investigation would shed more light on the nature of the transition to advanced mathematical thinking as well as the cause and effect relationship between the proposed prerequisites and the learning outcome.

We also recommend that the effects of study orientation are investigated more in depth. We believe that this area is more important than the results of this thesis indicate. While we did find that study orientation only had a very weak correlation with the grades in the different subjects, our data and previous research indicate that the influence may be more visible if we look at a single learning situation. Our variable called active meaning orientation was a mixture of the ideas of many different researchers. If we are more narrowly looking at a single task, we might be able to tease out which specific ideas are most important in that setting. When we are looking at success in a nine week long course, there are very many parameters at play which is likely to muddy the picture.

We find this area interesting for several reasons. Changing a student's orientation to studying or approach to a single task may not have as big an effect as teaching them the mathematical prerequisites to the task, but it may take less work to achieve. Also, if the student is able to acquire a fruitful approach to learning, they may find it valuable in many different situations – both inside and outside of the educational system.

Part VI

APPENDICES

11

QUIZ NUMBER 2: PROOF VALIDATION

This is the second test in the battery of tests administered to the students of Analysis 0. This test was prefaced with an example exercise:

> *Assume that you have already shown that x and x^3 are continuous in zero, and that you now want to show that x^2 is continuous in zero.*
>
> *What do you think of the following two proofs?*
>
> **Proof 1:** x^3 and x are continuous in zero, so $\frac{x^3}{x} = x^2$ must also be continuous in zero.
>
> **Proof 2:** x is continuous in zero, so $x \cdot x = x^2$ must also be continuous in zero.

along with an instructional text:

> In connection with this quiz, it is okay to apply results which are external to the quiz, regardless of whether you have seen them proven in a course or not. But the result has to be true! In this case, it is true that the product of continuous functions is continuous, but it is not true that a fraction with a continuous function in the numerator and denominator always is continuous. Take the example $\frac{1}{x}$. You therefore ought to discard the first proof, but accept the second.

The students were free to spend as much time as they needed to read this.

Once the students were ready, they started the quiz. They had a total of ten minutes to complete the quiz and had been told that correct answers gave 1 point, wrong answers -1 and “I don’t know”/no answer 0 points.

11.0.1 *Quiz items*

Question 1

One of the following proofs is valid and the other is not. Which of the two do you accept?

Proof 1: Regardless of which real number $x \in \mathbb{R}$ you put into the function $f(x) = x^2 - 4x + 7$, the result will always be positive. The reason is that

$$x^2 - 4x + 7 = (x - 2)^2 + 3$$

and a real number squared is always positive.

Proof 2: Regardless of which real number $x \in \mathbb{R}$ you put into the function $g(x) = x^2 + 6x + 5$, the result will always be positive. The reason is that

$$x^2 + 6x + 5 = (x + 3)^2 - 2^2$$

and a real number squared is always positive.

- ○ Proof 1
- ○ Proof 2
- ○ I don't know

Question 2

Theorem x^2 is positive for all $x \in [1, 10]$.

What do you think of the following proof of the theorem?

Proof: Set $f(x) = x^2$

Then $f(1) = 1$ and $f(10) = 100$

As f is continuous, the intermediate value theorem gives that no c between 1 and 10 can exist such that $f(c) = 0$

Hence, the function does not cut the x-axis, and thus is must be positive on the whole interval.

For reference, here is the intermediate value theorem, as you know it from MatIntro

Intermediate Value Theorem: If a function is continuous in the closed interval $[a, b]$, and $f(a)$ and $f(b)$ have different signs, then there exists a $c \in]a, b[$ such that $f(c) = 0$.

- ○ The proof is valid/has small flaws which are easily corrected
- ○ The function f is not continuous, so the proof is not valid.
- ○ The intermediate value theorem does not apply in this case, so the proof is not valid.
- ○ The theorem is false, so the proof cannot be valid.
- ○ I don't know.

Question 3

Below are two theorems and a proof. Which theorem is being proven?

Theorem 1: Regardless of how small a positive real number x, you look at, there will always exist a natural number n such that $\frac{1}{n} < x$.

Theorem 2: There exists a natural number n such that $\frac{1}{n}$ is strictly smaller than all positive real numbers x.

Proof: Let x be a positive and real number. As the natural numbers are not upwardly bounded, there exists a natural number n, which is bigger than $\frac{1}{x}$. The conclusion now follows from the standard rules for fractions as

$$n > \frac{1}{x} \Leftrightarrow x > \frac{1}{n}$$

- ○ Theorem 1
- ○ Theorem 2
- ○ None of the theorems
- ○ Both of the theorems
- ○ I don't know.

Question 4

Theorem: There exists a real number x such that $x^4 = x + 1$

What do you think of the following proof of the theorem?

Proof: Set $f(x) = x^4 - x - 1$

$f(1) = -1$ and $f(2) = 13$

As f is continuous, the intermediate value theorem gives that there exists a c between 1 and 2 such that $f(c) = 0$

c has the desired property.

For reference, here is the intermediate value theorem, as you know it from MatIntro

Intermediate Value Theorem: If a function is continuous in the closed interval $[a, b]$, and $f(a)$ and $f(b)$ have different signs, then there exists a $c \in]a, b[$ such that $f(c) = 0$.

- ○ The proof is valid/has small flaws which are easily corrected
- ○ The proof is invalid. You have to introduce a completely new idea to make it work
- ○ The theorem is false, so the proof cannot be valid.
- ○ I don't know.

Question 5

Theorem: The sequence $a_n = \sqrt{n}$ tends to infinity as $n \to \infty$

What do you think of the following proof of the theorem?

Proof: We know that $a < b \Rightarrow \sqrt{a} < \sqrt{b}$

$n < n + 1$ so $\sqrt{n} < \sqrt{n+1}$ for all n

This shows that $\sqrt{n} \to \infty$ for $n \to \infty$

- ○ The proof is valid/has small flaws which are easily corrected
- ○ The proof is invalid. You have to introduce a completely new idea to make it work
- ○ The theorem is false, so the proof cannot be valid.
- ○ I don't know.

Question 6

Theorem: The equation $x^4 - 4x^3 = -30$ has no solutions

What do you think of the following proof of the theorem?

Proof: Consider the function $f(x) = x^4 - 4x^3$. As f is a polynomium of degree 4 with a positive coefficient in front of x^4, $f(x) \to \infty$ for x going towards both positive and negative infinity.

As f is differentiable, the smallest value of $f(x)$ must be in a point where $f'(x) = 0$.

$f'(x) = 4x^3 - 12x^2$, so $f'(x) = 0$ for $x = 0$ and $x = 3$.

$f(0) = 0$ and $f(3) = -27$, so the smallest value of $f(x)$ must be in $x = -27$. Hence f can never reach -30.

- ○ The proof is valid/has small flaws which are easily corrected
- ○ The proof is invalid. You have to introduce a completely new idea to make it work
- ○ The theorem is false, so the proof cannot be valid.
- ○ I don't know.

Question 7

Theorem: If f is a continuous and real function, and $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

What do you think of the following proof of the theorem?

Proof: We prove the theorem in the two cases $f(x) \geq 0$ and $f(x) < 0$.

If $f(x) \geq 0$, then $f(x) = |f(x)|$, and thus $\int_a^b f(x)\,dx = \int_a^b |f(x)|\,dx$.

If $f(x) < 0$, then $\int_a^b f(x)\,dx \leq 0$. In contrast to this $|f(x)| \geq 0$, so $\int_a^b |f(x)|\,dx \geq 0$.

Thus we see in both cases that $\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$

- ○ The proof is valid/has small flaws which are easily corrected
- ○ The proof is invalid. You have to introduce a completely new idea to make it work
- ○ The theorem is false, so the proof cannot be valid.
- ○ I don't know.

12

QUIZ NUMBER 3: USE OF DEFINITIONS

This is the third test in the battery of tests administered to the students of Analysis 0. After each option is noted the number of points this option gave when scoring.

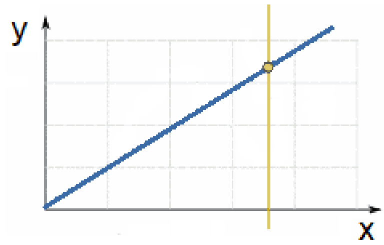
12.0.1 *Quiz items*

Question 1

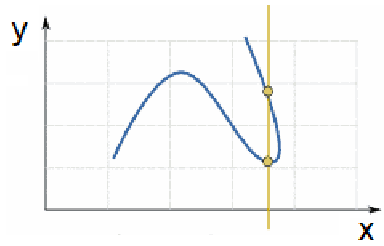
Definition: A function $f : X \to Y$ is a relation between elements of X and elements of Y such that each element of X is related to exactly one element of Y.

Intuitively, we think of functions as processes, which send each x to the y it is related to.

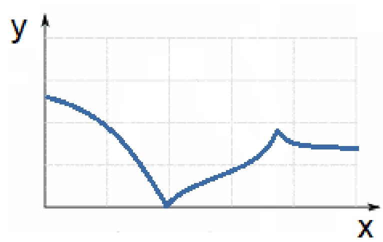
Examples: The following is a function, as all x values are related to exactly one y-value:



The following is not a function, as there exists an x-value, which is related to two y-values



- Does there exist a function, which sends all real numbers different from 0 to themselves, but sends 0 to -1? Yes (1 point) / no (-1 point) / I don't know (0 points)
- Does there exist a function, which sends all positive numbers to 1, all negative numbers to -1 and 0 to 0? (1 point) / no (-1 point) / I don't know (0 points)
- Does there exist a function, which has the following as its graph?



(1 point) / no (-1 point) / I don't know (0 points)

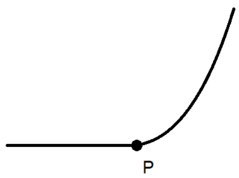
Question 2

Definition: A tangent in a point P is a straight line, which touches the curve in the point P, and matches the slope of the curve in the point P.

Five students were asked how many tangents the following function has in $P = (0, 0)$

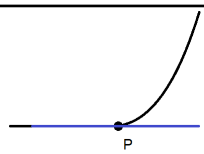
$$f(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Afterwards, they were asked to draw the tangent(s) if possible.



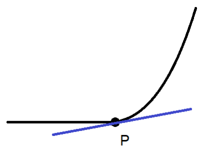
Which of the students answered correctly?

○ There is only one tangent:



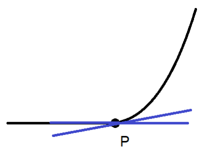
(1 point)

○ There is only one tangent:



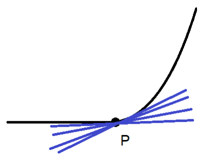
(-1/4 point)

○ There are two tangents:



(-1/4 point)

○ There are infinitely many tangents:



(-1/4 point)

○ The function does not have a tangent in that point (-1/4 point)

○ I don't know (0 points)

Question 3

In the course Linear Algebra you learned about determinants. Below is a number of true statements about determinants.

Which of the following was presented to you as a *definition* of the determinant of $n \times n$ matrix?

☐ The determinant is the function from the space of $n \times n$ matrices to $\mathbb{F}$ which satisfies

- $\det(a_1 ... b_k + c_k ... a_n) = \det(a_1 ... b_k ... a_n) + (a_1 ... c_k ... a_n)$ where a_k, b_k and c_k are the columns of the matrix.
- If you multiply a column of a matrix by a scalar, the determinant of the matrix will be multiplied by the same scalar.
- If two columns of a determinant are the same, the determinant of the matrix is zero.
- The determinant of the identity matrix is 1.

☐ The determinant of a triangular matrix is the product of the entries on the diagonal. The determinant of all other matrices can be found by performing column operations until the matrix is triangular.

☐ The determinant of a 2×2-matrix is given by $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

The determinant of bigger matrices can be found by using the Laplacian expansion equation after the i'th row: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

Or after the j'th column: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij}) a_{ij}$

☐ $\det(AB) = \det(A) \cdot \det(B)$

☐ $\det(A) = \sum_{\sigma} sign(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$

☐ The determinant is the function we use to determine if a matrix is invertible.

☐ $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

☐ I don't know

(3 points if 1 or 5 is chosen and none of the others are, -3 point if 4 or 6 is chosen, in all other cases: 0 points)

Question 4

In this question, you will work with two definitions, which have been invented with this particular quiz in mind:

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *totally unbounded*, if it is true for any $M > 0$ that $f(x) > M$ for all $x \in \mathbb{R}$.

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *totally bounded*, if it is true for any $M > 0$ that $f(x) < M$ for all $x \in \mathbb{R}$.

For each of the statements below, circle the correct option

- e^x is totally unbounded (-1/3 point) / totally bounded (-1/3 point) / both totally unbounded and totally bounded (-1/3 point) / neither totally unbounded nor totally bounded (1 point) / I don't know (0 points)
- $-\sin(x)$ is totally unbounded (-1/3 point) / totally bounded (-1/3 point) / both totally unbounded and totally bounded (-1/3 point) / neither totally unbounded nor totally bounded (1 point) / I don't know (0 points)
- There exists no totally unbounded functions (1 point) / a single totally unbounded function (-0.25 points) / infinitely many totally unbounded functions (-0.75 points) / I don't know (0 points)
- There exists no totally bounded functions (-0.75 points) / a single totally bounded function (-0.25 points) / infinitely many totally bounded functions (1 point) / I don't know (0 points)
- If f is totally bounded and g is totally bounded, then $f + g$ is totally bounded (1 point) / totally unbounded (-1/3 points) / sometimes totally bounded sometimes not (-1/3 points) / not totally bounded (-1/3 points) / I don't know (0 points)

Question 5

In this question you will work with some definitions, which are generally accepted:

Definition: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is called the whole numbers .

Definition: $\mathbb{N} = \{1, 2, 3, ...\}$ is called the natural numbers.

Definition: A whole number $n \in \mathbb{Z}$ is called even if $\frac{n}{2}$ is a whole number.

And one which is invented for this quiz in particular

Definition: A natural number $n \in \mathbb{N}$ is called super-even if an even number of digits in n are even.

For each of the questions below, circle the correct option

- Is 0 even? Yes (1 point) / no (-1 point) / I don't know (0 points)
- Is 2612 super-even? Yes (-1 point) / no (1 point) / I don't know (0 points)
- Is 201 super-even? yes (1 point) / no (-1 point) / I don't know (0 points)
- What is the smallest super-even number? 0 (-1/5 points) / 1 (1 point) / 2 (-1/5 points) / 4 (-1/5 points) / 10 (-1/5 points) / 20 (0 points) / the smallest super-even number is not on the list (-1/5 points) / I don't know (0 points)
- Is the product of two super-even numbers always super-even? Yes (-1 point) / no (1 point) / I don't know (0 points)

13

QUIZ NUMBER 4: PROOFS INVOLVING THE FORMAL DEFINITION OF LIMITS

This is the fourth test in the battery of tests administered to the students of Analysis 0.

13.0.1 *Quiz items*

Question 1

Which of the following do you accept as a proof of the fact that $f(x) = 2x + 2$ tends to 4 as x tends to 1?

- ☐ Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x-1| < \delta$

$$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

- ☐ For arbitrary $\varepsilon > 0$ we can choose $\delta = 2\varepsilon$. Then

$$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\varepsilon = \delta$$

- ☐ Theorem: Let f and g be real functions and let $a \in \mathbb{R}$ be a constant. Then
  - $\lim_{x\to 1}(f(x) + g(x)) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x)$
  - $\lim_{x\to 1} af(x) = a \lim_{x\to 1} f(x)$

  Hence it follows that

$$\lim_{x\to 1}(2x + 2) = 2\lim_{x\to 1} x + 2 = 2 \cdot 1 + 2 = 4$$

- ☐

ε	f	f
0.1	4.2	3.8
0.01	4.02	3.98
0.001	4.002	3.998
0.0001	4.0002	3.9998
0.00001	4.00002	3.99998

  The system obviously continues, so the limit must be 4.

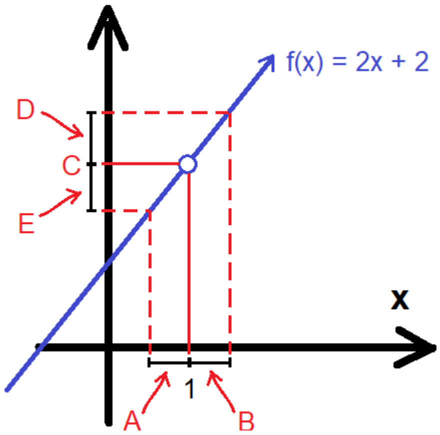
- ☐ I don't know

(-1 point if number two is chosen, 1 point if one is chosen and two is not, otherwise 0 points)

After answering this question, the students were told that the first option was correct while the second was not. Then they were allowed to quit the quiz if they had no idea why. If they had an idea of what was going on, they were encouraged to continue to the next questions:

Question 2

The image below is an illustration of the proof of the fact that $f(x) = 2x + 2$ approaches 4 when x tends to 1



Match the letters on the picture with the signs in the proof:

Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x - 1| < \delta$

$$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

- A: epsilon / delta / 4 / 2 / "I don't know"
- B: epsilon / delta / 4 / 2 / "I don't know"
- C: epsilon / delta / 4 / 2 / "I don't know"
- D: epsilon / delta / 4 / 2 / "I don't know"
- E: epsilon / delta / 4 / 2 / "I don't know"

Question 3

Here is the proof that $f(x) = 2x + 2$ approaches 4 when x tends to 1 again:

Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x - 1| < \delta$

$$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

If you were to use the same method to show that $f(x) = -10x - 3$ approaches -103 as x approaches 10, what would be the obvious delta to use? (open answer 1 point for right answer, -1 for wrong answer)

Question 4

Here is the proof that $f(x) = 2x + 2$ approaches 4 when x tends to 1 again:

> Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x - 1| < \delta$
>
> $$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

In the proof above, we had to make a choice of δ. Would the idea of the proof still have worked, if we had set $\delta = \varepsilon/3$?

- ○ Yes
- ○ No
- ○ I don't know

Question 5

Here is the proof that $f(x) = 2x + 2$ approaches 4 when x tends to 1 again:

> Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x - 1| < \delta$
>
> $$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

Which of the following phrases best sums up the proof?

- ○ For arbitrary ε, we set $\delta = \varepsilon/2$. Then $|x - 1| < \delta$ implies $|f(x) - 4| < \varepsilon$
- ○ For all $x \in \mathbb{R}$ with $|x - 1| < \delta$, it will be true that $|f(x) - 4| = |2x + 2 - 4| < 2\delta = \varepsilon$
- ○ For arbitrary $\varepsilon > 0$ we can show that $|f(x) - 4| < \varepsilon$
- ○ Given $\varepsilon > 0$, we can show that all x which satisfy $|x - 1| < \delta$ will also satisfy $|2x - 2| < 2\delta$
- ○ I don't know

Question 6

Here is the proof that $f(x) = 2x + 2$ approaches 4 when x tends to 1 again:

> Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x - 1| < \delta$
>
> $$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

Why do we start the proof by letting epsilon be given? (open answer: 2 points for answers which shows understanding of the connection to "for all", "for any" or similar. 1 point for answers of the type "otherwise I would not have anything to work with when choosing delta")

Question 7

Here is the proof that $f(x) = 2x + 2$ approaches 4 when x tends to 1 again:

> Let $\varepsilon > 0$ be given. If we set $\delta = \varepsilon/2$, then for all $x \in \mathbb{R}$ with $|x - 1| < \delta$
>
> $$|f(x) - 4| = |2x + 2 - 4| = |2x - 2| = 2|x - 1| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

One of your friends thinks he has found a problem with the proof. If he sets $\varepsilon = 1$ and $x = 2$, then the last line gives that $|f(2) - 4| < 1$. But $f(2) = 6$ so

$$|f(2) - 4| = |6 - 4| = 2$$

Which is not smaller than 1.

What would you say to your friend?

(Open answer. 1 point for rejecting the x based on the delta in this case. 0 points for anything else)

Question 8

Finish the proof of the theorem

$$\frac{1}{x} \text{ tends to 0 when } x \text{ tends to } \infty$$

Proof: [Blank], [blank]. Then [blank]

$$|\frac{1}{x} - 0| = |\frac{1}{x}| \text{ [blank] [blank]}$$

For each of the blank spaces the students were able to choose from a drop down menu. We list the options for each blank below:

- Blank 1: "Given a $K = 0$", "For any epsilon > 0", "For any $x > 0$" and "Given that x is a real number"
- Blank 2: "we can choose a $K > 1/\text{epsilon}$", "we can set $x = 2K$", "we can choose an epsilon > 0" and "we can set $K = 2$ epsilon"
- Blank 3: "for all epsilon > 0", [leave blank], "for $K >$ epsilon" and "for all $x > K$
- Blank 4: "$= 1/2K$", "$< 1/K$", "$< \varepsilon/2$" and "$= 1/\varepsilon$"
- Blank 5: "$< K$", "$> \varepsilon$", "$< \varepsilon$" and "$> K$"

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