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A Study and Research Path

on vectors in mathematics and physics

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UNIVERSITY OF COPENHAGEN

Department of Science Education

Master Thesis

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A Study and Research Path

on vectors in mathematics and physics

Supervisor: Carl Winsløw

Submitted on: 6 August 2018

Abstract

This thesis gives a description of the development of the notion of vectors in the scientific fields of mathematics and physics respectively. It includes a description of the interrelation between the notion in the two fields, because it gives some explanations why high school students struggle with the notion of vectors. Also the development of the notion of vectors in mathematics and physics in the context of high school teaching will be described. This has contributed to an understanding of why mathematics and physics, that historically are highly interrelated, have separated more and more over the years.

The examination of curricula, written exam problems, and textbooks have revealed that the mathematical notion of vectors is divided into an algebraic and a geometric approach. Since vectors are used to model the two or three dimensional spaces in physics, the geometric approach is more useful in physics than the algebraic. However, the geometric branch in mathematics does generally not include the approaches that are useful in physics.

These findings have been used to design a Study and Research Path on vectors. The idea was to combine mathematics and physics in the introduction to vectors, in order to utilise the motivations from physics and to make the applications in physics more obvious. The design was tested in a first year high school class, but it did not turn out as expected. None of the students developed a notion of vectors that was useful in the application to the physical problem they were asked to solve. However, the test of the design showed that it is highly relevant to keep working on alternative ways of teaching.

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Contents

Appendices

1. Introduction

Mathematics and physics are two branches of science that have developed very closely throughout most of their history. Multiple important results in mathematics have roots in physical problems, while many theories in physics have had crucial benefits from mathematical developments. In the 17^{th}, 18^{th}, and 19^{th} centuries it was often difficult to separate mathematicians from physicists and mathematical results from physical results. Well-known examples are Isaac Newton (1642-1727) and his work on fluxions that is one of the most important contributions to calculus or Joseph Fourier (1768-1830) and his work on heat conduction that led to the mathematical concept of Fourier series (Katz, 2009). Though the sciences of mathematics and physics have a long common (and sometime inseparable) history, it is well-known that in an educational context mathematics and physics suffer from an unfortunate (and escalating) disconnection. A majority of mathematics teachers will be of the opinion that mathematics is taught for the mathematics itself, even though one of the purposes of the teaching of mathematics in Danish high schools is to obtain the qualifications necessary for further educations that require mathematics (Danish Ministry of Education, 2017a). And mathematics *is* important in other science, e.g. physics, but when it is taught without consideration of its applications, the students will have a hard time applying the mathematics correctly to physical problems. As a consequence, the necessary mathematical concepts are taught in both mathematics and physics at the same time in different versions, and the disconnection is even more distinct (Orton & Roper, 2000). This issue is the first thing that motivated this thesis.

A recent model for teaching designs addresses the problem with very bounded teaching subjects, where students are presented to theories and results one after the other, sometimes without any connection, by their teachers without any invitation to ask questions themselves. The model is called Study and Research Paths and is meant to let the students work with their learning process with less directions from their teacher. Teaching that is organised in this way is very suitable for interdisciplinary work, where the students can work with problems that can only be solved by drawing on and developing knowledge in more than one subject. An interest in this design model and its opportunities and limitations in relation to interdisciplinary teaching is the second issue that motivated this thesis.

In August 2017 the Danish Ministry of Education implemented a new reform of the Danish high school. It implied a lot of changes in the organisational structure and in the curricula. In mathematics one of the changes concerned the topic *vectors*. Before, the teaching of vectors was reserved to students studying mathematics at A-level, but by the reform vectors were added to the C-level curriculum. The possible pedagogical challenges this restructuring can cause is the third issue that has motivated this thesis.

Like differential calculus and Fourier series, the notion of vectors is a mathematical concept that has evolved on the border of mathematics and physics. Furthermore, it is a necessary concept in both mathematics and physics in high school. However, the problematic disconnection of mathematics and physics is also present in relation to the teaching of vectors. This thesis has two parts that will both investigate this problem; a theoretical and an empirical. The aim of the theoretical part is to give a thorough description of the development of the notion of vectors that have appeared in the mathematical and physical communities respectively and the development of the notion of vectors that have been taught in high school mathematics and high school physics respectively, whit the purpose of detecting the origin of the disconnection of the two subjects in the context of teaching.

The aim of the empirical part of the thesis is to design a Study and Research Path on vectors in mathematics and physics, and test if it is beneficial for the students to encounter the concept in a way where mathematics and physics are incorporated from the beginning. In the process of designing, the theoretical findings will be used to avoid the usual disconnected encounter with vectors.

I Theoretical framework

2. The Anthropological Theory of Didactics

The theoretical framework for this thesis is the anthropological theory of didactics (henceforth abbreviated as ATD). The theory was launched by the French didactician Yves Chevallard in the 1980's. ATD builds on the assumption that "doing, teaching, learning, diffusing, creating, and transposing mathematics, […], are considered as human activities taking place in institutional settings." (Bosch & Gascón, 2014, p. 68). Hereby, an important part of research in the field of mathematical education conducted in the framework of ATD is to describe the institutional settings in which a teaching sequence is designed or in which a textbook is written.

One important step in the description is to analyse the path from the official mathematical knowledge to the curriculum. The researcher analyses the curriculum in the light of the official mathematical knowledge to see how it is organised. In a didactical study researchers use an explicitly stated *epistemological reference model* (henceforth abbreviated ERM) as a framework. The ERM is necessary in order to be able to make generalisable results, because empirical studies of didactical phenomena have a lot of uncontrollable effects and interpretation barriers that stem from the fact that teaching and learning in general are very subjective activities. An ERM makes it possible to generalise results by giving objective descriptions of the mathematical knowledge that is a part of the study which would otherwise be interpreted subjectively by the individual reader.

The aim for the theoretical part of this thesis is to construct an ERM that describes the organisations of the scholarly notion of *vectors* and the teaching topic *vectors*. Also the relations and links between the two organisations will be included. The model will be serving as the theoretical reference in the empirical study in the second part of the thesis.

Before the ERM can be constructed, some basic notions from ATD will be introduced and elucidated. The first notion is *the didactic transposition,* which describes the path from the scholarly mathematical knowledge through the curriculum to the students. This is of course important in order to elucidate the relations and links between the two notions of vectors, in the mathematical community and in a high school context respectively, and it will be described in section 2.2. The second important notion from ATD that will be described is *praxeologies* and especially *mathematical praxeologies*. This is a model that helps researchers to analyse how the mathematical knowledge is organised in the curriculum, textbooks, written exams, etc. A praxeological analysis of the notion of vectors in Danish high schools will be the foundation of the ERM. The notion of praxeologies will be described in section 2.3.

The notions mentioned above provides a framework for a theoretical analysis of the organisation of the mathematical content that is relevant in this thesis. Furthermore, the model *study and research paths* within ATD provides the framework for the empirical part of the thesis. The model can and will be used for both design *and* analysis issues, and it will be presented in section 2.4 and used in the empirical part of the thesis.

The next subsection will introduce the first important notion in the theoretical framework; the didactic transposition.

2.1 The didactical contract

The notion of the didactical contract is not developed within ATD, but it has turned out to be a necessary part of the theoretical framework in the analysis of the data material in the empirical part of the thesis, which is why it has to be described.

The didactical contract is an important notion in didactics, that covers the implicit expectations and commitments between the teacher and the students. It is a set of unwritten rules, that varies depending on the teacher and the students in a class. The didactical contract can both affect the teaching and learning positively and negatively. A positive effect of the didactical contract is that the students know exactly what is expected from them in specific situations. A negative effect is that the students are only able to learn, when the teaching proceeds exactly as they are used to, since an alternative way of teaching will be a breach of the didactical contract. Furthermore, the didactical contract can affect students' approach to different problems. A classic example is the problem of the age of the captain. The problem reads: "A captain owns 26 sheep and 10 goats. How old is the captain?". A general tendency across countries is that pupils in elementary school give the answer 36, even though the question does not make sense at all (Winsløw, 2006, p. 145-150). In this case pupils are so tied to the didactical contract, that they do not reflect on the problem before they give an answer. Some of these negative issues had a negative impact on the work with the SRP in the test class. These will be described in section 8.4.

2.2 The didactic transposition

By the word *mathematics* different things can be referred to. One thing is the mathematical theories, theorems, and results, that has developed over time, from the old Egyptians to the ancient Greeks to Newton and Leibniz to Cauchy etc. to all the new mathematical results that are produced and published on a daily basis nowadays; the mathematical theories, theorems, and results, new as well as old, that mathematicians in general agree on across countries.

Nowadays new mathematical results are mostly produced by professional mathematicians and the "official" mathematics described above lives mostly in a scholarly environment. Henceforth, the term *scholarly knowledge* will be used, when referring to *mathematics* as described above; the results, theorems, and theories that are generally agreed on by mathematicians.

Another meaning that can be attached to the word *mathematics* is the teaching subject *mathematics*. What this term precisely contain will then depend on the context, e.g. if it is a subject in primary school or high school and whether the high school is general, business oriented or technical oriented. In didactics the "school mathematics" is divided into two, the first one is the official description of the teaching subject that contains curricula, textbooks etc. This part is called *knowledge to be taught*. The other part is the mathematics as it is actually taught in the class, which is called *taught knowledge*. The difference between the terms and their interrelation will be elaborated on below.

The mathematics contained in curricula, textbooks, and written exams are not transferred directly from the scholarly community into a high school context without modifications. In the process of constructing these official documents a lot of choices have been made. A selection of the most important topics has to be picked out from the huge collection of all the mathematics that has ever been discovered. Within the chosen topics, the most important, useful, and suitable notions have to be picked out. Additionally, a selection of problems and solving methods within each topic is made. All these decisions and more are made as a part of the process of transforming the scholarly mathematics into the teaching subject of mathematics that is both accessible and useful for the students. The process is very complicated and involves a lot of different people with different agendas and professional backgrounds. The process where the mathematical notions are modelled to suit a high school context can be hard to describe, since is it very complex. However, the words *transforming* and *transposing* are sometimes used. Neither of the words are satisfactory as a description in their own right, but together they do almost capture both the "movement" from a scholarly context to a high school context *and* the process of changing, fitting and modelling the mathematical content. Regardless of the words used to describe it, the process of transforming and transposing the scholarly mathematical knowledge into the teaching subject is very important in didactic research (Chevallard & Bosch, 2014).

Likewise, is the process where the curriculum and other official documents, e.g. written exams etc., are interpreted by the teacher and transformed into concrete teaching sequences. These two – the process of constructing a teaching subject from some scholarly knowledge and the process of designing concrete teaching sequences from this, respectively – constitute *the didactic transposition*. Sometimes the model includes an additional process, namely the path that the knowledge undergoes on the way from the teaching situation into the student's catalogue of available knowledge. This process is interesting, since the goal for most teaching situations is that students "learn something". "To learn something" is very vaguely stated and difficult to grasp, but the reason for this is, that it is a hard task to define when somebody has "learned something". To characterise some knowledge as "learned" or "available" different aspects have to be taken into account. Students need to be familiar with both the objects (definitions, properties, and theorems) but also the "tool aspect" of the knowledge (how to use the mathematical concepts in exercises etc.). Furthermore, the students need to be able to use their knowledge without specific guidelines on how to do it, and they have to be able to blend new knowledge into their old knowledge (Robert, 2012).

The didactic transposition can be depicted like in Figure 1.

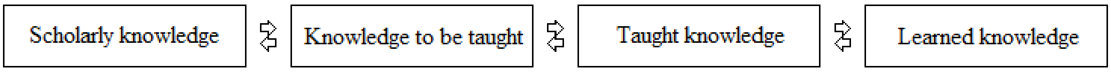


Figure 1 – The didactic transposition, simple version of illustration used in (Chevallard & Bosch, 2014, p. 171)

The first step describes the process of selecting from, transforming and transposing the *scholarly knowledge* into the *knowledge to be taught*. In this step curricula, guiding written exam problems, and textbooks are produced. Different people contributes to this process, both scientist, teachers and those working with the design and production of curricula. This group of people is called the *noosphere*. In other words, the noosphere is the collection of people who “’think about teaching” and participate in the process of picking out the mathematical subjects important and relevant to pupils or students at a given level of education and the transformation of the scholarly knowledge into a degree of difficulty that suits the target group (Chevallard & Bosch, 2014, p. 170).

The next step in the didactic transposition is made by the teachers and takes the *knowledge to be taught* to the *taught knowledge*. The process comprises an interpretation of the official documents and choices on the teaching material and teaching designs. It is the process of organising and carrying out the teaching sessions through which the students (hopefully) achieve the goals described in the curriculum. Though the starting point – the *knowledge to be taught* – is the same for everyone in the same context of education, the *taught knowledge* varies depending on the teacher and to some extend the students. It depends on the focus in the chosen textbooks, the students’ abilities, etc.

The last step is a process that takes place in the students or groups of students and concerns the transposition from the *taught knowledge* to the *learned knowledge*. It describes how the students receive the taught knowledge and to what extend the knowledge is available to them afterwards.

The arrows go both left and right in the diagram. The reason for this is that students and teacher may influence the production of the curriculum. An example of the mutual influence is the *modern mathematics* that was implemented in the curriculum of mathematics in a lot of European education systems in 1970’s and 1980’s. The *modern mathematics* renewed and reformed mathematics into a more abstract and formal discipline. The main reason for this was to reduce the gap between university mathematics and high school mathematics. A problem was, that the *modern mathematics* caused both students and teachers a lot of trouble. As a consequence, mathematics was changed gradually during the following years. This time from an abstract to a more concrete and application-oriented approach.

The didactic transposition can be divided into two – the *external* and the *internal* didactic transposition. The external didactic transposition is a name for the first step in Figure 1. It is the transformation from the *scholarly mathematics* to the *school subject mathematics*. The internal didactic transposition is the second step in Figure 1. It describes the adjustments made by the teacher, when the mathematical knowledge described in the curriculum is organised in a way that fits the respective group of students.

This division shows both the division in the people involved in the two processes, but also that the interface between the players in the two processes is often very small. The teachers do most often not focus on *how* the scholarly knowledge is selected, transformed and transposed into the curriculum. This is in spite of the fact that the creation and organisation of curricula are often influenced by tradition and practical reasons more than didactical or intrinsic mathematical reasons (Winsløw, 2006, p. 19).

The mathematical topic of interest in this thesis is *vectors*. Both the external and internal didactic transposition of this topic will be treated. Roughly, the external didactic transposition of vectors will be covered by section 3, while the internal didactic transposition of vectors will be covered in the second (the empirical part) of the thesis.

The next subsection will describe the framework in which the mathematical organisation of the knowledge on vectors will be analysed; praxeologies.

2.3 Praxeologies

In ATD "mathematics is seen as a human activity of *study of types of problems*." (Barbé, Bosch, Espinoza & Gascón, 2005, p. 236). The first step in mathematics education research is to construct a model of the mathematical activities that includes both the practical and theory-based ones. The notion of *praxeologies* provides a model for these activities in the framework of ATD. The aim for a praxeological description of knowledge is to combine practical and theoretical aspects in *one* model. The word *praxeology* is built from the words 'praxis' and 'logos'. Here 'praxis' is referring to the "know-how" relating to the subject and 'logos' is the theoretical thinking and reasoning behind and the discourse on it.

A praxeology is composed of two blocks – a praxis block and a theoretical block – and each block is again divided into two parts.

The praxis block contains information about the practical part of a subject. This information is divided into "types of tasks" and "techniques". "Types of tasks" are the different kinds of problems and the "techniques" are the "ways of solving" the problems or in a broader sense "ways of doing" (see Figure 2 (a)).

The 'logos'/theoretical block gives justification to and theoretical description of the praxis block. The two parts are "technology", which is discourse on the techniques in the praxis block, and "theory" that is the theoretical foundation of the "technology" part of the block (see Figure 2 (b)).

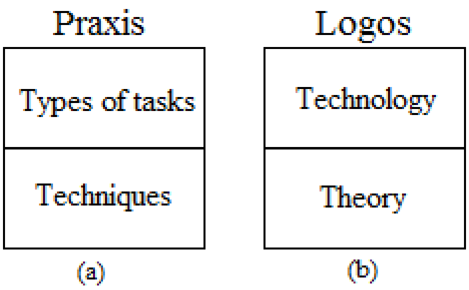


Figure 2 – Structure of a praxeology

The interrelation between and the mutual dependence of the two blocks is described by Chevallard in the following: “[…] no human action can exist without being, at least partially, “explained”, made “intelligible”, “justified”, “accounted for”, in whatever style of “reasoning” such an explanation or justification may be cast. *Praxis* thus entails *logos*, which, in turn, backs up *praxis*” (as cited in Bosch & Gascón, 2014, p. 68).

Praxeologies are useful to model mathematical activities in particular, but they can also be used to model other activities, e.g. the didactical activity of creating and describing a mathematical praxeology. Especially the mathematical praxeologies are useful for this thesis, and they will be described more thoroughly in the following.

2.3.1 Mathematical praxeologies

A mathematical praxeology is, like any other praxeology, divided into the four T’s: “types of tasks”, “techniques”, “technology” and “theory”.

The “types of tasks” are denoted with T and the techniques are denoted with τ. Together they constitute the praxis block. The logos block is composed of the technology part, which provides a discourse of the techniques τ. This is denoted with θ. The technology θ is justified by the theory in the logos block. The theory is denoted with Θ. The praxeology can now be written in a very compact way: $[T, \tau, \theta, \Theta]$ (see Figure 3).

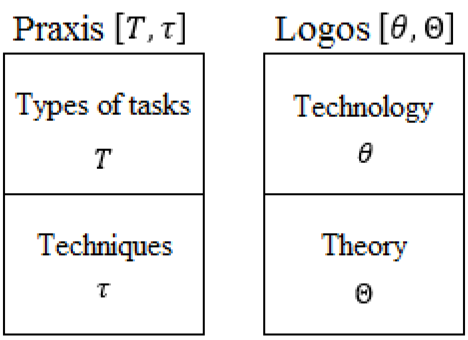


Figure 3 - Structure of mathematical praxeology

The praxis block is determined by the techniques, i.e. the types of tasks T in a praxis block is determined by the technique τ. For example the task $t \in T$ (shown in Figure 4) can be solved by the three different techniques, τ^A, τ^G, and τ^C (also shown in Figure 4).

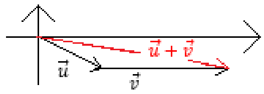
t: For $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ determine $\vec{u} + \vec{v}$	
τ^A: Algebraic technique	$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \Rightarrow \vec{u} + \vec{v} = \begin{pmatrix} 2+4 \\ -1+0 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$
τ^G: Geometric technique	
τ^C: Computer assisted technique	$u:=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $v:=\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ $u+v$ $\quad \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

Figure 4 – Task, t, and different techniques,τ^A, τ^G, and τ^C

The task t is contained in three different praxis blocks, $[T^A, \tau^A]$, $[T^G, \tau^G]$, and $[T^C, \tau^C]$. Each of these contains other tasks that can be solved by the three techniques respectively. For example both $[T^A, \tau^A]$ and $[T^C, \tau^C]$ contains the task shown in Figure 5.

t_1: For $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ determine $\vec{u} + \vec{v}$	
τ^A: Algebraic technique	$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{pmatrix}$, and $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \vec{v} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow$ $\vec{u} + \vec{v} = \begin{pmatrix} 1+4 \\ 2+3 \\ 3+2 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}$
τ^C: Computer assisted technique	$u:=\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $v:=\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ $u+v$ $\quad \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$

Figure 5 – Task $t_1 \in [T, \tau^A], [T, \tau^C]$ but $\notin [T, \tau^G]$

Also the task shown in Figure 6 is contained in $[T^A, \tau^A]$ and $[T^C, \tau^C]$.

t_2: For $\vec{u} = \binom{u_1}{u_2}$ and $\vec{v} = \binom{v_1}{v_2}$ determine $\vec{u} + \vec{v}$	
τ^A: Algebraic technique	$\vec{u} = \binom{u_1}{u_2}, \vec{v} = \binom{v_1}{v_2} \Rightarrow \vec{u} + \vec{v} = \binom{u_1 + v_1}{u_2 + v_2}$
τ^C: Computer assisted technique	$u := \begin{bmatrix} u1 \\ u2 \end{bmatrix}$ → $\begin{bmatrix} u1 \\ u2 \end{bmatrix}$ $v := \begin{bmatrix} v1 \\ v2 \end{bmatrix}$ → $\begin{bmatrix} v1 \\ v2 \end{bmatrix}$ $u+v$ → $\begin{bmatrix} u1+v1 \\ u2+v2 \end{bmatrix}$

Figure 6 – Task $t_2 \in [T, \tau^A], [T, \tau^C]$ but $\notin [T, \tau^G]$

Neither t_1 nor t_2 are contained in the praxis block $[T, \tau^G]$, since neither four dimensional vectors nor abstractly given vectors can be drawn in a coordinate system. On the other hand, the task shown in Figure 7 is contained in $[T, \tau^G]$ but neither $[T, \tau^A]$ nor $[T, \tau^C]$.

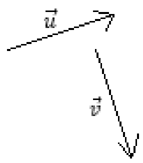
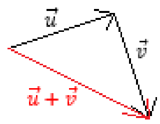
t_3: Given geometrical vectors $\vec{u}$ and $\vec{v}$ determine $\vec{u} + \vec{v}$  	
τ^G: Geometric technique	

Figure 7 – Task $t_3 \in [T, \tau^G]$ but $\notin [T, \tau^A], [T, \tau^C]$

The task t_3 is neither contained in $[T, \tau^A]$ nor $[T, \tau^C]$ since the algebraic and computer assisted techniques cannot handle geometric vectors independent of coordinate systems.

A praxeology is sometimes called a *mathematical organisation* (henceforth abbreviated as MO). Different praxeologies that contains the same type of task can be collected, and the collection will be called a *punctual* MO. All the praxeologies in a punctual MO have the same technique. If different punctual MO's are described by the same technology, they can be collected, and the collection is called a *local* MO. If again some local MO's share the theoretical discourse, the can be collected, and the collection is called a *regional* MO.

A praxeological analysis of the mathematical topic *vectors* will be made in section 3 and presented as the ERM in section 4.

The next section will describe the design/analysis model Study and Research Path, that will be used in the empirical part of this thesis.

2.4 Design model: Study and Research Paths

ATD provides a model for designing and analysing teaching that is called *study and research paths* (henceforth abbreviated as SRP). The model is, like the whole theory itself, introduced by Chevallard and the central point in the model is to focus on the degree of autonomy with which the students work. This is due to the assumption most didacticians agree on: that learning is not a process of transferring knowledge from teacher to student but instead the process of constructing knowledge – a process taking place within the individual student (Winsløw, 2006, p. 105).

An important feature of the model is, that it differentiates between the process of "study" and the process of "research". The "study" part is referring to the process of consulting and investigating already existing knowledge. This relates to the idea and assumption in ATD that teaching is taking place in an institutional setting, where a lot of knowledge is already provided to the students through books, the Internet etc. The "research" part is, on the other hand, problem solving activities, where the students work with exploration of challenging problems.

Studies show that "study" is often given a low priority compared to the "research" part, and that students in research activities most often work with problems raised by the teacher (Winsløw, Matheron & Mercier, 2013). As a design model, SRP proposes that more focus is laid on the "study" part, since this encourages students to work with a higher level of autonomy than they normally do, when they work in the "research" phase with questions posed by the teacher.

As an analysis tool, the model can be used in any kind of teaching situation where the students work with questions posed by either themselves or (as is most common) the teacher. The notion *SRP* then refers to the paths that students follow, when they work with the questions through study and research.

When using the model as a designing tool the outcome are teaching sequences that are called SRPs. The purpose of the SRPs is to encourage the students to work autonomously with questions posed by themselves instead of the teacher. These SRPs are motivated or generated by a question that is called the *generating question*. To decide whether a question is qualified as a generating question, the teacher must conduct an a priori analysis of it. During this analysis, the teacher puts herself in the place of the students, and tries to figure out how they would work with the question. The a priori analysis includes an analysis of the media that can help the students to answer the question, and in the end a description of the path of sub-questions, derived questions, partial and final answers the students are expected to pose. The a priori analysis can also be used to refine the generating question and to estimate the possible learning outcome.

A SRP (here in the meaning of a teaching sequence) can be used for different purposes and have different learning goals for the students. It can be conducted in an interdisciplinary setting or in a monodisciplinary setting. Often generating questions in interdisciplinary SRPs are more open while generating questions in monodisciplinary SRPs are more targeted.

The advantages of an interdisciplinary SRP with an open generating question are far-reaching. The question can seem more relevant, realistic and motivating for the students to work with and it will potentially combine different disciplines in a way that reflects how "real" scientific work is conducted. On the other hand, the predominant disadvantage is that open SRPs demand a huge overview from the involved teacher/teachers (often the interdisciplinary reach out of the field that a single teacher master, and therefore more than one teacher have to be involved), but it can also be very hard to predict the directions for the students' paths.

On the other hand, the more targeted, and potentially monodisciplinary, SRPs are more useful in everyday teaching, since they can lead the students towards the knowledge that is prescribed in the curriculum. Though the generating questions for targeted SRPs are more focused they can still be large such that they will call for derived questions and partial answers. This will potentially show off the interrelation and connection between the topics that often appear separated or disconnected when they are taught in the classic way, where topics are presented neatly in a row, one after the other.

When students work on a SRP, from a generating question Q, they make one or more of the following moves (Winsløw et al., 2013):

1. The activity of "study", where already existing knowledge is examined. This can be any "official" knowledge including books, the Internet and the knowledge that is available from topics that have been studied prior to the SRP.
2. The activity of "research", where the students create answers to the generating question through their own reasoning. It is also the move where possible answers are justified, also through reasoning.
3. Derivation of new questions, that comes in two categories:
    a. Sub-questions, that give partial answers to Q. These are denoted Q_1, Q_2 etc.
    b. Derived questions, that can either be motivated by the original question or by answers to the original question. The derived questions are not directly related to the original question in the sense that an answer to a derived question does not contribute to the answer to Q. Derived questions are denoted Q^*.

Though the three moves are distinguishable, they are very rarely made separately, but are closely linked. For example, the justification of an answer found in the study move has to be justified by some sort of critical reasoning, while possible answers often will be posed in the "research" move after consulting existing knowledge in the "study" move. New questions will most likely evolve from some sort of answers, e.g. answers to the partial questions.

In the empirical part of this thesis the design format SRP is crucial. First of all, it has been used in order to design a teaching sequence. This sequence has been tested empirically in a test class, and the data that came out will be analysed in the light of the analysis model SRP and the praxeological analysis of the mathematical organisation of vectors, that will be carried out in the next section.

3. Vectors

In this section the praxeological organisation of the mathematical topic *vectors* will be investigated. This includes both the scholarly knowledge, the knowledge to be taught, and the interrelation between these two. Furthermore, the notion of vectors used in physics will be investigated and included in this praxeological analysis in order to be able to design an interdisciplinary SRP on vectors in mathematics and physics.

The praxeological organisation found and described in this section will constitute the foundation of the analyses in the empirical part of the thesis. In order to guide the investigations of the praxeological organisation of vectors in mathematics and physics and in order to address the issues of lacking cooperation between mathematics and physics in high school teaching, some research questions have been posed. These are presented in section 3.1.

3.1 Research Questions (I)

As it was briefly mentioned in the introduction one of the issues that has motivated this thesis is the interrelation of mathematics and physics, that is fruitfully practiced in the scientific fields but almost never practiced successfully in high school teaching. Behind this issue hides a complex structure of interrelations (see Figure 8). Here it is depicted with the origin in the notion of vectors. The structure contains the relations between the scientific fields of mathematics and physics, the high school subjects mathematics and physics, the scientific field of mathematics and the high school subject mathematics, and the scientific field of physics and the high school subject physics.

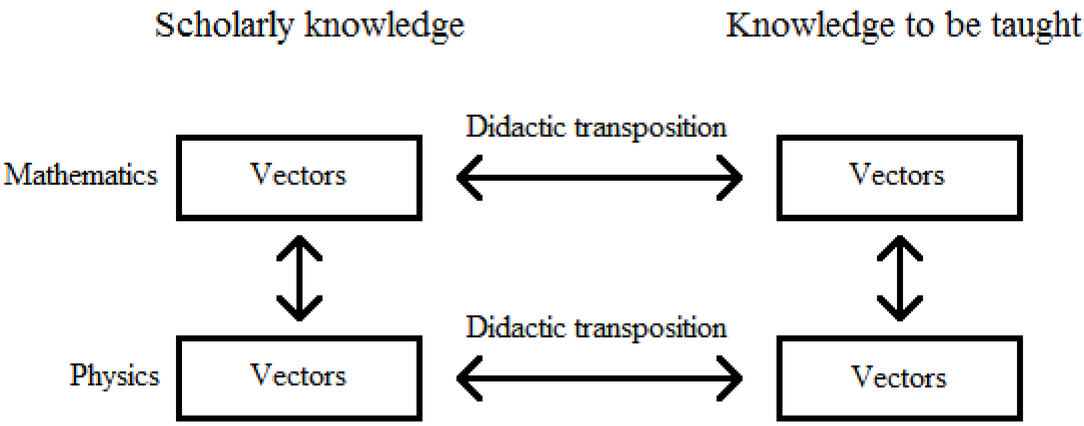


Figure 8 – Four relations between scientific fields and school subjects (here regarding the topic vectors*)*

In addition to the four arrows in Figure 8, the development over time plays a role, since neither the scientific fields nor the high school subjects have been static over time.

An analysis of the relations and their development over time, focusing on the organisation of vectors, will be the theoretical framework for the SRP-design in the empirical part of the thesis.

In order for the analysis of the relations to be useful, some research questions have been posed. These will serve the purpose of guiding the examination of the relations described above. The questions are categorised in two categories. The first category deals with the scholarly knowledge in mathematics and physics respectively and their interrelated development. The second category deals with the knowledge to be taught in mathematics and physics respectively and the historical development in mathematics and its relation to physics. The research questions in the two categories are presented in Table 1.

Research questions 1	Research questions 2
$RQ^I_{1,1}$: How has the mathematical scholarly knowledge on vectors developed in relation to physics?	$RQ^I_{2,1}$: How has the knowledge to be taught on vectors in mathematics developed in relation to the mathematical scholarly knowledge?
$RQ^I_{1,2}$: How is the scholarly knowledge on vectors organised in mathematics?	$RQ^I_{2,2}$: How has the knowledge to be taught on vectors in mathematics developed in relation to the knowledge to be taught in physics?
$RQ^I_{1,3}$: How is the scholarly knowledge on vectors organised in physics?	$RQ^I_{2,3}$: How is the knowledge to be taught on vectors currently organised in mathematics?
	$RQ^I_{2,4}$: How is the knowledge to be taught on vectors currently organised in physics?

Table 1 – Theoretical research questions

The research questions with the primary lower index number 1 will be treated in section 3.2 and the research question with the primary lower index number 2 will be treated in section 3.3. The research questions have the upper index number I, which is there to distinguish them from the research questions that will be posed in the empirical part of the thesis. These research questions will have the upper index number II.

3.2 The scholarly knowledge on vectors

This section will give answers to the research questions $RQ^I_{1,1}$, $RQ^I_{1,2}$, $RQ^I_{1,3}$. Section 3.2.1 will deal with $RQ^I_{1,1}$, and it will focus on how a theory of vector analysis developed on the border of mathematics and physics. Another focus is the interplay between the geometric and the algebraic approaches.

Section 3.2.2 will deal with $RQ^I_{1,2}$ focusing on the difference between $\mathbb{R}^2/\mathbb{R}^3$ and arbitrary abstract vector spaces. Furthermore, a few physical applications of vector spaces different from $\mathbb{R}^2/\mathbb{R}^3$ will be mentioned.

Section 3.2.3 will deal with $RQ^I_{1,3}$ and describe how the theory of vector analysis is used in physics. Again the interplay between the geometric and the algebraic approaches to the theory of vectors plays an important role.

3.2.1 Historical perspective

The notion of vectors has a long story that has played out on the border of mathematics and physics. Some of the important contributions will be described in the following.

The parallelogram of forces

In physics, the need for a theory on vectors emerged in the seventeenth century from an increasing interest in new physical quantities such as *force* and *velocity* (Crowe, 1967, p. 1). These quantities are what we nowadays call *vector quantities*. Beforehand, the notions of main interest were, what we now call scalar quantities, such as *mass* and *distance*. The "new" quantities differ from the old ones by having both magnitude *and* direction. As early as in the ancient Greece, velocities were composed by the use of the "parallelogram of velocities" (see Figure 9).

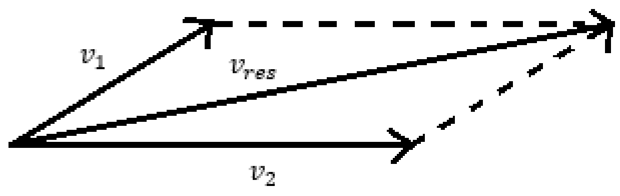


Figure 9 – The parallelogram of velocities. v_1 and v_2 being the components and v_{res} being the resulting velocity

In the seventeenth century the parallelogram was very common in publications as a method for composing (or adding) vector quantities. However, this did not mean that everyone at that time was aware of the vector theory that was hiding behind it and it is very unlikely, that the idea of the parallelogram of velocities or forces stimulated further works on vectors. Though the idea did not directly entail any results, it had an important influence, since it is an obvious example that vector theory can be used to model physical tasks (Crowe, 1967, p. 2). In this period, the "vector property" of interest was primarily addition, and the fact that the sum of two vectors were again a vector. Furthermore, it is remarkable that the approach at this time was entirely geometric.

In this initial stage, vector addition appeared as a technique in the praxis block of the praxeological organisation of the physical knowledge on composition of vector quantities (see Figure 10).

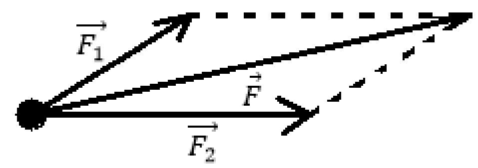
T: What is the resulting force, F on a particle acted on by the two forces F_1 and F_2?	
τ^{PoF}: "The parallelogram of forces"	

Figure 10 – Praxis block including geometric vectors in the technique

The theoretical block consisted of empirical results showing that velocities/forces was composed by this rule.

Leibniz and a geometry of situation

Another important contribution to the development of a theory on vectors came from the German mathematician and physicist Gottfried Wilhelm Leibniz (1646-1716), who concerned himself with the problem of constructing a *geometry of situation*. The idea was to create a system that would make spatial analysis easier and more direct. He was looking for a mathematical system that would do for "situation" (as he wrote) what algebra did for magnitude (Crowe, 1967, p. 3). Leibniz' attempt to define vector-like objects was based on congruence of sets of points. The basic idea was to identify sets of points having some fixed distance to each other. Some of the geometric objects, that Leibniz was using his new premature vectors to operate on, were planes, lines and spheres. Leibniz can be said to have constructed a system in which coordinates plays an important role, and the pioneering idea was that geometric entities were represented by symbols. From these symbols calculations should be carried out algebraically. Though the idea was great, Leibniz' system had some flaws, when it is compared to the modern system of vectors. Leibniz' objects could neither be added nor subtracted nor multiplied, and these are important properties for the system to be useful. Though Leibniz did not manage to accomplish this project, his ideas motivated and inspired other mathematicians to work on similar ideas (Crowe, 1967, pp. 4-5), which is why his attempt deserves to be mentioned here.

In terms of praxeologies, Leibniz searched for techniques to solve geometric tasks. The technology and theory were supposed to build on the new objects that he wanted to develop. However, his theory, and thereby technology, did not provide efficient techniques for all the relevant problems (e.g. addition, subtraction, and multiplication).

Geometrical representation of complex numbers: Wessel and Gauss

Most of the following work on vectors was related to complex numbers and the justification and representation of these (Crowe, 1967, p. 5). A lot of mathematicians worked on this matter, among others Caspar Wessel (1745-1818), a Norwegian/Danish mathematician, and the German mathematician/astronomer/physicist/geodesist Carl Friedrich Gauss (1777-1855). Wessel and Gauss did both discover a geometric representation of complex numbers around the millennial change, but Wessel's publication was not noticed until its republication in French 100 years later. From Wessel's memoir it appears that the question he was working with was the following (as cited in Crowe, 1967, p. 6):

> How may we represent direction analytically; that is, how shall we express right lines so that in a single equation involving one unknown line and others known, both the length and the direction of the unknown line may be expressed.

Wessel dealt, among other things, with the addition of straight lines. He stated the following (as cited in Crowe, 1967, p. 7):

> Two straight lines are added if we unite them in such a way that the second line begins where the first one ends, and then pass a right line from the first to the last point of the united lines. This line is the sum of the united lines.

What Wessel provides here, is a technique for solving tasks as shown in Figure 11.

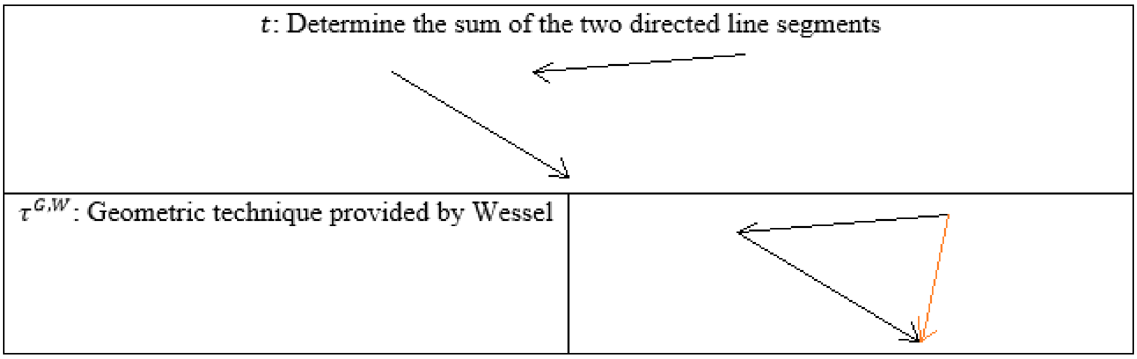


Figure 11 – Task solvable by the technique $\tau^{G,W}$

As mentioned, Gauss worked on the justification and representation of complex numbers just like Wessel did. By virtue of his well-established name in the mathematical community, Gauss managed, contrary to Wessel, to publish and spread his idea in 1831. Gauss's main focus was on the chase for a concept similar to the geometric representation of complex numbers, that could be used to describe and work with geometry in three dimensions (Crowe, 1967, pp. 8-9).

Hamilton and quaternions

Different mathematicians searched for the mathematical entities that could represent "higher dimensional complex numbers", but one of the most successful and earliest attempts was provided by the Irish mathematician and physicist William Rowan Hamilton. He was looking for numbers (in the beginning triplets, corresponding to the tuples that described ordinary complex numbers) that he hoped had some specific properties. The properties that he looked for are described in the following (Crowe, 1967, p. 28):

> 1. The associative property for addition and multiplication. Thus if N, N' and N'' are three such numbers, then $N + (N' + N'') = (N + N') + N''$ and $N(N'N'') = (NN')N''$.
>
> 2. The commutative property for addition and multiplication. $N + N' = N' + N$ and $NN' = N'N$.
>
> 3. The distributive property. $N(N' + N'') = NN' + NN''$.
>
> 4. The property that division is unambiguous. Thus if N and N' are any given complex numbers, it is always possible to find one and only one number X (in general, a number of the same form as N and N') such that $NX = N'$.
>
> 5. The property that the new numbers obey the law of the moduli. Thus if any three triplets combine so that
>
> $$(a_1 + b_1 i + c_1 j)(a_2 + b_2 i + c_2 j) = a_3 + b_3 i + c_3 j$$
>
> then
>
> $$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = (a_3^2 + b_3^2 + c_3^2)$$
>
> 6. The property that the new numbers would have a significant interpretation in terms of three dimensional space.

These properties are all satisfied by "two-dimensional" complex numbers except from the last one. Instead it satisfies the corresponding property, that is has significant interpretation in terms of *two* dimensional space. The above can be read as a first sketch of the axioms that Hamilton thought that a system of vector analysis should obey. Instead of a system of triplets, Hamilton discovered the quaternions, that are elements of the form $w + ix + jy + kz$. This is a system of quadruples that obeys all the above properties except from the commutativity of the multiplication. Hamilton named the real part, w, the scalar of the quaternion and the imaginary part, $ix + jy + kz$, the vector of the quaternion. He denoted a quaternion Q in the following way: $Q = SQ + VQ$, read as *the quaternion equals the sum of the scalar part of the quaternion (SQ) and the vector part of the quaternion (VQ).* Hamilton demonstrated the use of the symbols in an example: If two quaternions are given, $\alpha = xi + yj + zk$ and $\alpha' = x'i + y'j + z'k$ (both scalar parts are 0), then $S.\alpha\alpha' = (xx' + yy' + zz')$ and $V.\alpha\alpha' = i(yz' - zy') + j(zx' - xz') + k(xy' - yx')$. These two parts correspond to the negative of the modern scalar product and the modern vector product respectively (Crowe, 1967, p. 32).

Compared to the scalar product and vector product in modern vector theory Hamilton's quaternions are simpler, because for the modern scalar product, the associative law for multiplication is not relevant, since $\vec{a} \cdot \vec{b} \cdot \vec{c}$ does not make any sense and neither the fourth nor the fifth axiom are satisfied by the modern vectors. Regarding the modern vector product, both the associative and the commutative properties, and again the fourth and fifth axiom are not satisfied. Though the quaternions are simpler (in the sense that they satisfy more of the "wanted" properties) than modern vectors, they are also less innovative. In Hamilton's work, the axioms that the vector analysis was wanted to obey were the most important guidelines. In this period vectors were mostly represented algebraically instead of geometrically, though they were used for geometric purposes.

Grassmann

Simultaneously with Hamilton the German mathematician and physicist Hermann Günther Grassmann (1809-1877) developed another system of vector analysis. Though the Grassmannian vector analysis has major similarities with the modern vector analysis, and he demonstrated its usefulness in physical applications, his work was not spread and appreciated by his contemporaries. This is mostly due to the fact that he did not have a name in the mathematical community back then, but also because his principal work on vector analysis had a very complex, abstract, and philosophical structure, which made it difficult to read, even for mathematicians.

Grassmann's ideas of vector analysis was briefly introduced in the essay *Theorie der Ebbe und Flut* that he wrote as a part of his application for a position as a teacher at the University of Berlin in 1840. Four years later, in 1844, the ideas were elaborated and published in *Ausdehnungslehren*.

From the preface of *Ausdehnungslehren* it is revealed how the inspiration to the theory comes from geometric considerations (as cited in Crowe, 1967, p. 56):

> The first impulse came from the consideration of negatives in geometry; I was accustomed to viewing the distances AB and BA as opposite magnitudes.

This shows that though Grassmann's ideas initially showed up in a more physical work on tides, his ideas were originally purely geometric in nature.

The most fundamental operation in Grassmann's system of vector analysis is addition. The sequence of thoughts that led to the definition of addition is described in the preface of *Ausdehnungslehren* in the following way (as cited in Crowe, 1967, pp. 56):

> Arising from this idea was the conclusion that if A, B, C are points of a straight line, then in all cases $AB + BC = AC$, this being true whether AB and BC are directed in the same direction or in opposite directions (where C lies between A and B). In the latter case AB and BC were not viewed as merely lengths, but simultaneously their directions were considered since they were oppositely directed. Thus dawned the distinction between the sum of lengths and the sum of distances which were fixed in direction.

From this idea addition is defined similarly for distances that are not necessarily directed in the same or opposite directions.

Also multiplication of vectors is dealt with by Grassmann. Like in the modern vector analysis his system contains two different products. To Grassmann the geometric product of two vectors (similar to the modern vector product) is the most important compared to the linear product of two vectors (similar to the modern scalar product). The geometric product is defined in the following way (as cited in Crowe, 1967, p. 61):

> By the *geometrical product of two vectors*, we mean the surface content of the parallelogram determined by these vectors; we however fix the position of the plane in which the parallelogram lies. We refer to two surface areas as geometrically equal only when they are equal in content and lie in parallel planes.

This product is similar to the modern vector product in a couple of ways, but is does also have one important difference. The numerical value of the two products are the same, and they will also have the same sign in both Grassmann's and the modern vector analysis. Furthermore, they are both distributive and anti-commutative. The difference between the two is the nature of the product. In the modern vector product, the result is again a vector, but the result of Grassmann's geometric product is a directed area (Crowe, 1967, p. 62).

The product that corresponds to the modern scalar product in the Grassmannian theory is called *the linear product*. It is defined in the following way (as cited in Crowe, 1967, p. 63):

> By the *linear product* of two vectors we mean the algebraic product of one vector multiplied by the perpendicular projection of the second onto it.

These two products provide some techniques that are similar to modern ones. However, the logos block of the praxeological organisation of the modern vector product is different form the logos block of Grassmann's geometric product. The praxeological organisations of the modern scalar product and Grassmann's linear product are very similar.

As mentioned Grassmann's ideas were initially presented in a physical application in connection with his study of tides. This might have affected the process of development that the vector analysis went through, though Grassmann's ideas were initially mathematically motivated (Crowe, 1967, p. 60). In the preface to *Ausdehnungslehren* Grassmann described how he had managed to carry out the calculation in Lagrange's publication *Méchanique analytique* (a work that he had studied in connection with his own *Theorie der Ebbe und Flut*) ten times shorter with his new analysis. He stated the following about the usefulness of his theory (as cited in Crowe, 1967, p. 57):

> […] I feel entitled to hope that I have found in this new analysis the only natural method according to which mathematics should be applied to nature, and according to which geometry may also be treated, whenever it leads to general and to fruitful results.

Grassmann provided new techniques to solve some of the tasks that had been solved by other, and more extensive, methods before.

Though Grassmann was influenced by both physics and geometry, his ideas were fundamentally different from both the "parallelogram of forces"-tradition and the "geometrical justification of complex numbers"-tradition, since he did work *conceptually* on addition of lines, and not just taking a geometrically determined line (the diagonal) as a representative of the resultant of two forces or representing the sum of two complex numbers as a line respectively (Crowe, 1967, p. 58). In Grassmann's theory the notion *vector* described the distances with fixed lengths, and these were the objects of interest, contrary to Hamilton's theory, where the object of interest was the quaternions, that *contained* the notion of *vectors*.

Through his work with physical issues Grassmann did in addition develop vector calculus, and later in his career his system of vector analysis contributed to his work on electrodynamics (Crowe, 1967). These are examples of how Grassmann's work on vectors and physics respectively was highly interrelated.

The modern vector analysis

The work on vector theory in the subsequent period was mostly inspired by Hamilton, though Grassmann's theory was equally well-developed. This was most likely because Hamilton was more established as a mathematician than Grassmann. In the period from 1865 to 1880 different mathematicians worked on Hamilton's ideas, and one of the most important contributions came from the Scottish mathematician Peter Guthrie Tait (1831-

1901) who focused on a development of quaternions as a tool for research in physics (Crowe, 1967, p. 117).

Inspired by Tait's work the Scottish mathematician and physicist James Clerke Maxwell (1831-1879) developed and presented the theory of electricity and magnetism in his publication *Treatise on electricity and magnetism* in 1873, which is one of the most important in physics in the 19th century. Maxwell presented his famous equations that describes electromagnetism, using quaternionic notation, but he claimed that the quaternionic method did not provide a satisfactory system. This opinion was shared by other mathematicians, and from this viewpoint the quaternionic system was improved and the modern system of vector analysis was developed on its foundation (Crowe, 1967, pp. 137-139).

The modern system of vector analysis is ascribed to the American mathematician Josiah Willard Gibbs (1839-1903) and the English mathematician Oliver Heaviside (1850-1925). These two developed two almost identical systems independently which is why they both need to be mentioned. Gibbs introduced his work on the new theory in *Elements of vector analysis*, that was published in two parts in 1881 and 1884 respectively. Gibbs was a professor in mathematical physics at the University of Yale, and prior to the publication of his work, Gibbs had given a course in vector analysis with applications to both electricity and magnetism (Crowe, 1967, pp. 153-154). Gibbs developed his system by extracting essentials from among other Maxwell's theory on quaternions, and he was moreover inspired by Tait. It has also been discussed whether Gibbs had read Grassmann, because his system is very similar to the Grassmannian, but this cannot be known with certainty (Crowe, 1967, pp. 153-154).

The path of Heaviside's development is almost identical to Gibb's, which can also explain why the systems that they presented in the 1880's were so similar. Like Gibbs, Heaviside engaged in the study of electrical theory, and he got acquainted with quaternions and vectors through Maxwell's *Treatise on electricity and magnetism* (Crowe, 1967, pp. 160-162). The two very similar versions of vector analysis make up the Gibbs/Heaviside system.

By the middle of the 1880's "vector analysis" was divided into two different approaches. The first one was the quaternionic system introduced by Hamilton, and the other was the Gibbs/Heaviside vector system, that had emerged from a mix of Grassmannian and Hamiltonian ideas. Pioneers from each of the two systems, vectors and quaternion, fought for the diffusion of the respective approaches.

In 1910 the Gibbs/Heaviside vector system was dominating, which is why that version is used today. All the work presented above was mostly dealing with Euclidean vectors, meaning vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. The theory was generalised and by the end of the nineteenth century the modern definition of an abstract vector space was given by the Italian mathematician Guiseppe Peano (1838-1932), but the full development of the concept from an axiomatic approach was not made until the twentieth century (Katz, 2009, p. 865).

The praxeological view on the development has shown how vectors have been included in different organisations. Some of them have been more physical and some of them have been more mathematical. However, the outcome is a system of vector analysis for two and three dimensional spaces that is defined with mathematical precision, and is used to model countless physical situations e.g. in mechanics and electrodynamics. Figure 12 shows how the early traditions in the development of vectors are reflected in the modern system of vector analysis.

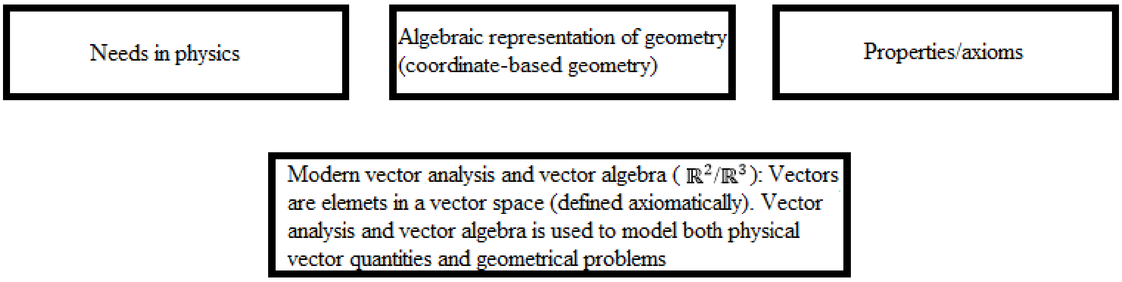


Figure 12 – The relation between the early traditions in the development of vectors and their relation to the modern vector analysis

The development of vectors is, as described in the introduction, only one of the numerous examples of a very close and fruitful interrelation between mathematics and physics. A study of the interrelation between mathematics and physics, and the implications on the teaching and learning of them is given by Constantinos Tzanakis. His study provides three scenarios in which a development of concepts in mathematics and physics can happen. The first of these is defined in the following way (Tzanakis, 2016, p. 4):

> *Parallel development*: The physical problems asking for solution and the formulation of appropriate mathematics (concepts, methods, or theories) evolve in parallel.

This is almost an exact description of what happened during the development of vector analysis.

The didactical implications, that this interrelation should have, according to Tzanakis are the following (Tzanakis, 2016, p. 3):

> […] learning mathematics or physics includes not only the “polished products” of the associated intellectual activity, but also the understanding of implicit motivations, the sense-making actions and the reflective processes of scientists, which aim to the construction of meaning.

This citing provides justification for this thorough review of the development of vectors in relation to the purpose of this thesis.

3.2.2 Definition and applications

The purpose of this section is to answer $RQ_{1,2}^{I}$ about the mathematical organisation of the scholarly knowledge on vectors ($MO_{scholarly}$). The introductory move in the examination of the organisation of the scholarly knowledge will be a description of *the algebraisation of mathematics*, since this have affected the development of the theory of vectors heavily.

The development of mathematical vectors is only one example of the algebraisation of mathematics. The general process has been studied by, among others, the American historian Michael Sean Mahoney (1939-2008). He describes *the algebraisation of mathematics* as the transition from "an old, traditional, geometric mode" to "a new, in many ways revolutionary, algebraic mode." (Mahoney, 1980, p. 1). The algebraic mode is described by three characteristics: (1): "Characterised by the use of an operative symbolism, that is, a symbolism that not only abbreviates the words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates", (2) "Deals with mathematical relations rather than objects. […] The subject of modern algebra is the structures defined by relations […]", and (3) "It is free of ontological commitment. Existence depends on consistent definitions within a given axiom system, and mutually compatible mathematical structures live in peaceful co-existence within mathematics as a whole. In particular, this mode of thought is free of the intuitive ontology of the physical world". Furthermore, it is characterised as an "abstract mode of thought, in contrast to an intuitive one" (Mahoney, 1980, p. 1).

The French mathematicians François Viète (1540-1603) and René Descartes (1596-1650) were two of the pioneers of the algebraisation. One of the findings, that can be ascribed to the algebraisation, is complex numbers. In the beginning algebra was about "relations among quantities" in a new symbolic form, but it changed into being about "relations among other objects of knowledge" (Mahoney, 1980, p. 7).

The algebraisation is relevant to consider in many different mathematical fields, but one of them is the development of vector algebra. Regarding vectors, the algebraisation has two levels. The first level is constituted by the motivation for the development of a vector analysis. Among others Leibniz, Gauss, Hamilton, and Grassmann were motivated by a wish to develop a mathematical model of geometry, containing objects that could be operated on directly. This level corresponds to the two characteristics (1) and (2) in Mahoney's description and it was one of the focus points in the previous section. On the second level the properties of the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are generalised, and the axiomatic definition of an abstract vector space is given. This level corresponds to the characteristic (3) in Mahoney's description, and the process of generalising the theory and the connection between the special cases $\mathbb{R}^2/\mathbb{R}^3$ and a general vector space will be the main subject below.

When Peano gave the definition of an abstract vector space, the study of vectors and their properties changed from concerning only two and three dimensional Euclidean vectors (those that can be represented as tuples or triplets of numbers, or geometrically as arrows) to concerning elements in any space that satisfy the abstract definition. Taking the scalars from a field $\mathcal{F}$, the definition reads as follows (Halmos, 1958, pp. 3-4):

> DEFINITION: A vector space is a set $\mathcal{V}$ of elements called vectors satisfying the following axioms.
>
> (A) To every pair, x and y, of vectors in $\mathcal{V}$ there corresponds a vector $x + y$, called the sum of x and y, in such a way that
>
> (1) addition is commutative, $x + y = y + x$,
>
> (2) addition is associative, $x + (y + z) = (x + y) + z$,
>
> (3) there exists in $\mathcal{V}$ a unique vector 0 (called the origin) such that $x + 0 = x$ for every vector x, and
>
> (4) to every vector x in $\mathcal{V}$ there corresponds a unique vector $-x$ such that $x + (-x) = 0$
>
> (B) To every pair, α and x, where α is a scalar and x is a vector in $\mathcal{V}$, there corresponds a vector αx in $\mathcal{V}$, called the product of α and x, in such a way that
>
> (1) multiplication by scalars is associative, $\alpha(\beta x) = (\alpha\beta)x$, and
>
> (2) $1x = x$ for every vector x.
>
> (C) (1) Multiplication by scalars is distributive with respect to vector addition, $\alpha(x + y) = \alpha x + \alpha y$, and
>
> (2) multiplication by vectors is distributive with respect to scalar addition, $(\alpha + \beta)x = \alpha x + \beta x$.

As described in the previous section, the whole axiomatisation was initially motivated by the study of Euclidean vectors, which obviously satisfies the axioms. These are elements in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively, and can be represented algebraically by tuples, e.g. $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ or $\vec{a} = (a_1, a_2)$ for $\vec{a} \in \mathbb{R}^2$ or $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ or $\vec{b} = (b_1, b_2, b_3)$ for $\vec{b} \in \mathbb{R}^3$. Or they can be represented geometrically by arrows, either independent of a coordinate system (see Figure 13) or in a coordinate system (see Figure 14).

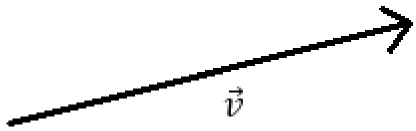


Figure 13 – Geometrically represented vector independent of coordinate system

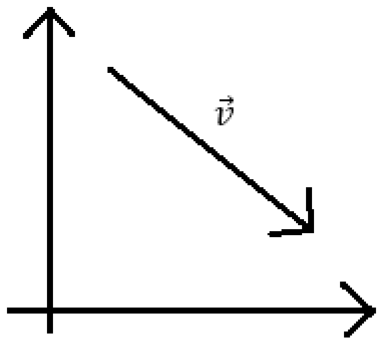


Figure 14 – Geometrically represented vector in coordinate system

Other examples of mathematical vectors are functions in function spaces, that are added pointwise and multiplied by scalars pointwise, or polynomial rings $\mathbb{F}[x]$, where the elements are polynomials:

$$f(x) = r_0 + r_1 x + \cdots + r_{n-1} x^{n-1} + r_n x^n$$

with the $r_0, r_1, \ldots, r_n \in \mathbb{F}$, where $\mathbb{F}$ is a field. The elements in these vector spaces are hardly representable in any other way than the very abstract $f \in \mathbb{F}[x]$ for example.

As it has been described in the previous section, the question of multiplication of vectors was very important. Initially the motivation for the development of a vector analysis was primarily a wish for a system to carry out spatial analysis. Therefore the vector space of main interest was $\mathbb{R}^3$. On $\mathbb{R}^3$ two products can be defined, the scalar product and the vector product. The scalar product of two vectors is, as the name reveals, not a vector but a scalar. The scalar product is defined on every Euclidean space. The vector product of two vectors is again a vector, but because of the geometric property of chirality it can only be defined on the specific Euclidean vector space $\mathbb{R}^3$. In physics the important property of the vector product $\vec{v} \times \vec{w}$ is that it is perpendicular to both $\vec{v}$ and $\vec{w}$. This property is utilised in both rotational mechanics and electromagnetism.

In the theory of vector spaces the scalar product can be generalised, and the generalisation is called an *inner product*. The inner product is defined in the following way (Halmos, 1958, p. 121):

DEFINITION. An inner product in a (real or complex) vector space is a (respectively, real or complex) numerically valued function of the ordered pair of vectors x and y, such that

(1) $$(x, y) = \overline{(y, x)}$$

(2) $$(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$$

(3) $$(x, x) \geq 0; \quad (x, x) = 0 \text{ if and only if } x = 0$$

A vector spaces that has an inner product is called an inner product space. Since the scalar product on $\mathbb{R}^2$ and $\mathbb{R}^3$ satisfy the definition these are both inner product spaces. A well-known feature of Euclidean vectors is that the angle between two of them can be measured, and that the scalar product can be used to calculate it. By Cauchy-Schwarz' inequality the notion of "angle between two vectors" can be generalised to any of the Euclidean spaces, $\mathbb{R}^n$ (Halmos, 1958, p. 126):

(2) In the Euclidean space $\mathbb{R}^n$, the expression

$$\frac{(x, y)}{\|x\| \cdot \|y\|}$$

gives the cosine of the angle between x and y

In general, the inner product is used to define the length of a vector (Halmos, 1958, p. 121):

In an inner product space we shall use the notation

$$\sqrt{(x, x)} = \|x\|;$$

the number $\|x\|$ *is called the* norm *or* length *of the vector* x.

Likewise, for the distance between vectors (Halmos, 1958, p. 125):

(1) In any inner product space we define the distance $\delta(x, y) = \|x - y\| = \sqrt{(x - y, x - y)}$

Since angles and orthogonality are closely related notions, the property that two vectors can be orthogonal is also attached to inner product spaces only.

The generalisation of Euclidean spaces to higher (possibly infinite) dimensions is unified in the theory of Hilbert spaces. The inner product-structure provides a sort of geometric intuition to infinite dimensional vector spaces. Hilbert spaces have important applications in physics, e.g. in the mathematical description of quantum mechanics.

A description of the scholarly mathematical knowledge on vectors has been given. In the view of this description $RQ^{I}_{1,2}$ about the organisation of the scholarly knowledge on vectors in mathematics can now be answered.

Because of the algebraisation the scholarly knowledge on vectors in mathematics is organised around the algebraic properties *addition of vectors* and *multiplication of a vector by a scalar*, which are the defining properties. In this organisation the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are only a diminutive part of the whole theory (see Figure 15). However, the structure of $\mathbb{R}^2/\mathbb{R}^3$, where a product of vectors (the scalar product) can be defined, has been crucial in the generalised theory as well. The inner product structure generalises some of the geometric properties that $\mathbb{R}^2$ and $\mathbb{R}^3$ have. These geometric properties are exactly the properties that make $\mathbb{R}^2/\mathbb{R}^3$ a suitable model for the plane/space. The generalised inner product structure makes it reasonable to talk about geometric properties such as length, angles, and distances in higher dimensions.

In $\mathbb{R}^3$ the vector product is an important property, but since it cannot be generalised it does not have any pronounced position in the organisation of the scholarly mathematical knowledge.

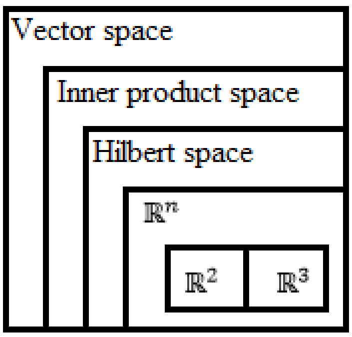


Figure 15 – Nested structure of vector space-properties

3.2.3 Vectors in physics

The purpose of this section is to answer $RQ^{I}_{1,3}$ about the physical organisation of the scholarly knowledge on vectors ($PO_{scholarly}$).

Vectors are used to model physics in a lot of different fields. The initial application of $\mathbb{R}^3$-vector analysis was in mechanics as a model for vector quantities such as velocity, force, acceleration, etc. Later on, when the mathematical concept was further developed, vectors were applied in electromagnetism. In Table 2 some of the important algebraic properties of $\mathbb{R}^3$-vector analysis are listed together with the physical applications of them.

Algebraic property	Physical application	List of symbols
Mechanics		
Scalar multiplication	$\vec{F} = m \cdot \vec{a}$	$\vec{F}$: the force that implies the acceleration $\vec{a}$ on a particle/body with the mass m.
Vector addition	$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y}$ $\vec{a} = \overrightarrow{a_x} + \overrightarrow{a_y}$ $\overrightarrow{F_{res}} = \overrightarrow{F_1} + \overrightarrow{F_2}$	$\vec{v}$: the "resulting" velocity of a particle/body having $\overrightarrow{v_x}$ as its horizontal velocity and $\overrightarrow{v_y}$ as its vertical velocity. $\vec{a}$: the "resulting" acceleration of a particle/body having $\overrightarrow{a_x}$ as its horizontal acceleration and $\overrightarrow{a_y}$ as its vertical acceleration. $\overrightarrow{F_{res}}$: the resulting force on a particle/body acted on by the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$.
Scalar product	$W = \vec{F} \cdot \Delta\vec{s}$	W: the work done by a force $\vec{F}$ on a particle/body over a displacement $\Delta\vec{s}$.
Vector product	$\vec{L} = \vec{r} \times \vec{p}$ $\vec{\tau} = \vec{r} \times \vec{F}$	$\vec{L}$: the angular momentum of a particle with linear momentum $\vec{p}$ rotating around an axis in the distance $\vec{r}$ from it. $\vec{\tau}$: the moment of force on a particle caused by a force $\vec{F}$ and the lever arm vector $\vec{r}$
Electromagnetism		
Vector product and scalar multiplication in combination	$\vec{F} = \vec{B} \times \vec{I} \cdot l$	$\vec{F}$: the magnetic force on a current-carrying wire of length l carrying a current $\vec{I}$ in a magnetic field $\vec{B}$
Vector product, vector addition and scalar multiplication in combination	$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$	$\vec{F}$: the Lorentz force (combined electric and magnetic force) on a point charge q moving with velocity $\vec{v}$ in the presence of an electric field $\vec{E}$ and a magnetic field $\vec{B}$

Table 2 - Some applications of $\mathbb{R}^3$-vectors in physics

The table shows a selection of cases where vectors are used to model physical correlations. It is the geometric properties of $\mathbb{R}^3$-vectors that are crucial in the modelling. For example, the formula $W = \vec{F} \cdot \Delta\vec{s}$ expresses the fact that it is only the component of the force that is parallel to the displacement that contributes to the work. Another example could be the formula $\vec{F} = \vec{B} \times \vec{I} \cdot l$, where the geometric property of chirality of $\mathbb{R}^3$, expressed by vectors, is used to determine the direction of the force $\vec{F}$. As a consequence of the general and abstract definition of a vector space, a mathematical $\mathbb{R}^3$-vector can be represented by infinitely many arrows all having a fixed length and direction. This property is very crucial in mathematics. In physics, on the other hand, the geometric interpretation of this property can be misleading, since the forces F_1 and F_2 on Figure 16 will affect the block in two completely different ways, even though the arrows represent the exact same mathematical vector.

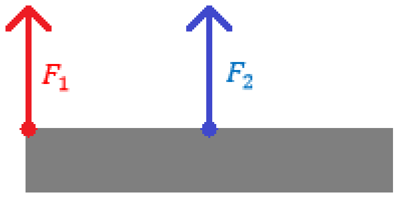


Figure 16 – Two physical forces represented by the same mathematical vector

Another example of how the algebraic properties are less crucial than the geometric consequences they imply in $\mathbb{R}^3$ is the property of commutativity of vector addition. Algebraically this property is described by the following: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$. However, the important implication of this property in physics is, that the red path ($\vec{v}$ followed by $\vec{w}$, written as $\vec{v} + \vec{w}$) and the blue path ($\vec{w}$ followed by $\vec{v}$, written as $\vec{w} + \vec{v}$) on Figure 17 end in the same place.

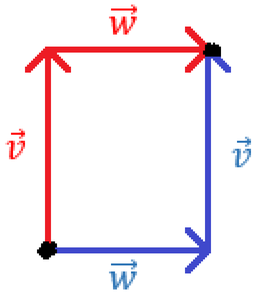


Figure 17 - Physical interpretation of the commutative property of vector addition

As it has been mentioned, the more abstract Hilbert spaces are used in the modelling of quantum mechanics. Since neither Hilbert spaces nor quantum mechanics are a part of the high school physics curriculum nor in the scope of this thesis, the modelling of quantum mechanics with the theory of Hilbert spaces will not be described further.

In physics vectors are found in the technique part of the praxis block in the praxeological organisation. The technology part of the logos block will give a discourse on the model that vector analysis provides. In physics the more theoretical parts of the theory of vector analysis, such as the algebraic property of a vector space etc., are secondary. Instead it is the geometric properties of $\mathbb{R}^3$ that makes it useful in the modelling of physical phenomena such as force, velocity, acceleration, angular momentum, etc.

3.3 The knowledge to be taught on vectors

This section will give answers to the research questions $RQ^I_{2,1}$, $RQ^I_{2,2}$, $RQ^I_{2,3}$, and $RQ^I_{2,4}$, which are all regarding the knowledge to be taught. Section 3.3.1 will deal with the historical development of $MO_{to\ be\ taught}$ which includes both $RQ^I_{2,1}$ and $RQ^I_{2,2}$. From a mathematical point of view, it will focus on how the theory of vectors is covered from both a geometric and an algebraic point of view, and how this is reflected in textbooks and written exam problems. From a more interdisciplinary view it will focus on the role that physics has played in the mathematical organisation of vectors. Furthermore, some sub-questions will be guiding the praxeological analysis:

${RQ_2}^I_1$: How are vectors introduced and defined?
${RQ_2}^I_2$: What types of tasks are found in the praxis block?
${RQ_2}^I_3$: What techniques are found in the praxis block?
${RQ_2}^I_4$: How is the theory part of the logos block organised?
${RQ_2}^I_5$: How do the $MO_{to\ be\ taught}$ differ from the $MO_{scholarly}$?

Section 3.3.2 will deal with $RQ^I_{2,3}$ where the crucial task is to answer the above sub-questions in order to give a praxeological description.

Section 3.3.3 will deal with $RQ^I_{2,4}$ focusing on how mathematics has generally been heavily reduced in the organisation of the knowledge to be taught in physics the past years.

About the structure of the analysis and the materials used

The analysis of the mathematical organisation of vectors will be divided into two periods; the historical and the current. The historical period is taken to be the years from 1935 to 2005. The starting point is chosen to be 1935, because vectors did not appear in the mathematics curricula before this year. The year 2005 is taken as the end of the historical period because the mathematical organisation of vectors in the 2005-curriculum is almost identical to the mathematical organisation of vectors in the 2013-curriculum. These two curricula are included in the current period, because the material that will give access to the mathematical organisation of vectors in the 2017-curriculum is still very limited. Furthermore, the 2005/2013-curricula are very similar to the 2017-curriculum, which is why these are drawn into the analysis of the current mathematical organisation. The purpose of the analysis of the historical period is to describe the *development* of the mathematical organisation. Therefore, the analysis of the historical period is again divided into smaller periods. The periods are bounded by the years that the Danish high school changed reform, i.e. the first period is 1935-1953, because the Danish Ministry of Education passed a high school reform in 1935 and again in 1953.

The material that will be used in the analyses are the official documents describing the curricula, and the guidelines for interpretation of the curricula (these have only been accessible for the years 2013 and 2017). Additionally, the written exam problems that relates to the theory of vectors will be used in the description of the mathematical organisation of vectors, since these will enlighten (especially) the praxis blocks in the respective praxeological organisations. For the questions about the more theoretical organisation textbooks will be used. For each period, the textbooks that were predominant and/or characteristic for the period are drawn in. Since the written exam problems are comparable across reforms it has seemed reasonable to give a review of the findings in the written exam problems that include vectors *before* the presentation of mathematical organisation of vectors in the respective periods.

The historical development of the mathematical organisation of vectors will be treated in section 3.2.1 and the current mathematical organisation will be treated in section 3.2.2. Both sections will include the links to physics that might be mentioned in both curricula, guidelines for interpretation of curricula, written exams, and textbooks.

The analysis of the organisation of vectors in physics is also divided into the same two periods as the analysis of the organisation in mathematics. In physics the material is unfortunately highly restricted, mostly because the material has not been as easy accessible as the corresponding material in mathematics. The official documents that have been available are curricula from 2013 and 2017, together with the guidelines for interpretation of the curricula, and the written exams from the period 2010-2017. A few textbooks from the historical period will be drawn into the analysis of the organisation of vectors in physics. The analysis of the current organisation of vectors will only include one textbook.

The historical development of the organisation of vectors in physics will be treated together with the historical development of the organisation of vectors in mathematics in section 3.3.1 and the current organisation in physics will be treated in section 3.3.3. Section 3.3.3 will include the links to mathematics that will be mentioned in both curricula, guidelines for interpretation of curricula, and textbooks.

3.3.1 Historical development

A French study conducted by B. A. Cissé and Jean-Luc Dorier in 2014 (Cissé & Dorier, 2014) that covers the period from 1852-2002 have showed how vectors have moved from the border of mathematics and physics into a more algebraic context, focusing on the axiomatic structure of $\mathbb{R}^2/\mathbb{R}^3$, with applications in geometric problems. A similar transfer can be observed in a Danish context and it will be enlightened throughout the following.

General structure of the written exam problems

Vectors have appeared in written exams since 1966. Across reforms the vector problems that appear in the written exams can be divided into three categories that are listed and described in Table 3.

Category	Description
Analytical-geometrical vector-problems	Problems regarding geometric configurations that are coordinate-based. Vectors are used in the formulation of the problem and required in the solution. The problems can be two or three dimensional. The geometric properties of the vector spaces $\mathbb{R}^2/\mathbb{R}^3$ are utilised.
Vector algebra-problems	Problems regarding vectors without connection to a geometric configuration. These problems are solved by purely algebraic manipulations with vectors. the algebraic structure of $\mathbb{R}^2/\mathbb{R}^3$ are utilised. These problems can include coordinates or they can be coordinate-free.
Vector function-problems	Problems involving vector functions.

Table 3 – The three categories of written exam problems including vectors

In this thesis the last category will not be paid any attention because it is out of scope. Instead a thorough examination of the two other categories will be given.

The category *analytical-geometrical vector-problems* contains a huge number of different types of tasks that are solved by a lot of different techniques. Instead of listing all the different techniques (as it is possible for the vector algebra-problems) it is more illustrative to divide the analytical-geometrical vector-problems in subcategories depending on the technology. These are presented and described in Table 4.

θ^I	The technology of techniques to determine ***intersections*** between e.g. two lines, line and plane, two planes, or circle and line
θ^A	The technology of techniques to determine ***angles*** between e.g. two lines or two planes
θ^P	The technology of techniques to determine the ***projection*** of e.g. point on line
θ^R	The technology of techniques to determine ***representations*** (equations and parametric representations) of planes and lines, and determination of direction/normal vectors from equations or parametric representations of lines and planes

θ^D	The technology of techniques to determine ***distance*** between e.g. point and plane or point and line
θ^T	The technology of techniques to determine ***tangents*** and ***tangent planes***
θ^Δ	The technology of techniques to carry out calculations on ***triangles*** and ***quadrangles*** (area of them and angles in them)

Table 4 – Technologies of techniques to solve analytical-geometrical vector-problems

Each of the local MO's that are organised around the technologies in Table 4 contain punctual MO's organised around different techniques. The techniques that include vectors are identical to techniques from the algebraic MO, and the technology provides a discourse on the vector spaces $\mathbb{R}^2/\mathbb{R}^3$ as a model for geometric objects in $\mathbb{R}^2/\mathbb{R}^3$.

A further discussion and some examples of problems will be given in the sections about the respective periods.

The other important category is *vector algebra-problems*. As it has been described above the vector algebra-problems are divided into two subcategories; those that are coordinate-free and those where coordinates are included. Both categories utilise some techniques that are not restricted to vector problems. These techniques will be categorised as "ordinary algebra" techniques and are found in Table 5.

Ordinary algebra techniques (no vectors)	
τ^{eq}	Solve equation
$\tau^{2\ eq}$	Solve two equations with two unknowns
$\tau^{3\ eq}$	Solve three equations with three unknowns
$\tau^{quadratic\ eq}$	Solve quadratic equation
$\tau^{vertex\ of\ parabola}$	Determine the vertex of a parabola by $x = -\frac{b}{2a}, y = -\frac{d}{4a}$
$\tau^{reduction}$	Reduce an equation
$\tau^{3rd\ deg.pol.}$	Solve a third degree polynomial

Table 5 – "Ordinary algebra" techniques that do not involve vectors

Problems in the subcategory without coordinates can be solved by combining the techniques from Table 5 above and Table 6 below:

Vector techniques (no coordinates)	
$\tau^{distributive(+,\cdot)}$	Use the distributive law of vector sum and scalar product. $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{c}$
$\tau^{\lvert\cdot\rvert,\cdot}$	Use the identity $\lvert\vec{a}\rvert^2 = \vec{a}^2 (= \vec{a} \cdot \vec{a})$
$\tau^{\perp,\cdot}$	Use that $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$
$\tau^{\angle,\cdot,\lvert\cdot\rvert}$	Use the relation between scalar product, lengths of vectors and angle between vectors. $\vec{a} \cdot \vec{b} = \lvert\vec{a}\rvert \cdot \lvert\vec{b}\rvert \cdot \cos(v)$

$\tau^{proj,\lvert\cdot\rvert,\cdot}$	Use the relation between length of projection vector, numerical value of the scalar product and the length of the vector that is projected on. $\lvert\overrightarrow{b_a}\rvert = \frac{\lvert\vec{a}\cdot\vec{b}\rvert}{\lvert\vec{a}\rvert}$
$\tau^{\lvert\cdot\rvert\ of \perp vector}$	Use the identity $\lvert\hat{\vec{a}}\rvert = \lvert\vec{a}\rvert$
$\tau^{par,\lvert\cdot\rvert,\angle}$	Use the relation between the lengths of and the angle between the two vectors that span a parallelogram. $A_{parallelogram} = \lvert\vec{a}\rvert \cdot \lvert\vec{b}\rvert \cdot \lvert\sin(v)\rvert$

Table 6 – Techniques to solve vector algebra-problems with no coordinates

The problems in the second subcategory can be solved by combining the techniques in Table 7.

Vector techniques (coordinates)	
τ^{sum}	Use the coordinate formula for addition. $\binom{a_1}{a_2} + \binom{b_1}{b_2} = \binom{a_1 + b_1}{a_2 + b_2}$
$\tau^{difference}$	Use the coordinate formula for subtraction. $\binom{a_1}{a_2} - \binom{b_1}{b_2} = \binom{a_1 - b_1}{a_2 - b_2}$
$\tau^{scalar\ mult}$	Use the coordinate formula for scalar multiplication. $k\binom{a_1}{a_2} = \binom{ka_1}{ka_2}$
$\tau^{scalar\ prod}$	Use the formula for the scalar product. $\binom{a_1}{a_2} \cdot \binom{b_1}{b_2} = a_1 b_1 + a_2 b_2$
$\tau^{\lvert\cdot\rvert}$	Use that $\lvert\vec{a}\rvert = \sqrt{{a_1}^2 + {a_2}^2}$
$\tau^{\wedge}$	Use that $\vec{a} = \binom{a_1}{a_2} \Rightarrow \hat{\vec{a}} = \binom{-a_2}{a_1}$
τ^{proj}	Use the coordinate formula for the projection vector. $\overrightarrow{b_a} = \binom{b_{a_1}}{b_{a_2}} = \frac{a_1 b_1 + a_2 b_2}{{a_1}^2 + {a_2}^2} \cdot \binom{a_1}{a_2}$
$\tau^{\perp}$	Use that $\binom{a_1}{a_2} = \vec{a} \perp \vec{b} = \binom{b_1}{b_2} \Leftrightarrow a_1 b_1 + a_2 b_2 = 0$
$\tau^{\parallel}$	Use that $\binom{a_1}{a_2} = \vec{a} \parallel \vec{b} = \binom{b_1}{b_2} \Leftrightarrow a_1 b_2 - a_2 b_1 = 0$
$\tau^{\angle,\cdot}$	Use that $a_1 b_1 + a_2 b_2 = \sqrt{{a_1}^2 + {a_2}^2} \cdot \sqrt{{b_1}^2 + {b_2}^2} \cdot \cos(v)$
$\tau^{\angle,det}$	Use that $a_1 b_2 - a_2 b_1 = \sqrt{{a_1}^2 + {a_2}^2} \cdot \sqrt{{b_1}^2 + {b_2}^2} \cdot \sin(v)$
τ^{det}	Use the coordinate formula for determinant. $\det(\vec{a}, \vec{b}) = a_1 b_2 - a_2 b_1$
$\tau^{parallelogram}$	Use the relation of the area of the parallelogram spanned by two vectors and their coordinates. $A_{parallelogram} = \lvert a_1 b_2 - a_2 b_1\rvert$
$\tau^{\overrightarrow{AB}}$	Use the coordinate formula for the vector from point $A(a_1, a_2)$ to point $B(b_1, b_2)$. $\overrightarrow{AB} = \binom{b_1 - a_1}{b_2 - a_2}$

Table 7 – Techniques to solve vector algebra-problems with coordinates

Examples of problems, applications of the techniques, and a further discussion will be given in the sections about the respective periods.

In the 2017-reform a new type of problems appeared in the guiding exam problems, where the approach is a little more geometric, though the problems are not categorised as analytical-geometrical vector-problems. Therefore, they have been put in the category with the vector algebra-problems. Some techniques that can possibly be used to solve them, are shown in Table 8.

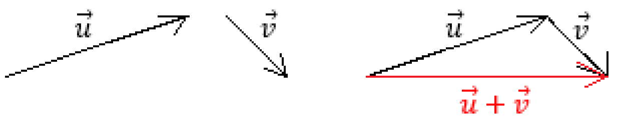
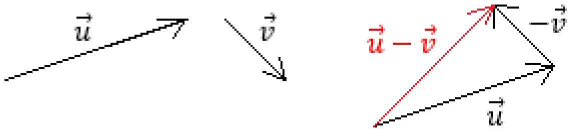
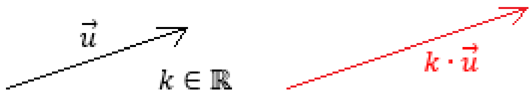
Geometric techniques	
$\tau^{geom\ sum}$	
$\tau^{geom\ diff}$	
$\tau^{geom\ scal\ mult}$	

Table 8 – Geometric techniques to solve the vector-algebra problems with a more geometric approach

1935-1953

During this period vectors were included in the curriculum, but they did not appear in the written exams. The curriculum was divided into two, *arithmetic and plane geometry* (in a broader meaning of the word *arithmetic* including also naive algebra, e.g. equations) and *stereometry* (spatial geometry) (Petersen & Vagner, 2003, pp. 187-188). Each part contained a list of the topics that should be covered, and under *arithmetic and plane geometry* the two topics "The composition and decomposition of vectors" and "Velocity in linear and curvilinear movement (in the plane). Acceleration in linear and circular movement" (as cited in Petersen & Vagner, 2003, p. 188) appear. The second of them is contained in kinematics, and will also be found in the physics curriculum rather than in the mathematics curriculum nowadays. A similar organisation was detected in France in the beginning of twentieth century (Cissé & Dorier, 2014, p.3).

The introduction of, motivations for, and definition of vectors will be described in the view of the textbook "Lærebog i matematik", that was prevalent in that time (Petersen & Vagner, 2003, p. 193). Unfortunately, the first edition from 1937 have not been accessible. Instead the fourth edition, from 1949, is used.

The last chapter in the first book was dedicated to vectors. The opening example, that introduces and motivates the notion of vectors, comes from physics (Juul & Rønnau, 1949, p. 216):

Ex. 1. In the physics curriculum from the middle school we have dealt with forces. To the determination of a force belongs three things: the size of the force, the direction (the line of action) of the force, and the point of action of the force. We depict a force as a line segment, which length denotes the size of the force, and which direction (denoted by an arrow) denotes the direction of the force, and which initial point is the point of action of the force. If two forces have the same point of action, the can be composed according to the rule of the parallelogram of forces (fig. 139) [see Figure 18]. The two forces K_1 and K_2 (components) can be replaced by the force R (the resultant). The correctness of this rule can be realised by experiments. Inversely, the resultant R can be decomposed in the forces K_1 and K_2. Directional quantities, e.g. forces, are called vectors, while quantities, that are not directional, e.g. area, volume, are called scalars. What we learn in the following about vector, applies to vectors in the same plane.

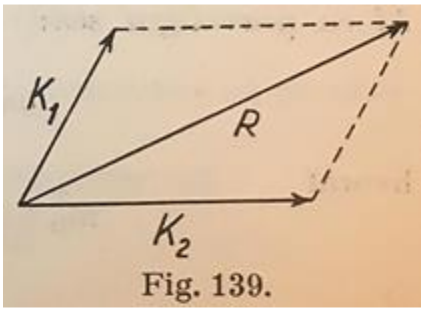


Figure 18 – Fig. 139 from Ex. 1 in (Juul & Rønnau, 1949, p. 216)

From this example a couple of interesting points can be highlighted.

1. Physical applications were important in the teaching of vectors in this period.
2. No distinction is made between the mathematical objects that are characterised as vectors and the quantities in physics that *behave* like vectors (vector quantities). This book puts equality sign between *vectors* and *vector quantities*.

This geometric composition of forces is a technique. It has already been described in section 3.2.1, but in the context of $MO_{to\ be\ taught}$ it will be denoted $\tau^{geom\ sum}$.

The definition of a vector is given right after the example shown above (Juul & Rønnau, 1949, p. 216):

A and B are two given points on a straight line l. If a point is moving on l from A against B, l is said to be run through in the direction AB. The line segment AB that goes from A to B is called a *vector*. […] The vector AB is denoted $\overrightarrow{AB}$.

The definition is supplemented by an illustration, that is shown in Figure 19. From the definition and Figure 19 it is seen that no distinction between the concrete directed line segment AB and the vector $\overrightarrow{AB}$ is made.

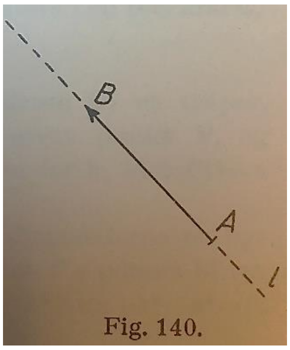


Figure 19 - Figure illustrating the definition of a vector (in Juul & Rønnau, 1949, p. 216)

The notion of "point bounded vector" does not correspond to anything from the scholarly mathematical organisation of vectors. However, the notion "point bounded vector" is more useful than the correct mathematical definition to model the situation described in section 3.2.3, where two forces are represented by the same mathematical vector, but the effects of the forces are different because the points of application are different. In the terminology of (Juul & Rønnau, 1949) the two vectors with different "points of application" are different *point bounded* vectors. The notion "free vector" in (Juul & Rønnau, 1949) is what corresponds to the scholarly notion of vectors, though the definition is way more geometric in (Juul & Rønnau, 1949) than the scholarly one. When defining a vector as a parallel displacement, the notion appears more dynamical, than the more static definition based on vector space properties, as in the scholarly definition. This reinforces the connection between vectors and their application in physics.

Remarkable, especially compared to the present mathematical organisation of the knowledge to be taught on vectors, is that vectors in (Juul & Rønnau, 1949) are not represented by coordinates at all. When calculations involve coordinates it is always the coordinates of points, e.g. the end points of vectors.

The very physical approach to vectors can explain some of the details in the mathematical organisation of vectors in this period. First of all, it explains why it was only "The compound and decomposition of vectors" that was represented in the curriculum, since these are the only properties of vectors that are relevant in kinematics. Secondly, it will explain why vectors were removed from the curriculum in the succeeding period from 1953 to 1961, where kinematics was no longer a part of the mathematics curriculum.

Exercises and problems are not a part of the content in (Juul & Rønnau, 1949), which means that it is difficult to make a detailed praxeological analysis, but a possible praxeological organisation of the knowledge to be taught on vectors is the following: The types of tasks are different physical problems regarding vector quantities (mostly velocity), the techniques are centred around geometric composition and decomposition of vectors, the technology is a justification of vectors as a model for physical vector quantities, and the theory is the definition of vectors as mathematical objects (see Figure 20).

Praxis	Logos
Calculations with physical vector quantities	Vectors as a model for physical vector quantities
Composition and decomposition of vectors	Mathematical theory of vectors

Figure 20 – Praxeological organisation of vectors during the period 1935-1953

1953-1961
As mentioned above vectors were absent from the curriculum, so the period will not be paid further attention.

1961-1971
This period is interesting because vectors were added to the curriculum again, and because vectors appeared for the first time in a written exam in this period. In 1961 the *modern mathematics* was implemented in the Danish mathematics curriculum through "Den røde Betænkning". The purpose was, like in France and the rest of Europe (Cissé & Dorier, 2014 pp. 5-6), to narrow down the gap between the scholarly knowledge and the knowledge to be taught. The gap had grown, since neither the topics in the curriculum nor the organisation of the knowledge to be taught had changed much over the years, while the mathematical community had made great progress in the scholarly knowledge during the same period. Two important changes in the scholarly knowledge had made the gap grow, because they had not affected the knowledge to be taught yet. The first one was the importance of *sets*, the second was the heavily algebraisation of almost every mathematical field, that was described in section 3.2.2. An attempt to implement the changes, that were made in the scholarly knowledge, in the knowledge to be taught was made in the 1961-reform.

"Den røde Betænkning" gave the following purpose of the mathematics teaching in high school (as cited in Petersen & Vagner, 2003, pp. 236-237):

> To let the students get acquainted with a number of fundamental mathematical notions and ways of thinking, to evoke their sense of clarity and coherence in mathematical argumentation and expression forms, to seek a development of their fantasy and inventiveness, to train them in the treatment of concrete problems, including the execution of numerical calculations, and to make them familiar with applications of mathematics within other scientific fields.

This purpose found its expression among other places in the mathematical topics that were included in the curriculum. Up to the 1961-reform, high school students had mostly been acquainted with different geometric topics, functions including calculus, equations, and some arithmetic. From 1961 the most fundamental notion in Danish high school mathematics became *sets*, which highly reflected the scholarly knowledge. Also equivalence relations, logic, and algebra were new important notions in the curriculum. Vectors, and especially the vector space structure of $\mathbb{R}^2$ and $\mathbb{R}^3$, became the fundamental notion in geometry in both two and three dimensions. The curriculum stated the following about plane geometry for first year students (as cited in Petersen & Vagner, 2003, p. 237):

> Plane geometry. The right angled coordinate system. Change of coordinates. Vectors and their coordinates. Calculations with vectors, including the scalar product of two vectors. The analytic representation of the straight line. Distances and angles. The analytic representation of the circle. The area of triangles and parallelograms. Definition and the analytic representation of parabola, ellipse, and hyperbola. Mappings of the plane on itself: Parallel displacement, rotation, reflection, multiplication and composition of these mappings. Affinity.

And the following about spatial geometry for first year students (as cited in Petersen & Vagner, 2003, pp. 237-238):

> Spatial geometry. The right angled coordinates system. Vectors and their coordinates. Calculation with vectors including the scalar product of two vectors. The analytic representation of the straight line. The analytic representation of the plane. Distances and angles. The equation of the sphere. Spherical coordinates. The spherical distance between two points (the law of cosines). Polyhedrons, Euler's polyhedron formula, the regular polyhedrons. Volume of prism, pyramid, cylinder of revolution, cone of revolution, and sphere; area of spherical triangles. Congruence and symmetry.

Furthermore, vectors were mentioned in the paragraph in the curriculum about applications of infinitesimal calculus (as cited in Petersen & Vagner, 2003, p. 238):

> Applications of infinitesimal calculus. Determination of the range of a function and the conditions of the functions monotony. Simple examples of determination of the asymptotic properties of a function. Drawing of plane curves determined by explicitly given functions or by parametric representations. The velocity vector, speed, acceleration vector […]

In the light of the curriculum alone, the context that it put vectors in, and the list of notions from vector theory that should be covered, it is obvious how vectors were mostly a tool for geometric purposes in the period 1961-1971, though it was also mentioned as an application of infinitesimal calculus. This is distinct from the purpose in the period 1935-1953, where vectors were used only as a tool for physical purposes (kinematics). Due to the lack of the corresponding materials from physics, it is hard to know whether vectors were included in the physics curriculum in this period.

Contrary to the period 1935-1953 this period provides a collection of written exam problems that contribute to giving an insight in the organisation of the praxis block. The written exam problems can, as described above, be divided into three categories. Problems in each category appeared in this period. The first vector problem ever was an analytical-geometrical vector problem from the exam in 1966. The problem will be presented below and referred to as Problem 1. It will be included in an analysis of the whole category of analytical-geometrical vector-problems later. The first vector function-problem appeared in 1967, but since vector functions are out of scope of this thesis, the analysis will not include the category of vector function-problems.

The first vector algebra-problem appeared in 1970 (and it was actually the *only* vector algebra-problem in this period). It will be presented below and referred to as Problem 3. It will be included in an analysis of the whole category of vector algebra-problems later. The division of the written exam problems and a review of different textbooks from this and the following periods shows that the praxeological organisation of vectors in mathematics can be divided into two regional MO's. The first one is the one organised around the geometric properties of $\mathbb{R}^2/\mathbb{R}^3$, $[T, \tau, \theta, \Theta^G]$. The other one is organised around the algebraic properties of $\mathbb{R}^2/\mathbb{R}^3$, $[T, \tau, \theta, \Theta^A]$. In this period the praxis block of $[T, \tau, \theta, \Theta^G]$ is predominant, while the textbooks reveals that the predominant logos block is taken from $[T, \tau, \theta, \Theta^A]$.

Analytical-geometrical vector-problems

Problem 1 – the first analytical-geometrical vector-problem from 1967 (Petersen & Vagner, 2003):

> In a coordinate system in the plane a quadrangle $ABCD$ is given. The side AD is situated on the line given by the equation
>
> $$x + 3y + 4 = 0$$
>
> and the side AB is situated on the line given by the equation
>
> $$11x - 8y + 44 = 0$$
>
> The point B is situated on the second axis (y-axis), $\overrightarrow{BC}$ has the coordinates $(6, -\frac{1}{2})$, and the point D is situated on the perpendicular bisector of the diagonal AC.
>
> Determine the area of the quadrangle and the angles B and D.

The problem is analytical-geometrical, since it is closely tied to a geometric configuration. Referring to the general organisation of the analytical-geometrical vector-problems (Table 4) this problem belongs under the technology θ^{Δ} (the technology of techniques to carry out calculations on ***triangles*** and ***quadrangles***). Problem 1 contain three tasks:

t_1: Determine the area of the quadrangle
t_2: Determine the angle B
t_3: Determine the angle D

t_2 and t_3 are contained in the same type of task. As it has been described the vector techniques in $[T, \tau, \theta, \Theta^G]$ are similar to some of the techniques in $[T, \tau, \theta, \Theta^A]$. For example t_1 is solved by combining some techniques from $[T, \tau, \theta^R, \Theta^G]$ that are not directly related to vectors and the technique $\tau^{parallelogram}$ from $[T, \tau, \theta, \Theta^A]$. The tasks t_2 and t_3 are similarly solved by a combination of some techniques from $[T, \tau, \theta^R, \Theta^G]$ that are not directly related to vectors and the technique $\tau^{\angle}$ from $[T, \tau, \theta, \Theta^A]$.

A closer look into the analytical-geometrical vector-problems from this period reveals that the tasks are most often in either $[T, \tau, \theta^{\Delta}, \Theta^G]$ (like Problem 2) or in $[T, \tau, \theta^{\mathrm{I}}, \Theta^G]$ (tasks regarding intersections). An example could be the following problem from 1968, that will be referred to as Problem 2 (Petersen & Vagner, 2003):

> In the plane is given a coordinate system with the origin O. The line given by the equation $y = \alpha x + q$ intersects the parabola given by the equation $y = x^2$ in the points P and Q.
>
> Show that the scalar product $\overrightarrow{OP} \cdot \overrightarrow{OQ}$ depends on q but is independent of α.
>
> About two points A and B on the parabola is given that $\overrightarrow{OA} \cdot \overrightarrow{OB} = 6$. Determine the coordinates to the intersection between the line AB and the secondary axis (y-axis).
>
> Determine the smallest value that $\overrightarrow{OS} \cdot \overrightarrow{OT}$ can take when S and T are arbitrary points on the parabola.

In addition to these two local MO's, the regional MO $[T, \tau, \theta, \Theta^G]$ contains other types of tasks that are solved by different techniques and justified by different technologies, but generally the problems are not as easy to categorise into a small bunch of types of tasks, as exam problems are nowadays. This variety in types of tasks require a well-developed logos block in order to be able to model many different mathematical configurations with vectors.

Vector algebra-problems

Problem 3 – the first vector algebra-problem from 1970 (Petersen & Vagner, 2003):

> In an oriented plane two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given. The vectors satisfy
>
> $|\boldsymbol{a}| = 1$ *and* $\boldsymbol{b} = 2\hat{\boldsymbol{a}}$
>
> In the following R is denoting the set of real numbers and N the set of positive whole numbers.
>
> Determine the set M_1 of numbers $t \in R$ that satisfies
>
> $$|t\boldsymbol{a} + \boldsymbol{b}| = 6$$
>
> Determine the set M_2 of tuples $(s, t) \in R \times R$, $(s, t) \neq (0,0)$, that satisfies
>
> $$s\boldsymbol{a} + t\boldsymbol{b} \perp \boldsymbol{a} - \boldsymbol{b}$$
>
> Determine the set M_3 of tuples $(s, t) \in R \times R$ that satisfies
>
> $$(s + 1)\boldsymbol{a} + \boldsymbol{b} \perp \boldsymbol{a} + (t + 1)\boldsymbol{b}$$
>
> Determine the set M_4 that satisfies
>
> $$M_4 = M_2 \cap M_3$$
>
> Determine the set M_5 of tuples $(s, t) \in N \times N$ that satisfies
>
> $$(s, t) \in M_2 \ \wedge \ 0 < |s\boldsymbol{a} + t\boldsymbol{b}| < 10$$

Problem 3 is clearly different from the two analytical-geometrical vector-problems, since it does not require any geometric interpretation of the vectors or the set-up in general. The problem can be solved by application of the algebraic properties alone. As it has been mentioned it is the only problem in the MO $[T, \tau, \theta, \Theta^A]$, but it is very interesting to examine for two reasons. First of all, the appearance of exam problems from this MO have increased over the years, and the past years they have appeared almost equally frequent as the geometric vector problems. Secondly, this specific problem, Problem 2, is almost an embodiment of the *modern mathematics* where sets played an important role. Since this was a general tendency across mathematical topics, and a style that was quickly abandoned in the vector problems, this detail will not be elaborated on. The tasks in Problem 3 are solved by different techniques that have one thing in common; they are all applications of the algebraic properties of $\mathbb{R}^2$.

As an example the first task

> t_1: Determine the set M_1 of numbers $t \in R$ that satisfies $|t\boldsymbol{a} + \boldsymbol{b}| = 6$

is solved by a combination of the techniques $\tau^{|\cdot|,\cdot}$, $\tau^{\perp,\cdot}$, and $\tau^{distributive(+,\cdot)}$ that are vector techniques and $\tau^{quadratic\ eq}$ that is the "ordinary algebraic" technique of solving a quadratic equation. The solution is sketched on Table 9.

$\lvert t\boldsymbol{a}+\boldsymbol{b}\rvert^2 = (t\boldsymbol{a}+\boldsymbol{b})\cdot(t\boldsymbol{a}+\boldsymbol{b})$	$\tau^{\lvert\cdot\rvert,\cdot}$
$\lvert t\boldsymbol{a}+\boldsymbol{b}\rvert^2 = t\boldsymbol{a}\cdot t\boldsymbol{a}+t\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot t\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}$	$\tau^{distributive(+,\cdot)}$
$\lvert t\boldsymbol{a}+\boldsymbol{b}\rvert^2 = t^2\lvert\boldsymbol{a}\rvert^2+t(\boldsymbol{a}\cdot\boldsymbol{b})+\boldsymbol{b}^2$	$\tau^{\lvert\cdot\rvert,\cdot}$
$\lvert t\boldsymbol{a}+\boldsymbol{b}\rvert^2 = t^2\cdot 1+t\cdot 0+\lvert 2\hat{\boldsymbol{a}}\rvert^2$	$\tau^{\perp,\cdot}$
$\lvert t\boldsymbol{a}+\boldsymbol{b}\rvert^2 = t^2+4$	Use the given information
$6^2 = t^2+4$	Use the given information
$t = \pm\sqrt{32}$	$\tau^{quadratic\ eq}$

Table 9 – A sketch of how the techniques $\tau^{|\cdot|,\cdot}$, $\tau^{\perp,\cdot}$, and $\tau^{distributive(+,\cdot)}$ are used to solve the algebraic task t_1

According to the written exams vectors were mostly a tool to be used in analytical-geometrical problems, or a tool in the application of infinitesimal calculus in the vector function-problems. From 1970 the students were also tested in their knowledge of the algebraic structure of $\mathbb{R}^2$ as a vector space. Vectors had obtained status in the written exams as objects in their own right, and not just as tools for geometric purposes.

On the organisation of the exam problems

The geometric branch of the mathematical organisation of vectors is the most frequently represented in the written exams. The types of tasks vary a lot, which requires a well-developed logos block. On the other hand, it is difficult to conclude too much about the algebraic tasks, since only one single problem in this category appeared in this period. However, it is worth noticing that the vector algebra-problem is free from coordinates.

Textbook

A review of one of the textbooks of that time will reveal more about the logos blocks in the praxeological organisation of the topic vectors. In this period the prevalent textbook was "Matematik I" by Erik Kristensen and Ole Rindung (Petersen & Vagner, 2003, p. 240). It was written in 1962 and to this day it has iconic status among many mathematics teachers as a treasure, because of its very concise and brief style. In addition to the prevalence of the book, the fact that both authors participated in the activities that led to the reform in 1961 (Petersen & Vagner, 2003, p. 236) serves as a justification of the use of exactly this book in the analysis of the mathematical organisation of vectors in this period.

The book covered all the topics for the first year at high school. The first chapters in the book have the headings "I. Sets and statements" and "II. Sets of numbers" (Kristensen & Rindung, p. V). On the foundation of these, the third chapter deals with vectors. It contains the subsections shown in Table 10.

Chapter	Subsections
III. Vectors	- The notion of vector - Parallel displacements - Addition and subtraction of vectors - Multiplication of vector by a number - Decomposition of a vector in given directions - The coordinates of vectors - The length of a vectors, the equation of the circle - The scalar product of two vectors - Geometric interpretation of the scalar product - Projection of vector on vector - Orthogonal vector - Rotation of the coordinate system - Other applications of the orthogonal vector

Table 10 – Subsections in chapter III. Vectors in "Mathematics I" (Kristensen & Rindung, 1962, pp. V-VII)

The significant status of set theory shows from the table of content, since the two introductory chapters are dedicated to this. It is also reflected in the organisation of the knowledge to be taught about vectors.

When looking into the first paragraph of the vector chapter, the definition of a vector is established. It is centred around the notions of direction of half-lines, oriented line segments, and equivalence relations. The definition of a vector in "Mathematics I" is stated in the following way "Any set on the form (4.1) is called a *vector*; and any oriented line segment that is contained in the set $\overrightarrow{AB}$, is said to *represent the vector* $\overrightarrow{AB}$." (Kristensen & Rindung, 1962, p. 54). Here (4.1) is the following statement:

$$\overrightarrow{AB} = \{\overline{PQ} | \overline{PQ} \equiv \overline{AB}\}$$

In this the notation $\overline{AB}$ means the directed line segment from A to B. The meaning of the equivalence sign, $\equiv$, and the definition of equivalence has been given formally on the previous page (Kristensen & Rindung, p. 53):

> We call two oriented line segments $\overline{AB}$ and $\overline{CD}$ equivalent and we write
>
> $$\overline{AB} \equiv \overline{CD}$$
>
> when the two line segments are unidirectional and have the same length. It is obvious that
>
> (1) $\overline{AB} \equiv \overline{AB}$
> (2) $\overline{AB} \equiv \overline{CD} \Rightarrow \overline{CD} \equiv \overline{AB}$
> (3) $(\overline{AB} \equiv \overline{CD}) \wedge (\overline{CD} \equiv \overline{EF}) \Rightarrow (\overline{AB} \equiv \overline{EF})$

The role of the directed line segment, $\overline{AB}$, is also stated very explicitly as a "representative of the vector $\overrightarrow{AB}$" (Kristensen & Rindung, 1962, p. 54). Without further explanations of the idea of vectors as an equivalence class that can be *represented* by directed line segments, or a clarification of the differences between the vector itself and its representative, a new notation for these abstract vectors is introduced: "Often we will use a single letter as the symbol for a vector, and we use small bold letters: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}$ etc." (Kristensen & Rindung, 1962, p 54).

According to this book, a vector *is* "an equivalence class of directed line segments". These can be represented by something that in everyday language would be called arrows (though this word is not used in the books).

In the next paragraph the interrelation between vectors and parallel displacements is established and defined very formally (Kristensen & Rindung, 1962, p. 55): "$\mathrm{Q} = \mathrm{p_a}(\mathrm{P}) \Leftrightarrow \overline{\mathrm{PQ}}$ is a representative of $\mathbf{a}$". The definition is supplemented with the illustration shown in Figure 21.

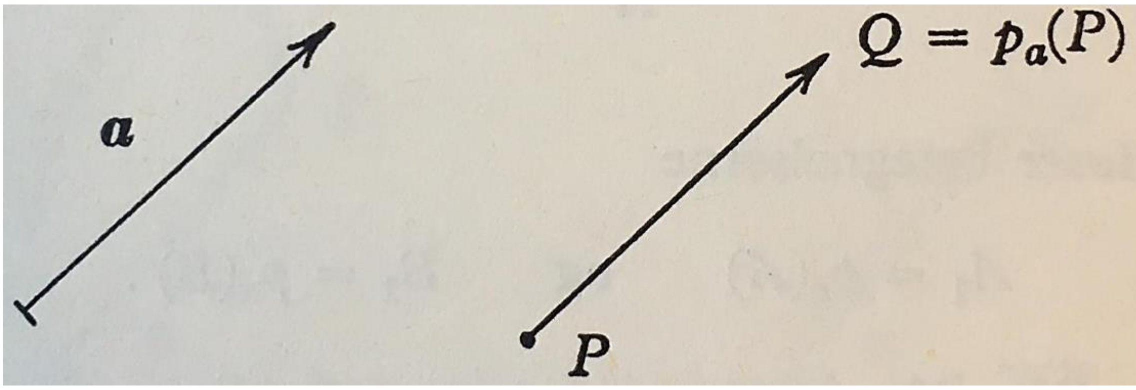


Figure 21 – Illustration of the relation between vectors and parallel displacements (Kristensen & Rindung, 1962, p. 55)

Here P is an arbitrary point, $\boldsymbol{a}$ is a given vector, and $Q = p_a(P)$ is the unique point that makes $\overline{PQ}$ a representative for $\boldsymbol{a}$. The relation is defined in the following way "p_a is called *the by $\boldsymbol{a}$ determined parallel displacement*" (Kristensen & Rindung, 1962, p. 55).

In this a vector is identified with a parallel displacement, and in order to prove that the sum of two vectors is again a vector, the book shows that a composition of parallel displacements (seen as functions) is again a parallel displacement.

In the beginning of the subsection "Addition and subtraction of vectors", the sum of two vectors is constructed geometrically in two ways, by the "polygon-method" to the left in Figure 22, and by the "parallelogram-method" to the right in Figure 22.

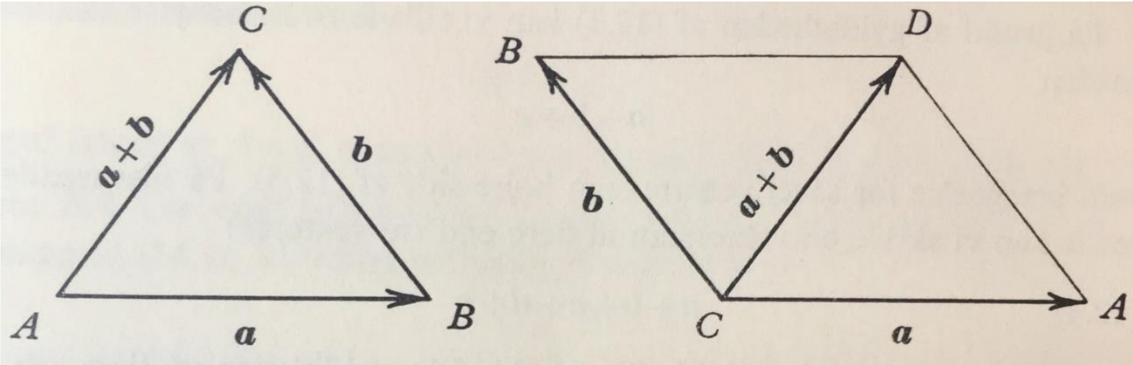


Figure 22 – Constructions of the sum of two vectors in (Kristensen & Rindung, 1962, p. 61)

After the geometric construction, an example establishes (very briefly) an opportunity of application of vectors in physics in the following way (Kristensen & Rindung, 1962, p. 61):

> 11.2 example. According to a theorem in the mechanical physics forces are compounded following a rule that is called "the parallelogram of forces". This physical theorem shows that the geometric notion of vector combined with the definition of vector addition given above is suitable as a description of the physical notion of forces –Vectors can be applied as a mathematical model of many other physical notions (velocity, acceleration, field strength, and others). These are objects, that can be described by arrows and that can be compounded in a way that corresponds to vector addition.

To a person engaging in teaching in the modern high school this example can seem superficial and not easily accessible, mostly because the physical notions are not necessarily covered yet, by the time vectors are taught in mathematics.

The application example appears in a section about addition and subtraction of vectors. The following sections deals with multiplication of a vector with a scalar. As an example both commutativity and associativity of vector addition is proven formally, but the geometric/physical interpretations of these results are never mentioned.

By looking into the scholarly knowledge, it reveals that the properties that are treated in these paragraphs are exactly the axioms that constitutes the definition of a vector space. Furthermore, it is noteworthy that most of the examples and exercises prove algebraic properties of vectors. For example, the following (Kristensen & Rindung, 1962, p. 64):

> 15.5 example. The following holds
> $$-(\boldsymbol{a}+\boldsymbol{b}) = (-\boldsymbol{a}) + (-\boldsymbol{b})$$
> because we have
> $$(\boldsymbol{a}+\boldsymbol{b}) + \big((-\boldsymbol{a}) + (-\boldsymbol{b})\big) = \boldsymbol{a} + (-\boldsymbol{a}) + \boldsymbol{b} + (-\boldsymbol{b}) = \boldsymbol{o}$$

Or the following exercise (Kristensen & Rindung, 1962, p. 64):

15.6 exercise. Prove the formula

$$-(\boldsymbol{a}+\boldsymbol{b}) = (-\boldsymbol{a}) + (-\boldsymbol{b})$$

From this remark it is reinforced, that the approach to vectors in this period is more algebraic and focusing on the algebraic properties of the vector space $\mathbb{R}^2$ instead of the geometric properties that are the ones that makes $\mathbb{R}^2$ a useful model in geometry and physics. This can seem to be conflicting with the prevalence of geometric tasks in the written exams. However, the algebraic approach shows how the *modern mathematics* has taken the knowledge to be taught closer to the scholarly knowledge.

Almost 20 pages into the chapter, the coordinate system and coordinates are introduced. So far the vectors have been "living freely" in an unspecified space, and everything has been treated purely as algebraic objects obeying specific rules. The definition of the unit vectors from points, $E = (1,0)$ and $F = (0,1)$, is given as the first thing in the paragraph "The coordinates of vectors". From a theorem proven in the previous paragraph about the uniqueness of the decomposition of vectors in given directions, it is proven that the coordinates of a vector are uniquely determined in a given coordinate system.

The very abstract nature of the book does again show in the notation used. It says (Kristensen & Rindung, 1962, p. 71):

> We could make some formulas clearer by denoting the coordinates (x, y) for a vector in the following way: $\binom{x}{y}$. We will permit ourselves to use this notation alternating with the usual notation (x, y).

It requires a high level of abstraction from first year students in order to separate the vector (x, y) from the point (x, y), but again this is closely related to the scholarly version of vectors (and mathematics in general) where the notation is more or less arbitrary compared to the underlying concept. Therefore, an alternation between notations is easy for a mature mathematician, that identifies mathematical objects by their properties and not their notation, but for a student that is just learning a new concept, the notation plays an important role.

Throughout the section about the coordinates of vectors some of the notions that have been treated previously are related to the corresponding results in coordinate representation. Furthermore, it is interesting to take a look at the examples that appear in this section. One example is (Kristensen & Rindung, 1962, p. 74):

> 25.4 example. A and B are two points given by coordinates (a_1, a_2) and (b_1, b_2) respectively. We want to determine the coordinates of the midpoint M of the line segment AB. Since $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$ we have, when O is the origin of the coordinate system

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$
$$= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$

> Since $\overrightarrow{OM}$, $\overrightarrow{OA}$, and $\overrightarrow{OB}$ are position vectors they have the same coordinates as the corresponding points M, A, and B. By using theorem (24.1) we get that M has the coordinates
>
> $$\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right)$$

Theorem (24.1) has given the sum, the difference, and the product of a scalar and a constant when the vectors are given by coordinates. In this example the algebraic properties of $\mathbb{R}^2$ are linked to the geometric interpretation. The example shows (again abstractly in the sense that the coordinates do not have concrete values) how vectors can be used to solve analytical-geometrical problems. In this particular example how the mid-point of the line segment AB can be determined by the use of the coordinates of the point A and B. The next paragraph continues with the geometric application by relating the length of a vector, the distance formula, and the equation of the circle.

In the paragraph about the scalar product some properties of the scalar product are proven. The subsequent paragraph deals with the geometric interpretation of the scalar product, where the applications in physics are again mentioned in an example (Kristensen & Rindung, 1962, p. 82):

> 32.9 example: The scalar product has many important applications in physics, e.g.:
>
> If a particle, that is acted on by a constant force given by the vector $\boldsymbol{a}$, is displaced that is given by the vector $\boldsymbol{b}$, the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the work done by the force during the displacement.

In (Kristensen & Rindung, 1962) the primary logos block is coming from the MO $[T, \tau, \theta, \Theta^A]$. A lot of the tasks are more abstract in character, compared to exercises in a modern textbook, e.g. proving activities instead of concrete calculations. In (Kristensen & Rindung, 1962) vectors are established as objects, that can be represented equivalently in different ways.

As it has been shown, the main purpose of vectors in the written exams during this period is in the analytical-geometrical problems, but in (Kristensen & Rindung, 1962) it is primarily the algebraic properties of vectors that are of the main interest. A praxeological description of the mathematical organisation of vectors has to be divided into a geometric approach and an algebraic approach. These two are shown in Figure 23 and Figure 24 respectively.

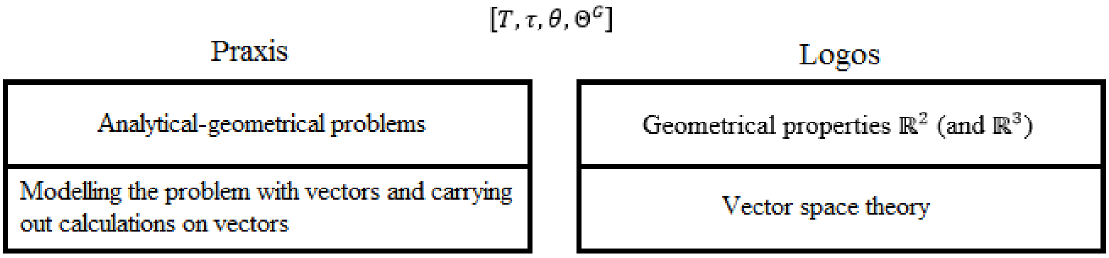


Figure 23 – $[T, \tau, \theta, \Theta^G]$

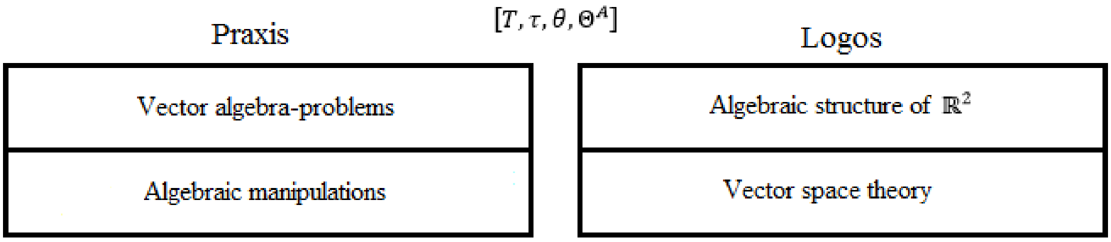


Figure 24 – $[T, \tau, \theta, \Theta^A]$

Vectors in physics

Since neither the physics curriculum nor the written exams have been available, the praxeological organisation of vectors in physics will be described in the light of one of the prevalent physics textbooks of this period. The textbook series that has been chosen is "Lærebog i fysik" by Mogens Pihl and Henning Storm. The series consists of three books, Textbook for physics I, Textbook for physics II, and Textbook for physics III. The first edition of the books was published in the years 1963-1965. The first edition has been inaccessible, and therefore the second edition has been used. The second edition was published in the years 1966-1970.

In the preface of "Lærebog i fysik" a comment on the use of vectors is made. It reads: "In the last chapters of the book the use of *the notion of vectors* is heavy, since it is covered early in the mathematics teaching." (Pihl & Storm, 1966, p. VIII). The notion of vectors is used for the first time in chapter VII that is dedicated to Newton's laws. It is introduced through the parallelogram of forces, that is assumed to have been covered in primary school. It reads: "For a body in rest forces can be regarded as *vectors* in the sense that the equilibrium does not change it, the two arbitrary forces are substituted by their vector sum." (Pihl & Storm, 1966, p. 98). The rest of Newton's laws are introduced using the notion of vectors.

In the next chapter, "VIII. Work and mechanical energy in the field of gravity" (Pihl & Storm, 1966, p. 101), the notion of *work* is defined as the scalar product of force and displacement. In a footnote the scalar product is described in the following way: "The scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ can be defined as the product of the projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ (calculated with sign) and $|\boldsymbol{b}|$. It holds that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ and $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$." (Pihl & Storm, 1966, p. 101). This definition differs from the primary mathematical definition, because it is more geometric, than the definition based on coordinates. Furthermore, a physical/geometrical interpretation of the distributive law is given (Pihl & Storm, 1966, p. 101):

> Nothing prevents the body from being affected by other forces than $\boldsymbol{F}$. Is it for example affected by the two forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ these will do the work $\boldsymbol{F_1} \cdot \boldsymbol{r}$ and $\boldsymbol{F_2} \cdot \boldsymbol{r}$ respectively. The sum of these two works is called the work done by the two forces. It equals
>
> (8,2) $\boldsymbol{F_1} \cdot \boldsymbol{r} + \boldsymbol{F_2} \cdot \boldsymbol{r} = (\boldsymbol{F_1} + \boldsymbol{F_2}) \cdot \boldsymbol{r}$,
>
> since the scalar product is distributive. *The work done by two forces equals the work done by their resultant*.

After some chapters dealing with scalar quantities, comes "X. The magnetic induction field" (Pihl & Storm, 1966, p. 133). In this chapter the following about magnetic induction is stated: "We will *assume* that there in *every point of a magnetic field exist a vector* $\boldsymbol{B}$ *that is called the magnetic induction*." (Pihl & Storm, 1966, p. 134). It is further described how magnetic inductions can be added as vectors, and that this can be illustrated by experiments. Also the chapter about steady state-current utilises vectors.

The second volume of the textbook series, "Lærebog i fysik II", uses vectors too. In the preface, the standpoint regarding the use of mathematics in the book is given. It reads (Pihl & Storm, 1969, p. VII):

> In the endeavour of giving the presentation a clear and accessible shaping we have utilised the valuable support that mathematics affords. It is our experience that the use of the language of mathematics can have a deterrent effect, if it appears sporadic, while a systematic application gives confidence in the understanding, which is a definite pedagogical advantage. This presupposes that mathematical notions and symbols are given clear physical content, in which we have endeavoured.

This citing reveals the highly mathematised style that the authors have chosen. The first chapter in the book deals with astronomy. In the first section the right angled coordinate system is introduced. This is done by the use of the position vector $\overrightarrow{OP}$ corresponding to a point P, and by the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ (and $\boldsymbol{k}$ in three dimensions). The position vector is used in the chapters "I. Astronomy" and "II. The kinematic description of the propagation of waves" (Pihl & Storm, 1969). In the third chapter "III. The kinematic description of the motion of particles" (Pihl & Storm, 1969) the use of vectors is extended. In the section about motion in two dimensions the velocity of a particle is defined as the derivative of the position vector with respect to time. The acceleration of the particle is defined in a similar way.

This series of textbooks reveal a highly mathematised style where vectors plays a crucial role in physics. A problem is, that the notion of vectors in mathematics is very algebraic and mostly coordinate-free, when it is more geometric and coordinate-based in physics. This issue can have caused students major problems with the transfer of the notion of vectors from mathematics to physics and the other way around.

1971-1984

The *modern mathematics* caused both students and teacher, because of the abstract and formal approach. Similar problems were detected in France (Cissé & Dorier, 2014, pp. 6-8). The reform in 1971 changed the purpose of the mathematics teaching slightly and removed some of the topics in order to make room for an optional topic, that the teacher and/or students could choose. Except from these minor changes, the curriculum stayed almost the same as in 1961. The new purpose was (as cited in Petersen & Vagner, 2003, p. 265):

> To let the students get acquainted with a number of fundamental mathematical notions, ways of thinking, and methods, to train the students in applications of mathematical notions, ways of thinking, and methods for formulation, analysis, and solution of problems within different fields, to practise clearness and coherence in proofs and expression form, to develop fantasy and inventiveness, and to give an understanding of and the ability to analyse the ways that mathematics is applied within other fields critically.

Though some topics were removed from the curriculum, the level of abstraction and formality was not lowered, but as shown in the citing above, the focus on application was strengthened. One of the major changes concerned vectors, since spatial geometry, and thereby three dimensional vectors, was abandoned.

Analytical-geometrical vector-problems

The problems in this category are very similar to the ones in the previous reform, though the written exams do of course not contain spatial geometry-problems. It is still primarily tasks in the MO $[T, \tau, \theta^{\Delta}, \Theta^{G}]$. An interesting example of a problem from this category in this period is the following, that will be referred to as Problem 5 (Petersen & Vagner, 2003):

In an oriented plane a proper vector $\boldsymbol{a}$ is given. About a quadrangle $ABCD$ it is given that

$\overrightarrow{AB} = \boldsymbol{a}, \overrightarrow{BC} = \boldsymbol{a} + \widehat{\boldsymbol{a}}$ and $\overrightarrow{CD} = -2\boldsymbol{a} + 2\widehat{\boldsymbol{a}}$

Determine the degree measure of the angles between the diagonals in the quadrangle.

In this example there are no coordinates that gives indication of where in the coordinate system the quadrangle is situated.

Vector algebra-problems

The more interesting changes in exam problems occur in the vector algebra-problems. First of all, they appeared more frequently than before 1971, and furthermore the *modern mathematics* and its focus on sets and logic was heavily toned down in them, compared to the vector algebra-problem (Problem 3) from 1970 that. Some examples of vector algebra-problems will be given. The first one will be referred to as Problem 5 (Petersen & Vagner, 2003):

A coordinate system is given. Determine the numbers t, such that

$\boldsymbol{a}(2t + 5, -t)$ and $\boldsymbol{b}(t^2 + 2t + 1{,}4t + 4)$

are proper vectors, that are parallel.

The second will be referred to as Problem 6 (Petersen & Vagner, 2003):

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy

$$(\boldsymbol{a} + \boldsymbol{b})^2 = 13,$$

$$(2\boldsymbol{a} - \boldsymbol{b})^2 = 7, \text{ and}$$

$$(\boldsymbol{a} + 2\boldsymbol{b})^2 = 28.$$

Determine the lengths of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and the degree measure of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

The third will be referred to as Problem 7 (Petersen & Vagner, 2003):

In an oriented plane, a proper vector $\boldsymbol{v}$ is given.

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given by

$$\boldsymbol{a} = 2\boldsymbol{v} - 3\widehat{\boldsymbol{v}} \text{ and } \boldsymbol{b} = \boldsymbol{v} + 2\widehat{\boldsymbol{v}}$$

Determine the number t such that the length of the vector $\boldsymbol{a} + t\hat{\boldsymbol{b}}$ is smallest.

All these problems are vector-algebra problems but the differ in the level of abstraction. Though the vectors in Problem 5 are given abstractly in the sense that the coordinates depends on t the two vectors are still somewhat closely related to the a concrete coordinate system that is more geometric in appearance. This problem is furthermore interesting, since this type of task has appeared frequently in the more recent written exams. It is solved by the technique $\tau^{\parallel}$ (see Table 11) in combination with the "ordinary algebraic" techniques $\tau^{reduction}$ and $\tau^{3rd\ deg.pol.}$.

$(2t+5)\cdot(4t+4) - t\cdot(t^2+2t) = 0$	$\tau^{\parallel}$
$t^3 + 10t^2 + 29t + 20 = 0$	$\tau^{reduction}$
$t = -1, t = -4, t = -5$	$\tau^{3rd\ deg.pol.}$

Table 11 – The technique $\tau^{\parallel}$ used on Problem 5

In Problem 6 the vectors are not directly tied to the coordinate system, since the coordinates are not given, but the vectors are still given by the scalar product of different linear combinations of them with themselves, which is closely tied to the geometric property *length*. The problem is solved by a combination of the techniques $\tau^{distributive,(+,\cdot)}$, $\tau^{|\cdot|,\cdot}$, $\tau^{\angle,\cdot,|\cdot|}$, and $\tau^{3\ eq}$ (see Table 12). These are all very algebraic, which means that the geometric properties are not necessarily concerned by the students when the problem is solved.

$\begin{cases}(\boldsymbol{a}+\boldsymbol{b})^2 = 13\\(2\boldsymbol{a}-\boldsymbol{b})^2 = 7\\(\boldsymbol{a}+2\boldsymbol{b})^2 = 28\end{cases} \Leftrightarrow \begin{cases}\boldsymbol{a}^2+\boldsymbol{b}^2+2\boldsymbol{ab} = 13\\4\boldsymbol{a}^2+\boldsymbol{b}^2-4\boldsymbol{ab} = 7\\\boldsymbol{a}^2+4\boldsymbol{b}^2+4\boldsymbol{ab} = 28\end{cases}$	$\tau^{distributive(+,\cdot)}$
$\begin{cases}\boldsymbol{a}^2+\boldsymbol{b}^2+2\boldsymbol{ab} = 13\\4\boldsymbol{a}^2+\boldsymbol{b}^2-4\boldsymbol{ab} = 7\\\boldsymbol{a}^2+4\boldsymbol{b}^2+4\boldsymbol{ab} = 28\end{cases} \Leftrightarrow \boldsymbol{a}^2 = 4, \boldsymbol{b}^2 = 3, \boldsymbol{ab} = 3$	$\tau^{3\ eq}$
$\boldsymbol{a}^2 = \lvert\boldsymbol{a}\rvert^2 = 4 \Rightarrow \lvert\boldsymbol{a}\rvert = 2$ $\boldsymbol{b}^2 = \lvert\boldsymbol{b}\rvert^2 = 3 \Rightarrow \lvert\boldsymbol{b}\rvert = \sqrt{3}$ $\boldsymbol{ab} = 3 \Rightarrow v = \cos\left(\frac{3}{4\cdot 3}\right)$	$\tau^{\lvert\cdot\rvert,\cdot}$ and $\tau^{\angle,\cdot,\lvert\cdot\rvert}$

Table 12 – A sketch of how the techniques $\tau^{distributive\ (+,\cdot)}$, $\tau^{|\cdot|,\cdot}$, $\tau^{\angle,\cdot,|\cdot|}$, and $\tau^{3\ eq}$ are used to solve Problem 6

In Problem 7 the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given without any connection or reference to the geometric context. It is solved by a combination of the purely algebraic techniques $\tau^{distributive\ (+,\cdot)}$, $\tau^{|\cdot|,\cdot}$, $\tau^{|\cdot|\ of \perp vector}$, and $\tau^{vertex\ of\ parabola}$ (see Table 13).

$\left\|\boldsymbol{a}+t\widehat{\boldsymbol{b}}\right\|^2=\boldsymbol{a}^2+t^2\boldsymbol{b}^2+2t\boldsymbol{ab}$	$\tau^{\|\cdot\|,\cdot}$ and $\tau^{distributive(+,\cdot)}$
$\left\|\boldsymbol{a}+t\widehat{\boldsymbol{b}}\right\|^2=4\|\boldsymbol{v}\|^2+9\|\widehat{\boldsymbol{v}}\|^2+t^2(\|\boldsymbol{v}\|^2+4\|\widehat{\boldsymbol{v}}\|^2)+2t(2\|\boldsymbol{v}\|^2-6\|\widehat{\boldsymbol{v}}\|^2)$	Use the given information and $\tau^{reduction}$
$\left\|\boldsymbol{a}+t\widehat{\boldsymbol{b}}\right\|^2=(5t^2-8t+13)\|\boldsymbol{v}\|^2$	$\tau^{reduction}$ and $\tau^{\|\cdot\|\ of \perp vector}$
Vertex in: $t=\frac{8}{10}$	$\tau^{vertex\ of\ parabola}$

Table 13 – A sketch of how the techniques $\tau^{distributive\ (+,\cdot)}$, $\tau^{|\cdot|,\cdot}$, $\tau^{|\cdot|\ of \perp vector}$, and $\tau^{vertex\ of\ parabola}$ are used to solve Problem 7

The types of tasks in Problem 6 and Problem 7 have not appeared in the written exams in the recent period.

On the organisation of the exam problems

The geometric branch of the mathematical organisation of vectors is the most frequently represented in the written exams. The types of tasks vary a lot, which requires a well-developed logos block. On the other hand, it is difficult to conclude so much about the algebraic tasks, since only one single problem in this category appeared in this period. However, it is worth noticing that the vector algebra-problem is free from coordinates.

In this period the algebraic branch of the mathematical organisation of vectors is more frequently represented than in the previous period. Furthermore, the number of vector algebra-problems exceeded the number of analytical-geometrical vector-problems in this period.

The number of different types of tasks in the analytical-geometrical vector-problems is reduced compared to the previous period, but except from that, the nature of the problems is similar to the analytical-geometrical vector-problems in the period 1961-1971.

The category of vector algebra-problem is also easy to divide into types of tasks, and the number of different types of tasks is of course larger than in the previous period (where only one problem in this category appeared).

Textbook

In this period the predominant textbook was still the series “Matematik” from Kristensen and Rindung. As it has been mentioned it was republished several times over the years. In the examination of the mathematical organisation of vectors in this period the seventh edition of “Matematik I” from 1976 will be used. The table of content is not exactly the same as in the first edition. The first chapter is “Sets and statements”, the second chapter is “Real numbers”, the third chapter is “Powers, slide ruler, and logarithms”, and the fourth chapter is “Vectors” (Kristensen & Rindung, 1976, p. V).

There are some important changes from the first edition to the seventh edition. One of the most interesting of these is, that the notion of *arrow* is defined strictly from the notion of oriented line segments (Kristensen & Rindung, 1976, p. 76):

> A line segment that is equipped with an ordering after which the end points are denoted *initial point* and *terminal point* respectively is called an *oriented line segment* or an *arrow*. The arrow that is determined by the tuple (A, B) is denoted $\overline{AB}$.

Then the notion of arrow is related to the notion of parallel displacement and equivalence of arrows is defined from the relation between parallel displacements and arrows (Kristensen & Rindung, 1976, p. 77):

> Two arrows $\overline{AB}$ and $\overline{CD}$ are equivalent when and only when the parallel displacement $A \curvearrowright B$ is the same as the parallel displacement $C \curvearrowright D$.

After this the notion of vector is defined (Kristensen & Rindung, 1976, 77):

> To a given parallel displacement that takes an arbitrary point Q to Q_p corresponds a class of equivalent arrows $\overline{AA_p}$, $\overline{BB_p}$, $\overline{CC_p}$,… (fig. 2.II). The set of the arrows that in this way corresponds to a parallel displacement is called a *vector*.

This definition comes with an illustration that is shown in Figure 25.

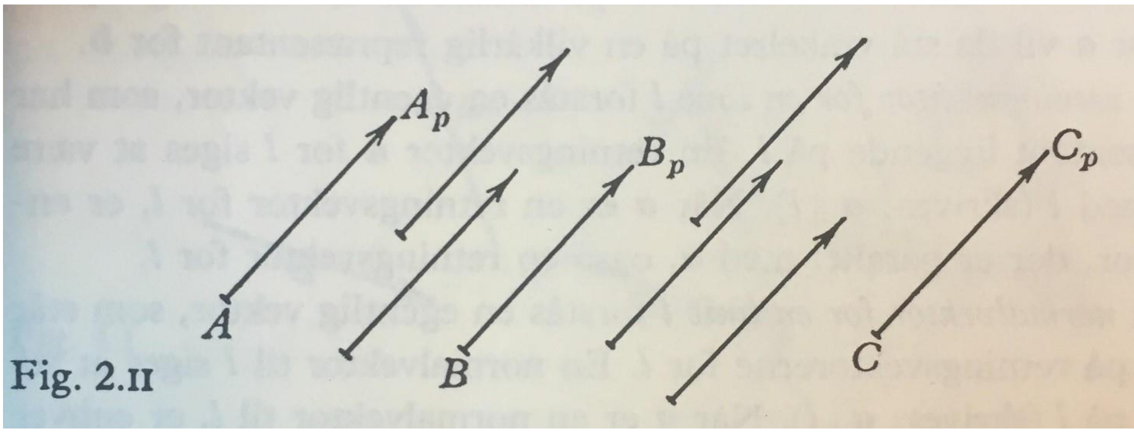


Figure 25 – A class of equivalent arrows representing the same parallel displacement P in (Kristensen & Rindung, 1976, p. 77)

Vectors in physics

In this period the same textbooks written by Henning Pihl and Mogens Storm were still used in the physics teaching. Since the written physics exams from this period has been unavailable, the $PO_{to\ be\ taught}$ is therefore assumed to be similar to the one in the previous period.

1984-2005

In this period some more radical changes were made. First of all, the calculator meant that some particular skills became unnecessary to practise. However, this did not influence the topic *vectors* to the same extend as some of the other topics. Furthermore, the nature of the problems posed in written exams changed. Contrary to what had been done before, the problems were more often related to everyday problems. The purpose was to show the students that mathematics was applicable. (Petersen & Vagner, 2003, pp. 272-273).

During this period, the curriculum contained the following about vectors (Petersen & Vagner, 2003, p. 271):

> Plane and spatial geometry. The coordinate system. Vectors in the plane and space, the coordinates of vectors. Calculations with vectors, including scalar product of two vectors. Orthogonal vector, vector product. Projection of vector on vector. Analytic description of point sets in the plane including line, circle, and half plane. Distance between points and between point and line. Intersection between lines and between line and circle. Sine, cosine, and tangent. Calculations on sides and angles in triangles. Area of triangle and parallelogram. Analytic description of point sets in space, including straight line, plane, and sphere. Distance, angle and intersection between two point sets in space.

Compared to the 1971-reform the 1984-reform did only change the vector part by adding spatial geometry again and removing vector functions.

Analytical-geometrical vector-problems

During this period the organisation of the analytical-geometrical vector problems begin to look very similar to the recent organisation. In the previous period most of the problems were geometry problems contained in the MO determined by the technology θ^{Δ}. In this period, the problems started spreading over the different local MOs determined by the technologies in Table 4.

An example from the early years of this period is the following problem from 1988, that will be referred to as Problem 8 (Petersen & Vagner, 2003):

> In an oriented plane is given a vector $\boldsymbol{a}$ with the length 6.
> The quadrangle $ABCD$ is given by
> $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{AD} = -\boldsymbol{a} + \frac{2}{3}\hat{\boldsymbol{a}}$ and $\overrightarrow{BC} = -\frac{1}{2}\boldsymbol{a} + \frac{5}{3}\hat{\boldsymbol{a}}$
> Determine $\overrightarrow{DC}$ given by $\boldsymbol{a}$ and $\hat{\boldsymbol{a}}$.
> Determine $\angle A$ and $\angle D$ in the quadrangle $ABCD$.
> Determine the area of the quadrangle $ABCD$.

This problem is contained in the MO determined by the technology θ^{Δ}. Another problem from later in the period is the following from 1999, that will be referred to as Problem 9 (Petersen & Vagner, 2003):

> I a coordinate system in space is given a point $P(5,-1,4)$ and a plane α given by the equation
>
> $$\alpha: 2x - 2y + z + 2 = 0$$
>
> Determine the distance from the point P to the plane α.
>
> A line l passes through P and has the direction vector $\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
>
> Determine the coordinates of the intersection between l and α.
> Determine the acute angle between l and α.
> A plane β passes through the points $A(0,0,4)$, $B(2,0,0)$, and $C(1,1,4)$.
> Determine the equation of the plane β.
> The planes α and β are both tangent planes to a sphere K. The centre of the sphere, its points of tangency with α and β, and the point P are situated on a straight line.
> Determine an equation of the sphere K.

This problem contains tasks from different local MOs. The first sub-question is contained in the MO determined by θ^{D}, the second is in the MO determined by θ^{I}, the third is in the MO determined by θ^{A}, and the last two are contained in the MO determined by θ^{R}. In general, the analytical-geometrical problems started including a lot of different sub-questions solved by techniques in a lot of different local MOs in this period. This idea of combining a lot of different types of tasks in one extensive analytical-geometrical vector-problem is still used in the current mathematical organisation.

Vector algebra-problems

In this period the most abstract vector algebra-problems (like Problem 8) start appearing very rarely. Problems that are similar to Problem 7, where vectors are given by different geometric properties such as length, appear frequently like in the previous period. The vector algebra-problems where the vectors are given by coordinates starts appearing more and more frequent. An interesting example of a coordinate-based vector algebra-problem is shown below. The problem is from 1991 and will be referred to as Problem 10 (Petersen & Vagner, 2003):

> In a coordinate system in the plane two vectors are given
> $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
> Determine the angle between the vectors $\vec{a}$ and $\vec{b}$.
> Determine the value of the number t, such that $\vec{a} + t\vec{b}$ is perpendicular to $\vec{a}$.

> Determine the value of the number t, such that the length of the vector $\vec{a} + t\vec{b}$ is as small as possible.

This problem is interesting, since the vectors are given by coordinates, which is the most concrete representation of a vector in a written exam. The first sub-question could easily have appeared in the recent period, but the two last sub-questions would not.

Textbook

In the wake of a reform of the primary school, where mathematics became more experience-based than rigorous, the textbooks for high school mathematics changed into a more application-oriented focus. One of the new textbooks containing the theory of vectors that were used in this period was "Funktioner og vektorer: Teori og redskab" by Steffen Jensen and Karin Sørensen (Vagner & Petersen, 2003, pp. 266-267). The book was published in 1981 and had a very special and new structure, where the first part of the book covered the theory and the second part gave examples of applications.

The introduction to the notion of vectors is given in the following way (Jensen & Sørensen, 1981, p. 31):

> We will in this chapter introduce a mathematical quantity, that is characterised by having both a direction and a numerical value, a vector, and some computation rules of vectors. From physics a number of quantities being characterised by a direction and a numerical value is known.

After this introduction an example is given (Jensen & Sørensen, 1981, p. 31):

> 1.1 Example The physical notion velocity is an example of a quantity that is determined by a number (the speed) and a direction; that a particle has a particular velocity means that it is moving by a given speed in a given direction. Forces are in the same manner determined by a numerical value and a direction.

This example shows how the physical applications plays a role in the introduction to vectors in the mathematical organisation in this period. The definition of a vector shows signs of being the successor of the Kristensen and Rindung books. First the notion of an arrow is defined (Jensen & Sørensen, 1981, p. 31):

> 1.2 Definition By an arrow is understood a line segment of a given length equipped with a given direction. One says that the line segment is oriented. If A and B are two points in the plane we will denote the arrow from A to B by $\overline{AB}$

The next definition is of the notion *vector*. The definition says (Jensen & Sørensen, 1981, p. 32):

> 1.4 Definition By a vector is understood the set of unidirectional arrows having the same length. Every arrow in the set is said to be a representative of the vector.
> As symbols for vectors lower-case letters with an arrow over them are usually used: $\vec{a}$, $\vec{b}$, … (In some books lower-case letters set up in bold-face).

Before this definition both the notions of unidirectional and opposite arrows are defined. Of course most of the notions that are defined are the same as in the books by Kristensen and Rindung, but generally Jensen and Sørensen are using more geometric illustrations in definitions and proofs. An example could be the following theorem and the corresponding proof (Jensen & Sørensen, 1981, pp. 41-43):

> 2.2 Theorem For arbitrary vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ the associative law holds
> $$\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$
> Proof: The first figure [see Figure 26] shows that
> $$\left(\vec{a}+\vec{b}\right)+\vec{c}=\overrightarrow{AB}$$
> and the second figure [see Figure 27] shows that
> $$\vec{a}+\left(\vec{b}+\vec{c}\right)=\overrightarrow{AB}$$
> With that the desired is shown.

The first figure that is mentioned in the proof is shown on Figure 26.

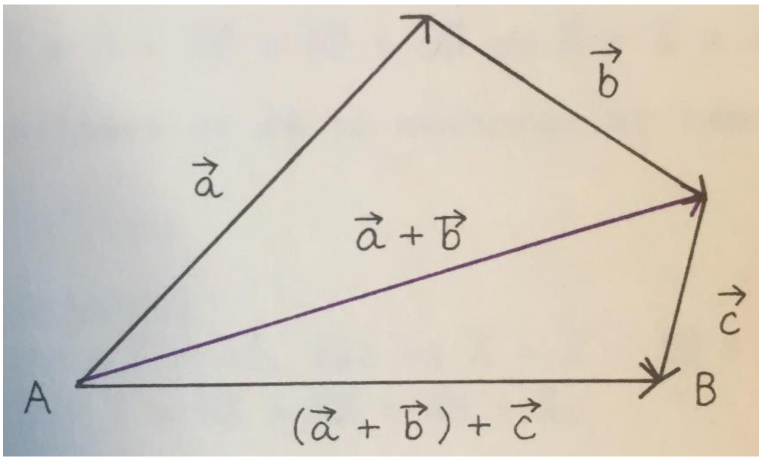


Figure 26 – The first figure mentioned in the proof of the associative law in (Jensen & Sørensen, 1981, p. 41)

Figure 27 shows the second figure that is mentioned in the proof.

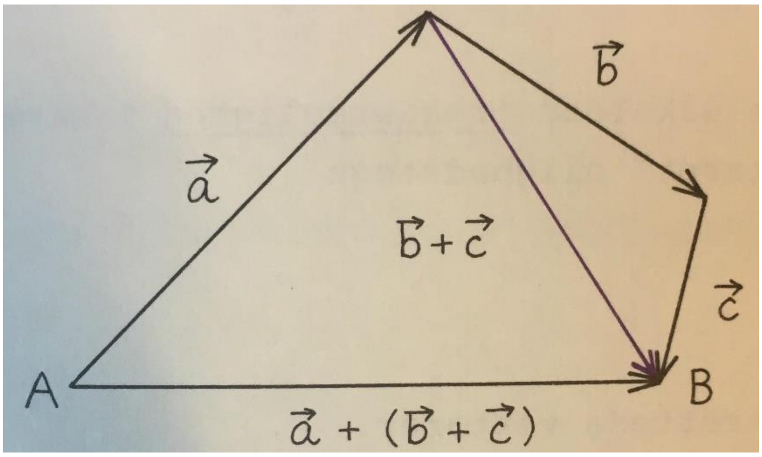


Figure 27 – The second figure in the proof of the associative law in (Jensen & Sørensen, 1981, p. 43)

Also in the discussion of the parallelogram spanned by two vectors the approach is more geometric than in Kristensen and Rindung (Jensen & Sørensen, 1981, p. 87):

> 9.9 Example We consider two proper non-parallel vectors $\vec{a}$ and $\vec{b}$.
> The two vectors span a parallelogram $ABCD$, which area we want to determine.
> The base of the parallelogram is $|\vec{a}|$ and its height is the length of the projection of $\vec{b}$, $\overrightarrow{b_1}$, on $\hat{\vec{a}}$ (see the figure [see Figure 28]).
> The wanted area is then
> $|\vec{a}||\overrightarrow{b_1}| = |\hat{\vec{a}}||\overrightarrow{b_1}| = |\hat{\vec{a}} \cdot \overrightarrow{b_1}|$, since $\vec{a} \parallel \overrightarrow{b_1}$
> $= |\hat{\vec{a}} \cdot \vec{b}| = |\det(\vec{a}, \vec{b})|$

The figure mentioned in the example is shown in Figure 28.

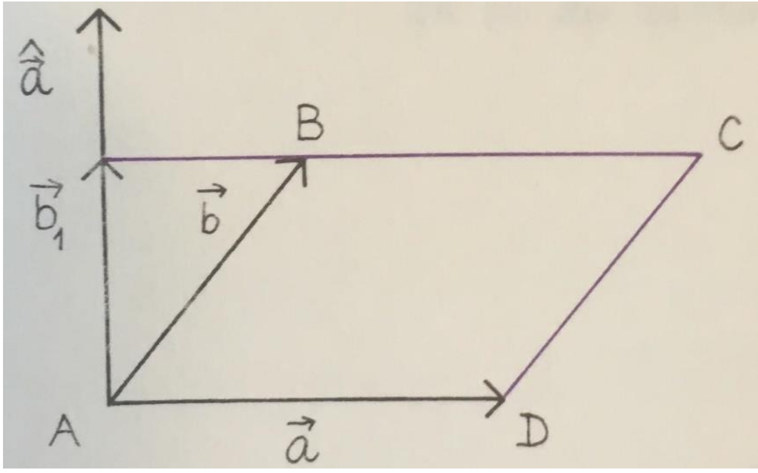


Figure 28 – Figure that illustrates example 9.9 in (Jensen & Sørensen, 1981, p. 87)

In the application part of the book physics is important. One of the examples is the definition of work, which was also mentioned in Kristensen and Rindung. However, in this book it is described similarly as in the physics book by Pihl and Storm. Another interesting example is the following (Jensen & Sørensen, 1981, p. 353):

> A 54 A body is affected by a force that in a given coordinate system has the coordinates
> $$\vec{F} = \binom{4}{5}$$
> The body is moving from the point having the coordinates $(-5,-2)$ to the point having the coordinates $(7,3)$. F is measured in N and the distance in m. Determine the work that the force makes.

Theoretically, the idea of this application is good, but as it has been discussed earlier, physicist do not use the coordinate representation of vector quantities such as force, which means that this is an example of how mathematics teaching and physics teaching is very disconnected in praxis, even though the mathematics textbook tries to include physical applications.

"Funktioner og vektorer: Teori og redskab" is still algebraic in its approach to vectors, but contrary to Kristensen and Rindung, the authors give more illustrations linking the algebraic properties to a geometric interpretation. Furthermore, the style of this textbook puts more importance into examples of applications in for example physics. However, some of the examples are a bit synthetic, and therefore not helpful in the purpose of interrelate the notion of vectors in mathematics and physics.

Vectors in physics

During this period, some different approaches to the teaching of vector quantities appear in different textbooks. The following will examine the two very different textbooks "Fysikkens spor" and "Fysik for 3.G: Hverdag, videnskab og verdensbillede". The first one is written by Claus Christensen, Carsten Claussen, and Bjørn Felsager and published in the first edition in 1990. The second is written by Esper Fogh and Knud Erik Nielsen and published in 1991. The two books take completely different positions regarding the use of vectors.

"Fysik for 3.G" presents the theory of vector quantities without the use of vectors. In the preface an explanation of this choice is given (Fogh & Nielsen, 1991, preface):

> Physics at high level is to some extend chosen by students without a knowledge of vectors. Therefore, we have chosen a presentation that does not assume such knowledge. Out of consideration of those who prefer to use vectors we have put "grey frames" containing a parallel presentation of the content in the language of vectors.

One example of this "parallel presentation" is found under the heading "3. The energy of mechanics" (Fogh & Nielsen, 1991, p. 56). Here the notion of *work* is defined in the ordinary text in the following:

> Work is force times displacement. A constant force F, that is affecting an object, that is displaced by Δs in the direction of the force, does the work:
> $$\Delta A = F \cdot \Delta s$$
> The SI unit of work is $N \cdot m$, that is valled J (joule).

Right next to this definition, a "grey frame" gives an alternative vector definition (Fogh & Nielsen, 1991, p. 56):

> The vector formulation of the definition of work:
> If the force $\vec{F}$ displaces a particle with $\overrightarrow{\Delta s}$, it does the work:
> $$\Delta A = \vec{F} \cdot \overrightarrow{\Delta s}$$
> $$= |\vec{F}| \cdot |\overrightarrow{\Delta s}| \cdot \cos(v)$$
> where v is the angle between $\overrightarrow{\Delta s}$ and $\vec{F}$.

In this particular example, the two definition do actually differ, by the fact that the one without vectors is only working, when force and displacement have the same direction. The definition that utilises vectors can also be used in situations where force and displacement have an angle between them.

Aside from the two-sided definition of work, there are only two other "grey frames" in the book, both in kinematics. The first one describes how movement in two dimensions can be represented by the use of vectors and the other one describes how the notion of position vectors can be used to represent circular motion.

Contrary to "Fysik for 3.G" the authors of the book "Fysikkens spor" have chosen to use vectors to describe directional quantities (Christensen, Claussen, & Felsager, 1990, p. 7):

> We have chosen to describe directional physical quantities by vectors. The few necessary prerequisites are presented in appendix A.

This citing is interesting, because it shows how the use of vectors is something that textbook authors needed to take a stand on during this period, contrary to the previous periods, where the use of vectors was the automatic choice.

The appendix, that is used to present the necessary prerequisites, spends two pages defining vector addition, multiplication of a vector by a scalar, the scalar product, and the vector product. Vector addition is introduced by geometric addition by the parallelogram-rule, but also the polygon-rule. Both descriptions are equipped with illustrations, linking the algebraic properties to the geometric properties. The description of multiplication by a scalar is also illustrated by its geometric interpretation, and additionally a physical example, where the momentum of a particle is given as the product of its mass (that is a scalar quantity) with its velocity (that is a vector quantity).

The definition of the scalar product is also linked to a physical example, namely the definition of work. And the definition of the vector product is linked to the physical notion of *spin*, that is the vector product of the direction vector and the momentum vector.

In this period the teaching of vectors in physics depends on the teacher's standpoint and the choice of textbook, because some prefer more mathematised presentation of vector quantities, while some prefer a presentation where words instead of mathematical notation describes the difference between vector quantities and scalar quantities.

3.3.2 Current mathematical organisation

This period covers the three reforms from 2005, 2013, and 2017. In 2005 the curriculum contained the following about vectors (Danish Ministry of Education, 2005):

> Proportion calculations in similar triangles and trigonometric calculations in arbitrary triangles, vectors in two and three dimensions given by coordinates, applications of vector based coordinate geometry for plane and spatial geometric problems.

In 2013 the curriculum contained the exact same about vectors. In addition to the curriculum, the guidelines for interpretation of the curriculum will be used in the analysis of the mathematical organisation. It reads the following about vectors (Danish Ministry of Education, 2013a):

> The students are required to master the computation rules for vectors and the operations such as
> - to find the orthogonal vectors to a given vector in the plane
> - to determine the scalar product
> - to determine the determinant between two vectors and to be able to interpret this number
> - to determine the cross product between two spatial vectors and to be able to interpret this vector
> - to find the projection of a vector on a vector
>
> The analytical geometry is covered in both two and three dimensions as a vector based coordinate geometry, where the students at a written exam are required to be able to
> - set up and rearrange equations for circles and spheres and to be able to determine tangents and tangent planes

> - translate back and forth between equation and parametric representation of lines in the plane
> - determine the equation of planes and parametric equations of lines in space
> - determine potential intersections between lines, between lines and planes, and between lines and circles, lines and sphere respectively
> - detemine angles between lines, between lines and planes, and between two planes
> - determine distances between points and in the plane: the distance from point to line and in space from point to plane.

The first requirements reveal some of the organisation of the algebraic branch while the last requirements reveal some of the organisation of the geometric branch. They are also reflected in the written exam problems.

The 2017-curriculum describes the organisation of the theory of vectors in the following way (Danish Ministry of Education, 2017a):

> Vectors in two dimensions given by coordinates, including scalar product, determinant, projection, angles, area, line, circle, intersections, and distance calculations, and application of vector based coordinate geometry for plane geometric problems, including trigonometric problems
>
> Vector functions, graphical paths of trajectories, including determination of tangents, and applications of vector functions.

It is noteworthy that vector functions are added to the curriculum again after around 30 years of absence. However, it will not be paid more attention, since it is out of scope of this thesis.

In the guideline for interpretation of the curriculum the properties and applications of vectors mentioned in the curriculum are elaborated on under the heading “geometry and vectors” (Danish Ministry of Education, 2018a):

> Vectors in the curriculum are serving different purposes. Through their work with vectors the students are developing their numeracy, conceptual knowledge, and algebra, when they set up and solve geometric problems in and outside the coordinate system. Vector algebra will contribute to the students’ maintenance and development of the algebraic and calculation skills, that they have from primary school, while they learn something quite new and study the geometric notions in depth. The introduction to the notion of vectors should be given by an alternation between construction and calculation, and an alternation between paper/pencil-activities and computer assisted activities

Furthermore, the guidelines for interpretation of the curriculum describes what is required from first year students (Danish Ministry of Education, 2018a):

> The students are expected to be able to operate with the notions zero vector, unit vector, position vector, connection vector, orthogonal vector, and they are required to be able to determine the angle between vectors, including the handling of orthogonal and parallel vectors. Furthermore, they are expected to be able to apply the simple transition formulas, that are necessary to handle obtuse angles between vectors. Furthermore, the students are expected to be able to carry out calculations in right-angled triangles from the trigonometric formulas, that can be deduced from the unit circle. In calculation and in geometric interpretation by construction the students are required to be able to use the elementary operations of vectors (addition, subtraction, and 'multiplication by a constant') and the other operations: to determine the length of a vector, the orthogonal vector to a given vector, the scalar product of two vectors, and the determinant between two vectors, the angle between two vectors, and the projection of a vector on a vector. Similarly, the students are required to be able to handle calculations involving the laws of cosine and sine, that can be deduced from the scalar product and determinant respectively, and that can simplify the calculation in some trigonometric problems. Furthermore, the height, the median, and the bisector in triangles are assumed well-known.

For the students on second year there are the following additional requirements (Danish Ministry of Education, 2018a):

> On B-level the calculations with vectors are extended to containing the part of the analytical geometry, that deals with the analytical description of the objects line and circle. In this phase as well, it is important as a support of the students' conceptual knowledge, that the teaching varies between construction and calculation and between paper/pencil activities and activities with mathematical CAS-tools. The students should obtain skills and competences in setting up and rewriting equations of circles (completing the square) and determining the equations of circle tangents and rewriting back and forth between the equation and the parametric representation of a straight line. Furthermore, the students are required to be able to determine the intersection between lines and between lines and circles and angles between lines and distance from point to line. Connected with the angle between lines is also the relation between angle of inclination (with the first axis) and the slope of a straight line.

This shows that the division into an algebraic and a geometric branch is still valid in the new reform. Compared to the reforms in the historical periods the algebraic branch is more concrete since it is mostly based on coordinates instead of a more abstract coordinate-free approach, as it was prevalent in the 60's, 70's, and 80's. A new feature is, that the students are required to be able to geometrically construct the sum and the difference of vectors, and the multiplication of a vector by a constant.

The geometric branch looks very similar to especially the one in the previous period. The students will still primarily be working with lines and circles (not planes, since spatial geometry was removed with the 2017-reform), and angles, representations, intersections, distances, and tangents. An examination of the written exam problems will elaborate both the algebraic and the geometric branch.

Analytical-geometrical vector-problems

The analytical-geometrical vector-problems from the current period are primarily concerning three dimensional configurations. Often the problems are equipped with a sketch showing the situation. An example from 2011 is shown below. The problem will be referred to as Problem 11 (Danish Ministry of Education, 2018c):

> A sphere in a coordinate system in space has its centre in $C(0,0,5)$, and the point $P(2,-1,3)$ is situated on the sphere.
>
> a) Determine an equation of the sphere and determine an equation for the tangent plane in P.
>
> Another tangent plane to the sphere is given by the equation
>
> $$\alpha: 3x + 6y - 6z + 3 = 0$$
>
> b) Determine the coordinates of the point of tangency Q between the sphere and α.

The problem is equipped with the illustration shown in Figure 29.

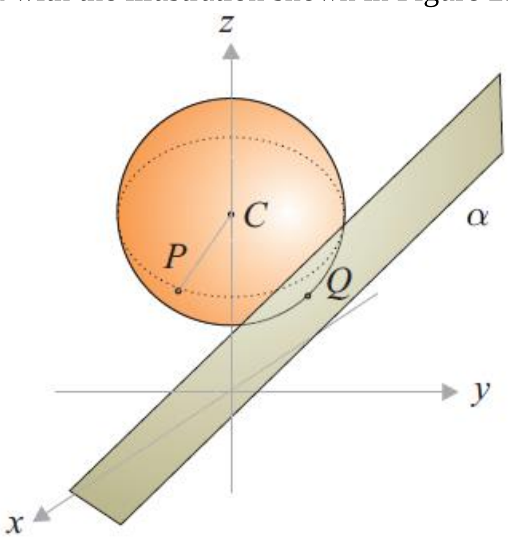


Figure 29 – Sketch of the situation in Problem 12 (Danish Ministry of Education, 2018c)

The style of exam problems, where the text is equipped with a figures, is new compared to the historical periods. Except from that, analytical-geometrical vector-problems are very similar in style compared to problems that appeared in the end of the previous period, where multiple technologies are invoked in the same problem.

Vector algebra-problems

Since 2005 the vector algebra-problems have always been coordinate-based and they are mostly two dimensional. They are often posed in the part of the exam that do not allow any aids. The tasks can easily be categorised in a handful of different types of tasks. They are solved by combinations of the vector techniques $\tau^{|\cdot|}$, τ^{proj}, $\tau^{\perp}$, $\tau^{\parallel}$, τ^{det}, $\tau^{parallelogram}$, and $\tau^{\overrightarrow{AB}}$, and the ordinary algebra techniques τ^{eq} and $\tau^{quadratic\ eq}$. Some examples are shown below. The first one is from 2010 and will be referred to as Problem 12 (Danish Ministry of Education, 2018c):

> In a coordinate system two vectors are given by
>
> $\vec{a} = \begin{pmatrix} t-1 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ t \end{pmatrix}$
>
> where t is a number.
>
> a) For $t = 4$ determine the angle between $\vec{a}$ and $\vec{b}$.
> b) Determine the values of t for which $\vec{a}$ and $\vec{b}$ are parallel.

Sub-question a) is solved by the technique $\tau^{\angle}$ in combination with τ^{eq}. This problem is posed in the part of the exam that allows aids. A sketch of the solution is shown in Table 14.

$3 \cdot 3 + 2 \cdot 4 = \sqrt{3^2+2^2} \cdot \sqrt{3^2+4^2} \cdot \cos(v)$	$\tau^{\angle}$
$v = \cos^{-1}\left(\frac{3 \cdot 3 + 2 \cdot 4}{\sqrt{3^2+2^2} \cdot \sqrt{3^2+4^2}}\right) = 19{,}44°$	τ^{eq}

Table 14 – A sketch of how the techniques $\tau^{\angle}$ and τ^{eq} are used to solve a) in Problem 12

Sub-question b) is solved by the technique $\tau^{\parallel}$ in combination with $\tau^{quadratic\ eq}$. The solution is shown in Table 15.

$(t-1) \cdot t - 2 \cdot 3 = 0$	$\tau^{\parallel}$
$t^2 - t - 6 = 0$	$\tau^{reduction}$
$t = -2 \wedge t = 3$	$\tau^{quadratic\ eq}$

Table 15 – A sketch of how the techniques $\tau^{\parallel}$ and $\tau^{quadratic\ eq}$ are used to solve b) in Problem 12

In the guiding exam problems from the 2017-reform a new type of task showed up. In this task the algebraic approach is mixed with the geometric approach. The problem is shown below, and will be referred to as Problem 13 (Danish Ministry of Education, 2018c):

> The figure [see Figure 30] shows representatives for three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.
>
> a) Draw a representative of the vector $2 \cdot \vec{a} + \vec{b} - \vec{c}$.
>
> If convenient use the enclosed.

The figure that is mentioned is shown in Figure 30.

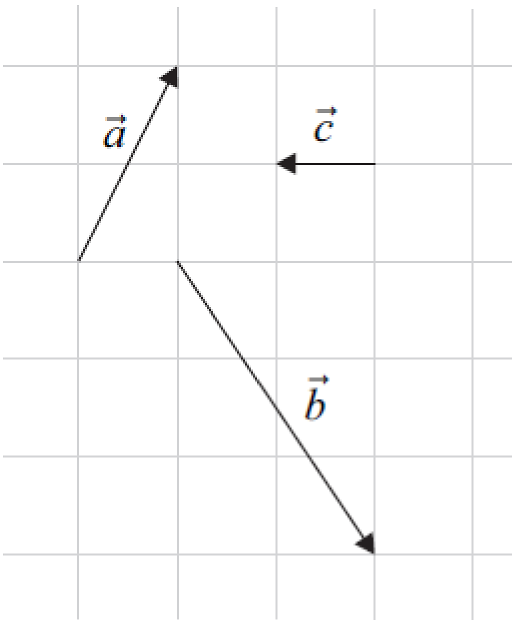


Figure 30 – Figure from Problem 14

Since this type of task is new in the context of written exams, it is not unambiguous how this task is solved. It can either be solved by the algebraic coordinate-based techniques τ^{sum}, $\tau^{difference}$, and $\tau^{scalar\ mult}$, if the coordinates are determined from the figure. Alternatively, a more geometric solution could be given by the use of a combination of the three geometric/algebraic techniques $\tau^{geom\ sum}$, $\tau^{geom\ diff}$, and $\tau^{geom\ scal\ mult}$. The geometric solution is shown in Figure 31.

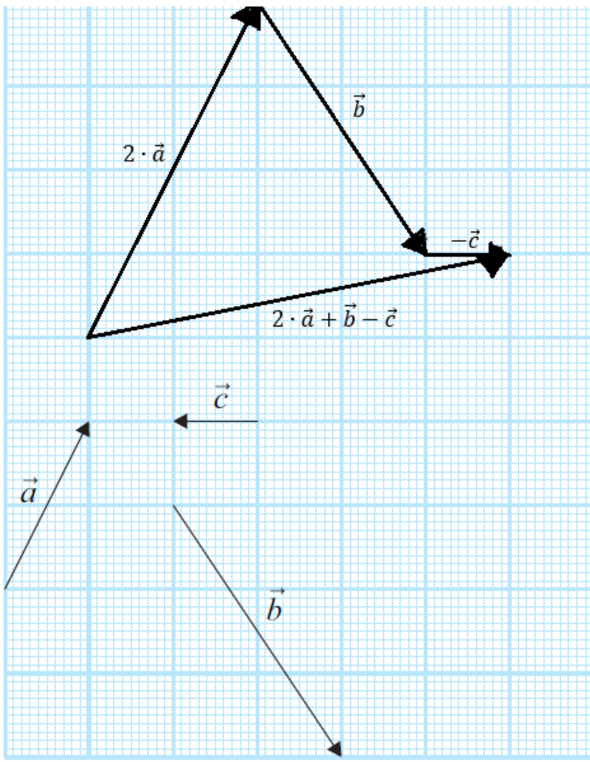


Figure 31 – Geometric solution to the algebraic/geometric problem from the guiding exam problems from the 2017-reform

Textbook

In the recent period a lot of different textbooks have been used. One of the most prevalent is "Gyldendals Gymnasiematematik" written by Flemming Clausen, Gert Schomacker, and Jesper Tolnø and the second edition, that will be used here, was published in 2012. In this textbook, the need for the notion of vectors is established by the following example (Clausen, Schomacker, & Tolnø, 2012, p. 91):

> When one needs to report the position of a school, a church, or an ancient monument, both a distance and a direction is needed: That the way to school is 5 km long, does only tell, that the school *S* is situated within a circle with the home *H* as centre and 5 km as the radius (figure 401). On the other hand, the following information tells exactly where the school is situated: The school is situated 3,5 km from *H* north by east (figure 402).

The figures that are mentioned in the example are shown in Figure 32, corresponding to figure 401 and Figure 33, corresponding to figure 402.

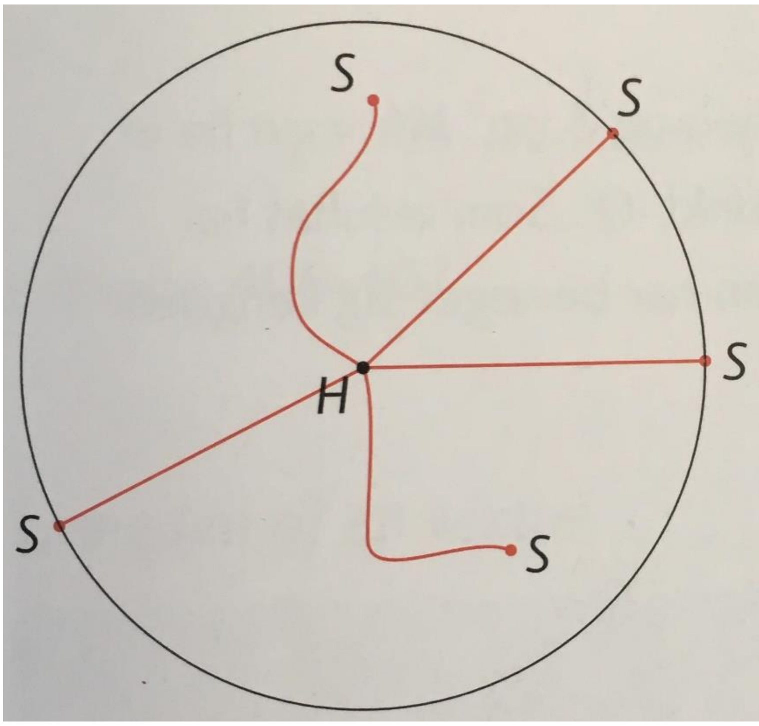


Figure 32 – Figure 401 from (Clausen et al., 2012, p. 91)

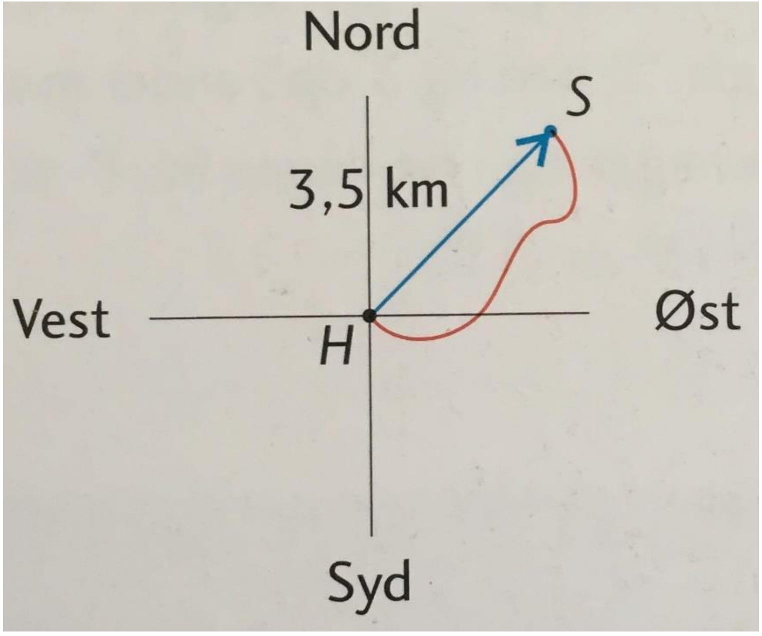


Figure 33 – Figure 402 from (Clausen et al., p. 91)

An additional example is given. This is even more physical, since it is centred around the velocity of a particle. After these two examples, the notion of vector is defined (Clausen et al., 2012, p. 91):

> A length together with a direction is called a *vector*. In a coordinate system a length and a direction can be denoted by a set of coordinates, e.g. $\vec{v} = \binom{2}{3}$.
> A *representative* of a vector $\vec{v}$ is obtained by choosing a point P and from this moving "2 horizontally and 3 vertically". By doing this, one ends in a point Q. The arrow from P to Q represents the vector.

This definition is equipped with two figures that show a representative of vector $\vec{v}$. There are two interesting remarks that can be made in relation to this definition. The first one is, that this book introduces the chapter on vectors with an example that serves the purpose of motivating the theory. This is different from many of the textbooks that were used in the previous periods, where the motivation was given *after* the definition. The other interesting remark, that makes the treatment of vectors very different from the ones in the previous periods, is the introduction to coordinates on the very first paper. This shows how the algebraic branch of the current mathematical organisation of vectors is coordinate-based alone. Almost all the definitions, formulas, theorems, and proofs are coordinate-based, and the same counts for exercises.

Furthermore, this book differs from the previous textbooks by having an extensive part about the applications to analytical-geometrical vector-problems, which have not been taking up a lot of space in the other books.

3.3.3 Current physical organisation

This analysis of the current physical organisation of vectors include the curricula from 2013 and 2017 and the respective guidelines for interpretation.

None of the curricula mention vectors and the guidelines for interpretation from 2013 states that (Danish Ministry of Education, 2010):

> It is not a requirement that the traditional vector formulation is used in the description of motion in two dimensions. It can even, in many cases, be an advantage to carry out calculations coordinate-wise. Aside from this, the description must match the students' mathematical preconditions.

This citing shows how the physics teaching is almost fully detached from mathematics, at least regarding the teaching and use of vectors in physics.

Written exams

In the written exams throughout the years vector-like objects have appeared in the disguise of arrows. An example could be the following from the written exam in August 2012 (Danish Ministry of Education, 2018c):

A Segway has the maximal speed of $5{,}6\,\frac{\mathrm{m}}{\mathrm{s}}$.

a) How much time will a Segway use to travel 800 m, when it is driving at its maximal speed?

A Segway is driving with the speed $4{,}2\,\frac{\mathrm{m}}{\mathrm{s}}$. The Segway is braking with a constant acceleration at the magnitude of $4{,}7\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

b) How long is the braking distance for a Segway in this situation?

A Segway is driving at constant speed up a hill that has an inclination of $8{,}0°$ from horizontal. The total mass of the Segway with driver is 145 kg. The air resistance, friction in bearings, and the rolling resistance gives a force with the magnitude 130 N that is facing backwards.

c) Sketch in on appendix 1 arrows that show the magnitude and direction of the forces that affect the Segway while it is driving uphill.

The mentioned appendix 1 is shown on Figure 34.

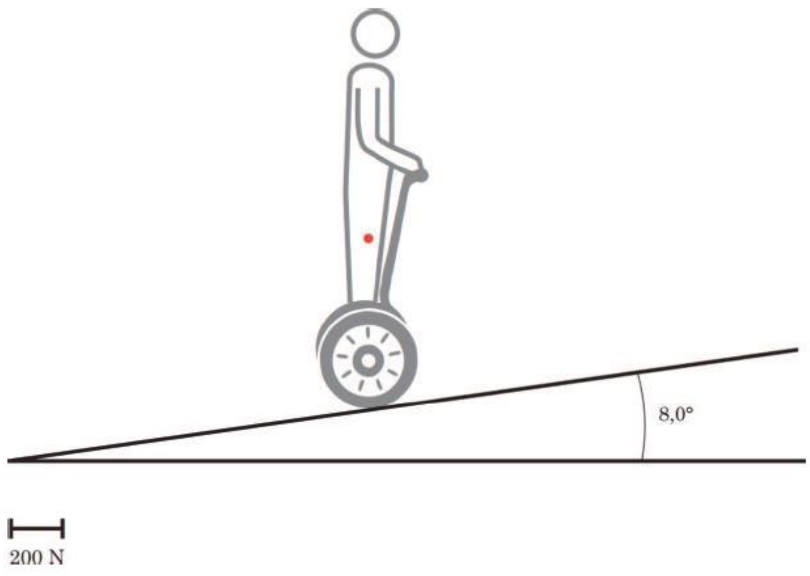


Figure 34 – Appendix 1 that is needed for the vector-related physics problem

This type of physics tasks, that involves vectors has appeared in written exams at least since the 1990's, and it is the only type of task that includes vectors.

Textbook

Though the guidelines for interpretation of the curriculum is almost advising against using vectors in the treatment of motion in two dimensions one of the most prevalent textbook in physics, “Vejen til fysik A”, that is written by Knud Erik Nielsen and Esper Fogh and published in the first edition in 2007, *do* mention how two-dimensional motion can be described by the use of vectors. This is done by creating the coordinate function $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. From this both velocity and acceleration on coordinate functions are derived. However, most calculations are made by the use of coordinates instead of vectors.

In the chapter about forces and movement vectors play a more important role. After a short introduction to the notion of forces, a section about the composition of forces introduces the notion of vectors (Nielsen & Fogh, 2011, p. 212):

> When we describe the effect of a force on a particle we need to specify both the direction and the magnitude of the force. In praxis this is done by representing the force by an arrow. The direction of the arrow gives the direction of the force and the length of the arrow gives the magnitude. Such an arrow is called a *vector*, here a *force vector*. When a particle is affected by multiple forces at the same time, we can determine the magnitude and direction of one single force, that has the same effect as the forces altogether. This is called the *resultant of* the single forces or their *resulting force*.

In the following the rule for composition of forces and the parallelogram of forces are described. Also a brief general description of vectors is given. However, vectors are mostly used in this book to depict especially the directions of different forces. Calculations are generally made with scalars. This is in line with the use of vectors in the written exam problems.

4. The Epistemological Reference Model

The examination of the mathematical organisation and the physical organisation of the scholarly knowledge on vectors, and the mathematical organisation and the physical organisation of the knowledge to be taught on vectors has shown how these four organisations are woven together in an inseparable constellation (see Figure 35).

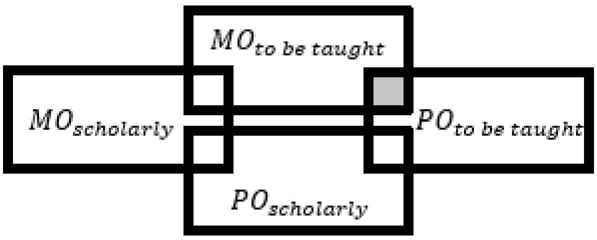


Figure 35 – The interrelation between the different organisations described in section 3

As shown on Figure 35 it is especially the interrelation between $MO_{to\ be\ taught}$ and $PO_{to\ be\ taught}$ (marked in grey) that is important to the empirical part of the thesis, but alto the interrelations $MO_{scholarly} \leftrightarrow PO_{scholarly}$, $MO_{scholarly} \leftrightarrow MO_{to\ be\ taught}$, and $PO_{scholarly} \leftrightarrow PO_{to\ be\ taught}$ are necessary to include in the ERM.

The relation $MO_{scholarly} \leftrightarrow PO_{scholarly}$

The relation between these two organisations is important in order to explain the potential in an interdisciplinary approach to the teaching of vectors in high school. As it has been shown in section 3, these two have developed along parallel tracks and in close interrelation. The two primary focus points in $MO_{scholarly}$ are the algebraic representation of the geometry of $\mathbb{R}^2/\mathbb{R}^3$ and the generalisation of the geometric properties of $\mathbb{R}^2/\mathbb{R}^3$ to higher dimensions. Furthermore, but not so important to this thesis, some of the tasks in $MO_{scholarly}$ are to use the most general theory of vector spaces to prove results in spaces (such as function spaces etc.) that do not have a geometric structure.

In $PO_{scholarly}$ the mathematical theory of vector spaces is used to carry out calculations in a compact manner (as it was described by e.g. Grassmann, who was able to shorten down extensive calculations). It is used in both classical mechanics to model vector quantities such as velocity, acceleration, and forces, or quantities from rotational mechanics such as angular momentum or moment of force, in electrodynamics to model vector quantities such as current or magnetic field, or in quantum mechanics in the shape of Hilbert spaces.

The potential in the relation between $MO_{scholarly} \leftrightarrow PO_{scholarly}$, in the context of high school teaching, is to provide motivation for the theory of vectors by the various applications. Furthermore, the development of the relation between $MO_{scholarly} \leftrightarrow PO_{scholarly}$ shows an example of how the algebraisation and generalisation of a theory have implied new results in physics, e.g. some of the findings in electrical theory as mentioned in section 3.2.1. Though $MO_{scholarly}$ gives the algebraic representations and methods a higher priority than the more geometric/visual representations and methods, it is mostly the geometric approach (that was also one of the initial approaches in mathematics) that is used for practical purposes in physics. This circumstance can beneficially be taken into account in an analysis of where problems in an interdisciplinary approach to vectors in high school comes from.

The relation $MO_{scholarly} \leftrightarrow MO_{to\ be\ taught}$

This relation, and especially the development of it, is important in order to understand and justify the $MO_{to\ be\ taught}$. Furthermore, a deeper understanding of this relation can help to explain why high school students struggle with the notion of vectors. As it has been shown in section 3.3.1 the relation has gone through a development illustrated in Table 16.

Period	Relation between MO's/PO's	Techniques
1935-1953	$MO_{to\ be\ taught}$ very far from $MO_{scholarly}$ but almost containing the $PO_{to\ be\ taught}$ of that time.	$\tau^{geom,comp}$, $\tau^{geom,decomp}$
1961-2005	$MO_{to\ be\ taught}$ close to the praxis and the logos block of the algebraic structure of the part of $MO_{scholarly}$ that contains the vector space $\mathbb{R}^2(/\mathbb{R}^3)$ combined with the praxis block the geometric applications of $\mathbb{R}^2(/\mathbb{R}^3)$ in $MO_{scholarly}$. $MO_{to\ be\ taught}$ and $PO_{to\ be\ taught}$ outdistancing from each other.	$\tau^{distributive(+,\cdot)}$, $\tau^{\lvert\cdot\rvert,\cdot}$, $\tau^{\perp,\cdot}$, $\tau^{\angle,\cdot,\lvert\cdot\rvert}$, $\tau^{proj,\lvert\cdot\rvert,\cdot}$, $\tau^{\lvert\cdot\rvert\ of \perp vector}$, $\tau^{par,\lvert\cdot\rvert,\angle}$, τ^{sum}, $\tau^{difference}$, $\tau^{scalar\ mult}$, $\tau^{\lvert\cdot\rvert}$, $\tau^{\wedge}$, τ^{proj}, $\tau^{\perp}$, $\tau^{\parallel}$, τ^{det}, $\tau^{parallelogram}$, $\tau^{\overrightarrow{AB}}$
2005-2017	$MO_{to\ be\ taught}$ containing primarily the coordinate based praxis block of the part of $MO_{scholarly}$ that contains the vector space $\mathbb{R}^2$ and the praxis block of its geometric applications. $MO_{to\ be\ taught}$ and $PO_{to\ be\ taught}$ very far from each other, especially because $PO_{to\ be\ taught}$ is almost non-existing.	τ^{sum}, $\tau^{difference}$, $\tau^{scalar\ mult}$, $\tau^{\lvert\cdot\rvert}$, $\tau^{\wedge}$, τ^{proj}, $\tau^{\perp}$, $\tau^{\parallel}$, τ^{det}, $\tau^{parallelogram}$, $\tau^{\overrightarrow{AB}}$ (In guiding exam problems from 2017 also $\tau^{geom,sum}$, $\tau^{geom,diff}$, $\tau^{geom,scal,mult}$)

Table 16 – Development of the relations between the organisations of vectors

The change from the period 1961-2005 to the period 2005-2017 has happened gradually, and the table shows how both the more abstract part of the praxis block (the coordinate-free tasks and techniques) has been removed while most of the logos block has been removed from $MO_{to\ be\ taught}$ too. Along with the reduction of the content in the $MO_{to\ be\ taught}$ some of the techniques have become instrumented and together with the higher focus on written exams that has caused a minimised focus on the logos block the justification and understanding of vectors as objects has almost disappeared.

The relation $PO_{scholarly} \leftrightarrow PO_{to\ be\ taught}$

This relation is not the most necessary to describe in the ERM, but one thing is important to include. As it has been described the scholarly physics is highly mathematised, which the development of vector analysis has also a part of. In the periods 1961-1971 and 1971-1984 the $PO_{to\ be\ taught}$ showed that this mathematisation of physics was also valid in high school physics. In these periods a lot of two and three dimensional mechanics was coordinate based and calculations were carried out with vectors. However, teachers experienced how students struggled with the heavy mathematised physics, and $PO_{to\ be\ taught}$ gradually abandoned vectors (and mathematics) more and more. A consequence of this might be a vicious circle where mathematics teachers and physics teachers and the mathematics and physics to be taught interacted less and less due to the students lack of interdisciplinary skills that was caused by less focus on it and so on and so forth.

The current $MO_{to\ be\ taught}$

The current $MO_{to\ be\ taught}$ is composed of the regional MO's $[T, \tau, \theta, \Theta^A]$ and $[T, \tau, \theta, \Theta^G]$. The praxis block of the algebraic branch is primarily coordinate based and consists of the tasks that can be solved by the techniques τ^{sum}, $\tau^{difference}$, $\tau^{scalar\ mult}$, $\tau^{|\cdot|}$, $\tau^{\wedge}$, τ^{proj}, $\tau^{\perp}$, $\tau^{\parallel}$, τ^{det}, $\tau^{parallelogram}$, and $\tau^{\overrightarrow{AB}}$. In the 2017-reform the geometric techniques $\tau^{geom\ sum}$, $\tau^{geom\ diff}$, and $\tau^{geom\ scal\ mult}$ to solve algebraic tasks are required to be mastered. The logos block of the algebraic branch has a lower priority, and is also primarily coordinate based. The applications and the motivation comes from the praxis block of the MO $[T, \tau, \theta, \Theta^G]$ and is very similar to the corresponding organisations from previous periods. Furthermore, it is more explicitly treated in current textbooks, compared to e.g. (Kristensen & Rindung, 1966) or (Jensen & Sørensen, 1981).

The current $PO_{to\ be\ taught}$

The current $PO_{to\ be\ taught}$ is very limited. Only one type of task is included in the written exams (t: draw the arrows that represents the forces that acts on a given body), and most often these tasks are easier solved by methods that are not vector-based.

The logos block has contained vectors as a model for vector quantities but this approach has gradually been abandoned, and the guidelines for interpretation of the 2013-curriculum recommended that the presentation of motion in two dimensions (which is one of the most obvious applications of vectors) is made without the use of vectors. This means that the $PO_{to\ be\ taught}$ does not promote the interrelation or recommend an interdisciplinary approach to vectors and vector quantities, mostly because the mathematicised physics teaching has gradually been abandoned.

The relation $MO_{to\ be\ taught} \leftrightarrow PO_{to\ be\ taught}$

As it has been mentioned this relation is the most important in the ERM. The relation suffers, primarily because the techniques that potentially *could* be useful in $PO_{to\ be\ taught}$ ($\tau^{geom\ sum}$ and $\tau^{geom\ diff}$) have a low priority in $MO_{to\ be\ taught}$. Instead most of the techniques in $M_{to\ be\ taught}$ are coordinate-based, and these techniques are not included in $PO_{to\ be\ taught}$.

A general observation made in the examination of the organisation of vectors in mathematics and physics was, that it is primarily the geometric properties of $\mathbb{R}^2/\mathbb{R}^3$ that are useful in physics, but that the techniques in the regional MO $[T, \tau, \theta, \Theta^G]$ are not included in $PO_{to\ be\ taught}$. Furthermore, the geometric interpretations of the algebraic properties and the techniques $\tau^{geom\ sum}$, $\tau^{geom\ diff}$, and $\tau^{geom\ scal\ mult}$ are useful in physics, but they have not been prioritised in the $MO_{to\ be\ taught}$ before the 2017-reform. Instead the $MO_{to\ be\ taught}$ have preferred the coordinate-based techniques over the geometric techniques.

II The Study and Research Path

5. Design

The backbone in the empirical part of this thesis is an interdisciplinary Study and Research Path in mathematics and physics on vectors. The aim for the SRP is to introduce the students to vectors in a setting, that focuses more on the geometric and physical interpretation of the properties that vectors have, than on the algebraic properties themselves.

The rest of section 5 will describe the process of designing the SRP. It will contain the context of the teaching sequence, some of the considerations that has been taken into account during the designing process, the purpose and learning goals of the teaching sequence, and the final SRP that has been the outcome of the a priori analysis of the generating question.

5.1 Context

As described in the introduction, one of the issues that has motivated this thesis is the new reform that was implemented by the Danish Ministry of Education in August 2017. Therefore, the target group for the SRP will be a Danish first year class at STX. Because of the interdisciplinary approach to vectors, the test class should preferably include students that are interested in physics and maybe even considering to upgrade physics to A-level. This requirement is due to the fact that no two-dimensional quantities are included in the C- and B-level physics curricula. Based on these requirements a test class was chosen.

5.2 The test class

The test class has mathematics at A-level, physics at B-level, and chemistry at B-level as their primary disciplines. It consists of 27 students, 10 of them males and 17 females. They entered high school in August and spent the period from August 7th to November 3rd in temporary classes. During this period all the first year students attended (intentionally) identical (or at least comparable) basic training courses in all subjects. After three months of basic training courses the students chose their main subjects and based on these choices the classes were reorganised. From November 6th, the class has been in the same constellation as it was on April 3rd when the test was launched.

5.2.1 Mathematical topics covered

The basic training course in mathematics was based on a compendium, that was used across the whole year group. The compendium covered some different mathematical topics. They are all listed in Table 17, because the whole content might have affected the students' work with the SRP:

Topic	Content
Variables	• Variables • Independent and dependent variable
Variable relations	• Coordinate system • The four representations of relations of variables • Translation between representations of relations of variable
Linear functions	• Linear regression • Linear functions
Functions	• What is a function?

Table 17 – Topics covered in the basic training course in mathematics

Roughly speaking, the mathematical content in the basic training course is collected in a regional MO based on the theory of functions, Θ^{fct}. The MO $[T, \tau, \theta, \Theta^{fct}]$ contains different local MO's, one of the primary being the one collected under the technology of translation between the four representations (graph, table, equation, and language). This technology will be denoted $\theta^{trans,rep}$. The local MO $[T, \tau, \theta^{trans,rep}\Theta^{fct}]$ contains different punctual MO's collected under different techniques, e.g. $\tau^{g\rightarrow t}$ (the translation between graph and table), $\tau^{e\rightarrow t}$ (the translation between equation and table), $\tau^{e\rightarrow g}$ (the translation between equation and graph), etc..

Nested in the MO $[T, \tau, \theta, \Theta^{fct}]$ is the MO $\left[T, \tau, \theta, \Theta^{lin,fct}\right]$ that is collected under the theory of linear functions, which is a part of the theory of more general functions ($\Theta^{lin,fct} \in \Theta^{fct}$). The regional MO $\left[T, \tau, \theta, \Theta^{lin,fct}\right]$ contains a local MO collected under the technology of linear regression, $\theta^{lin,reg}$. The local MO $\left[T, \tau, \theta^{lin,reg}, \Theta^{lin,fct}\right]$ contains different punctual MO's $\left[T, \tau^{eq}, \theta^{lin,reg}, \Theta^{lin,fct}\right]$ and $\left[T, \tau^{CAS}, \theta^{lin,reg}, \Theta^{lin,fct}\right]$, where the two techniques τ^{eq} and τ^{CAS} are the manual and computer assisted techniques to carry out linear regression. A sketch of the praxeological structure of the basic training course is shown in Figure 36.

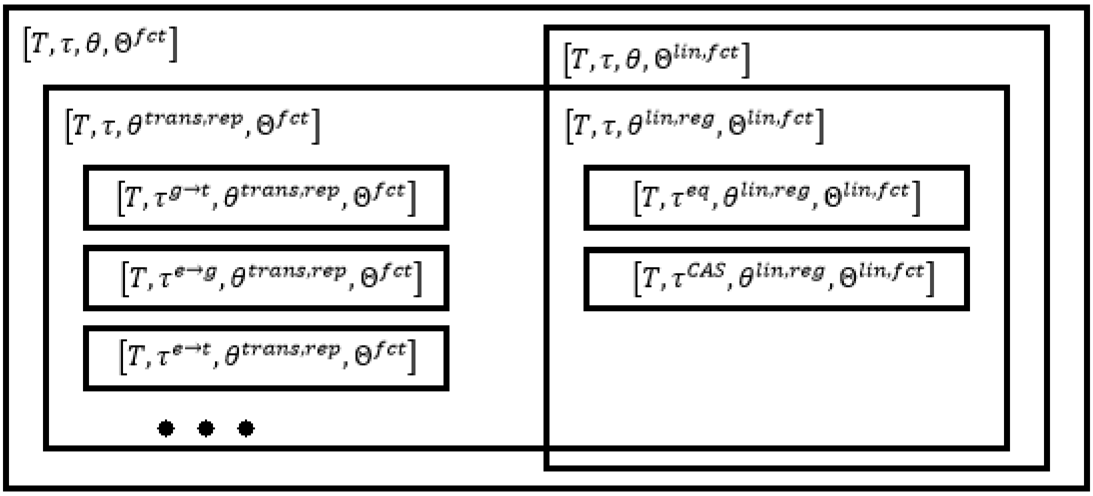


Figure 36 – Sketch of the praxeological organisation of the mathematical content in the basic training course

During the period from November 6th to March 23rd the class worked with different mathematical topics. Again they are all listed in Table 18, because they might have affected the students' work with the SRP:

Topic and duration	Content
Variable relations, functions, growth (25 lessons)	• The hierarchy of arithmetic operations • Manipulation with symbols • Inverse proportionality • Absolute value • Piecewise defined function • Composite functions • Inverse function • Absolute and relative growth • Index numbers • Percent and computation of rate of interest • Logarithmic functions • Exponential functions and exponential growth • Exponential regression
Statistics (15 lessons)	Descriptive statistics • Simple statistical descriptors • Simple representations of data with and without CAS • Examples of the use of statistics
Variable relations, functions, growth (12 lessons)	• Power functions and growth based on power functions

	• Power regression • Transformation of data • Description of graphs

Table 18 - Topics covered in mathematics after the basic training course and until the test of the teaching design

The topic 'Variable relations, functions, growth' that is covered over two periods extends the MO $[T, \tau, \theta, \Theta^{fct}]$. The topic 'Statistics' is contained in another regional MO $[t, \tau, \theta, \Theta^{stat}]$. A more detailed structure of this MO is not relevant for this study, and will not be dealt further with.

It will also be necessary to mention the technique of plotting data. In mathematics it has been taught and used in TI-Nspire in connection with regression. This technique will be denoted $\tau^{plot\ Nspire}$. In physics a similar technique has been taught in Excel. This will be denoted $\tau^{plot\ Excel}$. Both techniques are relevant in the analysis of the design.

5.2.2 The students' mathematical abilities

The mathematics teacher is experienced in teaching classes with the same combination of main subjects as the test class (mathematics at A-level, physics and chemistry at B-level). The teacher claimed, that in comparison to previous years' first year science-classes, the test class' mathematical abilities were lower. However, she did also claim that a decline in first year students' abilities has been a tendency over the past years. An additional claim was, that two circumstances in the new reform can have affected the students' progress negatively compared to previous years. The first problem that is claimed to be caused by new structure is that the students cannot necessarily build new mathematical knowledge on the same foundation. This seems to be conflicting with the idea of a common basic training course building on the same compendium, but however, the teacher has *observed* that the students' mathematical foundation varies depending on what teacher they have been taught by during the basic training course. The second problem is that the basic training course did not cover the same amount of material that has been covered by this teacher in the sciences classes through the first three months the previous years.

5.2.3 Organisation of ordinary mathematics lessons

The test class has been using the CAS-tool TI-Nspire since the beginning of November. They have not been instructed in how to use Excel in the mathematics lessons, but some of the students might be familiar with it from primary school. Furthermore, they have used it at least once in the basic training course in physics.

For most mathematics lessons the students have had a homework consisting of a few problems that they had to solve. In the beginning of the lessons the students were asked if they had difficulties solving the homework, and the potential difficulties and questions would be discussed.

5.2.4 Physical topics covered

Like in mathematics, the students attended a basic training course in physics during the first three months. The material that constituted the basic training course in physics across the year group was collected on a web page. The electronic compendium covered a list of topics. The ones that might have affected the students' work with the SRP are listed in Table 19:

Topic	Content
What is physics?	• SI base units • Definition of velocity • Mandatory experimental exercise: Velocity of bubbles • Excel template: linear regression for 3 data sets in same graph • Guidelines for journals • Alternative experiment: Energy of rolling ball • Alternative experiment: Bouncing ball

Table 19 – Topics covered in the basic training course in physics

For the purpose of this thesis, the important topic is mostly the first one, where the students encounter the notion of velocity for the first time. The definition of velocity is given in the following way (Stenhus Gymnasium, 2017):

> Figure 3.1 shows a (t, s)-graph of a linear motion. In the time span from t_1 to t_2 the position changes from s_1 to s_2. For this time span we define the average velocity $v_{average}$ by the formula
>
> 3.1 $v_{average} = \frac{s_2 - s_1}{t_2 - t_1}$
>
> From this we see that the SI-unit of average velocity is m/s:
>
> 3.2 $[v_{average}] = \frac{m}{s}$

The figure that is referred to is shown in Figure 37.

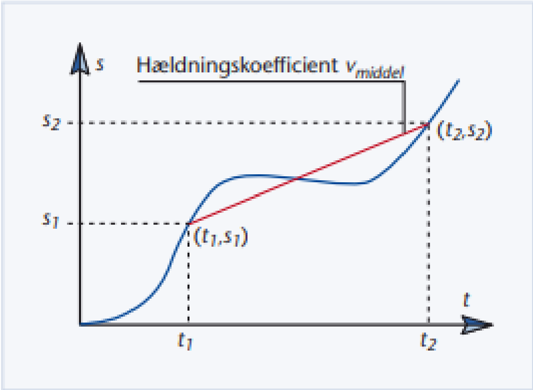


Figure 37 – Figure 3.1 from the basic training course-material (Stenhus Gymnasium, 2017)

Furthermore, the notions *momentary velocity* and *speed* are introduced. The experiments in the rest of the topic are centred around the notion of velocity.

During the period from November 6th to March 23rd the class worked with different topics. The ones that might have affected the students' work with the SRP are listed in Table 20:

Topic and duration	Content
Motion with respect to position, speed, and acceleration (8 lessons)	• Length • Time • SI base units • Speed • Experiment: Free fall • Force (gravitation) • Acceleration (gravitation) • Energy • LoggerPro (for data collection and analysis)
Energy including repetition from the basic training course (5 lessons)	• Experiment: Free fall • Potential energy • Kinetic energy

Table 20 – Topics covered in physics after the basic training course and until the test of the teaching design

Again it is mostly the first topic that is relevant to this thesis. A relevant part of the material that was covered is another definition of speed (Claussen, Both, Hartling, 2011, p. 15):

> **Speed** The speed v of an object is the travel distance per time unit. The speed is calculated by dividing the total distance by the time, that the ride takes:

$$v = \frac{\text{strækning}}{\text{varighed}}$$

The SI unit is: $\mathrm{m/s}$.

At first glance, the topics “Force (gravitation)” and “Acceleration (gravitation)” could seem to be relevant, but neither of these focus on direction or anything else that relates to vectors.

5.2.5 NV (the basic training course in natural sciences)

The topics covered in the basic training course in natural sciences have been reviewed in order to see if they contained anything that could be relevant in the SRP, but everything was either covered by the topics in mathematics or physics.

5.2.6 Social environment

The class has a good social environment. The students seem to care about each other, e.g. by noticing who is attending and who is absent. There is a comfortable atmosphere, and the students help each other with both practical issues and questions regarding mathematics. Generally, the students have a positive attitude towards group work and they are conscientious in that context.

Two students were repeaters. Both had entered high school one year before the others but started over in August 2017.

5.3 The purpose of the teaching sequence

The SRP has two main purposes. The first one is to create a necessity of the notion *vector* in the students, that will motivate their autonomous study and research. The second purpose is to reinforce the tie between mathematical vectors and physics, which will be done by changing the focus from an algebraic coordinate-based approach (which is the most common) to a combination of a more geometric but still coordinate-based approach.

Additionally, the teaching sequence have some more specific learning goals, that will make sure that it covers some of the notions in the curriculum and the guidelines for interpretation of the curriculum, that were presented in section 3.3.2. The learning goals are, that the students know the general notion of vector, the notions of zero vector, position vector, connection vector (vector from one point to another), orthogonal vector, and angle between vectors. Furthermore, the students are desired to be able to add vectors (both algebraic from coordinates and geometric), subtract vectors (both algebraic from coordinates and geometric), multiply a vector by a constant, determine the length of a vector, determine the coordinates of the vector from one point to another, determine the scalar product between two vectors, determine the determinant of two vectors, and determine the area of the parallelogram spanned by two vectors.

Some of the more praxis-oriented learning goals can be represented by tasks and techniques. These are presented in Table 21.

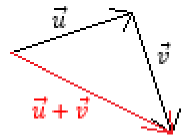
Task	Technique
Determine the sum of two vectors (geometric)	
Determine the sum of two vectors (algebraic)	$\vec{u}=\binom{u_1}{u_2}, \vec{v}=\binom{v_1}{v_2}, \vec{u}+\vec{v}=\binom{u_1+v_1}{u_2+v_2}$, (τ^{sum})
Multiply a vector by a number	$\vec{u}=\binom{u_1}{u_2}, k \in \mathbb{R}, k \cdot \vec{u} = k \cdot \binom{k \cdot u_1}{k \cdot u_2}$, $(\tau^{scalar\ mult})$
Determine the length of a vector that is given by coordinates	$\vec{u}=\binom{u_1}{u_2}, \lvert\vec{u}\rvert = \sqrt{{u_1}^2+{u_2}^2}$, $(\tau^{\lvert\cdot\rvert})$
Determine the coordinates of a vector from one point to another (algebraic)	$A(a_1,a_2), B(b_1,b_2), \overrightarrow{AB}=\binom{b_1-a_1}{b_2-a_2}$, $(\tau^{\overrightarrow{AB}})$
Determine the scalar product of two vectors (algebraic)	$\vec{u}=\binom{u_1}{u_2}, \vec{v}=\binom{v_1}{v_2}, \vec{u}\cdot\vec{v}=\binom{u_1}{u_2}\cdot\binom{v_1}{v_2}=u_1v_1+u_2v_2$, $(\tau^{scalar\ prod})$
Determine the determinant of two vectors (algebraic)	$\vec{u}=\binom{u_1}{u_2}, \vec{v}=\binom{v_1}{v_2}, \det(\vec{u},\vec{v})=u_1v_2-u_2v_1$, (τ^{det})
Determine the area of the parallelogram spanned by two vectors (algebraic)	$\vec{u}=\binom{u_1}{u_2}, \vec{v}=\binom{v_1}{v_2}, A_{parallelogram} = \lvert\det(\vec{u},\vec{v})\rvert = \lvert u_1v_2-u_2v_1\rvert$, $(\tau^{parallelogram})$

Table 21 – Learning goals in terms of tasks and techniques

In addition to the purely mathematical learning goals that has been described above, the students are desired to be able to apply especially the techniques $\tau^{\lvert\cdot\rvert}$, $\tau^{\overrightarrow{AB}}$, $\tau^{parallelogram}$, and τ^{sum} to the physical setting, that will be the framework of the SRP.

5.4 Considerations

During the designing process a lot of considerations have been made. First of all, it has not been easy to find examples of Study and Research Paths, or teaching designs in general, that combines mathematics and physics in the teaching of vectors. Examples of teaching sequences combining vectors in mathematics and forces in physics are available (e.g. (Doerr, 1996)), but the notion of forces is only in the A-level curriculum, and will certainly not have been covered within the first months of the first year. The test class had briefly touched the notion when dealing with motion in one dimension, but it had not been enough to contribute in an interdisciplinary teaching sequence on vectors.

The lack of SRPs on vectors in mathematics and physics has meant, that the main work has been put into the process of coming up with ideas for a suitable topic that contains the relevant use of vectors and some physics that is not too complicated for first year students.

The original aim was to design a Study and Research Path that could guide the students to the notion of vectors by themselves for example by offering a collection of media, that would be the object of study in the SRP. Since the notion of vectors is very complicated, new and abstract compared to the other topics the students had already been working with so far, this idea was abandoned.

Instead, the idea of a motivating setting with a problem that would be solvable with the use of vectors, and an introduction to vectors made by the teacher in the middle, was preferred.

In the preliminary phases of the designing process, it was considered whether the physics teacher should be involved and to what extension. Time pressure and the physics teacher's absence from work due to illness caused the decision not to use the physics teacher. Another reason for this decision was, that the eight mathematics lessons seemed to be a suitable amount of time to carry out a teaching sequence that should introduce vectors.

Table 22 shows how the lessons are spread over the two weeks:

	Week 1	Week 2
Monday	Day off due to Whit Monday	No mathematics lessons
Tuesday	Lesson 1: 11^{50}-12^{45}	Lesson 5: 11^{50}-12^{45}
Wednes-day	No mathematics lessons	No mathematics lessons
Thursday	Lesson 2: 11^{50}-12^{45}	Lesson 6: 11^{50}-12^{45}
Friday	Lesson 3+4. First 3: 8^{15}-9^{10}. Ten minutes' break. Second lesson: 9^{20}-10^{15}	Lesson 7+8. First lesson: 8^{15}-9^{10}. Ten minutes' break. Second lesson: 9^{20}-10^{15}

Table 22 – Distribution of mathematics lessons

In the process of designing the SRP, the theoretical study of the development of vectors in mathematics and physics respectively was taken into account. Since the theoretical study showed, that the algebraic branch of the mathematical organisation of vectors is very far from the physical organisation of vectors, one of the requirements in the generating question is to use the coordinate-based techniques on a physical task.

Furthermore, the geometric techniques from mathematics will be introduced in a physical setting as well, such that the students get used to representing vector quantities from physics by vectors. Also the geometric technique, $\tau^{geom\, sum}$, will be devoted attention in a physical context.

5.4.1 The final generating question and SRP

The result of the considerations that were made in the preliminary phase of the designing process is a generating question in the setting of a problem that is constructed for the purpose of the teaching sequence, but formulated in an authentic framework. The setting is that the students are recruited by Malaysia Airlines to assist them in the investigations of a plane-crash. The generating question in the first part of the investigation is:

$Q_{0,I}$: What has happened to the aeroplane?

In addition to this, the students get some different pieces of information, that can be used (see Appendix A):

- A map showing the intended route from departure to arrival
- The distance from the starting point to the ending point
- The duration of the flight under normal circumstances
- A spreadsheet containing the position (sets of x- and y-coordinates) of the aeroplane every minute during the flight

The students will then work with this question during two lessons (Lesson 1 and Lesson 2). As a built-in task the students will have to produce a poster, and present their results orally to the rest of the class.

In Lesson 3+4 the students will have a break from the investigations, to be introduced to the theory of vectors by their teacher. These two lessons are classical "whiteboard-teaching"-lessons, but to keep the connection to the aeroplane-problem the examples that are chosen to illustrate vector quantities are related to velocity and aviation. As an exercise, the students are asked to consider what happens to an aeroplane that is flying in crosswind. This example is used to illustrate geometric vector addition. Furthermore, the students are asked to work with an interactive "game-like" exercise, where they have to land an aeroplane safely while flying in a very strong crosswind (see Figure 38) (Bourne, M. (2017, August 21)).

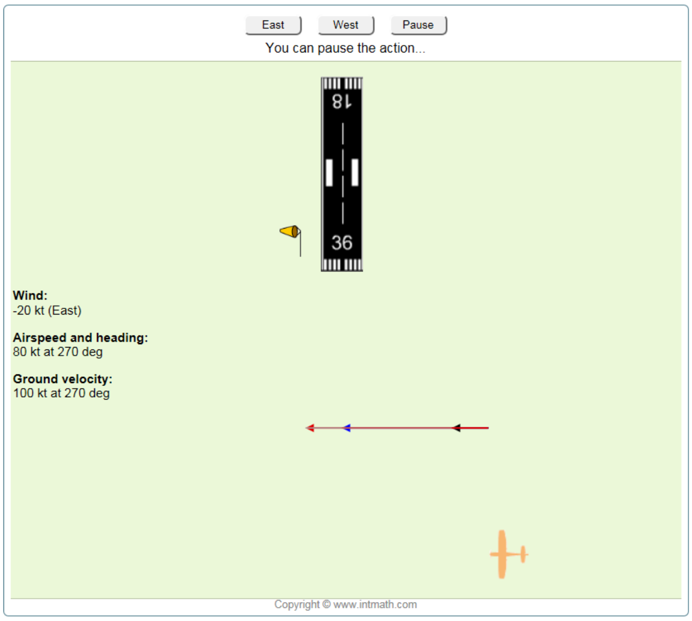


Figure 38 – Game to illustrate vector addition by the example of velocity and crosswind

After the introduction to vector theory, the students will get additional information in lesson 5 and a more specific generating question to lead the second part of the investigations. The question is:

$Q_{0,II}$: How big is the area that has to be searched in order to find the aeroplane?

The additional information, that is meant to call for the application of vector theory, is:

- The amount of fuel and an estimate on how far this amount can take the aeroplane
- Wind conditions during the last part of the flight

The handouts that are made to guide the students are found in Appendix A.

5.5 Lesson plan

While working with the generating questions, the students are divided into nine groups containing three persons each. The groups were formed by the teacher beforehand.

Time	Time accumulated	Students	Teacher
Lesson 1			
5 min.	5 min.	Listening	(+ researcher) introducing the project
5 min.	10 min.	Asking questions about the project	Dividing groups, handing out materials, asking questions about the project
35 min.	45 min.	Working on the problem and writing logbooks	Circulating the classroom to get an overview of the process
10 min.	55 min.	Preparing the poster and oral presentation	Circulating the classroom to get an overview of the process
Lesson 2			
15 min.	15 min.	Presenting posters and asking questions	Listening, directing order etc.
20 min.	35 min.	Working on the problem, updating posters, preparing presentations and writing logbooks	Circulating the classroom to get an overview of the process
20 min.	55 min.	Presenting posters and asking questions	Listening, directing order etc.
Lesson 3+4			
55 min.	55 min.	Listening and asking questions	Presenting vector theory on the whiteboard (see section 5.4.1 for more details)

10 min.		Break	Break
55 min.	110 min.	Listening and asking questions	Presenting vector theory on the whiteboard (see section 5.4.1 for more details)
Lesson 5			
5 min.	5 min.	Listening	Handing out materials
50 min.	55 min.	Working on the problem and writing logbooks	Circulating the classroom to get an overview of the process
Lesson 6			
50 min.	50 min.	Writing reports	Circulating the classroom to get an overview of the process
5 min.	55 min.	Listening and asking questions	Briefing students about feedback procedure for Lesson 7+8
Lesson 7+8			
35 min.	35 min.	Reading and discussing reports and preparing feedback	Circulating the classroom to get an overview of the process
15 min.	50 min.	Giving/receiving feedback	Circulating the classroom to get an overview of the process
10 min.		Break	Break
15 min.	65 min.	Giving/receiving feedback	Circulating the classroom to get an overview of the process
15 min.	80 min.	Giving/receiving feedback	Circulating the classroom to get an overview of the process
20 min.	100 min.	Adjusting reports in view of feedback	Circulating the classroom to get an overview of the process
10 min.	110 min.	Listening and asking questions	(Researcher) rounding of the project

Table 23 – Lesson plan

5.5.1 The teaching sequence about vector theory

Lesson 3+4 was planned to be a break from the plane crash-investigations. Instead of working independently with the generating question, the students were given a presentation of some, hopefully useful, theory on vectors.

In the teaching sequence vectors will be defined and the motivation will be the study of physical quantities having both magnitude and direction. The properties and notions that will be introduced are: geometric vectors, the coordinates of vectors, the zero vector, lengths of vectors, the sum of vectors from coordinates, scalar multiplication, geometric vector-addition, opposite vector, geometric vector-subtraction, the difference of two vectors from coordinates, position vector, coordinates of a vector from a point A to a point B, scalar product, angle between two vectors, orthogonal vectors, the orthogonal vector, determinant, parallelogram spanned by two vectors, and area of parallelogram spanned by two vectors.

These notions are exactly the notions that were included as learning goals for the teaching sequence.

6. A priori analysis

In this section an a priori analysis of the generating questions will be presented. This represents the paths that the students are expected to follow. Since the SRP is separated into two parts, one before and one after the introduction to vectors, there will be two Q&A-diagrams. The first diagram will represent the expected questions and answers coming from the information presented in the first hand out (see Appendix A.1.1). The second diagram will represent the expected questions and answers coming from the students' new knowledge about vector theory and the second hand out (see Appendix A.2).

6.1 Q&A-diagram for the first part of the SRP

The generating question for the first part of the SRP is the following:

$Q_{0,I}$: What has happened to the aeroplane?

An a priori analysis of this question leads to the expected path described below (see also Figure 39):

Q_1: How does the route, that the aeroplane followed, look?
$Q_{1,1}$: What is remarkable about the route?
$Q_{1,1,1}$: Why did the aeroplane make a "loop"?
$Q_{1,1,2}$: Why did the plane suddenly make the change in course in minute 56?
$Q_{1,2}$: How far from Farawayistan Airport is the aeroplane, when it disappears from the radar?
$Q_{1,2,1}$: How is the distance to Farawayistan Airport calculated from the x- and y-coordinates?
$Q_{1,3}$: How far from Neverland Airport is the aeroplane, when it disappears from the radar?
Q_2: By what speed/velocity did the aeroplane fly during the flight?
$Q_{2,1}$: How is speed/velocity calculated from position and time?
$Q_{2,1,1}$: How is the fact that the aeroplane is moving in two directions taken into account, when calculation speed/velocity?
$Q_{2,2}$: How fast does an Airbus a320 fly?
$Q_{2,3}$: What can be derived about the flight from its velocities?
$Q_{2,3,1}$: Why was the speed lowered by the end of the monitored part of the flight?

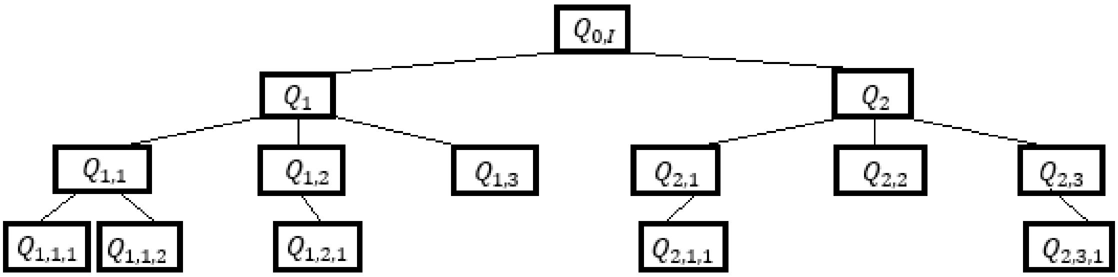


Figure 39 – Tree diagram showing the a priori analysis of the first part of the SRP

Based on the prerequisites from the mathematics teaching the question Q_1 is very likely to be posed (either implicitly or explicitly) and answered by making a scatter plot in Excel or TI-Nspire (technique τ^{plot}). A scatter plot of the data is shown in Figure 40.

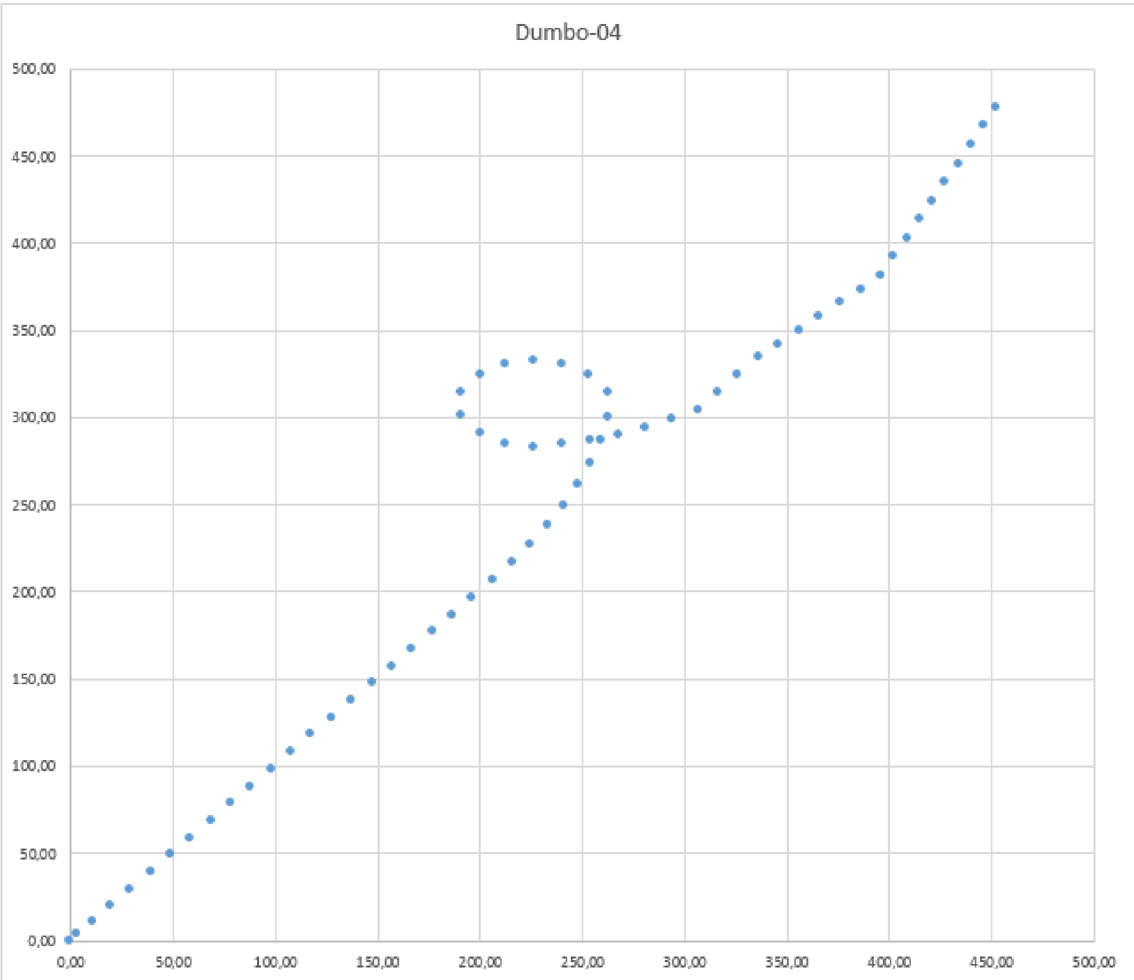


Figure 40 – Scatter plot of the data that is given to the students

A scatter plot of the data will be denoted A_1, since it is the answer to the question Q_1. The scatter plot reveals some remarkable moves that the aeroplane has made. These moves are the “loop” (the red mark on Figure 41) and the two sharp changes in direction (the green and the yellow mark on Figure 41).

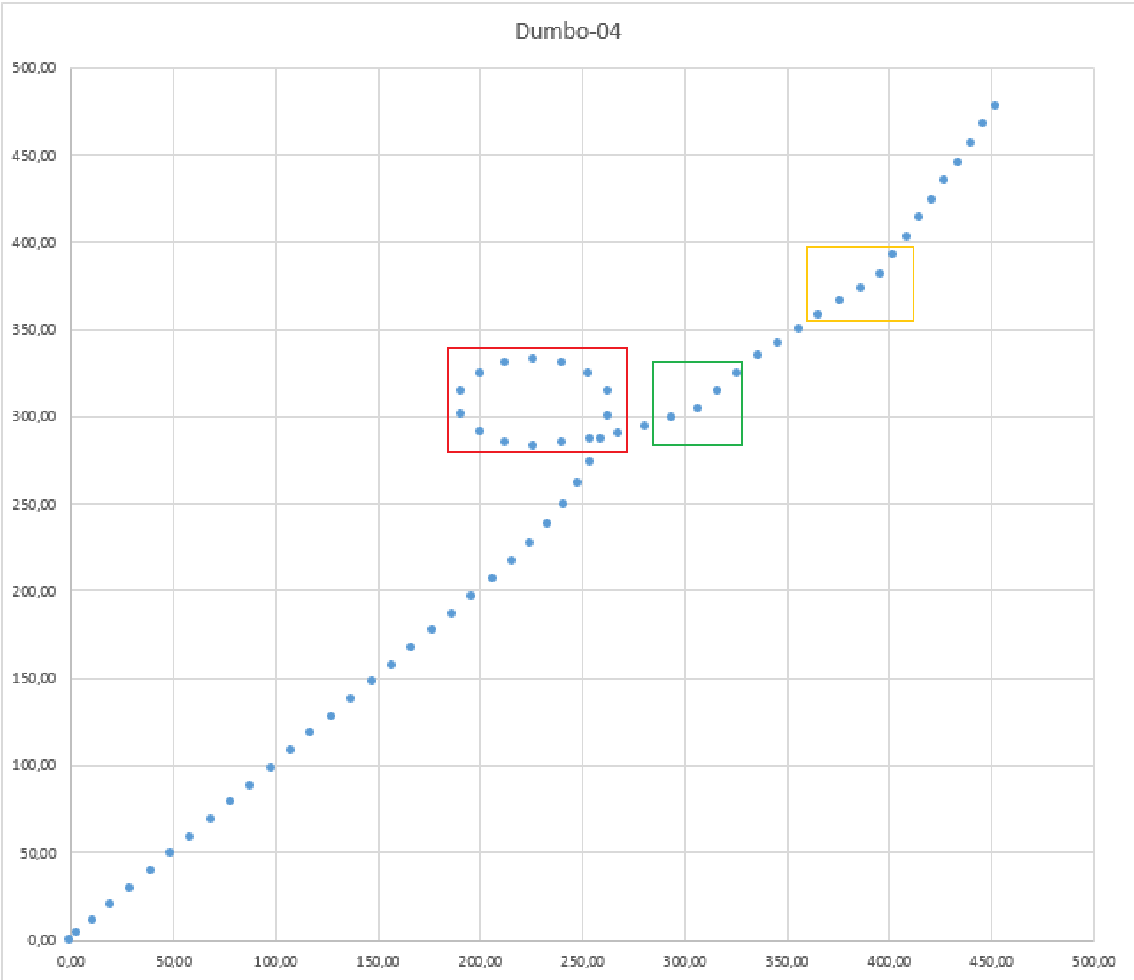


Figure 41 – Scatter plot with remarking details marked

Since the students are mostly used to work with linear, exponential, and power functions all or some of these irregularities will be mentioned in the answer to the question $Q_{1,1}$. The answer will be denoted $A_{1,1}$. Most likely the students will ask the question $Q_{1,1,1}$ and if their answer to $Q_{1,1}$ includes the rapid change in direction after 56 minutes maybe also the question $Q_{1,1,2}$. All three “irregularities” (the red, the green, and the yellow marks on Figure 41) are put in the data set to make the students consider the difference between speed and velocity, since this will produce a need for the theory of vectors.

The questions $Q_{1,1,1}$ and $Q_{1,1,2}$ are not directly necessary for the students to pose and answer in order to answer the generating question, but however they are in the diagram because the students will most likely wonder where these irregularities come from. Some students might search the internet for reasons why aeroplanes make these strange moves.

Whatever the students might find on the internet will be denoted $A_{1,1,1}$ and $A_{1,1,2}$ respectively. Some groups might include wind conditions in $A_{1,1,2}$, which would be good for the introduction to vectors and for the second part of the SRP.

Another branch of questions rising from Q_1 is about the distance that the aeroplane has travelled. This is highly relevant for the generating question to be answered, and it seems very likely that the students will pose the question $Q_{1,2}$ (or similar, e.g. $Q_{1,3}$, or both). Since the students have not worked with motion in two dimensions the question $Q_{1,2}$ will probably lead to the sub-question $Q_{1,2,1}$ about how the travelled distance is calculated from the x- and y-coordinates. Some students will probably know about the Pythagorean theorem and use this to calculate the distance, otherwise it will probably be found on the internet and applied. The Pythagorean theorem will be denoted $A_{1,2,1}$ and the answer $\sqrt{452{,}78^2 + 477{,}50^2} = 658{,}04$ will be denoted $A_{1,2}$. Some groups might answer $Q_{1,3}$ by subtracting $658{,}04$ from 1000 ($1000 - 658{,}04 = 341{,}96$). Other groups might answer it by calculating the coordinates of Neverland Airport and then carry out the calculation: $\sqrt{(707{,}11 - 452{,}78)^2 + (707{,}11 - 477{,}50)^2} = 342{,}64$.

The questions with primary index number 1 are all about distance. The other main branch in the first part of the SRP that has primary index number 2 and all the questions are about velocity. These questions are important for the transition into the theory of vectors, since the introductory example will be about velocities. However, it will probably only be the fastest working groups that make it that far in the first lesson.

The students have worked with both speed and velocity shortly, and therefore the question Q_2 is likely to be posed. Since the students have only touched the notions velocity and speed briefly some groups might need to look up the formula for velocity – they ask the question $Q_{2,1}$. This question will either be answered by notes from their physics teaching or by searching the internet for the formula. The formula $v = \frac{\Delta s}{\Delta t}$ will be denoted $A_{2,1}$. Since the students have not worked with motion in two dimension they will need to look up how this formula is translated into the two dimensional setting that they work in. They will probably find out that velocities in two dimensions can be calculated in the directions of the axes, such that $v_x = \frac{\Delta s_x}{\Delta t}$ and $v_y = \frac{\Delta s_y}{\Delta t}$, and maybe also that the *speed*, $|v|$, is calculated from $|v| = \sqrt{{v_x}^2 + {v_y}^2}$. These formulas are denoted $A_{2,1,1}$. From these formulas the students will be able to answer Q_2. The answer is that the aeroplane keeps the speed $835\frac{km}{h}$ for most of the flight, but that it lowered twice. The first time (about minute 51) to $765\frac{km}{h}$ and the second time (about minute 57) to $745\frac{km}{h}$. This answer will be denoted A_2.

Some groups might want to compare the velocities, that they have calculated to the normal speed of an Airbus a320 to see if there is something remarkable about the velocities.

On the internet the students might find that an Airbus a320 has the normal speed $835\,\frac{km}{h}$ and that the maximal speed is $900\,\frac{km}{h}$. This answer will be denoted $A_{2,2}$. The students might want to compare the calculated velocities to the normal speed of an Airbus a320. They will probably ask the question $Q_{2,3}$. The answer to this question will be that the aeroplane flies at normal speed for the first 50 minutes and after that it is lowered. This answer is denoted $A_{2,3}$. Some groups might ask why the speed is lowered ($Q_{2,3,1}$). However, this question cannot be answered from the information that is provided.

This Q&A-diagram for the first part of the SRP might be quite ambitious, and all the questions and answers will probably not be posed by all groups. Hopefully, the class as a whole will cover both some of the "distance"-branch (the questions with primary index number 1) and some of the "velocity"-branch (the questions with primary index number 2), because this will constitute a good foundation of the discussion of vectors.

After the first part of the SRP the class will be taught some vector theory, that they will need to apply in order to answer the generating question in the second part of the SRP. The content of the teaching sequence has been described in section 5.5.1.

6.2 Q&A-diagram for the second part of the SRP

The second part of the SRP will be directed by the following question together with the additional information described in section 5.4.1:

$Q_{0,II}$: How big is the area that has to be searched in order to find the aeroplane?

An a priori analysis of this question leads to the expected path described below (see also Figure 42):

Q_3: How can vector theory be applied to calculate the area that has to be searched in order to find the aeroplane?
$Q_{3,1}$: Which two vectors should form the parallelogram?
Q_4: How can the information about cross wind be used?
$Q_{4,1}$: How are wind conditions taken into account, when calculating the velocity of an aeroplane?
$Q_{4,1,1}$: What is the "true airspeed" of the aeroplane, when it is flying in the cross wind?
Q_5: How can the information about the amount of fuel be used?
$Q_{5,1}$: For how many kilometres will the remaining fuel last?
$Q_{5,1,1}$: How big a distance has the aeroplane been travelling from take-off and until it disappears from the radar?

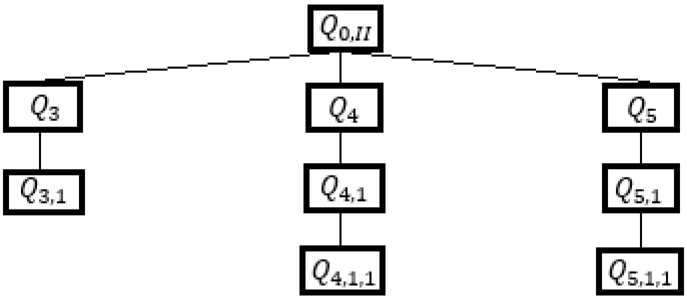


Figure 42 - Tree diagram showing the a priori analysis of the second part of the SRP

Since the students have been taught vector theory prior to the second part of the SRP they will most likely pose question Q_3 or something similar. In the light of the teaching sequence on vectors the answer to question Q_3 will probably be that they can use the area of the parallelogram spanned by two vectors. This answer will be denoted A_3. An obvious sub-question to Q_3 and A_3 will be $Q_{3,1}$. The answer to this question will depend on the assumptions that the students make, but the questions Q_4 and Q_5 are very likely to be posed in order to make the assumptions explicit. A possible sub-question to Q_4 is $Q_{4,1}$. When answering this question some of the groups will maybe search the internet and find out that pilots operate with more than one speed measure, namely "true airspeed" and "groundspeed" respectively. Some groups might have encountered this distinction already in the first part, or have considered it in connection with the little computer game that was incorporated in the teaching sequence. A possible answer to $Q_{4,1}$ could be that the velocity of an aeroplane can be measured relative to either the ground (where the velocity is lower in head-wind and higher in tail-wind) or to the surrounding air (where the velocity is independent of wind-conditions). This answer will be denoted $A_{4,1}$. With this information the students will probably ask $Q_{4,1,1}$ about the true airspeed. To answer this question the students will need to use the technique τ^{sum} that they have been taught in the teaching sequence. The answer that is produced by the use of this technique will be denoted $A_{4,1,1}$. From the answers to the sub-question a possible answer to Q_4 could be that the information about the crosswind can be used to determine the directions of the two vectors that span the parallelogram that they want to determine the area of. This answer will be denoted A_4.

An obvious sub-question to question Q_5 is $Q_{5,1}$. This is very likely to be posed by most of the groups. For this question to be answered the groups will need to answer the sub-question $Q_{5,1,1}$. This question can be answered by an application of their new knowledge on vector theory, since the distance can be calculated as the sum of the lengths of the vectors from point to point (by the technique $\tau^{|\cdot|}$). The answer to this question is that the aeroplane had travelled around 870 km before it disappeared from the radar. This answer is denoted $A_{5,1,1}$. From this answer the students will be able to answer $Q_{5,1}$ by subtracting the number 870 km from the maximal distance that the total amount of fuel can take the aeroplane. In the light of the answers $A_{5,1,1}$ and $A_{5,1}$ the question Q_5 can be answered. The information about the amount of fuel can be used to determine the lengths of the vectors which span the parallelogram that they want to determine the area of.

This a priori analysis of the generating questions shows how $Q_{0,I}$ will plant a need for the theory of vectors in the students by considerations about velocity, and how they will be able to apply vector theory both in the calculations of the travelled distance in the second part, in the considerations about true airspeed and groundspeed, and when they determine the area where the search for the crashed aeroplane should be made as the area of a parallelogram spanned by two vectors.

7. Methodology

This section will describe the circumstances of the execution of the teaching sequence and the collection and presentation of the data material.

7.1 Execution of the SRP

The teaching of the SRP and the built in teaching sequence was assigned to the teacher that was usually responsible for the mathematics teaching in the test class. This decision has both advantages and disadvantages. The first advantage is, that the students will most likely behave in a way that is relatable to an ordinary teaching situation, when they are taught by their ordinary teacher. Another advantage of assigning the teaching to another person is, that it will free the researcher to make and note down the necessary observations.

A disadvantage of this decision is, that when the responsibility of the teaching of the sequence is assigned to another person it is more difficult to control concrete teaching situations. Furthermore, the teacher will need a very thorough introduction to the teaching sequence, the a priori analysis of the generating questions, the potential of the generating question, and instructions in how to act on the different moves that the students will possibly make.

The teaching design was discussed with the teacher both beforehand and during the two weeks of teaching, but section 8 will show how this was not done carefully enough. Furthermore, the consequences of this problem will be presented.

7.2 Data collection

In the design different kinds of assignments were included. This counted logbooks, reports, and a poster. These assignments constitute the data material, that will be analysed in section 8. It would have been preferable to be able to include audio recordings in the analysis in addition to the written data material, but the recording of the conversations did generally fail in most cases.

Requirements for, the nature of, and volume of each of the four types of data material are described below:

- Logbooks: Each group handed in logbooks after both the first and the second lesson. The logbooks contained the thoughts, questions, and calculations that the groups had worked with during each lesson. The logbooks are found in Appendix B, which include 17 items, since Group 3 did not hand in the second logbook.
- Poster: Each group handed in a poster in the second lesson. The poster was supporting an oral presentation of the results that the groups had found so far. The posters did not contain any results or calculations that cannot be found in the logbooks, and have therefore not been attached to the appendix.
- Reports: Each group handed in a report by the end of the sixth lesson. The report presented all the results that they had found, including calculations and arguments. After a feedback-session with two other groups and the study of a guiding solution to the generating questions (see Appendix A.3) the groups rewrote the reports and handed them in. Both the first and the second version of the reports are found in Appendix C, that contains 18 items.
- Audio recordings: The students were asked to record their conversations during their work with the questions. A lot of groups did this, but it turned out to be impossible to share the files in most cases. One group (group 7) managed to record their work with the first generating question in the first lesson and share the file. This recording has been supporting the a posteriori analysis of the SRP made by group 7.

7.3 Detection of questions that are not explicitly posed

Though the students are requested directly to pose questions in their logbooks, they do not always do this explicitly. However, it is still possible to detect the underlying questions from the answers that they give.

An example of an implicitly posed question is found in the first logbook by group 4 (see Appendix B.6):

> By drawing the graph on TI-Nspire we have found out that the aeroplane flew in a circle (i.e. that it has been flying forwards, backwards, and forwards again). The beginning of the circle is in the point $(259.33; 286.77)$. The end is in the point $(267.77; 289.37)$. We can also see that the aeroplane crashed in the point $(452.78; 477.5)$. We can see that after the point $(396.26; 381.2)$, the aeroplane leans towards the left on the route.

No questions are posed before the group states this, but it is clearly the answer to the two questions “How does the route, that the aeroplane followed, look?” and “What is remarkable about the route?”, since they do at first make a scatter plot of the data to get a visual overview of the situation, and afterwards point out some of the coordinates in which the aeroplane makes some remarkable changes in direction.

Similar interpretations of the text in logbooks and reports are made over and over again to detect the implicitly stated questions in order to get an overview of the realised SRPs.

7.4 Diagrams and notation

The groups’ realised SRPs will be presented in section 8. The presentation will include the list of sub-questions they have posed (explicitly or implicitly) but also the more visually illustrative tree diagrams. In section 2.4 a distinction between sub-questions and derived questions were made. It has not been possible to include this distinction in the a posteriori analysis of the groups’ SRPs. A discussion of this issue is included in section 8.6.

The sub-questions are denoted on the form $Q_{x,x,x,x,x}$. Questions that do only have numbers instead of the x’s are questions that are found in the a priori analysis. When some of the groups pose questions that are not found in the a priori analysis, they contain letters in them. As an example, the sub-question $Q_{2,1}$ is a part of the a priori SRP, while the question $Q_{2,a}$ is a part of the SRP of some of the groups, but not of the a priori SRP. Both $Q_{2,1}$ and $Q_{2,a}$ are sub-questions to the question Q_2. By this method of indexing, the questions can be bundled.

In Appendix D, a list of all the questions is found. This list will hopefully give the reader an overview of the questions and their relations if necessary.

8. Data and a posteriori analysis

The following section will describe how the teaching sequence developed in reality. Not all of the lesson plans were followed to the letter all the way through the two weeks caused by both time pressure and the fact that the students did not apply their knowledge on vectors as expected in the second part of the SRP. The deviations and revised lesson plans will be described in section 8.1.

In section 8.2 some empirical research questions will be posed. These research questions will guide the analysis of the students’ work with the generating question. Section 8.3 is dedicated to a description and analysis of the SRPs that the students followed during the teaching sequence. The realised SRPs will be compared to the expected SRP that was described in section 6.1 and 6.2.

8.1 The realisation of the teaching sequence

Teaching is rarely following the lesson plan exactly to the point, and the same applies for the test of this design. In order to give an overview of what happened in the classroom, the following section will contain tables that presents the activities that took place during each of the lessons. The tables contain three columns. One reports the duration of the activity, one reports the students' actions, and one reports the teacher's actions. Each section will furthermore contain a more elaborated description of the activities if it is necessary.

It will be obvious from the following sections, that some of the lesson plans needed a revision, caused by deviations from the previous lesson's lesson plans. These revised lesson plans will be included in the end of the subsections where it is relevant after a description of the realised teaching.

8.1.1 Lesson 1

Time	Students	Teacher
2 min.	Listening and marking attendance	Registering attendance
5 min.	Listening	(+ researcher) introducing the project
5 min.	Listening and rearranging	Dividing groups and handing out the first hand out
10 min.	Reading, preparing themselves to get started and asking questions	Answering questions
5 min.	Trying to download data file and helping the other groups to get the data file	Organising sharing of the data file
30 min.	Working in groups on the task and writing logbooks. Some are working on the posters	Circulating the classroom to get an overview of the process
1 min.	Asking questions about the posters	Conferring the researcher for decision

Table 24 – Activities in Lesson 1

The internet connection was very unstable which made the process of downloading the data file very complicated. The teacher suggested the students to share the data file in their intern Facebook group, which was done. Finally, all the groups had the data file downloaded to at least one computer, and the investigations could start. However, most groups had lost five minutes from the time that should have been spent on the task. Furthermore, some of the groups ran into more troubles. The data file was provided in Excel format and most of the groups wanted to work in TI-Nspire. Some of the groups, especially those working on MacBooks, spent a lot of time trying to replace the decimal point in Excel, which is a comma, with the decimal point required in TI-Nspire, which is a dot. This problem was unexpected and took time away from the actual work with the task.

Though the hand out told the students very explicitly to finish their posters within the first lesson, most of the groups did not manage to do so. Some groups asked for five minutes in the beginning of Lesson 2 to finish the poster. The request was agreed upon.

Due to the time pressure, that was caused by the issues described above, the plan for the second lesson needed a revision. The revised plan for Lesson 2 is shown in Table 25.

Time	Time accumulated	Students	Teacher
5 min.	5 min.	Finishing posters	Circulating the classroom to get an overview of the process
15 min.	20 min.	Presenting posters	Listening, directing order etc.
20 min.	40 min.	Working on the problem, updating posters, preparing presentations and writing logbooks	Circulating the classroom to get an overview of the process
15 min.	55 min.	Presenting posters and asking questions	Listening, directing order etc.

Table 25 – Revised plan for Lesson 2

8.1.2 Lesson 2

Time	Students	Teacher
2 min.	Listening and marking attendance	Registering attendance and asking for questions regarding the homework
5 min.	Finishing posters	Circulating the classroom to get an overview of the process
2 min.	Arranging posters on the whiteboard and preparing for presentations	Directing the arrangement of the posters
10 min.	Presenting the posters	Listening and directing the order
35 min.	Working in groups on the task and writing logbooks	Circulating the classroom to get an overview of the process. Conferring the researcher regarding second presentation
1 min.	Finishing and handing in logbooks	Circulating the classroom to get an overview of the process

Table 26 – Activities in Lesson 2

In the beginning of the lesson the teacher used a few minutes to ask and note down if the students had questions for the homework that had been posted on the intranet. The questions were not asked or answered in this lesson, but postponed until after the project. Before the students were ready to present their posters almost ten minutes had passed.

Though 15 minutes were set aside for the presentations the students did only use ten minutes. It had not been clear to the students that they should ask questions and be a part of the presentations given by the other groups.

Since the quality of the first presentations was not as high as hoped for and since the students had not had enough time to work continuously on the task for a longer period the second poster presentation was skipped. The decision was made by the researcher.

8.1.3 Lesson 3+4

Time	Students	Teacher
1 min.	Listening and marking attendance	Registering attendance
2 min.	Listening and asking questions about a homework problem	Answering questions
4 min.	Listening and asking questions (not directly related to the teaching subject)	Moving on to the teaching sequence
18 min.	Listening, taking notes, and asking questions (related to the teaching subject)	Teaching vector theory on the whiteboard
8 min.	Playing with the applet on the internet	Circulating the classroom to get an overview of the process
22 min.	Listening, taking notes, and asking questions (related to the teaching subject)	Teaching vector theory on the whiteboard
10 min.	Break	Break
1 min.	Asking questions (not directly related to the teaching subject)	Answering questions
54 min.	Listening, taking notes, and asking questions related to the teaching subject)	Teaching vector theory on the whiteboard

Table 27 - Activities in Lesson 3+4

While a problem with the lack of chairs in the classroom was fixed the teacher spent two minutes to answer a question about a homework problem, that a majority of the students had had difficulties answering. The homework problem had no connection to the project.

When the students were finally settled the teaching sequence could start. Regarding the order in which things were presented, the plan was followed to the letter, but it took more time than expected.

In order to cover all of the theory in the teaching sequence about vectors, the lesson plan for Lesson 5 needed a revision. The teacher claimed that it could possibly be done in 30 minutes. The revised plan for Lesson 5 is shown in Table 28:

Time	Time accumulated	Students	Teacher
30 min.	30 min.	Listening and asking questions	Teaching vector theory on the whiteboard
25 min.	55 min.	Working on the problem and writing logbooks	Circulating the classroom to get an overview of the process

Table 28 – Revised plan for Lesson 5

8.1.4 Lesson 5

Time	Students	Teacher
2 min.	Listening and marking attendance	Registering attendance and checking up on home work
12 min.	Listening and asking questions about the homework problems	Answering questions about the homework problems
47 min.	Listening, taking notes, and asking questions related to the teaching subject)	Teaching vector theory on the whiteboard

Table 29 - Activities in Lesson 5

Like for the other lessons the teacher had posted some homework for the students on the intranet. This time the problems were related to the vector theory that they had been presented to in the previous lessons. After the attendance registrations the teacher spent 12 minutes answering the students' question regarding the homework problems.

After this the teacher continued the teaching sequence, but it took another 47 minutes to finish, which meant that the students could not get started on the second part of the investigations before Lesson 6.

Since the students had not had time in Lesson 5 to get started on the second part of the SRP, the plan for Lesson 6 needed a revision. The revised plan for Lesson 6 is shown in Table 30:

Time	Time accumulated	Students	Teacher
55 min	55 min.	Working with the problem and writing reports	Circulating the classroom to get an overview of the process

Table 30 – Revised plan for Lesson 6

8.1.5 Lesson 6

Time	Students	Teacher
2 min.	Listening and marking attendance	Registering attendance
10 min.	Reading hand out	Handing out the second hand out and waiting while the students read
40 min.	Working in groups on the task and writing reports. Asking questions	Circulating the classroom to get an overview of the process and answering questions

	about the requirements for the content in the reports	
3 min.	Listening	Briefing students about the feedback procedure for next day

Table 31 - Activities in Lesson 6

The students started out by reading the second hand out. When they had finished the students worked on the problem and their reports, but it was hard for them to grasp what they were expected to write in the reports. Some of the groups asked a lot of questions in order to make the requirements clearer and more specific.

From the observations made during Lesson 6, it was decided to let the students read a solution of the task after the feedback session and before they worked through the reports again in order to rewrite them. The revised plan for Lesson 7+8 is shown in Table 32:

Time	Time accumulated	Students	Teacher
20 min.	20 min.	Reading reports and preparing feedback	Circulating the classroom to get an overview of the process
10 min.	30 min.	Giving feedback to the first group	Circulating the classroom to get an overview of the process
10 min.	40 min.	Giving feedback to the second group	Circulating the classroom to get an overview of the process
10 min.	50 min.	Giving feedback to the third group	Circulating the classroom to get an overview of the process
10 min.		Break	Break
25 min.	75 min.	Reading and discussing the handed out solution	Circulating the classroom to get an overview of the process
25 min.	110 min.	Adjusting the reports in view of feedback and solution	Circulating the classroom to get an overview of the process
10 min.	120 min.	Listening and asking questions	(Researcher) rounding off the project

Table 32 – Revised plan for Lesson 7+8

8.1.6 Lesson 7+8

Time	Students	Teacher
1 min.	Listening and marking attendance	Registering attendance
4 min.	Sending and downloading reports	Directing the students
1 min.	Listening	Giving instructions on the feedback procedure
15 min.	Reading reports and preparing feedback	Circulating the classroom to get an overview of the process
15 min.	Giving and receiving feedback in groups to/from other groups	Circulating the classroom to get an overview of the process
2 min.	Listening	Handing out the guiding solution

17 min.	Reading the hand out	Circulating the classroom to get an overview of the process
10 min.	Break	Break
45 min.	Working in groups on the final report and handing in	Circulating the classroom to get an overview of the process
10 min.	Listening	(Researcher) rounding off the project

Table 33 – Activities in Lesson 7+8

It had not been arranged how the reports should be shared in the feedback groups. Therefore, it took five minutes before the students could start reading and discussing feedback. After 15 minutes the students started talking about irrelevant subjects, so the preparation of the feedback was shortened down with five minutes. When the students had been in feedback groups for 15 minutes, they did again start to talk about irrelevant subjects. Therefore, the solution was handed out, and the students spent the last 17 minutes before the break reading the and discussing the solution.

After the break the students made some adjustments to their reports and handed in the final versions.

8.2 Research Questions (II)

From an analysis of the collected data, and in the light of the realised SRPs, the following questions will be answered:

${RQ_1}^{II}$: What questions do the groups develop from the generating questions $Q_{0,I}$ and $Q_{0,II}$?

${RQ_2}^{II}$: How and why do the realised SRPs deviate from the expected SRPs?

${RQ_3}^{II}$: What praxeological organisation of vectors reveals from the groups' work with the second generating question ($Q_{0,II}$)?

8.3 The realised SRPs

Since the class was divided into nine groups during the test of the design, the data material is extensive. Some interesting parts of the SRPs have been picked out and analysed. These parts can either be representative for more groups or show how the design and the a priori analysis has been on point or off compared to the students' realised SRPs. The parts that are picked out will be analysed with the purpose of answering the research questions.

8.3.1 Group 4

This group has been picked out for three reasons. The first reason is, that they work with scales relating to the map that was handed out in the first part of the SRP. Group 7 did also work with scales. The second reason is that they calculate the "expected speed" of the aeroplane from the expected distance and time. Group 1, 3, and 7 did something similar in the first or second logbook The last reason is, that the second report made by this group contains a misunderstanding in the praxeological organisation of vectors.

Logbook 1
From the first logbook the following questions reveal:

$Q_{1,2,a}$: In what part of the flight did the aeroplane crash?
Q_1: How does the route, that the aeroplane followed, look?
$Q_{1,1}$: What is remarkable about the route?
$Q_{1,1,1}$: Why did the aeroplane make a "loop"?
Q_a: Why did the aeroplane crash after 65 minutes?
$Q_{0,II,a}$: Where did the aeroplane crash?

It seems like the overall question for this group in the first lesson is Q_1, however it is not posed explicitly in the logbook. Neither are the questions $Q_{1,2,a}$ and $Q_{1,1}$, and it seems like $Q_{1,2,a}$ and partly $Q_{1,1}$ are answered without plotting the data. The group gives the following answers to these two questions in the logbook (see Appendix B.6):

$A_{1,2,a}$: "We have found out, that it crashed on the last third of the flight. We have found out that it crashed on the last 350 km of the flight. $90 - 65 = 25$ min. It is approximately $1/3$. Then we measured $1/3$ of 20 cm $= 6{,}666$ cm. Then we found that $6{,}5$ cm in the map $= 350$ km. The aeroplane would have crashed after 650 km."

$A_{1,1}$: "We have found out that it is following the intended route for the first 23 min. and after that it starts to lurch." (It seems like this answer is given before the group has plotted the data). "By drawing the graph on TI-Nspire we have found out that the aeroplane flew in a circle (i.e. that it has been flying forwards, backwards, and forwards again). The beginning of the circle is in the point $(259.33; 286.77)$. The end is in the point $(267.77; 289.37)$. We can also see that the aeroplane crashed in the point $(452.78; 477.5)$. We can see that after the point $(396.26; 381.2)$, the aeroplane leans towards the left on the route."

The questions $Q_{1,1,1}$, Q_a, and $Q_{0,II,a}$ are stated explicitly at the end of the logbook as the questions that the group want to work with in Lesson 2.

On the poster the group has marked a circle around the point where the aeroplane disappeared.

Logbook 2
In logbook 2 the group tries to answer the three questions that they posed in the end of logbook 1. After that they (implicitly) poses the following question:

$Q_{2,a}$: What speed corresponds to the expected time and distance?

The group gives the following answer to this question in the logbook (see Appendix B.7):

$A_{2,a}$: "Distance $= 1000$ km $= 1000 \cdot 1000 = 1000000$ m. Time $= 90$ min. $= 90 \cdot 60 = 5400$ s. Speed $= \frac{distance}{time} \rightarrow$ Speed $= \frac{1000000 \text{ m}}{5400 \text{ s}} \rightarrow$ Speed $= 185{,}185\ \frac{\text{m}}{\text{s}}$."

The group wants to calculate this speed in order to compare it to the actual speed of the aeroplane. They pose the following question as the primary for the next lesson:

Q_2: By what speed did the aeroplane fly during the flight?

The tree diagram of the first part of the SRP of group 4 is shown in Figure 43.

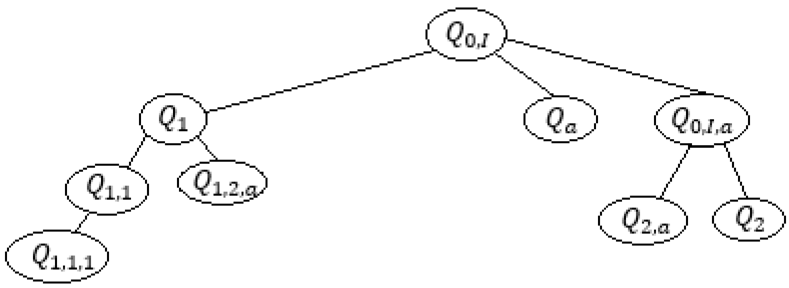


Figure 43 – Tree diagram showing the first part of the SRP of group 4

Report 1
In the first report the group works with the following (implicitly posed) questions:

Q_5: How can the information about the amount of fuel be used?
$Q_{5,a}$: How much longer than the expected distance can the aeroplane travel on the additional fuel?
$Q_{5,b}$: How much did the aeroplane weigh?
Q_4: How can the information about cross wind be used?
$Q_{4,a}$: Where would the aeroplane have ended if the cross wind had affected it from exactly minute 50?

Questions $Q_{5,a}$ and $Q_{5,b}$ are answered by the following (see Appendix C.7):

$A_{5,a}$: "[…] it can fly 1300 km. Therefore, it can fly 300 km more."
$A_{5,b}$: "The maximal start weight of an airbus 320 is 77 t with passengers. I this case the fuel tank was not filled up […] which is why the aeroplane will weigh 74 t with passengers."

The question $Q_{4,a}$ is interesting, since the answer would potentially involve some of the vector theory that they had been taught prior to this second part of the SRP. The answer given in the report is (see Appendix C.7):

$A_{4,a}$: "We have calculated the coordinate where the aeroplane would have ended if the wind had affected it from exactly the 50th minute. $326{,}25 + 3{,}33 = 329{,}58$. $324{,}03 - 32{,}33 = 291{,}7$. But on the graph it shows that it got off course later…"

Report 2
In the second report, that is made after the students have read the handed out solution, they reproduce some of the results that are presented in the guiding solutions (see Appendix A.3). The first two calculations are about the speed and the acceleration, but they are not that interesting, since they are just copied from the hand out. However, an interesting misunderstanding appears in the third calculation they make. They want to show how the vector from a point A to a point B can be found, but the calculation is the following (see Appendix C.8):

"By using the coordinates, where A is x and B is y, we can use the vector formula to find the vector of AB: $AB(\text{vector}) = \binom{b_1 - a_1}{b_2 - a_2}$. $AB(\text{vector}) = \frac{226{,}77 - 225{,}02}{237{,}77 - 233{,}51} = \frac{1{,}75}{4{,}26} = 0{,}4$."

The tree diagram of the second part of the SRP of group 4 is shown in Figure 44.

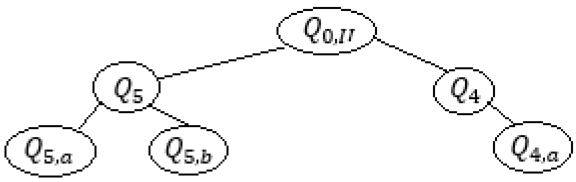


Figure 44 – Tree diagram showing the second part of the SRP of group 4

8.3.2 Group 6

This group has been picked out for two reasons. The first one is, that they used the internet a lot, e.g. to look up specifications for the particular type of aeroplane and other conditions that they thought could be relevant. That this way of working with the design is represented, shows the importance *and* potential of a very, very detailed a priori analysis of a generating question. Also the groups 1, 3, 4, 7, and 8 used the internet to look for specifications of the aeroplane or possible reasons for an aeroplane crash. The second reason is, that this group made a sketch of the wind vector in the first report. The groups 7 and 9 did also make a sketch of the wind vector in the first report.

Logbook 1

In the first logbook this group poses (some explicitly and some implicitly) the following questions (see Appendix B.10):

Q_1: How does the route, that the aeroplane followed, look?
$Q_{1,1}$: What is remarkable about the route?
$Q_{3,1,a}$: How can it be determined where the aeroplane crashed?
$Q_{3,1,a,a}$: How far can an aeroplane glide?
$Q_{0,II,b}$: Where did the aeroplane land?
Q_0: What happened to the aeroplane?

The first four questions, are the questions that the group worked with during the first lesson. They are all posed implicitly. The first question, Q_1, is answered by plotting the data points in TI-Nspire (A_1). $Q_{1,1}$ is answered by looking at the scatter plot. The answer is:

$A_{1,1}$: "The route should have been a straight line from Farawayistan to Neverland, but by the scatter plot we can read that the aeroplane has been flying in a circle during the flight."

The other two details that were described in the a priori analysis are not mentioned. The question $Q_{3,1,a}$ is answered by a possible guess:

$A_{3,1,a}$: "The aeroplane might have crashed straight ahead of where it disappeared from the radar."

The last question that the group managed to work with in the first lesson was $Q_{3,1,a,a}$. The answer to this is:

$A_{3,1,a,a}$: "An aeroplane can glide for 150 km."

This answer is equipped with a link to a web-page, where a pilot can be asked questions about aviation. On this page the group has found this question and an answer to it.

At the end of the logbook the group poses the questions $Q_{0,II,b}$ and Q_0 explicitly. These are the questions that they want to work with during the second lesson.

Logbook 2
In the second lesson the group worked with some sub-questions relating to the questions they posed in the end of the first lesson (see Appendix B.11):

$Q_{0,I,a}$: Why did the aeroplane not follow the intended route?
$Q_{3,1,a,a,a}$: What is the lift-drag ratio for an Airbus a320 when it is turning?

As the answer to $Q_{0,I,a}$ the group takes the following guess:

$A_{0,I,a}$: "We read about the aeroplane a320. [web-link]. We could read that this aeroplane model has a defect that in 2008 37 times turned off the electronic screens and instruments in the cockpit. This means that the pilots do not know where they should fly or by what speed they fly. This can have caused the remarkable route. In the article it says that they in previous cases have tried to turn around and return home. Maybe our pilots tried that, but found out that they were too far from land that they could neither orientate towards Farawayistan."

The web-link that the groups has put in the logbook is an article about this defect and the consequences that it has had.

The second question, $Q_{3,1,a,a,a}$, is not answered because of a lack of time. The question is posed with the explanation: "We have looked at these two links in order to try to calculate the lift-drag ratio for an a320 when it is turning and by this number to reduce the area where the aeroplane could have landed." Before this explanation the group has put two links to pages where the lift-drag ratio is described.

The tree diagram of the first part of the SRP of group 6 is shown in Figure 45.

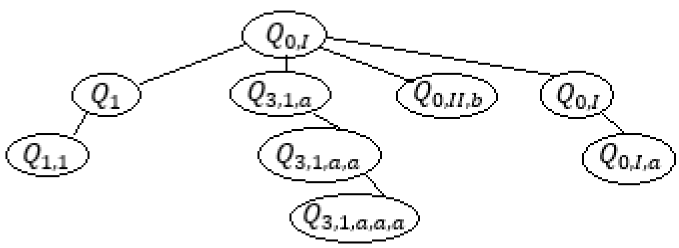


Figure 45 – Tree diagram showing the first part of the SRP of group 4

Report 1
In this first report the group works with the following questions (that are both posed implicitly, see Appendix C.11):

$Q_{4,b}$: How has the cross wind affected the route of the aeroplane?
Q_2: By what velocity did the aeroplane fly during the flight?

As a part of the answer to $Q_{4,b}$ the group states that wind is blowing with the velocity $1{,}95\,\frac{\text{km}}{\text{min.}}$ in the direction that they show on a sketch (see Figure 46).

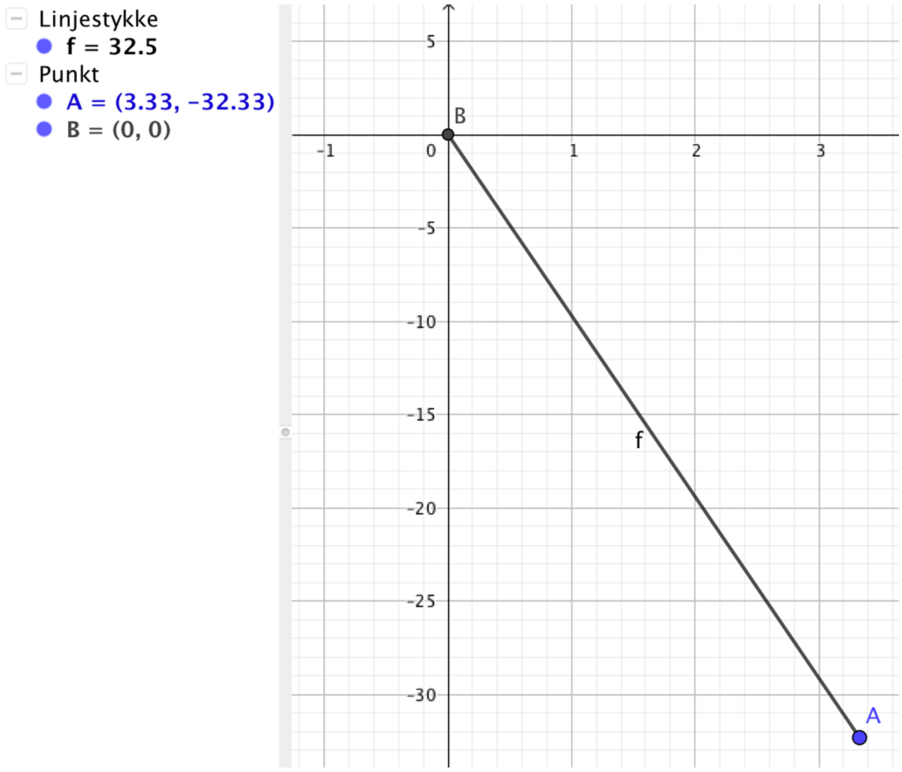


Figure 46 – A sketch of the wind vector made by group 6 (see Appendix C.11)

Report 2
In the second report the group does primarily follow some of the calculation made in the guiding solution that was handed out prior to their work with the second report. They did not work with anything interesting in connection to vectors (see Appendix C.12).

The tree diagram of the second part of the SRP of group 6 is shown in Figure 47.

Figure 47 – Tree diagram showing the second part of the SRP of group 6

8.3.3 Group 7

This group has been picked out for three reasons. The first one is, that they use the Pythagorean theorem to determine the distance before they have been taught vector theory. None of the other groups did anything similar. The second reason is, that they try to apply some different techniques from the vector theory they have been taught autonomously in the first report, and that they do not just reproduce results from the third hand out (see Appendix A.3) in the second report. The third reason is, that the data material for this group includes an audio recording from the first lesson.

Logbook 1

During the first lesson the work of this group is divided into two parts. The first part does not involve the data set at all, while the second part is all about investigating the data set. This division was detected from the audio recording. The questions posed for the first part are:

$Q_{5,1,1}$: How big a distance has the aeroplane been travelling from take-off and until it disappears from the radar?
Q_2: By what speed did the aeroplane fly during the flight?
$Q_{5,1,1,a}$: What does this distance correspond to on the map?
$Q_{5,1,1,a,a}$: What is the scale of the map?

This group starts the investigations without opening the data set. They start by asking question $Q_{5,1,1}$. In order to answer this question the students ask the question Q_2. The answer to this is given in the logbook (see Appendix B.12):

A_2: "$\frac{1000 \text{ km}}{1{,}5 \text{ h}} = 666{,}667 \frac{\text{km}}{\text{h}}$"

From this, the group calculates the answer to $Q_{5,1,1}$:

$A_{5,1,1}$: "$\frac{13}{12} \cdot 666{,}667 = 722{,}223$"

The speed is multiplied with $\frac{13}{12}$ because $65 \text{ min.} = \frac{13}{12}$ h (detected from the audio recording). After this, the group makes some measurements on the map that is included in the hand out, in order to answer question $Q_{5,1,1,a,a}$ (detected from the audio recording). The answer is:

$A_{5,1,1,a,a}$: "$\frac{1000}{20} = 50$". Where the group have measured the 1000 km-distance to be equal to 20 cm on the map.

After this the group gives the answer to $Q_{5,1,1,a}$:

$A_{5,1,1,a}$: "$\frac{722,23}{50} = 14{,}4445$. The radio contact closed down after $14{,}5$ cm on the paper."

The work with the data set is not included in the first logbook.

Logbook 2
In the second lesson the group has worked with the questions:

$Q_{1,2}$: How far from Farawayistan Airport is the aeroplane, when it disappears from the radar?
$Q_{1,1,1,a}$: Why did the aeroplane first disappear from the radar 36 minutes after it made the "loop"?

Neither of the questions are posed explicitly, and from the answer to $Q_{1,2}$ it reveals that the formulation is maybe a little different. The answer is:

$A_{1,2}$: "We have calculated how far the aeroplane would have reached if it had not made a loop, and determined how far it has reached by saying $\sqrt{475^2 + 450^2}$ =how far it has reached."

This answer is equipped with a little sketch, that is shown in Figure 48.

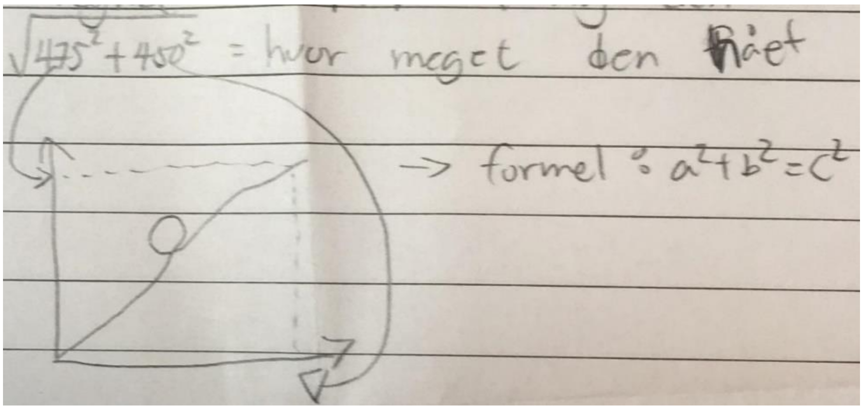


Figure 48 – A sketch connected to a calculation made by group 7 (see Appendix B.13)

The tree diagram of the first part of the SRP of group 7 is shown in Figure 49.

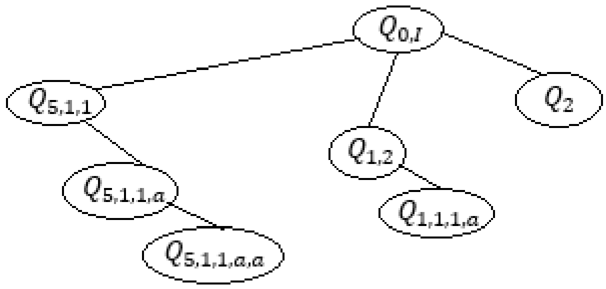


Figure 49 – Tree diagram showing the first part of the SRP of group 7

Report 1
In the first report the group works with the following questions:

$Q_{1,a}$: How far did the aeroplane travel after it made the “loop”?
Q_3: How can vector theory be applied to calculate the area that has to be searched in order to find the aeroplane?
$Q_{4,d}$: How far has the aeroplane travelled in the strong wind?
$Q_{4,c}$: What is the length of the wind vector?
$Q_{4,d,a}$: How far would the aeroplane have travelled if the strong wind had not been there?
$Q_{4,d,a,a}$: What is the area of the triangle spanned by the wind vector and the distance that the aeroplane would have travelled if the strong wind had not been there?
$Q_{4,d,a,b}$: [A question of the area of a circle that is drawn. Due to a bad picture resolution it is impossible to figure out the thoughts behind the circle and its area]

Neither of the questions are posed explicitly, and it is only a few calculations that are included in the report. The answer $A_{1,2}$ is elaborated in the report:

$A_{1,a}$: “The length of the flight: 654 km”

The number 654 is the result of $\sqrt{475^2 + 450^2}$, that was written in Logbook 2. The answer to the question $Q_{1,a}$ is the following:

$A_{1,a}$: “The length after the aeroplane made a “loop”: 394 km”.

The number 394 is $\sqrt{267{,}77^2 + 289{,}37^2}$ (or something similar) where the two numbers in the square root are the coordinates of the aeroplane after it had finished the “loop”. After this, the answer to $Q_{4,d}$ is given:

$A_{4,d}$: “$654 - 394 =$How much has the aeroplane travelled in the strong wind$= 260$ km”

After this the question $Q_{4,c}$ is answered:

$A_{4,c}$: $\begin{bmatrix} 3.33 \\ -32.33 \end{bmatrix} \frac{\text{m}}{\text{s}}$ then is the length $\sqrt{(3.33)^2 + (-32.33)^2} = 32.501$ km”

And $Q_{4,d,a}$is answered:

$A_{4,d,a}$: “$\sqrt{260^2 - (32.5)^2} = 257.961$ is how much the aeroplane would fly if the wind was not there.”

From this answer the question $Q_{4,d,a,a}$ can be answered:

$A_{4,d,a,a}$: “The area of the triangle: $\frac{258 \cdot 32.5}{2} = 4192.5$ cm^2”

And to the answer to the question $Q_{4,d,a,b}$ is the following:

$A_{4,d,a,b}$: “The area of the circle: $260^2 \cdot \pi = 67600 \cdot \pi = 212371.66$ cm^2”

Report 2
Contrary to the majority of ‘Report 2’s this report does contain some interesting actions. First of all, the group changed the area that they wanted to calculate, to a parallelogram. This is most likely because of the hand out, but though the hand out did contain calculations of the area, this group makes their own calculations. In the report group 9 writes the following (see Appendix C.18):

> We assume that the aeroplane started to fly downwards after the last point on the graph. From this assumption, we have drawn a parallelogram from the following calculations. $\sqrt{56{,}51^2 + 96{,}23^2 = 111{,}60 \text{ km}}$ is the vertical vector. $\sqrt{15{,}38^2 + (-149{,}23)^2 = 150 \text{ km}}$ corresponds to the influence form the wind on the direction of the aeroplane. From the parallelogram, we have calculated the area of the parallelogram: $\frac{150 \cdot 111{,}6}{2} = 8370$ km^2. We have moved the orange triangle in the appendix up, because we think that the aeroplane changed its course after the last coordinate: the green triangle.

The appendix that is mentioned is shown on Figure 50

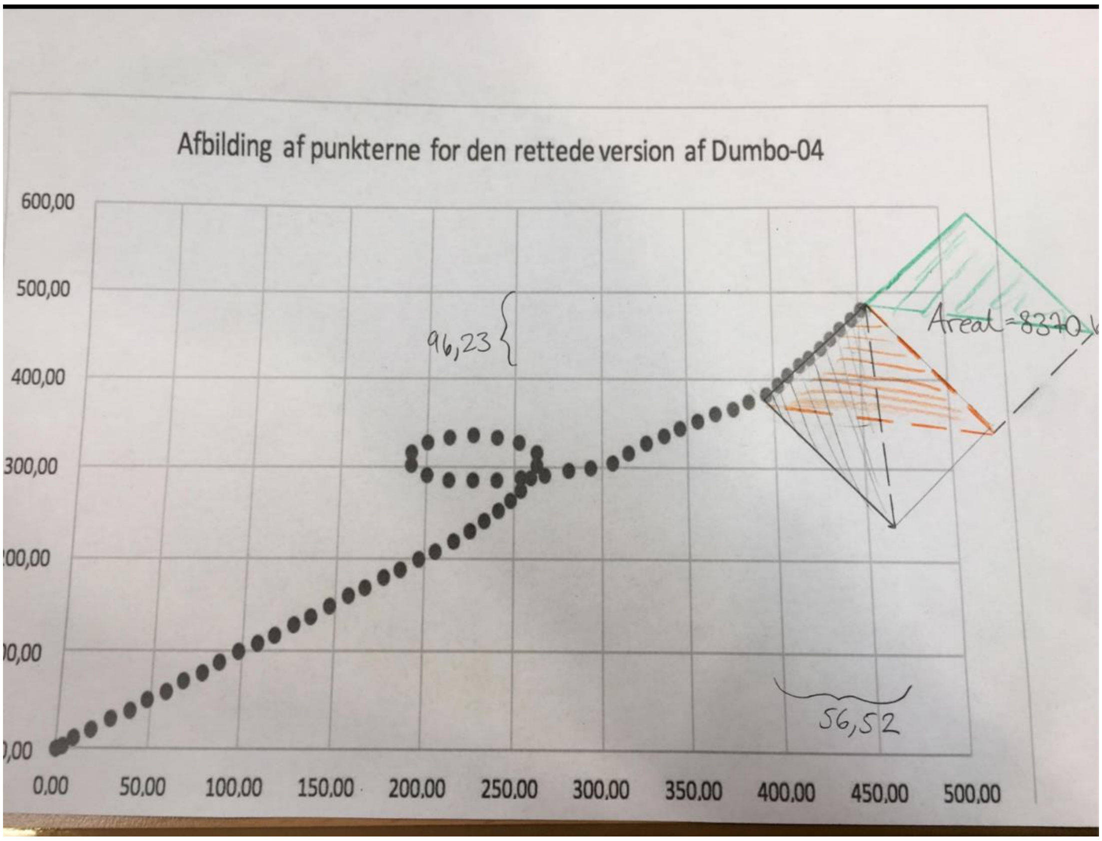


Figure 50 – Appendix in Report 2 made by group 7

The tree diagram of the second part of the SRP of group 7 is shown in Figure 51.

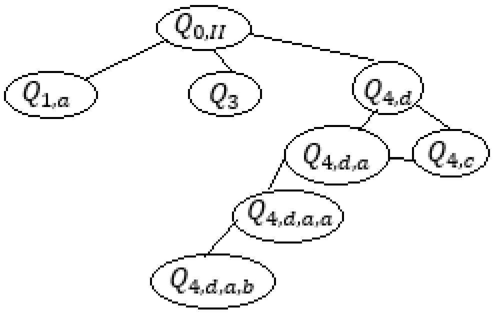


Figure 51 – Tree diagram showing the second part of the SRP of group 7

8.3.4 Group 9

This group has been picked out for two reasons. The first one is their very first encounter with the generating question, because they cling to the techniques that they have used to solve tasks with prior to this project. Group 7 did something similar when they first started working with the data set. The second reason is, that they are the only group that is primarily working with velocity in the first lesson.

Logbook 1
This group makes an interesting remark in the beginning of their logbook. They write: ”We start by applying the things that we have been taught. E.g. linear regression, bar chart in Excel”. This is a significant example of the didactical contract; since they have not been taught anything new relating to this project, they assume that the teacher must except them to use some techniques that they are already familiar with. These actions are not motivated by questions relating to the generating question. After this they work with the following questions:

Q_2: By what speed is the aeroplane flying during the flight?
$Q_{2,3}$: What can be derived about the flight from its velocities?

The group has not included any calculations in their logbook, and they do not give an explicit answer to Q_2, but $Q_{2,3}$ is answered by the following:

$A_{2,3}$: “We could conclude, by looking at the time and calculating the $\frac{\mathrm{km}}{\mathrm{h}}$ on the flight, that there has been weather-related conditions that has caused the crash. E.g. turbulence.”

By the end of the logbook they posed an additional question that they want to answer in the next lesson. This is the question:

$Q_{2,3,a}$: Did the aeroplane increase its velocity when it started crashing compared to the velocity when it was heading towards Neverland? And has it influenced where the aeroplane has crashed?

Logbook 2
In the second lesson the group works with the question that they posed in the first logbook ($Q_{2,3,a}$). Before they give the answer to the question they implicitly pose the following sub-question:

$Q_{2,1}$: How is velocity calculated from position and time?

The group does not show any calculations, and no numbers are included in the answer. Instead they give the following answer:

$A_{2,1}$: “We used a velocity formula, to calculate whether the velocity increased, that we found on the home page [web-link].”

And from this, they give the answer to the original question $Q_{2,3,2}$:

$A_{2,3,a}$: "By this we found that the velocity increased after 23 minutes."

No further comments are added to this, and this answer is wrong, since the velocity is steady until minute 51. In this situation it is very unfortunate that their work was not audio recorded, since it is very difficult to analyse the path they have followed, when they do not include any calculations or more elaborated answers to the questions.

The tree diagram of the first part of the SRP of group 9 is shown in Figure 52.

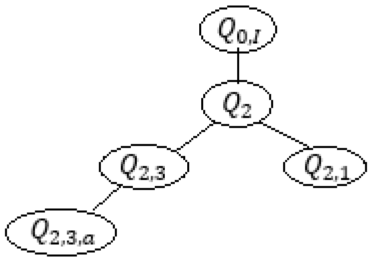


Figure 52 – Tree diagram showing the first part of the SRP of group 9

Report 1

In the first report the group has implicitly posed the following questions:

Q_1: How does the route, that the aeroplane followed, look?
$Q_{4,b}$: How has the cross wind affected the route of the aeroplane?
Q_5: How can the information about the fuel be used?
$Q_{4,c}$: What is the length of the wind vector?

The first question (Q_1) is answered by a scatter plot from TI-Nspire. The group has not made any comments on it. The second question $Q_{4,b}$ is answered by the following:

$A_{4,b}$: "By reading the coordinates one can see that the strong wind has affected the aeroplane from around $56 - 57$ min., since the x-values are not increased with around 10 per min. but by $6 - 7$ per min."

The question Q_5 is answered by the following:

A_5: "We assume that the aeroplane is using more fuel when the strong wind starts to create turbulence."

After this the last question is answered:

$A_{4,c}$: "To find out where the aeroplane has crashed we determine the length of the vector of the wind velocity. It is 32."

The answer is equipped with an illustration, that is shown in Figure 53.

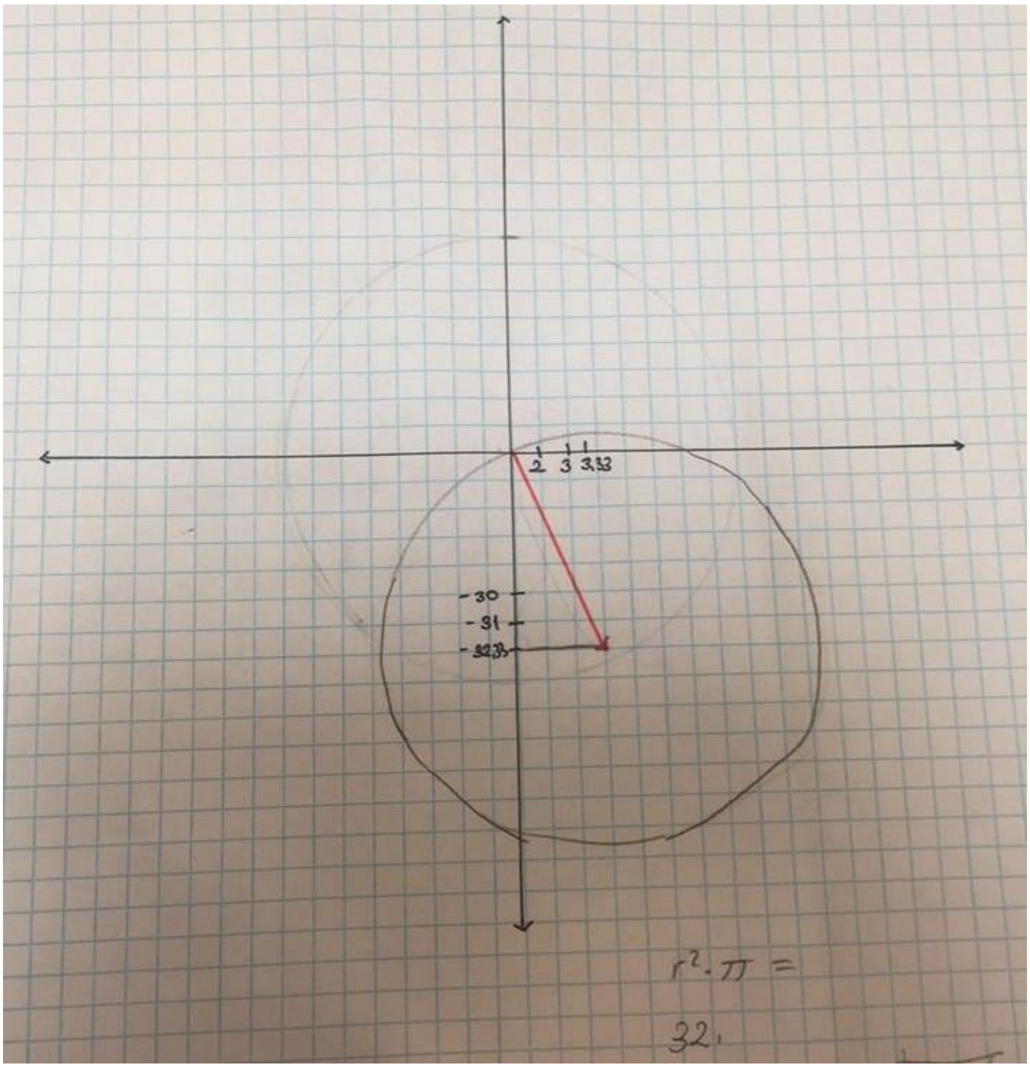


Figure 53 – Illustration from Report 2, group 9

Report 2

In the second report this group is following the handed out solution closely, and the work does not contribute to the answer to the research questions, and therefore the second report will not be described further.

The tree diagram of the second part of the SRP of group 9 is shown in Figure 54.

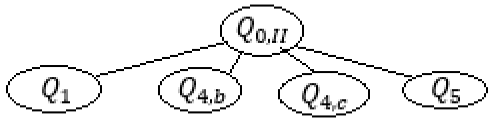


Figure 54 – Tree diagram showing the second part of the SRP of group 6

8.4 The impact of the didactical contract

It became clear within the first ten minutes of the first lesson, that this teaching design was a major breach on the didactical contract in the test class. Both the teacher and the researcher were asked a lot of questions in the category "What do you want us to do?". The questions were mostly raised in the first lesson where the generating question was more open in nature than the second generating question. But also in the second part of the SRP, the students had a hard time grasping the purpose of the teaching sequence. This issue ended up taking a lot of time from the actual work with the problem. A consequence that can possibly be ascribed to it, is the fact that most groups did not ask as many questions as expected, because the students spend much time being frustrated that the format of the teaching did not look similar to anything they were used to from the mathematics lessons. Another consequence was, that the groups did not write down all of their questions even though they were explicitly requested to do that.

Furthermore, many of the groups' actions were highly affected by the didactical contract. Some different examples support this claim. One of them comes from an exchange of words in group 7 in the first lesson. After discussing shortly what they are expected to do, one of the students reads aloud from the hand out that they are asked to analyse the flight based on its coordinates. The answer from one of the other group members is the following question: "How do you analyse mathematics?". This citing shows how a part of the didactical contract in the mathematics lessons is that doing mathematics does not involve the action of analysing.

Another example is found in the first logbook by group 9. The logbook is introduced by an interesting comment: "We start by trying the things that we have learned. E.g. linear regression, bar chart in Excel." This citing shows how the students immediately link a data set containing a list of x- and y-coordinates to linear regression in the context of mathematics. The bar chart is maybe a leftover from primary school or from some other subject at high school.

The last example that will be mentioned here, in order to support the statement, that the class was highly restricted by the didactical contract, showed up in the majority of the logbooks. A lot of the groups used the expected distance and the expected time to calculate "the speed". Group 4 and group 9 were aware that the "expected speed" was not equal to the actual speed during the flight, but the other three groups that worked with the speed of the aeroplane did not realise, or at least mention, that there is a difference between the "speed" that they calculate from the expected distance and time and the actual speed of the aeroplane. This might be a consequence of the didactical contract again, because they are used to problems where the information that is given is necessarily included in the solution.

8.5 Comparison of the a priori and the realised SRPs

The realised SRPs deviate a lot from the a priori SRP on several points. The first and most obvious difference is, that most of the groups asked fewer questions than expected. There are two primary reasons for this. The first one is the general time pressure that has been described in section 8.1 and the second is that the students struggled with the framework of the teaching sequence, that deviated a lot from what they were used to as described in the previous section.

In the first part of the SRP, that is generated from question $Q_{0,I}$, most groups asked the questions relating to the $Q_{1,1}$-branch. Only group 1 and group 9 did not include a scatter plot in their logbooks nor comment on any of the remarkable changes that can be observed from such a plot (see Appendix B.1 and B.16). However, group 9 wrote that they had tried to make a linear regression, which indicates that they *did* actually make a scatter plot, but just did not include it in the logbook. In relation to the question $Q_{1,1}$ the majority of the groups pointed out the loop as being remarkable, but only a few groups mention the two rapid changes, that was described in the a priori analysis as a part of the answer to $Q_{1,1}$. The groups 1, 3, 7, and 9 did not state anything about the loop or any other remarkable details. The answer to the question $Q_{1,1}$ is most thorough in the logbook from group 4. They pointed out the coordinates of both the beginning and the end of the loop, and the last point where the aeroplane makes a rapid change in direction. Though group 4 made a remark on the change in direction in minute 56, they do not ask the question $Q_{1,1,2}$. The rapid changes in direction were put in the data set to make the students consider the wind conditions. It was unfortunate that only a few groups pointed out these points, since it removed a potential self-detected need for the vector techniques $\tau^{geom\ sum}$ and $\tau^{geom\ diff}$ in the second part of the SRP.

The $Q_{1,2}$-branch is also well-represented in the realised SRPs. All the groups 1, 2, 3, 4, and 7 ask questions about the distance that the aeroplane has travelled in the first part of the SRP. These questions and especially the struggles that the students would most likely meet when they tried to answer them where crucial in the teaching design, since they were assumed to create a need for the vector techniques $\tau^{\overrightarrow{AB}}$ and $\tau^{|\cdot|}$. Instead the groups did either resort to some of their available techniques from their work with calculation of percentage, that they had worked with in the mathematics lessons (see Table 18), and techniques from calculations with scales, which has not been detectable in neither the mathematics teaching nor the physics teaching, but might be techniques from either another subject in high school or a technique that they have acquired in primary school. The calculation of percentage technique was used by group 1, 3, 4, and 7. These groups did all make their calculations from the information about expected time and expected distance and did not include the data set. The groups 4 and 7 brought in scales in their calculations (see section 8.3.? and 8.3.?). Neither of the groups realised, that they had not calculated the actual distance, which turned out to be crucial in the second part of the SRP, and for the potential of the learning outcome of the SRP. The reason is, that one of the obvious applications of the theory of vectors, that the students should come across, was the use of the techniques $\tau^{\overrightarrow{AB}}$ and $\tau^{|\cdot|}$ to calculate the travelled distance, and eventually the distance from the point where the radar contact to the two airports was interrupted.

Another consequence of students ignoring the data set in the calculation of speed was, that the students did not pose so many of the questions from the Q_2-branch of the SRP. The questions $Q_{2,3}$ and $Q_{2,3,1}$ might have been another clue leading the students to considerations of wind conditions, that might have motivated the teaching of vector theory, like the question $Q_{1,1,2}$.

In the second part of the SRP, the students had been taught vector theory, and the a priori analysis of the second generating question suggested a couple of obvious applications of vector theory based on the new pieces of information that were included in the second hand out (see Appendix A.2).

Especially in the second part of the SRP the a priori and the a posteriori analyses differ. It is highly restricted what can be said about the second part of the students' SRPs, because they did only have very short time to work with the second generating question. The few things that *can* be said will be presented in the following. First of all, there are two groups that do not at all include anything related to vectors in their answers to the second generating question. All seven groups that tried to use the vector theory they have been taught works with the wind vector. Some groups draw it either in TI-Nspire or by pencil. This counts group 4 (see Figure 46), group 8 (see Appendix C.15), and group 9 (see Figure 53). Other groups tried to do some calculations on the coordinates based on the wind vector, e.g. group 4, that calculated where the aeroplane would have ended if the wind had started affecting the aeroplane in exactly the 50th minute as it was described in section 8.3.2. A similar observation was made by group 5 (see Appendix C.10). The way that group 7 deals with the information about the wind is interesting. As it has been described in section 8.3.3, they correctly calculate the length of the wind vector. After that they tried to calculate the distance that the aeroplane would have travelled if the wind had not been there. These calculations were also described in section 8.3.3. In the report the two numbers 32,5, that is the length of the wind vector, and 258 that is the distance that the aeroplane would have travelled constitute the two sides enclosing the right angle in a right-angled triangle. This shows that the group did implicitly pose the question Q_3, which was very crucial in order for the technique $\tau^{parallelogram}$ to be applied. However, they decide to use a right-angled triangle as the area instead, and they did not use the technique $\tau^{parallelogram}$, but the formula for the area of a triangle, that is something coming from their primary school teaching.

Only a few groups worked with the travelled distance in the second part of the SRP, though the information about the amount of fuel had seemed to highly suggest that, in the a priori analysis of the question. A possible reason that they did not work with the travelled distance in the second part is, that most of them had already made some of the calculations in the first part. Even though the calculations were not exact they were not aware of that, and they kept believing that they had made the right computations, which might be a reason why they did not return to it in the second part where they had the vector techniques to carry out the calculations correctly.

The vector techniques that were used autonomously in the second part of the SRP are primarily the basic technique of drawing a vector from its coordinates and calculating the length of a vector ($\tau^{|\cdot|}$). A few groups have tried to conclude something about the coordinates of the aeroplane by adding the coordinates of the wind vector, to the x- and y-coordinates of the aeroplane.

In the second report, the students have read a guiding solution, which makes it difficult to detect their praxeological organisation of vectors. However, the second reports from group 4 and group 9 are interesting. In the second report made by group 4 (see Appendix C.8), they tried to show how the vector from one point to another is calculated. At first they write the formula: "$AB(\text{vector}) = \left(\frac{b1-a1}{b2-a2}\right)$". After this they substitute the coordinates of two points into the formula and make the following calculation: "$AB(\text{vector}) = \frac{226,77-225,02}{237,77-233,51} = \frac{1,75}{4,26} = 0,4$". This reveals a misunderstanding of the notation, where the two numbers on top of each other is interpreted as being divided by each other, and not as the coordinates of a vector. Except from this, the calculations in the second report are primarily reproductions of the calculations shown in the hand out.

Group 7 did actually try to calculate the area of a parallelogram spanned by two vectors in order to determine the size of the area in which the aeroplane might be found. However, they did not use the technique $\tau^{parallelogram}$, but instead they made the calculation $\frac{150 \cdot 111,6}{2} = 8370 \text{ km}^2$.

All in all, the realised SRPs differ heavily from the a priori SRP. Some of the reasons are (1) that the students did not have enough time to work with the generating questions, (2) that the teaching sequence on vectors was too disconnected from the SRP, and (3) that the students had a hard time grasping what was expected from them, which has something to do with the didactical contract, but also with the quality of the generating question, that will be discussed below.

8.6 The quality of the generating question

The previous section has showed how the work with the SRP did not make the students develop a correct and useful praxeology of vectors. Neither did the first part of the SRP evoke the needs for vector theory that were expected from the a priori analysis.

Three main reasons have to be mentioned in relation to the unsuccessful outcome of the SRP. The first one is the complexity of the setup. When the SRP was designed a lot of considerations were made, among others whether the data set should be provided in Excel or TI-Nspire, whether the units of the data set and the wind vector could be different, and whether it should include the dimension of altitude. The decisions were made, building on the assumptions that the students would easily be able to transfer the data set from Excel to TI-Nspire if they wanted, that they could easily convert the units, and that, even though it took away an important part of the realistic setup, it would be easier for the students to deal with a two dimensional problem, and that it would not produce problems to leave out the altitude. However, all three of these issues *did* cause problems. The first one took away time from a lot of the groups. The second caused some mistakes in the calculations in the second part of the SRP, and the last made some groups spend time focussing on some parts of the SRP that did not contribute to anything vector-related. As an example group 6 can be mentioned. They spend almost a whole lesson trying to calculate how long it would take for the aeroplane to crash. Furthermore, the complexity of aviation made it difficult in both the a priori and a posteriori analysis to distinguish sub-questions, that would contribute directly to the answer to the generating questions, and derived questions, that are motivated by sub-questions or answers to sub-questions. At first glance it can seem like a minor problem, that sub-questions and derived questions cannot be distinguished, but it is a symptom of a very problematic issue, that is one of the main reasons why this SRP is not good. The problem is, that an answer to the generating questions require that the students know what assumptions and simplifications that have been made during the designing process. Since this was not a part of the material that was handed out, the students were not able to answer the question fully. The implicit assumptions made it hard for the students to know what techniques they should apply, where they should apply them, and how they should apply them.

Another problem is, that the teaching sequence in the middle was not adjusted to the physical setting in which the rest of the project took place. Despite a few initiatives, such as focusing on aviation when introducing geometric addition and the little game where the students worked implicitly with geometric vector addition when they tried to land an aeroplane in cross wind, the teaching sequence kept the algebraic approach to vectors that is normally used. Also the amount of points in the data set can have made it difficult for the students to know how the vector techniques could be applied. In the teaching sequence they had only been working with either two points or two vectors at a time, but suddenly they had a list of 65 points. Certainly it required some skills of how to manipulate a spreadsheet to work with this amount of points that the students did not have. This can possibly be another reason why they did not apply the vector technique $\tau^{\overrightarrow{AB}}$.

9. Discussion

This thesis has worked with some different issues. First of all, the interrelation between vectors in mathematics and physics has been examined. Both the development of the interrelation between the scholarly notion of vectors in the two fields, and the development of the interrelation between the knowledge to be taught on vectors in the two fields have been examined. The investigations were guided by some research questions. It showed that the scholarly notion of vectors in mathematics and in physics evolved in a very close relation in the beginning, when the motivation for the development of a theory of vectors was still primarily coming from physical problems. At that time the geometric approach to vectors was prevalent, since this approach is closer to the one used in physics. This close relation can be used as an inspiration in the teaching of vectors in high school, if it utilised, that the need for vector analysis came from physics in the beginning.

The next phase in the development was the first step in the separation of the physical notion of vectors from the mathematical notion of vectors. In this phase an algebraic approach became prevalent, and vectors were beginning to be algebraic objects defined by axioms alone. The geometric properties of vectors, that were crucial in the relation to physics receded into the background, while the algebraic properties where developed. The next step was an axiomatisation of vector analysis, and a generalisation to the abstract notion of vector spaces. Though algebra and mathematics were dominating in this phase, applications of the more and more general vector notion in physics was still an important issue.

Nowadays, the scholarly notion of vectors is centred around the general definition of vector spaces, but one of the most important special cases is Hilbert spaces, that in some sense generalises geometry to higher (and possibly infinite) dimensions. This shows how the geometric properties are still relevant, though the mathematical definition had been through a heavy algebraisation.

In physics it is almost only the geometric properties of $\mathbb{R}^2/\mathbb{R}^3$ that are relevant. Sometimes the algebraic properties can even be misleading, when they are interpreted in a physical setting. It is worth noticing this difference, because it is often reflected in the organisation of the knowledge to be taught in high schools, and it can contribute to some of the struggles that the students are dealing with.

In the context of high school teaching, the notion of vectors has also moved from the border of mathematics and physics, where it was situated in the period from 1935-1953, to being almost fully separated in the two subjects in the current organisation. One of the reasons is the entry of the modern mathematics, where the knowledge to be taught was attempted to be neared the scholarly notion. In this period, that lasted from 1961-1984, vectors were primarily treated as algebraic objects that satisfied certain properties. The majority of the exercises dealt with proofs of the properties, and most of it was done with abstract symbols and no coordinates. However, the written exams, that constitute a great part of the praxis block in the praxeological organisation, where primarily containing analytical-geometrical vector-problems. These are the problems that utilise the geometric interpretation of the algebraic properties.

In the same period physics was generally highly mathematised in high school. Among other things, this included the representation of directional quantities by vectors. However, the use of vectors in physics was not particularly compatible with the very algebraic notion of vectors that was practised in mathematics. This is exactly the reflection of some of the problems that the scholarly notions of vectors in mathematics and physics suffer from.

During the next period, from 1984-2005, the mathematics teaching in high school started being more and more application-oriented. Also in the teaching of vectors this sneaked into the textbooks, though some of the applications that were mentioned was artificial and not reflecting any actual physical application. The coordinate-based techniques took up more and more space while the analytical-geometrical applications where also dealt with in the textbooks. The written exam problems where divided into the vector-algebra problems and the analytical-geometrical problems. The vector algebra-problems were more and more often coordinate-based, and the analytical-geometrical vector-problems became very extensive and involved multiple different technologies. At the same time, it started to be more and more prevalent in physics to introduce directional quantities without the use of vectors. This might be caused by the unsuccessful attempts of interrelating mathematics and physics during the modern mathematics.

In the recent period, the mathematical organisation of vectors has looked very similar to the one in the previous period. However, the introduction of vectors in textbooks does often use examples from physics as a motivation. Also the focus on coordinates have extended from the period 1984-2005 to the recent period. Modern textbooks introduce coordinates right away, where historical books did most often spend a lot of pages dealing with coordinate-free algebraic properties.

Regarding the modern organisation of vectors in physics, the guidelines for interpretation of the curriculum from 2013 recommended to avoid the use of vectors when dealing with motion in two dimensions.

The examination of the organisation of the knowledge to be taught on vectors in mathematics revealed a division into a geometric and an algebraic branch that has been valid since 1961. In the beginning the algebraic branch was prevalent in the logos block, while the geometric branch was prevalent in the praxis block. As the years went by, the algebraic branch entered the praxis block more and more, while the geometric branch entered the logos block. The algebraic branch of the logos block was almost coordinate-free in the beginning, but in the modern organisation, it is almost fully coordinate-based.

The empirical part of the thesis wanted to examine if and how the theoretical findings could contribute to a Study and Research Path on vectors, that would introduce the notion of vectors in a way, that would make it useful in both mathematics and physics. The ideas that were used are (1) to motivate the introduction of vectors by a physical problem, (2) to focus on geometric properties in mathematics, such as geometric addition of vectors and the interpretation of the numerical value of the determinant of two vectors as the area of the parallelogram spanned by them, and (3) to make physical calculations on coordinates.

A SRP was designed on the foundation of the three ideas above. The a priori analysis showed, that the generating question would most likely invoke some sub-questions that would motivate the notion of vectors, and after a short teaching sequence, the second generating question would most likely invoke sub-questions that required the application of some of the vector techniques that the students had been taught.

During the execution of the SRP a lot of troubles showed up. The collection of audio recordings did generally fail, which made the data material strictly limited. Also the time became an important player, since different issues made it pass faster than expected. The lesson plans needed extensive revisions, which changed the work with the SRP. An important consequence was, that the students did not have enough time to try and apply their newly acquired vector theory to the data set. Before the revision of the reports, that the students had to make, it was necessary to hand out a guiding solution, to make sure that the students did learn something. A consequence of this was, that the second report was almost useless for the analysis, because most of the groups did just reproduce fragments of the hand out.

The analysis of the data material was guided by the research questions (${RQ_1}^{II}$-${RQ_3}^{II}$). It showed how the students ask fewer questions than they were expected to, that they did actually ask some of the questions that were expected to motivate the teaching of vector theory, but that they answered them by other methods. This is possibly the reason why the vector techniques were not used to calculate the travelled distance or the velocity that the aeroplane was flying at. One of the flaws in the design was that the teacher was not involved enough. Instead of letting the students pass the wrong results of the travelled distance or the velocity, the teacher should have been instructed in how to make the students consider their answers. By that, they would maybe have realised, that they needed the theory of vectors to solve the question.

From the data material, especially the reports, it became clear, that the students had not developed a praxeology of vectors, that was useful in the SRP. They might have been able to solve some of the vector-algebra problems from written exams, with the techniques that they were taught in the sequence about vectors.

If a similar SRP should be used in connection with the theory of vectors, some extensive adjustments have to be made. Generally, it would be preferable that the students have engaged in study and research paths before they get such a complex generating question. An idea would also be to sort out the conversion of unit, transfer of data from one computer program to another, and some of the presentations or assignments, such that the students can focus on the core of the SRP, namely the vector theory.

It would also be important to make it clearer what details that are relevant and which that are not, since aviation is generally very complex. E.g. it should be explicitly stated, that the height dimension should not be taken into account. It would also be an idea to introduce the students to the notion of "true airspeed" (that is the speed of the aeroplane relative to the surrounding air) and "groundspeed" (that is the speed of the aeroplane relative to the ground), since these notions will be useful to invoke the notion of vectors in relation to velocity and wind conditions.

As it has already been mentioned, the role of the teacher should have been prioritised a lot more. The teacher should have been used to make the students realise when they produced incorrect results, that kept them from having the maximal benefit of the teaching sequence.

Last but not least, a citing "How do you analyse mathematics?" from one of the students underline the importance of alternative teaching. If students never do anything else than reproducing the results and techniques that their teacher have shown them, they will not be able to use their knowledge outside of the mathematics classroom. If this scenario can be prevented, by working with interdisciplinary problems, it is clearly worth it to keep developing a SRP like the one that has been designed in this thesis to make it work in practice.

10. Conclusion

The aim for this thesis was to examine the development of vectors in mathematics and physics respectively, together with their interrelation and its effects on the teaching and learning of vectors in high school. The analysis of the two teaching subjects should constitute the foundation of the design of a Study and Research Path on vectors in mathematics and physics. The teaching design was later tested in a test class and the data has been analysed in order to determine whether it had given the students some available knowledge on vectors, that hopefully could be used in both mathematics and physics.

The praxeological analysis of the physical organisation of the knowledge to be taught on vectors showed that vectors in physics are mostly regarded as geometric objects that are represented by an arrow. Furthermore, the coordinate representation of vectors can be used to represent motion in two dimensions. This was the prevalent approach to motions in two dimensions through many years, but in the recent periods, this approach has been advised against in the guidelines for interpretation of the curriculum.

The praxeological analysis of the mathematical organisation of vectors showed, that, except from the very early teaching of vectors in the period from 1935-1953, the approach has been divided into two branches. The first branch is the algebraic, where $\mathbb{R}^2$ and $\mathbb{R}^3$ are treated from the view of a vector space that is defined by axioms. The other branch is the geometric, where $\mathbb{R}^2$ and $\mathbb{R}^3$ are treated from a more geometric point of view. This means that the special geometric properties, that are not general in every vector space, are utilised in order to solve different geometric problems. In both branches vectors are mostly represented in a more algebraic way, either abstractly by a symbol representing a vector or by coordinates. In the early years of the teaching of vectors, the theory block in the praxeological organisation was primarily algebraic, while at the same time the analytical-geometrical vector-problems in the written exams were highly overrepresented compared to the vector algebra-problems. Gradually, the vector algebra-problems started to show up more frequently in the written exams, while the geometric branch took up more and more space in the logos block in the praxeological organisation.

While mathematics was gradually toned down in the physics teaching, examples and motivation from physics, or a combination of physics and geometry, started to increase in importance in the textbooks and analytical-geometrical vector-problems in the written exams.

These uncoordinated developments in mathematics and physics respectively might be one of the reasons why the students struggle in applying the vector theory that they are taught in mathematics when they are taught physics.

On the foundation of these analyses, the SRP was designed. The purpose was to ask the students an initial generating question being physical in nature, that would create a need for vector theory in order to carry out calculations on coordinates. After the need had been created, the students should be taught vector theory in a more classical teaching situation. The next part of the SRP would then be initiated by a second generating question. The purpose of this question was to invoke the newly learned vector theory, in order to answer it. A SRP like this should leave the students with a feeling that the theory of vectors is useful in physics, and that vectors in mathematics are objects that can be utilised in real problems.

The analysis of the teaching showed that a more physical approach to a new mathematical subject was motivating to the students, though they were still frustrated with the alternative way of working, but that the design contained too many problems to create a useful praxeology of vectors.

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Appendix A

Hand outs

A.1 First hand out, Tuesday 3 April 2018

Assistancemelding til 1w

Kort overblik

Flyselskabet Malaysian Airlines er kommet i problemer. De har mistet kontakten til et af deres fly, Dumbo04, på en rute fra Langbortistan til Ønskeøen. Dette er meget, meget uheldigt, set i lyset af katastrofen i 2014, hvor et fly forsvandt på en tur fra Kuala Lumpur i Singapore til Beijing i Kina. Sagen har været massivt dækket i pressen siden den dag flyet forsvandt (se f.eks. den vedlagte artikel). Det er endnu ikke fundet og først efter tre år er det lykkedes eksperter at give et fornuftigt bud på hvor flyet er styrtet ned.

Denne gang vil Malaysian Airlines meget gerne have tingene under kontrol hurtigst muligt, så det ikke bliver lige så dramatisk som MH370-sagen fra 2014. Derfor beder de om assistance fra jer. Med udgangspunkt i et sæt af baggrundsoplysninger om flyveturen, skal I forsøge at få et overblik over flyvningen. Dette overblik skal præsenteres og dokumenteres i en **rapport**, der skal afleveres **torsdag d. 12. april**. Undervejs i forløbet skal I understøtte hinandens arbejde, ved at fremlægge jeres foreløbige resultater og derefter modtage feedback.

Oplysninger om turen

Distance: Turen fra Langbortistan til Ønskeøen er ca. 1000 km lang, og foregår over hav det meste af vejen (se det vedlagte kort).

Tid: Under normale vejrforhold tager det 90 min. at tilbagelægge de 1000 km.

Maskine: Dumbo-04 er en Airbus a320

Position: Flyets koordinater (i hhv. x- og y-retning) registreres én gang i minuttet. Efter 65 minutter flyvning mister kontroltårnet radarkontakten med flyet og siden er der ingen oplysninger om flyets position.

Opgave: På baggrund af de ovenstående oplysninger samt et datasæt bestående af flyets koordinater skal I analysere flyveturen. Datasættet får I udleveret i Excel, men det kan overføres til TI-Nspire og analysere dér, alt efter hvilket program i foretrækker.

<u>Arbejdsgang for første del af undersøgelsen</u>

Tirsdag d. 3. april: I arbejder i grupper med datasættet. Som dokumentation skal I føre en logbog, hvor I noterer hvilke observationer I gør, hvilke spørgsmål/teorier I arbejder med samt hvilke svar/konklusioner I finder. Ved timens afslutning skal I have et svar/en konklusion klar sammen med et spørgsmål, som I kunne tænke jer at arbejde videre med torsdag d. 5. april.

Produktkrav for timen: Følgende skal være **klar til præsentation** ved timens afslutning tirsdag d. 3. april (præsentationen foregår torsdag d. 5. april) 1. Jeres bedste bud på et resultat, med begrundelse 2. Et spørgsmål, som I kunne være interesserede i at arbejde videre med i timen Begge dele skal præsenteres på en poster, som kan understøtte en mundtlige præsentation. **Desuden:** Jeres logbog sendes i Lectio til UG ved timens afslutning tirsdag d. 3. april

Torsdag d. 5. april: Timen begynder med at I, sammen i grupperne, præsenterer jeres svar og spørgsmål (dette skal ske mundtligt, men understøttet af posteren). Grupperne beslutter sig i fællesskab for et eller to spørgsmål, som skal undersøges i løbet af timen. I skal stadig dokumentere jeres arbejde i en logbog.

Produktkrav for timen: Følgende skal være **klar til præsentation** 20 min. før timens afslutning torsdag d. 5. april 1. Jeres besvarelse/forsøg på en besvarelse af spørgsmålet 2. Noget I undrer jer over, som kunne være interessant at undersøge nærmere **Desuden:** Jeres logbog sendes i Lectio til UG ved timens afslutning torsdag d. 5. april

A.2 Second hand out – Thursday 12 April 2018

Assistancemelding til 1w – del 2

Kort overblik

Malaysian Airlines er meget glade for det arbejde I har udført indtil videre. I den sidste fase af undersøgelserne er målsætningen at bestemme arealet af det område, hvor der skal ledes efter det nedstyrtede fly.

Som nævnt tidligere, skal dette præsenteres i en rapport, som skal afleveres i slutningen af timen **torsdag d. 12. april**. Rapporten skal være målrettet Malaysian Airlines, og skal indeholde alle de resultater/konklusioner I har fundet sammen med velargumenterede begrundelser for disse.

Fejlrettelse

Ved nærmere eftersyn har det vist sig, at der er en lille fejl i det udleverede datasæt. Det drejer sig om y-koordinaten i minut 53. Den er *ikke* $357{,}31$, men i stedet $357{,}53$.

Oplysninger om turen

Vejr: Vejrforholdene på turen er blevet undersøgt, og det viser sig, at flyet efter ca. 50 min. (dvs. *efter* det har lavet sit ”loop”) er havnet i en meget kraftig vind, $\begin{pmatrix} 3{,}33 \\ -32{,}33 \end{pmatrix} \frac{m}{s}$.

Brændstof: Under normale omstændigheder vil et fly som Dumbo-04 bruge ca. 4 ton brændstof på en tur på 1000 km. Der vil dog være tanket ca. 8 ton brændstof, så piloterne kan være næsten helt sikre på at redde sig ud af en eventuel nødssituation. Det viser sig desværre at der er sket en fejl under tankningen og Dumbo-04 har kun haft 5 ton brændstof med. Det betyder, at den maksimale flyvedistance for flyet er 1300 km.

Arbejdsgang for anden del af undersøgelsen

Torsdag d. 12. april: Med udgangspunkt i jeres tidligere arbejde, samt de nye informationer I har fået, skal I arbejde med at forsøge at bestemme arealet af det område, hvor der skal ledes efter flyet.

Sideløbende skal I udarbejde en rapport til Malaysian Airlines, som på en tydelig og velargumentet måde præsenterer de resultater/konklusioner I har fundet igennem de sidste to uger. (Dette kan sandsynligvis gøres på 2-4 sider inkl. grafer/figurer, udregninger m.m.)

Da det skriftlige produktkrav er en rapport, skal I ikke skrive logbog, men til gengæld skal I optage lyd fra dagens arbejde, medmindre at det er *helt umuligt*.

Rapporten skal afleveres ved timens afslutning i en afleveringsmappe i Lectio.

Fredag d. 13. april: I disse to timer, skal hver gruppe give feedback til to andre gruppers rapporter, samt modtage feedback på deres egen rapport fra de to grupper.

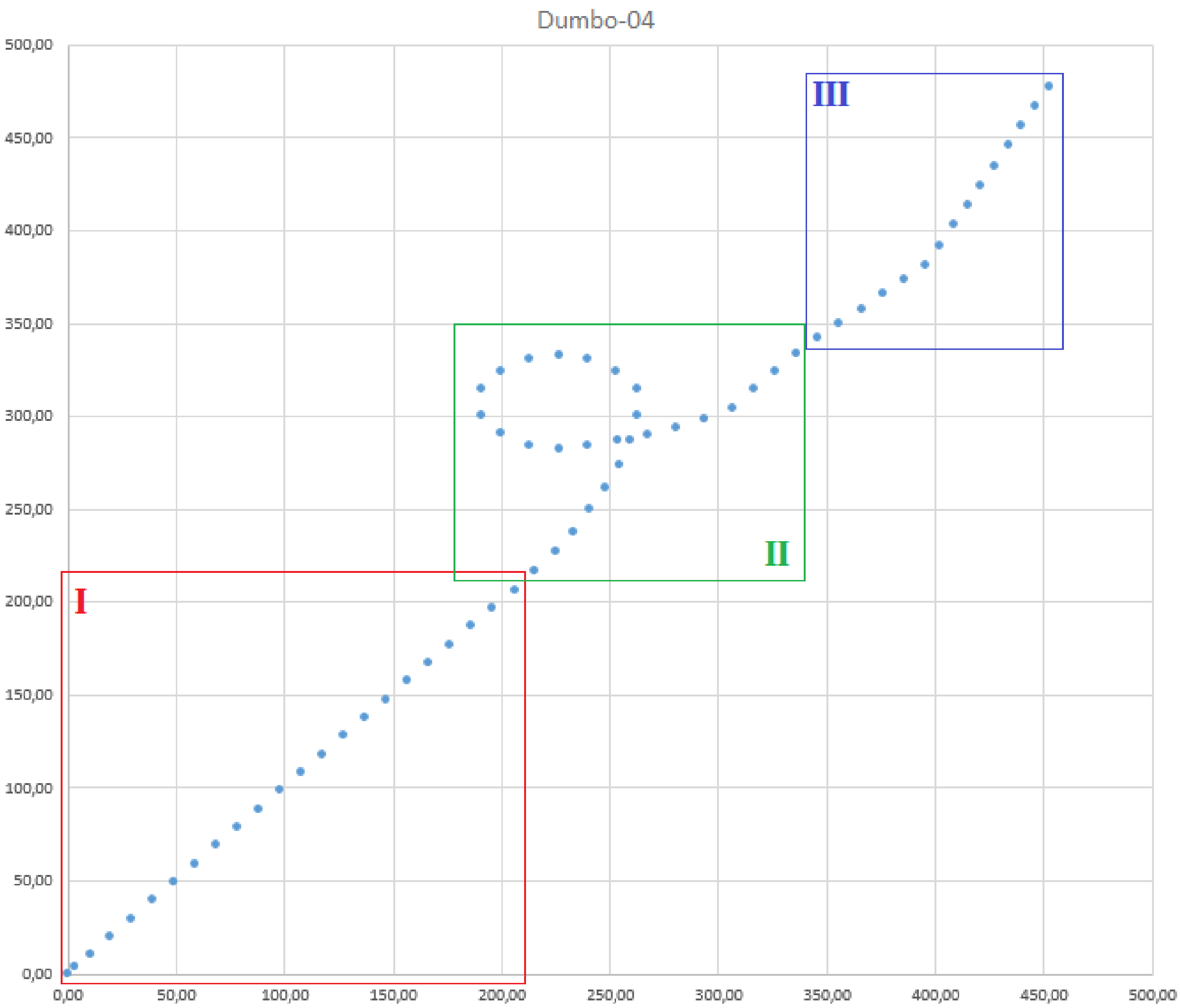
Når I har modtaget og givet feedback, skal I rette jeres rapport til ved at inddrage det relevante feedback og genaflevere den.

Tidsplan:

- Alle grupper læser, diskuterer og udarbejder feedback til to andre grupper (ca. 35 min.)
- Første rapport gives feedback af to grupper (ca. 15 min.)
- Anden rapport gives feedback af to grupper (ca. 15 min.)
- Tredje rapport gives feedback af to grupper (ca. 15 min.)
- Tilretning af rapporter på baggrund af feedback samt genaflevering (ca. 20 min)

A.3 Third hand out – Friday 13 April 2018

Hvad er der sket med Dumbo-04?



Figur 1 – Turen inddelt i sektioner

Første del af turen (I på Figur 1)

Efter at have accelereret op til en passende hastighed, holder Dumbo-04 sin kurs med en konstant fart indtil minut 22 (se Figur 2). Denne fart er på $13{,}90\,\frac{km}{min.}$ eller $834{,}10\,\frac{km}{t}$. Farten er bestemt ved formlen:

$$|v| = \sqrt{v_x^2 + v_y^2}$$

Hvor gennemsnitshastighederne (over et tidsinterval) i hhv. x- og y-retningerne er bestemt ved at benytte formlen for hastighed:

$$v = \frac{\Delta s}{\Delta t} = \frac{s_{slut} - s_{start}}{t_{slut} - t_{start}}$$

Et konkret eksempel på en udregning af gennemsnitshastigheden i det første minut i x-retningen:

Vi ønsker at bestemme gennemsnitshastigheden i x-retningen i det første minut. Vi ved at $\Delta t = 1\ min.$, da vi modtager et signal hvert minut. Vi bestemmer Δs_x:

$$\Delta s_x = s_{x,slut} - s_{x,start} = 3{,}5\ km - 0{,}0\ km = 3{,}5\ km$$

Og vi kan nu bestemme gennemsnitshastigheden, v_x:

$$v_x = \frac{3{,}5\ km}{1\ min.} = 3{,}5\,\frac{km}{min.}$$

For at omregne til $\frac{km}{t}$ skal værdien ganges med faktoren $60\,\frac{min.}{t}$.

Gennemsnitshastighed	
v_x	v_y
[km/min.]	[km/min.]
3,50	3,50
7,00	7,00
9,00	9,00
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83

Figur 2 – Gennemsnitshastigheder for de første 23 min.

Farten $834{,}10\,\frac{km}{t}$ er normalt for en Airbus a320, så det tyder på, at alt var under kontrol til og med minut 22.

Vi kan desuden se at Dumbo-04 er nået ca. $292\ km$ væk fra Langbortistan Lufthavn, hvilket er den samme afstand, som det flyet har tilbagelagt, da flyet har fløjet efter en ret linje. Denne afstand kan bestemmes ved brug af vektorregning, ved at bestemme længeden af stedvektoren til punktet med koordinaterne for flyets position i minut 22:

I minut 22 er flyet i punktet P med koordinatsættet $P(206{,}27; 206{,}27)$. Vi har altså stedvektoren:

$$\overrightarrow{OP} = \begin{pmatrix} 206{,}27 \\ 206{,}27 \end{pmatrix}$$

Denne vektor har længden:

$$\left|\overrightarrow{OP}\right| = \sqrt{206{,}27^2 + 206{,}27^2} = 291{,}71$$

Anden del af turen (II på Figur 1)

I minut 23 begynder Dumbo-04 at krænge mod nord og dermed ud af kurs. Det ser altså ud til at der er opstået problemer, men ser man på de gennemsnitlige hastighedsændringer over intervallerne – altså gennemsnitsaccelerationerne – vil man observere, at de ser forholdsvist kontrollerede ud. På baggrund af dette er det sandsynligt at flyet stadig har været i kontrol.

Gennemsnitsaccelerationerne (over et tidsinterval) i hhv. x- og y-retningerne er bestemt ved formlen:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_{slut} - v_{start}}{t_{slut} - t_{start}}$$

Et konkret eksempel på en udregning af gennemsnitsaccelerationen i minut 23 i x-retningen:

> Vi ønsker at bestemme gennemsnitsaccelerationen i x-retningen i minut 23. Vi ved at $\Delta t = 1\ min.$, da vi modtager et signal hvert minut. Vi bestemmer Δv_x:
>
> $$\Delta v_x = v_{x,slut} - v_{x,start} = 9{,}65\frac{km}{min.} - 9{,}83\frac{km}{min.}$$
> $$= -0{,}18\frac{km}{min.}$$
>
> Og vi kan nu bestemme gennemsnitsaccelerationen, a_x:
>
> $$a_x = \frac{-0{,}18\frac{km}{min.}}{1\ min.} = -0{,}18\frac{km}{min.^2}$$

Gennemsnitsacceleration	
a_x [km/min.^2]	a_y [km/min.^2]
...	...
-0,18	0,17
-0,55	0,50
-0,61	0,50
-0,68	0,50
-0,80	0,50
-0,93	0,50
-1,16	0,50
-1,61	0,50
-3,31	0,50
-9,65	-4,00
-2,89	-4,00
-1,21	-4,00
0,00	-4,00
1,21	-4,00
2,89	-4,00
9,65	-3,90
9,65	3,90
2,89	4,00
1,21	4,00
0,00	4,00

Figur 3 – Gennemsnitsaccelerationer for minut 23 til minut 42

Grundet den ”kontrollerede” måde flyet er kommet på afveje, er det sandsynligt, at flyet har haft problemer med de instrumenter, som bruges til navigation. Noget lignende er sket før, f.eks. med flyet Aeroperú Flight 603, som havde problemer med fart- og højdemålere under en flyvning i 1996 (se evt. https://www.youtube.com/watch?v=FUKGqBlKQvA).

Da vi ved at brændstofmængden kan have givet flyet problemer, er det relevant at vide, hvor lang en ”ekstrastrækning” flyet har tilbagelagt ved at lave sit ”loop”. Fra flyet kommer på afveje, og til det havner i den stærke vind, har det tilbagelagt ca. 389 km. Denne strækning kan bestemmes ved at lægge længderne af vektorerne mellem koordinat-punkterne sammen.

Hvis vi har koordinatsættene for to punkter efter hinanden – f.eks. koordinatsættet for minut 23 (vi kalder punktet for A) og koordinatsættet for minut 24 (vi kalder punktet for B) – kan vi finde vektoren $\overrightarrow{AB}$ ved at benytte formlen:

$$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

Herefter kan vi finde ud af hvor stor en afstand flyet har tilbagelagt fra minut 23 til minut 24 ved at bestemme længden af denne vektor:

$$\left|\overrightarrow{AB}\right| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

Ved at gøre dette for alle minutterne fra minut 23 til minut 50 og lægge alle disse længder sammen, kan vi bestemme den afstand flyet har tilbagelagt undervejs i sit loop og indtil det ryger ind i den stærke vind.

Lægger vi i stedet længderne sammen for *alle* minutterne, kan vi bestemme hvor mange kilometer flyet har fløjet i alt.

Ved at lægge alle vektorernes længder sammen, kan vi se at flyet har tilbagelagt ca. $681\ km$ fra det lettede og indtil det kommer ind i den stærke vind.

Desuden ved vi, at flyet i minut 50 er ca. $474\ km$ fra Langbortistan Lufthavn. Dette er igen bestemt ved at finde længden af stedvektoren, ligesom ovenfor.

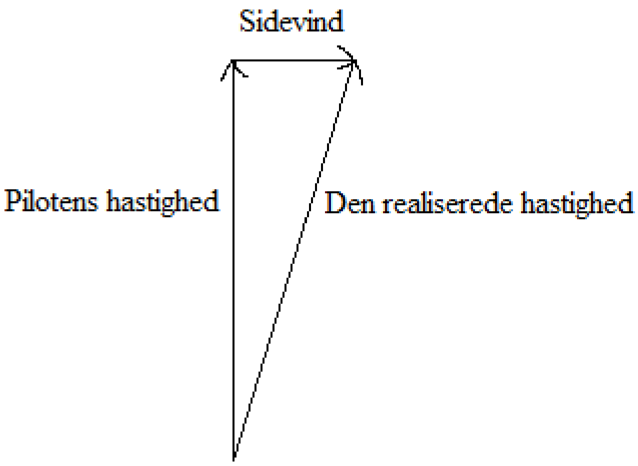
Tredje del af turen (III på Figur 1)

Her har vi information om vindforholdene. Gennemsnitshastighederne lige inden flyet kommer ind i den stærke vind, er det samme, som før det begyndte at gå galt for Dumbo-04. Det er altså sandsynligt at piloten har haft opfattelsen af, at han var på rette kurs igen.

Hvis piloten har problemer med instrumenterne i cockpittet, kan han have problemer med at vide hvor han flyver og hvor hurtigt han flyver.

Når et fly flyver i sidevind er der forskel på, den hastighed piloten har indstillet flyet til at flyve med og så den hastighed flyet reelt set flyver med, når sidevinden er i spil (se Figur 4).

Den hastighed vi kan bestemme ud fra datasættet er den realiserede hastighed. Men når vi kender vinden, kan vi udregne den hastighed piloten har indstillet flyet til at flyve med. Dette gøres ved at trække sidevindsvektoren fra den realiserede hastighedsvektor.



Figur 4 – Hastigheder med sidevind

For at kunne udregne pilotens indstillede hastighed, omregnes vindhastigheden til enheden $\frac{km}{min.}$, da datasættet indeholder enhederne km og $min.$:

For at komme fra $\frac{m}{s}$ til $\frac{km}{min.}$ skal vi gange med faktoren:

$$\frac{60\,\frac{s}{min.}}{1000\,\frac{m}{km}}$$

For at bestemme vindens hastighed i $\frac{km}{min.}$ skal vi altså gange hastighedsvektoren med konstanten (skalaren) $\frac{60\frac{s}{min.}}{1000\frac{m}{km}}$:

$$\begin{pmatrix}3{,}33\\-32{,}33\end{pmatrix}\frac{m}{s}\cdot\frac{60\,\frac{s}{min.}}{1000\,\frac{m}{km}}=\begin{pmatrix}0{,}20\\-1{,}94\end{pmatrix}\frac{km}{min.}$$

Vi kan nu udregne pilotens hastighed, ved at trække vinden fra. Hvis vi antager at vinden bliver ved med at være lige så stærk resten af turen, så vil pilotens hastigheder de sidste 15 min. ligge på hhv. $\begin{pmatrix}9{,}83\\9{,}83\end{pmatrix}\frac{km}{min.}$ fra og med minut 51 til og med minut 56, og $\begin{pmatrix}6{,}08\\12{,}64\end{pmatrix}\frac{km}{min.}$ fra og med minut 57 til og med minut 65.

Pilotens hastighed i f.eks. minut 53 bestemmes ved at benytte formlen:

$$v_{pilot}=\begin{pmatrix}v_{pilot,1}\\v_{pilot,2}\end{pmatrix}=v_{fly}-v_{vind}$$
$$=\begin{pmatrix}v_{fly,1}\\v_{fly,2}\end{pmatrix}-\begin{pmatrix}v_{vind,1}\\v_{vind,2}\end{pmatrix}=\begin{pmatrix}v_{fly,1}-v_{vind,1}\\v_{fly,2}-v_{vind,2}\end{pmatrix}$$

Og ved at indsætte værdierne fås:

$$v_{pilot}=\begin{pmatrix}10{,}03-0{,}2\\7{,}89-(-1{,}94)\end{pmatrix}=\begin{pmatrix}9{,}83\\9{,}83\end{pmatrix}$$

Pilotens gennemsnitshastigheder	
v_x [km/min.]	v_y [km/min.]
...	...
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
9,83	9,83
6,08	12,64
6,08	12,64
6,08	12,64
6,08	12,64
6,08	12,64
6,08	12,64
6,08	12,64
6,08	12,64
6,08	12,64

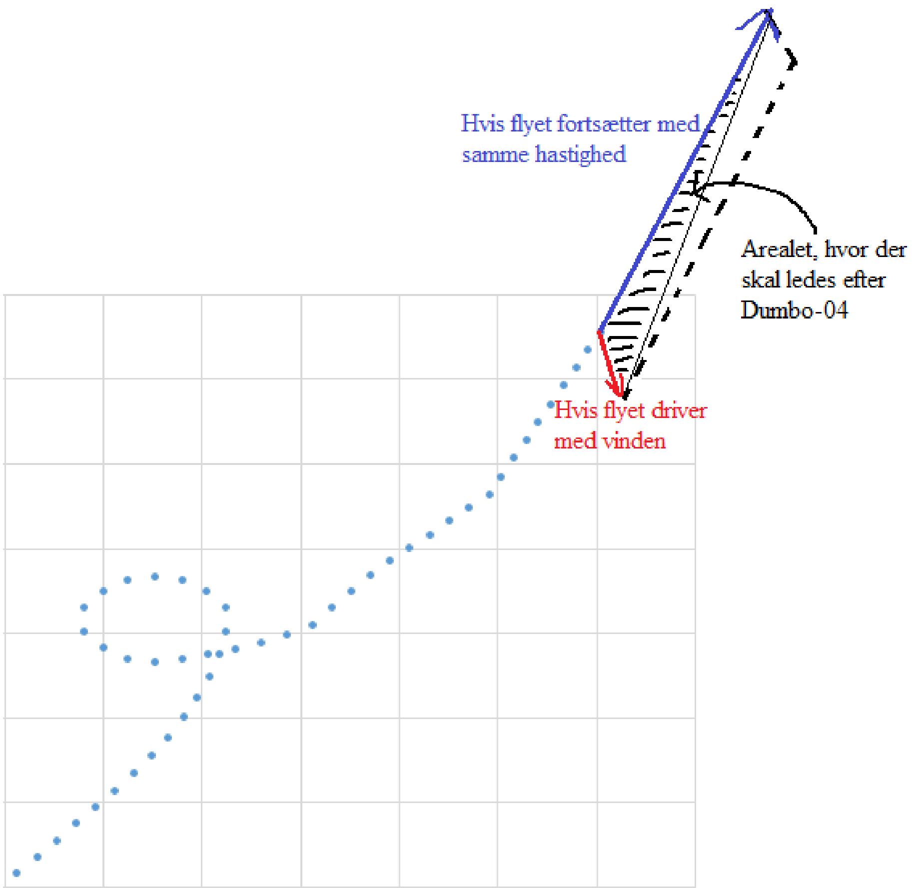
Figur 5 – Gennemsnitshastigheder for minut 51 til minut 65

Vi kan altså se, at piloten holder samme hastighed de første minutter, efter at flyet er kommet ind i den stærke vind. I minut 57 ændres hastigheden igen, men holdes konstant indtil radarkontakten ophører.

<u>I hvor stort et område skal vi lede efter Dumbo-04?</u>

Vi kan vælge at afgrænse afsøgningsområdet ved to grænsesituationer. De to grænsetilfælde kan hver repræsenteres ved en vektor, med en længde svarende til den distance flyet flyver inden det lander. Da kan vi bestemme arealet af afsøgningsområdet som arealet af den ene halvdel af det parallelogram, der udspændes af de to vektorer.

I den ene grænse kan vi forestille os, at flyet holder den samme hastighed, som det fløj med umiddelbart inden radarkontaktens afbrydelse, indtil det løber tør for brændstof. Som den anden grænse kan vi antage, at flyets motorer slukkede, da radarkontakten ophørte, og at flyet svæver med vinden, indtil det lander (se Figur 6).



Figur 6 – Parallelogram udspændt af de to grænsetilfælde

Som det første bestemmes den distance, som flyet har brændstof til. Dette gøres ved først at bestemme den distance flyet allerede *har* tilbagelagt. Det gøres ved at lægge længderne af alle vektorerne mellem koordinatpunkterne sammen (metoden er beskrevet i boksen på side 4). Denne udregning giver os, at flyet har tilbagelagt en distance på $869\ km$. Da flyet kan flyve maksimalt $1300\ km$, kan det altså nå $(1300 - 869) = 431\ km$.

Som det næste skal vi bestemme den vektor, som er parallel med den retning flyet flyver i lige før radiokontakten mistes og som har længden $431\ km$:

Hvis flyet fortsætter med den samme hastighed: På ét minut kommer tilbagelægger flyet en distance svarende til længden af hastighedsvektoren $\begin{pmatrix} 6{,}28 \\ 10{,}70 \end{pmatrix} \frac{km}{min.}$. Den bestemmes ved at benytte formlen for længden af en vektor:

$$|\vec{v}| = \sqrt{v_1^1 + v_2^2}$$
$$= \sqrt{6{,}28^2 + 10{,}70^2} = 12{,}41$$

Vi skal nu bestemme den, hvilken konstant, k, vi skal gange denne vektor med, for at den får længden 431:

$$k = \frac{431\ km}{12{,}41\ km} = 34{,}73$$

Denne konstant er desuden et udtryk for, hvor længe flyet maksimalt vil kunne flyve. Vi skal altså gange hastighedsvektoren op med konstanten $k = 34{,}73$ for at få den vektor, som beskriver den retning og distance flyet maksimalt kan flyve på det resterende brændstof:

$$\overrightarrow{s_{blå}} = \begin{pmatrix} 6{,}28 \\ 10{,}70 \end{pmatrix} \frac{km}{min.} \cdot 34{,}73\ min. = \begin{pmatrix} 218{,}10 \\ 371{,}61 \end{pmatrix}\ km$$

Den blå vektor på Figur 6, skrives altså $\begin{pmatrix} 218{,}10 \\ 371{,}61 \end{pmatrix}\ km$.

Som det næste bestemmes den vektor, som svarer til den retning og distance, som flyet kan drive med vinden, hvis motorerne er stoppet, da radarkontakten ophørte. På baggrund af følgende antager vi, at flyet kan glide 150 km, og at det med det samme glider i samme retning som vinden.

A: Såfremt vi antager at alle motorer pludselig holdt op med at producere fremdrift samtidigt, så vil et moderne passagerfly kunne svæve ganske langt. Hvor langt afhænger af en række faktorer så som – hvad er hastigheden af flyet når motorerne går ud, hvad er højden på flyet, har flyet medvind eller modvind (dette har betydning for flyets "ground speed" og dermed glide distance), er "high drag" devices så som "landing gear" ude… etc. For at give dig nogle grove ca tal – så vil en moderne passager jet have et typisk glidetal på omkring "15". Et glide tal betyder at for hver 1 enhed højde flyet taber –vil flyet kunne bevæge sig 15 enheder frem. Flyvning forgår ofte i ca. 10 km højde – så regnestykket bliver ret simpelt… 150 km kan flyet glide.

Figur 7 – Et bud på flyets glidelængde: https://spoerg-piloten.dk/hvis-en-boeing-777-eller-737-altsa-et-fly-i-passagerfly-storrelse-oplevede-at-alle-motorer-stopper-med-at-virke-hvordan-vil-flyet-sa-falde-og-hvor-lang-tid-ville-det-kunne-holde-sig-i-luften/

Endnu en gang skal vi bestemme længden af den vektor, som beskriver den retning og distance flyet tilbagelægger, mens det glider.

Hvis flyet glider med vinden: Som det første bestemmer vi længden af vindvektoren, for at bestemme koordinaterne til en parallel vektor med længde 150 km:

$$|\overrightarrow{v_{vind}}| = \sqrt{0{,}2^2 + (-1{,}94)^2} = 1{,}95$$

Vi bestemmer konstanten, k_1, som vi skal gange op med, for at få en parallel vektor, med længden 150 km:

$$k_1 = \frac{150}{1{,}95} = 76{,}92$$

Vi kan nu bestemme den vektor, som beskriver den tilbagelagte strækning:

$$\overrightarrow{s_{rød}} = \begin{pmatrix} 0{,}2 \\ -1{,}94 \end{pmatrix} \cdot 76{,}92 = \begin{pmatrix} 15{,}38 \\ -149{,}23 \end{pmatrix}$$

Den røde vektor på Figur 6 skrives altså $\begin{pmatrix} 15{,}38 \\ -149{,}23 \end{pmatrix}$ km.

Som det sidste, kan vi nu bestemme arealet af parallelogrammet udspændt af de to vektorer. Dette gøres ved at benytte formlen:

$$A_{parallelogram} = |\det(\overrightarrow{s_{blå}}, \overrightarrow{s_{rød}})|$$
$$= \left|\begin{vmatrix} 218{,}10 & 15{,}38 \\ 371{,}61 & -149{,}23 \end{vmatrix}\right| = |218{,}10 \cdot (-149{,}23) - 15{,}38 \cdot 371{,}61| = |-38262{,}42| = 38262{,}42$$

Da de to vektorer er grænsetilfældene, skal vi kun lede efter flyet i den halvdel af parallelogrammet, som er tættest på det punkt, hvor radarkontakten ophørte. Vi får arealet af dette område (som er en trekant) ved at dividere med 2:

$$A_{afsøgningsområde} = \frac{38262{,}42}{2} = 19131{,}21$$

Dvs. at arealet af det område, som skal afsøges, er på ca. 19131 km^2.

Appendix B

Logbooks

B.1 Logbook 1, group 1

Louises projekt omkring Malaysia flyet.

3. April 2018

Vi kan se hvor stor en procentdel af turen flyet aflagde på den tid der er gået.

Vi kan finde dens fart $\frac{m}{s}$

For at finde dens efterfølgende svævetid, hvis flyet gik i ”stå”.

Undersøge dens tsunami flyvetrick.

Næste gang

Hvor meget

B.2 Logbook 2, group 1

Louises projekt omkring Malaysia flyet.

3. april 2018

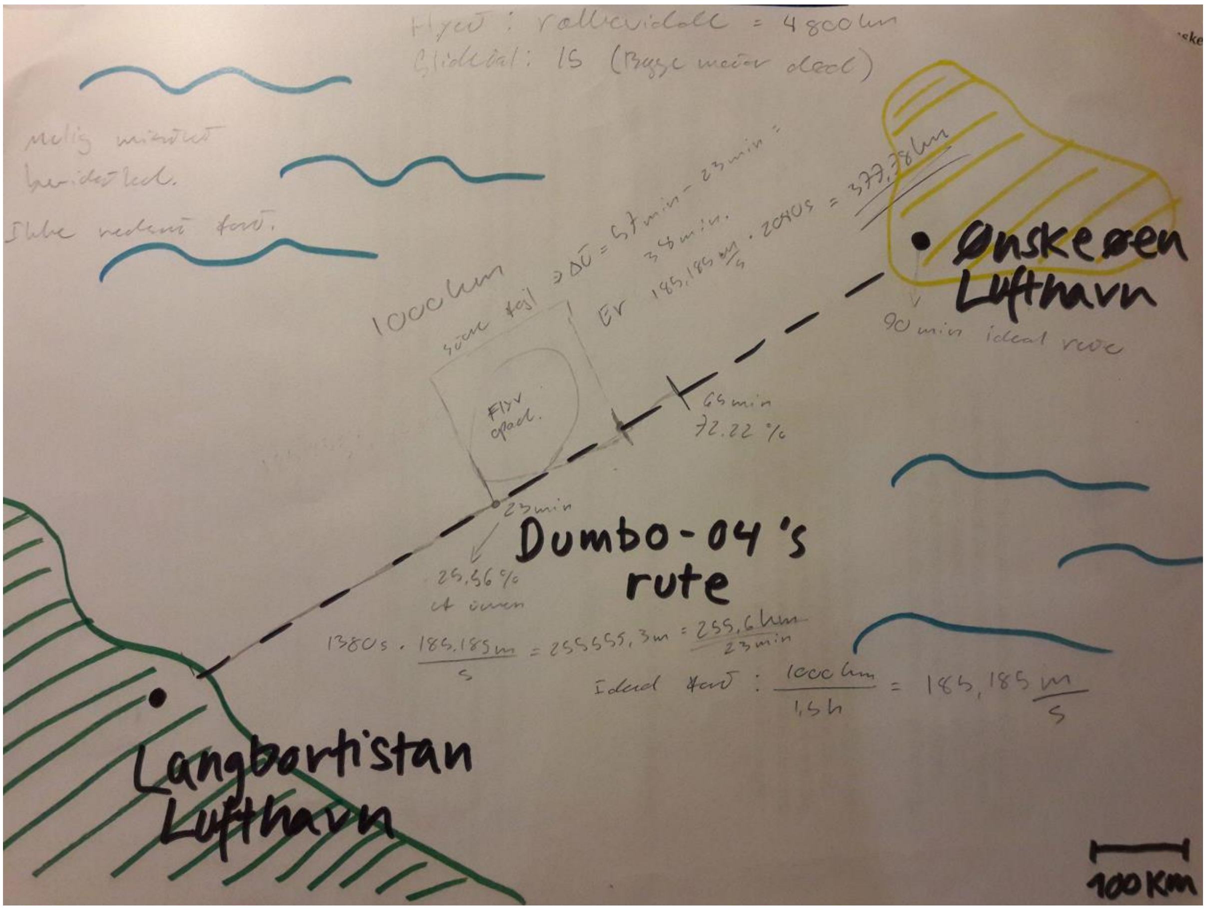
Vi kan se hvor stor en procentdel af turen flyet aflagde på den tid der er gået.

Vi kan finde dens fart $\frac{m}{s}$

For at finde dens efterfølgende svævetid, hvis flyet gik i "stå".

Undersøge dens tsunami flyvetrick.

5. april 2018



Det tager 90min (5400s) at flyve 1000km (1.000.000m)
Det er en gennemsnitsfart på $\frac{1000000m}{5400s} = \frac{5000}{27} = 185{,}1852\,\frac{m}{s}$ => $185{,}1852 \cdot 3600 = 666666{,}7\,\frac{m}{h}$

Dvs. at flyet flyver med ca. 667 $\frac{km}{h}$

Indtil det 23. minut, flyver flyet u besværet ifølge dets placering på koordinaterne.
23 minutter er $\frac{23min}{90min} \cdot 100\% = 25{,}55556\%$

Dvs. at de har nået omkring 25.6 % af turens forløb, før der begynder at være vanskeligheder.

Bevidsløshed

Hvis man går ud fra at piloterne måske er blevet bevidstløs her, ligesom i en artikel vi læste om, så er det jo ikke er i stand til at styre flyet. Men hvis det stadigvæk har en form for auto-/fartpilot sat til, sænkes farten ikke.

Så fra det 23. minut til det 57. minut, hvor der sker forstyrrelser på ruten, og de bl.a. laver et loop, vil det altså sige, at med en fart på 185,1852$\frac{m}{s}$ over en tid på $57 - 23 = 34$ minutter,

$$185{,}1852\frac{m}{s} \cdot (34 \cdot 60)s = 377777{,}8m \Rightarrow 378\ km$$

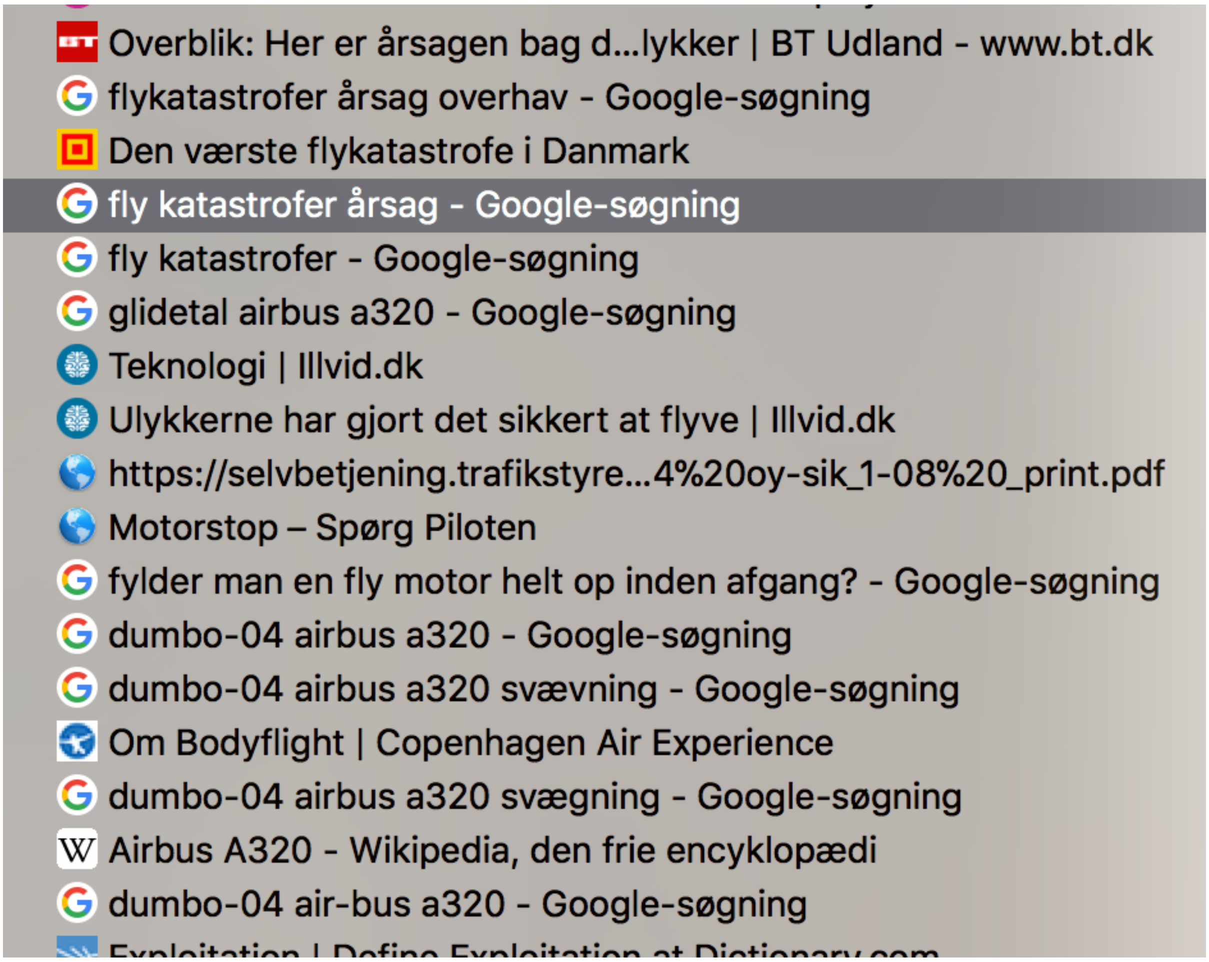
har de fløjet en længde på 378 km.

Den ekstra flyvetid kan have medfølger i hvor langt flyet kunne have være nået i enhver retning. Fly bliver ikke tanket helt op, kun omkring det nødvendige til selve ruten, og måske lidt ekstra for en sikkerhedsskyld. Men ellers ville en for fyldt tank være skyld i en større brug af brændstof, fordi flyet er tungere.

Storm

I et andet eksempel, fandt vi at et fly havde lavet en sådanne cirkel manøvre, for at øge flyvehøjden, fordi de mødte en storm. Det samme kan være grunden til at de laver den her, og det efterfølgende ujævne flyvemønster, kan være grundet at de stadigvæk påvirkes af selve stormen.

Søgehistorik



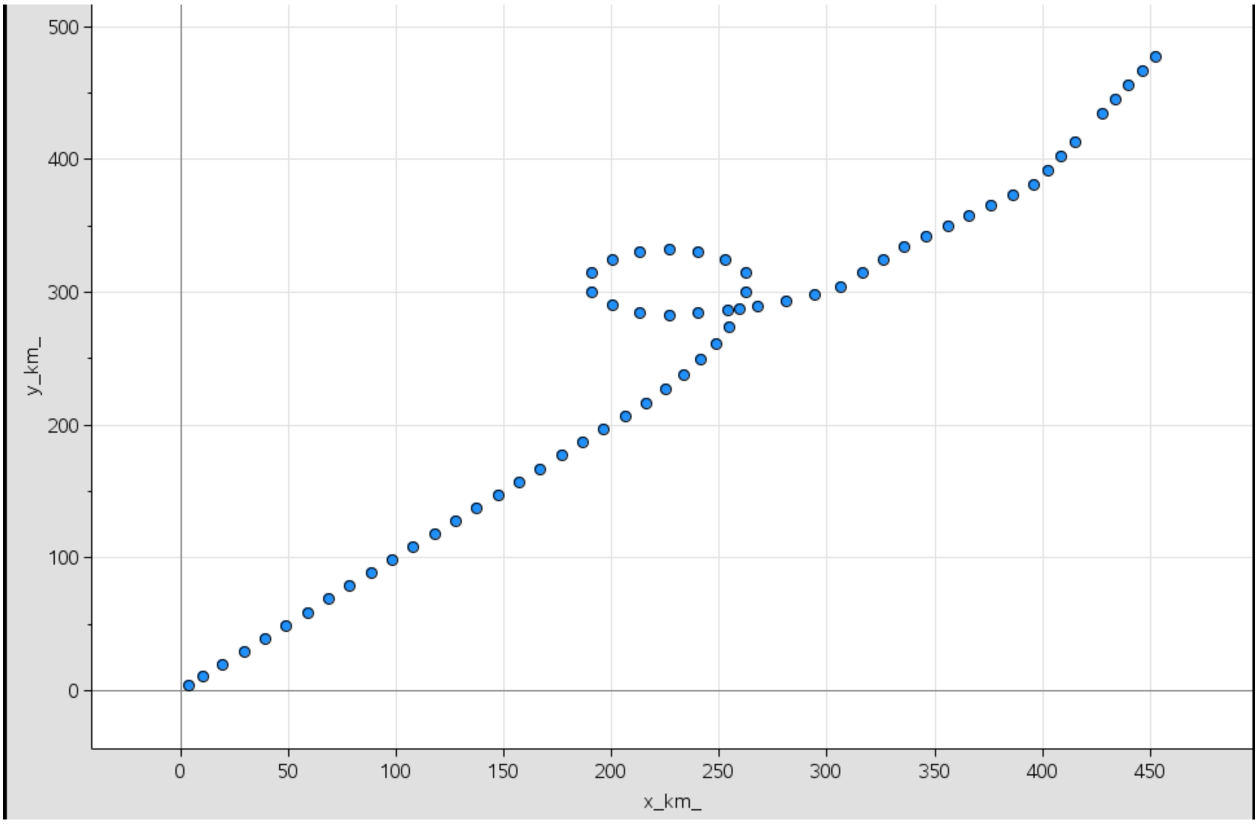
B.3 Logbook 1, group 2

[redacted]

Gruppe 2: Logbog

Tirsdag d. 3 april 2018:

Ud fra vores graf som vi lavede ud fra datasættet, kan vi konkludere at flyet ikke er fløjet i en lineær bane gennem hele turen, men har lavet et loop undervejs. Vores tanker er at flyet enden er blevet ramt af et vindstød, eller har prøvet at undgå at styrte ind i noget.



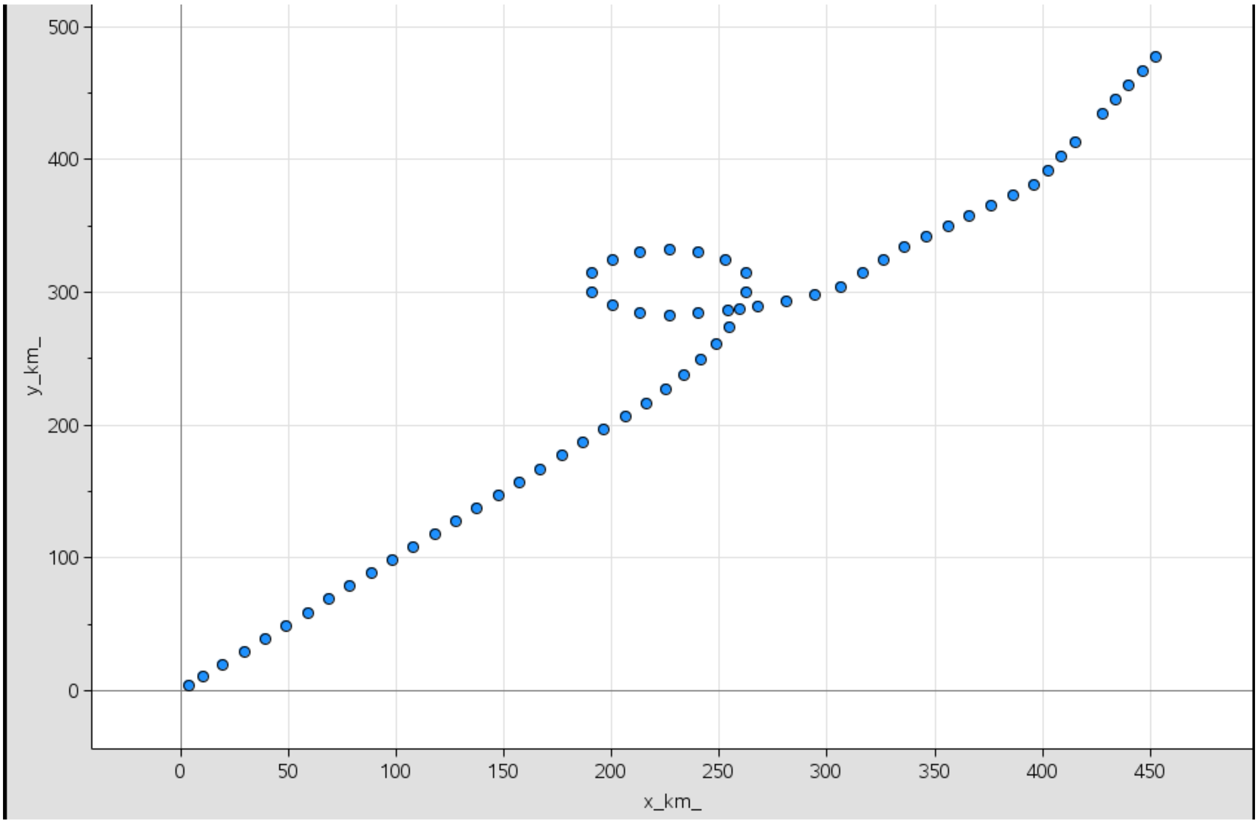
Til næste gang vil vi gerne undersøge, hvorfor flyet har lavet dette loop.

B.4 Logbook 2, group 2

[redacted]

Gruppe 2: Logbog

<u>Tirsdag d. 3 april 2018:</u>
Ud fra vores graf som vi lavede ud fra datasættet, kan vi konkludere at flyet ikke er fløjet i en lineær bane gennem hele turen (kun de første 23 minutter), men har lavet et loop undervejs. Vores tanker er at flyet enden er blevet ramt af et vindstød, eller har prøvet at undgå at styrte ind i noget.



Til næste gang vil vi gerne undersøge, hvorfor flyet har lavet dette loop.

<u>Torsdag d. 5 april 2018:</u>
Vi valgte ikke at besvare vores spørgsmål fra tirsdags, da vi ikke ville kunne bruge vores oplysninger om turen. I stedet valgte vi at finde et nyt spørgsmål:

"Hvor lang og hvor lang tid har flyet fløjet før det lavede loopet?"

Vi fandt ud af at flyet er fløjet i 25,5% af tiden, før de lavede loopet. Dvs. det har fløjet 255 km.

[redacted]

På de første 23 minutter kan man se at flyet har fløjet fint, hvilket tyder på at vejret har været fint.

B.5 Logbook 1, group 3

LOGBOG _[redacted]

Flyet ligger nord for Ønskeøen eftersom y-koordinatet stiger.

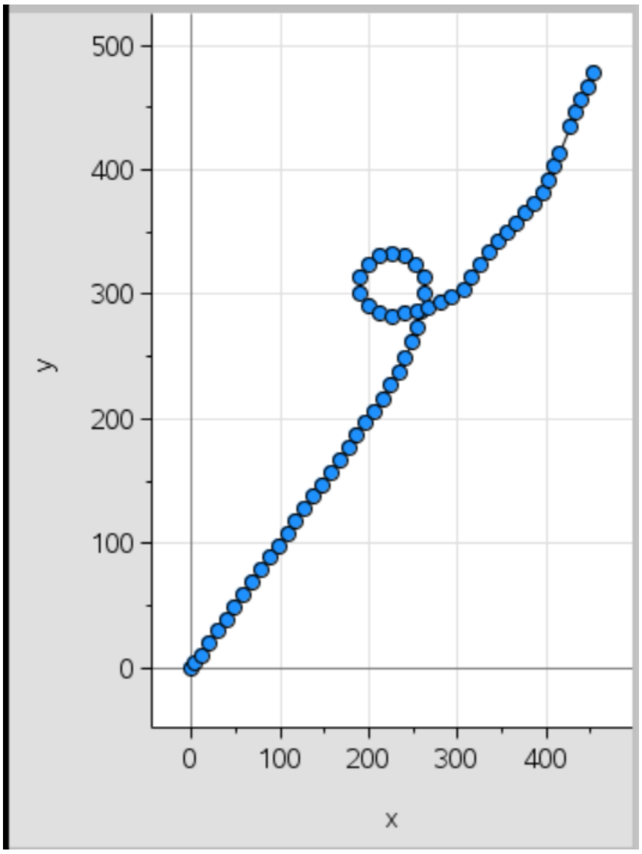
Flyet kan ligge 3800km længere ud fra Ønskeøen, da motoren i alt kun holder til at 4800km ifølge en AirBus320a's spefikationer. - Svævetiden vil tilføje endnu mere distance.

Flyets hastighed har vi regnet ud til at være 666,667km/t.

Derudover fandt vi ud af at flyet manglede 27,7% resterende af turen også svarende til 277km. Altså vil der gå noget tid, før at de vil opdage, at de er langt fra ruten og land, og der vil blive gjort noget ved retningen

Hvad vil vi finde ud af til næste gang:

Undersøge nærmere og se om vi kan komme tættere på, hvilket område flyet kunne være.



B.6 Logbook 1, group 4

Logbog d. 3/4-2018 Matematik forløb Gruppe 4

Vi har fundet ud af at den styrtede ned på den sidste tredjedel af flyveturen.

Vi har fundet ud af at den flyver lige efter ruten de første 23 minutter og derefter begynder den at slingre.

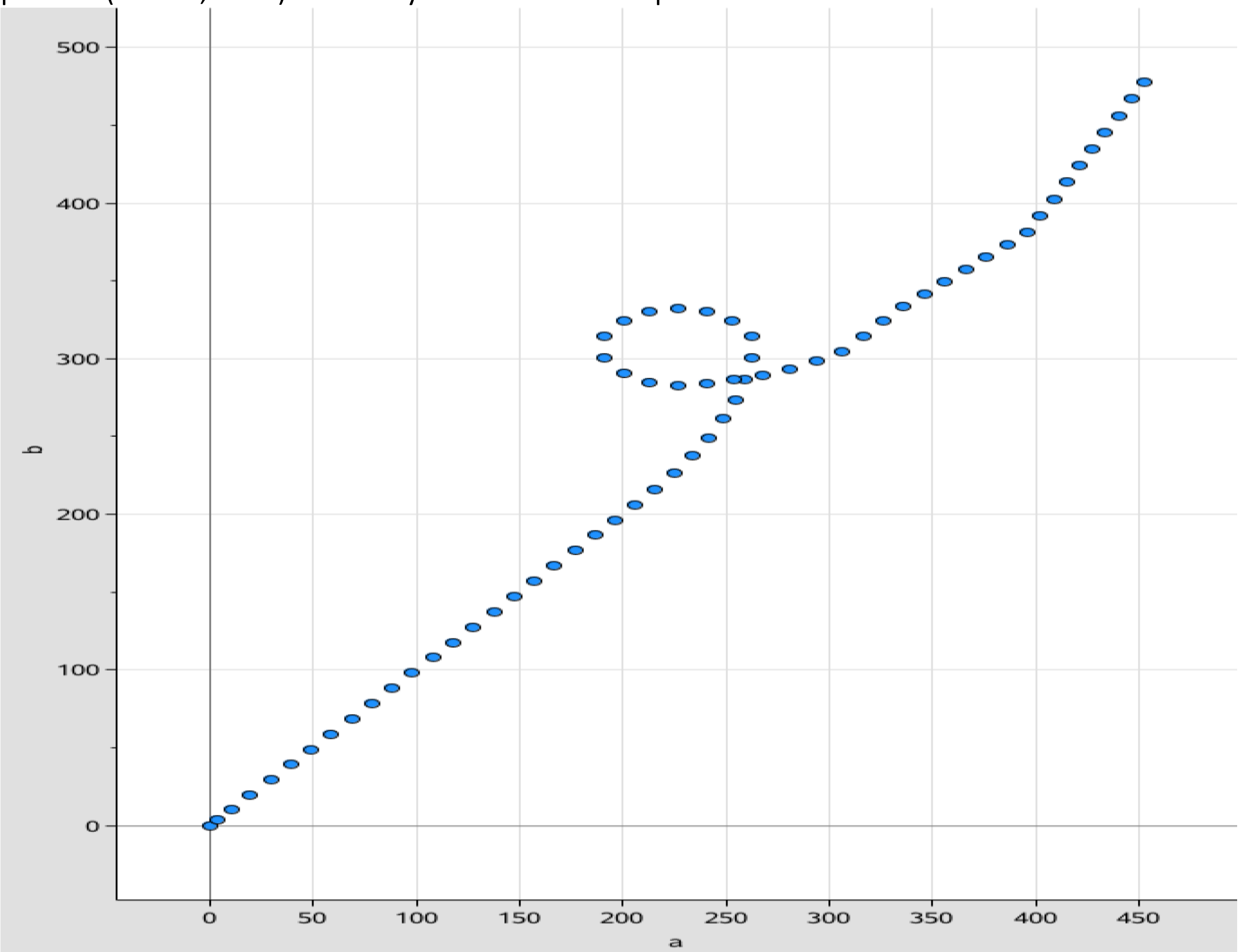
Vi har fundet ud af at den styrtede ned på de sidste 350 km af flyveturen.

90-65= 25 min
Det er ca. 1/3

Så målte vi 1/3 af 20 cm= 6,666 cm

Så fandt vi ud af at 6,5 cm på kortet = 350 km. Flyet vil altså have styrtet ned efter 650 km

Ved at tegne grafen på TI-Nspire har vi fundet ud af, flyet i en cirkel (dvs. den har fløjet frem, tilbage og frem igen). Starten på cirklen sker i punktet (259.33;286.77). Slutningen sker i punktet (267.77;289.37). Så kan vi også se, at flyet styrter ned i punktet (452.78;477.5). Vi kan se, at efter punktet (396.26;381.2) hælder flyet mere til venstre på ruten.



De spørgsmål vi gerne vil arbejde videre med er:

- Hvorfor fløj flyet tilbage (lavede en cirkel)?
- Hvorfor styrtede flyet ned efter 65 min?
- Hvor styrter flyet ned?

B.7 Logbook 2, group 4

Logbog d. 3/4-2018 Matematik forløb Gruppe 4

Vi har fundet ud af at den styrtede ned på den sidste tredjedel af flyveturen.

Vi har fundet ud af at den flyver lige efter ruten de første 23 minutter og derefter begynder den at slingre.

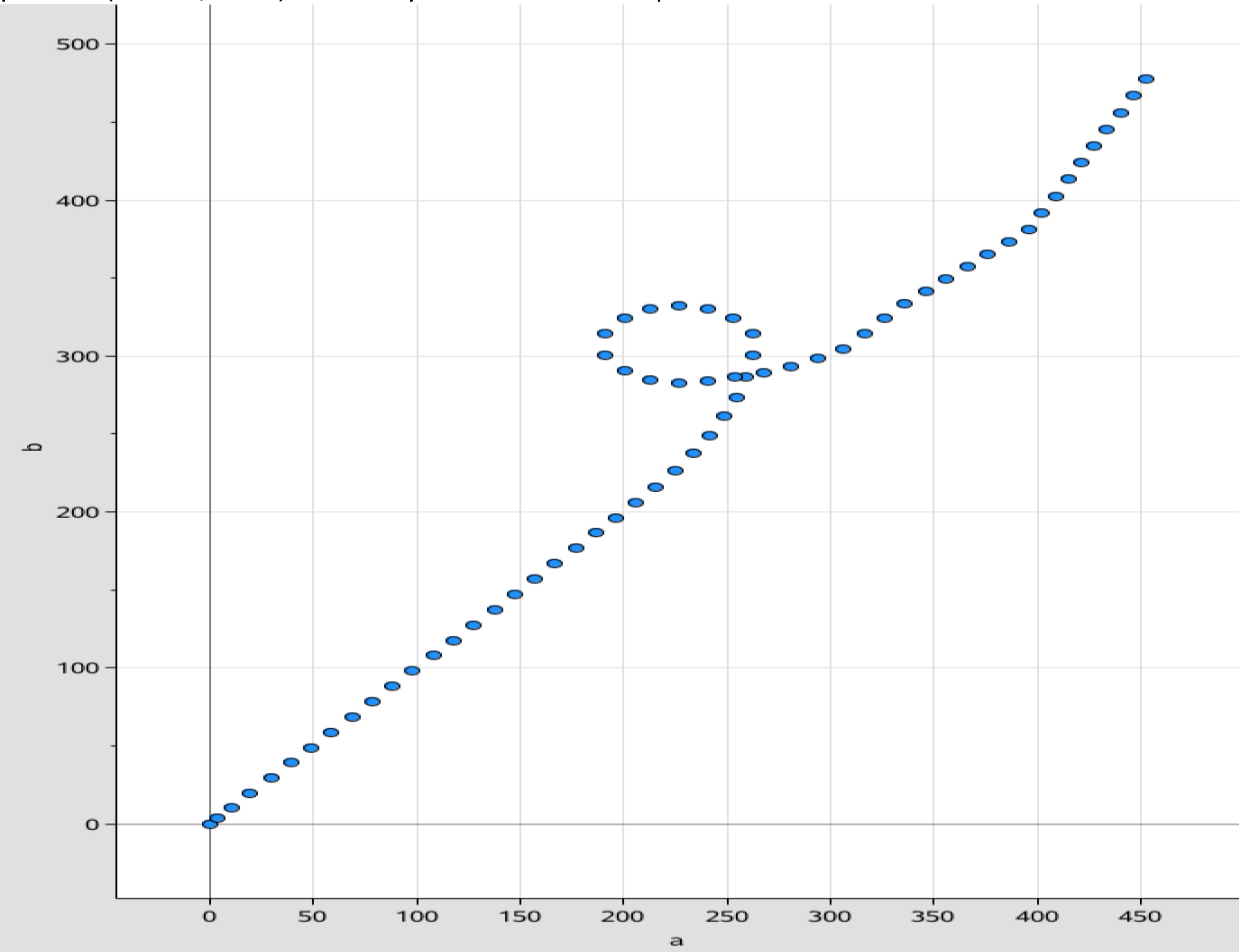
Vi har fundet ud af at den styrtede ned på de sidste 350 km af flyveturen.

90-65= 25 min
Det er ca. 1/3

Så målte vi 1/3 af 20 cm= 6,666 cm

Så fandt vi ud af at 6,5 cm på kortet = 350 km. Flyet vil altså have styrtet ned efter 650 km

Ved at tegne grafen på TI-Nspire har vi fundet ud af, flyet i en cirkel (dvs. den har fløjet frem, tilbage og frem igen). Starten på cirklen sker i punktet (259.33;286.77). Slutningen sker i punktet (267.77;289.37). Så kan vi også se, at flyet styrter ned i punktet (452.78;477.5). Vi kan se, at efter punktet (396.26;381.2) hælder flyet mere til venstre på ruten.



De spørgsmål vi gerne vil arbejde videre med er:

- Hvorfor fløj flyet tilbage (lavede en cirkel)?
- Hvorfor styrtede flyet ned efter 65 min?
- Hvor styrter flyet ned?

5/4-18
Vi har fundet ud af, at et almindeligt passagerfly som Dumbo_04 har en svævetid på ca. 150 km (hvor der ikke er medregnet vejr og andet, der kan nedsætte svævetiden), når motoren er gået i stå. Det har vi fundet ud af ved følgende hjemmeside: https://spoerg-piloten.dk/hvis-en-boeing-777-eller-737-altsa-et-fly-i-passagerfly-storrelse-oplevede-at-alle-motorer-stopper-med-at-virke-hvordan-vil-flyet-sa-falde-og-hvor-lang-tid-ville-det-kunne-holde-sig-i-luften/

Til det første spørgsmål om hvorfor flyet fløj tilbage, tænker vi, at det kan være fordi at piloterne i cockpittet har indset, at det af en eller anden ukendt årsag ikke kan nå hele vejen ud til Ønskeøen. Derfor er de vendt om, men da det indser der også er for langt tilbage til at de kan nå det, drejer de tilbage på kurs igen.

Der kan igen være mange årsager til, at flyet styrtede ned, men ca. 50% af de flystyrt, der sker i moderne tid er forårsaget af tekniske problemer, hvilket vi også tænker kan have været tilfældet her. Vi tænker, at de har slukket motoren for eventuelt at flyet ikke skulle gå op i brænd og derfor har radarne mistet forbindelsen til flyet.

Vi kunne godt tænke os at beregne flyets fart ud fra den tid, det brude have taget og så perspektivere til den tid, det faktisk har taget for at finde et mere præcist område, flyet er styrtet ned i og evt. en årsag:

Distance = 1000 km = 1000 * 1000 = 1 000 000 m
Tid = 90 min = 90 * 60 = 5400 sek.

Fart = $\frac{distance}{tid}$ → Fart = $\frac{1\,000\,000\ m}{5400\ sek}$ → Fart = 185,185 $\frac{m}{sek}$

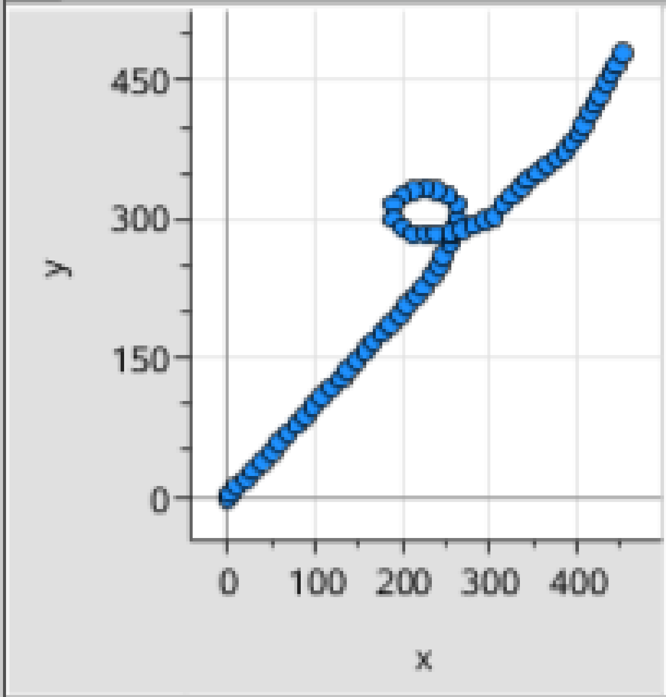
Vi vil i næste time finde ud af, hvor hurtigt Dumbo_04 har fløjet og derefter beregne mere præcist, hvor det er styrtet ned henne og evt. af hvilke årsager.

B.8 Logbook 1, group 5

<Opgavenavn>

	A	B x	C y	D	E	F
=						
1		0.	0.			
2		3.5	3.5			
3		10.5	10.5			
4		19.5	19.5			
5		29.33	29.33			
6		39.16	39.16			

A1



Log bog

vi tænker at der har været nået turbulens i luften så flyet har været kommet ud af kurs, men er komet på rette vej igen, derfor ligger koordinaterne mere stabilt.

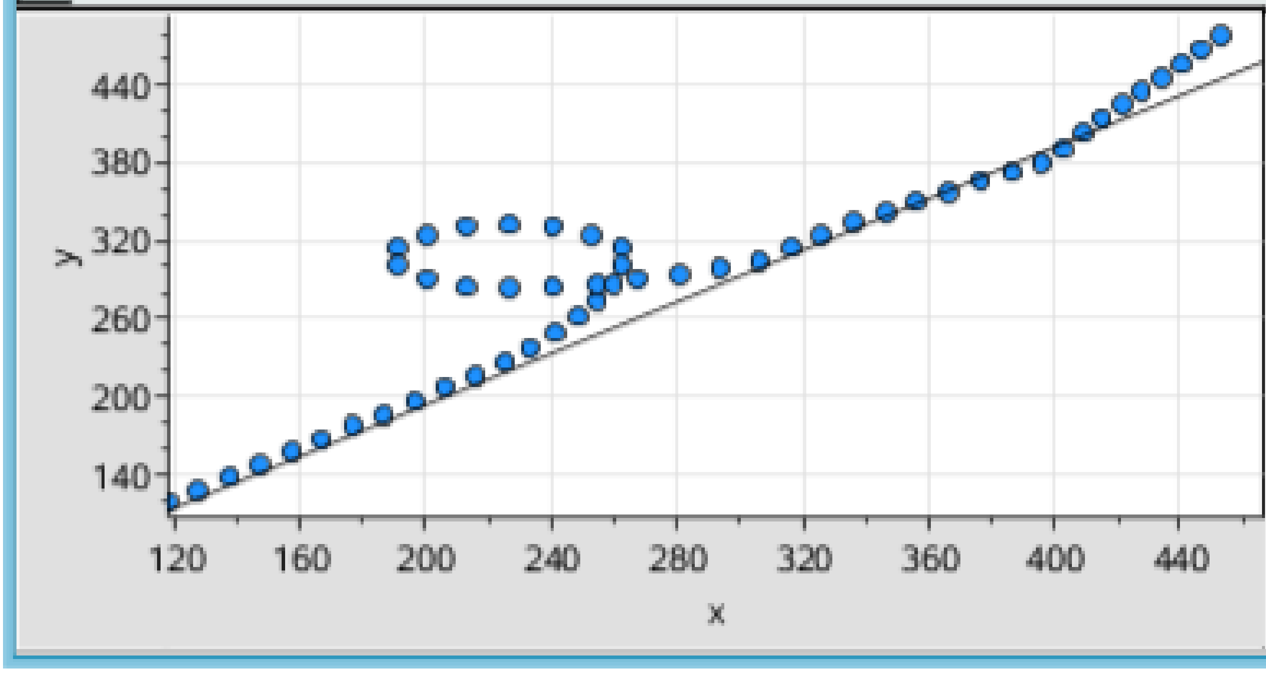
vi vil undersøge hvor længe der vil gå før flyet når sin destination.

B.9 Logbook 2, group 5

<Opgavenavn>

	A	B x	C y	D	E	F
=						
1		0.	0.			
2		3.5	3.5			
3		10.5	10.5			
4		19.5	19.5			
5		29.33	29.33			
6		39.16	39.16			
7		48.99	48.99			

A1



Log bog

vi tænker at der har været nået turbulens i luften så flyet har været kommet ud af kurs, men er komet på rette vej igen, derfor ligger koordinaterne mere stabilt.

vi vil undersøge hvor længe der vil gå før flyet når sin destination.

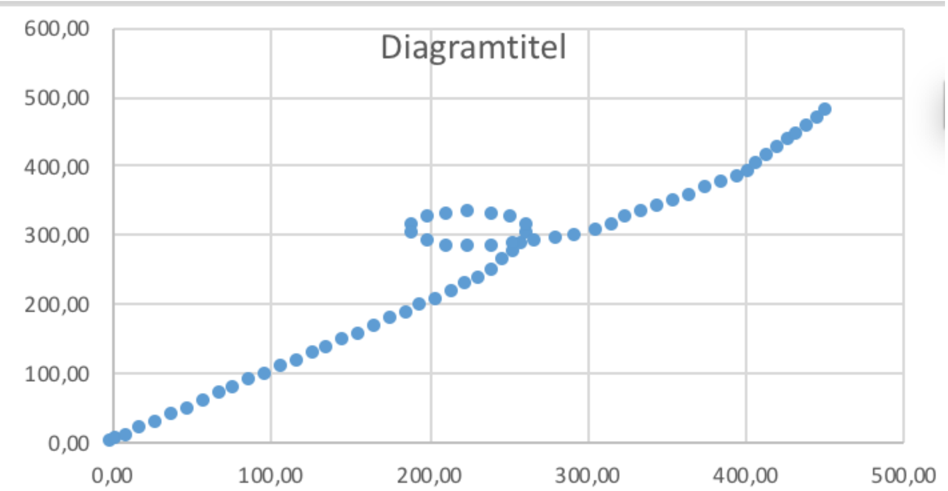
vi har regnet ud at flyet har brugt 14 min i cirklen og derfor har fløjet hele turen på 104 minutter

på hele turen.

vi er ved at regne ud hvordan hvor lang tid flyet har brugt på turen hvis det ikke er fløjet linært.

B.10 Logbook 1, group 6

Den egetnlige rute skulle være en lige linje fra langbortistan til ønske øen.



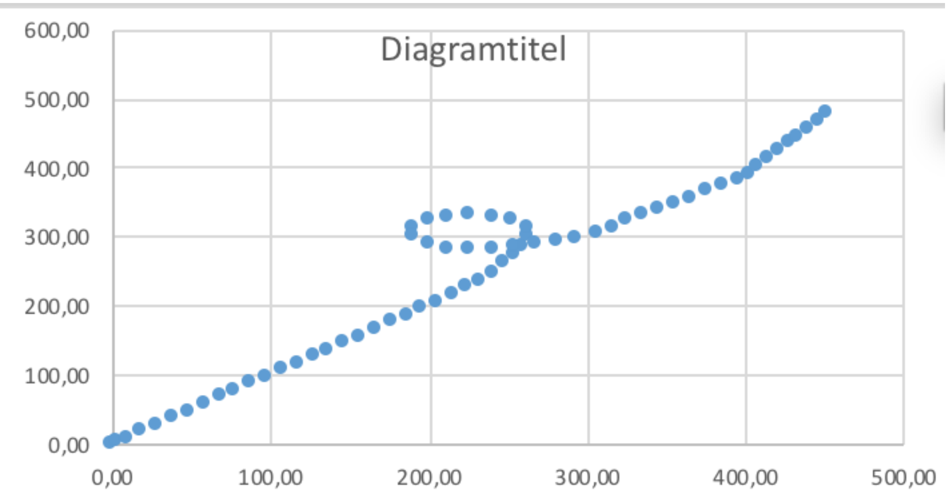
Men ved hjælp af et punktdiagram kan vi aflæse at flyet undervejs er fløjet i en cirkel.
Flyet er måske styrtet ned lige fremme for hvor vi mistede kontakten

Et fly kan cirka svæve 150 km
https://spoerg-piloten.dk/hvis-en-boeing-777-eller-737-altsa-et-fly-i-passagerfly-storrelse-ople-vede-at-alle-motorer-stopper-med-at-virke-hvordan-vil-flyet-sa-falde-og-hvor-lang-tid-ville-det-kunne-holde-sig-i-luften/

Hvor er flyet landet?
Hvad er der sket?

B.11 Logbook 2, group 6

Tirsdag 3/4 2018
Den egetnlige rute skulle være en lige linje fra langbortistan til ønske øen.



Men ved hjælp af et punktdiagram kan vi aflæse at flyet undervejs er fløjet i en cirkel.
Flyet er måske styrtet ned lige fremme for hvor vi mistede kontakten

Et fly kan cirka svæve 150 km
https://spoerg-piloten.dk/hvis-en-boeing-777-eller-737-altsa-et-fly-i-passagerfly-storrelse-oplevede-at-alle-motorer-stopper-med-at-virke-hvordan-vil-flyet-sa-falde-og-hvor-lang-tid-ville-det-kunne-holde-sig-i-luften/

Hvor er flyet landet?
Hvad er der sket?

Torsdag 5/4 2018
Vi læste om flyet a320.
https://ing.dk/artikel/livsfarlig-fejl-slukker-instrumenter-pa-airbus-320-fly-90024
Her kunne vi læse at det fly før har en defekt der i 2008 37 gange har slukket de elektroniske skærme og instrumenter i cockpittet.
Det betyder piloterne ikke ved hvor de skal flyve hen eller hvor hurtigt de flyver. Det kan være grunden til flyets bemærkelsesværdige rute.
I artiklen står der også at de i tidligere tilfælde har forsøgt at vende om at flyve hjem. Det kan være vores piloter også prøvede det, men de var så tilpas langt væk fra land at de ikke kunne orientere sig tilbage til Langtbortistan.

http://www.pilotfriend.com/training/flight_training/aero/gliding.htm

https://aviation.stackexchange.com/questions/14425/how-can-the-glide-ratio-in-a-balanced-turn-be-estimated?utm_medium=organic&utm_source=google_rich_qa&utm_campaign=google_rich_qa
Vi har kigget på disse to links for at prøve at regne glidetallet ud for et a320 når det drejer og derved indskrænke området flyet kunne lande.

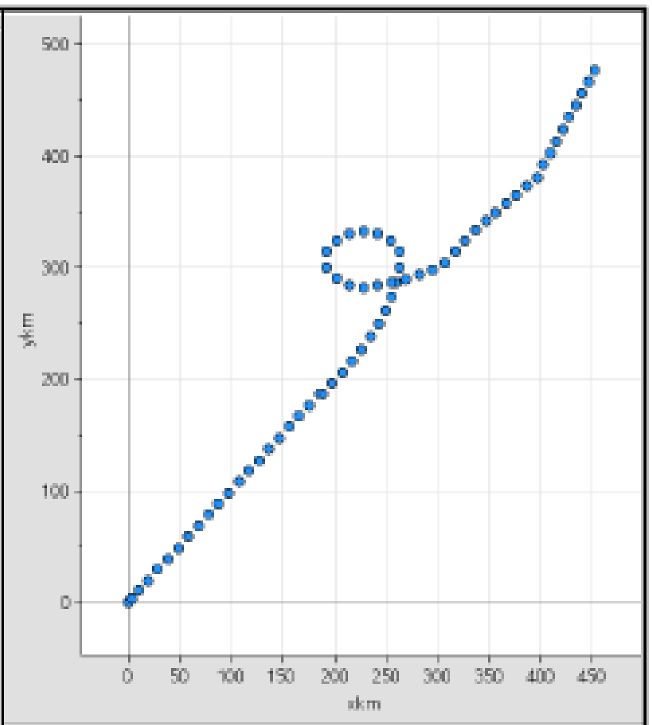
Vi havde ikke nok tid til at undersøge det ordentligt. Det vil vi gøre hvis vi skulle arbejde med det igen.

B.12 Logbook 1, group 7

<Opgavenavn>

	A tid	B xkm	C ykm	D	E	F
=						
1	0	0.	0			
2	1	3.5	3.5			
3	2	10.5	10.5			
4	3	19.5	19.5			
5	4	29.33	29.33			
6	5	39.16	39.16			
7	6	48.99	48.99			
8	7	58.82	58.82			
9	8	68.65	68.65			
10	9	78.48	78.48			
11	10	88.31	88.31			
12	11	98.14	98.14			
13	12	107.97	107.97			
14	13	117.8	117.8			
15	14	127.63	127.63			
16	15	137.46	137.46			
17	16	147.29	147.29			
18	17	157.12	157.12			
19	18	166.95	166.95			
20	19	176.78	176.78			
21	20	186.61	186.61			
22	21	196.44	196.44			
23	22	206.27	206.27			
24	23	215.92	216.27			
25	24	225.02	226.77			
26	25	233.51	237.77			
27	26	241.32	249.27			
28	27	248.33	261.27			
29	28	254.41	273.77			
30	29	259.33	286.77			
31	30	262.64	300.27			
32	31	262.64	314.27			
33	32	252.99	324.24			

B11 88.31



Efter 24 minutter, begyndt det at gå galt

$\frac{1000 km}{1,5 h} = 666,667 \, km/h$

$\frac{13}{12} \cdot 666.667 \cdot 722.223$

$\frac{1000.}{20} \cdot 50.$

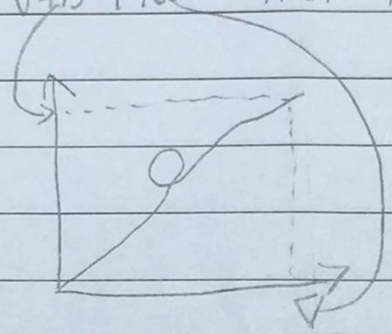
722.223/50 • 14.4445

Det mister kontakt efter 14.5 cm på papiret

B.13 Logbook 2, group 7

Torsdag:

Vi tegnet den stor papir ting, og regnet ud hvor lang vil flyet have nået hvis den ikke lavet en cirkle, og regnet ud hvor lang den nået, med at sig $\sqrt{475^2+450^2}$ = hvor meget den nået



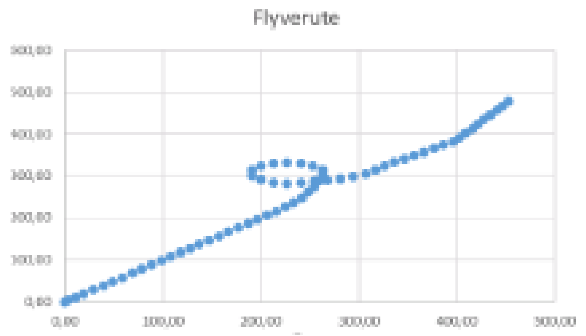
→ formel: $a^2+b^2=c^2$

og så undersøgte vi hvorfor mistet flyet først kontakt 36 min efter den lavet en sirkle.

B.14 Logbook 1, group 8

Opgave 1

Flyverute:



Spørgsmål:

Hvorfor viser grafen at piloten laver et lupe?

Logbog:

Først har vi fundet vores data og derefter har vi brugt dataen til at lave et kordinatsystem, for flyvemaskinens rute.

Vi har spekuleret over, hvor flyvet har styrtet ned og hvordan kantakten er brudt.

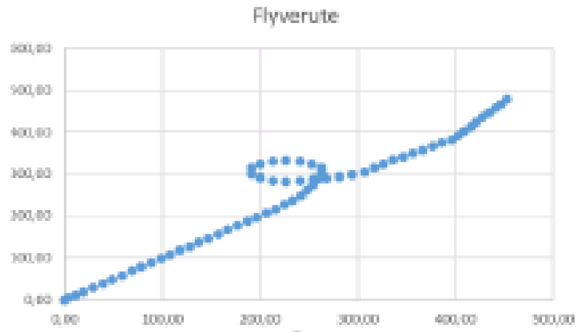
	A tid_min	B x_km	C y_km	D	E	F
=						
1	0	0	0			
2	1	3.5	3.5			
3	2	10.5	10.5			
4	3	19.5	19.5			
5	4	29.33	29.33			
6	5	39.16	39.16			
7	6	48.99	48.99			
8	7	58.82	58.82			
9	8	68.65	68.65			
10	9	78.48	78.48			
11	10	88.31	88.31			
12	11	98.14	98.14			
13	12	107.97	107.97			
14	13	117.8	117.8			
15	14	127.63	127.63			
16	15	137.46	137.46			
17	16	147.29	147.29			
18	17	157.12	157.12			
19	18	166.95	166.95			
20	19	176.78	176.78			
21	20	[illegible]	[illegible]			

A1 0

B.15 Logbook 2, group 8

Opgave 1

Flyverute:



Spørgsmål:

Hvorfor viser grafen at flyvet laver et loop?

Logbog:

Først har vi fundet vores data og derefter har vi brugt dataen til at lave et kordinatsystem, for flyvemaskinens rute.

Vi har spekuleret over, hvor flyvet har styrtet ned og hvordan kantakten er brudt.

	A tid_min	B x_km	C y_km	D	E	F
=						
5	4	29.33	29.33			
6	5	39.16	39.16			
7	6	48.99	48.99			
8	7	58.82	58.82			
9	8	68.65	68.65			
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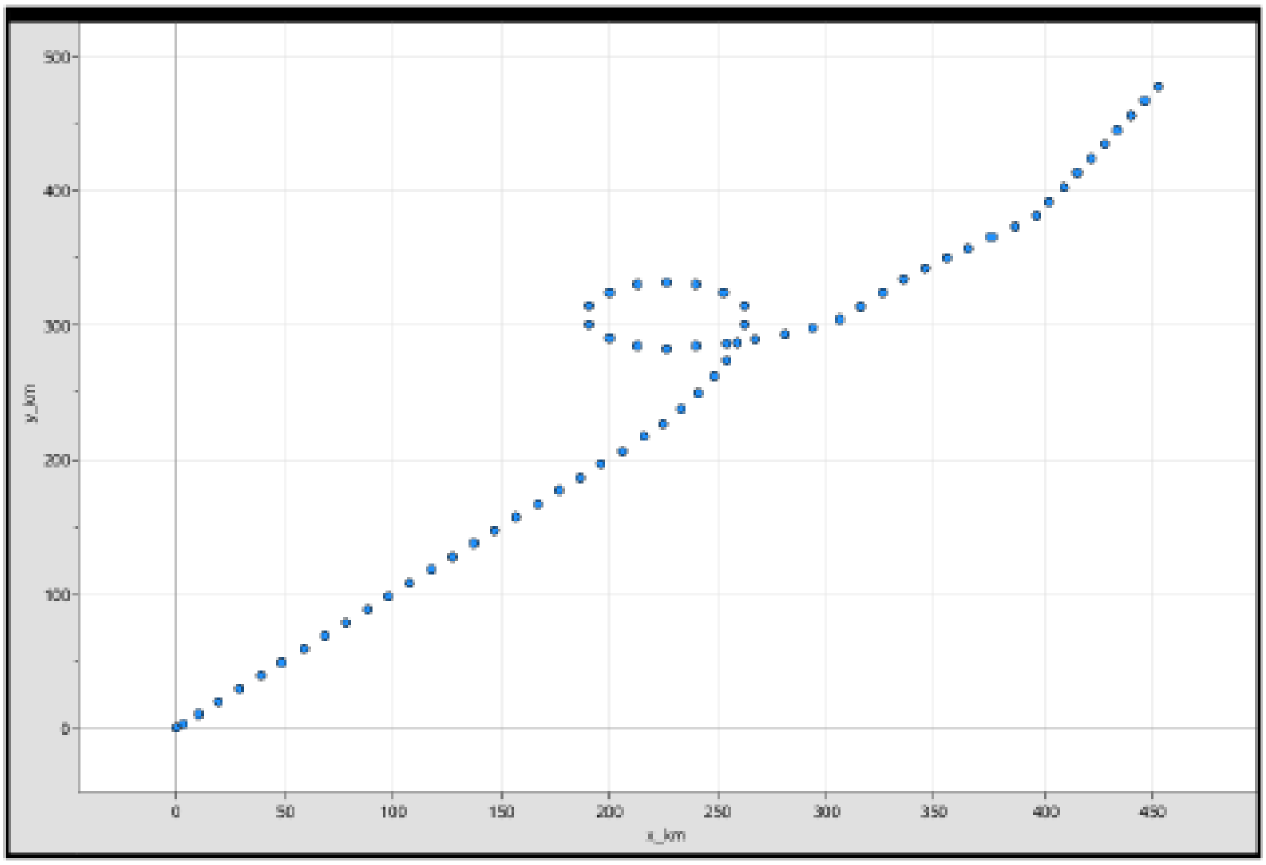
19	18	166.95	166.95			
20	19	176.78	176.78			
21	20	186.61	186.61			
22	21	196.44	196.44			
23	22	206.27	206.27			
24	23	215.92	216.77			
25	24	225.02	226.77			

A25 24



Årsager til flystyrt:

Ud fra grafen kunne loopet kunne skyldes en teknisk fejl i en af de venstre motere og for at der skal skabes en ligevægt, slukkes begge motere også prøver man at tænde dem på samme tid for at få moterene til at virke.

Vi kan også ud fra vores graf se at de sidste punkter ligger tættere på hinanden og derfor kan man gå ud fra at flyvet har tabt fart.

B.16 Logbook 1, group 9

[redacted]

Logbog:
Vi starter med at prøve med de ting lært om.
Fx: lineær regression, søjle diagram inde på Excel.
Vi kunne konkludere, ved at se på tiden, og ved at udregne km/t på flyveturen, at der har været vejrmæssige forhold, som har påvirket styrtet. F.eks. turbulens.

Spørgsmål: Har flyveren, fået en højere hastighed da det begyndte at styrte, end da det var i gang med at flyve imod ønskeøen? Og har det påvirket hvor flyet er styrtet ned?

B.17 Logbook 2, group 9

[redacted]

Logbog:
Vi starter med at prøve med de ting lært om.
Fx: lineær regression, søjle diagram inde på Excel.
Vi kunne konkludere, ved at se på tiden, og ved at udregne km/t på flyveturen, at der har været vejrmæssige forhold, som har påvirket styrtet. F.eks. turbulens.

Spørgsmål: Har flyveren, fået en højere hastighed da det begyndte at styrte, end da det var i gang med at flyve imod ønskeøen? Og har det påvirket hvor flyet er styrtet ned?

- Vi brugte en hastighedsformel for at finde ud af om hastigheden blev højere som vi fandt på hjemmesiden: http://fysikleksikon.nbi.ku.dk/h/hastighed/
- Drmed fandt vi ud af at hastigheden blev højere efter 23 minutter

Appendix C

Reports

C.1 Report 1, group 1

[redacted]

Louises projekt omkring Malaysia flyet.

3. april 2018

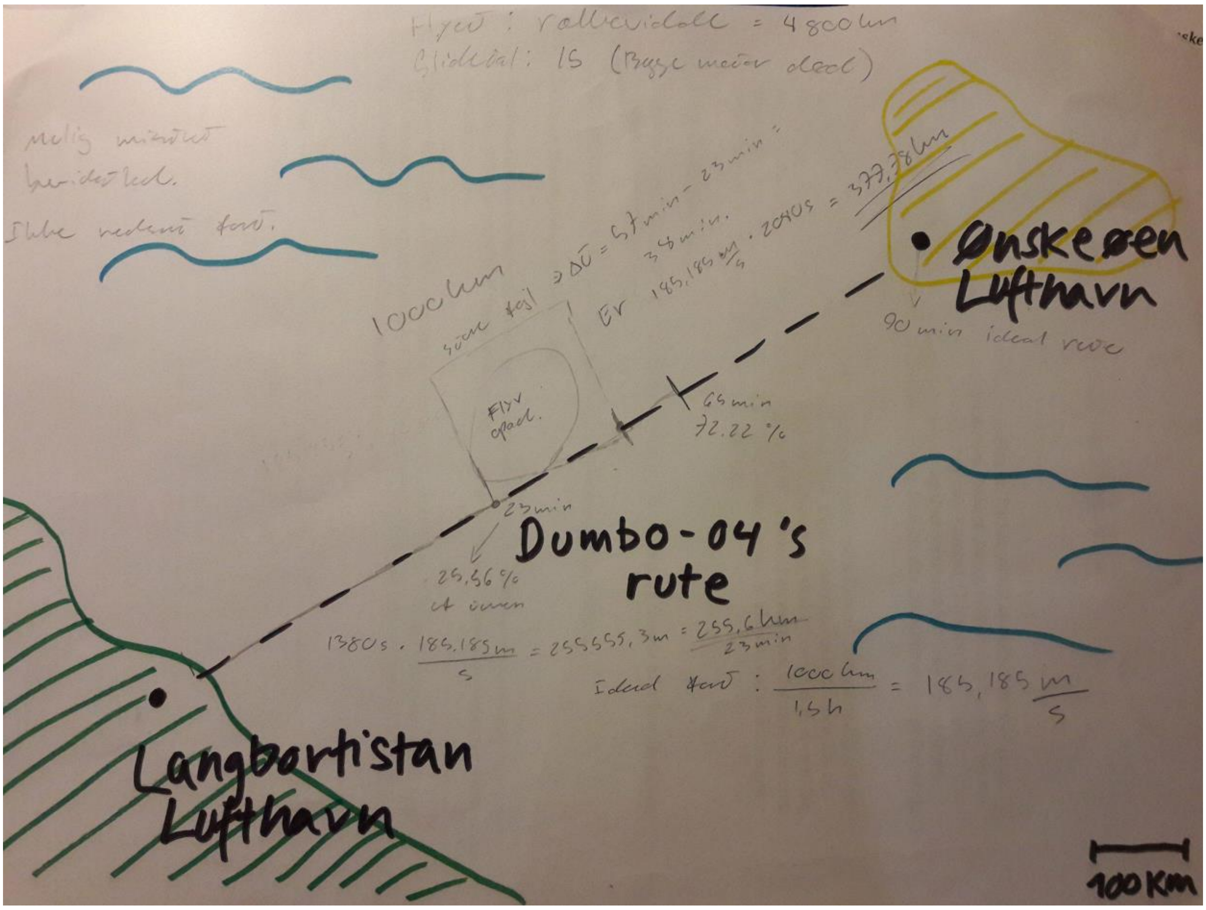
Vi kan se hvor stor en procentdel af turen flyet aflagde på den tid der er gået.

Vi kan finde dens fart $\frac{m}{s}$

For at finde dens efterfølgende svævetid, hvis flyet gik i ”stå”.

Undersøge dens tsunami flyvetrick.

5. april 2018



Det tager 90min (5400s) at flyve 1000km (1.000.000m)
Det er en gennemsnitsfart på $\frac{1000000m}{5400s} = \frac{5000}{27} = 185{,}1852\,\frac{m}{s}$ => $185{,}1852 \cdot 3600 = 666666{,}7\,\frac{m}{h}$

Dvs. at flyet flyver med ca. 667 $\frac{km}{h}$

Indtil det 23. minut, flyver flyet u besværet ifølge dets placering på koordinaterne.
23 minutter er $\frac{23min}{90min} \cdot 100\% = 25{,}55556\%$

Dvs. at de har nået omkring 25.6 % af turens forløb, før der begynder at være vanskeligheder.

[redacted]

Bevidsløshed

Hvis man går ud fra at piloterne måske er blevet bevidstløs her, ligesom i en artikel vi læste om, så er det jo ikke er i stand til at styre flyet. Men hvis det stadigvæk har en form for auto-/fartpilot sat til, sænkes farten ikke.

Så fra det 23. minut til det 57. minut, hvor der sker forstyrrelser på ruten, og de bl.a. laver et loop, vil det altså sige, at med en fart på $185{,}1852\frac{m}{s}$ over en tid på $57 - 23 = 34$ minutter,

$$185{,}1852\frac{m}{s} \cdot (34 \cdot 60)s = 377777{,}8m => 378\ km$$

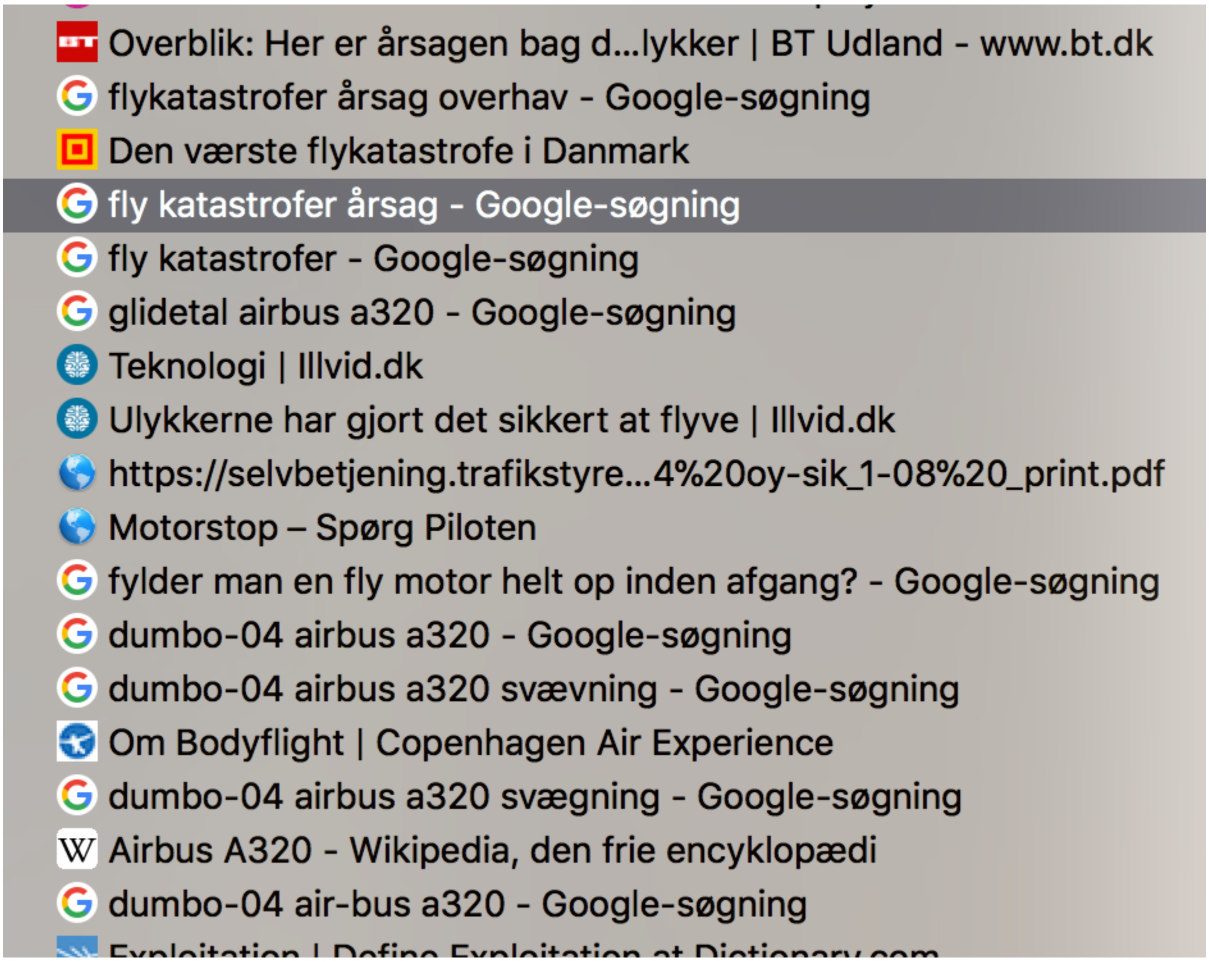
har de fløjet en længde på 378 km.

Den ekstra flyvetid kan have medfølger i hvor langt flyet kunne have være nået i enhver retning. Fly bliver ikke tanket helt op, kun omkring det nødvendige til selve ruten, og måske lidt ekstra for en sikkerhedsskyld. Men ellers ville en for fyldt tank være skyld i en større brug af brændstof, fordi flyet er tungere.

Storm

I et andet eksempel, fandt vi at et fly havde lavet en sådanne cirkel manøvre, for at øge flyvehøjden, fordi de mødte en storm. Det samme kan være grunden til at de laver den her, og det efterfølgende ujævne flyvemønster, kan være grundet at de stadigvæk påvirkes af selve stormen.

Søgehistorik



Hvis man går ud fra at farten er konstant, har vi fundet ud af ruten fløjet og den resterende længde flyet kan flyve.

Længde 1: 2min ->

Længde fløjet = Gennemsnitsfart · Tid gået

185.1852 $\frac{m}{s}$ · 3660 s = 677.78 km

Flyvelængde tilbage = Maks flyvelængde - Længde fløjet.

1300 km · 677,78 km = 622,22 km

Loop = Gennemsnitsfart · Looptid

185.1852 $\frac{m}{s}$ · 900 s = 166.67 km

Rute fløjet = Længde fløjet - Loop

677.78 km - 166.67 km = 511.11 km

C.2 Report 2, group 1

[redacted]

Louises projekt omkring Malaysia flyet.

3. april 2018

Vi kan se hvor stor en procentdel af turen flyet aflagde på den tid der er gået.

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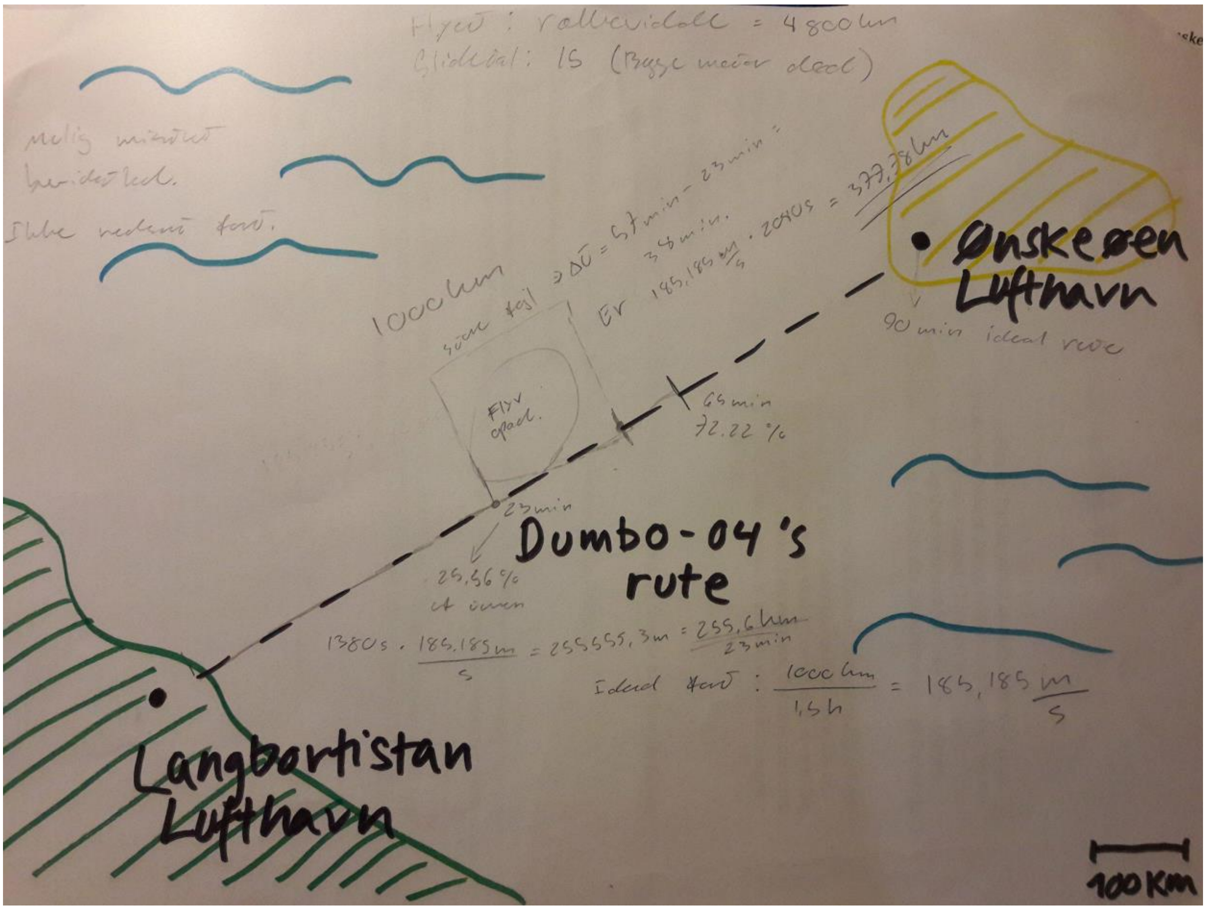

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Bevidsløshed

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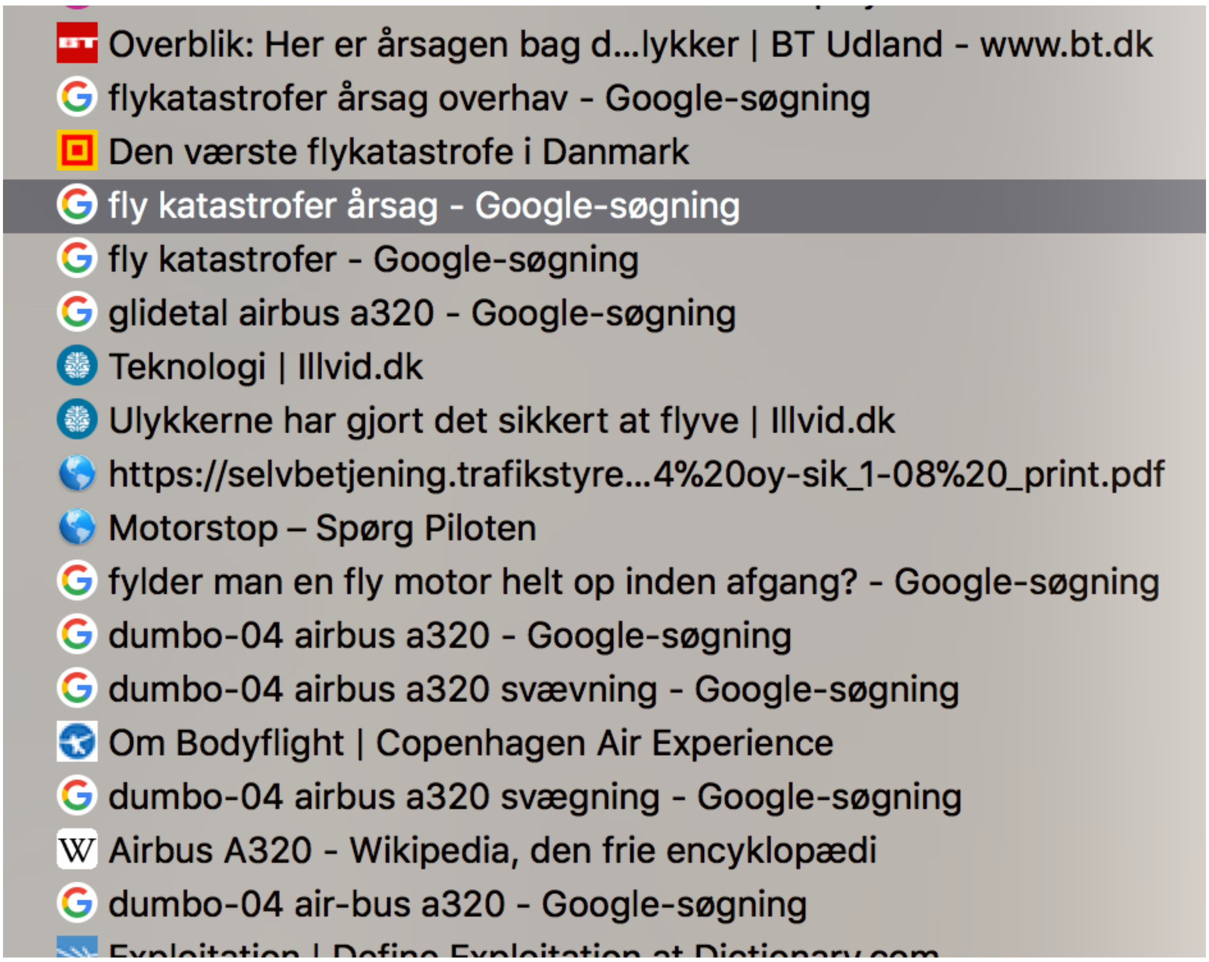

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Rute fløjet = Længde fløjet - Loop

677.78 km - 166.67 km = 511.11 km

Og så kan man lave en radius på resterende længde der er mulig at flyve, og fra hvor flyet forsvinder.

Og vha. vinden. Kan man komme endnu tættere på hvilken retning man brude lede.

[redacted]

Den sorte boks: http://illvid.dk/transport/fly/den-sorte-boks-hvordan-virker-den

Siden den sorte boks stopper med at sende signal,

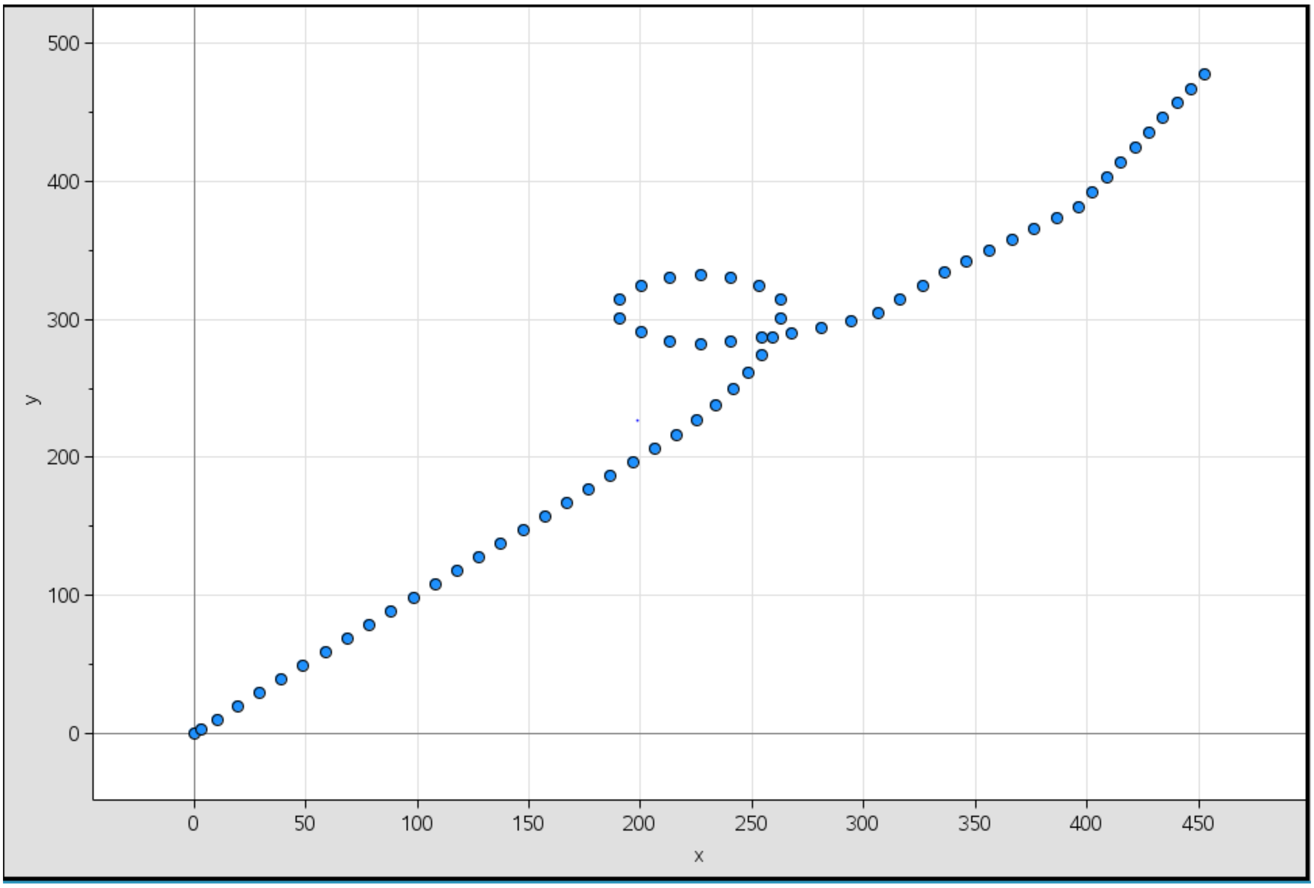
C.3 Report 1, group 2

[redacted]

Rapport til Malaysia Airline:

<u>Tirsdag d. 3 april 2018:</u>

Ud fr datasættet som vi fik, har vi lavet en graf som viser Dumbo-04's fly tur. Ud fra grafen kan vi konkludere, at flyet ikke er fløjet i en lineær bane gennem hele turen (kun de første 23 minutter), men har lavet et loop undervejs.



<u>Torsdag d. 5 april 2018:</u>

Vi fandt ud af at flyet er fløjet i 25,5% af tiden, før det lavede loopet. Flyet har fløjet ca. 350 km.

[redacted]

Torsdag d. 12 april 2018:

Vi har fundet ud af at flyet i alt har tilbagelagt ca. 800 km, hvilket vil sige at den stadig havde brændstof nok til at tilbagelægge 500 km. (Hvilket vil have været nok til at få den fløjet hen til målet).

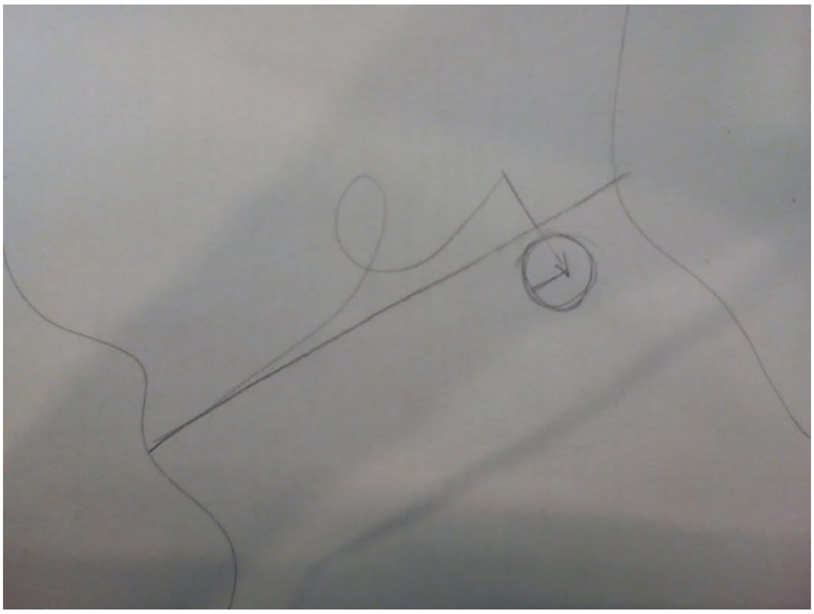
$452^2 + 477^2 = \sqrt{431833}$ (Udregning for samlede længde uden loop)

$250^2 + 250^2 = \sqrt{125000}$ (Udregning for samlede længde før loop)

Da den flyver lineær og perfekt i starten har vi antaget at den har fløjet i godt vejr, og efter er den havnet i en storm.

Vi har læst at et fly der flyver i ca. 10 km højde, vil være ca. 16,5 minutter om at falde til jorden. Hvis vi antager at vindmodstanden er konstant (3.33,-32,33) $\frac{m}{s}$ vil flyet havde været havnet ca. 32 km sydøst for hvor det sidst udsendte et signal. Vi ved at hvis flyets motor er gået ud, så kan den flyve ca. 150 km videre.

Loopet kunne skyldes at piloten så en storm, og valgte at cirkulere for at se om stormen aftog, og derefter køre ind i den. Et lyn eller noget andet kunne havde slukket motoren, hvilket ville kunne forklare det stoppende signal.



(Skitse)

C.4 Report 2, group 2

[redacted]

Rapport til Malaysia Airline:

Tirsdag d. 3 april 2018:

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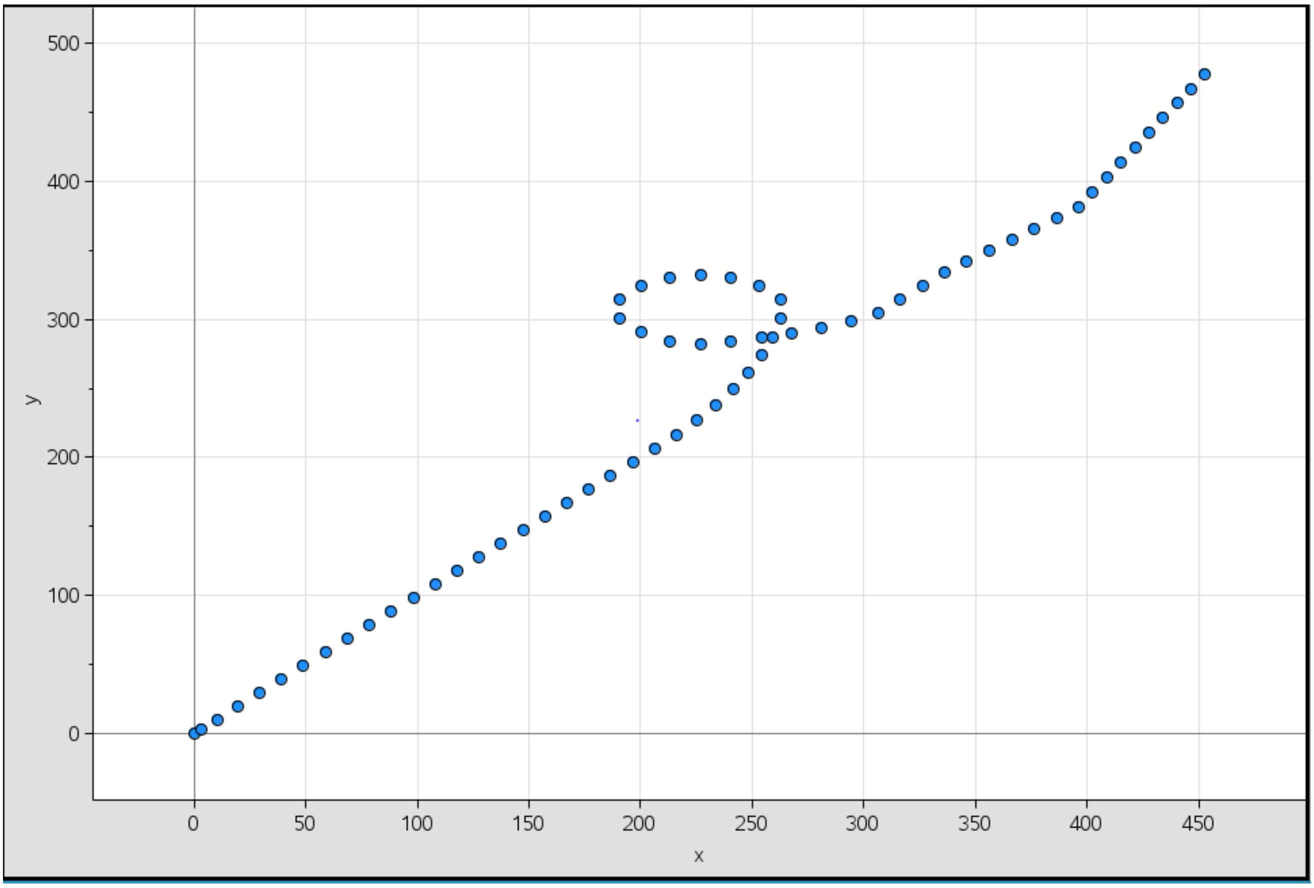

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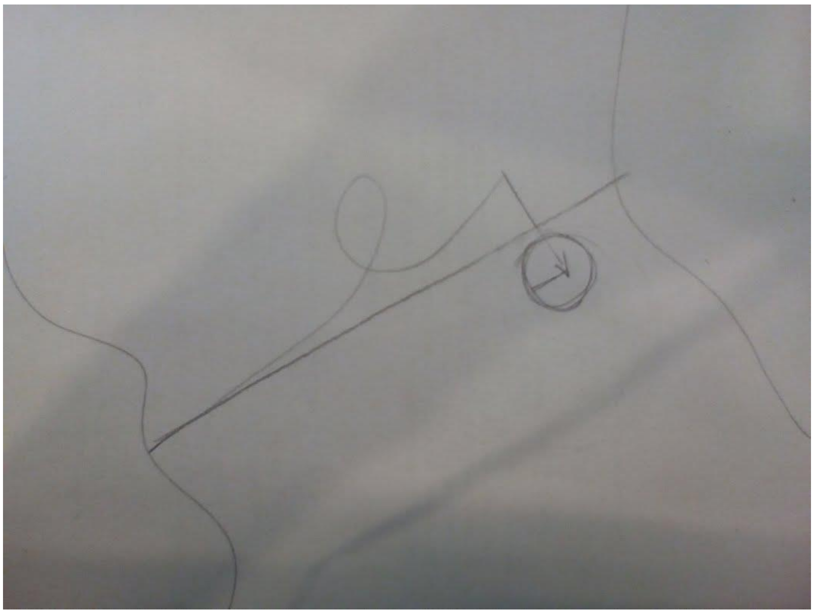


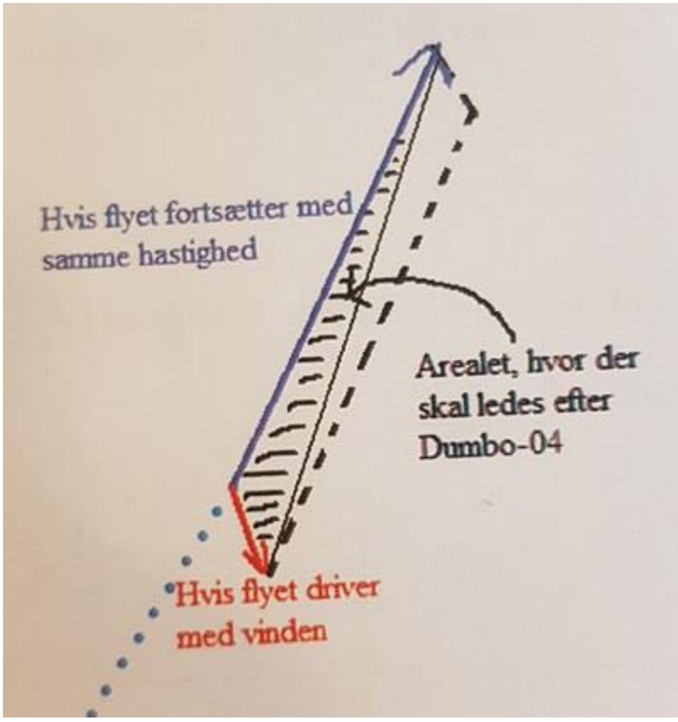
(Skitse)

[redacted]

Fredag d. 13 april:

Hvis vi antager at flyet efter sidst udsendte data, blev ved med at flyve i en konstant retning i samme fart til der ikke var mere brændstof tilbage (431 km) og at vinden fik flyet til at svæve 150 km, ville man kunne tegne et parallelogram og finde ud af at arealet af det område der skal undersøges er på ca. 19131 km^2

$=\begin{Vmatrix} 218{,}10 & 15{,}38 \\ 371{,}61 & -149{,}23 \end{Vmatrix} = \begin{pmatrix} 218{,}1 & 15{,}38 \\ 371{,}61 & -149{,}23 \end{pmatrix} = |-38262{,}42| = 38262{,}42 = \frac{38262{,}42}{2} = 19131{,}21\ km^2$

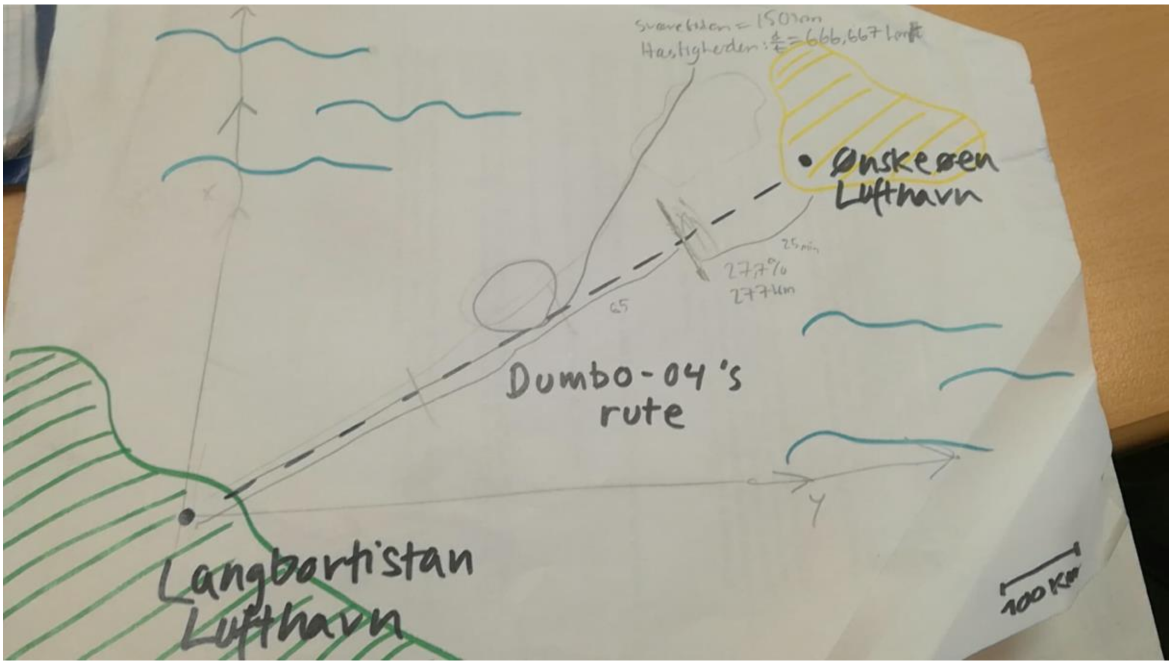


Det burde være hele parallelogrammet der skulle undersøges!

C.5 Report 1, group 3

RAPPORT

Malaysian Airlines fly Airbus A320 skulle have fløjet fra Langbortistan Lufthavn til Ønskeøen, men er på vejen derhen kommet i problemer.

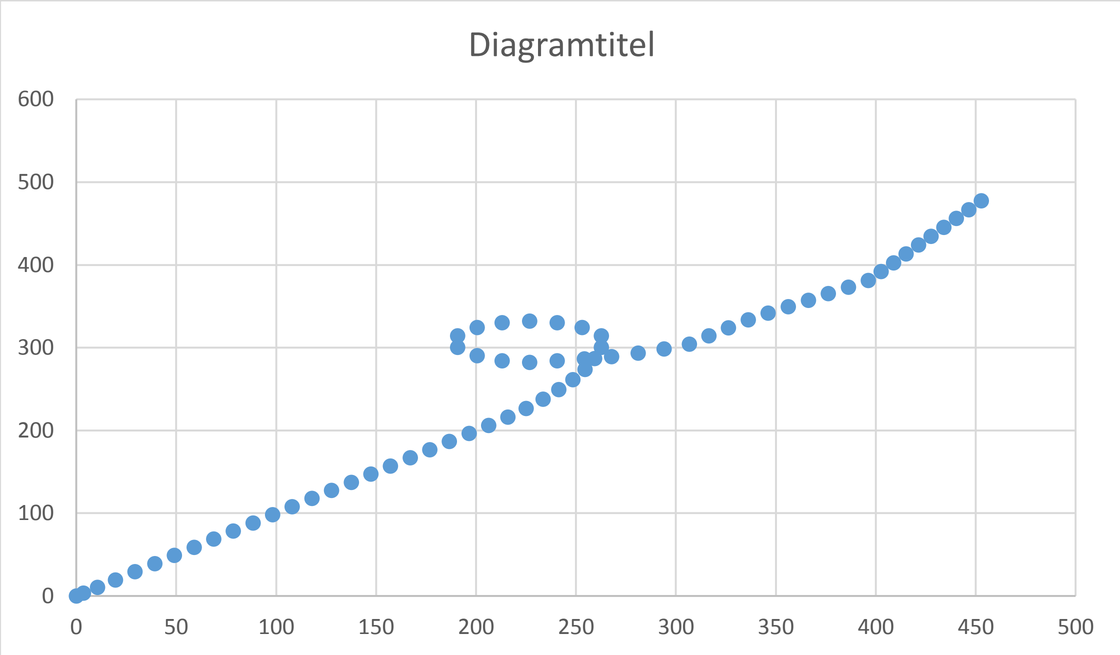


Efter 65 min er der blevet fløjet … km og det der svarer til 2,86 tons brændstof. Ud af de 5 tons brændstof den i alt havde var 2,86 tons blevet brugt efter de 65 min, hvilket vil sige at der var 2,14 tons tilbage.

C.6 Report 2, group 3

RAPPORT [redacted]

Malaysian Airlines fly Airbus A320 skulle have fløjet fra Langbortistan Lufthavn til Ønskeøen, men er på vejen derhen kommet i problemer.



Første del af turen:

Farten indtil minut 22 er på 843,1 km/t, hvilket er normalt for en Airbus a320, da dens spefikationer siger at den maks kan flyve med 903 km/t.

Længden for den første del af turen er regnet til at være: $\sqrt{206{,}26^2 + 206{,}26^2} = 291{,}71km.$

Anden og tredje del af turen:

Længden for den anden del af turen tog vi fra datasættet, men hvis man skulle regne det ville man tage fra punkt til punkt af den givet rute: f.eks. fra minut 24 $\binom{225{,}02}{226{,}77}$ til minut 25 $\binom{233{,}51}{237{,}77}$ ville man bruge vektorformlen mellem to punkter som hedder: AB$\binom{b_1 - a_1}{b_2 - a_2}$ så det ville hedde $\binom{233{,}51 - 225{,}02}{237{,}77 - 226{,}77}$, som giver $\binom{8{,}49}{11}$, hvor vi herefter tager kvadratroden: $\sqrt{8{,}49^2 + 11^2} =$ 13,9km hvilket vil sige at fra minut 24 til minut 25 ville der være 13,9km.

Men hele ruten svarer til 869,24km.

Selve hele flyet kunne flyve 1300km i alt, hvilket vil sige at Airbus a320 ville have 1300km-869,24km = 430,76km længere at have fløjet efter mistet kontakt.

C.7 Report 1, group 4

Malaysia airlines

Flytypen er en airbus 320

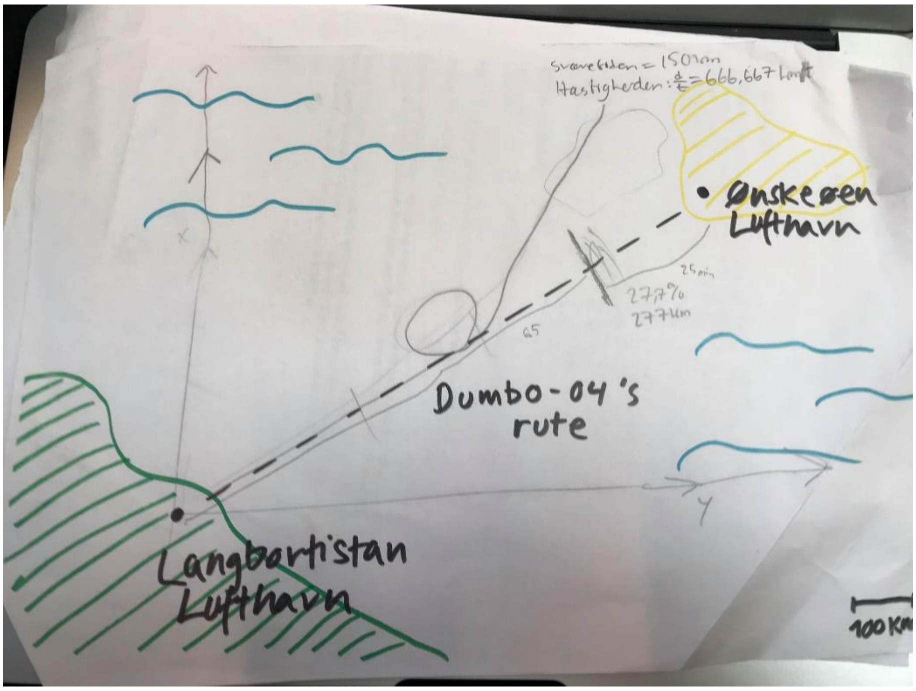
Der blev kun tanket 5 ton brændstof på i en tank der kunne rumme 8 ton = den kan flyve 1300 km. Den kan derfor flyve 300 km længere.

Der er kraftig vind (3,33   -32,33) m/s som har ramt flyet efter 50 min (efter den har lavet sit loop)

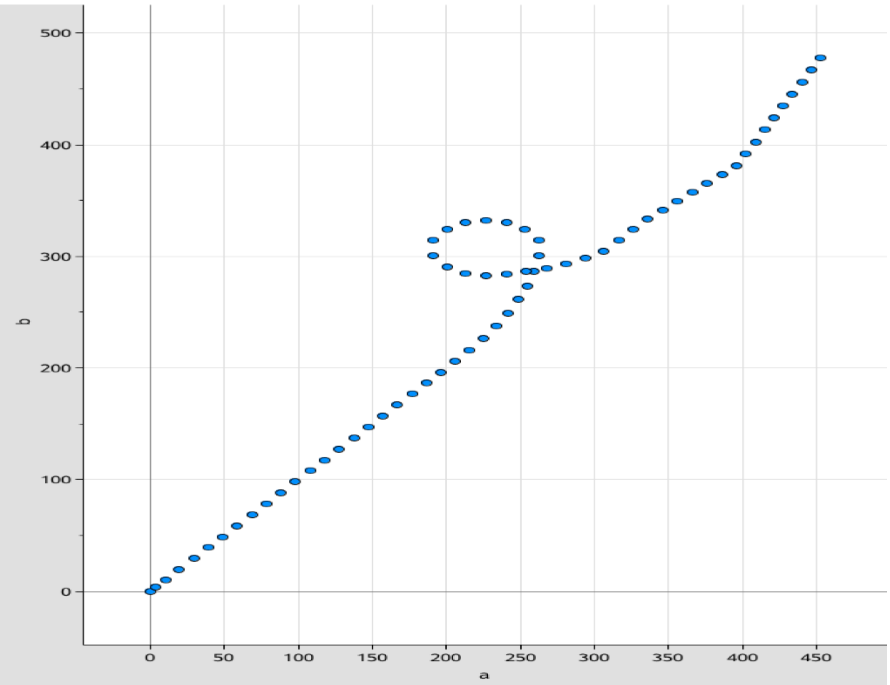
Maks startvægt på en airbus 320 er 77 ton med passagerer. I vores tilfælde blev tanken ikke tanket helt op (der blev tanket 5 ton og ikke 8) derfor vil flyet veje 74 ton med passagerer.

1000000 m/5400 sek = 185,2 m/s (som er farten på flyet)

Her ses ruten for flyet på et kort:



Her ses ruten for flyet i et diagram:



Vi har regnet ud de koordinater hvor flyet ville være endt hvis vinden ramte på præcis 50 min.

326,25 + 3,33 = 329,58

324,03 - 32,33 = 291,7

Men på grafen viser den at det er senere at den kommer ud af kurs….

C.8 Report 2, group 4

Malaysia Airlines

Flytypen er en Airbus 320

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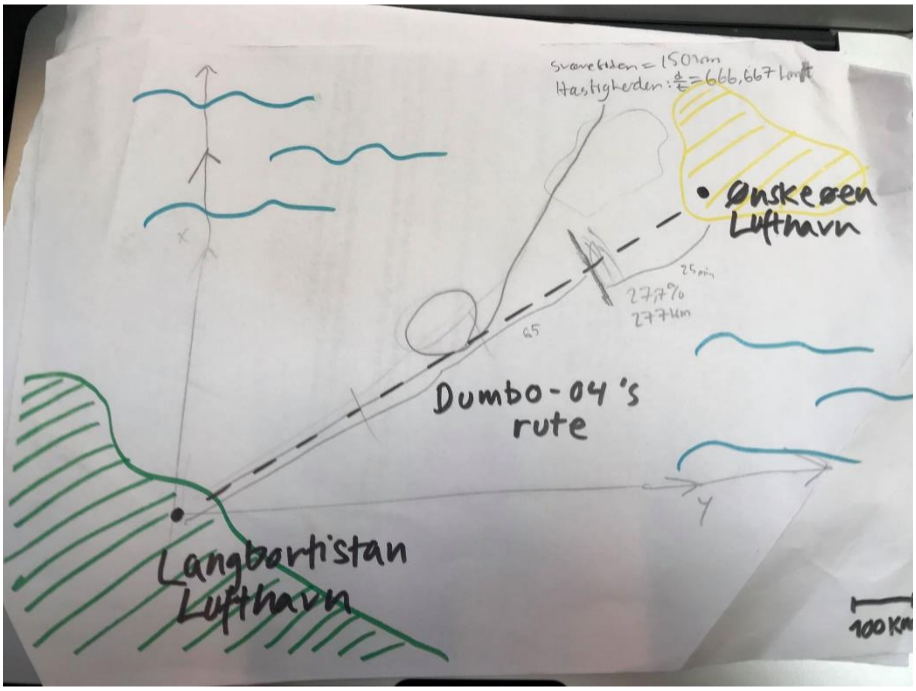

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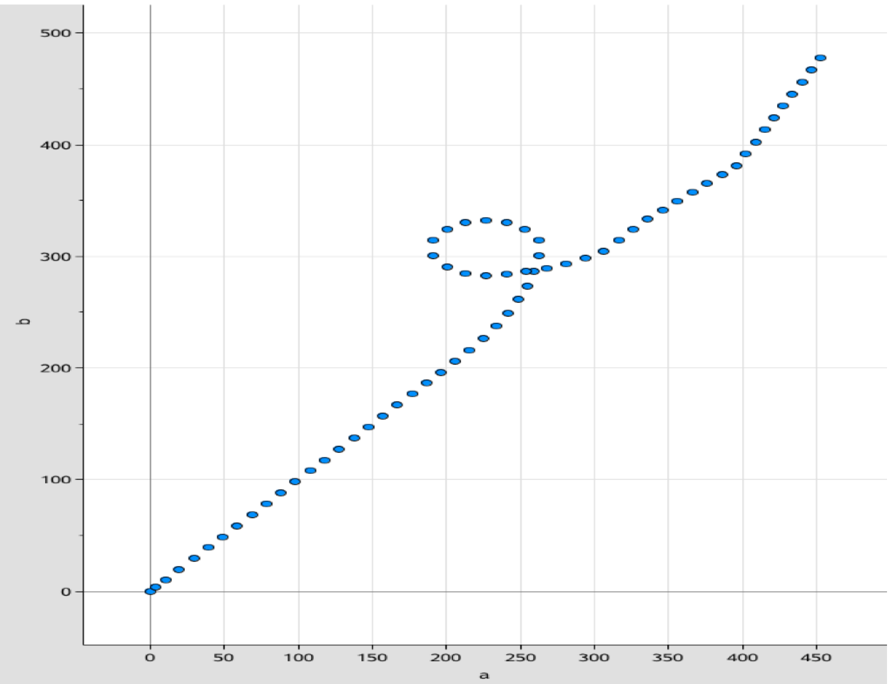

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326,25 + 3,33 = 329,58

324,03 - 32,33 = 291,7

Men på grafen viser den at det er senere at den kommer ud af kurs….

Ved at bobservere koordinaterne, kan man se, at koordinaterne er ens indtil minut 22. Det vil sige, at den indtil minut 22 har holdt sin kurs. Vi bestemmer farten ved hjælp af vektorformlen:

$|v| = \sqrt{v(x)^2 + v(y)^2}$

$|v| = \sqrt{9{,}83^2 + 9{,}83^2}$ = 13,90 km/min. For at få km /min i km/t, ganger vi med 60:
13,90 km/min * 60 = 834,10 km/t.

For at finde et mere nøjagtigt sted, hvor flyet er styrtet ned, beregner vi gennemsnitsacceleratio-nen ved hjælp af vektorformlen:

$a = \frac{\Delta v}{\Delta t} = \frac{v(slut) - v(start)}{t(slut) - t(start)}$

Δt = 1 min

Δv = 9,65 km/min-9,83 km/min = -0,18 km/min

$a = \frac{-0{,}18\ km/min}{1\ min}$ = -0,18 km/min^2

Ved at bruge koordinaterne, hvor A er x og B er y, kan vi bruge vektorformlen for at finde vektoren for AB:

AB (vektor) = $(\frac{b1-a1}{b2-a2})$

AB (vektor) = $\frac{226,77-225,02}{237,77-233,51}$ = $\frac{1,75}{4,26}$ = 0,4

Vi finder afstanden vha. vektorformlen:

AB (vektor) = $\sqrt{(b1-a1)^2+(b2-a2)^2}$

AB (vektor) = $\sqrt{(226,77-225,02)^2+(237,77-226,77)^2}$ = $\sqrt{3,06\ +\ 121}$ = $\sqrt{124,06}$ = 11,13

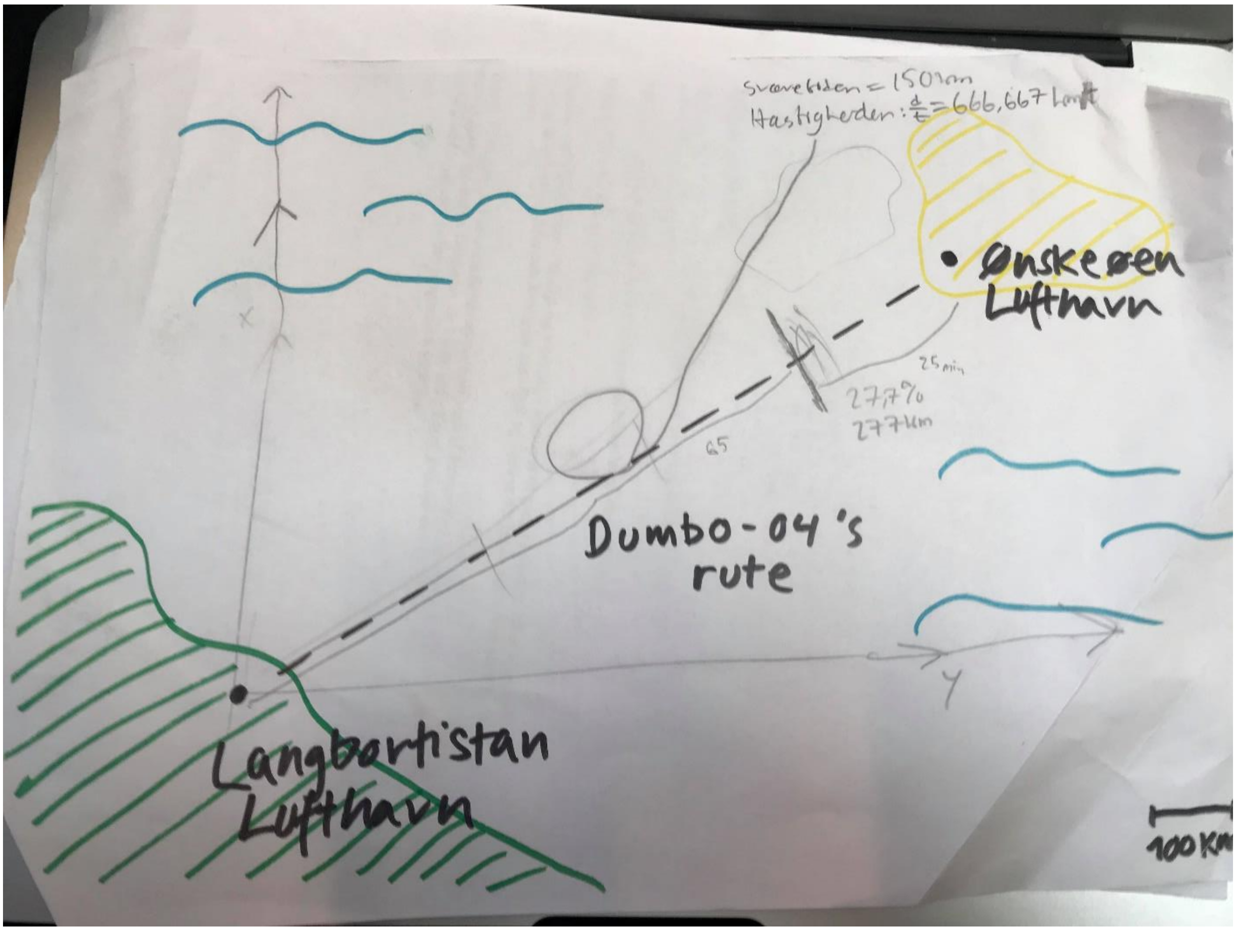
C.9 Report 1, group 5

Malaysian Arilines

Malaysian Airlines flyver fra Langtbortistan lufthavn til Ønskeøens lufthavn. En tur der tager 90 minutter i det rigtige vejr.

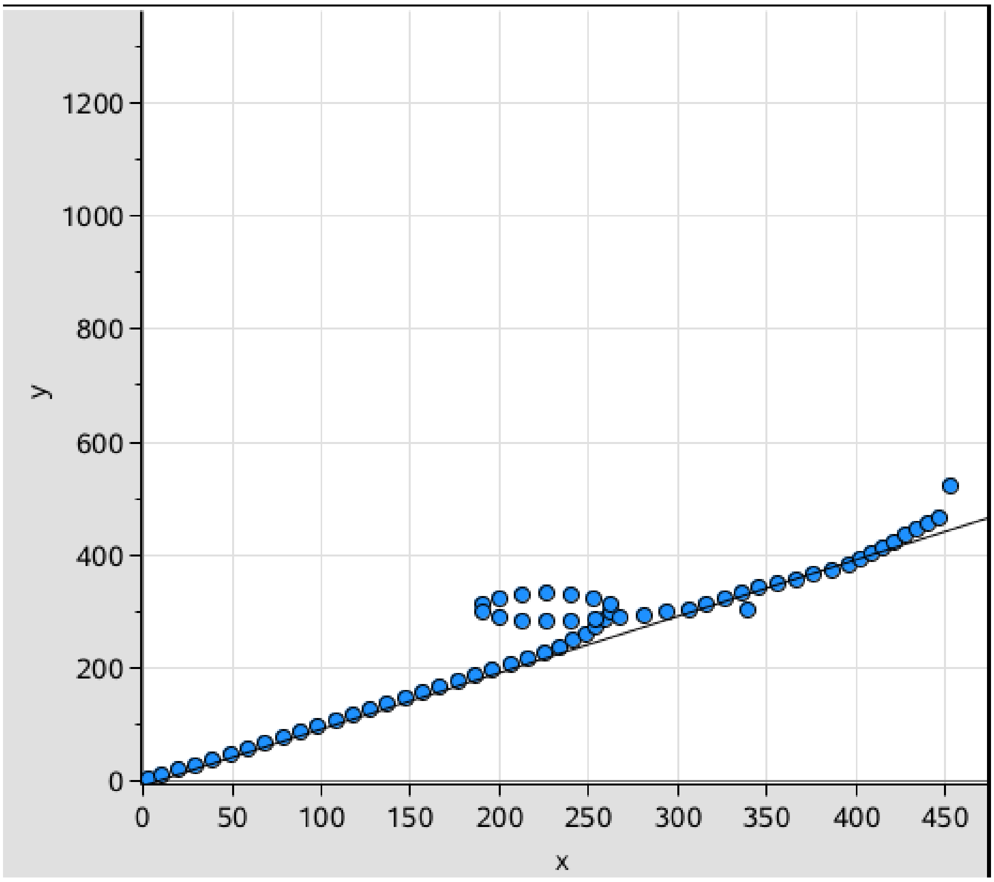
Turen er 1000 km lang. Flyet hedder Dumbo-04 og er en Airbus a320 model som kan have 8 tons brændstof i tanken. Men bruger kun 4 ton på turen til ønskeørnen. Ved en fejl er flyet kun blevet tanket med 5 ton brændstof.

Flyet skal efter planen flyve i en linenær linje til ønske øen.



Malaysian Airlines opdatere flyet koordinater hvert minut. Men efter 65 minutter mister kontrolcentret signalet med flyet.

Flyets koordinater er:



Her ser vi flyet koordinater og det loop som flyet laver et 23 minutter af turen.

Vi har udregnet af flyet mister 14 minutter i loop så der for er flyet nye tid til det ankommer 104 minutter hvis flyet altså retter op og holder kursen resten af turen:

$$14 + 90 = 104$$

Ud fra oplysningerne skulle flyet havne i en meget kraftig vind ved ca. 50 min. Hvis flyet havde ramt den kraftige vind, præcis i det 50 min. skulle det ud fra vektorens kordinater ende ved punktet der skiller sig ud, ved at være under den rette linje.

Dog kan man ved at aflæse på grafen, se at flyet ved ca. 50 minut, kan havde ramt en kraftig vind, da tallene på x-aksen indtil da er steget med ca. 10 km/min, men nu stiger med ca 6-7 km/min.

C.10 Report 2, group 5

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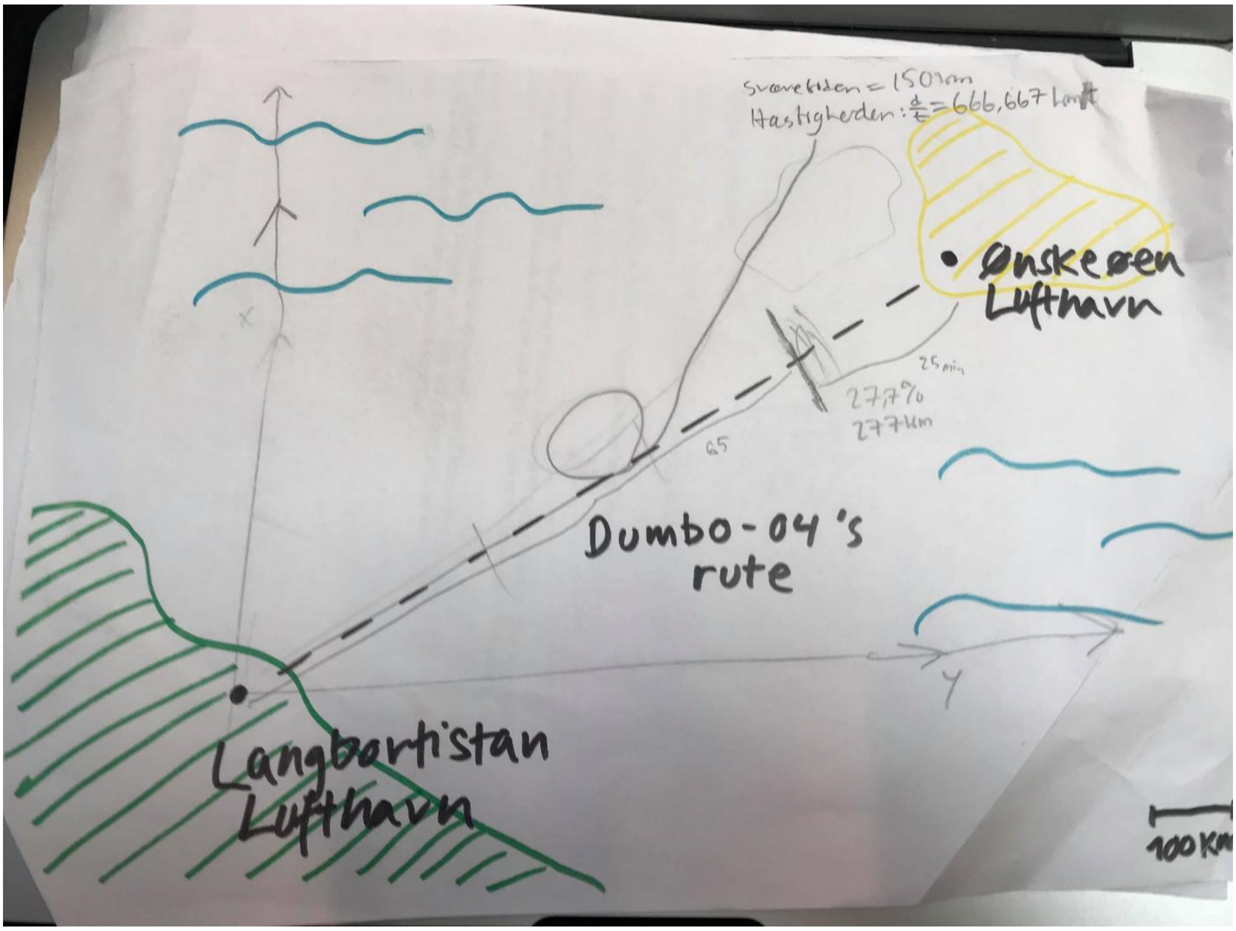

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Flyets koordinater er:

Her ser vi flyet koordinater og det loop som flyet laver et 23 minutter af turen.

Vi har udregnet af flyet mister 14 minutter i loop så der for er flyet nye tid til det ankommer 104 minutter hvis flyet altså retter op og holder kursen resten af turen:

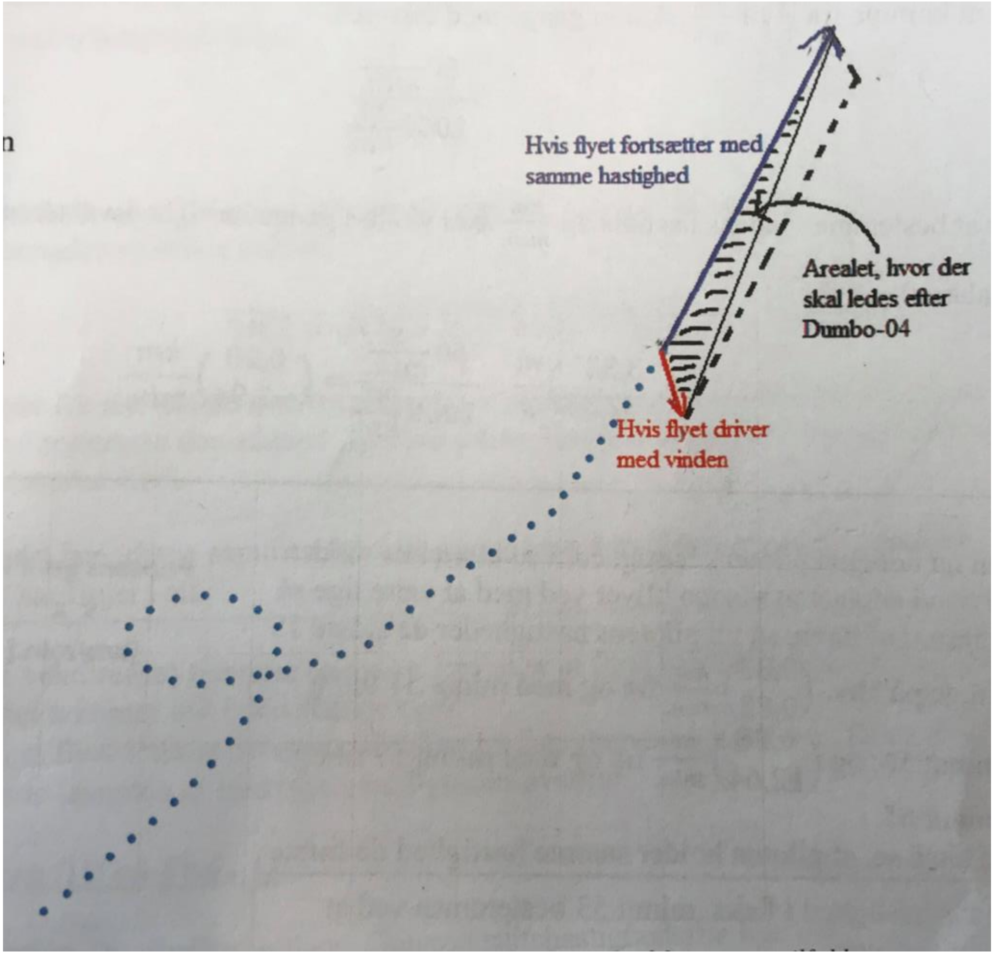
$$14 + 90 = 104$$

Ud fra oplysningerne skulle flyet havne i en meget kraftig vind ved ca. 50 min. Hvis flyet havde ramt den kraftige vind, præcis i det 50 min. skulle det ud fra vektorens kordinater ende ved punktet der skiller sig ud, ved at være under den rette linje.
Dog kan man ved at aflæse på grafen, se at flyet ved ca. 50 minut, kan havde ramt en kraftig vind, da tallene på x-aksen indtil da er steget med ca. 10 km/min, men nu stiger med ca 6-7 km/min.

Vi regner vores vind hastighed om fra m/s til til km/min

$$\begin{pmatrix} 3{,}33 \\ -32{,}33 \end{pmatrix} \frac{m}{s} \cdot \frac{60 \frac{s}{min}}{1000 \frac{m}{km}} = \begin{pmatrix} 0{,}20 \\ -1{,}94 \end{pmatrix} \frac{km}{min}$$

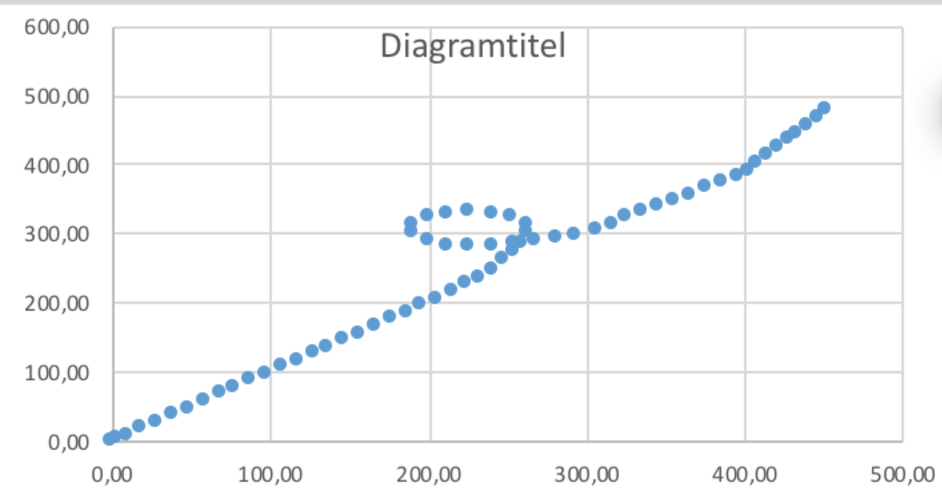
Vi antager at vindhastigheden er den sammen resten af turen.



På billede ser vi det område flyet kan lande i. Vi antager at piloten har synes han holdte den rette kurs, og derfor bare er fløjet lige ud indtil han løb tør for brændsel. Hvis vi også tænker på at vinden har haft indflydelse på flyet rute tænker vi at flyet ligger et sted i det midterste Felt

C.11 Report 1, group 6

Den egetnlige rute skulle være en lige linje fra langbortistan til ønske øen.



Men ved hjælp af et punktdiagram kan vi aflæse at flyet undervejs er fløjet i en cirkel.
Flyet er måske styrtet ned lige fremme for hvor vi mistede kontakten

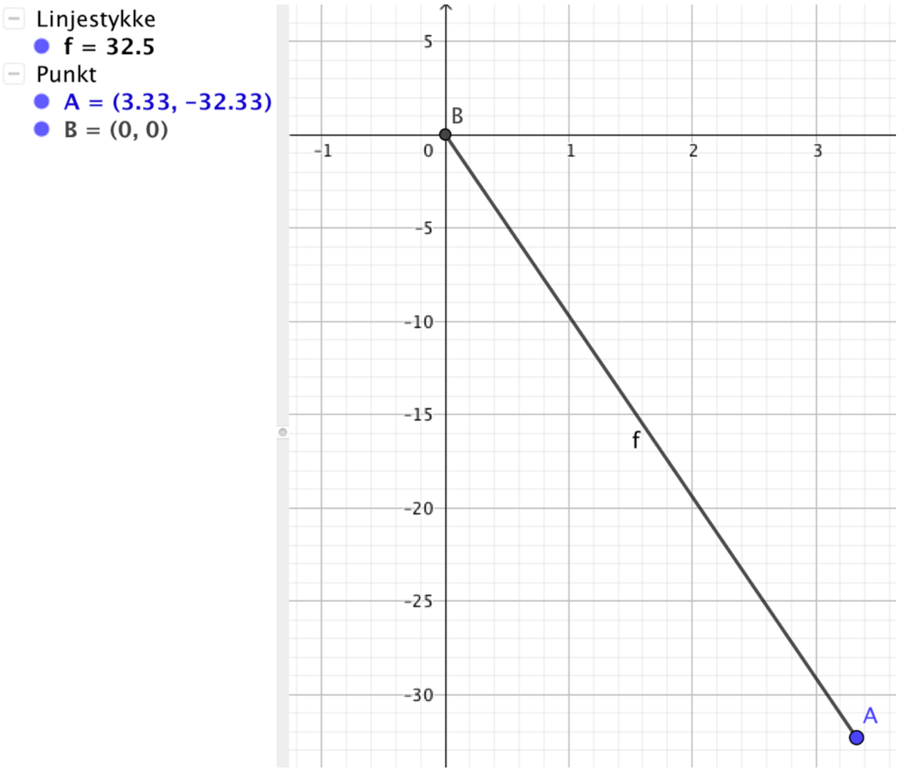
Et fly kan cirka svæve 150 km
https://spoerg-piloten.dk/hvis-en-boeing-777-eller-737-altsa-et-fly-i-passagerfly-storrelse-oplevede-at-alle-motorer-stopper-med-at-virke-hvordan-vil-flyet-sa-falde-og-hvor-lang-tid-ville-det-kunne-holde-sig-i-luften/

Vi læste om flyet a320.
https://ing.dk/artikel/livsfarlig-fejl-slukker-instrumenter-pa-airbus-320-fly-90024
Her kunne vi læse at det fly før har en defekt der i 2008 37 gange har slukket de elektroniske skærme og instrumenter i cockpittet.
Det betyder piloterne ikke ved hvor de skal flyve hen eller hvor hurtigt de flyver. Det kan være grunden til flyets bemærkelsesværdige rute.
I artiklen står der også at de i tidligere tilfælde har forsøgt at vende om at flyve hjem. Det kan være vores piloter også prøvede det, men de var så tilpas langt væk fra land at de ikke kunne orientere sig tilbage til Langtbortistan.

http://www.pilotfriend.com/training/flight_training/aero/gliding.htm

https://aviation.stackexchange.com/questions/14425/how-can-the-glide-ratio-in-a-balanced-turn-be-estimated?utm_medium=organic&utm_source=google_rich_qa&utm_campaign=google_rich_qa
Vi har kigget på disse to links for at prøve at regne glidetallet ud for et a320 når det drejer og derved indskrænke området flyet kunne lande.
Vi havde ikke nok tid til at undersøge det ordentligt. Det vil vi gøre hvis vi skulle arbejde med det igen.

Blæser med 1.95 km/m den retning.



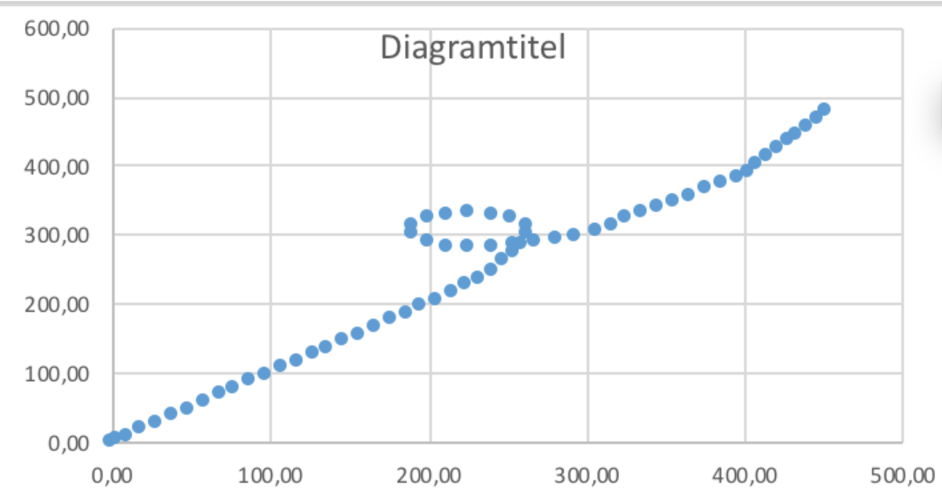
Vier i gang med et udregne hastigheden flyet har fløjet med og hvilken effekt faktoren har haft på flyet rute.

C.12 Report 2, group 6

[redacted]
Matematik
Stenhus Gymnasium

Dumbo-04

Den egentlige rute skulle være en lige linje fra Langbortistan til Ønske øen.



Men ved hjælp af et punktdiagram kan vi aflæse at flyet undervejs er fløjet i en cirkel.
Flyet er måske styrtet ned lige fremme for hvor vi mistede kontakten

Et fly kan cirka svæve 150 km
https://spoerg-piloten.dk/hvis-en-boeing-777-eller-737-altsa-et-fly-i-passagerfly-storrelse-oplevede-at-alle-motorer-stopper-med-at-virke-hvordan-vil-flyet-sa-falde-og-hvor-lang-tid-ville-det-kunne-holde-sig-i-luften/

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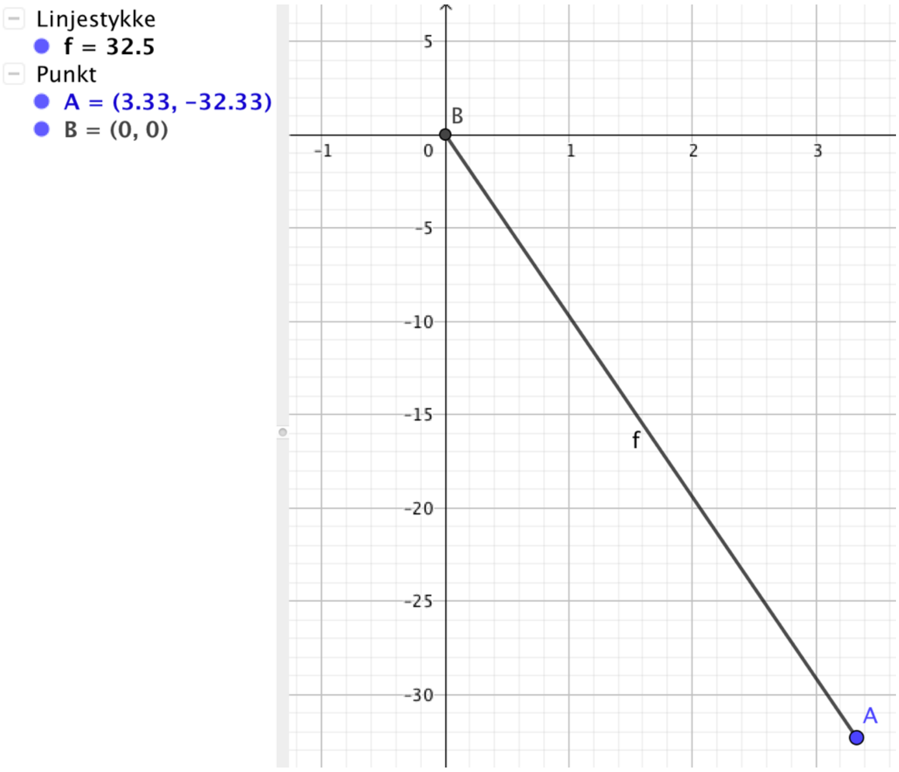
Det betyder piloterne ikke ved hvor de skal flyve hen eller hvor hurtigt de flyver. Det kan være grunden til flyets bemærkelsesværdige rute.
I artiklen står der også at de i tidligere tilfælde har forsøgt at vende om at flyve hjem. Det kan være vores piloter også prøvede det, men de var så tilpas langt væk fra land at de ikke kunne orientere sig tilbage til Langtbortistan.

http://www.pilotfriend.com/training/flight_training/aero/gliding.htm

https://aviation.stackexchange.com/questions/14425/how-can-the-glide-ratio-in-a-balanced-turn-be-estimated?utm_medium=organic&utm_source=google_rich_qa&utm_campaign=google_rich_qa
Vi har kigget på disse to links for at prøve at regne glidetallet ud for et a320 når det drejer og derved indskrænke området flyet kunne lande.
Vi havde ikke nok tid til at undersøge det ordentligt. Det vil vi gøre hvis vi skulle arbejde med det igen.

Blæser med 1.95 km/m den retning.

$\begin{pmatrix} 0.2 \\ -1.94 \end{pmatrix} \frac{\text{Km}}{\text{min}}$



Området hvor man skal lede efter Dumbo-04

Vi har udregning i Excel ark at flyet har tilbagelagt en distance på 869 Km. Flyet kan maksimalt flyve 1300 km, med det brændstof det har fået tanket. Så vi kan derfor finde ud af hvor langt flyet kan komme videre efter vi har mistet kontakten.

$$1300 - 869 = 431\text{Km}$$

Flyet kan derfor flyve 431 km efter at man har mistet kontakten med det. Derfor kan vi lave en parallel med den retning flyet flyver lige inden radiokontakten mistes.

Vindhastigheden har vi fået at vide, at den er $\begin{pmatrix} 3.33 \\ -32.33 \end{pmatrix} \frac{m}{s}$, vi skal omregne enheden til $\frac{Km}{min}$, da vores Excel ark indeholder enhederne Km og min:

Når man skal omregne fra $\frac{m}{s}$ til $\frac{km}{min}$ skal vi gange med

$\frac{60\frac{s}{min}}{1000\frac{m}{km}}$

For at bestemme vindens hastighed i $\frac{km}{min}$, ganger vi hastighedsvektoren med konstanten $\frac{60\frac{s}{min}}{1000\frac{m}{km}}$

$$\begin{pmatrix} 3.33 \\ -32.33 \end{pmatrix} \frac{m}{s} \cdot \frac{60 \frac{s}{min}}{1000 \frac{m}{km}} = \begin{pmatrix} 0.20 \\ -1.94 \end{pmatrix} \frac{km}{min}$$

C.13 Report 1, group 7

Fly

Længdeaf flyvetur: 654 km

Længdenaf efter flyvet lavet en cirkel: 394 km

654−394=Hvor meget flyet har flyet med stærk vind= 260 km

$\begin{bmatrix} 3.33 \\ -32.33 \end{bmatrix} \frac{m}{s}$ så er længde $\sqrt{(3.33)^2+(^-32.33)^2}$ ▸ 32.501 km

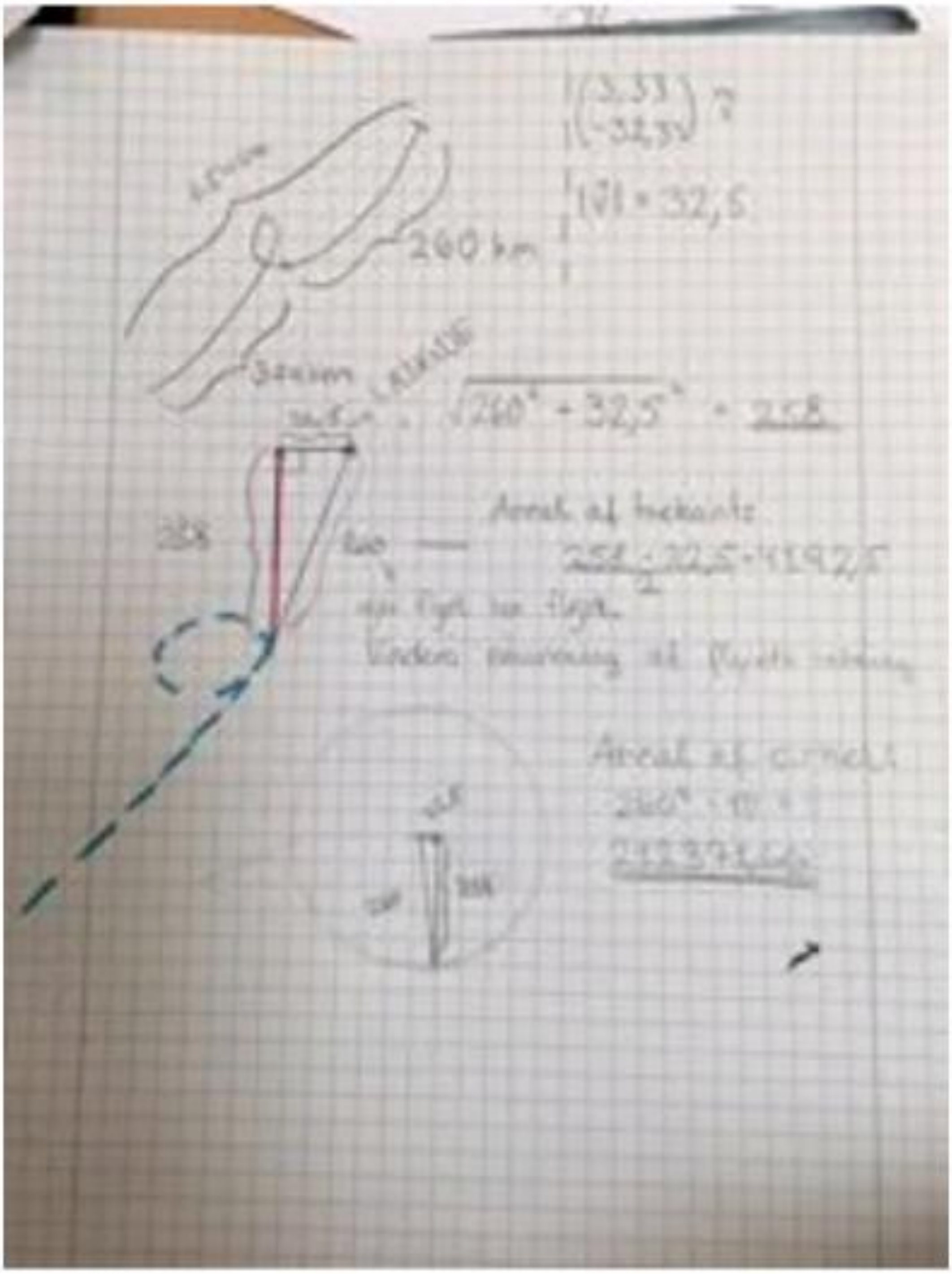
$\sqrt{260^2-(32.5)^2}$ ▸ 257.961 er hvor meget flyet vil flyve hvis ikke vindenvaar der.

trekankens areal: $\frac{258 \cdot 32.5}{2}$ ▸ $4192.5\ cm^2$

cirkelen areal: $260^2 \cdot \pi$ ▸ $67600 \cdot \pi = 212371.66\ cm^2$

[redacted]

1.w



C.14 Report 2, group 7

GRUPPE 7 [redacted]

Vi går ud fra at flyet begyndte at flyve ned af efter det sidste punkt på grafen.
Ud fra dette, har vi tegnet et parallelogram ud fra følgende beregninger:

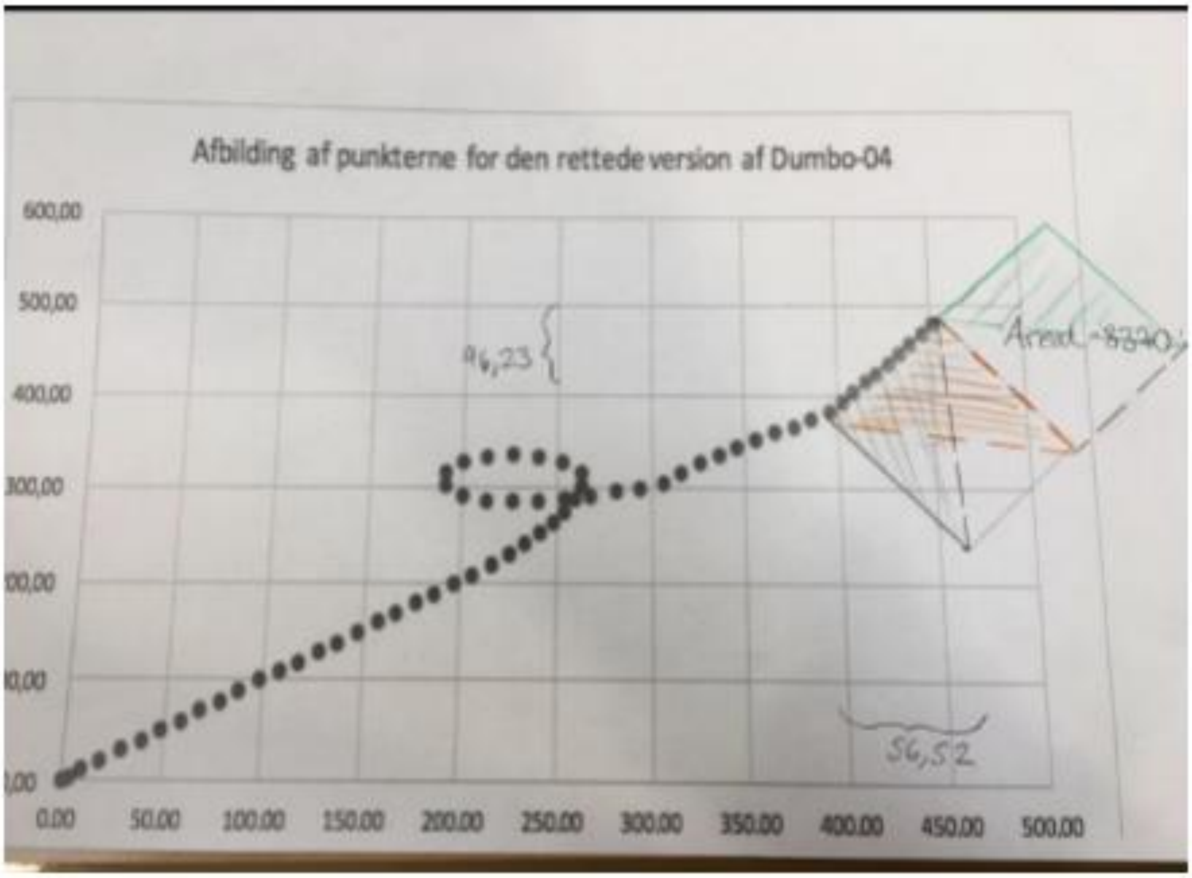
$\sqrt{56{,}52^2 + 96{,}23^2 = 111{,}60\ km}$ er den lodrette vektor

$\sqrt{15{,}38^2 + (-149{,}23)^2 = 150}$ svarer til vindens påvirkning på flyets retning.

Ud fra parallelogrammet, har vi udregnet arealet af parallelogrammet:

$\frac{150 \cdot 111{,}6}{2} = 8370\ km^2$

Den orange trekant på det vedlagte bilag har vi rykket længere op, da vi mener at flyet ændrede kurs efter sidste koordinat: den grønne trekant.



C.15 Report 1, group 8

Rapport (Assistancemelding)

Hypotese:

Der kunne være sket en teknisk fejl, i en af de venstre motorer.

En anden hypotese vi har foretaget os er at vejret har påvirkede dombo-04 og dermed har flyet foretaget et loop.

Oplysninger om flyveturen:

Vejrforholdene er blevet undersøgt og det viser sig at flyet efter ca 50 min er havnet i en meget kraftig vind $\begin{pmatrix} 3,33 \\ -32,33 \end{pmatrix} \frac{m}{s}$

Senere har man fundet ud af at der er sket en fejl under benzintakningen af flyet Dumbo-04. Normalt skal et fly have 8 ton brændstof til en tur på 1000 km, for at være sikker på at flyet kan holde ruten og at flyet ikke løber tør for brændstof. Der er dog sket en fejl ved benzintankningen og derfor har flyet Dumbo-04, fået 5 ton brændstof tanket i stedet for 8 ton.

Resultater:

Maks km= 1300 km

Flyveruten= 1000 km

$\sqrt{452,78^2 + 477,5^2}$ = 658,04 km

$\frac{658,04}{1000}$ = 0,63805 → $0,63805 \cdot 100$= 63,805%

63,8% af 1000 km → $\frac{63,8 \cdot 1000}{100}$ = 688 km

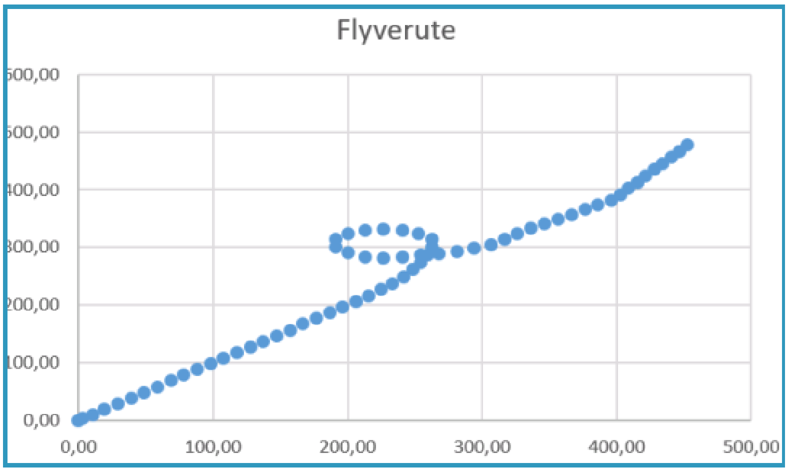
1300 km - 688 km = 612 km

612+ svævelængden (150 km) = 762

Flyet kan styrte ned fra en radius af 762 km inklusive svævefasen

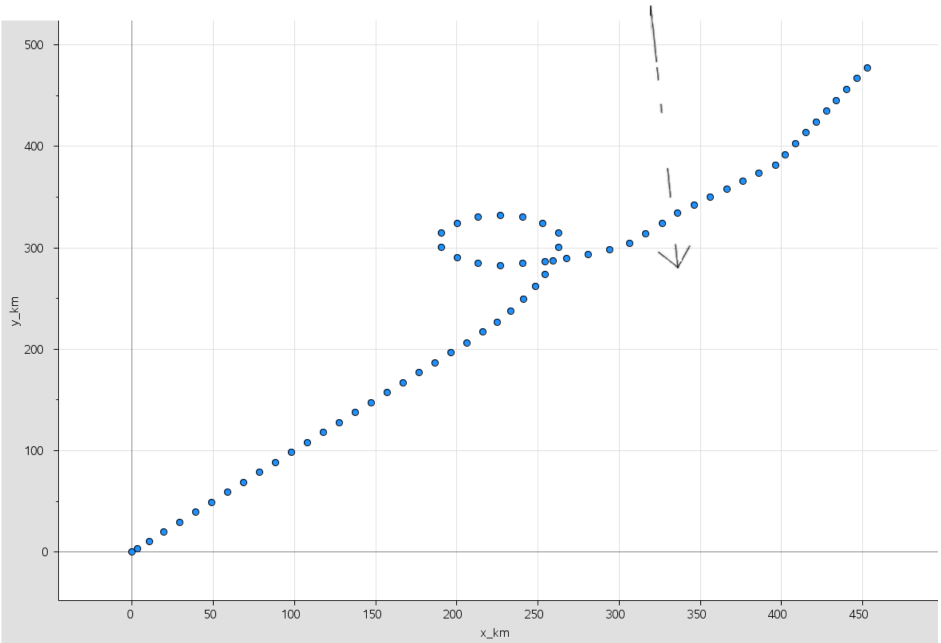
Grafer:

Grafen over flyruten for Dumbo-04.



Der er foregået en fejl i det udleverede datasæt og derfor er y-koordinatet i min 53 ikke 357,31 og i stedet 357,53 og i grafen vises der også vindretningen efter det foretagende loop.

Bevis ses på grafen.



C.16 Report 2, group 8

Rapport (Assistancemelding)

Hypotese:

Der kunne være sket en teknisk fejl, i en af de venstre motorer.

En anden hypotese vi har foretaget os er at vejret har påvirket Dumbo-04 og dermed har flyet foretaget et loop.

Oplysninger om flyveturen:

Vejrforholdene er blevet undersøgt og det viser sig at flyet efter ca. 50 min er havnet i en meget kraftig vind $\begin{pmatrix} 3{,}33 \\ -32{,}33 \end{pmatrix} \frac{m}{s}$

Senere har man fundet ud af at der er sket en fejl under benzintakningen af flyet Dumbo-04. Normalt skal et fly have 8 ton brændstof til en tur på 1000 km, for at være sikker på at flyet kan holde ruten og at flyet ikke løber tør for brændstof. Der er dog sket en fejl ved benzintankningen og derfor har flyet Dumbo-04, fået 5 ton brændstof tanket i stedet for 8 ton.

Resultater:

Vi udregner vektoren $\left[\overrightarrow{AB}\right]$, hvor punkt A er lufthavnen og punkt B er, hvor man har mistet kontakt til flyet dumbo-04.

Maks km= 1300 km

Flyveruten= 1000 km

$\sqrt{452{,}78^2 + 477{,}5^2}$ = *658,04 km*

$\frac{658{,}04}{1000}$ = *0,63805* → $0{,}63805 \cdot 100$= *63,805%*

63,8% af 1000 km → $\frac{63{,}8 \cdot 1000}{100}$ = *688 km*

1300 km - 688 km = 612 km

612+ svævelængden (150 km) = ***762***

Udregning af loopet fra 29. min til 44. min.:

$(13{,}9\ km \cdot 10) + 14\ km\ + (13{,}89\ km \cdot 5) = \mathbf{222{,}45\ km}$

Udregning af antal km flyet Dumbo-04 har fløjet:

$658{,}04\ km + 222{,}45\ km$= **880,49 km**

Udregning af hvor mange km flyet Dumbo-04 kan flyve, før man har mistet kontakten til flyet:

$1300\ km\ - 880{,}49\ km = \mathbf{419{,}51}$

Arealet flyet Dumbo-04 kunne søges efter:

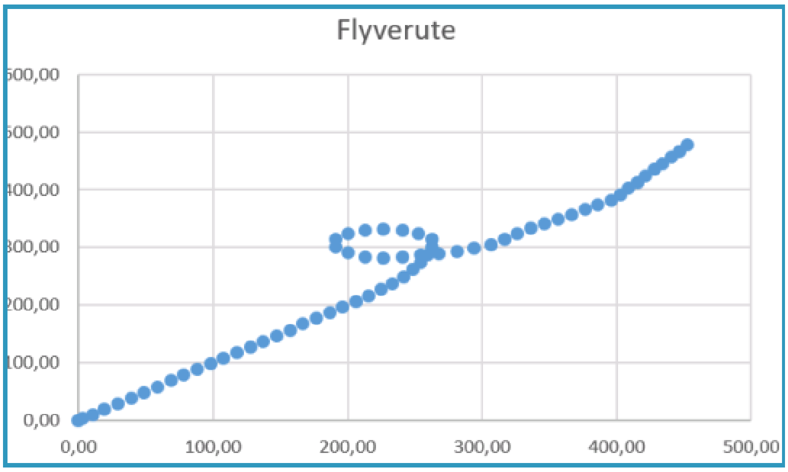
$419{,}51^2 \cdot 3{,}14 = \mathbf{552604{,}34\ km^2}$

$\frac{552604{,}34}{4} = \mathbf{138151{,}08\ km^2}$

Vi dividerer med 4, for at indsnævre søgefeltet, da det ikke er så relevant at lede bag ved flyets rute

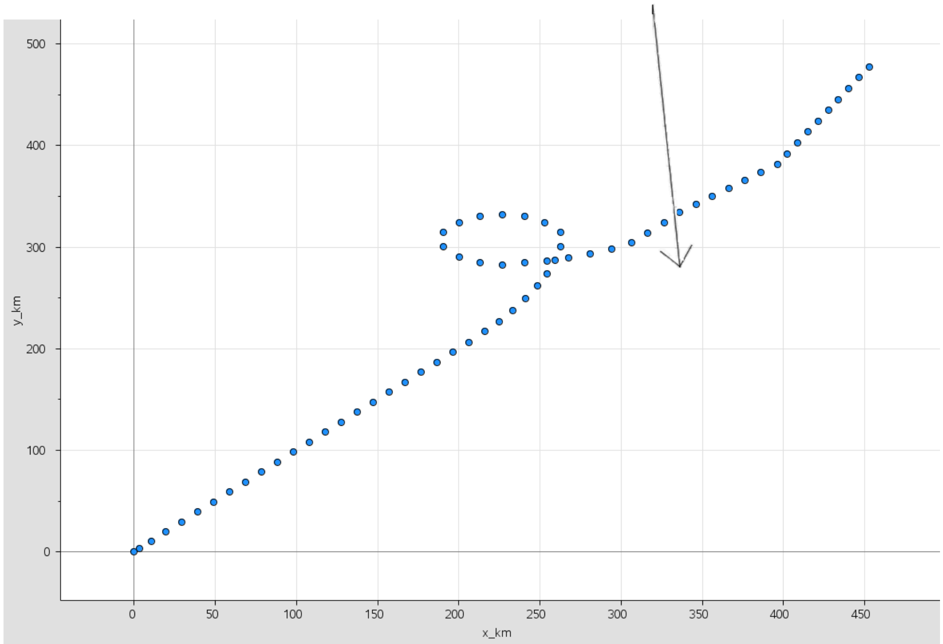
Grafer:

Grafen over flyruten for Dumbo-04.



Der er foregået en fejl i det udleverede datasæt og derfor er y-koordinatet i min. 53 ikke 357,31 men i stedet 357,53. Grafen viser det nye koordinat og vindretningen efter det foretagende loop.

Bevis ses på grafen.



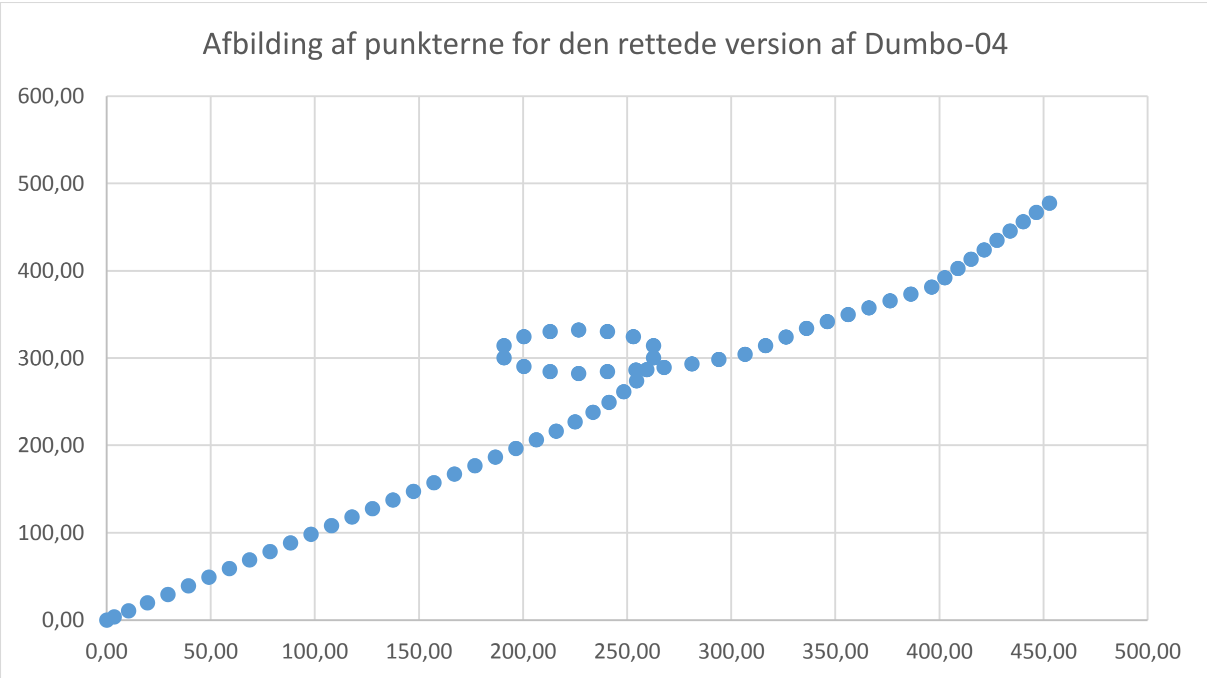
C.17 Report 1, group 9

Rapport

Koordinater for Dumbo-04

Tid [min,]	x [km]	y [km]
0	0,00	0,00
1	3,50	3,50
2	10,50	10,50
3	19,50	19,50
4	29,33	29,33
5	39,16	39,16
6	48,99	48,99
7	58,82	58,82
8	68,65	68,65
9	78,48	78,48
10	88,31	88,31
11	98,14	98,14
12	107,97	107,97
13	117,80	117,80
14	127,63	127,63
15	137,46	137,46
16	147,29	147,29
17	157,12	157,12
18	166,95	166,95
19	176,78	176,78
20	186,61	186,61
21	196,44	196,44
22	206,27	206,27
23	215,92	216,27
24	225,02	226,77
25	233,51	237,77
26	241,32	249,27
27	248,33	261,27
28	254,41	273,77
29	259,33	286,77
30	262,64	300,27
31	262,64	314,27
32	252,99	324,27
33	240,45	330,27
34	226,70	332,27
35	212,95	330,27
36	200,41	324,27
37	190,76	314,27
38	190,76	300,37
39	200,41	290,37
40	212,95	284,37
41	226,70	282,37
42	240,45	284,37
43	254,20	286,37

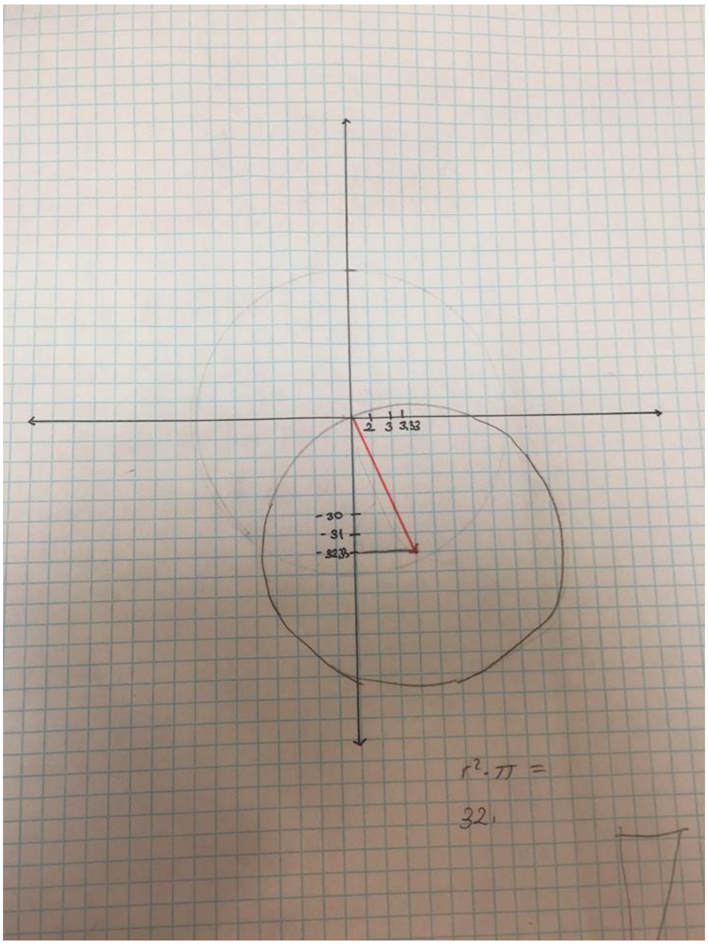
44	267,77	289,37
45	281,08	293,37
46	294,05	298,37
47	306,59	304,20
48	316,42	314,20
49	326,25	324,03
50	336,08	333,86
51	346,11	341,75
52	356,14	349,64
53	366,17	357,53
54	376,20	365,42
55	386,23	373,31
56	396,26	381,20
57	402,54	391,90
58	408,82	402,60
59	415,10	413,30
60	421,38	424,00
61	427,66	434,70
62	433,94	445,40
63	440,22	456,10
64	446,50	466,80
65	452,78	477,50



Ved at aflæse på koordinatsættene kan man se, at den kraftige vind påvirker flyet ved ca. 56-57 minutter, da x-værdierne ikke stiger med ca. 10 pr. minut. men med 6-7 pr. minut.

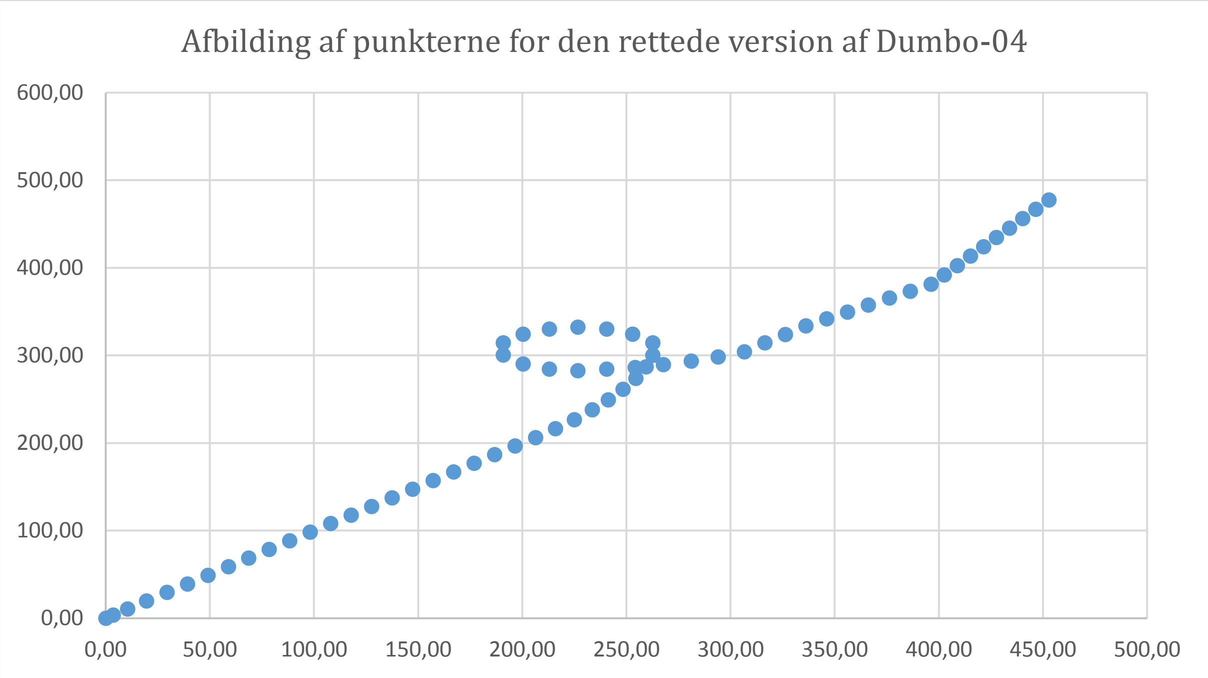
Vi antager at flyet bruger mere brændstof når den kraftige vind begynder at skabe turbulens.

For at finde ud af hvor flyet er styrtet, finder vi først længden af vektoren for vindens hastighed. Den er på 32,



C.18 Report 2, group 9

Rapport



Man kan inddele grafen i 3 faser, og derefter undersøge enkelte fase.

Ved at aflæse på koordinatsættene kan man se, at den kraftige vind påvirker flyet ved ca. 56-57 minutter, da x-værdierne ikke stiger med ca. 10 pr. minut. men med 6-7 pr. minut.

Vi antager at flyet bruger mere brændstof når den kraftige vind begynder at skabe turbulens.

Hypotese: Har flyveren, fået en højere hastighed da det begyndte at styrte, end da det var i gang med at flyve imod ønskeøen? Og har det påvirket hvor flyet er styrtet ned?

Dumbo-04 holder sin kurs med en fart indtil minut 22,

Vi inddeler grafen i tre faser. I den første fase forløber flyveturen som planlagt. Det er først i den anden fase at der går kage i den. Derfor vil vi vælge at fokusere på den anden fase og den tredje fase.

Man kan aflæse på punkterne at Dumbo-04 krænger mod nord når der er gået 23 minutter. Der opstår gennemsnitlige hastighedsændringer over intervallerne.

$$a = \frac{\Delta v}{\Delta t} = \frac{v_{slut} - v_{start}}{t_{slut} - t_{start}}$$

Vi indsætter derefter værdierne.

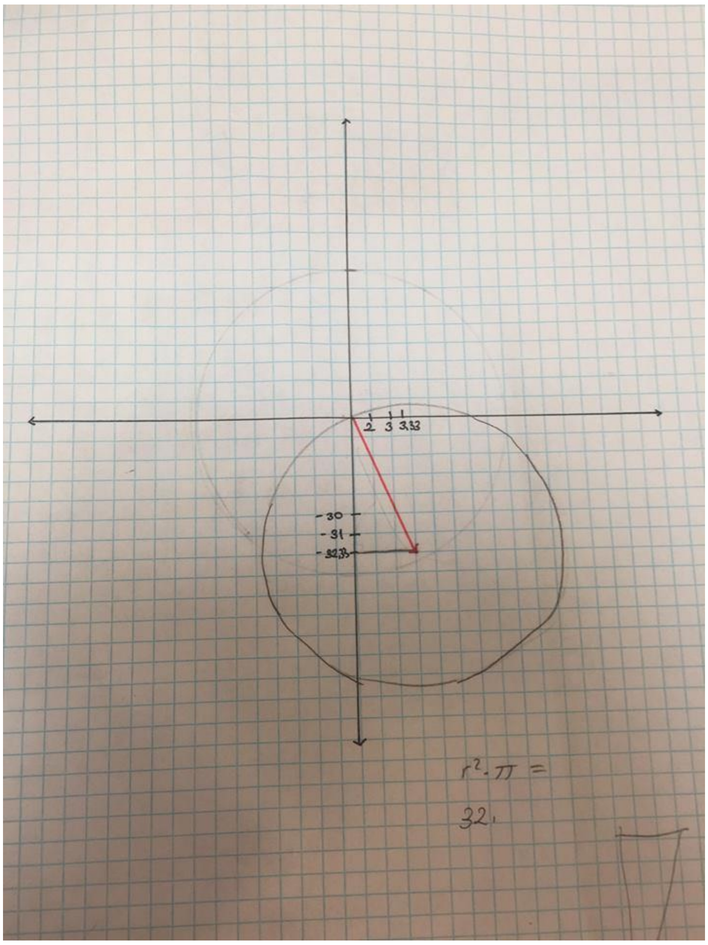
$$\Delta v_x = v_{x,start} = 9{,}65\frac{km}{min} - 9{,}83\frac{km}{min} = -0{,}18\frac{km}{min}$$

Gennemsnitsaccelerationen i minut 23 aftager med 0,18 $\frac{km}{min}$.

Derefter bestemmer vi gennemsnitsaccelerationen, a_x:

$$a_x = \frac{-0{,}18\,\frac{km}{min}}{1\ min} = -0{,}18\,\frac{km}{min^2}$$

$\overrightarrow{AB} = \begin{pmatrix} b_1 - & a_1 \\ b_2 - & a_2 \end{pmatrix}$



Appendix D

Lists of questions

$Q_{0,I}$: What has happened to the aeroplane?
$Q_{0,I,a}$: Why did the aeroplane not follow the intended route?
$Q_{0,II}$: How big is the area that has to be searched in order to find the aeroplane?
$Q_{0,II,a}$: Where did the aeroplane crash?
$Q_{0,II,b}$: Where did the aeroplane land?

Q_1: How does the route, that the aeroplane followed, look?
$Q_{1,1}$: What is remarkable about the route?
$Q_{1,1,1}$: Why did the aeroplane make a "loop"?
$Q_{1,1,1,a}$: Why did the aeroplane first disappear from the radar 36 minutes after it made a "loop"?
$Q_{1,1,2}$: Why did the aeroplane suddenly make the change in course in minute 56?
$Q_{1,2}$: How far from Farawayistan Airport is the aeroplane, when it disappears from the radar?
$Q_{1,2,1}$: How is the distance to Farawayistan Airport calculated from the x- and y-coordinates?
$Q_{1,2,a}$: In what part of the flight did the aeroplane crash?
$Q_{1,3}$: How far from Neverland Airport is the aeroplane, when it disappears from the radar?
$Q_{1,a}$: How far did the aeroplane travel after it made the "loop"?

Q_2: By what speed/velocity did the aeroplane fly during the flight?
$Q_{2,1}$: How is speed/velocity calculated from position and time?
$Q_{2,1,1}$: How is the fact that the aeroplane is moving in two directions taken into account, when calculation speed/velocity?
$Q_{2,2}$: How fast does an Airbus a320 fly?
$Q_{2,3}$: What can be derived about the flight from its velocities?
$Q_{2,3,1}$: Why was the speed lowered by the end of the monitored part of the flight?
$Q_{2,3,a}$: Did the aeroplane increase its velocity when it started crashing compared to the velocity when it was heading towards Neverland? And has it influenced where the aeroplane has crashed?
$Q_{2,a}$: What speed corresponds to the expected time and distance?

Q_3: How can vector theory be applied to calculate the area that has to be searched in order to find the aeroplane?
$Q_{3,1}$: Which two vectors should form the parallelogram?
$Q_{3,1,a}$: How can it be determined where the aeroplane crashed?
$Q_{3,1,a,a}$: How far can an aeroplane glide?
$Q_{3,1,a,a,a}$: What is the lift-drag ratio for an Airbus a320 when it is turning?

Q_4: How can the information about cross wind be used?
$Q_{4,1}$: How are wind conditions taken into account, when calculating the velocity of an aeroplane?
$Q_{4,1,1}$: What is the "true airspeed" of the aeroplane, when it is flying in the cross wind?

$Q_{4,a}$: Where would the aeroplane have ended if the cross wind had affected it from exactly minute 50?
$Q_{4,b}$: How has the cross wind affected the route of the aeroplane?
$Q_{4,c}$: What is the length of the wind vector?
$Q_{4,d}$: How far has the aeroplane travelled in the strong wind?
$Q_{4,d,a}$: How far would the aeroplane have travelled if the strong wind had not been there?
$Q_{4,d,a,a}$: What is the area of the triangle spanned by the distance that the aeroplane would have travelled if the strong wind had not been there and the wind vector?

Q_5: How can the information about the amount of fuel be used?
$Q_{5,1}$: For how many kilometres will the remaining fuel last?
$Q_{5,1,1}$: How big a distance has the aeroplane been travelling from take-off and until it disappears from the radar?
$Q_{5,1,1,a}$: What does this distance correspond to on the map?
$Q_{5,1,1,a,a}$ What is the scale of the map?
$Q_{5,a}$: How much longer than the expected distance can the aeroplane travel on the additional fuel?
$Q_{5,b}$: How much did the aeroplane weigh?

Q_a: Why did the aeroplane crash after 65 minutes?