INSTITUT FOR NATURFAGENES DIDAKTIK
KØBENHAVNS UNIVERSITET

The Nature of Mathematics Given Physicalism

Magnus Vinding
Bachelorprojekt - Matematik

Vejleder: Mikkel Willum Johansen

IND's studenterserie nr. 69, 2018

INSTITUT FOR NATURFAGENES DIDAKTIK, www.ind.ku.dk
Alle publikationer fra IND er tilgængelige via hjemmesiden.

IND's studenterserie

32. Henrik Egholm Wessel: Smartphones as Scientific Instruments in Inquiry Based Science Education (2013)
33. Nicole Koefoed: Et didaktisk design om definition, eksistens og eksakt værdi af bestemt integral (2013)
34. Trine Louise Brøndt Nielsen: From Master's programme to labour market – A study on physics graduates' experience of the transition to the labour market (2013)
35. Rie Hjørnegaard Malm: Becoming a Geologist – Identity negotiations among first year geology students (2013)
36. Mariam Babrakzai Zadran: Gymnasiealgebra I et historisk perspektiv – Matematiske organisationer I gymnasiealgebra (2014)
37. Marie Lohmann-Jensen: Flipped Classroom – andet end blot en strukturel ændring af undervisningen? (2014)
38. Jeppe Willads Petersen: Talent – Why do we do it? (2014)
39. Jeanette Kjølbæk: One-dimensional regression in high school (2015)
40. Anders Wolfsberg: A praxeological investigation of divergence – Exploring challenges of teaching and learning math-in-physics (2015)
41. Asger Brix Jensen: Number tricks as a didactical tool for teaching elementary algebra (2015)
42. Katrine Frovin Gravesen: Forskningslignende situationer på et førsteårskursus I matematisk analyse (2015)
43. Lene Eriksen: Studie og forskningsforløb om modellering med variabelsammenhænge (2015)
44. Caroline Sofie Poulsen: Basic Algebra in the transition from lower secondary school to high school (2015)
45. Rasmus Olsen Svensson: Komparativ undersøgelse af deduktiv og induktiv matematikundervisning (2016)
46. Leonora Simony: Teaching authentic cutting-edge science to high school students(2016)
47. Lotte Nørtoft: The Trigonometric Functions - The transition from geometric tools to functions (2016)
48. Aske Henriksen: Pattern Analysis as Entrance to Algebraic Proof Situations at C-level (2016)
49. Maria Hørlyk Møller Kongshavn: Gymnasieelevers og Lærerstuderendes Viden Om Rationale Tal (2016)
50. Anne Kathrine Wellendorf Knudsen and Line Steckhahn Sørensen: The Themes of Trigonometry and Power Functions in Relation to the CAS Tool GeoGebra (2016)
51. Camilla Margrethe Mattson: A Study on Teacher Knowledge Employing Hypothetical Teacher Tasks - Based on the Principles of the Anthropological Theory of Didactics (2016)
52. Tanja Rosenberg Nielsen: Logical aspects of equations and equation solving - Upper secondary school students' practices with equations (2016)
53. Mikkel Mathias Lindahl and Jonas Kyhnæb: Teaching infinitesimal calculus in high school - with infinitesimals (2016)
54. Jonas Niemann: Becoming a Chemist – First Year at University
55. Laura Mark Jensen: Feedback er noget vi giver til hinanden - Udvikling af Praksis for Formativ Feedback på Kurset Almen Mikrobiologi (2017)
56. Linn Damsgaard & Lauge Bjørnskov Madsen: Undersøgelsesbaseret naturfagsundervisning på GUX-Nuuk (2017)
57. Sara Lehné: Modeling and Measuring Teachers' praxeologies for teaching Mathematics (2017)
58. Ida Viola Kalmark Andersen: Interdiciplinarity in the Basic Science Course (2017)
59. Niels Andreas Hvitved: Situations for modelling Fermi Problems with multivariate functions (2017)
60. Lasse Damgaard Christensen: How many people have ever lived? A study and research path (2018)
61. Adonis Anthony Barbaso: Student Difficulties concerning linear functions and linear models (2018)
62. Christina Frausing Binau & Dorte Salomonsen: Integreret naturfag i Danmark? (2018)
63. Jesper Melchjorsen & Pia Møller Jensen: Klasserumsledelse i naturvidenskabelige fag (2018)
64. Jan Boddum Larsen, Den lille ingeniør - Motivation i Praktisk arbejdsfællesskab (2018)
65. Annemette Vestergaard Witt & Tanja Skrydstrup Kjær, Projekt kollegasparring på Ribe Katedralskole (2018)
66. Martin Mejlhede Jensen: Laboratorieforsøgs betydning for elevers læring, set gennem lærernes briller (2018)
67. Christian Peter Stolt: The status and potentials of citizen science: A mixed-method evaluation of the Danish citizen science landscape (2018)
68. Mathilde Lærke Chrøis: The Construction of Scientific Method (2018)
69. **Magnus Vinding: The Nature of Mathematics Given Physicalism (2018)**

***IND's studenterserie** omfatter kandidatspecialer, bachelorprojekter og masterafhandlinger skrevet ved eller i tilknytning til Institut for Naturfagenes Didaktik. Disse drejer sig ofte om uddannelsesfaglige problemstillinger, der har interesse også uden for universitetets mure. De publiceres derfor i elektronisk form, naturligvis under forudsætning af samtykke fra forfatterne. Det er tale om studenterarbejder, og ikke endelige forskningspublikationer.*
Se hele serien på: www.ind.ku.dk/publikationer/studenterserien/

The Nature of Mathematics Given Physicalism

Thesis project, University of Copenhagen

Written by Magnus Vinding, sph746

Supervised by Mikkel Willum Johansen

Abstract

This project aims to explore the nature of mathematics given a physicalist ontology. In particular, it seeks to explore whether and in what way mathematics can be accommodated within, and as nothing over and above, the physical world. The project takes point of departure in a physicalist account of the nature of mathematics presented in three articles by László Szabó, and proceeds to discuss the main problems and objections faced by this account in order to assess the plausibility of physicalist accounts of the nature of mathematics.

It is concluded that it indeed does seem possible to accommodate mathematics within a physicalist ontology and, more than that, that a physicalist account of the nature of mathematics in fact seems most plausible all things considered. This has unexpected and intriguing implications for the nature of mathematics, as such an account breaks down the widely accepted dichotomy between "mathematics" on the one hand and "the physical world" on the other. By extension, it also has the implication that mathematical knowledge is not fundamentally different from other kinds of knowledge of the physical world, and thus that belief in the universality of mathematics rests, in one sense at least, on an inductive assumption.

Table of Contents

Introduction

The question for the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in which direction it will find its final solution nor even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization.[1]

— Hermann Weyl

It is peculiar that mathematicians of all people, those who are engaged with questions concerning truth and consistency, accept a fundamentally inconsistent position with respect to mathematics and refuse to provide a satisfying answer to what the nature of mathematical truth in fact is.[2]

— Mikkel Willum Johansen & Henrik Kragh Sørensen

Why care about the nature of mathematics?

This is a question to which many reasonable answers can be given. For one, we use mathematics for many important purposes, from constructing buildings to modeling the future of the universe, which makes it seem quite relevant to know what the nature of this "thing" that we use for these many purposes indeed is, including how reliable it is. This may be considered somewhat of an "applied" reason to explore the question, and one that we should arguably all find compelling given the widespread application of mathematics.

Another reason we can give for exploring the question may be considered more of a "pure" one, namely that we want to understand the nature of mathematics for its own sake; because exploring and answering this question is of value in itself. This may be the answer the philosopher of mathematics prefers to give. Alternatively, we may wish to explore it because the nature of mathematics has direct implications for the very practice of mathematics itself, which may be the main reason why the pure

1 As quoted in (Mollin, 2010, p. 31).

mathematician would, or at least should, care about the question. As an example can be mentioned that the question of whether the continuum hypothesis has an actual answer depends on our view of the nature of mathematics. The continuum hypothesis was proven undecidable given the ZF axioms by Paul Cohen in 1963-64 (Cohen 1963; 1964), and given a formalist view of mathematics, this undecidability can be considered a final and satisfying answer (at least given the ZF axioms). Platonists, however, would seem bound to the position that the question *does* have an ultimate answer, and that Cohen's proof merely tells us that we need to explore other axiomatic systems in order to settle the matter (cf. Johansen & Sørensen, pp. 39-40). Thus, also for the pure mathematician, one may even say *especially* for the pure mathematician, considerations concerning the nature of mathematics are of great relevance.

Finally, a much less commonly invoked, yet no less compelling reason may be given, one that perhaps appeals most of all to the arch philosopher seeking to understand the very nature of existence itself. For over the entire course of the history of philosophy, philosophers have grappled with the question concerning the ultimate nature of the world, apparently without much success. Then, increasingly over the last few centuries, loose philosophical speculation has been replaced with evermore precise explanations provided by the sciences, concerning everything from the origin of the universe to the nature of biological life. Modern physics especially, with its precise and successful mathematical predictions on scales ranging from the smallest quantum event to galactic superclusters and beyond, seems to make a mockery of the philosopher's attempts to establish the ultimate nature of the world.

Or does it? For if our answer to the question concerning the ultimate nature of the world is that it is "all physics", and that physics ultimately is written in terms of mathematics, our quest for understanding the ultimate nature of the world would seem to then force us to ask what the nature of *this*, i.e. mathematics, is. In this way, having started out fumbling around in the realm of philosophy only to be referred onward to the realm of physics, the quest for the ultimate nature of reality appears to have returned us to a "philosophical question" in full, that question being: what is the nature of mathematics? To say that we understand the nature of the world in mathematical terms, yet that we do not know what mathematics itself is seems to leave our understanding of the world incomplete at best.

Why Physicalism?

---

2 My own translation; original version: "Det er forbavsende, at netop matematikere, der jo beskæftiger sig med spørgsmål om sandhed og konsistens, accepterer en grundlæggende inkonsistent holdning til matematikken og nægter at give et dækkende svar på, hvad matematisk sandhed egentlig er." (Johansen & Sørensen, 2014/2015, p. 93).

With this motivation, indeed this entire list of compelling motivations, for exploring the nature of mathematics laid down, what remains to be clarified with respect to the motivation for this thesis project is why one would seek to explore the nature of mathematics within the framework of a physicalist ontology — the doctrine that "everything is physical; all facts supervene on, or are necessitated by, the physical facts." (cf. Szabó 2017, p. 1).

To answer this question, it is perhaps worth taking a step back to answer another question that arguably seems more apt from our present point of view, namely why physicalism has *not* been the prevailing framework in which people have tried to address this question in the past? And one may here argue that a prominent reason is that for almost the entire history of the philosophy of mathematics, physicalism was, given what people knew at the time, far from being as appealing a framework in which to explain the nature of mathematics as it is at the present moment — a point in time whose uniqueness readily escapes us. For instance, it can be easy to forget, or at least fail to appreciate, that the structure of DNA was not discovered until 1953, and to forget what a revolution in our worldview scientific discoveries like this have effected over the last several decades; how differently people viewed the world before scientific revolutions of this sort compared to after.

For while it is now uncontroversial to say, in scientific circles at least, that we understand biology, including the human organism, to ultimately be all about physical processes in the physical world, this was *not* the prevailing view in the past. Just as Newton revolutionized our understanding of the world with his unification of the mechanics of the heavens and Earth by suggesting that the force that makes an apple fall is also what keeps the planets in orbit, modern science has now unified, or at least bridged the gap once thought to exist between living matter and dead matter, and explained the difference as a matter of degree, or one could say complexity and composition, rather than kind. We now understand that the phenomenon called life is not made of some different kind of stuff than, say, oceans and mountains; it is not animated by some *Élan vital* or life force, as was commonly — and admittedly rather naturally — supposed in the past. Perhaps most significantly, and also more recently, this unifying physicalist revolution in our understanding has finally also come to encompass our understanding of the human mind itself — its thoughts, emotions, mathematical intuitions, etc. — which modern neuroscience suggests, most counterintuitively, is mediated by physical processes in the brain. Again, this is far detached from how most people, including highly educated ones, used to think about the mind.[3]

3 See for instance (Pinker, 1997).

In sum, the story of uncovering the relationship we now understand to exist between physical, chemical, and biological systems and laws — e.g. that hereditary information in biological organisms is stored in the molecule DNA, whose properties is ultimately explainable in terms of quantum mechanics; that thoughts and emotions are mediated by delicate physical processes in the brain; etc. — is really quite a recent one. And thus it should be no surprise that pre-20th century philosophizing, including philosophizing about the nature of mathematics in particular, tended not to reflect this recently acquired understanding. This would seem to answer the question concerning the relative neglectedness of this framework in the past.

What remains somewhat mysterious, however, is why physicalism would seem so neglected today in discussions concerning the nature of mathematics, given that modern science arguably points us toward such a worldview on virtually every page of our textbooks. Indeed, one can argue that the question above concerning why one would take point of departure in a physicalist ontology is really put the wrong way around, as the question should perhaps rather be: why would one *not* take point of departure in a physicalist ontology when trying to explore the nature of mathematics given that our best understanding of the world appears to point us toward such an ontology?

Or to simply answer the original question — *why take point of departure in a physicalist ontology?* — in brief: because that seems the most reasonable thing to do in light of what we know about the world at this point in time. We appear to have a coherent story that holds that biology, including the human brain, ultimately supervenes on chemistry, which in turn supervenes on physics. Indeed, that these different branches of science merely represent different levels of description on which we explain the same physical reality. To believe otherwise seems tantamount to saying that our modern scientific worldview gets something very fundamental fundamentally wrong, which seems a strong and not easily justifiable claim. Thus, as hinted above, one can argue that, in light of our modern understanding of the world, the burden of explanation really lies upon those who wish to negate physicalism rather than on those who accept it.

Outline of the Project

This project aims to do exactly what has been motivated above, namely to explore the nature of mathematics given a physicalist ontology — in particular, to examine whether and in what way the nature of mathematics can be accommodated within a physicalist ontology. The project takes point of departure in a physicalist account of the nature of mathematics presented in three articles by László

Szabó (Szabó 2003; 2012; 2017), and then proceeds to discuss the implications of this view, including its strengths and weaknesses, in order to assess whether mathematics indeed can be accommodated within a physicalist ontology.

László Szabó's Physicalist Account of the Nature of Mathematics

László Szabó is a Hungarian physicist and philosopher who defends a physicalist view of the nature of mathematics, which he has presented and argued for in the two journal articles, "Formal systems as physical objects: a physicalist account of mathematical truth" (Szabó 2003) and "Mathematical facts in a physicalist ontology" (Szabó 2012), as well as in the book chapter, "Meaning, Truth, and Physics" (Szabó 2017). Collectively, these articles will serve as the foundation for the present examination of whether and how mathematics can be accommodated within a physicalist ontology, and thus I shall begin, in this chapter, by presenting the argument Szabó makes in these articles.[4]

The Physico-Formalist View of Mathematics

In his own words, Szabó's aim is to "naturalize mathematics" (Szabó 2012, p. 10). His view rests on the doctrines of physicalism, formalism, and empiricism, which he defines in the following way:

> *Physicalism:* everything is physical; all facts supervene on, or are necessitated by, the physical facts.
> *Empiricism:* genuine information about the world can be acquired only by a posteriori means.
> *Formalism:* logic and mathematics are thought of as statements about manipulations with meaningless symbols.
> (Szabó 2017, p. 1)

Szabó assumes these doctrines as initial premises without arguing for them, other than remarking that "[…] they are legitimate philosophical positions", and then seeks to discuss the consequences that follow concerning the nature of mathematics (Szabó 2017, p. 2). However, Szabó maintains that these

4 This presentation of Szabó's argument does not present all the arguments made in the articles above, as these articles, especially (Szabó 2017), also make arguments that are not directly relevant to Szabó's argument concerning the

three doctrines are not independent, and that one need only assume physicalism to arrive at them, as he argues that the two other doctrines, empiricism and formalism, are ultimately corollaries of physicalism (Szabó 2017, p. 2). The reason Szabó considers empiricism an implication of physicalism will be explained in greater detail below, but as for why he thinks formalism follows from physicalism, he writes:

> Physicalism denies the existence of mental and abstract entities; consequently, there is no room left for any kind of platonism or mentalism in the philosophy of logic and mathematics. Therefore, […] formalism […] seems to be the only account for logic and mathematics that can be compatible with physicalism.
> (Szabó 2017, p. 2)

Szabó calls his view the physico-formalist interpretation of mathematics, and views it as "[…] a reflection to the following fundamental problem: If physicalism is true, then the logical/mathematical facts must be necessitated by the physical facts of the world." (Szabó 2017, p. 2). In other words, if *everything* is physical, and if *all* facts are necessitated by physical facts, then this must also hold true of mathematics and mathematical facts respectively: mathematics must be physical, and mathematical facts must be necessitated by physical facts. This, Szabó argues, raises the crucial question of *how* — "How can the logical and mathematical facts be accommodated in a purely physical ontology?" (Szabó 2017, p. 2). A question that, in turn, forces us to first provide a clear answer to the question: "What is it that has to be accounted for within a physicalist ontology? What are the logical/mathematical facts?" (Szabó 2017, p. 2).

Szabó here, as mentioned above, takes formalism as his starting point, and quotes a definition of formalism from David Hilbert: "Mathematics is a game played according to certain simple rules with meaningless marks." (Szabó 2017, p. 2). That is, mathematical truths are just truths about what theorems can be derived from what formal systems, where the statements in the formal system are neither true nor false in and of themselves, and the process of derivation is viewed as "[…] a mechanistic operation with meaningless strings of symbols." (Szabó 2017, p. 3).

---

nature of mathematics per se. Also, it should be noted that there is a large overlap between the three articles, which all argue for the same view of the nature of mathematics in a similar way.

Having provided this answer to the question of what it is he is trying to account for within a physicalist ontology, Szabó goes on to formulate his physico-formalist thesis:

> **The physico-formalist thesis:** A formal system should be regarded as a physical system which consists of signs and derivational mechanisms embodied in concrete physical objects, concrete physical configurations, and concrete physical processes.
> (Szabó 2017, p. 3)

Therefore, Szabó argues, a mathematical truth such as that a particular theorem can be derived from a given formal system is a truth that "[…] expresses an *objective fact* of the formal system as a particular portion *of the physical world*." (Szabó 2017, p. 3).

Szabó argues for this claim in three steps, where the first is to assert the premise that a formal system can be represented in a physical system. As an example, he describes a CD in a laptop which contains a program that lists the theorems of a formal system, and considers it commonly accepted that "[…] in the 'computer + CD' system we have 'a physical representation of the formal system' in question." (Szabó 2017, p. 3).

Next, as the second step, Szabó argues that "All mathematical facts can be thought of as a physical fact in some physical representation." (Szabó 2017, p. 4). The reason he gives for this is that all the mathematical truths we know and ever will know are known via representations in the concrete physical world — in the form of the states of computers and brains, say (Szabó 2017, p. 4). Finally, in the third step, Szabó argues that the formulations in the two preceding statements are not entirely accurate, as there is no real representation of anything in the first place. There are just the concrete physical objects themselves:

> Actually, there is nothing to be "represented"; there is nothing beyond the flesh and blood formal systems. That is to say, a formal system "*as* formal system" is a part of the physical reality. Consequently, any statement about a formal system – including a statement like "$\Sigma \vdash A$" – is a statement of a physical fact; and it has exactly the same epistemological status as any other statements about the physical world.
> (Szabó 2017, p. 4)

The Physico-Formalist View and Physical Theories

Szabó is careful to distinguish his physico-formalist view from the immanent or physical realism of the kind defended by John Stuart Mill, according to which mathematical concepts and propositions refer to features of the physical world in a direct way. On the physico-formalist view, in contrast, mathematical concepts and propositions do not carry any such meaning per se (Szabó 2012, p. 2).

As an example, Szabó considers the Pythagorean theorem. On Mill's view, this theorem reflects a general truth about real triangles in the physical world, whereas the physico-formalist view merely views it as a provable theorem within the formal system that is Euclidean geometry. On the physico-formalist view, this still does express something about the physical world, yet "[…] this something has nothing to do with the physical triangles and rigid bodies." Instead, it is "[…] a property of the physical system consisting of the signs and the derivation mechanisms [of Euclidean geometry]." (Szabó 2017, p. 6). And whether Euclidian geometry then describes (aspects of) the physical world is considered a separate, empirical question — a question of physics.

More generally, Szabó draws a distinction between two different kinds of truth, where truth in the first sense, Truth1, concerns whether a theorem is true within a particular formal system, while Truth2 concerns whether a theorem, when combined with a certain semantics of a physical theory, holds true of the physical world, and proceeds to clarify the distinction with the following example:

> For example, "The electric field strength of a point charge is kQ/r^2" is a theorem of Maxwell's electrodynamics—one can derive it from the Maxwell equations. (This is a fact of the formal system consisting of the symbols and the derivation rules.) On the other hand, according to the semantics relating the symbols of the Maxwell theory to the empirical terms, this sentence corresponds to an empirical fact (about the point charges). Truth1 and Truth2 are independent, in the sense that one does not imply the other.
> (Szabó 2003, pp. 118-119)

Yet Szabó stresses that a distinguishing feature of the physico-formalist view of mathematics is that even truths of the first kind, truths concerning formal systems and nothing else, are about physical objects, and thus that, according to this view, "[…] even Truth1 is of empirical nature, the factual content of which is rooted in our experiences with respect to the formal system itself." (Szabó 2003, p. 124).

The Nature of Mathematical Knowledge According to Szabó

This leads us to Szabó's contention that empiricism — the view that "genuine information about the world can be acquired only by a posteriori means" — is an implication of physicalism, and to the implications of the physico-formalist thesis for the nature of mathematical knowledge more generally (Szabó 2017, p. 2). As we have seen above, Szabó argues that statements about formal systems are statements about the physical world, and thus, according to Szabó, such statements have "[…] exactly the same epistemological status as any other statements about the physical world." (Szabó 2017, p. 4). This status being: we have to observe them in order to know them. Thus, on Szabó's account: "Mathematics is, in this sense, empirical science." (Szabó 2003, p. 124).

Szabó argues that this view has an important and unexpected implication for the nature of deduction in general, namely that deduction is a form of induction — a form of fallible generalization from particular instances — as deductive inferences, on this view, always generalize based on the observations of the outcomes of particular physical processes (Szabó 2003, p. 124). This then raises the problem of explaining the near-universal conviction that mathematical knowledge is not inductive and not, in contrast to inductive knowledge, uncertain (Szabó 2012, p. 11). Szabó quotes British philosopher A.J. Ayer's statement of this problem, a problem that, according to Ayer, faces any empiricist view of the nature of mathematical knowledge:

> For whereas a scientific generalization is readily admitted to be fallible, the truths of mathematics and logic appear to everyone to be necessary and certain. But if empiricism is correct[,] no proposition which has a factual content can be necessary or certain. Accordingly[,] the empiricist must deal with the truths of logic and mathematics in one of the following ways: he must say either that they are not necessary truths, in which case he must account for the universal conviction that they are; or he must say that they have no factual content, and then he

> must explain how a proposition which is empty of all factual content can be true and useful and surprising. […]
>
> If neither of these courses proves satisfactory, we shall be obliged to give way to rationalism. We shall be obliged to admit that there are some truths about the world which we can know independently of experience; […]
>
> (Ayer, 1952, p. 72)

As hinted above, Szabó chooses the first of the two possible responses proposed by Ayer: since, according to Szabó, logical and mathematical truths express objective facts about certain parts of the physical world, our knowledge of such facts is "[…] synthetic, *a posteriori*, not necessary, and not certain." (Szabó 2017, p. 6). As a first response to the problem of why mathematical knowledge then appears necessary and certain, Szabó argues that many aspects of our everyday knowledge seem necessary and certain, although they are obtained via inductive generalization:

> Break a long stick. We are "sure" about the outcome: the result is a shorter stick. This regularity of the physical world is known to us from experiences. The certainty of this knowledge is, however, no less than the certainty of the inference, say, from the Euclidean axioms to the height theorem.
>
> (Szabó 2012, p. 11)

Both these observations, Szabó argues, rest equally on an underlying regularity of the physical world, and, although it may be very small, there is uncertainty in both cases: "[…] our knowledge about the deductive (physical) process, the outcome of which is the height theorem, is uncertain, no matter how many times we repeat the observation of this process." (Szabó 2003, p. 124). Szabó further argues that since logical and mathematical truths express objective facts about certain parts of the physical world, it follows that they can be, in Ayer's words, true and useful and surprising, and that they can, indeed *must*, be discovered, like other facts of nature (Szabó 2017, p. 6). Finally, Szabó notes that:

> The fact that the flesh and blood formal systems usually are simple physical systems of relatively stable behavior, like a clockwork, and that the knowledge of logical and mathematical

> truths does not require observations of the physical world external to the formal systems explains the universal *illusion* that logical and mathematical truths are necessary, certain and *a priori*.
> (Szabó 2017, p. 7)

In other words, although mathematical facts are not necessary, in one sense at least, they are nonetheless necessitated by physical facts (the core premise of Szabó's argument), and these facts happen to be highly regular. In this way, physicalism can, according to Szabó, also "[…] resolve the long-standing debate surrounding the truth-of-reasoning versus truth-of-facts dichotomy." (Szabó 2017, p. 7). He explains this proposed resolution to the rationalist-empiricist debate in the following way:

> The age-long rationalist–empiricist debate is based on the delusion that reasoning can deliver us truth of higher degree of certainty than inductive generalization. As we have seen, mathematical and logical truth is nothing but knowledge obtained through inductive generalization from experiences with respect to a particular physical system, the formal system itself. Since mathematical and logical derivation is reasoning par excellence, one must conclude that there is no higher degree of certainty than the one available in inductive generalization.
> (Szabó 2012, p. 11).

Or as he puts it elsewhere: "[…] we must draw the epistemological conclusion: There is no *higher* degree of certainty than available from experience." (Szabó 2017, p. 7).

The Objectivity of Mathematics

Szabó explicitly clarifies whether and how mathematics is objective on the physico-formalist view, whereby he also proposes an answer to the age-old question of whether mathematics is invented or discovered. He argues that formal systems are physical objects created by humans, and as such, "[…] they are not eternal and not readily given to us." (Szabó 2017, p. 7). Yet he also argues that, in another sense, given that we have specified a certain formal system, the particular properties of this (physical) system are not up to us, and thus, according to Szabó, mathematical truths are objective in that they

"[…] express objective facts of a particular part of the physical world, namely, the facts of the formal systems themselves." (Szabó 2012, p. 11).

So we are, according to Szabó, nonetheless constrained, as we cannot just create anything with arbitrary properties. We cannot, for instance, choose whether a set of axioms we have formulated are consistent or not, nor can we formulate an arbitrarily long list of axioms with the finite time and finite computational power available to us. As Szabó puts it: "The objective features of physical reality predetermine what can be created and what cannot." (Szabó 2017, p. 7), a statement he explains with the following analogy:

> For example, even if we assume that there are no polyvinyl chloride molecules in the universe, except the ones created by mankind, the laws of nature admit the possibility of their existence and predetermine their properties. Similarly, the laws of nature predetermine what kinds of formal system can exist. In this sense, the logical and mathematical facts are eternal and independent from us. They are contingent as much as the laws of nature are contingent.
> (Szabó 2017, p. 7)

Having presented László Szabó's physicalist account of the nature of mathematics, we shall now proceed to undertake a critical examination of it. This we shall do by in turn raising potential problems and objections that may speak against it, and examine whether these problems can be addressed within a physicalist framework.

Does Physicalism Deny the Existence of the Mental?

As mentioned in the previous chapter, Szabó maintains 1) that physicalism denies the existence of the mental, and 2) that formalism, in part for this first reason, "[…] seems to be the only account for logic and mathematics that can be compatible with physicalism." (Szabó 2017, p. 2). These two claims shall be scrutinized in this chapter.

Concerning the first claim, one may object that Szabó's denial of the existence of the mental appears self-undermining, since all our knowledge, including our mathematical knowledge, arguably is known *in* consciousness, *as* mental phenomena. Thus, to deny the existence of the mental seems inconsistent with our experience and knowledge of mathematics, and indeed the existence of our minds in the first place. The question is whether Szabó's argument, and a physicalist account of mathematics more generally, can be saved from this objection.

The first thing to note here is that it is indeed misleading when Szabó writes that physicalism denies the existence of mental entities, since this claim is not true of all physicalist positions that have been proposed. A counter-example is the physicalist position defended by British philosopher Galen Strawson (Strawson, 2006), sometimes referred to as Strawsonian physicalism.[5] This view entails the explicit denial of Szabó's claim above, as it holds the existence of the mental, i.e. consciousness, to be undeniable, while also maintaining that the world conforms to description in terms of the equations of physics, and that everything that exists is "[…] spatiotemporally (or at least temporally) located." (Strawson, 2006, p. 3). Thus, Szabó's claim that physicalism denies the existence of the mental appears unwarranted, and if he were to justify it, he would need to explain why physicalist positions that do not deny the existence of the mental, such as Strawsonian physicalism, are untenable.

---

5 A view that can be traced back to the view expressed by Bertrand Russel in his *Analysis of Mind* (Strawson, 2018).

Does Physicalism Exclude Non-Formalist Accounts of Mathematics?

Since Szabó rests his claim that physicalism implies formalism on a premise he does not justify, it is natural to doubt this claim as well — that only formalism is consistent with physicalism, which indeed seems a strong and problematic claim. For example, one may argue that, according to Strawsonian physicalism, mentalist views of mathematics can be as valid as, indeed equivalent to, Szabó's physico-formalist view. For instance, one intuitionist definition of mathematics defines it as: "[…] the mental activity which consists in carrying out constructs one after the other." (Campbell, 1984, p. 187). And according to Strawsonian physicalism, there need not be a conflict between this definition of mathematics and then Szabó's view that formal systems (which on his view comprise all of mathematics) "[…] should be regarded as a physical system which consists of signs and derivational mechanisms embodied in concrete physical objects, concrete physical configurations, and concrete physical processes." (Szabó 2017, p. 3). On Strawsonian physicalism, "the physical" and "the mental" are ultimately just two different modes of description that pertain to the same reality (Strawson, 2006, p. 6). The mental activity of carrying out constructs one after the other *is* a concrete physical process on the Strawsonian view, and hence describing mathematics in terms of the former can be fully compatible with describing it in terms of the latter. Thus, in light of Strawsonian physicalism, it appears false to claim, as Szabó does, that only a non-mental formalist account of mathematics is consistent with physicalism.

Beyond that, it should also be noted that there are physicalist accounts of the nature of mathematics which do not appear to rest on or support formalism. One such example is the explicitly physicalist account of mathematics defended by philosopher John Bigelow in the book *The Reality of Numbers: A Physicalist's Philosophy of Mathematics* (Bigelow, 1988), which maintains that universals have real existence in the concrete physical world, and holds these universals to be the core subject of mathematics: "Mathematics is the theory of universals" (Bigelow, 1988, p. 13). I shall not delve into Bigelow's view here, but merely let it suffice to note that Bigelow's view also constitutes a counter-example to Szabó's claim that non-mental formalism seems the only account of mathematics that can be compatible with physicalism. So as in the case of claiming that physicalism entails the non-existence of the mental, Szabó here again makes a non-obvious claim that he does not justify, and he again fails to note that others have defended views that contradict what he claims.

Where does this leave us with respect to Szabó's physicalist account of the nature of mathematics, as well as physicalist accounts of the nature of mathematics more generally? Concerning the first objection above — that the denial of the mental seems self-undermining — it seems that one can save physicalist accounts of mathematics by adopting a version of physicalism that does not deny the existence of consciousness, such as Strawsonian physicalism. This would also seem to save Szabó's particular account from this objection. For although Szabó claims that physicalism denies the existence of the mental, this denial does not seem crucial for his physico-formalist thesis that formal systems should be regarded as concrete physical objects (cf. Szabó 2017, p. 3). That is, Szabó's account does not seem to rest on the claim that the physical is not mental, as Strawson and a significant number of other physicalist philosophers maintain it indeed is (Strawson, 2006, p. 6; Strawson, 2018).

As for the second point — that Szabó does not justify his claim that only formalism seems consistent with physicalism — one can also argue that this omission does not, in itself, count as a reason to reject Szabó's view. For the fact that Szabó does not justify his claim that *only* formalism is consistent with physicalism does not suggest that formalism is not indeed both a consistent and compelling account of the nature of mathematics given physicalism, or that it might not indeed be the most compelling one. What we have seen above is merely that 1) Szabó's argument for this claim rests on an unjustified premise (i.e. that physicalism denies the existence of the mental), and 2) that one can point to examples of physicalist accounts of the nature of mathematics that do not rest on a formalist conception of mathematics. Concerning this latter point, one can argue that it is a strength for physicalism in general that there seems to be a diversity of paths one can pursue toward providing a physicalist account of the nature of mathematics, suggesting that even if the particular account provided by Szabó falls short, this need not imply the failure of physicalist views of the nature of mathematics altogether.

Whether Szabó's account is indeed satisfying, specifically whether formalism is a satisfying account of what mathematics is, is what shall concern us in the following chapter.

Is Formalism a Satisfying Account of What Mathematics Is?

As we have seen, Szabo's account rests on a formalist definition of mathematics: "*Formalism:* logic and mathematics are thought of as statements about manipulations with meaningless symbols." (Szabó 2017, p. 1). Yet it can be disputed whether this is indeed a satisfying account of that which we call "mathematics".

There are various criticisms that can be raised against the formalist definition of mathematics, i.e. against whether this definition indeed sufficiently covers what we refer to as "mathematics". One such criticism is historical: what we today recognize as rigorously formalized mathematics only emerged in relatively recent times, arguably not before the 19th century (cf. Lützen, 2011, pp. 234-235), so would a formalist definition of mathematics imply that people were not doing mathematics before that? Were Archimedes, Newton, and Euler not doing mathematics on this definition? If so, can this definition of mathematics be considered at all sensible?

A similar such criticism is that the formalist conception of mathematics can appear rather detached from the actual practice of mathematicians, not only in the past, before mathematics became increasingly formalized, but also in this day and age, even as mathematicians do aspire to a high degree of formalization. For example, Field's medal winner William Thurston, in describing his own experience as a working mathematician in the late 20th century, emphasized the many-faceted process that modern mathematics is. He described the furthering of human understanding as a, if not *the* prime goal of modern mathematical practice, and underscored the importance of the variety of means mathematicians use to acquire this understanding — diagrams, ordinary human language, mental models, etc. — as well as the many motivations that animate what they do, such as deeper mathematical understanding, institutional incentives, and prestige (Thurston, 1994, pp. 164-165; pp. 171-172).

Elaborate studies of mathematicians' perspectives on their own research indicate that Thurston was not alone in this view, as mathematicians in general report that seemingly non-formalist aspects of mathematical practice, such as material representations and social processes, play a crucial role in modern mathematical practice (see e.g. Johansen & Misfeldt, 2018). In other words, if we ask practicing mathematicians themselves, a much more nuanced and many-sided view of mathematics seems to emerge compared to what the simple formalist definition captures.

Yet, interestingly enough, Thurston himself nonetheless still seemed very close to preferring a formalist definition of mathematics as most satisfying:

> Mathematicians generally feel that they know what mathematics is, but find it difficult to give a good direct definition. It is interesting to try. For me, "the theory of formal patterns" has come the closest, but to discuss this would be a whole essay in itself.
> (Thurston, 1994, p. 162)

So in spite of the many-faceted picture of modern mathematical practice that Thurston himself paints, a picture that indeed finds strong support in more systematic surveys of said practice, he still felt that something close to a formalist definition of mathematics is the most accurate. What are we to make of this? More generally: can the formalist definition perhaps be considered satisfying after all?

Two Senses of the Term "Mathematics"

Some terminological clarification and discussion seems called for here. In particular, it seems necessary to distinguish two very different senses of the term "mathematics". For just as there is a clear distinction between "physics" as the science and practice conducted by physicists, and then "physics" as the object that physicists describe, i.e. the physical world, we should be careful not to confuse "mathematics" as *the science and practice performed by* mathematicians, with "mathematics" in the sense of *the object* that mathematicians study.[6]

With this distinction in mind, it seems possible to make some sense of the apparent conflict above. For one, we can start by conceding that a formalist conception of mathematics obviously does not capture what we mean by mathematics in the first sense above. What mathematicians *do* as practicing mathematicians clearly cannot be exhaustively described merely as "making statements about manipulations with meaningless symbols", as there is evidently much more to the practice of mathematicians than this, even if it may be considered an essential part. Yet this is a separate matter from the question of what the object of the science of mathematics is. Can mathematics in this sense be "thought of as statements about manipulations with meaningless symbols"? At a first glance, this at

6 I use the term "object" in a wide sense, and I use this term on purpose, since defining this "object" in more particular terms, e.g. "a body of theorems", can be deemed question begging: it risks assuming too much from the outset about "the thing" (broadly construed) mathematicians study.

least seems plausible, one reason being that the objections mentioned above concerning how mathematics was not explicitly formalized in the past, and the fact that mathematical practice has many different facets, many of which appear rather remote from purely formalized mathematics, are not necessarily in tension with this definition of the object of mathematics.

In response to the objection that we cannot be satisfied with a definition of mathematics that renders pre-19th century mathematics "not real mathematics", one can say that Szabó's formalist definition in fact does no such thing. For although mathematicians did not think about what they were studying in purely formal terms in the past, it remains true, one may argue, that what they were studying still *could be thought of* in these terms; that the object of study that these mathematicians were concerned with at least could be formalized in principle.[7] And the same may be said in response to the objection that mathematicians still do not — at least not always, and perhaps only rarely — think about mathematics in purely formal terms, and that their practice involves much more than merely making statements about the manipulation of meaningless symbols. For even if mathematicians do not think about the object of their studies purely in these terms, it may nonetheless be maintained that this object always can, if not in practice then at least in principle, be thought of in these terms; i.e. that it conforms to formalization. Indeed, this seems close to the view expressed by Thurston himself, who was careful to point out the many diverse ways in which mathematicians think about and practice mathematics, while still seeming to prefer a formalist definition of mathematics.

One may, of course, question whether it can at all be defended to draw a distinction between the object mathematicians study and the practice of mathematicians. Yet it seems implausible to say that it cannot, since saying that the object and the practice are indistinguishable is tantamount to saying that someone who studies the practice of mathematicians is necessarily also a mathematician, which is clearly not the case. Unlike sociologists and historians of mathematics, mathematicians do not study and write papers about the practice of mathematicians, not primarily at least. They clearly study something else. What is true to say, however, is that what mathematicians study, and how, is by no means independent of the practice of (other) mathematicians, since the particular formal systems (or formal*izable* systems) any given mathematician chooses to study, and how, is no doubt strongly influenced by the practice of other mathematicians. In this way, the sentiment expressed above indeed does contain a grain of truth: although we can reasonably distinguish between the object mathematicians study and the practice of mathematicians, it is true to say that the two are closely related.

Satisfying for What Purpose?

Whether a given definition of a term is satisfying will, of course, depend on what our aim is. Thus, it seems worth clarifying for what purpose it is that we need this definition of mathematics to be "satisfying" or "sufficient" in this context: it is for the purpose of clarifying whether mathematics can be accommodated in the physical world. And with respect to this aim, it can be argued that the formalist definition is satisfying, at least to the extent that all of that which we recognize as part of the object of the science of mathematics indeed can be formalized. For if we can account for how formal systems can be accommodated within a physicalist ontology, as Szabó attempts to, and we believe all of mathematics can be formalized, at least in principle, it seems that we have thereby accounted for how all of mathematics can, at least in principle, be accommodated within a physicalist ontology.

This may be considered somewhat analogous to showing something about one algebraic object (comparable to showing that formal systems can be accommodated within a physicalist ontology), then establishing that this object is in fact isomorphic to another class of objects (comparable to establishing that all of that which we recognize as an object of mathematics can be expressed in a formal system), and thus generalize our original conclusion to this broader class (analogous to drawing the conclusion: all of that which we recognize as mathematics can be accommodated within a physicalist ontology).

Clarifying whether and, at least roughly, how such physicalist accommodation is possible is, it should be remembered, precisely what Szabó attempts to do, and also our primary concern in this project. In other words, Szabó's aim (as well as ours) pertains to a question of fundamental ontology — can we account for mathematics within a purely physicalist ontology? The aim is not to give a detailed account of how mathematics as we now practice it emerged, which would, of course, require a more elaborate, multidisciplinary approach than the one Szabó pursues. Szabó's core claim is *that* formal systems are physical objects created by humans (Szabó 2017, p. 7). He does not attempt to fill in the details, and thus refrains from saying anything about questions such as how humans create these formal systems (or at least formal*izable* systems), what enables us to make them, and with what purpose we do it. Such a project would involve contributions from various fields, including evolutionary biology, cognitive science, and the social sciences (cf. Johansen, 2010), as the work any given mathematician is able and motivated to do depends on a myriad of factors, including psychological, economic, and historical ones.

---

7 By analogy, one could say that even though pre-20th century physicists did not think about what they were studying as being a quantum field, what they were studying could nonetheless, according to modern physics at least, be described in those terms.

Such a multidisciplinary account would not be in tension with Szabó's project, however, but rather complementary to it. Indeed, the project of placing the object of mathematics within a physicalist ontology itself arguably comprises a centerpiece of this larger project of investigating mathematics within a scientific framework. For these more detailed, empirical investigations — indeed naturalistic accounts of mathematics in general — arguably all, more or less tacitly, rest on a physicalist framework, and thus these investigations, and the project of naturalizing mathematics more generally, can hardly be considered complete in the absence of a physicalist account of the nature and place of the object of the science of mathematics itself.

Formalism and the Effectiveness of Mathematics in the Sciences

An objection that has been raised against the formalist conception of mathematics, and thus against the claim that formalism can be considered a satisfying view of mathematics, is that it faces a problem in explaining why manipulations with meaningless symbols should have so many real-world applications. Specifically, as Stewart Shapiro skeptically asks: "Why are mathematical games so useful in the sciences?" (Shapiro, 2000, p. 146). Or to use the words of Eugene Wigner (Wigner, 1960), how can we reconcile the (purportedly) "unreasonable effectiveness of mathematics" with formalism?

A first thing worth noting here is that, even if we were to concede that mathematics is unreasonably effective at describing the natural world, it is not clear why this should pose a problem for the relatively weak claim that mathematics *can always be thought of* as being about manipulations with meaningless symbols (i.e. that all mathematics is, in principle, formalizable), compared to the stronger claim that mathematics *only* can be thought of in this way, for which it is more plausible that such effectiveness poses a problem. Whether there indeed is such an effectiveness is then the next question.

For the premise that mathematics is all that effective in the sciences in general has in fact been strongly disputed. For example, economist Vela Velupillai paints the opposite picture of the utility of mathematics for the science of economics in his article "The unreasonable ineffectiveness of mathematics in economics" (Velupillai, 2005). Similarly, philosopher of science Roberto Poli has presented a series of lectures entitled "The unreasonable ineffectiveness of mathematics in cognitive sciences" (Poli, 1999); mathematician Jeremy Gunawardena has spoken about "The unreasonable ineffectiveness of mathematics in computer engineering" (Gunawardena, 1998); and mathematician and Kyoto Prize winner Israel Gelfand, whose mathematical work contributed both to physics and biology, is quoted as saying:

> Eugene Wigner wrote a famous essay on the unreasonable effectiveness of mathematics in natural sciences. He meant physics, of course. There is only one thing which is more unreasonable than the unreasonable effectiveness of mathematics in physics, and this is the unreasonable ineffectiveness of mathematics in biology.
> (As quoted in Petrovici, 2016, p. 82)

So it seems that Shapiro's general claim that mathematics is "so useful in the sciences" may be greatly exaggerated; it at least appears to be according to a number of scientists and mathematicians.

Yet we still seem left with the mystery of why mathematics is unreasonably useful within the particular science of physics, as Wigner argued it is, and Gelfand seemed to concede. Interestingly, Szabó himself takes up this supposed objection against formalism, and responds as follows:

> Let me start with mentioning that it is not mathematics *alone* by which the physicist can predict what will happen, but physical axioms and mathematics together. The physical axioms are determined by empirical facts. More exactly, the physicist, keeping, as long as possible, the logical and mathematical axioms fixed, *tunes* the physical axioms such that the theorems *derivable* from the unified system of logical, mathematical, and physical axioms be compatible with the empirical facts. Consequently, the employed logical and mathematical structures in themselves need not reflect anything about the real world in order to be useful.
> (Szabó, 2017, p. 14)

This fits with an aspect of Szabó's view we saw described in an earlier section ("The Physico-Formalist View and Physical Theories"), namely that we can talk about formal systems and truths about these in and of themselves, what he calls Truth1, and then we can talk about which formal systems that, when combined with the semantics of a certain physical theory, happen to describe the physical world, a description that constitutes truth in a very different sense, i.e. Truth2. And Szabó's point is that it is false to claim that mathematical statements — i.e., on his account, statements about formal systems — *per se* describe the physical world. Rather, we only get "mathematics that describes the physical world" after we have gone through a process of carefully refining and selecting, based on empirical investigation, the mathematics/formal system that, when combined with empirically determined physical axioms, turns out to approximate aspects of the physical world reasonably well.

Thus, on this view, to say that "mathematics describes the physical world miraculously well" is akin to claiming that "words describe the world around us miraculously well". In the case of the latter "miracle", we realize that words only "describe reality so well" because we over thousands of years have developed a language for this very purpose of describing the world — among other purposes, of course — *and* because we carefully choose to put the right words of this language together in the right ways. There is hardly any miracle to be found in this story, and a similarly unmiraculous story can arguably be told about the effectiveness of mathematics within physics too.

Indeed, the general statement that "words describe the world so well" is barely accurate or even that meaningful. As hinted above, only *the right words within a given language put together in the right ways* can reasonably be claimed to describe reality well. And the same may be said of the effectiveness of mathematics within physics: it indeed makes little sense to claim that mathematics, or formal systems, *in general* describe the physical world. Echoing Szabó's claims above, only a meticulously chosen mathematical theory can, when combined with physical axioms and applied to a carefully chosen aspect of physical reality, be deemed effectively descriptive. This is, for example, evident from the fact that classical mechanics rests on Euclidian geometry while general relativity rests on non-Euclidian geometry, which has very different mathematical properties. Or, to take another example, that classical mechanics models reality as fundamentally continuous while quantum mechanics models it as fundamentally discrete. Just as we must be careful to pick the right words to describe the world, we must be careful — which, among other things, means being guided by empirical investigation — in order to be able to pick the right mathematical theory that describes a given aspect of physical reality.

Historian of mathematics Jesper Lützen proposes a similar answer to the conundrum of the unreasonable effectiveness of mathematics within physics in his article "The Physical Origin of Physically Useful Mathematics" (Lützen, 2011). Lützen argues that the usefulness of the formal systems we employ to describe the physical world is not that unreasonable in light of a closer examination of the historical development of mathematics — a development, Lützen notes, in which formalization has been an important, yet in a sense post hoc part (cf. Lützen, 2011, pp. 234-235). For physics, Lützen argues, has influenced the development of mathematics strongly since antiquity and up to modern times, an influence that includes which axioms and aspects of mathematics that mathematicians have chosen to investigate, which renders the usefulness of this mathematics appear a lot less unreasonable than Wigner supposed (Lützen, 2011, pp. 237-238).[8]

In light of these considerations, it does not seem like the effectiveness of mathematics within physics poses a problem for the formalist conception of the object of mathematics — especially not for the weaker formulation: that the object of the science of mathematics *can be thought of* as manipulation with meaningless symbols — since formal systems are not effective at describing the physical world in general. Such effectiveness only emerges after we have selected particular formal systems and combined them with empirically determined physical axioms and parameters in order to make the resulting combined theory as effective as possible at describing a particular part or aspect of the physical world. In light of this selection process aimed at finding mathematics that effectively describes the physical world, we should arguably not be that surprised that we have, at least in many cases, found something close to what we have actively been searching for.

In sum, in order to say whether a given definition of mathematics is satisfying, it is necessary to clarify what sense of the term "mathematics" we want covered in a satisfying way (e.g. "the practice of mathematicians" versus "the object that mathematicians study"), and in order to determine what "satisfying" means in this context, it seems that we further need to clarify what purpose we need the definition to be satisfying for. And when we specify this, we find that Szabó's definition indeed plausibly can be considered satisfying for the purpose that concerns us here. For the purpose of clarifying whether and how (at least roughly) the object of the science of mathematics can be accommodated within a physicalist ontology, it seems satisfying to adopt a definition of mathematics that describes it as something that is, or at least always can be, "thought of as statements about manipulations with meaningless symbols" (Szabó 2017, p. 1), as it seems plausible that everything that could reasonably be considered part of the object of mathematics is amenable to formalization. Thus, as argued above, by accounting for how formal systems can be accommodated within a physicalist ontology, one will arguably thereby have accounted for how the object of mathematics proper can be accommodated within a physicalist ontology, and thus also demonstrated the superfluousness of

---

8 Interestingly, Lützen also comments on the formalist philosophy of mathematics, where he, as I read him, comes very close to the view expressed by Thurston: that mathematics, as a science, is clearly not practiced in accordance with a naive formalist conception of mathematical practice, where one thinks purely in terms of manipulations with meaningless symbols, yet he still seems to agree that mathematics is formalizable, and views formalization as a kind of hygiene we can subject our mathematical theories to after they have been developed, cf. the section "Formalism as hygiene" in (Lützen, 2011, pp. 234-235).

invoking more than the physical world in order to make room for the object of the science of mathematics.[9]

---

9 It should be noted again, though, that the possibility of a physicalist account of the nature of mathematics need not, as argued in the previous chapter, be predicated on formalism.

Is Szabó's Account of the Nature of Mathematical Knowledge Satisfying?

In this final chapter we shall explore the conclusions Szabó draws concerning the nature of mathematical knowledge. In particular, we shall examine whether the conclusion that mathematical knowledge is empirical and inductive constitutes a reason to reject Szabó's view.

Does Mathematical Knowledge Really Appear Certain?

As mentioned earlier, the main objection that seems to arise for any empiricist account of the nature of mathematical knowledge is the one raised by A.J. Ayer: if mathematical knowledge is inductive and uncertain, why does it, according to Ayer, appear otherwise to everyone? If mathematical truths "[…] are not necessary truths, […] [we] must account for the universal conviction that they are." (Ayer, 1952, p. 72).

Various replies can be given to this objection. A reply Szabó himself does not pursue is to question whether mathematical knowledge indeed does appear so certain in the first place. For example, most students of mathematics will probably recognize the feeling of being certain of some supposedly valid deduction only to later realize that it was, in fact, invalid. Or, perhaps more saliently for social creatures like ourselves, many of us have observed a fellow student or colleague publicly display such deductive certainty during a lecture only for them to realize that their certainty was entirely unwarranted.

And examples of this kind of unwarranted deductive certainty can also be found among the greatest of mathematicians publishing in the most esteemed of journals. For example, Carl Friedrich Gauss published an alleged proof of the fundamental theorem of algebra in 1799, a proof that was widely recognized as the first rigorous proof of this theorem for many years, including by Gauss himself, although the proof was in fact incomplete; it was not completed before 1920, by Alexander Ostrowski (Smale, 1981, pp. 4-5). A similar example would be the alleged proof of the four-color theorem published by Alfred Kempe in 1879 (Kempe, 1879), a proof that was believed to be valid by many mathematicians until, a full 11 years later, it was shown invalid by Percy Heawood (Ball, 1892/1905, p. 46). For a more modern example, we may take Andrew Wiles' proof of Fermat's last theorem, which he first presented at a conference in June 1993 (Kolata, 1993). Yet despite Wiles' belief that his proof was valid, it turned out, only two months later, that it was in fact incomplete, and it took him more than a year to complete it; the final proof that eventually satisfied the mathematical community was published in two papers in May 1995 (Scientific American, 1999).

In light of these examples, it is difficult to make sense of the conviction that mathematical knowledge is certain, as these examples make such a conviction appear wholly unwarranted. One may, of course, object that mistaken proofs and deductions do not count as *real* mathematical knowledge, but that they are merely believed, incorrectly, to constitute mathematical knowledge, and that *real* mathematical knowledge indeed is certain. Yet this objection can be used to defend the certainty of any kind of knowledge, and thus fails to distinguish mathematical knowledge from other forms of knowledge. For instance, one could also say that *real* knowledge about physics indeed is certain, and that all the mistaken convictions we may harbor about physics do not constitute *real* knowledge. To make such a post hoc move is to betray what we mean when we say that a given form of knowledge constitutes "certain knowledge". For to say that the knowledge in a given field is certain is to say that all of that which is believed to be knowledge in that field is certain — not merely some smaller subset that remains after we have removed those claims that turned out to be false.

The view that mathematical knowledge is uncertain is indeed, contrary to Ayer's claim that mathematical truths seem certain to everyone, not a rare one among mathematicians. For example, mathematician Reuben Hersh writes:

> We do not have absolute certainty in mathematics; we may have virtual certainty, just as in other areas of life. Mathematicians disagree, make mistakes and correct them, are uncertain whether a proof is correct or not. […]
> The actual experience of all [foundationist] schools—and the actual daily experience of mathematicians—shows that mathematical truth, like other kinds of truth, is fallible and corrigible.
> (Hersh, 1979, p. 6)

Indeed, it has been estimated that half of all published mathematical proofs contain errors (Weber et al, 2014, p. 51). More than that, all the mathematical knowledge we know of and remember should arguably also be considered fallible for the reason that it is stored in our human memory, which we know to be fallible; indeed, modern psychology has revealed that our memory is unreliable to an even greater extent than we usually realize (cf. Shaw, 2016).

In light of these considerations, it seems reasonable to conclude that Szabó is right that (what we consider) mathematical knowledge is not, in fact, certain and infallible, and that, to the extent most people believe that it is, they are simply mistaken. Indeed, in light of these considerations, what seems in need of explanation is arguably the conviction that mathematical knowledge is certain, not the contention that it is not.

Is Mathematical Knowledge Uniquely Certain?

Having cut the rumored certainty of mathematical knowledge down to size in this way, one may proceed to compare the certainty we have in (that which we consider) mathematical knowledge with the certainty we have concerning other forms of knowledge, such as that found in other sciences. For even if mathematical knowledge is not absolutely certain, might it still not be far more certain than the knowledge found in other sciences, and thus pose a problem for the empiricist view that mathematical claims are (also) based on induction, i.e. based on generalization from observations of particular instances?

As we have seen, Szabó argues that it is not, which he does with reference to the common sense knowledge that when we break a stick, we are sure that we get a shorter stick, and that this certainty is no smaller than the certainty we have in the most certain of mathematical deductions (Szabó, 2012, p. 11). It seems difficult to disagree with Szabó's argument here, not because we are not highly confident about established mathematical truths, but because we are indeed also highly confident than when we break a stick, we get a shorter one. And it does not seem difficult to point to (more) facts from the empirical sciences that appear equally certain. For example, in physics: that when we drop a rock close to the surface of Earth, it falls toward the ground with approximately constant acceleration; in biology: that the molecule DNA is found in all humans; in economics: that the United States has a higher GDP than Greenland (it had last time I checked, and to believe it still does rests on inductive inference).

These claims are all based on induction, on generalizations of observations of particular instances, yet this does not imply that we are not highly confident about them — roughly as confident, one can argue, as we are about the most certain of mathematical beliefs. And hence the claim that mathematical beliefs cannot be based on induction because our certainty about such beliefs is greater than our certainty about (what is commonly acknowledged to be) inductive claims does not seem justified. As the vast body of established knowledge found within the sciences demonstrates, inductive claims can also reasonably be assigned a very high credence.

Indeed, direct studies of the mathematical literature also casts doubt upon the view that mathematical knowledge is more certain than knowledge obtained in other sciences, as it has been found that mathematics papers are not significantly less likely to contain errors than papers from other disciplines (Weber et al, 2014, p. 51). Another fact worth noting is that what we deem mathematical knowledge is often accepted based on an overtly inductive foundation rather than a (directly) deductive one. Take, for example, the aforementioned proof for Fermat's last theorem by Wiles. Most mathematicians will agree that we now know that this theorem is "correct" (or rather: that it is provable within ZFC). Yet the source of this knowledge is, for most mathematicians, not deductive, as only relatively few mathematicians understand this proof in much depth. Thus, most mathematicians' knowledge that Fermat's last theorem is provable within ZFC is ultimately based on trust in the evaluations of Wiles proof carried out by others, not their own deductive powers. Such trust is arguably fully warranted, yet it should be noted that such trust rests overtly on induction: we seem to have been able (for the most part) to trust the judgment of other professional mathematicians so far, and thus we likely can in this case as well.[10]

Eternal Truths About Concrete Physical Systems

If we are to zoom out a bit and try to evaluate the claim that mathematical claims are based on induction, does it then really seem plausible to say that the fact that 2 + 2 = 4 is merely an uncertain generalization? After all, once a mathematical result has been established, is it not, unlike the seemingly more contingent laws of physics, eternally true?

A thing worth clarifying in this context is what we mean by "eternally true". For if we say something true about this particular moment in time, e.g. "the Moon orbits Earth right now, in 2018", then this will presumably remain true forever, even if the more general claim "the Moon orbits Earth" will not always be true. Thus, like any historical fact about the past, the fact that I found 2 + 2 to equal 4 a few seconds ago will presumably remain true in the future. What is not clear, however, is whether a physical system that embodies a mathematical claim like this, as well as more complicated mathematical claims — more complicated formal systems and facts about them — will also exist in the future, and indeed whether it will even be physically possible for them to exist in the indefinite future. For example, if we are to believe cosmologists, the accelerating expansion of the universe will eventually mean that stable atoms can no longer exist (cf. Krauss, 2012, p. 188), which would likely render the physical embodiment of much of what we now recognize as mathematical facts impossible.

In this way, given physicalism, it may in fact not be the case that it will always be possible to demonstrate that 2 + 2 = 4. Or, to phrase it more generally, it may not always be the case that some proven theorem will always be provable within a given formal system in which it has once been proven, as the existence of this given formal system itself may not always be physically possible. At the very least, it seems that such unrealizability would be the case in the hypothesized big bang singularity, where the entire universe is thought, at least by some cosmologists (cf. Hawking, 1996), to have been confined to a single, informationless point, which would mean that mathematical truths of the sort mentioned above would not be provable, even in principle, throughout all time and space.

In this way, it does indeed seem plausible that the validity of mathematical truths is based on induction — that it rests on uncertain generalization based on particular instances. On Szabó's account, any statement of mathematical fact is derived from observation of a particular physical instantiation of a formal system, and *given that* the same formal system can be physically realized repeatedly, *then* we always get the same results[11] — otherwise, these realizations would not, in fact, comprise instantiations of the exact same rules and axioms, and thus not instantiate *the same* formal system (cf. Szabó, 2017, pp. 4-5). Yet whether the assumption of such repeated physical realizability itself holds true is, like the question concerning whether the laws of nature will remain the same in the future, an uncertain, inductive question about the physical world. And to say that mathematical truth rests on induction in this sense seems quite plausible.

Is Mathematical Knowledge "Necessary"?

The final claim that shall concern us in this chapter is the claim Szabó makes about the necessity, or absence thereof, of mathematical knowledge. For instance, he writes that it is a "[…] universal *illusion* that logical and mathematical truths are necessary […]" (Szabo, 2017, p. 7). However, it is not obvious

10 The same point is made at greater length in (Weber et al, 2014).

11 It should be noted that mathematics and physics — and any other science for that matter — are not fundamentally different in this regard, as we also can, and indeed do, make conditional generalizations of this kind in the realm of physics. For example, we can say that *if* such and such laws and initial conditions will apply at some particular future time, *then* such and such will also apply. The only difference between mathematics and theoretical physics, then, is, on the physicalist view, *the kind of* conditional generalization they make.

what the term “necessary” means in this context.[12] For, as noted earlier, Szabó also writes that mathematical facts indeed are necessary, yet presumably in a very different sense of the term, as he writes that “[…] if there are logical/mathematical facts, they must be understood in terms of the physical reality; they must be *necessitated* [emphasis mine] by the physical facts of the world.” (Szabo, 2012, p. 1).

In one place, Szabó seems to define “necessary truths” in the first sense, the sense in which he thinks mathematical truths are not necessary, as “truths completely demonstrated by reason” (Szabó, 2003, p. 122), which is itself a problematic definition, since it is predicated on a clear definition of what it means to be “completely demonstrated by reason”, and what reason entails more generally; itself a profound philosophical question. Szabó does not specify this explicitly, yet, upon reading the relevant passages by Szabó, it seems most plausible, in my view, that what he means by “completely demonstrated by reason”, and thus by “necessary truth” (of this kind), is “a truth we can obtain without performing any empirical observation”, cf. Szabó’s note on “the truth-of-reasoning versus truth-of-facts dichotomy” in (Szabó, 2017, p. 7). Thus, what Szabó means by a “necessary truth” is, by all appearances, essentially an *a priori* truth, which fits well with Szabó’s explicit statement that mathematical knowledge is, on his account, not *a priori* but *a posteriori*.

In order to assess whether this claim of Szabó’s — the claim that mathematical knowledge is *a posteriori* rather than *a priori* — is plausible, it seems worth first clarifying what he means, or at least what he does not mean, by these terms. For it may perhaps be tempting to think of *a priori* knowledge as being “that which we can know *before* we open our eyes and look at the world” or indeed before any information from the external senses reaches us more generally (cf. the literal meaning of *a priori*, “from the earlier”, as well as how the concept of *a priori* knowledge is often explained). Yet this is manifestly not how Szabó defines this term, as he views all knowledge that is derived from observation, in any sense of the term — not just observation of the external world — as being *a posteriori* knowledge. And Szabó’s point is that our mathematical knowledge, including that obtained while doing deductive reasoning with our eyes closed, is the product of an observation of physical processes: processes going on in our brain which instantiate the rules of the formal system we are examining. In this way, one can say Szabó is challenging and expanding our common sense conception of what counts as an observation of a physical process. Or one could simply say he has internalized physicalism and

12 There are various definitions of “necessary truth”, and it is not obvious that they are equivalent — for instance, “a truth that is true in all possible worlds” and “a truth whose negation is impossible”. Furthermore, there is also a notion of so-called epistemological necessity, which has been defined as “a necessity that can be deduced from a thinker's other beliefs” (cf. Bunnin & Jiyuan, 2004, p. 218).

our modern view of what our thought processes, including those thought processes we deem deductive and mathematical, in fact are; namely, processes that conform to description in physical terms.

So does Szabó's claim that we do not know mathematical truths before we have observed the outcome of a physical process seem plausible? It seems difficult to say why it should not, especially in light what we now know about our cognition in neuroscientific terms. The most compelling objection is arguably the one Szabó raises himself: that our mathematical derivations seem more reliable than (what we usually recognize as) observations of physical processes. Yet, as we have seen above, and as Szabó himself does with his analogy of breaking a stick, one can question whether this is indeed the case. And the best analogy to employ for such questioning is perhaps a computer simulation, or indeed just the output of a pocket calculator. For nobody seems to dispute that the outcome of the calculations of a pocket calculator is anything but the result of physical processes, and yet we generally trust the results of our pocket calculators — often much more so than we trust our own minds, at least with respect to problems where they have a distinct advantage, such as multiplication with big numbers. Therefore, it seems false to claim that the status of the mathematical claims derived by our brain alone have a fundamentally different epistemological status, at least in terms of their certainty, than do statements derived from observation of (overtly) physical systems. Indeed, on Szabó's account, our knowledge of mathematics is derived via a process that is closely analogous to the observation of a pocket calculator or a simulation. The only difference is that the notional calculator in our minds is installed directly in the brain, from which we can read its calculations directly, without opening our eyes or ears, and hence it is not obvious to us that it is, in fact, also physical.

To sum up this chapter, it seems that Szabó's conclusions about the nature of mathematical knowledge are indeed tenable. That mathematical knowledge, or at least what we consider mathematical knowledge, is uncertain appears vindicated by the fact that mathematicians make mistakes, not only in the laboratory of their heads and notebooks, but also in their published proofs. And not just in rare cases, but rather often — again, no less than half of all published proofs were estimated to contain errors (Weber et al, 2014, p. 51). That mathematics is empirical, in the sense that it is the result of observations of physical processes, appears plausible in light of what we know about our cognition in neuroscientific terms, and is not, contrary to what one might object, rendered implausible in the least by the reliability of our deductive reasoning — both in terms of the confidence we have, as well as the regularity we find, in such reasoning — since we find roughly the same level of reliability in our observations of the outcome of transparently physical processes such as the calculations of pocket calculators and personal computers. And if we accept that mathematical knowledge is empirical in this sense, it follows that it is not "necessary", and hence not *a priori*, in Szabó's sense of these terms; i.e. that mathematical knowledge is not, indeed cannot, be acquired in any other way than by observing the outcome of a physical process. Finally, it also follows that mathematical knowledge is inductive, in that we cannot confidently assume that a physical process that instantiates the derivation of a given mathematical truth from a given formal system can always be realized.[13] Indeed, as we have seen above, such an assumption may well be false, which is not, needless to say, the same as saying that we have good reason to start doubting whether familiar mathematical results will be physically possible to prove tomorrow, any more than conceivable doubts about the universality of, say, gravity gives us good reason to doubt that gravity will still work tomorrow.

In light of these conclusions, one may argue that Szabó's conclusions about mathematical knowledge do not constitute a reason to reject his physicalist account of the nature of mathematics, but rather the opposite.

13 Indeed, one can reasonably ask how it could possibly be otherwise, as we arguably never observe *anything but* particular states of the world. For instance, in any given moment, all we know is arguably the particular state of our conscious mind, based on which we infer conclusions about things elsewhere. From this perspective, one can also question whether even platonism could escape this inductive conclusion. For even if we were to posit the existence of a supra-physical platonic realm (which seems unnecessary if all our ideas, including our mathematical ones, can plausibly be accommodated within a physicalist ontology), we would still seem to have the problem of justifying any belief we may have in the eternity of this realm, let alone the persistence of our access to it. Who is to say that this realm will persist and that it will be accessible to us tomorrow? To say that it will seems an inductive assumption as well. And, to then turn the tables on platonism, if we never know of mathematical objects through anything but concrete realizations of them, why believe they have non-concrete existence in an abstract realm? To posit the existence of such a realm would seem superfluous and without justification.

Conclusion

The aim of this project has been to examine the nature of mathematics given physicalism: can mathematics be accommodated within a physicalist ontology, and if so, how? For this purpose, we took point of departure in the particular account proposed by László Szabó. So what can we conclude in light of our examination of this view?

First of all, we can conclude that Szabó's argument required some modification and specification in order for it to seem tenable. Most glaringly, perhaps, it required modification so as to avoid the seemingly self-refuting move of denying the existence of consciousness. This did not require us to give up on physicalism, but merely to adopt a version of physicalism that does not deny the existence of consciousness. Such modification did not, however, appear to fundamentally change, much less defeat, Szabó's argument, as his argument does not require the assumption that consciousness is unreal. One thing it does defeat, however, is Szabó's rather loose argument for claiming that *only* a formalist conception of mathematics is consistent with physicalism; as we have seen, mentalist views of mathematics need not be incompatible with versions of physicalism that hold consciousness to be real.

The next thing we examined was then whether a formalist conception of mathematics can indeed satisfyingly cover that which we refer to as "mathematics". And here it seemed that some specification was required, in two ways. First, we needed to clarify that the sense of "mathematics" we and Szabó are most interested in examining, and hence what we need to define clearly, is *the object* of the science of mathematics; not the practice of mathematicians. Second, we needed to clarify the definition of this object, as Szabó defines formalism in the following, somewhat ambiguous manner: "*Formalism*: logic and mathematics are thought of as statements about manipulations with meaningless symbols." (Szabó, 2017, p. 1). This definition can be interpreted to be either a strong or a weak claim. The strong claim is that "mathematics *can only* (validly) be thought of as manipulations with meaningless symbols", where the weak claim is the much more modest "mathematics *can always* (validly) be thought of as manipulations with meaningless symbols". Szabó appears to interpret his definition in the strong sense, yet as we saw above, this strong claim seems questionable, and, more than that, it is also unnecessary, since we only seem to need the much weaker and more plausible "can always" claim in order for Szabó's argument to run through.

Finally, we examined the conclusions Szabó draws with respect to the nature of mathematical knowledge: that mathematical knowledge is empirical, uncertain, unnecessary, and inductive. Here, some specification again seemed necessary, as, for example, with respect to the claim that mathematical knowledge is unnecessary, as well as the claim that it is inductive; it is inductive in the sense that it is an uncertain, inductive matter whether the same formal system and the derivations of its theorems can be physically realized throughout all space and time. And given such specification, we found, in short, that these conclusions of Szabó's also seem plausible. What we consider mathematical knowledge is indeed overtly fallible, and the objection that mathematical knowledge derived by our minds seems too reliable for it to result from observations of a physical system appears contradicted by the reliability of observations of physical systems such as pocket calculators.

Thus, given that we modify and specify Szabó's argument in the way we have done here, the resulting argument for how mathematics can be accommodated within a physicalist ontology does indeed seem a plausible one. We should, of course, be careful not to draw too strong conclusions, as we have not examined all possible objections one could raise against Szabó's thesis. Yet we arguably have raised the most compelling such objections, and, in the face of these, the argument indeed does seem to stand up to scrutiny (modulo, again, a few modifications which nonetheless leave the physicalist premise wholly intact). Hence, in light of our examination of Szabó's physicalist account of the nature of mathematics, we do not appear to have any compelling reason to reject his core thesis: that mathematics *can* be accommodated within a physicalist ontology. That all of what we consider mathematics *can be thought of* as manipulations with meaningless symbols, and that such manipulation plausibly can be accounted for as nothing over and above the workings of the physical world.

Yet it should again be noted that Szabó's physicalist view of the nature of mathematics is but one such view, and that there are, as we have seen, others on offer (cf. Bigelow, 1988), implying that, however promising we may consider Szabó's particular account, the superset that is all physicalist views of the nature of mathematics must be considered more promising still. And if we zoom out and behold this superset, it seems defensible to say that the most plausible account of the nature of mathematics is to be found here. For as mentioned in the introduction, the natural sciences appear to give us strong reasons to accept a physicalist ontology, and the question is then what compelling reasons we might have for not adopting such an ontology with respect to the nature of mathematics. Do we, when exploring the nature of mathematics, find any reasons that point us away from the physicalist ontology which the natural sciences so strongly point us toward? And this is exactly the question to which we just answered in the negative: in light of our examination of Szabó's physicalist account, we failed to find any such reasons. With respect to the certainty and lawfulness of mathematics, it seems that the same certainty and lawfulness can also be found in the behavior of systems that we all recognize to be purely physical, which undermines the perhaps strongest objection against a physicalist view of mathematics — namely, that mathematical certainty and lawfulness is unphysical.

Therefore, in light of the strong reasons the natural sciences give us to favor a physicalist ontology, as well as the apparent absence of compelling reasons against it, we may tentatively conclude that the opposite of this "it is unphysical" claim actually seems more plausible, not just with respect to the certainty and lawfulness of mathematics, but with respect to mathematics altogether. That is: that the nature of mathematics is, most plausibly, wholly physical. That everything we recognize as mathematics is, most plausibly, nothing over and above the states and workings of the physical world.

This conclusion, even as it is phrased with the important qualifier "most plausibly", has rather revolutionary implications for our view of the nature of mathematics. For it casts strong doubt upon the sensibility of a tacit and deep-rooted dichotomy that is commonly assumed[14] to exist between mathematics and the physical world, indeed between the abstract and the concrete more generally. Here is David Hume, for instance:

> That the square of the hypotenuse is equal to the square of the two sides, is a proposition which expresses a relation between these figures. That three times five is equal to the half of thirty, expresses a relation between these numbers.
> Propositions of this kind are discoverable by the mere operation of thought, without dependence on what is anywhere existent in the universe. Though there never were a circle or triangle in nature, the truths demonstrated by Euclid would for ever retain their certainty and evidence.
> (Hume, 1772/1993, Cause and Effect, Part I)

Hume here almost seems to endorse an ontological dualism — a widely accepted one, no doubt — between mathematics on the one hand and the physical world, the totality of that which is "anywhere

14 As one textbook put it (my own translation): "[…] where, for instance, the natural sciences are concerned with physically existing things, the mathematical sciences are about objects that are defined and formal." (Johansen & Sørensen,

existent in the universe", on the other. Yet such a dualism cannot be maintained on a physicalist ontology (which is, of course, not to say that the distinction between mathematics and physics cannot be maintained, cf. Szabó's two different senses of truths about the physical world). According to any physicalist view of mathematics, mathematical propositions, as well as all other things that are "discoverable by the mere operations of thought", are exactly dependent on something, somewhere existent in the universe. Indeed, on physicalism, mathematical propositions are existent and discoverable in *nothing but* concrete states of the universe (since concrete physical states are *all* that exist according to physicalism). And for this very reason, Hume's final claim — that Euclid's truth will forever retain their evidence — may well also, as we have seen, turn out to be false on the physicalist view, since it may well be the case that familiar theorems will not "for ever" be possible to prove, even in principle. This is another way in which physicalism is clearly at odds with our commonly accepted wisdom about mathematics, as the universality of mathematics is not commonly thought to rest on an inductive assumption about the continued realizability of certain physical structures.

In sum, it seems plausible that we do not need to split the world into two, or otherwise invent new worlds beyond the physical world, in order to make room for mathematics, as it seems plausible — indeed "most plausible", I have argued — that all of what we call "mathematics" exists as nothing over and above the physical world. This has radical implications for our view of mathematics, e.g. that mathematical truths express facts about concrete, spatiotemporally located states, and that the assumed universality of these truths, in one sense at least, rests on induction. Finally, that such an accommodation of mathematics within the physical world appears plausible also has implications for our view of physicalism, as it would appear to render the physicalist ontology itself even more plausible than it already did.

2014/2015, p. 20). Original version: "[…] hvor fx naturvidenskaberne beskæftiger sig med fysisk eksisterende ting, handler de matematiske videnskaber om objekter, som er definerede og formelle." (Johansen & Sørensen, 2014/2015, p. 20).

Acknowledgments

I should like to thank Mikkel Willum Johansen for advising me during the process of writing this project. In particular, I want to thank him for providing critical suggestions and discussion along the way that greatly improved the direction and content of this project, as well as for enthusiastically encouraging me and waving me on as I came and said that I had this desire to write about what mathematics is. I am grateful that I have been given such total freedom and received nothing but support and helpful feedback.

I also wish to send my thanks to David Pearce for making me think deeper about the nature of mathematics, and to Sille for supporting me and keeping my spirits high.

Bibliography

Ayer, A. J. (1952). *Language, Truth and Logic*. New York: Dover Publications.

Ball, W. W. R. (1892/1905). *Mathematical Recreation & Essays*. London: Macmillan Publishers Ltd.

Bigelow, J. (1988). *The Reality of Numbers: A Physicalist's Philosophy of Mathematics*. Oxford New York: Clarendon Press Oxford University Press.

Bunnin, N. & Jiyuan Y. (eds). (2004). *The Blackwell Dictionary of Western Philosophy*. Blackwell Publishing. Blackwell Reference Online [accessed 23 May 2018]: http://www.blackwellreference.com/public/book.html?id=g9781405106795_9781405106795

Campbell, D. (1984). *Mathematics: People, Problems, Results*. Belmont, Calif: Wadsworth Internat.

Cohen, P. J. (1963). The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the United States of America,* 50(6), pp. 1143-1148.

Cohen, P. J. (1964). The independence of the continuum hypothesis, II. *Proceedings of the National Academy of Sciences of the United States of America*, 51(1), pp. 105-110.

Gunawardena, J. (1998). The unreasonable ineffectiveness of mathematics in computer engineering. Talk delivered at category theory seminar in Sydney. Title and date listed at [accessed 23 May 2018]: http://web.science.mq.edu.au/groups/coact/seminar/cgi-bin/past-talks.cgi?year=1998

Hawking, S. (1996). The Beginning of Time. Public lecture. Retrieved from [accessed 23 May 2018]: http://www.hawking.org.uk/the-beginning-of-time.html

Hersh, R. (1979). Some Proposals for Reviving the Philosophy of Mathematics. *Advances in Mathematics*, 31, pp. 31-50.

Hume, D., Norton, D. & Norton, M. (1738/2000). *A Treatise of Human Nature*. Oxford New York: Oxford University Press.

Hume, D. (1772/1993). *An Enquiry Concerning Human Understanding*. Indianapolis: Hackett Publ Co.

Johansen, M. W. (2010). Naturalism in the Philosophy of Mathematics. PhD dissertation, University of Copenhagen.

Johansen, M. W. & Sørensen, H. K. (2014/2015). *Invitation til matematikkens videnskabsteori.* Samfundslitteratur.

Johansen, M. W. & Misfeldt, M. (2018). Material representations in mathematical research practice. Unpublished manuscript.

Kempe, A. B. (1879). On the Geographical Problem of the Four Colours. *American Journal of Mathematics*, 2(3), pp. 193-220.

Kjeldsen, T. (2011). *Hvad er matematik*. Kbh: Akademisk.

Kolata, G. (1993, June 24). At Last, Shout of 'Eureka!' In Age-Old Math Mystery. *The New York Times*, p. 1. Retrieved from [accessed 23 May 2018]: https://www.nytimes.com/1993/06/24/us/at-last-shout-of-eureka-in-age-old-math-mystery.html

Krauss, L. M. (2012). *A Universe from Nothing: Why There Is Something Rather than Nothing*. New York: Free Press.

Lützen, J. (2011). The Physical Origin of Physically Useful Mathematics. *Interdisciplinary Science Reviews*, 36(3), pp. 229–243.

Mollin, R. (2010). *Advanced number theory with applications*. Boca Raton: CRC Press.

Müller-Hill, E. (2009). Formalizability and Knowledge Ascriptions in Mathematical Practice. *Philosophia Scientiæ*, 13(2), pp. 21-43.

Petrovici, M. (2016). *Form Versus Function: Theory and Models for Neuronal Substrates*. Switzerland: Springer.

Pinker, S. (1997). *How the Mind Works*. New York: Norton.

Poli, R. (1999). The unreasonable ineffectiveness of mathematics in cognitive sciences. Lecture series delivered at McGill University. Abstract retrieved from [accessed 23 May 2018]: http://www.math.mcgill.ca/rags/seminar/poli.txt

Scientific American. (1999, October 21). Are mathematicians finally satisfied with Andrew Wiles's proof of Fermat's Last Theorem? Why has this theorem been so difficult to prove? *Scientific American*. Retrieved from [accessed 23 May 2018]: https://www.scientificamerican.com/article/are-mathematicians-finall/

Shapiro, S. (2000). *Thinking About Mathematics: The Philosophy of Mathematics*. New York: Oxford University Press.

Shaw, J. (2016). *The Memory Illusion: Remembering, Forgetting, and the Science of False Memory*. London: Random House Books.

Smale, S. (1981). The Fundamental Theorem of Algebra and Complexity Theory. *Bulletin of the American Mathematical Society*, 4(1), pp. 1-36.

Stoljar, D. (2001/2017). “Physicalism”. *The Stanford Encyclopedia of Philosophy* (Winter 2017 Edition), Edward N. Zalta (ed.). Retrieved from [accessed 23 May 2018]: https://plato.stanford.edu/archives/win2017/entries/physicalism/

Strawson, G. (2006). Realistic Monism: Why Physicalism Entails Panpsychism. *Journal of Consciousness Studies*, 13(10-11), 2006, pp. 3-31.

Strawson, G. (2018). The Consciousness Deniers. The New York Review of Books. Retrieved from [accessed 23 May 2018]: http://www.nybooks.com/daily/2018/03/13/the-consciousness-deniers/

Szabó, L. (2003). Formal System as Physical Objects: A Physicalist Account of Mathematical Truth. *International Studies in the Philosophy of Science,* 17, pp. 117-125. Retrieved from [accessed 23 May 2018]: http://www.tandfonline.com/doi/pdf/10.1080/0269859031000160568?needAccess=true

Szabó, L. (2012). Mathematical Facts in a Physicalist Ontology. *Parallel Processing Letters*, 22, 1240009, [12 pages]. Retrieved from [accessed 23 May 2018]: http://philsci-archive.pitt.edu/9894/7/LESzabo-math_in_physical-preprint.pdf

Szabó, L. (2017). Meaning, Truth, and Physics. In G. Hofer-Szabó, L. Wronski (eds.), *Making it Formally Explicit*, European Studies in Philosophy of Science 6. (Springer International Publishing, 2017). Retrieved from [accessed 23 May 2018]: http://philsci-archive.pitt.edu/12891/13/LE_Szabo-Meaning-Truth-Physics-v5.pdf

Thurston, W. P. (1994). On Proof and Progress in Mathematics. *Bull. Amer. Math. Soc.* (N.S.) 30, pp. 161-177.

Velupillai, K. V. (2005). The unreasonable ineffectiveness of mathematics in economics. *Cambridge Journal of Economics.* 29(6), pp. 849-872.

Weber, K., Inglis, M., & Mejia-Ramos J. P. (2014). How Mathematicians Obtain Conviction: Implications for Mathematics Instruction and Research on Epistemic Cognition. *Educational Psychologist*, 49(1), pp. 36-58.

Wigner, E. P. (1960). The unreasonable effectiveness of mathematics in the natural sciences. Richard Courant lecture in mathematical sciences delivered at New York University, May 11, 1959.

C

o

m

m

u

n