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Trigonometry in upper secondary school context: identities and functions

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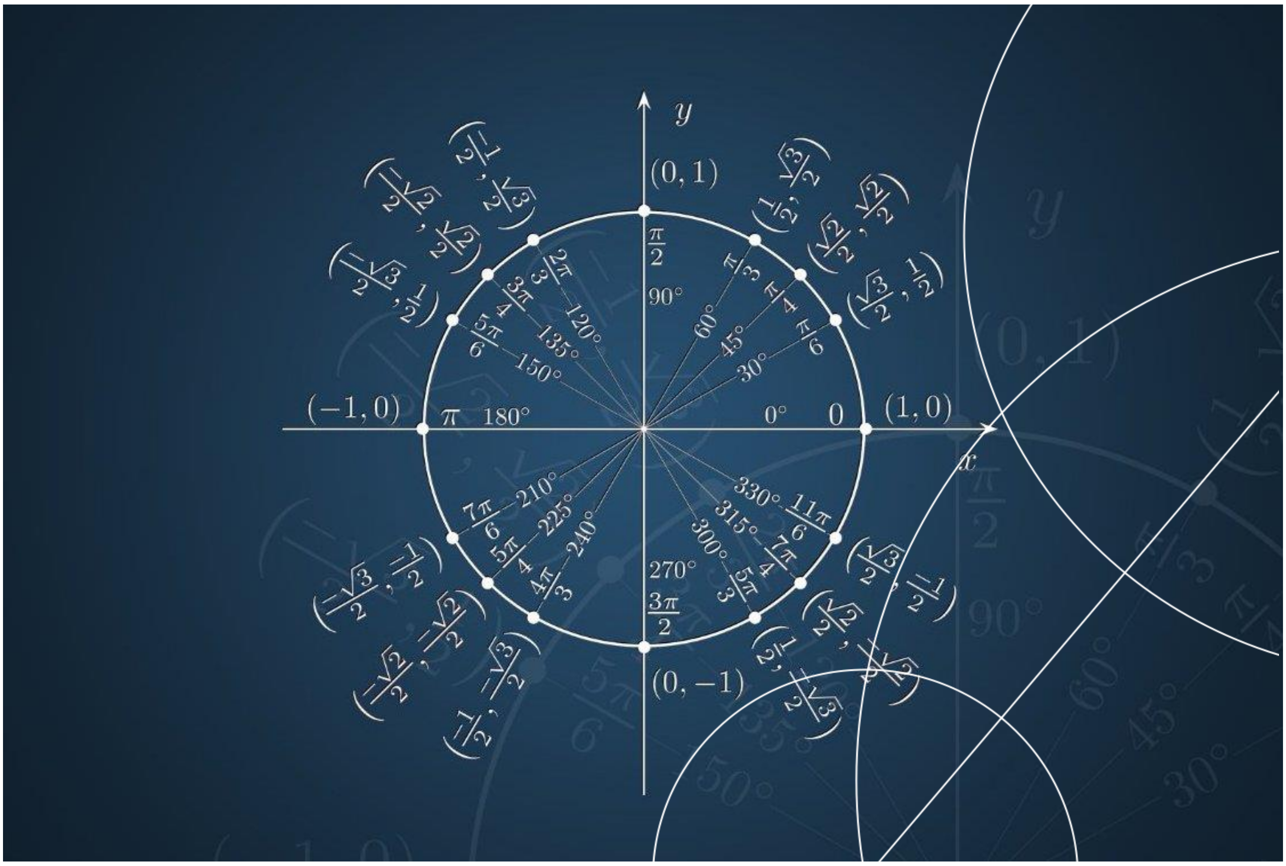
FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Master's thesis

Maria Anagnostou

Trigonometry in upper secondary school context: identities and functions



Academic advisor: Carl Winsløw

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Titel og evt. undertitel:	Trigonometry in upper secondary school context: identities and functions
Title / Subtitle:	Trigonometry in upper secondary school context: identities and functions
Subject description:	This thesis aims to investigate upper secondary school student knowledge on trigonometric identities and trigonometric functions.
Academic advisor:	Carl Winsløw
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Abstract:

In the last years of secondary school in Denmark, students transfer from triangle Trigonometry, to unit circle Trigonometry. To this transition, constructing and interpreting graphs of trigonometric functions is also added. The purpose of this thesis is to observe how students transfer between these trigonometric settings, which setting do they mostly work with and to investigate students' misconceptions related to $\sin(x)$ and $\cos(x)$.

The results of this thesis are based on observations of lessons and students' interviews in an upper secondary school in Denmark. It was found that some students encountered didactical obstacles when they tried to combine the old knowledge of the triangle setting, to the new knowledge of the unit circle or function setting. Moreover, most students found it challenging to convert between these settings. A main factor for this difficulty was the lack of prerequisite knowledge regarding functions. The results also revealed some misconceptions related to $\sin(x)$ and $\cos(x)$, such as identifying them to the y-axis and the x-axis respectively, when the students were working in the unit circle setting.

Acknowledgements:

I would first like to express my sincere thanks to my supervisor Carl Winsløw for providing me with all the necessary help and guidance through the writing of this thesis. He was always available to answer any of my questions and his suggestions were valuable.

Many thanks to the high school teachers Jeanette Kjølbæk and Lotte Nørtoft for giving me the opportunity to attend their classes and collect data. Their help has been extremely useful.

I would also like to thank my parents for their support all of those years.

Finally, a big thank you to Bryan for his continuous encouragement throughout my Master's program.

Contents:

1. Introduction:

There is a limited amount of studies connected to students' difficulties with Trigonometry. Due to the fact that students begin learning Trigonometry through right-angled triangles, later the unit circle and finally through graphs of trigonometric functions, they often see these three settings as isolated from each other. To this, the difficulty of different representations is added. In particular, the students should learn to transfer between a symbolic representation ($f(x) = \sin x$), a diagrammatic representation (where sine is represented as a wave) and a geometric representation (where sine is defined in terms of ratios of sides of right-angled triangles, or as the second coordinate of a point on the unit circle). Hence, students have to learn how to separately handle the different settings and representations, as well as how to transfer between them.

Bressoud argues that the right-angled triangle, which is the first contact of students to Trigonometry, works as a convenient shortcut in teaching and learning Trigonometry. According to him, the problem is that like most shortcuts, it creates more problems in the long run than the ones that it solves originally (as cited in Van Sickle, 2011, p.43). Indeed, students first learn how to handle sine and cosine as ratios of sides of a right-angled triangle, not being able to see the relation between the two continuously varying quantities, this of the angle and this of the trigonometric function. On the other hand, on the unit circle, the students can observe this relation, and they can use angles as inputs of sine and cosine (Dejarnette, 2014, p.21).

In this thesis, we will examine students' perceptions of sine and cosine in order to find possible misconceptions, including remains of the triangle setting. If some remains do exist, we would like to investigate if they become an obstacle in the students' perceptions of sine and cosine as trigonometric functions. Finally, we will investigate how students transfer between right-angled triangles, the unit circle and the graphs of sine and cosine and observe misconceptions related to these transitions.

1.1 Structure of the thesis:

After the introduction, in the second section, there is an outline of Trigonometry. There, we provide definitions and information about angles and the trigonometric functions sine and cosine. There is also a focus on trigonometric identities with respect to right-angled triangles, the unit circle and the graphs of sine and cosine.

In the third section, we refer to mathematical objects, settings and representations, as well as to the interplay between trigonometric settings. There are also presented prior research results, which are connected to students' difficulties with Trigonometry.

Section four, includes the research questions of this thesis, and in section five, there are presented some elements of the Theory of Didactical Situations, which will be used later in this thesis, to analyze the data which we have gathered. Section six presents the methodology, which we used to collect and analyze the data. Moreover, there is information on how we constructed the questionnaire, which the students used during the interviews. Also, the strategy which we followed during these interviews is presented.

This thesis results are presented in section seven. In order to facilitate the reading, we divided this section into seven subsections. The first three subsections include results related to students' explanations of specific trigonometric identities. The fourth subsection includes the ways which students handled the difference between a trigonometric identity and solving a trigonometric equation. Next, there are presented students' perceptions of angles. In the final two subsections, results regarding students' choices of trigonometric settings, as well as students' perceptions of sine, cosine and their inputs, are presented.

Finally, section eight is a discussion of most of this thesis' results. In section nine, there is the conclusion of this thesis, and the final section is the bibliography which was used.

2. Outline of Trigonometry:

2.1 Definitions of an angle:

An angle, or γωνία (gonía) in ancient Greek, or angulus in Latin, means literally "a little bending". One can indeed consider an angle to be measured on how much "bending" it appears to have. The most contemporary units of an angle are the degrees and the radians. By definition, a circle is divided into 360 equal parts, and each part is 1 degree (°). Hence, one circle has 360°. The $\frac{1}{60}$ of the degree is called minute ( )$'$ and the $\frac{1}{60}$ of the minute is called second ( )$''$. So, $20°19'10''$ is equal to $20 + \frac{13}{60} + \frac{10}{3600}$ degrees. In this section, some definitions of the angle are presented, as well as, how challenging it is to precisely define an angle.

Some define an angle as the figure formed by two distinct rays (half-lines) which have a common end-point. The rays are called the sides, or legs, or more rarely, the arms of the angle and the common endpoint is called the vertex of the angle. The interior of an angle is the area between those two rays and the exterior of an angle is the complement of the union of the angle and its interior. Even if the angle is constructed of line segments, and so the sides have finite length, the interior extends beyond them infinitely.

Looking back in history, Veronese defines an angle as a part of cluster of rays, bounded by two rays, just as a segment is a part of a straight line bounded by two points. Another definition wants the angle to be defined by rotating a ray about its endpoint, having the amount of rotation to be the angle's measurement. The starting position of the ray is called the initial side of the angle. The ending position of the ray is called the terminal side. An angle is in standard position when its vertex is at the origin of the cartesian system and its initial side is the positive x-axis. An angle is called positive when it is generated by counterclockwise rotation, whereas it is negative, when it is generated by clockwise rotation.

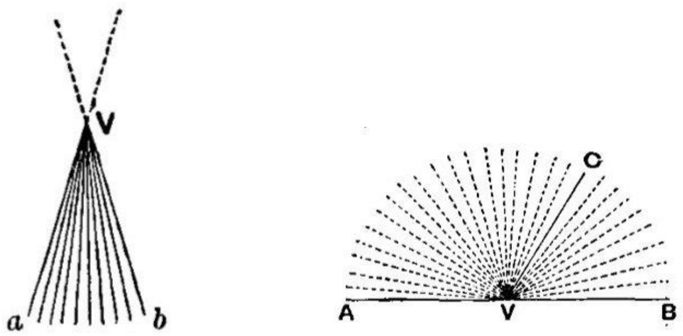


Figure 2.1: An angle and a flat angle, as defined by Veronese (image taken by Health, 1968).

So far, we can see that an angle can be defined by a quale (ποιόν), or a quantum (ποσόν), but it can also be defined as a relation (πρός τι) (Health, 1968, p.177). That is, because in a circle, a central angle's measurement is equal to the ratio of the length of the arc which subtends the angle, divided by the radius (see section 2.2). Euclid was categorized among those who believed that an angle is a relation. However, he also considered an angle to be a quantity, since he recognized that an angle is contained by the two lines forming it.

Whereas in the past some considered an angle with only one of the above definitions, nowadays we see that it is hard to only work with one of these and neglect the others. For example, it is difficult to argue that an angle is the space between two lines (quality), neglecting its measure of rotation (quantity), which is useful in, for example, comparing angles. Hence, one problem that arose for those who only thought of angles as a quality, was, how could angles then be compared or even bisected?

> "Further, the more and the less are the appropriate attributes of quality, not the equal and the unequal; if therefore an angle were a quality, we should have to say of angles, not that one is greater and another smaller, but that one is more an angle and another less an angle, and that two angles are not unequal but dissimilar. As a matter of fact, we are told by Simplicius, 538, 21, on Arist. de caelo that those who brought the angle under the category of quale did call equal angles similar angles." (Health, 1968, p.178).

2.2 Arc length:

The arc length is the distance between two points along a section of a curve (https://en.wikipedia.org/wiki/Arc_length). To find the length of a curve, we connect a finite number of points on the curve with line segments. The length of a continuously differentiable curve $\gamma : [a,b] \rightarrow \mathbb{R}^n$ is defined as $\ell = \int_a^b \| \gamma'(t) \| dt$, where $\|\cdot\|$ denotes the standard Euclidean norm of a vector $v(v_1, \ldots, v_n)$ in $\mathbb{R}^n$ given by $\|v\| = \sqrt{{v_1}^2 + \ldots + {v_n}^2}$. If we take a continuous curve $\gamma : [a,b] \rightarrow \mathbb{R}^2$ and divide [a,b] into n equal pieces $a = t_0, t_1, \ldots, t_{n-1}, t_n = b$ and connect $\gamma(t_0)$ to $\gamma(t_1)$, $\gamma(t_1)$ to $\gamma(t_2)$ etc. with line segments, we get the arc length $\ell = \sup \{\sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|\}$.

The length of an arc AB on a circle is denoted as $s = \frac{2\theta\pi r}{360°}$ (see figure 2.2), where θ is the measurement of the angle in degrees. (Especially for θ = 360°, the formula gives the circumference of the circle, s = 2πr). For the case of the unit circle, we have that $s = \frac{2\theta\pi}{360°}$.

Now, one radian, or the circular measure of an angle, as it used to be called, is the measure of an angle that subtends an arc of length equal to the radius (s = r or s = r = 1 for the unit circle). Hence, for the case of the unit circle and using the formula $s = \frac{2\theta\pi}{360°}$, we have that $1 = \frac{2 \cdot (1\text{rad}) \cdot \pi}{360°}$ and hence, 2π rad = 360°, or, 1 rad = $\frac{180°}{\pi} \approx 57.3°$ and we can easily convert from degrees to radians and conversely, using the formula $\frac{\theta}{180°} = \frac{\varphi}{\pi}$, where θ is the angle measurement in degrees and φ in the angle measurement radians.

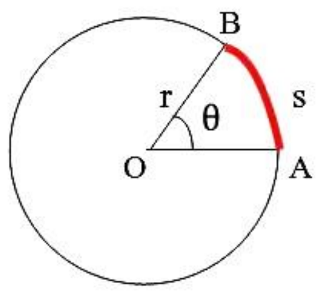


Figure 2.2: The length of the arc AB is $s = \frac{2\theta\pi r}{360°}$.

Back to the length of the arc for the case of the unit circle, and using the relation $\frac{\theta}{180°} = \frac{\varphi}{\pi}$, we get that s = φ. In other words, on the unit circle, the length of the arc is equal to the radian measurement of the angle.

2.3 The equality sign, equations vs. identities:

Robert Recorde introduced in 1557 the equality symbol "=" giving it the meaning of equality as we know it and use it today (as presented in Essien & Setati, 2006, p.47). The purpose was to use it as a shortcut and to stop repeating the phrase "is equal to". In Groza (1968) we can find why he used that specific symbol. As he explained, Recorde thought that no two things can be more equal, than two parallel straight lines (as cited in Essien & Setati, 2006, p.48). The equality symbol was of course represented long time before that, even from ancient times, but by different symbols. At the same time, many others have used the equality symbol to denote other things. For example, Descarted used it to indicate "±" and Euclides used it to express parallel lines. As Ball (1960) explains, Vieta and Schooten used "=" between two quantities to denote the difference between them (as cited in Essien & Setati, 2006, p.48).

The equality sign can express different things: Firstly, it can be used to define something, for example "define f to be f(x) = x + 2". Secondly, it can state that the two parts of the equality are equal or have the same quantity and thirdly, it can be used as a "give the answer sign" in arithmetics. Regarding the equivalence relation, it is a relation, which is symmetrical, reflexive and transitive.

An equation is a statement that asserts the equality of two expressions (https://en.wikipedia.org/wiki/Equation). Even though in French there is a distinguishing between an equality and an equation, with the second one necessarily containing one or more variables, in English, an equality and an equation are the same. We will not distinguish between an equality and an equation in this thesis.

There are two kinds of equations: the identities and the conditional equations. (source: https://en.wikipedia.org/wiki/Equation). An equation is called conditional when the solution set is a proper subset of the domain of the equation, or in other words, if the equation holds only for some values of the variable (source: https://en.wikipedia.org/wiki/Equation). When the solution set is the domain of the equation, the equation is called an identity or identity equation (https://brilliant.org/wiki/solving-identity-equations/). Finally, when the solution set is the empty set, our equation is a false statement, hence, we have a contradiction. If two equations $f(x) = 0$ and $g(x) = 0$ have the same solution set, they are called equivalent and we write: $f(x) = 0 \Leftrightarrow g(x) = 0$. Those definitions do not agree though, with Euler's identity ($e^{i\pi} + 1 = 0$) , an equation which always holds, but contains zero variables. So, for this thesis we will use another definition for identity: An identity is an equation which is always true (M.A.R.S., 2015, p.5). In other words, $\sin^2 x + \cos^2 x = 1$ is an identity and $2 = 2$ is also an identity. Both of them, as well as conditional equations (for example $2x = 2$) will be considered as equalities/equations.

2.4 Trigonometric functions and trigonometric identities:

"Trigonometry" comes from the Greek word "trigonometria" ("trigono" means "triangle" and "metro" means "to measure"), so the word translates as measurement of a triangle. Both the Egyptians and Babylonians are believed to have started to study triangles back in the 2nd millennium BC. The Egyptians used trigonometry to benefit from the land and to build the pyramids, whereas the Babylonian astronomers used trigonometry to study the rising and setting of stars, the motion of planets and eclipses. The Babylonians are also believed to have measured angles and to have divided the circle into 360°. The Greeks engaged with the calculation of the chords (taking an arc on a circle, a chord is the line that subtends the arc) and with measurements of sides and angles of triangles, as well as finding the relationships among those two.

There are six trigonometric functions: sine, cosine, tangent, cotangent, secant and cosecant. We will refer to the first two, as it is the aim of this thesis to focus on the high school curriculum. A trigonometic identity is an identity involving trigonometric functions.

2.4.1 The triangle:

Given the acute angle XOY and B a point of the side OY, we project from the point B onto the point A on the side OX, so that we get the right-angled triangle OAB.

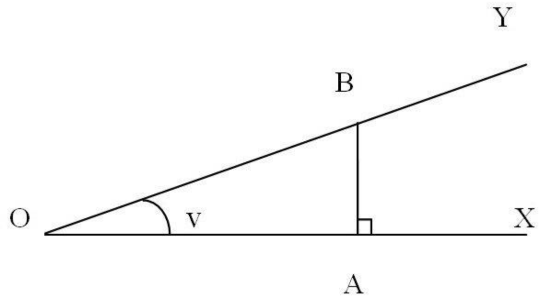


Figure 2.3: Defining trigonometric functions on the right-angled triangle OAB.

In the right-angled triangle OAB, define:

$\sin v = \frac{|AB|}{|OB|} = \frac{\text{opposite side}}{\text{hypotenuse}}$, $\cos v = \frac{|OA|}{|OB|} = \frac{\text{adjacent}}{\text{hypotenuse}}$,

$\tan v = \frac{|AB|}{|OA|} = \frac{\text{opposite side}}{\text{adjacent}}$, $\cot v = \frac{|OA|}{|AB|} = \frac{\text{adjacent}}{\text{opposite side}}$,

$\sec v = \frac{|OB|}{|OA|} = \frac{\text{hypotenuse}}{\text{adjacent}}$, $\csc v = \frac{|OB|}{|AB|} = \frac{\text{hypotenuse}}{\text{opposite side}}$.

Among the trigonometric functions there are some relationships between some of them.

Some reciprocal identities:

$$\sin v = \frac{1}{\csc v} \qquad \cos v = \frac{1}{\sec v} \qquad \tan v = \frac{1}{\cot v}$$

Some quotient identities:

$$\tan v = \frac{\sin v}{\cos v} \qquad \cot v = \frac{\cos v}{\sin v}$$

By using the Pythagorean theorem for acute angles on the above triangle, we have that $|AB|^2 + |OA|^2 = |OB|^2$ and by dividing both sides by $|OB|^2$, we get that $\sin^2 v + \cos^2 v = 1$, known as the Pythagorean identity. We can obtain two more Pythagorean identities, ($\tan^2 v + 1 = \sec^2 v$,

$\cot^2 v + 1 = \csc^2 v$), if we divide each term of our original Pythagorean identity with either $\cos^2 v$ or $\sin^2 v$.

Collectively, the reciprocal identities, the quotient identities and the Pythagorean identities are called the Fundamental identities.

2.4.2 The unit circle:

Now, we take an angle v with its vertex at the center of a unit circle. The terminal side of the angle intersects the unit circle on the point P(x,y). Then, using the above definitions, the x-coordinate on the unit circle is $\cos v$ and they-coordinate is $\sin v$, and since the radius of the circle is 1, we have that (x,y) = ($\cos v$,$\sin v$) (see figure 2.4). Now, using the fact that the equation of the unit circle is $x^2 + y^2 = 1$, we get that $\sin^2 v + \cos^2 v = 1$.

Since $\sin v = y$, the sine function is positive when y>0 (in Quadrants I and II) and negative when y<0 (in Quadrants III and IV). Similarly, since $\cos v = x$, the cosine function is positive when x>0 (in Quadrants I and IV) and negative when x<0 (in Quadrants II and III). The angles v and $-v$ cut the unit circle at two different points, P and P$'$. The two points have the same x-coordinate and opposite y-coordinates (see figure 2.4).

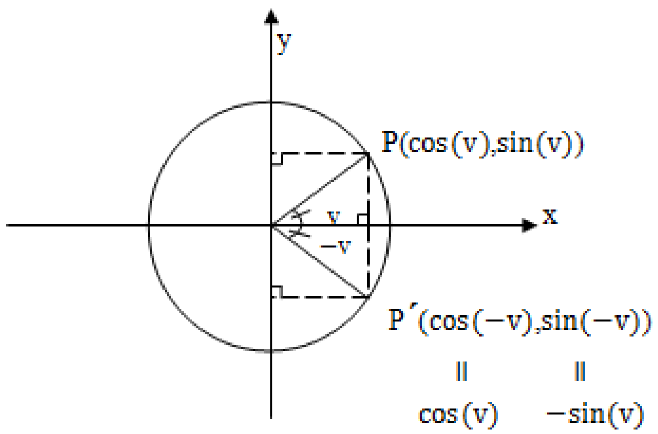


Figure 2.4: The points P and P$'$ on the unit circle have the same x-coordinate and the opposite y-coordinate.

Thus, $\cos(-v) = \cos(v)$ and $\sin(-v) = -\sin(v)$. Similarly, and with the help of the unit circle, we can find the sine and cosine of other angles:

	$\theta = -v$	$\theta = \pi + v$	$\theta = \pi - v$
$\sin\theta$	$-\sin v$	$-\sin v$	$\sin v$
$\cos\theta$	$\cos v$	$-\cos v$	$-\cos v$

A function f is defined to be a cofunction of a function g if f(u) = g(v), whenever u and v are complementary angles, which means that their sum is 90°. Sine and cosine are cofunctions of each other, which explains the "co" in "cosine".

Hence, we get the cofunction identities:
$\sin v = \cos(\frac{\pi}{2} - v)$ and $\cos v = \sin(\frac{\pi}{2} - v)$

They can be verified by taking sine and cosine of an angle in a right-angled triangle, or with the help of the unit circle, or one can simply observe that they hold by the use of the graphs of sine and cosine. Here, the proof by the use of the unit circle is presented:

Consider an angle v whose terminal side intersects the unit circle at the point $M(x_1, y_1)$ (see figure 2.5). Now we take the angle $\frac{\pi}{2} - v$, whose terminal side intersects the circle at the point $M'(x_2, y_2)$. The points M and M' are symmetrical with respect to the bisector of XOY, thus $x_1 = y_2$ and $y_1 = x_2$.

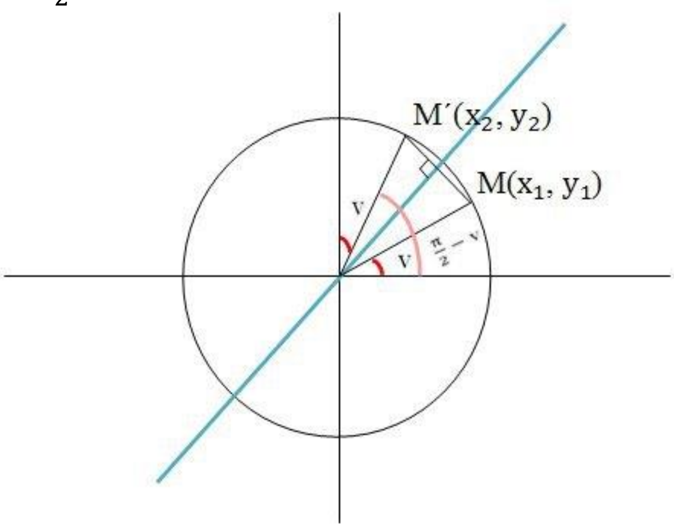


Figure 2.5: Proof of $\sin v = \cos(\frac{\pi}{2} - v)$ and $\cos v = \sin(\frac{\pi}{2} - v)$ using the unit circle.

2.4.3 Unwrapping the unit circle: the graphs of sine and cosine:

First, we take an angle θ in our unit circle. A right-angled triangle is formed, if we take the vertical line towards the y-axis. We have defined $\sin(\theta)$ to be the length of the triangle's vertical side and $\cos(\theta)$ to be the length of the horizontal leg of our triangle. This vertical and horizontal segment can now be transferred from the unit circle and be placed along the real number line. Repeating the same procedure, we get the graphs of the sine and cosine functions.

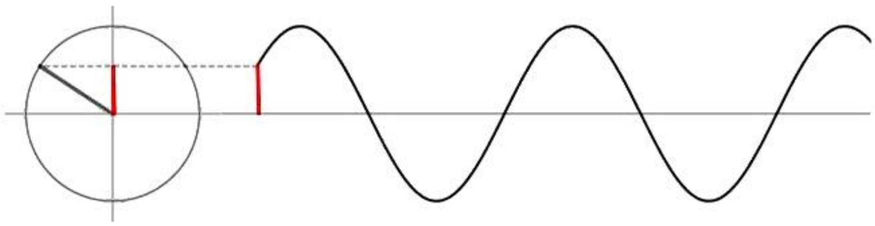


Figure 2.6: "Unwrapping" the vertical segments from the unit circle: The red vertical segment represents the sine of the given angle and its length is equal in both representations. To the right, the graph of the sine function is being formed.

If we observe those two representations, we see that as the sine is negative in the Quadrants III and IV, and positive in the Quadrants I and II, the y-coordinate of the sine function is also negative in $[\pi,2\pi]$ and positive in $[0, \pi]$. Similarly, for the cosine function, the x-coordinates are positive and negative in relation to in which quadrants the cosine is positive and negative in the unit circle.

We can recognize functions' properties through their graphical representations. For example, we observe that the sine and cosine functions are periodical, because they repeat on intervals of length 2π. This can also be expressed by the periodicity identities $\sin(2k\pi + \theta) = \sin\theta$ and $\cos(2k\pi + \theta) = \cos\theta$, $k\in\mathbb{Z}$. We also have that the sine function, being odd ($\sin(-x)= -\sin(x)$, is symmetric about the origin, whereas the cosine function, being even ($\cos(-x) = \cos(x)$), is symmetric about the y-axis.

2.5 Trigonometric Functions and Complex Numbers:

We define the complex numbers as $\mathbb{C} = \{x + iy \mid (x,y) \in \mathbb{R}^2\}$, $i^2 = -1$. A complex number $z = x + iy$ consists of the real part $Re(z) = x$ and the imaginary part $Im(z) = y$. In the Cartesian plane, the point (x,y) can be represented in polar coordinates as $(x,y) = (r\cos\theta, r\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, where θ is the angle formed by the radius, between the point and the x-axis.

The Complex plane is a coordinate system in which every point corresponds to a complex number $x + iy$. The horizontal axis is the real axis, the vertical axis is the imaginary axis and the complex number is a vector in the plane. Now, using polar coordinates, we have that the trigonometric form of a complex number z is, $z = x + iy = r(\cos\theta + i\sin\theta)$, where r is the length of the vector and θ is the angle made with the real axis. Points on the unit circle can now be given by complex numbers, as seen in figure 2.7. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ is a fundamental relationship between the trigonometric function and the complex exponential function. We can now easily deduce that $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

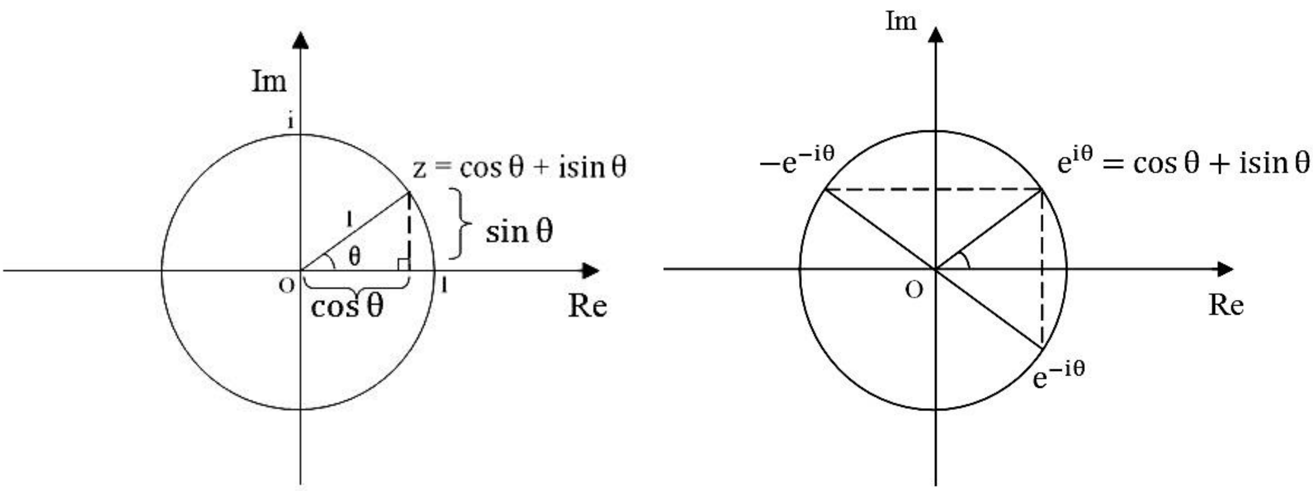


Figure 2.7: To the left the trigonometric form of a complex number z and to the right, its connection to the exponential function through Euler's Formula.

Furthermore, sine and cosine can be represented as power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The exponential function can also be represented as the power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Hence, ignoring convergence issues, we have that:

$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots = \cos x + i\sin x$, which gives the Euler's Formula.

Furthermore, we can use Euler's formula to deduce some trigonometric identities:

$e^{i(x+y)} = e^{ix} \cdot e^{iy} = (\cos x + i\sin x) \cdot (\cos y + i\sin y) =$
$\cos x \cdot \cos y + i\cos x \cdot \sin y + i\sin x \cdot \cos y - \sin x \cdot \sin y =$
$(\cos x \cdot \cos y - \sin x \cdot \sin y) + i(\sin x \cdot \cos y + \sin y \cdot \cos x)$.

But $e^{i(x+y)} = \cos(x + y) + i\sin(x + y)$, so if we first equate the real parts and then the imaginary parts, we get the angle sum identities:

$\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$ and

$\sin(x + y) = \sin x \cdot \cos y + \sin y \cdot \cos x$.

If in the first relation we take for example $y = \frac{\pi}{2}$, we get $\cos(\frac{\pi}{2} + x) = -\sin x$. Also, if we take y to be $-y$, we get the angle subtraction identities:

$\cos(x - y) = \cos x \cdot \cos y + \sin x \cdot \sin y$ and

$\sin(x - y) = \sin x \cdot \cos y - \sin y \cdot \cos x$

Now, if we take it a step further, by adding or subtracting $\sin(x+y)$ to $\sin(x-y)$, and setting $x + y = \theta$ and $x - y = \varphi$, we get the sum identities: $\sin\theta \pm \sin\varphi = 2\sin(\frac{\theta\pm\varphi}{2}) \cdot \cos(\frac{\theta\mp\varphi}{2})$.

Likewise, we get that $\cos\theta + \cos\varphi = 2\cos(\frac{\theta+\varphi}{2})\cdot\cos(\frac{\theta-\varphi}{2})$ and $\cos\theta - \cos\varphi = -2\sin(\frac{\theta+\varphi}{2}) \cdot \sin(\frac{\theta-\varphi}{2})$

Now, take $e^{i(x+x)} = e^{ix}\cdot e^{ix} = (\cos x + i\sin x)\cdot(\cos x + i\sin x) = \cos^2 x - \sin^2 x + 2i\cdot\cos x\cdot\sin x = (\cos^2 x - \sin^2 x) + i(2\cos x\cdot\sin x)$.

But $e^{i(x+x)} = e^{i2x} = \cos(2x) + i\sin(2x)$ and as before, if we equate the real parts together and the imaginary parts together, we get the double angle identities :

$\cos^2 x - \sin^2 x = \cos(2x)$ and $2\cos(x)\cdot\sin(x) = \sin(2x)$.

Now, for every complex number x and every integer n, we can use Euler's formula and get:

$(\cos x + i\sin x)^n = (e^{ix})^n = e^{i(nx)} = \cos(nx) + i\sin(nx)$, which is known as De Moivre's formula.

2.6 Solving Trigonometric Equations:

If θ is a solution of $\sin(x) = a$, then $x = 2k\pi + \theta$, or $x = (2k+1)\pi - \theta$, $k\in\mathbb{Z}$.

The proof is quite easy: On the unit circle take the points $P(\cos(\theta), \sin(\theta))$ and $P'(\cos(\pi-\theta), \sin(\pi-\theta))$. If θ is a solution of $\sin(x) = a$, (and so $\sin(\theta) = a$), then also $\sin(\pi-\theta) = a$. Hence, $x = 2k\pi + \theta$ or $x = 2k\pi + \pi - \theta$, $k\in\mathbb{Z}$.

Regarding the solution of $\cos(x) = a$, we take the points $P(\cos(\theta), \sin(\theta))$ and $P'(\cos(-\theta), \sin(-\theta))$ on the unit circle and working similarly, if θ is a solution of $\cos(x) = a$, then we have that $a = \cos(\theta) = \cos(-\theta)$ and hence, $x = 2k\pi + \theta$ or $x = 2k\pi - \theta$, $k\in\mathbb{Z}$.

3. *Literature related to students' difficulties with Trigonometry:*

3.1 Interplay between settings in Trigonometry:

To define what a setting is, we first need to define the notion of object. According to Godino and Batanero (1998, p.8), mathematical objects are abstract entities, which emerge from the socially shared systems of mathematical practices, and which are connected to a field of mathematical problems. For example, a straight line or a function are mathematical objects.

Duval argues that a mathematical object should not be confused by its representation. For instance, a triangle as a mathematical object should not be identified with a drawing of a triangle (as cited in Winsløw, 2003, p.272). Likewise, the object of a straight line should not be identified with a drawing of it, as there are different ways to represent it (for example, a symbolic representation could be y = x, and a diagrammatic representation could be a Cartesian graph where the line passes through the origin).

Moreover, Duval argues about the importance of transitioning between the different representations of a mathematical object (as cited in Winsløw, 2003, p.272). To see this, let us consider the unit circle as a mathematical object. Below in figure 3.1 there are three of its representations, as presented in Winsløw (2003, p.272). In the three images we see the geometric representation of the unit circle and two symbolic representations of it. Now, transitioning from the first to the third representation we can get the information 0≤t≤2π, whereas from the third to the second representation, we can observe that the radius of the circle is 1.

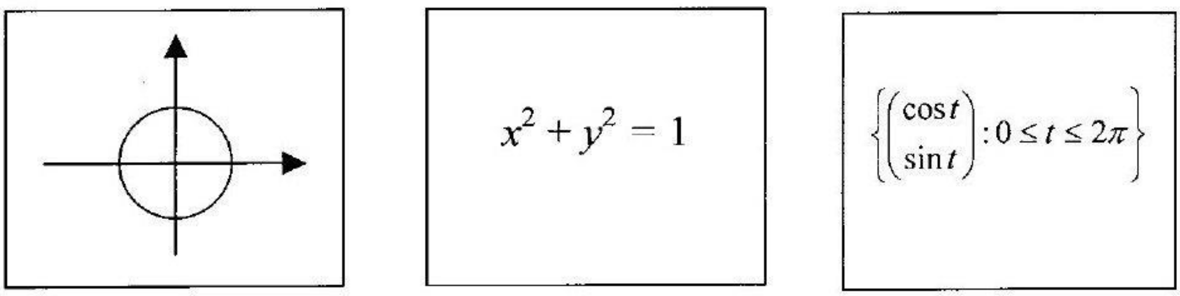


Figure 3.1: Three representations of the mathematical object of the unit circle. Image taken from Winsløw (2003, p.73).

The mathematical objects, together with the relations between them, their different formulations and the mental images with which they are associated, form a setting (Douady, 1985, p.39). Examples of settings are the circle setting, the triangle setting, the function setting, but many others as well. For example, in the setting of high school functions, we could for instance encounter differentiating, finding the domain of the function, find the value of a function for a given x, etc. The triangle setting can for example include everything from mental images of triangles, to diagrammatic representations (drawing triangles on paper or constructing triangles on CAS), symbolic representations ($\widehat{A} + \widehat{B} + \hat{C} = 180°$), etc.

When the student changes the setting with which he works, he can obtain different information for the same problem, which can help him see the problem from a different perspective and use a different technique to solve it. Douady (1985, p.40) refers to this change between settings, as "a translation (of a problem) from one setting to another", or as "an interplay between settings". Douady (1985, p.41) distinguishes three phases of IBS (interplay between settings): transfer and interpretation, imperfect links and improving the links and extending knowledge.
During transfer and interpretation, the student tries to solve the problem in the specific setting that he is working with. Using his mathematical background and experience, he can translate the problem into a different setting, creating links between the two settings. Some of them though, are not well-established, creating imperfect links. Now, when those links between the settings get improved, the student will be able to extend his knowledge, by making use of the new information he obtains from all the different settings.

In this thesis, we will encounter three settings, all related to Trigonometry: the triangle setting, the unit circle setting and the function setting (see figure 3.2). According to Demir and Heck (as presented in Winsløw, 2016), the first introduction to Trigonometry is usually made by using the triangle setting. Here, students calculate sine and cosine of the triangle's angles. In this setting, angles are between 0° and 90°. Later, the hypotenuse of the right-angled triangle is the radius 1 of the unit circle and the sine and cosine are defined as coordinates of the intersection of the radius with the unit circle. This is the unit circle setting. At this point, students are introduced to negative angles and angles over 90°, or over 360°, and students should also be able to visualize $\cos x$ and $\sin x$ as sides of a given right-angled triangle within the unit circle. Finally, through the function setting, students will recognize that sine and cosine are functions, interpret them in terms of their properties, such as their domain, period, symmetry etc. and realize that the input x of the function sin x, is a real number.

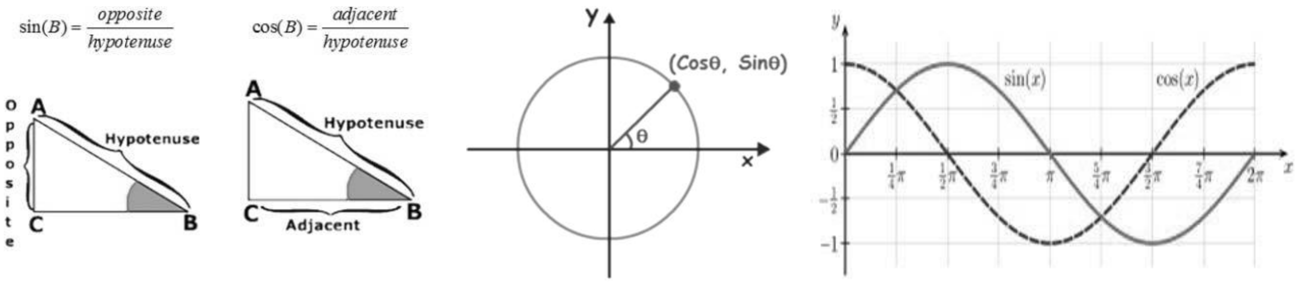


Figure 3.2: The trigonometric settings. Images from Winsløw (2016).

3.2 More students' difficulties with respect to Trigonometry:

Brown (2005) argues that many students have fragmented views of the sine and cosine functions, mainly because of a lack of prerequisite knowledge (such as knowledge of coordinates, distance from the axes, etc.), or because they do not consider sine and cosine as both ratios and numbers (as cited in Wescoatt, 2013, p.34). Thinking of sine or cosine as a ratio refers to the triangle setting, whereas thinking of it as a number, refers to the function or the unit circle setting. This implies that students who do not see a connection between the different trigonometric settings, have fragmented perceptions of sine and cosine. So, if for example a student only works in the triangle setting, he cannot explain the identity $\cos(-x) = \cos(x)$. On the other hand, he could explain it in the function setting, using the graph of cosine. Solving a trigonometric equation is another example. It cannot be solved using the triangle setting, but it can be solved using the sine or cosine graph and observe where the graph cuts the x-axis. Hence, it is useful that students transfer between settings, in order to handle the same problem from a different perspective.

In particular, regarding difficulties in the transition from the triangle to the unit circle setting, Akkoc (2008) and Moore (2013) argued that it can be due to the angle measure. In other words, the difficulty may lie in the connection between radians and degrees for measuring angles (as cited in Dejarnette, 2014, p.23). It is often that the students do not know whether they should use degrees or radians and why. Kupkova (2008, p.73) argues that radians are usually seen as an angle measure in high schools, and the students are not able to connect it to the length of an arc of the unit circle. To this, she adds that the way that students are introduced to radians is with respect to degrees. In other words, they learn the π = 180° equivalence, neglecting the fact that π = 3.14. Kupkova had conducted a small study among 44 Slovak college students, where she asked them if the number π = 3.14, has anything to do with the π that they see used in the trigonometric functions. All students marked their x-axes with $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π … , but 68% of the students saw no relation between π in those values and 3.14.

Another problem is this of the function notation. As students learn that sin x is a function, the properties of sine and cosine, etc., they may confuse the new information with the already existing rules of algebra. For example, they may think that sin x is a multiplication between s, i, n and x, just as ab is a multiplication between a and b. They could also get confused by the lack of parenthesis in sin x, also misinterpreting it for sin times x. Sajka (2003) argues that not understanding the symbolic notation of the new system could lead to not solving problems involving objects represented by this new system (as cited in Wescoatt, 2013, p.32).

Another difficulty appears to be that even though sine and cosine are presented to students through the sine and cosine graph, the students sometimes do not connect these to trigonometric functions. Breidenbach, Dubinsky, Hawk and Nichols (1992) stated that it is hard for students to see trigonometric functions as functions, because they are not expressed as algebraic formulae involving arithmetical procedures (as cited in Weber, 2005, p.91 and Dejarnette, 2014, p.22). Weber (2008, p.144) stated that a way to understand a trigonometric function is to compare it to taking a square root of a number. In other words, a trigonometric function can be thought as an operation applied to angles, having the angle as the input and a real number as an output (Weber, 2008, p.145).

Weber’s research in 2005 gives us information on students’ perceptions of sine and cosine. He investigated how students perceive trigonometric functions and how they can use these perceptions to explain the trigonometric functions' properties. He conducted a study, where two groups of college students from the same college were taught Trigonometry; the first group was taught in the traditional way, mainly following the textbook, and the second group was taught in an experimental way. The first group was taught by a college professor who was not participating in the study, whereas the second group was taught by Weber himself.

During the first lectures of the experimental class, the students formed small groups to work in. Firstly, they had to learn how to follow a procedure solving a trigonometric exercise, for example, calculating $\sin 20°$. Afterwards, the teacher would evaluate if the students had successfully completed the procedure. Then, the students would follow the same procedure multiple times, with the intention that they would begin to anticipate specific results without the need to complete all steps of the procedure. Afterwards, there were open discussions, where the students justified some of the results they obtained, without having completed the whole procedure. Such a question would be, if $\sin 23°$ or $\sin 37°$ is bigger and why. The students were expected to argue that $\sin 37°$ is bigger than $\sin 23°$, because its intersection with the circle was "higher up". (Weber, 2005, p.96). Furthermore, the students were asked to explain parts of the procedure (for example, why $\sin x$ can never be 2).

After the lectures, the students were asked a series of questions and then some of the students were interviewed. The purpose of the questions was to investigate how students would describe properties of trigonometric functions. The results showed that the students of the group taught in the traditional way could not construct geometrical representations on their own, which would be related to the given mathematical task. On the contrary, the students were expecting to be given the geometrical representation in addition to the mathematical task. Furthermore, they were not able to explain properties of trigonometric functions. On the other hand, the students of the group which was taught in the experimental way, were able to explain trigonometric functions' properties, mainly by the use of the unit circle setting. As it is relevant to this thesis, results from both groups will now be presented. We will name the standard group, "first group", and the experimental group, "second group".

To the question "describe $\sin x$", two of the four students of the first group described $\sin x$ as ratios of sides in a right-angled triangle. The other two students answered that it depends on whether they were given a triangle. If they were, then they would think of $\sin x$ as a ratio, but if they were asked to find the sine of a "known" angle, say $\sin 30°$ then they would already know the answer, since they had memorized the trigonometric ratios table. In the second group, all four students described $\sin x$ with respect to a geometric construction. Three of them used the unit circle and one used the right-angled triangle.

Four students from the first group and four from the second group were asked why $\sin x$ is a function. None of them understood the question, so it was rephrased as "How do you know that $\sin x$ can only have one value for a given x?". None of the first group was able to answer, even though two students stated with certainty that $\sin x$ is a function. After students were told that an operation was a function where each input had a unique output, three students of the second group pointed out that $\sin x$ only gives one answer, whereas the fourth student was not able to explain.

Moreover, when students were asked to explain why $\sin^2 x + \cos^2 x = 1$, only 12.9% of the students (4 out of 31) of the first group answered correctly using a right-angled triangle, and none of the students used the unit circle. In the second group, whereas 37.5% of the students (15 out of 40) gave a valid explanation, only 2 students preferred the unit circle over the other 13 who used the right-angled triangle.

4. Research Questions:

The Research Questions (RQ's) for this thesis are:

RQ1: How do students transfer between the three trigonometric settings? Which setting do they primarily choose to work with and why?

The aim of this research question is to investigate if students can use the three trigonometric settings (triangle setting, unit circle setting, function setting) or if they see them as disconnected. For example, which setting do they use explain that a trigonometric identity holds? In which setting do they explain the difference between a trigonometric identity and solving a trigonometric equation?

RQ2: To which extent do students perceive sine and cosine as functions? Which students' misconceptions connected to sine and cosine can be found?

In the context of RQ2, we are particularly interested in investigating how students perceive the functions sine and cosine. Do they realize, for example, that x is the input of the trigonometric function sine, whereas $\sin(x)$ is the output? We would also like to investigate if students express themselves in a proper mathematical way, by explicitly using the word "function" (for example, sine is a function because for every input there is only one output). Finally, we would like to gather all information which can be related to the input of sine and cosine. For example, to which trigonometric setting have the students connected the notion of angle?

The questions will be answered after analyzing the data gathered from the observations of lessons of two classes and from students' interviews of a Danish upper secondary school.

5. *Some elements of the Theory of Didactical Situations:*

5.1 Didactical and adidactical situations, and the didactical contract:

According to to TDS (Theory of Didactical Situations), a theory initiated by Guy Brousseau from the late 1960's, a didactical situation is a situation in which there is a direct or indirect manifestation of a will to teach (Brousseau, 1997, p.214). Brousseau states that the interplay between the teacher, the student and the mathematical content to be explored through the milieu is a didactical situation (as cited in Strømskag, 2015, p.13). The adidactical or pseudo-adidactical situation is a situation organized by the teacher and it has a didactical purpose. The students are intentionally left without the teacher's interventions and without his will, so that they work autonomously, interacting only with the milieu (Brousseau, 1997, p. 236).

During the teaching process, there are mutual expectations and behaviors from the teacher towards the students and conversely. All those expectations and responsibilities to one another form the didactical contract. A didactical contract can differ from classroom to classroom. We will now present a problem known as the "Age of the Captain", where the traces of the didactic contract are visible.

Some researchers at the Institute of Research on the Teaching of Mathematics (IREM of Grenoble) asked some 8-year-old students the following problem: "On a boat there are 26 sheep and 10 goats. How old is the captain?". 76 out of 97 students answered "36 years old". Naturally, there was a large amount of reactions towards the teachers, blaming them for their way of teaching, but as Brousseau argued (as cited in Brousseau, Sarrazy & Novotná, 2014, p.154) this was a matter of an "effect of the contract". Indeed, when the students were asked why they answered to such a "stupid" (using their own words) problem, they answered that they did so because the teacher asked for it.

For the purpose of this thesis, we will try to detect traces of the didactical contract during the students' interviews, or from the students' notebooks. We are interested in investigating how big an influence the didactical contract can be for students and to which extent it affects their answers and their choice of strategy.

5.2 Origins of didactical obstacles in TDS:

There are three types of obstacles according to Brousseau (1997, p.86): those of ontogenetic origin (or ontogenetic obstacles), of didactical origin (often seen as didactical obstacles) and of epistemological origin (or epistemological obstacles). The last two belong to the category of epigenetic obstacles (Manno, 2004, p.33).

The ontogenetic obstacle is the obstacle that the student faces due to his cognitive level. It can for example be that his cognitive level does not correspond to his age, due to slower mental development (Manno, 2004, p.33).

A didactical obstacle is a conflict in knowledge, stemming from the way of teaching (Ruthven, Laborde, Leach & Tibergien (2009), Manno, (2004)). For example, the number π is firstly introduced to students as 3.14. Later, the students learn that π is equal to π = 3.14159…., continuing with infinitely many decimals and not having a periodic decimal expansion. So, in the first situation, a student could write π as the rational number $\frac{314}{100}$, but in the latter situation, π is irrational. Moreover, later in school, the teacher for the first time introduces π as the ratio of a circle's circumference to its diameter, a definition which had been absent until that given moment. Gradually, the students will learn to use π without substituting it with 3.14. For example, if we consider the sine graph, some of the x-axis values are $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$. Hence we can see, that depending on the way of teaching, (for instance the students' grade can be a reason why), the teacher choses specific ways to teach and specific information with which he presents a mathematical topic. This results in a conflict of the old with the new knowledge.

> “This happens because when students learn a new mathematical concept, their minds are not blank slates, but they are filled with already existing knowledge, beliefs and experience. New knowledge is not simply added on, but it must be merged with the old knowledge. When the old knowledge contradicts the new knowledge, the old knowledge may become an obstacle to learning the new one.” (Sierpinska, 2003, p.83)

The notion of epistemological obstacle was firstly used by Bachelard in 1938, in the context of natural sciences. In TDS the term is used for those obstacles that everyone encounters with respect to the new mathematical knowledge. In other words they are unavoidable, because they stem by the nature of the knowledge. Bachelard argues that no one can, nor should escape them, because of their formative role in the knowledge being sought (as cited in Ruthven, Laborde, Leach & Tibergien, 2009, p.2).
In this thesis we will focus on students’ didactical and epistemological obstacles.

6. Methodology:

6.1 Collection of data and information about the lessons observed:

The data collected is gathered from Ordrup Gymnasium, a public upper secondary school in Denmark. The data is taken from attending lessons taught in two different classes and taught by two different teachers. From now on, we will refer to the classes as X and Y. The teacher of Class X has had four years of experience, whereas the teacher of Class Y has had three years of experience, having also written her Master's Thesis on trigonometric functions and how they are taught in secondary school. The observations were conducted in September and October of 2019.

The Danish students sit for national exams at the end of secondary school, so, all the teachers need to have taught the official curriculum by the end of secondary school. They do, however, have the freedom to teach with the order and speed they think is more suitable to the individual class. In other words, they do not necessarily follow the official material in the order it is found in the textbooks, as long as the students have been taught the necessary curriculum for the exams by the end of secondary school. So, how fast or how much in depth something is taught, depends entirely on the teacher. In our case, both Classes X and Y were taught the curriculum of the next grade (they were taught mathematics of the third grade, while they were on the second grade). Moreover, both classes consisted of students of stx level A, all at their second year of high school. The study line was independent for each student and did not have any relation to if the student attended Class X or Y. This means that not all students had the same courses and on the same level. Class X consisted of 28 students, 18 girls and 10 boys and Class Y of 24 students, 12 girls and 12 boys.

Every student had a laptop at his disposal and the textbook "Matematisk Formelsamling, stxA" (Mathematics Formula Collection) which was only used as a supplement, and in no case as their main source. The material which the students were taught was from the textbook A3stx

Mathematics, even though it was not used during the lessons. The students were using internet in the classroom and the CAS-tool Nspire, which they were not only able, but in fact required, to use. Most of the times the students worked with their laptops (Nspire, internet, Word), but significantly less with pen and paper.

Class X was taught Trigonometry in two lessons. The first lesson lasted an entire day of seven school hours and the second one, one school hour. Class Y had four lessons of Trigonometry. In total, both classes had eight school hours of Trigonometry. Finally, each lesson lasted between 35 to 45 minutes.

6.2 Methodology of data collection:

Data was gathered by participating in the lessons, as well as interviewing some of the students, after the Trigonometry lessons were completed. During all lessons, notes were taken that would afterwards help reconstruct the lesson as precisely as possible. The data was gathered from photos of the whiteboard, smartboard, the students' screens, their notebooks, video-recordings of the teacher and audio-recordings of the students. All interviews got audio-recorded and photos of what the students wrote were taken.

The interviews conducted included five students from Class X and six students from Class Y, who volunteered to be interviewed. Also, there were two more students from Class X, who did not show up. They were all informed that the interviews did not aim to test their knowledge, rather to investigate their ways of handling Trigonometry and trigonometric problems. They were also informed that their participation and answers would remain anonymous, something which seemed to encourage more students to volunteer. This is also the reason why we do not mention in which country Student 9 had finished the lower secondary school. Each student of Class X was interviewed for approximately 3-11 minutes, whereas each student of Class Y, for approximately 8-13 minutes. The length of the interviews depended on the amount of time that the students were able to be away from class, as the interviews took place at the same time as their last lesson of Trigonometry, in a different classroom.

6.3 Questionnaire:

Below are the questions conducted during the interviews. The students were given one page. On the front were the questions and on the back were the cosine and sine graphs.

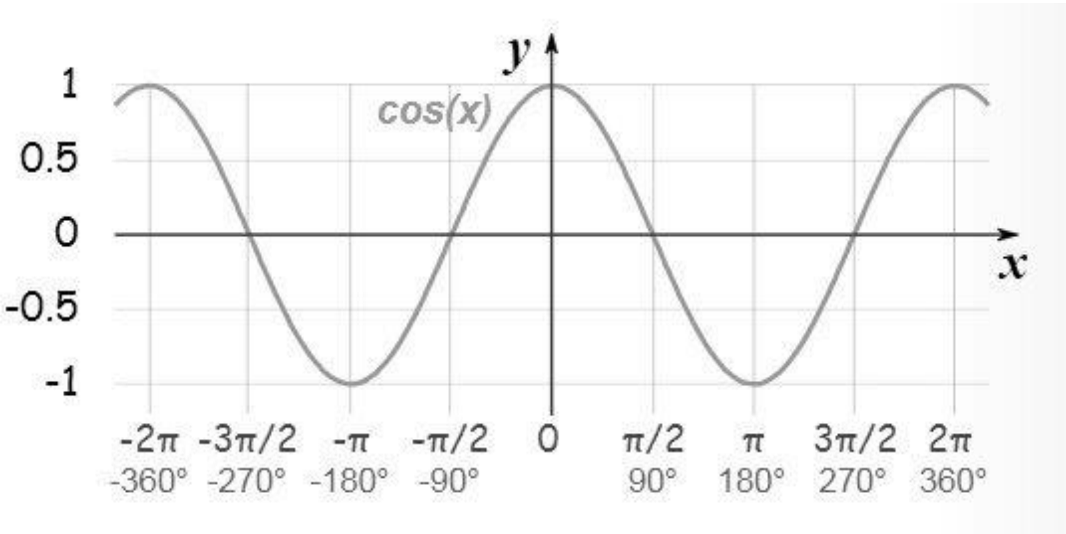
<u>Questions</u>

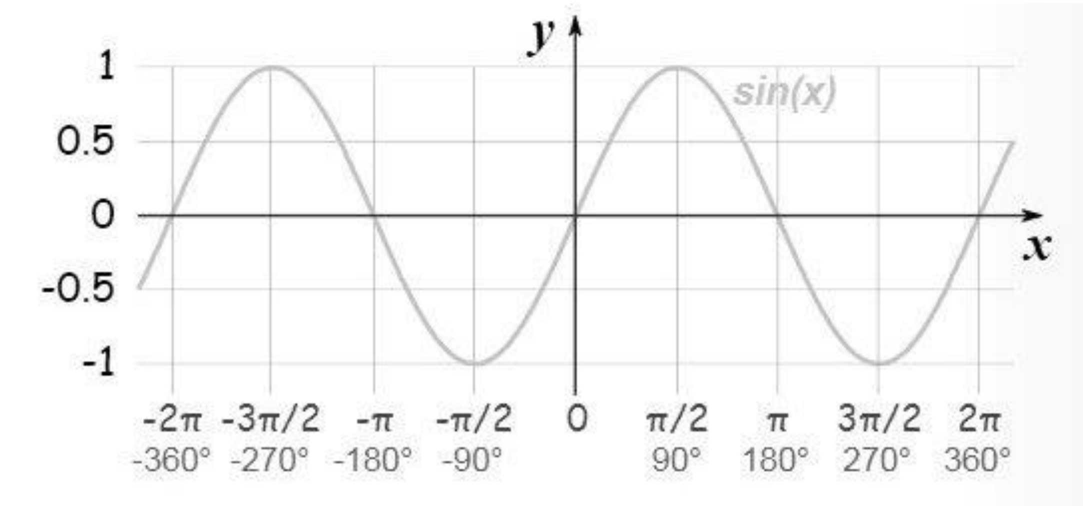
1. Can you explain these relations?
   - $\cos(x + 2\pi) = \cos(x)$
   - $\cos^2(x) + \sin^2(x) = 1$

2. What is the difference between these?
   - solve $\cos(x) = 0$
   - $\cos(\frac{\pi}{2}) = 0$

3. Apart from the unit circle, where else have you seen sine and cosine?
4. What does $\cos(-x) = \cos(x)$ mean graphically?
5. Do you prefer to work with radians or degrees? Why?
6. What is an angle?

Yes/no questions	Yes	No
7. $-390° = 30°$	◯	◯
8. $\frac{\sin x}{x} = \sin$	◯	◯
9. $\sin(-\frac{\pi}{2} + \pi) = \sin\left(-\frac{\pi}{2}\right) + \pi$	◯	◯
10. $8\pi = 2\pi$	◯	◯





6.4 Constructing the interviews' questions and an a-priori analysis of the interviews:

We will now explain how we constructed the questionnaire and we will also write an a-priori analysis for the interviews, including which answers we expect to get during the interviews. The questionnaire for the interviews consists of two parts: general questions about Trigonometry and yes/no questions. The yes/no questions were primarily made due to the short amount of time we knew there would be for each interview. In general, a part of the questions was designed so that they are similar to those answered and analyzed in class. For example, Question 1 uses trigonometric identities from the book “Matematisk Formelsamling stx”. Others were not similar, aiming to investigate the students’ understanding of trigonometric identities and functions, and their way of thinking in unfamiliar situations. Question 2 is such an example: Even though students were taught how to “solve $\cos(x) = 0$” and verify that “$\cos(\frac{\pi}{2}) = 0$”, we assume that they were not explicitly taught the difference between these two. All questions of the questionnaire aimed to answer the research questions.

For the construction of the questionnaire, as well as questions that we would like to ask the students during the interview, we borrowed some questions from Weber (2005), who had also interviewed students, investigating their perceptions on trigonometric functions.

- Why is it true that $\sin^2 x + \cos^2 x = 1$?
- Describe $\sin x$.
- Why is $\sin x$ a function?

The first bullet is a part of our Question 1, whereas the other two, even though they are not a part of the questionnaire, will be used during the interview to help the students answer Question 8 and Question 9.
The construction of Question 8 ($\frac{\sin x}{x} = \sin$), was strongly recommended by an upper secondary school teacher with 35 years of experience in Greek schools, who argued that a big amount of students reply that this is correct. We were curious to investigate if this would also happen with our students. It is expected that some students will answer “yes”, but it is also expected that they will change their answer to “no”, if they will be later asked what “sin” stands for, or, following Weber’s question, what is $\sin x$. Question 9, $\sin(-\frac{\pi}{2} + \pi) = \sin\left(-\frac{\pi}{2}\right) + \pi$, is also constructed to investigate how students perceive sine. It is expected that some students will try to calculate both parts of the equation to see if it holds. We would like to observe though, if without calculating both sides, they can explain that it is different for $+\pi$ to be, or not to be a part of the input of sine. In total, through Questions 8 and 9, we would like to investigate the students’ perceptions of the concept of function and its properties, as well as problems connected to function notation.

Question 1 aims to investigate how students perceive trigonometric identities. Even though it is uncertain that they know the term “identity”, the goal is to investigate which setting they will use in order to explain that a trigonometric identity holds. Those specific identities were chosen because they can be assumed to be well-known to students. In fact, from the list of trigonometric identities which is found in the book Matematisk Formelsamling stxA, $\sin^2 x + \cos^2 x = 1$ is the first identity, whereas $\cos(x + 2\pi) = \cos(x)$ is the second one. The fact that those two identities are the first ones on the list of the trigonometric identities in the book, suggests that it is more possible for students to recall them, because they will probably have spent more time engaging with them and trying to memorize them. This list includes in total seven trigonometric identities: $\sin^2 x + \cos^2 x = 1$, $\cos(x + 2\pi) = \cos(x)$, $\cos(-x) = \cos(x)$, $\cos(\pi - x) = -\cos(x)$, $\sin(x + 2\pi) = \sin(x)$, $\sin(-x) = -\sin(x)$ and $\sin(\pi - x) = \sin(x)$. We would also like to observe if there is a reason why students prefer a specific setting, and if they can transfer between the trigonometric settings. For example, the identity $\cos(x + 2\pi) = \cos(x)$ can be explained in the unit circle setting, arguing that a full revolution around the unit circle is 2π and so, the relation is true. Alternatively, the students can describe this identity by picking different values of x and observing that the relation holds. Another way this relation can be explained, is with the use of the graph setting. Looking at the cosine graph, the students can pick a specific value of x and observe that the identity holds. They can later try with another x and conclude that since the cosine graph repeats itself, the identity will always hold.

Regarding the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$, we expect that students will explain it by using the unit circle, since they did not use the triangle setting during this year's trigonometry lessons. A correct answer would be to pick a point (x,y) on the unit circle and connect it to the origin of the circle, where an angle, say v, is formed. Then by using the fact that $x = \cos v$ and $y = \sin v$, and that the equation of the unit circle is $x^2 + y^2 = 1$, the student will conclude that the Pythagorean identity holds. As explained above, even though the use of the triangle setting is not expected to be seen, a student may explain the identity for acute angles, using the same procedure as in section 2.4.1. He may also attempt to prove the relation by using specific angles and notice that the relation holds. Moreover, we might observe notation problems. We expect though, that most of the students will recall that $(\cos(x))^2 = \cos^2(x)$.

Question 3 was constructed to help students make the connection between the different settings. We would like to mention here, that it is not necessary that the questions are asked in the specific order of the questionnaire, and questions will be revisited during the interview (see also about the strategy of the interviews in the next section). So, the aim of Question 3 is to revisit the identity $\cos^2(x) + \sin^2(x) = 1$ of Question 1 and observe if the students can explain the identity by using the triangle setting.

To collect more data for RQ1, apart from Question 3 and the identity $\cos(x + 2\pi) = \cos(x)$ in Question 1, we also constructed Question 4. The common intention while constructing those two questions was to gather more information on how students transfer between the trigonometric settings. The explanation of Question 4 is similar to this of the identity $\cos(x + 2\pi) = \cos(x)$ in Question1. However what is different, is that students will specifically be asked to describe the identity $\cos(-x) = \cos(x)$ in the graph setting, and so, we also expect that some students' answers will refer to the fact that cosine is an even function, by explaining that the graph of cosine is symmetric with respect to the y-axis. Nevertheless, we assume that many students will first try to use the unit circle setting. A reason why we suspect that the graph setting will be harder for them, is that even though they had explained the trigonometric identities with the use of unit circle during the lessons, this did not happen for the function setting as well. It should also be mentioned that in Class X the teacher only mentioned the identities $\cos(x + 2\pi) = \cos(x)$ and $\sin(x + 2\pi) = \sin(x)$, and not any others. On the other hand, in Class Y, the students spent time to explain all the trigonometric identities which were included in their textbook (as presented in the previous page), by use of the unit circle.

So, the reason why the questionnaire includes the graphs of sine and cosine, is to save some time from the interview, since it is not certain that students would construct them on their own. Depending on the time we will have left for the interview, we might at first not show the graph to the student, to see if he can construct the graph on his own. In other cases though with more limited time, we will immediately provide the graph to the student. However, we do not think that graphing the sine and cosine graph on paper is something we should particularly focus on, since it is something the students usually do with the use of CAS. It is interesting though to observe, if there is time, if a student can roughly draw the graph, or if he has learnt to heavily rely on the CAS-tool.

Having given two identities in Question 1, Question 2 aims to investigate if the students find a difference between a trigonometric identity, or in other words a trigonometric equation which holds for a specific value ($\cos(\frac{\pi}{2})$ = 0), and a trigonometric equation to be solved (solve $\cos(x) = 0$). In case they see a difference, which setting do they use to explain it? It is expected that the students will find this comparison challenging, because it is not often that the teacher points out specifically the difference between the two. A student who sees the difference between those two, is expected to answer that $\cos(\frac{\pi}{2}) = 0$ is an equality that (always) holds for $\frac{\pi}{2}$. On the other hand, "solve $\cos(x) = 0$" has infinitely many solutions. A student, observing the function setting or the unit circle setting, can for example argue that the cases where $\cos(x)$ = 0, are at $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, …etc. Then, he can point out that the solutions are infinitely many, either by looking at the graph setting and observing that the cosine graph repeats itself, and so $\cos(x)$ equals 0 infinitely many times, or by looking at the unit circle and noticing that $\cos(x)$ will be equal to 0 infinitely many times, since he can rotate around the circle infinitely many times as well. Finally, we might ask the students if $\cos(\frac{\pi}{2})$ = 0 and $\cos(x)$ = 0 are equalities which are always true, in order to gather more data on how they perceive trigonometric equations and their solutions, and to additionally redirect their thinking in case they cannot explain the difference. However, to be clear, answering the above, would not constitute an answer to Question 2, as in Question 2 we are interested in explaining the difference between $\cos(\frac{\pi}{2})$ = 0 and solving $\cos(x)$ = 0.

Questions 5, 6, 7 and 10 were constructed to gain insight into the students' perception of an angle (quale, quantum or relation – see section 2.1), as well as how students consider an angle in each trigonometric setting. We are also interested in their explanations of their preference of radians or degrees, rather than merely which they choose, as well as if they have connected degrees and radians to a specific trigonometric setting. Will for example a student mention that he uses degrees in the triangle setting? How easy will he find it to change between radians and degrees? In particular about Question 6, it is expected that most students will answer that an angle is something that can be measured (for example, an angle can be 30°). Such an answer is connected to the first introduction in primary school about angles, where the students use the protractor to measure and draw angles. This is also connected to the triangle setting, where students are used to translate the triangle's angles as their measure (for example a usual task is to know the degrees of two angles of the triangle and find the degrees which correspond to the third angle). Another answer we expect is for students to relate an angle to its quale, in other words, to identify an angle with its figure. On the other hand, perceiving an angle as a relation would mean that in the unit circle, the students see a central angle as the length of the arc which subtends the angle.

Regarding Question 7 ($-390° = 30°$), it is expected that many students will answer yes. Knowing that a circle corresponds to 360°, they might add 360° to $-390°$. However, instead of $-30°$, they may think that $-390° + 360° = 30°$, which will be the reason why their answer will be "yes". It may seem as an easy enough question but firstly, it aims to investigate if the students, who have been heavily relying on the calculators and CAS-tools over the years, can perform a simple calculation on their own, and secondly, how they will react to the sight of a negative angle. We expect that those who will say "no", it will be because $-390°$ is not on the same location on the unit circle as 30°.

It is also expected that many students will say "yes", to $8\pi = 2\pi$ (Question 10), thinking that 8π rads is on the same place on the unit circle as 2π rads. It could also be that instead on focusing whether $8\pi = 2\pi$, they might observe that $\cos(8\pi) = \cos(2\pi)$, or that $\sin(8\pi) = \sin(2\pi)$, and conclude therefore that $8\pi = 2\pi$, confusing the input with the output of a trigonometric function. On the other hand, students who will try to answer this question using the function setting, they will answer "no", after placing 2π and 8π on the x-axis. Another way to think this question can be to substitute π with 3.14 on the equation and conclude that $8\pi \neq 2\pi$.

6.5 Strategy of conducting the interviews:

In general, the strategy which will be used during the interviews is to not give any answers for the questions. However, if a student does not answer anything, or if we see that he cannot elaborate further because he is very confused, or that he replies something which is not connected to the question, we may give some hints to help him continue and to also save some time for the interview. Moreover, if the student makes a mistake, we would like to give him hints towards the right direction, instead of telling him which his mistake is and how he should correct it. For example, if a student states that $\frac{\sin x}{x}$ = sin, we will not say that this is wrong, but we will give some hints to indicate that this is not correct. We can for instance ask if he would have replied the same, had the question included a parenthesis ($\frac{\sin(x)}{x}$ = sin). We may also ask him to describe $\sin x$ and sin and to find the difference between the two.

The questions which will be selected for each student will depend on the amount of time we will have for each interview. It is foreseen that some questions will be skipped due to lack of time. Furthermore, the questions for each student will additionally be selected depending on the previous student's answers. For example, if a student cannot explain any trigonometric identities, the strategy is to immediately move to the yes/no questions to save some time and get some data related to the other questions. On the other hand, if the student has some difficulties with, for example, explaining the identity $\cos(x + 2\pi) = \cos(x)$ by using the function setting, but he can explain it on the unit circle, he will be encouraged to try it afterwards in the function setting, so that we can record what these difficulties are. Also, we may afterwards pick a new identity for the student to explain in the graph setting, in order to collect more data on how the specific student uses the graph setting.

Another strategy is to try to have the student compare his answers in different questions. For example, if a student answers "yes" in Question 8 and "no" in Question 9, he will then be asked to compare those two and explain which is the difference. The intention of this is to investigate if the student will change his answer to one of the two questions, how his answer will be changed and how he will justify this change.

During each interview, the paper sheets that will be used from each student to answer or draw will be marked as Student 1 to Student 11, so that all possible data is collected. Students 1 to 5 are from Class X, whereas Students 6 to 11 are from Class Y. We would like to mention once more that Students 1 to 5 were interviewed for 3-11 minutes, whereas Students 6 to 11, for approximately 8-13 minutes, making the data for Students 6 to 11 more.

6.6 Methodology of data analysis:

For this thesis, we will especially focus on the students' interviews. We consider the data to be collected from the interviews to be rather important, as we will be able to interact with the students during the interviews and ask appropriate questions when it is needed, in order to gather more information about our research questions. The data to be analyzed, will be divided into categories to facilitate the analysis and presentation of the results. Each category will include an individual question of the questionnaire, when there is enough data to present for each question, or groups of questions from the questionnaire, when there is a smaller amount of data which is gathered for each question, or in case the questions are similar to each other. All data will be analyzed with respect to the existing literature about previous studies on students' difficulties with Trigonometry, which is presented in section 3, in order to investigate whether our results agree with these studies.

After transcribing the interviews, we will have to choose the data to be analyzed. We will collect three "types" of data to be analyzed. Firstly, the students' answers with the most details will be chosen, as it is our purpose to have as much valid interpretations of the data as possible, and the amount of details in the students' explanations will significantly help. Secondly, if the students' responses are as short as "yes/no" and this substitutes the only data which we will have gathered, we will try to see if this data is relevant to other parts of the students' answers and whether we could obtain a broader image of the students' perceptions. If on the other hand it is not relevant to other points of the students' answers and explanations, and there are not any additional information which could support the specific answer, we will not present those data as a part of the results. Thirdly, if a student's answer is unclear, and he does not explain his answer further, but on the other hand, there is some external information which can support his answer, we will include this data in the results. For example, a student might not be able to explain something, but he may have asked a question before the interviews, during the lesson, which could give us additional information in order to analyze his answer.

Furthermore, the geometric and graphical constructions which the students will draw during the interviews will be presented, in order to gain a better understanding on the students' explanations. During the analysis of the interview data, there will also be presented and analyzed data from the lessons, such as photos of students' screens and students' notebooks, photos of the whiteboard and from the schoolbook, which will be relevant to the student's interviews. It can for example be a common misconception between a student from the classroom and the student who is interviewed. The purpose of this is to gather relevant students' challenges all together, in order to present the results as organized as possible. Finally, there will be an effort to detect traces of the didactical contracts of the students, which could show how their recollections of the didactical contract have affected their answers.

7. *Results from students' interviews and from the classroom:*

In this section we will present and analyze some students' answers related to the RQ's. It should be noted that the language of the interviews was both Danish and English. When a student was not familiar with the English mathematical terms, then the conversation changed into Danish. For students who did not feel comfortable speaking in English, the interview happened completely in Danish. Everything which will be presented here is translated to English. We will not mention mistakes, which seem to be due to problems of communication in the two languages (for example a student said "dot" instead of "point").

Within […] are some personal notes, which will help the reader understand information, which is implied from the context of the whole dialogue. An example of this is if a student pointed at the graph and said "there", or if he had already mentioned some information, which he does not repeat thoroughly in the part of the dialogue which will be presented. Moreover, some details of the dialogues have been left out and as a result some parts have been slightly reformulated, aiming to facilitate the reading, but without changing the content of the dialogues. Finally, we would like to mention that Students 1 to 5 are from Class X, whereas Students 6 to 11 from Class Y.

The presentation of the results will follow the order of the questions of the questionnaire. The questions with sufficient data for analysis will be presented, together with relevant results from the class. The purpose of this combination is to get a broader image of the students' challenges on the specific situations.

7.1 The trigonometric identity $\cos(x + 2\pi) = \cos(x)$:

Question: “Can you explain the relation $\cos(x + 2\pi) = \cos(x)$?”

Out of the six students asked (Student 1, 2, 7, 9, 10, 11), five of them explained the relation with respect to the unit circle setting, and one with respect to the function setting. All explanations apart from Student’s 7 can be considered correct, according to section 6.4. Student 9, a student who had finished the lower secondary school in another European country, was the only one who chose the function setting to explain the identity. Student 7 was the only student who explained the identity in the unit circle setting, and then he was additionally asked to explain this identity in the function setting. We will now present the answers of those two students.

Student 9:

> “It is because cosine starts up here [she shows at the point (0,1)], and then because it moves 2π it comes up here [she shows the point (2π,1)] and if you say $\cos(x + 2\pi)$, it gives you the exact same graph as $\cos(x)$, because it is the same thing.”

The student can distinguish between the input and output of the function cosine, stating that $\cos(x + 2\pi)$, “gives you the exact same graph as” $\cos(x)$ (and not that x gives the same as x + 2π), referring to the fact that the two points have the same height on the cosine graph. Moreover, she drew the cosine graph on her own. However, due to writing on the x-axis the value “2π”, where the value “$\frac{5\pi}{2}$” should be, a mistake which she realized on her own, she was given the printed cosine graph on the back page of the questionnaire in order to save some time.

Student 7 seemed to have difficulties explaining the identity. When he was first asked if he could explain the relation, he did not reply anything. Then, he was asked if the relation is true for all x, but he said that he “really does not know”. We continued with other parts of the questionnaire and we returned to this question again after a while.

Interviewer: Back to the first question, can you explain $\cos(x + 2\pi) = \cos(x)$?

Student 7: We just had it in the classroom. You had … You start in the unit circle and 2π is 2 rounds,1π is 180° and then another π, it is another 180°, so you are back where you started.

To begin with, we cannot be certain whether the student did not know that there are 2π rads in a full circle, or if he meant something else. From his explanation afterwards, it seems like he knew the correct information, that is, that 2π corresponds to one full rotation around the unit circle. What is curious though about the student's answer, is that even though he explained the relation correctly in terms of the unit circle setting, his geometric representation of the unit circle is not corresponding to the explanation (see figure 7.1). We cannot know why this happened, but we believe that he might have recalled another geometrical representation of the unit circle during his last hour of Trigonometry, where the teacher explained some trigonometric identities with the use of the unit circle. So, the student might have remembered this different representation, which was used during the lessons to describe the trigonometric identity $\cos(-x) = \cos(x)$, and used it without thinking further if it matched the interview's question. This implies that he was under the effect of the didactic contract. This can also be seen by his first words "We just had had it in the classroom". Afterwards, the student was encouraged to explain the identity on the cosine graph.

Interviewer: Ok, so when we say $\cos(\frac{\pi}{2})$ we are here [showing on the graph], right?

Student 7: Yes.

Interviewer: Now if you say $\cos(\frac{\pi}{2} + 2\pi)$, where would you be on the graph?

Student 7: About here somewhere. [see in the graph: I put an arrow where he showed]. Because if you are at 90° and you go another 90° [so he was actually trying to find $\cos(\frac{\pi}{2} + \frac{\pi}{2})$], you are at 180° and you are around here.

In this part of the interview, we see a confusion on spotting $\cos(\pi)$ on the cosine graph, which implies that there is lack of prerequisite knowledge on translating the graph of a function. He seemed to think that $\cos(\pi)$ is the area between the cosine graph and the x-axis. This can also be seen by his first reaction: “About here somewhere”. Moreover, when he was asked about $\cos(\frac{\pi}{2} + 2\pi)$, but he said that “you are at 90° and you go another 90°”, it might seem at first that he could not convert between radians and degrees. However, on the first part of the dialogue, he knew that 1π corresponds to 180°, so we cannot draw a conclusion on why he was thinking of $\cos(\frac{\pi}{2} + \frac{\pi}{2})$ instead of $\cos(\frac{\pi}{2} + 2\pi)$.

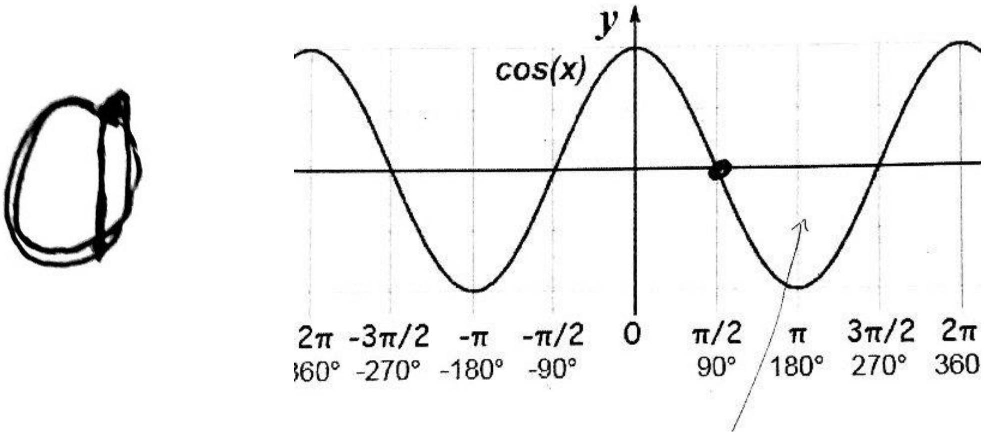


Figure 7.1: Representation of $\cos(x + 2\pi) = \cos(x)$ by Student 7, with respect to the unit circle setting and the graph setting.

7.2 The trigonometric identity $\cos^2 x + \sin^2 x = 1$:

Question: “Can you explain the relation $\cos^2 x + \sin^2 x = 1$?”

Six students (Students 1, 2, 4, 5, 8, 10) were asked to explain the above identity. None of them used the triangle setting. All six students used the unit circle to explained the identity. However, they all used the Pythagorean theorem in the right-angled triangle which was formed inside the unit circle, where the hypotenuse was 1. Thus, the students only explained the identity for acute angles. We will now present the answers of Student 4 and 1, where we gain an insight on how they view sine and cosine as trigonometric functions.

Student 4:

> Student 4: You know, if we have a triangle, we have the Pythagorean theorem, so we have that $\cos^2$ and $\sin^2$, which are the two sides, and then we have the long one [he means the hypotenuse], which is 1, because we know that the radius of the circle is 1, and that is why if we have a circle and then the radius is 1, then we have uhm…. $\cos^2$ here and then $\sin^2$ here [showing on air], and if we add those two together, it has to give 1.
>
> Interviewer: So is it always true?
>
> Student 4: Yes.

The student started speaking about a triangle. He did not mention though, that he was in the unit circle setting, as if it was self-evident. We understand this by his words: "because we know that the radius of the circle is 1". Also, $\cos^2$ and $\sin^2$ are not the sides of the triangle, but since he used the Pythagorean theorem, it seems that he knew that the power of two is due to applying the theorem, and not because $\cos^2$ and $\sin^2$ are the sides of the triangle.

What is perhaps the most important part in the student's answer, is that he does not seem to handle sine and cosine as functions. When he mentioned sine and cosine, he did not give an input for sine or cosine: "we have uhm…. $\cos^2$ here and then $\sin^2$ here, and if we add those two together, it has to give 1". So, according to the student, the sum of $\cos^2$ and $\sin^2$ is 1. But then, what are sin and cos? Had he written down an explanation, would have he given sine and cosine an input? Or does he think of sine and cosine standing alone, without an input, which hints a problem in perceiving sine and cosine as functions? Unfortunately, the time did not allow us to ask Student 4 about Questions 8 and 9, where we would have gathered more information.

Next, we will refer to the answer of Student 1, who seemed to identify the x-axis with $\cos(x)$, and the y-axis with $\sin(x)$. We found this misconception especially interesting, since we observed it from other students during the lessons as well. We will first present some of the observations on this type of mistake and then, we will refer to Student 1.

In figure 7.2, we can see an image from the school textbook, and next to it some students' drawings, which reveal the same misconception as with Student 1. We can observe that in both students' constructions, $\sin x$ and $\cos x$ are meant to be written outside the unit circle, functioning as the axes. Also, the students have "forgotten" an important property of the trigonometric functions sine and cosine. As it can be seen from their images, they have marked sine and cosine to be taking values above 1. Those two images are from students of Class Y, whereas Student 1, whose interview we will afterwards present, is from Class X. This shows that this did not only occur to students of the same class. Indeed, while student groups in Class X were trying to calculate different values of sine and cosine by the use of the unit circle and without the help of CAS, one student asked the other how they can calculate $\cos(90°)$. Then, the other student of the group replied that "cosine is the x-axis".

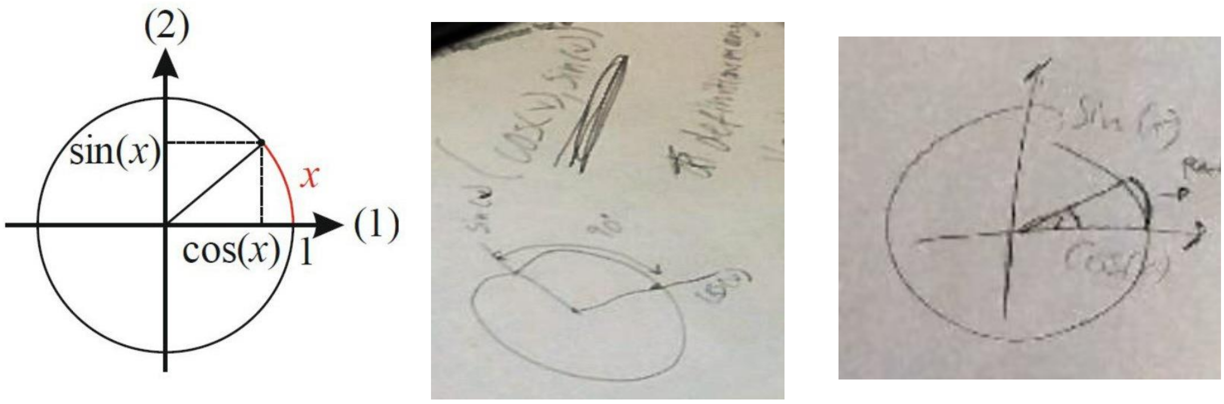


Figure 7.2: The first image is from the book Matematisk Formelsamling, whereas the other two from students from Class Y who seem to have perceived $\sin(x)$ and $\cos(x)$ as the axes.

On the other hand, during the lesson, Student 9 from Class Y constructed the unit circle on the whiteboard (figure 7.3). From her construction, we can clearly see that she did not confuse $\cos x$ to the x-axis and $\sin x$ to the y-axis. Firstly, $\sin x$ is not written near the y-axis so that there would be a confusion between the two and secondly, even though $\cos x$ is written just below the x-axis, we observe that the distance is marked with an arrow, restraining it from the origin of the circle to the vertical to the x-axis line.

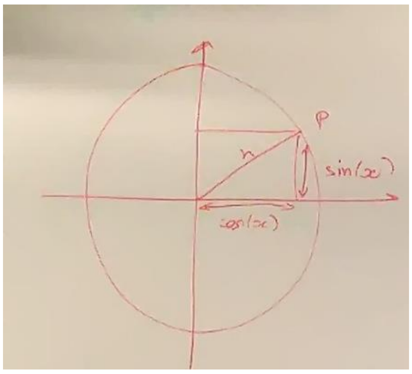


Figure 7.3: Construction of the unit circle by Student 9.

Student 1:

Interviewer: Can you explain the relation $\cos^2 x + \sin^2 x = 1$?

Student 1: I would explain this as you make a vector between the origin and the unit circle's circumference. Well, we know it should be 1. We know there has to be 1 radius, but when you take the cosine, because…., as a vector, to find the length of a vector, you take the x-axis to the power of two and, …, plus the y-axis to the power of two, and then if you take the square root of that, you get what should be c, if you take… as a hypotenuse in a triangle.

Interviewer: In a triangle. So this is the Pythagorean theorem, right?

Student 1: Yes.

Interviewer: So, is this something that is true only for some x?

Student 1: It should be true for all x.

From the student's answer we can see that he is confusing the x-axis with $\cos(x)$. He said "you take the x-axis to the power of two", when he tried to apply the Pythagorean theorem. He constructed the unit circle, (see figure 7.6-top left corner), but his drawing does not give any additional information. Had the students with this misconception changed to the function setting, they would have probably seen that this cannot be true, since the x-axis is something different than the graph of cosine (and hence different than $\cos(x)$).

Trying to understand why some students identified sin x and cos x to the axes, we will now present and analyze some of the classroom data. The teacher of Class Y introduced the unit circle to the students by drawing the same unit circle as the book's (figure 7.2-left image). However, when the teacher of Class X introduced the unit circle (see figure 7.4-left image), she did not specifically mark the distance which is equal to cos(v) and sin(v). The first time she marked that distance, was on the last hour of Trigonometry (figure 7.4-right image). The image is blurry, but it can hopefully be noticed that the teacher had marked the distance equal to cos(x). Moreover, she had from the start recommended that in order to find the value of sin(v), the students should "read" the y-axis, something which the students might have misinterpreted as "sin(v) is the y-axis". This could therefore be why some students from Class X misinterpreted cos(v) and sin(v) for the x-axis and the y-axis respectively.

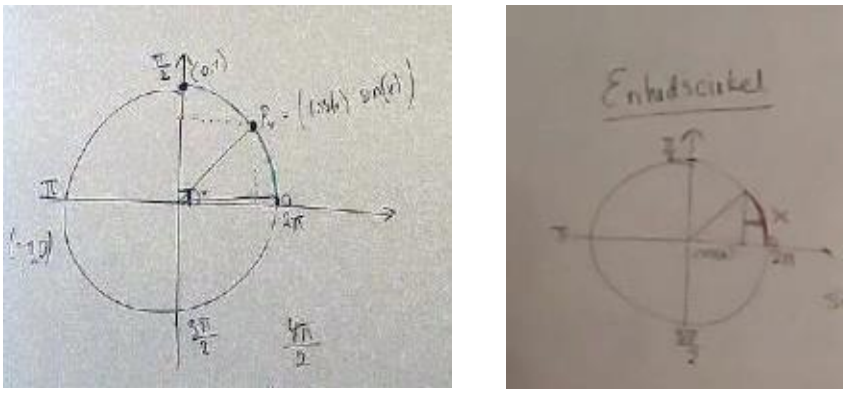


Figure 7.4: Images of unit circles by the teacher of Class X.

Regarding Class Y, we decided to analyze something that the teacher drew on the whiteboard, right after she had introduced the graphs of sine and cosine. First, the teacher gave the students the following table in figure 7.5 and another table with the same values and sin(x). She asked the students to work in groups and fill in the two tables with the help of the CAS-tool Nspire. In figure 7.5 to the bottom left, we can see a student's calculations of the cosine of different angles and plotting the points in the graph setting. Then, with the help of Nspire, another student sees the result of the cosine regression (figure 7.5 at the bottom right).

v	0	45°	90°		180°		270°	360°		495°		630°	
x				$\frac{3\pi}{4}$		$\frac{5\pi}{4}$			$\frac{5\pi}{2}$		3π		4π
cos(x)													

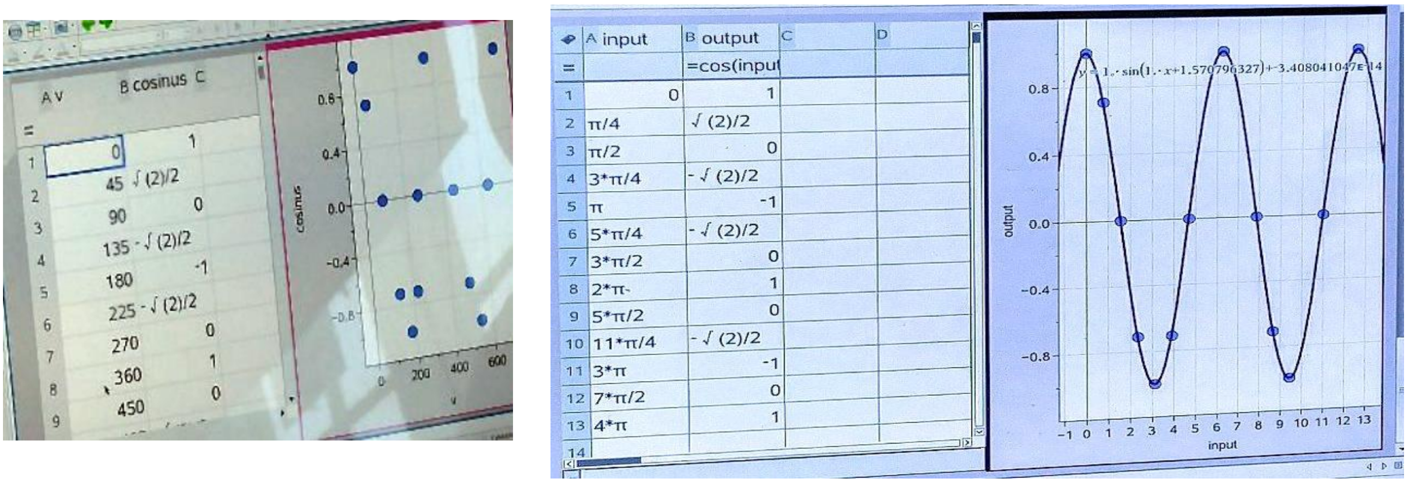


Figure 7.5: On top we see part of the exercise sheet given to Class Y. Down, we see two students' screens, while they are transferring from a symbolic to a graphical representation.

At that point, the teacher, wanting to explain to the students what the y-axis represented in this activity, she drew a coordinate system, referring to the exercise they have been working on. In this coordinate system, she marked the y-axis to be $\sin(x)$. She did not write anything for the x-axis. We assume that there is a possibility that some students remembered this image, which lead them to mark the y-axis in the unit circle setting as $\sin(x)$. However, there is a difference between the unit circle and the graph setting. In the graph setting, the y-axis represents the values which the function sine takes, because in the graph setting, the y-axis represents f(x). So our assumption is, that because in both settings sine's and cosine's range is $[-1,1]$, some students assumed that there must be a correspondence between other characteristics of the settings as well. This could be a reason why they perceived the y-axis as $\sin(x)$ in both cases.

This assumption, however, does not explain why those students also perceived the x-axis as $\cos(x)$ in the unit circle setting. In particular, in the function setting, the x-axis does not represent f(x) (so in our case, $\cos(x)$). Moreover, after drawing the first coordinate system, as already mentioned above, the teacher drew a second one, where $\cos(x)$ was written next to the y-axis. She was trying to explain that in the cosine or sine graph setting, the y-axis represents the $\cos x$ or $\sin x$ respectively.

Hence, we assume that this image could have misled some students into identifying sine to the y-axis in the unit circle setting. Had the students also observed, that the teacher, apart from drawing a coordinate system where the y-axis represented $\sin(x)$, additionally drew another coordinate system where the y-axis represented $\cos(x)$, they would have probably realized that there is not such a correspondence between the two settings. At the same time, it seems more likely that some of the students identified the axes with $\sin(x)$ and $\cos(x)$, because in the unit circle setting, they did not observe that $\sin(x)$ and $\cos(x)$ represent the specific distances between the vertices of the triangle (the triangle's sides). It is actually something common that when someone draws the unit circle, that he does not always mark the distance representing $\cos(x)$ with an arrow, and hence, the confusion of $\cos(x)$ being the x-axis is created. Regarding $\sin(x)$, which is usually written next to the other leg of the triangle, it is also common that in order to calculate it, we correspond it to the y-axis. For example, if we want to calculate $\sin(30°)$, we take the point $A(\cos(30°), \sin(30°))$ on the unit circle and we construct a right-angled triangle, where $AB = \sin(30°)$ and $CB = \cos(30°)$. Then, in order to calculate $\sin(30°)$, we correspond the height of AB to the y-axis, to find that it is 0.5. Hence, this correspondence seemed the most possible reason why some students identify $\sin(x)$ with the y-axis.

7.3 The trigonometric identity $\cos(-x) = \cos(x)$:

Question: "Can you explain graphically the relation $\cos(-x) = \cos(x)$?"

Ten students (St1, St2, St4, St5, St6, St7, St8, St9, St10, St11) were asked this question. Some of the students' answers which provide enough data to be analyzed are presented below. Most students were immediately given the cosine graph and were asked to explain the identity with respect to the graph setting. The reason why we gave the graph, as we have already mentioned, was to save time from the interview and also, for the question to be as clear as possible. Now, as already mentioned, even though we would like to collect data on whether students can draw their own geometrical constructions, it is more important for this thesis to focus on the explanation of trigonometric identities. Hence, whether we gave the students the cosine and sine graphs, depended on the amount of time we had for each interview. In total, the unit circle setting was the most preferred. In particular, some students insisted on using the unit circle, even though the question was asking them to use the graph of cosine.

I gave **Student 1** the cosine graph and he was asked to explain the relation using the function setting. He refused to do it, since he said that he gets more confused when he looks at it and he said that he would rather try on the unit circle. When he was asked why he prefers the unit circle, he said that it is easier for him to see the relation between $\cos x$ and $\sin x$.

> Student 1: … you know that in a point you are in a unit circle, you start by the x-axis, which is if you are at the right side of the second axis [he means the y-axis and we understand that he is trying to say that he is on the first quadrant], then it is going to be $\cos(x)$, and then… if it is up at this corner… [see figure 7.6]
>
> Interviewer: The quadrant.
>
> Student 1: Yes, the quadrant. Then it is going to be $\cos(x)$, $\sin(x)$and if it is down at this one it is going to be $\cos(x)$ and then it is $-\sin(x)$ [he means at the fourth quadrant].

Student 1: It is going to be $\cos(x)$, $\sin(x)$ [on the first quadrant-he means positive] and if it is down at this one [on the fourth quadrant] it is going to be $\cos(x)$ and then it is $-\sin(x)$.

Interviewer: Ok and why is that?

Student 1: Because the sine is about the y-axis and when we go under the x-axis, the y-axis [he means the y] has to be negative.

Here, the student did not explain the relation $\cos(-x) = \cos(x)$. Instead, he explained in which quadrants $\cos(x)$ and $\sin(x)$ are positive and negative. He also made a geometric construction to base his reasoning. Our assumption is that he perceived the identity $\cos(-x) = \cos(x)$, as $-\cos(x) = \cos(x)$, and he tried to explain in which quadrants cosine is positive and in which it is negative. If that was the case, then the student was not able to see the difference between the negative output $-\cos(x)$, and the negative input $-x$, giving the impression that the student thought, that like in the trigonometric identity $\sin(-x) = -\sin(x)$ the minus "goes out", as the function is odd, so does it happen in the case of $\cos(-x)$, constituting a general rule. The student refused to work in the function setting. Otherwise, it would perhaps have been easier for him to see that the cosine function is even. What is remarkable in his answer, is that even though in a previous part of the interview (p.45) he referred to $\sin(x)$ as the y-axis, at this point, he did not seem to do so, as he said that the sine is "about" the y-axis. This implied that he had made a correspondence between $\sin(x)$ and the y-axis, and not an identification. Hence, we assume that his perception of $\sin(x)$ includes both: "$\sin(x)$ is about the y-axis" (where we can only assume what the student meant) and $\sin(x)$ to be the y-axis.

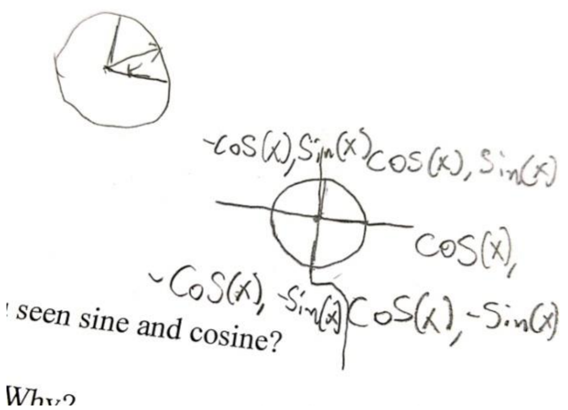


Figure 7.6: Student 1: To the left, the unit circle, and to right, the student's construction to explain the identity $\cos(-x) = \cos(x)$.

Student 2 was also immediately asked to describe the relation using the cosine graph, but he said that he prefers to work with the unit circle setting, since it "makes much more sense to him" and that "the graph is harder for him". Then he said: 'If you have for example π as an x, then you go half a way, and then minus half is the same". It is uncertain what the student thought is the same. He might have been referring to π and –π which are on the same place on the unit circle (see figure 7.7: the student marked with a dot the place where π and $-\pi$ lie). However, considering the following part of the dialogue, where he seems to distinguish the input of cosine from the output, we understand that he was referring to $\cos(\pi)$, being equal to $\cos(-\pi)$. Then, the student decided to continue his explanation by taking x = $\frac{\pi}{4}$. He said that $\cos(\frac{\pi}{4})$ and $\cos(-\frac{\pi}{4})$ would have the same value, explaining it by drawing a vertical line which connected the two points of the unit circle (see figure 7.7). When he was asked why they would have the same value, he said that he did not know why and he could not explain it further, but he knew that $\cos(\frac{\pi}{4})$ would have the same value as $\cos(-\frac{\pi}{4})$, whereas $\sin(\frac{\pi}{4})$ would not be equal to $\sin(-\frac{\pi}{4})$. We assume that the student relied on his class's didactical contract, by reproducing the knowledge that he had learnt during the lesson, without finding it necessary to explain why. Afterwards, the student was asked once more to try and explain the relation in the function setting (see figure 7.7).

> Interviewer: What about here [on the cosine graph]? For example, when we say $\cos(-x)$, can we pick an x and a $-x$ to begin with?
>
> Student 2: Well, uhm…. I mean, if we look at this, $\cos(-\pi)$, would be the same.. here [he means the same as $\cos(\pi)$ showing it on the graph].
>
> Interviewer: The same what?
>
> Student 2: The same… they would have the same... like… if I do like this [he draws a horizontal line which connects the two points], they would have the same… let's say it is -3, then it would be also -3.
>
> Interviewer: So the same height?
>
> Student 2: The same y-coordinate.

With some encouragement this student was able to justify the relation, both in the unit circle setting and the function setting. He did not notice in the graph setting though, nor did he recall, that $\cos x$ must lie between -1

and 1. He wrote −3 next to the cosine graph, trying to explain that the cosine function's range is the interval (3, −3). Had the student compared the unit circle with the function setting, and had he noticed that in the unit circle, cos(x) can only take values between −1 and 1, he would probably have corrected the range. However, except from this mistake, he could explain the identity in both settings. It seems though, that due to his influence of the didactical contract, he did not want to try using the function setting from the start, since they did not use graphs of trigonometric functions in class, to explain trigonometric identities.

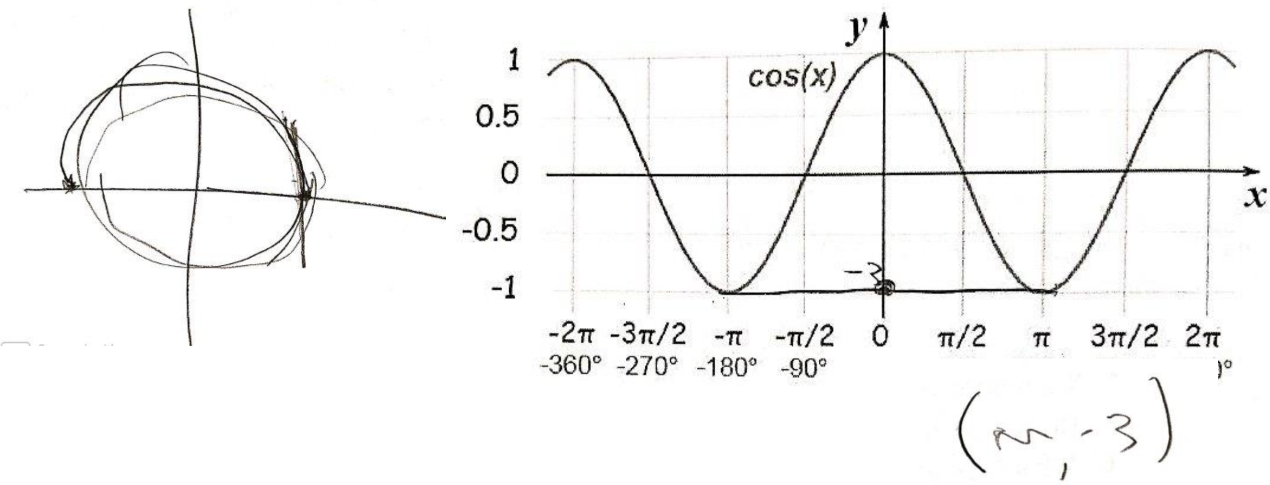


Figure 7.7: Explanation of the relation cos(−x)= cos(x) by Student 2, with the use of the unit circle and function setting.

Student 6:

> Interviewer: What about cos(−x) = cos(x)? Can you explain it using this cosine graph?
>
> Student 6: We have −x… I would say when it [cosine] goes down, it is negative, because it is minus, and when we have +cosine, then we are up here, so it goes up here. Uhm..
>
> Interviewer: So why is it equal? Why are those two equal?
>
> Student 6: Because like here, here it goes up, it is plus, but here it goes down, so it is minus.

Student 6 found it difficult to explain the identity in the function setting. However, he had just explained the same identity less than an hour before the interview in the classroom using the unit circle. We are not sure is his difficulty was due to lack of prerequisite knowledge regarding graphs of functions, or if there was an additional difficulty in transitioning between the trigonometric settings. He did not seem to realize that when the cosine graph decreases, this does not necessarily mean that $\cos(x)$, or $\cos(-x)$ is negative. Also, the fact that he perceived $\cos(-x)$ to be negative, without specifying for which x, leads us to the assumption that he was referring to the input $-x$ being negative. So firstly, there seems to be a challenge in distinguishing between the input and the output of a function and secondly, there is the misconception that $-x$ is necessarily negative.

Student 7:

> Interviewer: About the relation $\cos(-x) = \cos(x)$, can you explain it on the graph that you see here?
>
> Student 7: Can you repeat?
>
> Interviewer: I would like you to explain to me this relation, but not with the unit circle, but here on the graph.
>
> Student 7: I do not know how to do that.
>
> Interviewer: What about if you pick an x? Would you like to pick an x?
>
> Student 7: I would pick an x… 2?1?1?
>
> Interviewer: Yes, but you see though, on your x-axis here, there are some values, so maybe it is helpful if you pick from those.
>
> Student 7: Ah, you mean the degrees… 90°.
>
> Interviewer: Ok so where are you now?
>
> Student 7: Yes, but I do not know how to do this form. We learnt it in the circle.

Here, we see how persistent the student was in not attempting to answer the question. When I tried to encourage him to pick an x, his answer showed that he perceived the question as disconnected from the specific graph of cosine, which did not have $x = 1$ explicitly written on the x-axis. To be clear, the intervention aimed to help the student pick a number x, for which he could spot $\cos(x)$ on the graph. So, because we did not know at that point, if the student perceived π as 3.14, and so, he could realize where x = 1 is on the x-axis, we suggested that he picks a value of x which was already written on that specific graph. Until then, it seems that he had not realized the connection between x as a real number, and x as an input of the cosine function, where its value is included on the x-axis. It seems that when he said "Ah, you mean the degrees… 90°", he made the connection between x and degrees, due to the Cartesian graph which apart from radians, also included degrees (something which usually does not happen). We cannot say with certainty what the student's perception of the input of a trigonometric function was, but we assume that the student had either connected the input of cosine to degrees and not to real numbers, or that he had not understood that when I asked him to pick an x, I was referring to the input of cosine, so that he could explain the identity $\cos(-x) = \cos(x)$.

When the student said that he did not know how to answer because they had not done it in the classroom, we can observe his commitment to the class's didactical contract and that his responsibilities started and ended, with respect to the teacher's requests. It is the same student who we mentioned before, who tried to explain the identity $\cos(x + 2\pi) = \cos(x)$, but his unit circle construction actually served as a justification for another identity, which he had seen in the classroom a while before the interview. The traces of the didactical contract can also be seen by his words "We learnt it in the circle", which shows that the student found it sufficient to explain the relation with respect to the unit circle.

Student 8:

Interviewer: What does $\cos(-x) = \cos(x)$ mean graphically?

Student 8: I do not know why, but it feels like a tricky question.

Interviewer: Not really. First of all, is it true?

Student 8: No, it is not true, I think, I do not think it is.

Interviewer: It is true, for all x.

Student 8: Oh yes, of course, because it is the same length.

Interviewer: So when you want to explain it, how would you explain it?

Student 8: Unit circle.

Interviewer: Ok, but now we can try here on the graph.

Student 8: I really do not know how, because…

Interviewer: We have this relation which is always true. If you do not know how to start, pick an x.

Student 8: 2.

Interviewer: In our graph here, what x's do we have? Maybe pick one of these.

Student 8: Ok, π.

Interviewer: So we have that $\cos(-\pi) = \cos(\pi)$. Can you show that on the graph?

Student 8: … Because this [he shows the point $(-\pi,0)$] is the same as this $(\pi,0)$].

Interviewer: This here [I show$(-\pi,0)$]?

Student 8: Yes.

Interviewer: This [I show$(-\pi,0)$] or this [I show $(-\pi,-1)$]?

Student 8: If this shows $\cos(-x)$, then this would be this, is true, then they have to be the same [he did not point specifically on the graph].

Interviewer: Show me $\cos(\pi)$ on the graph.

Student 8: I do not know.

Interviewer: [I show him where $\cos(\pi)$ and $\cos(-\pi)$ are]

Student 8: This is what I meant before.

Interviewer: Ok, because you were showing me on the x-axis.

Student 8: Yes, but I meant those points.

Interviewer: So, why are they equal?

> Student 8: Because it is the same function that repeats itself, those two points are the same.
>
> Interviewer: So they have the same…..?
>
> Student 8: They have the same points of shifting of the x-axis, they have always the same y-value.

We should first mention that during the interviews, we found it helpful when the students did not know how to explain the identity, to ask them if that identity is always true. The students needed to learn by heart identities like the ones in our questionnaire, and they were told by the teachers to find those identities in their book Matematiske Formelsamling. In this situation though, the student did not think that the identity is always true. The intervention seemed to confuse him, so we told him that it is always true, in order to focus on the explanation on why the identity holds.

In the beginning of the interview, the student had first explained the identity $\sin^2 x + \cos^2 x = 1$, by drawing a unit circle. From this drawing (see figure 7.8), we observe that he did not confuse $\cos x$ and $\sin x$ with the x-axis and y-axis respectively. He had specifically marked with a point where the length of x-axis corresponding to cosx stops and $\sin(x)$ is represented with a vertical line, as the length of the leg of the triangle. So, when in the above dialogue he answers "because it is the same length", we assume he referred to the fact that the length of the x-axis corresponding to $\cos(x)$ and $\cos(-x)$ is equal. We did not insist on asking him to elaborate in this setting, as Question 4 aims to gather information for the function setting.

The student first said that he did not know how to work with the cosine graph. Then, he was encouraged to pick an x and later to pick an x from what he saw on the cosine graph that he was given. When the student showed the points $(-\pi,0)$ and $(\pi,0)$ on the x-axis, probably thinking that they are $\cos(-\pi)$ and $\cos(\pi)$ respectively, we interrupted again to investigate if he knew where $\cos(-\pi)$ and $\cos(\pi)$ are on the graph and to spot possible gaps in prerequisite knowledge. However, the student did not give a clear answer, so we intervened again asking him directly to show $\cos(\pi)$ on the graph. The student replied that he did not know, revealing the exact gap in the prerequisite knowledge. Nevertheless, even though the student showed the x-axis in the function setting instead of the points $\cos(-\pi)$ and $\cos(\pi)$, it is not true that he identified $\cos(x)$ to the x-axis in the unit circle setting, as we can see from figure 7.8.

To sum up, we have seen so far that Student 7 was not able to find where $\cos\left(\frac{\pi}{2}+2\pi\right)$ was. It was also Student 8 who had a similar difficulty, not being able to find $\cos(\pi)$ on the cosine graph.

Back to the analysis of the dialogue, I showed to the student where $\cos(\pi)$ and $\cos(-\pi)$ were on the graph. After this, the student explained that $\cos(-\pi) = \cos(\pi)$, adding that those two have the same y-value, using the fact that the function repeats itself. However, when the student mentioned that since the graph of the function repeats itself and thus, the two points are equal, it seems that he was specifically referring to $\cos(-\pi) = \cos(\pi)$, and not to the identity $\cos(-x) = \cos(x)$.

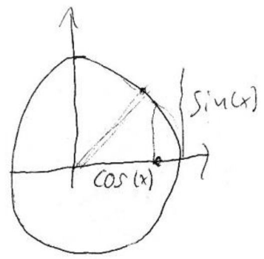


Figure 7.8: Student 8 constructed the unit circle to justify $\sin^2 x + \cos^2 x = 1$. We observe that his perception of $\sin(x)$ and $\cos(x)$ does not coincide with the axes.

Student 10 had several difficulties in answering Question 4. We will present small excerpts of the dialogue and analyze them afterwards, as the whole dialogue was hard to follow. First of all, the student could not remember that a point's coordinates on the unit circle were defined as $(\cos x, \sin x)$. She remembered the converse: $(\sin x, \cos x)$. So, as she was working in the unit circle setting, she perceived Question 4 as "explain why $\sin(-x) = \sin(x)$". Thus, when she said cosine, she thought of sine. When she was asked the question, she immediately looked at the unit circle, which she had drawn for a previous question (see figure 7.9).

Interviewer: Ok, what about Question 4?

Student 10: I guess this is when cosine is up here is positive [she shows at $\sin(\frac{\pi}{2})$] and down here [she shows that $\sin(-\frac{\pi}{2})$] is negative.

Interviewer: So you took x to be $\frac{\pi}{2}$?

Student 10: Yes.

Interviewer: So you saw those two points.

Student 10: Yes.

Here, the student found a counterexample to $\sin(-x) = \sin(x)$: Perceiving $\cos x$ as $\sin x$, and practically trying to explain the relation $\sin(-x) = \sin(x)$, which did not hold, she took x to be $\frac{\pi}{2}$ and disproved $\sin\left(-\frac{\pi}{2}\right) = \sin(\frac{\pi}{2})$, by stating that the left-hand side is negative, whereas the right-hand side is positive. She followed the same strategy when later in the dialogue I asked her to take $x = \pi$.

Interviewer: What about for another x, $x = \pi$ for example?

Student 10: So cosine is positive over here … this is the positive side and this is the negative side, if you know what I mean [she thinks of sine being positive in the 1st and 2nd quadrant].

Interviewer: Try $x = \pi$.

Student 10: Sorry?

Interviewer: [I write it explicitly for her $\cos(-\pi) = \cos(\pi)$] Is this true?

Student 10: No, because it says the same thing [as before].

Interviewer: Show me on the circle.

Student 10: These two are different, because cosine, when π is negative [she might have meant $\sin(-\pi)$] is over here and cosine positive is down here…..

Here, writing one more time the identity which she had to prove, I tried to draw her attention back to the original question. However, the student, having in her mind the false relation $\sin(-\pi) = \sin(\pi)$, she argued that it is the same case as with $\sin\left(-\frac{\pi}{2}\right) = \sin(\frac{\pi}{2})$, so those in the second equality cannot be equal either. At that point, I decided to write on the circle where 0, $\frac{\pi}{2}, \frac{3\pi}{2}$, 2π are. Furthermore, from her unit circle we can see that she also identified sine and cosine to the axes. I corrected "sin" to "cos" and reversely and I did not mention anything about the axes or the fact that the trigonometric functions did not have an input.

> Interviewer: [writing on the unit circle where 0, $\frac{\pi}{2}, \frac{3\pi}{2}$, 2π are] Cosine is this one [I showed her the x-axis, trying to not intervene in her choice of not writing an input].
>
> Student10: I will write that down. I guess it is always over here. [showing at $(-1{,}0)$]

After the intervention, the student explained that $\cos(-x) = \cos(x)$, by taking x = π and substituting in the equation. By "it", we assume that she meant $\cos(-\pi)$, as well as $\cos(\pi)$, because in the previous part of the dialogue, she seemed to distinguish the input from the output of the function: "cosine, when π is negative" (meaning $\cos(-\pi)$). The student though, did not give a complete answer, having only explained the identity for x = π.

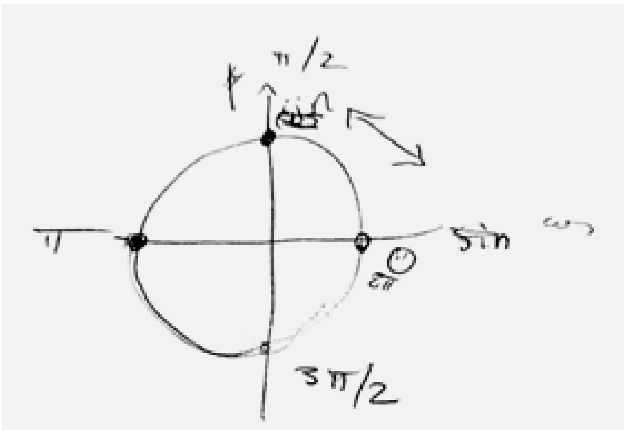


Figure 7.9: Construction of the unit circle by Student 10, which was used to explain the identity $\cos(-x) = \cos(x)$.

> Interviewer: Can you show me here on the cosine graph?
>
> Student 10: So this is the line… cosine? [pointing at the graph]
>
> Interviewer: Yes.

Student 10: Is it.. so it is these two [showing at $(-\pi,-1)$ and $(\pi,-1)$].

Interviewer: Did you take $-\pi$ and π?

Student 10: Yes. And you can see it is the same…. like… coordinates.

Interviewer: The same height, y-coordinate.

Student 10: They have different x-coordinates and the same y.

Even though at first the student could not recognize the graph of cosine, and even though at first she started explaining the identity using the unit circle, whereas the question specifically indicated the use of the function setting, the student was able to use the function setting. She picked $x = \pi$ and she showed where $\cos(\pi)$ and $\cos(-\pi)$ are on the cosine graph (see figure 7.10). This could have been though, due to that she had already done the same process of picking an x, using the unit circle a bit ago in the interview. Then, with some help, she concluded that those two points on the graph, have the same y-coordinates and different x-coordinates, implying, but not stating clearly, that $\cos(-x) = \cos(x)$.

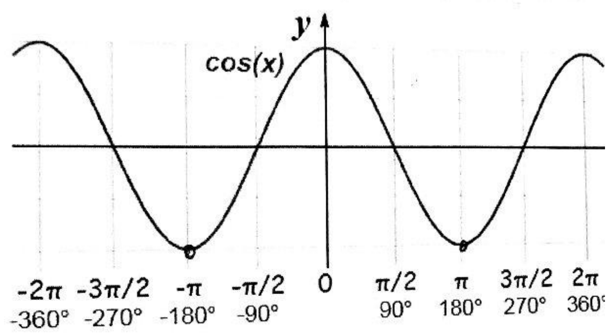


Figure 7.10: Explanation of the identity $\cos(-x) = \cos(x)$ using the graph of cosine by Student 10.

Student 11:

Interviewer: What about Question 4?

Student 11: I will draw my unit circle and this one means the same, because if you move x something, a number up, in the circle here, then you will have P, it is called cos(x), sin(x) and if you move the same number down, you will also get cos(x), and I think −sin(x), because you still move cosine up, so it will be the same, if you calculate it in radians. [see figure 7.11]

Student 11 described x as the length of an arc. In particular, she started by explaining the identity in the unit circle setting. She constructed a unit circle, where she marked an arc (see figure 7.11), and she defined the point where the arc ends, as $(\cos(x), \sin(x))$. Then she picked $-x$, drawing the corresponding length of the second arc and explained that the point where that arc ends, would also have $\cos(x)$ as the first coordinate. She then said that the y-coordinate would be $-\sin(x)$, but she wrote $\sin -(x)$ on the paper. We do not know if what she wrote was by accident and she meant to write $-\sin(x)$, as she also said. However, if this was not written by accident, we see a gap in the prerequisite knowledge about functions. Where she wrote the minus, she could either mean that it is a part of the output or the input. Also, in either of these two situations, "$-$" stands for -1. Had the student written -1 instead of "$-$", would she have committed the same error: "$\sin -1(x)$"?

So far, it seems that the student had explained the identity $\cos(-x) = \cos(x)$ with the use of the unit circle, since she picked x and $-x$ as inputs and she found that their outputs $\cos(-x)$ and $\cos(x)$ are equal. However, the student continued her explanation, by mentioning that "you still move cosine up, so it will be the same, if you calculate it in radians". By "move cosine up", it seems that she was referring to the projection from the point $(\cos(x), -\sin(x))$. to the x-axis. By "it", we understand that she meant "$\cos(-x)$ and $\cos(x)$". We assume that the reason why she said "if you calculate it in radians" was because she was thinking of length of arcs. This assumption is due to that during the first lesson of Trigonometry in Class Y, the radians were introduced with respect to the length of an arc in the unit circle. It could also be that she meant that both x and $-x$ should be in radians, in order for the equality $\cos(-x) = \cos(x)$ to hold.

For the following part of the dialogue, see the figure 7.11. I marked as (1), (2), (3), (4), the points which the student mentioned so that the dialogue is clear to the reader. I only marked the points after the interview was over.

Interviewer: Can you explain $\cos(-x) = \cos(x)$ using this graph?

Student 11: I am not sure I understand the question…. if I have to find…?

Interviewer: If you can explain this relation on this graph. Show me what it means.

Student 11: I am not sure.

Interviewer: Is this relation always true?

Student 11: When it is about angles… no with sine is different.

Interviewer: Ok, but now I am asking about cosine, if $\cos(-x) = \cos(x)$ is true for all x.

Student 11: Yes, I think so.

Interviewer: Can you show me on the graph?

Student 11: Yes…….

Interviewer: Pick an x.

Student 11: I pick this x here [point (1)] and we go down here [point (2)] and we pick this one [point (3)] and then if we do it like minus over here [point (4)], but it is at the same point as here [point (2)], so we go equally the same down here [she shows points (2) and (4)].

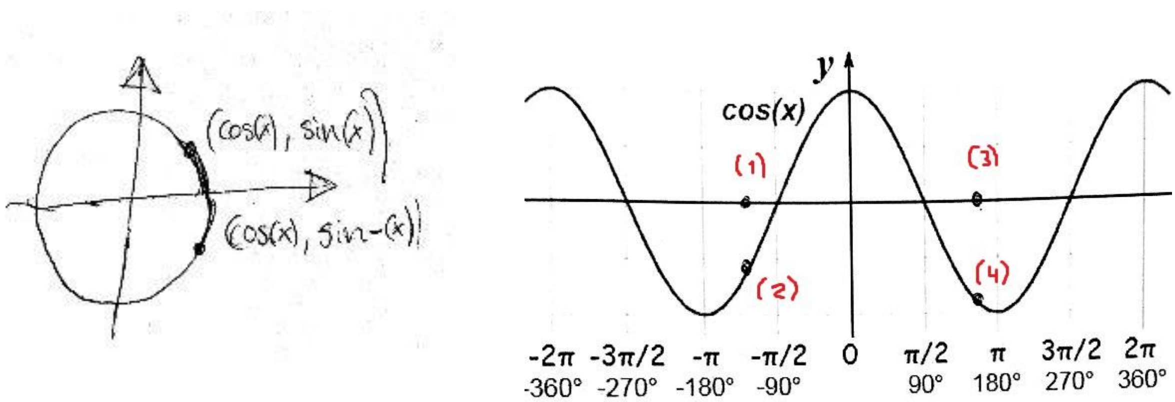


Figure 7.11: Explanation of the identity $\cos(-x) = \cos(x)$ by Student 11.

After the question “is the relation always true?”, the student seemed to have been confused. Instead of answering about the identity $\cos(-x) = \cos(x)$, she neglected cosine and it seems like she understood “is this relation $f(-x) = f(x)$ always true?”. This can be seen by her response “… no with sine it is different”. When we asked her if the relation is always true, she replied “When it is about angles…”, amplifying our assumption that she was additionally referring to other functions than only the trigonometric ones and hence, we consider her to have recognized cosine as a function.

The student seemed uncertain on how to interpret the trigonometric identity in the graph setting, so she was encouraged to pick an x in the graph setting. Then, the student explained the identity using the cosine graph, by marking four points in the function setting. Points (1) and (3) represent $-x$ and x, whereas (2) and (4) represent $\cos(-x)$ and $\cos(x)$. She then said that (4) and (2) “are at the same point, so we go equally the same down here”. What she meant was that (2) and (4) have the same y-coordinate and thus, $\cos(-x) = \cos(x)$. In total, Students 2, 7, 8 and 11 were asked to pick an x in order to explain the identity. Student 11 was the only one who did not substitute x with a number in order to justify the identity, even though she was given the same advice (“pick an x”) as the other students at Question 4.

7.4 The difference between a trigonometric identity and solving a trigonometric equation:

Question: "What is the difference between "$\cos(\frac{\pi}{2})$ = 0? and "solve $\cos(x)$ = 0"?

Student 2:

Interviewer: What about Question number 2? Is there a difference?

Student 2: Yes. Well, … what do you mean solve $\cos(x) = 0$? Oh… yeah! Those are the same… because… $\cos(\frac{\pi}{2}) = 0$, because if you look at the circle, then two times cosine is one circle, then you will adapt π at the x-axis, which is where cosine is 0. "Solve $\cos(x) = 0$", you have two options. It is both $\frac{\pi}{2}$ and one and a half as well [he means $\frac{3\pi}{2}$].

Interviewer: So, is it the same or not?

Student 2: I mean, it is the same, but there will also be another option. You know $\cos(x)$ solve would give me two results.

This dialogue has been slightly compressed, because the student repeated the same answer. Firstly, Student 2 seems to have meant π instead of "cosine", in his phrase "two times cosine is one circle". Also, he made a mistake by stating that $\cos \pi = 0$. Having already mentioned that he was working in the unit circle setting, his perception of the roots of $\cos(x) = 0$ seems to have been limited for $0 \leq x \leq 2\pi$. We assume that this happened because in the geometrical representation of the unit circle, only 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π are marked on the unit circle. However in Class X, where Student 2 belonged, the students had worked on finding the roots of trigonometric equations without the use of CAS and the teacher had mentioned that there exist more roots, depending on how many rounds around the unit circle one goes. Also, they were given the function $f(t) = 67.5 \cdot \sin(0.209 \cdot t - 1.57) + 70,\ 0 \leq t \leq 30$, where f(t) was the height of a Ferris wheel cart over the ground in meters and t was the time in minutes. The exercise they had had, asked them to find at which point in time the cart was 40 meters over the earth for the first time. The students had to use Nspire to solve the equation f(t) = 40. Then, they would find $t \approx 5.3$ or $t \approx 24.74$, where $0 \leq t \leq 30$.The answer would be $t \approx 5.3$, as this was the first time that the cart was 40 meters over the ground. Before using Nspire though, the students of Class X calculated different values of the trigonometric functions sine and cosine ($\cos(90°)$, $\sin(\pi)$, etc.) with respect to the unit circle. They had also worked in groups, solving trigonometric equations (for example, "solve $\cos(x) = 0$, solve $\sin(x) = 1$ etc.") with the specific instruction to consider how many solutions there are in each equation. So, Student 2 had practiced in class both in calculating different values of cosine, and in finding multiple roots of trigonometric equations. This is the reason why we consider that his answer revealed a didactical obstacle, where the old knowledge of angles not exceeding 360° conflicted with the newly introduced knowledge of angles exceeding 2π.

To sum up, the student's view of solving $\cos(x) = 0$ compared to the equation $\cos(\frac{\pi}{2}) = 0$, was that the first one would give an additional option, $\frac{3\pi}{2}$. His focus was on finding the equation's roots and verifying whether $\frac{\pi}{2}$ satisfied the equation $\cos(x) = 0$. He used the unit circle setting and this was probably why he did not consider any additional roots. This however does not mean that in case he had used the function setting, that he would have necessarily argued about the existence of more roots.

Student 3:

Interviewer: Do you see any difference between those two in Question 2?

Student 3: cos $(\frac{\pi}{2})$ is right, that is right.

Interviewer: So are they both right?

Student 3: Uhm, with that one, we go up 90°, $\frac{\pi}{2}$, yes that would be 0. The same if you solve $\cos x = 0$, yes, that should be $\frac{\pi}{2}$.

Interviewer: So are those completely the same?

Student 3: ….

Interviewer: How would you solve this? [solve $\cos(x) = 0$]

Student 3: For that one, I would say that would be $\frac{\pi}{2}$.

Interviewer: Only?

Student 3: Yes, 1 point something. It could also be a number, the π, divided by 2, which is … yeah .

Interviewer: But it can also have other… solutions?

Student 3: Yes, it can be $\frac{3\pi}{2}$ and you can add π, yes sure.

We see that the student did not directly reply with yes or no to the question. He might have perceived the question as "are those correct?", because he answered that "cos $(\frac{\pi}{2})$ is right", meaning that it is right that it is equal to 0. Then, trying to help him see the difference between the two, I asked him if they were both "right", using his own words. Instead of answering if solving $\cos(x) = 0$ is "right", he tried to solve it, giving only the root $\frac{\pi}{2}$. At this point, the student had not answered to the question if they were both "right", nor to my question if those two are completely the same, which was an effort to hint the existence of more roots. Then, I asked the student how he would solve $\cos(x) = 0$, but he again gave the root $\frac{\pi}{2}$. With the question "Only?", I tried once more to hint the existence of multiple roots. The student, having understood that he was expected to say something else, he substituted π to 3.14 and then divided it by 2, giving an alternative answer. What was surprising is that when the student was directly asked if solving $\cos(x) = 0$ can have more solutions, he answered that there are more solutions, and he explained how to get them by adding π. In the case where the student would have just answered that there are more solutions, without explaining how to find them, we would have thought that he was guided towards the answer. So, what is curious, is that even though the student knew that the solutions to this trigonometric equation are multiple, he did not say it until he was specifically asked.

Student 10:

> Interviewer: Is there a difference between those two in Question 2?
>
> Student 10: Uhm… I would say yes…… this one is a point [she shows at $\cos(\frac{\pi}{2}) = 0$], but the other one is a form for….. I am not sure.
>
> Interviewer: Before you solved something on Nspire, right?
>
> Student 10: Yes.
>
> Interviewer: So what do you expect to find?
>
> Student 10: This is like.. the coordinates you find when you say cosine and sine.

The student's perception of $\cos(\frac{\pi}{2}) = 0$ is that it is a point, but we are not sure if she thought of the unit circle, so she was talking about a point on the unit circle, or the function setting, so she was referring to a point on the graph of cosine. I tried to remind her that she had had similar tasks on solving trigonometric equations on Nspire. For example, in the last hour of Trigonometry, Class Y had the same exercise as in Class X about the Ferris wheel, as described before.

The student's answer was not clear: "it is like the coordinates you find when you say cosine and sine". She was either referring to graphing points (the solutions) of the form $(\theta,\cos(\theta))$ in the function setting with the use of CAS, or she could have been referring to the geometrical representation of the unit circle which students from both classes were using the whole time during their Trigonometry lessons. That unit circle was both in degrees and radians and some points of the unit circle were labeled with their coordinates (see figure 7.12).

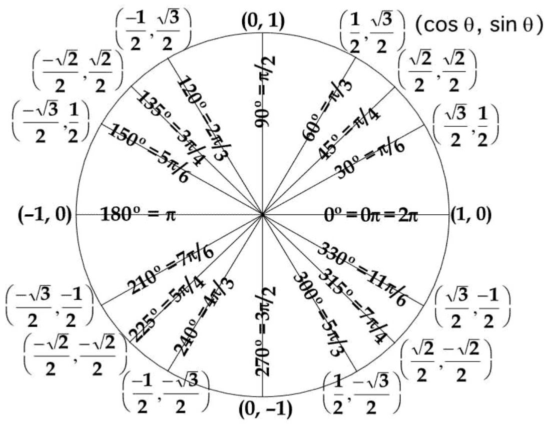


Figure 7.12: Illustration of the unit circle that students of both classes used during their Trigonometry lessons.

Student 11:

Interviewer: Ok, what about Question 2?

Student 11: Is this like in Nspire?

Interviewer: Yes.

Student 11: I am not really sure.

Interviewer: When you have to solve that cos(x) = 0, what do you expect to find?

Student 11: I would move this cosine into this side, I cannot remember where I end up being.

Interviewer: You get x equals something. How many x?

Student 11: I think we usually find a lot of x, because when you have cosine, you start at point one here and then you move… so if you have to find somewhere maybe minus half, it will be many times, because it will keep going, unless you define if it should stop here.

Interviewer: So it has many solutions.

Student 11: Exactly.

The student seemed to want a point of reference in order to answer the question. She wanted to relate our question to what she remembered from the classroom. The traces of the didactical contract were present. The student, having accepted the didactical contract of her classroom, she had developed a behavior and certain techniques to handle the mathematical content. In this case, it is clear that Nspire was something accepted, even required, by the class's didactical contract. Indeed, many students before joining the interviews, they asked if they should bring their laptops with.

In that moment, without knowing what the student meant exactly with "like Nspire", and without knowing which of the two ($\cos(\frac{\pi}{2})$ = 0 or solve $\cos(x)$ = 0) she connected to the Nspire, I answered yes, having in mind that in Class Y, they had solved trigonometric equations both with the use of Nspire and without. They had also used CAS to calculate different values of sine and cosine. We assume that the student's difficulty was the comparison between cos $(\frac{\pi}{2})$ = 0 and solving $\cos(x)$ = 0. So, we tried to help her elaborate on her thoughts, by asking her what she expected to find when she would solve $\cos(x)$ = 0. The purpose of this question was to help the student recall how many solutions a trigonometric equation has. The student continued to be confused, so I told her that x would equal something, and then I continued by asking her how many x those would be. That intervention helped the student recall information from the classroom, as she answered that she would find a lot of x and she explained it by the use of the unit circle. When she mentioned "minus half", she probably meant that by rotating $-\pi$, in other words by rotating half a unit circle, she would end up at $-\frac{\pi}{2}$, which is also a root for $\cos(x)$ = 0. The student also said "you start at point one here and then you move", trying to explain that we find the first root of the equation and then by rotating around the unit circle, we keep finding more roots ("it will be many times, because it will keep going"). Moreover, the student seems to have been aware that the roots will be infinitely many, unless there is a restriction ("unless you define if it should stop here"). To sum up, she did not refer to cos $(\frac{\pi}{2})$ = 0. Instead, she focused on describing how one finds the roots of $\cos(x)$ = 0, using the unit circle.

Overall, Student 7 was the only student who did not give any information related to this question. He said that he thought there is a difference, but he could not explain which. Students 1, 5, 6, 8 and 9 explained the difference between solving cos(x) = 0 and cos $(\frac{\pi}{2})$ = 0. For example, Student 1 used the unit circle and said that "there are two sides of the unit circle where it can be equal to 0, but he can also add 2π over and over again", whereas he referred to cos $(\frac{\pi}{2})$ = 0 as a specific point on the unit circle. Student 5 referred to x in "solve cos(x) = 0" as "an unknown number", in comparison to $\frac{\pi}{2}$ in "cos $(\frac{\pi}{2})$ = 0", where he referred to it as "a number". He then spotted the difference to be, that even though the first has multiple solutions, the second one has only one solution, $\frac{\pi}{2}$. Student's 9 answer was similar, by stating the difference between the first having multiple solutions and the second having "one specific answer". Student 8 immediately related solving cos(x) = 0 to Nspire and he said that he expected to find a point that belongs to the unit circle. When he was asked how many points, he said that they would be endless. However, when he was asked if cos $(\frac{\pi}{2})$ = 0 is something which is always true, he replied that he did not know. Student 6 was literal in his answer. He first said that the difference is "solve, $\frac{\pi}{2}$ and x". When he was asked for additional information, he said: "we solve cos(x) = 0 with the mathematical calculator, whereas the second one just stands as it is, without "solve"". Then the student was asked how many solutions would "solve cos(x) = 0" have. He said many. It seems like the student had understood the difference between the two, interpreting cos $(\frac{\pi}{2})$ = 0 as an equality which holds ("just stands as it is"), whereas solving cos(x) = 0, as an equation to be solved, and which has multiple solutions.

7.5 Angle perceptions:

Question: "What is an angle?"

Student 6:

> Interviewer: What is an angle?
>
> Student 6: It is how many degrees… Are we talking about the unit circle?
>
> Interviewer: No, generally. Whatever comes into your mind.
>
> Student 6: If we for example have a triangle, we can find the corner…so we have a specific angle, which shows how acute or obtuse or how sharp the angle, and it tells us how big the spread of the triangle is.

Even though at first the student thought of the triangle setting ("it is how many degrees"), he then switched into the unit circle setting, as it is the setting which he had recently been using in class. His first answer connects an angle to its quantity ("how many degrees"), but then it seems like his old knowledge about angles inside a triangle, overpowered the recently established knowledge of the unit circle, and he moved to the triangle setting. He considered an interior angle of a triangle and he said that that angle can be translated in terms of quality, in other words, "how acute or obtuse or how sharp the angle" is. By using the words "how big the spread of the triangle is", we assume that the student was referring to the quality of the triangle's angles.

Student 8:

> Interviewer: What is an angle?
>
> Student 8: An angle is a relation between two lines that lie one over the other, it is difficult to explain, but it is like.. how open it is. For example, there are three kinds of angles: obtuse, right and acute.
>
> Interviewer: Is this here [I showed the exterior angle in his drawing] an angle?

Student 8: Yes. This is also an angle.

To begin with, we cannot consider this student's perception of the angle as a relation (as described in section 2.1), as he did not describe an angle as a central angle within a circle and the measure of the angle to be the ratio of the length of the arc which subtends the angle to the circle's radius. In the case of the unit circle, we would also consider an angle's perception as a relation to be, that the measure of the angle equals the length of the arc which subtends the angle. However, Student 8 referred to an angle as a "relation", as he called it, between two lines, or, "how open it is", relating an angle to its quality. He also mentioned as an example that an angle can be obtuse, right or acute (see figure 7.13). When he was asked if the exterior of an angle is also an angle, he said yes.

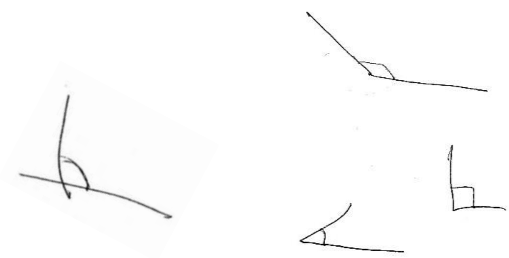


Figure 7.13: To the left, Student's 8 drawing, while defining what an angle is. To the right, types of angles by Student 8.

Student 9:

Interviewer: What is an angle?

Student 9: An angle is if you have, I do not know how to explain it, but imagine we have two lines crossing each other, then an angle would be the width difference between the two lines. So it is this point [she draws it]. So for example an angle could the edge of a table.

Student's 9 perception of an angle seems to be limited to its quality. She drew the figure of an angle (see figure 7.14) and she explained that the angle can be seen as the "width difference between the two lines". However, she did not mention anything about the measure of the angle. Had the student said "an angle would be the width difference which is contained between the two lines", we would have additionally considered her perception of an angle to be related to the quantity captured in between the two lines. Nevertheless, we have to admit that we do not know if the student additionally thought that an angle is related to quantity. The existence or absence of a word ("contained"), cannot be the only criterion for analyzing the student's perception of an angle, and we do not have more data to draw a concrete conclusion. Finally, the student gave an example, where the angle that she chose was a right angle, underlining again her focus on the quality of an angle.

The answer of Student 10 was similar to this of Student 9. She also said that an angle is the difference between two lines. Then she rephrased by saying that an angle is the space between the two lines, perceiving an angle as a quality.

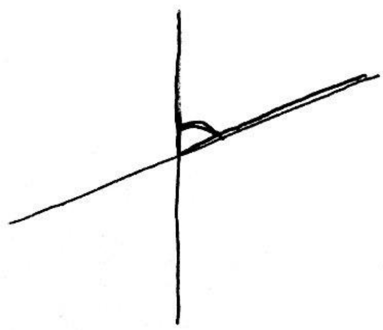


Figure 7.14: Construction of an angle by Student 9.

Student 11:

Interviewer: What is an angle?

Student 11: Can I draw it? We use the unit circle and we can see the angles if you for example move this [she shows a point on the unit circle, and marks the arc, see figure below], we call it radians I think, and then you can see an angle and we can use it to find other things with it, for example cosine and sine.

The student used the unit circle setting to answer the question. She seemed to perceive an angle as a relation, as she interpreted a central angle inside the unit circle as the length of the arc which subtends the angle. She also explained that if we move the chosen point on the unit circle (see figure 7.15-left image), the length of the arc would also change, and it would correspond to a different angle. She then said "we call it radians", probably referring to the length of the arc of the unit circle, being measured in radians. Moreover, the student mentioned that the angle can be used as an input to the trigonometric functions sine and cosine, making her the only student who had made this connection. In the figure below, the left image is of Student 11, whereas the right one is of the teacher of Class Y. We observe the similarity between the two images and that the teacher's highlights of the angle and the length of the arc, are what the student used in her image to show the relation between the angle and the length of the arc which subtends it.

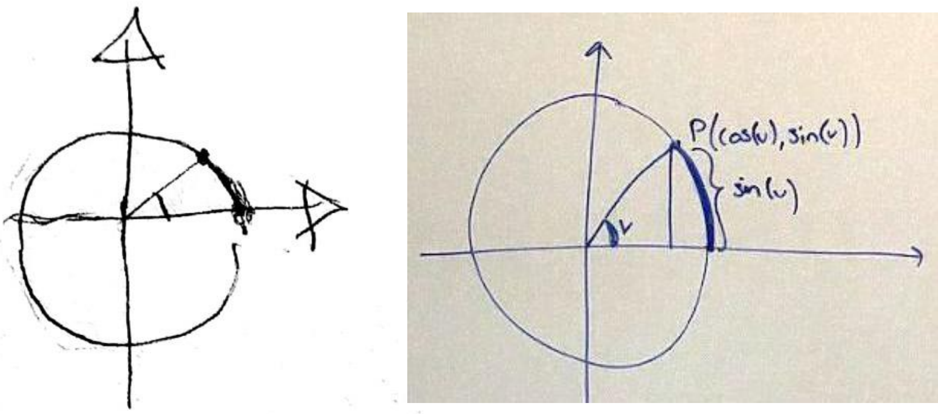


Figure 7.15: The relation between an angle and the length of the arc which subtends it. To the left, the drawing is by Student 11. The right one is drawn by the teacher of Class Y.

7.6 Unit circle or function setting?

Question: "−390° = 30°" and "8π = 2π": Yes/No answers

Students 1, 2, 6, 7, 8, 9, 10 and 11 were asked the above yes/no questions. Student 1 answered that 8π = 2π but he did not explain why. He also answered that −390° ≠ 30°, by adding 360° and concluding that −30° ≠ 30°, without further explanations. Student 2 miscalculated the sum of −390° and 360° to be 30°, but when he was told that he made a wrong calculation, he said that −390° ≠ 30°, but he did not explain why.

Student 7 did not answer to either of the two questions. Now, we will present the rest of the students' answers, which among others, give us information on which trigonometric setting students have connected to the notion of angle and which trigonometric setting they mostly used.

Student 6:

> Student 6: This $-390° = 30°$, I would say false.
>
> Interviewer: Because..
>
> Student 6: Because this is minus, so it cannot be plus, I think
>
> Student 6: About the tenth question, it is the same. We go many rounds around our circle, so it is the same**.**

The student wondered how a minus "can be" a plus and so, he rejected the idea of a negative angle being equal to a positive angle. He did not mention anything about adding 360°, and then observing that $-390°$ and 30° are not on the same location on the unit circle. We might have assumed that the student did not perceive this as equal, because even in the case where the two angles were coterminal, they would have not performed the exact same rotation. However, observing his other answer about "$8\pi = 2\pi$" being "the same", we understand that in the first situation, he did not answer "no", because of the rotation of the two angles, but because he encountered both a didactical and a epistemological obstacle. It is didactical, because of the way that he has been taught about angles. His first experience with angles in primary school probably was to construct an angle by using the protractor, or to use the protractor to find the measure of a positive angle. On the protractor no angle is marked to be negative. There are only marks of positive angles, starting either from the right or the left side of the protractor. It is epistemological, because it seems that he finds it difficult to perceive how something negative can be equal to something positive.

At the same time, he seems to have connected degrees to the triangle setting. We can understand this, because, firstly, he did not use the unit circle to analyze the question "$-390° = 30°$", but he did it for the equality "$8\pi = 2\pi$", and secondly, when he was asked to define what an angle is

(p. 71), his answer revealed a distinguishing between degrees and the unit circle.

> Interviewer: What does π stand for?
>
> Student 6: [It stands for] how many times we go around in our circle, 2π or 360° around. It is the same.
>
> Interviewer: But isn't π, 3.14?
>
> Student 6: Eh, yes, it is.
>
> Interviewer: So if it is, is 8π = 2π?
>
> Student 6: Yes…eh…
>
> Interviewer: 8 times 3.14 =….
>
> Student 6: Yes. you can say that.
>
> Interviewer: … = 2 times 3.14?
>
> Student 6: No, it is not the same if we multiply 8 with π and 2 with π. It does not give the same [result]. But… when you ask it like that [he means like in Question 7], I think of the circle, but it does not give the same.

Then, I tried to investigate Student's 6 perception of π. His answer shows that he had connected it to the unit circle setting, perhaps confusing it with the period of sine or cosine being 2π. The student was confronted with a contradiction between the old and the new knowledge about π, encountering a didactical obstacle. According to the old knowledge, $8\pi \neq 2\pi$, but according to the new knowledge, 8π could be "the same", using his words, as 2π, because of the same location that the two angles have on the unit circle. Then the student gave two answers: it is not "the same" if he substituted the equation with $\pi = 3.14$, but it is "the same" if he considers the unit circle. It should be mentioned, that even though in the end of the dialogue, his final answer was that "it does not give the same", he did not change his answer on the paper to "no".

Student 8:

> Student 8: About the −390° = 30°, it actually is true, I think so, because if you go 360° around, then you get 30° more, which would be 30°.
>
> Interviewer: This is −390° [I gave emphasis on the "minus", thinking that he misinterpreted the question for 390° = 30°].
>
> Student 8: Oh yes, that is actually true, so I will draw it to understand it better. We start here, go around and I make the 30° line… so it is not. I am in doubt, because I thought it is of course true, but now I see what you mean with the minus [see figure 7.16].
>
> Interviewer: So is it a yes or a no?
>
> Student 8: Yes, I still think it is true.
>
> Student 8: But 8π = 2π is not true, because 8 times 3.14 is not the same as 2 times 3.14

At first, the student did a miscalculation. Then he constructed a circle. By the "30° line", he referred to the angle that he drew, but he did not explain why he chose that angle to have its vertex on the circle (see figure 7.16). However, without excluding that this specific angle could be 30°, when we refer to angles in the unit circle setting, we draw angles which begin from the standard position on the coordinate system, that is, with the angle's vertex at the center of the circle and its initial side being the positive x-axis. Hence, it is curious that Student 8 drew the 30° angle on that place of the circle. Moreover, as we can see from figure 7.16, the angle does not "face" towards the circle, but towards the center of the circle and the angle's vertex seems to be on the circle, instead of the circle's center. This gives us the impression that that was a result of a didactical obstacle. We assume that the student's already existing knowledge from primary school where he used to draw angles with the protractor prevails the new knowledge about angles and rotation of angles in the unit circle setting. We believe so, because when someone uses the protractor using its right side to draw an angle, an angle would look like the angle that Student 6 has drawn. However, even if he considered the starting point of the 30° and −30° angles to be with the vertex at the center of the circle and the initial side to be on the negative x-axis, he could still have realized that the 30° angle is not the same as the −30° angle.

The student then answered that it is not true that −390° = 30°, but he was not certain either, because of the remark that I made about the minus. What happened is that in the start of the dialogue, I tried to put his attention to the fact that it was not written 390° = 30° on the paper, but −390° = 30°, thinking that he miscalculated. The student however, seems to have perceived my intervention about the minus, not as a problem on his calculation, but as something additional which he had to consider. We draw this assumption because he said that he thought that −390° = 30° is true, but "he sees what I meant with the minus". Had he realized that the problem was on the calculation, we assume that he would have tried to calculate it again. Then the student decided to neglect my comment about the minus, and to support his original answer, "yes".

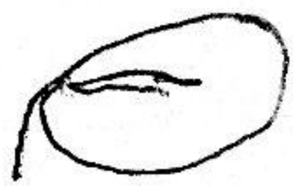


Figure 7.16: Explanation of −390° = 30° by Student 8.

Student 9:

Student 9: This is a no, 8π will do 8 times 3,14, but if you say 2 times π, it is 6 point something, so it does not give the same.

Interviewer: Ok.

Student 9: And about the 7^{th} question, is it both in degrees?

Interviewer: Yes, because they both have the (°).

Student 9: Ok, I would say no.

Interviewer: Because?

Student 9: Because −390° is not the same as 30°.

Interviewer: What if you had to write the correct answer to that, what would it be?

Student 9: I would say it is equal to 390°.

Interviewer: So $-390° = 390°$?

Student 9: Yes, because it is degrees and I do not think you can have a negative degree value, so if you solve an exercise and you get $-390°$, I would say it gives a 390° angle.

Interviewer: Do you prefer to work with radians or degrees and why?

Student 9: I mean, for this [she refers to her current Trigonometry lessons], it makes more sense to work with radians, but to me it is two different things. If you work with triangles, you work with degrees, but here it makes more sense to work with radians.

Interviewer: Why?

Student 9: Probably because that is the way I have learnt it. So I learnt that triangles are basically in degrees, because you calculate an angle with the degree. Here I have learnt it with radians, so it makes more sense to do it with radians.

In the first part of the dialogue, we see that the student recognized π as 3.14 and thus, she argued that $2\pi \neq 8\pi$. Afterwards, the student answered "no" to $-390° = 30°$, but she could not explain why, repeating that they are not the same. When I asked her what the correct answer would then be, she said it would be equal to 390°, showing at "$-390°$", arguing that there cannot be negative angles measured in degrees. The student, not being able to overcome the problem of negative angles measured in degrees, she constructed her own way of handling this problem, concluding that 390° is equal to $-390°$.

We believe that the problem originates from the way the student had formed a connection between radians and degrees, as we can also see later in the dialogue. She perceived degrees and radians as something which is not connected, and she said that she chooses degrees or radians depending on the setting: "If you work with triangles, you work with degrees, but here it makes more sense to work with radians". The reason why, as she explained, was because she had learnt it this way. Here, the effects of the didactical contract are evident, as the student used her experience from the classroom to justify her strategy of choosing between radians and degrees. Therefore, we assume that when she said that there cannot be a negative degree value, she did so, because she had connected degrees to the triangle setting. This is a didactical obstacle. When students first learn about angles, there is no reference to the existence of negative angles, or to angles which exceed 360°. Later, they learn about radians through the unit circle setting, connecting the radians to the length of an arc of the unit circle. Also, the connection between degrees and radians is usually made through the formula 2π rads = 360°. In Class Y, the teacher mentioned the relation of the radians to the length of an arc and she also mentioned the radians to degrees formula. So, it could be that even though the student had been aware of the conversion formula between degrees and radians, that she may not have actually made the correspondence ("To me it is two different things").

Finally, we would like to refer to the student's question whether $-390°$ and 30° are both in degrees. Even though this might seem surprising, since there is the degree sign on both angles, we assume that the student asked this, because there was a problem in Class Y when the students were using Nspire. In the exercise where they had to calculate different values of sine and cosine, many students had used degrees instead of radians as inputs. For example. in figure 7.17, the student to the right has the correct setting in Nspire, as the inputs should be in degrees. The screen to the left though, even though it has multiples of π as inputs, the setting is also in degrees. As a result, Nspire, when calculating $\sin(\frac{\pi}{2})$, does not consider $\frac{\pi}{2}$ as radians but as degrees, hence $\sin(\frac{\pi}{2}) = \sin(\frac{3.14}{2}) = \sin(1.57) = 0.027\ldots$

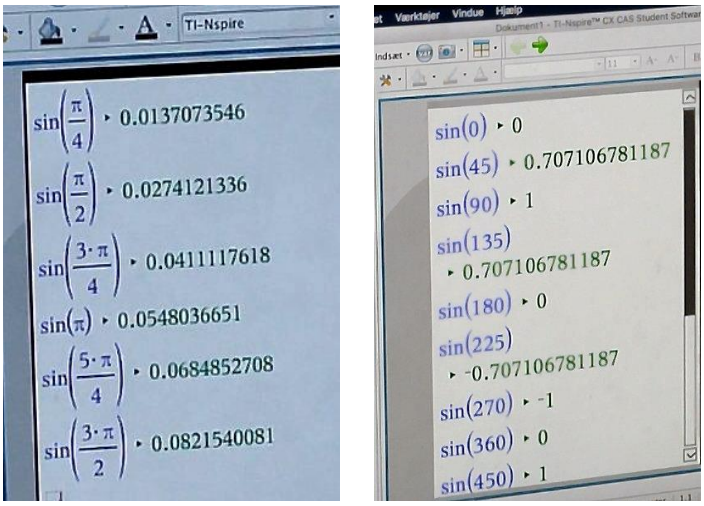


Figure 7.17: In both students' screens, the input is set to be in degrees in Nspire.

Student 10:

Student 10: I do not think that −390° = 30° is true, because how do you have ... it does not make sense….

Interviewer: Because of the minus or why?

Student 10: Uhm, yes, how can minus give plus? Is it because you go [she shows around with her finger]?

Interviewer: You go around.

Student 10: Aah ok, then it could be actually true. So if you have a circle and you go around and then it is the difference between…so this is a circle and that is 360°, so then it would be 30 if we see the difference between these two, then it would be true, but I am not sure I understand the question right.

Student 10 was confused at first, encountering the same epistemological obstacle as Students 6 before her: "how can minus give a plus?". Then, she argued that the equality holds, connecting it to the unit circle setting, and explaining the equation by rotating around the circle. The student moved from the algebraic to the geometric representation by constructing a circle, where again like Student 8, she did not draw the 30° angle on its standard position. It also appears that she had miscalculated

$-390° + 360° = 30°$, when she mentioned that the difference between “these two” would be 30°.

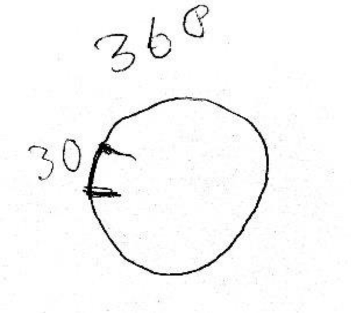


Figure 7.18: Explanation of why −390° = 30° by Student 6.

Student 11:

Student 11: I would say 8π = 2π is false.

Interviewer: So 8π rads ≠ 2π rads?

Student 11: I would say it is the same because we have 2π around the circle, so I think yes and no. Otherwise I would say false, it is two different numbers.

Student 11 initially perceived 8π and 2π as two different numbers, but at the same time, she was aware that one full rotation around the unit circle corresponds to 2π. My intention when I repeated the question, but mentioning the radians, was to shift her attention to the unit circle setting, where she could possibly have related it to radians. Indeed, the student took the hint and she answered that then they would be the same, referring to 2π and 8π having the same location in the unit circle setting. The reason why she first answered that 2π ≠ 8π could either be because she considered π to be 3.14 or because she thought of 2π and 8π as points on the x-axis in the function setting.

7.7 Perceptions of sine and cosine:

Question: "$\sin(-\frac{\pi}{2}+\pi) = \sin\left(-\frac{\pi}{2}\right)+\pi$" and "$\frac{\sin x}{x}$ = sin" :Yes/No answers

All students except Student 4 answered those two questions, among whom, most explained their answer. We will now present all students' answers related to the above questions, as we believe that they contain a big amount of information about their perceptions of sine, whether they consider it to be a function, and how they distinguish between the input and output of the sine function. It should be mentioned at this point, that the fact that in "$\frac{\sin x}{x}$ = sin" a parenthesis around the input is "missing", was not done on purpose. However, we consider it to have worked positively into gathering more information.

Student 1:

> Student 1: I would say yes [to $\frac{\sin x}{x}$ = sin], because if you divide it out, but x cannot be zero of course, because you cannot divide by zero. Actually, I do not know.
>
> Student 1: [about Question 9]…minus $\frac{\pi}{2}$, that is down here, plus π, then we would go up here, oh..
>
> Interviewer: Without thinking about the unit circle?
>
> Student 1: I would not say it is the same, because there is a parenthesis around it. I would not think they should be the same.

Student 1 wrote "yes" on his paper, explaining that $\frac{\sin x}{x}$ = sin due to the division of x by x, but he said that he did not know if that was the correct answer. He then started explaining $\sin(-\frac{\pi}{2}+\pi) = \sin(-\frac{\pi}{2})+\pi$ by using the unit circle. He started from the left-hand side of the equation and he first tried to estimate where the angle $-\frac{\pi}{2}+\pi$ would be on the unit circle. So, his first reaction was to transfer from an algebraic representation to a geometrical representation (this of the unit circle), where the input of sine was represented as a sum of angles in the unit circle setting. At that point I intervened because he had already revealed his strategy of commenting on the equality, by calculating both sides and observing if they were equal, so I tried to move his attention away from the unit circle. Then, the student answered that the equality does not hold, because in the left-hand side "there is a parenthesis around it", referring to the input of the function being inside the parenthesis. So, we observe that the deciding factor for his answer was the parenthesis. In the first case where there was no parenthesis, the student seems to have perceived $\sin x$ as a multiplication between sin and x, whereas in the second situation, the parenthesis made the distinction clear between what is a part of the input of sine and what is not.

Student 2:

> Student 2: This does not make any sense to me, I do not think that is true.
>
> Interviewer: Why?
>
> Student 2: Because dividing $\sin x$ with x, does not give you sine. It does not give any sense to me. I do not know if it is true, but I do not think that is true.
>
> Interviewer: It is not true. What about the next one?
>
> Student 2: No…. wait. No, I would not say that is true either.
>
> Interviewer: Because..?
>
> Student 2: Because you cannot just take this one [$+\pi$] and then put it outside the parenthesis.
>
> Interviewer: Why can't you just…[with emphasis on the "just"]?
>
> Student 2: Because it would not be the same. If you take sine, then you take sine to this whole number, but if you put this outside this…. box.. then you would take $\sin(-\frac{\pi}{2})$ and then…. add π, which is a different number.

Student 2 did not change representation in order to explain his answers. He said "no" to both of the questions. In the first case, the lack of parenthesis did not seem to confuse him and he seems to have perceived $\sin x$ as a function instead of a multiplication. However, he did not explain further why $\frac{\sin x}{x} \neq \sin$. Regarding the second equality, he used the word "just", to emphasize that if the input of sine changes, the output would also change, trying to express the difference between the input and the output of sine. This can also be seen by the fact that in the first equation, his perception of the x in the nominator and the one in the denominator was different: the one in the nominator was sine's input, whereas the one in the denominator was a number, and that is why he could not divide or do any other kind of algebraic manipulation between the two x. Hence, the student seems to have perceived sine as a function, distinguishing between the input and the output, where the input is a "whole number", or "a box", as he mentioned.

Student 3:

> Interviewer: What about $\frac{\sin x}{x} = \sin$?
>
> Student 3: It looks true, yes, just because x goes up…. $\sin x$…. yes that would be true. And the next one…
>
> Interviewer: Without thinking how it looks on the unit circle… just algebraically speaking, can you take something out of the parenthesis?
>
> Student 3: No?
>
> Interviewer: Because?
>
> Student3: Yes, because the values change. It does change. Because you have to save the sine off the π [referring to $\sin(-\frac{\pi}{2} + \pi) = \sin\left(-\frac{\pi}{2}\right) + \pi$].

We do not know what Student 3 had in mind when he said that "x goes up". He could have thought of the unit circle setting, where one draws an angle x (or the corresponding length x of the arc) by rotating counterclockwise from the starting position. This counterclockwise rotation could be what the student meant with x “going up”. It could also be that he had in mind the graph of sine in the function setting, but in that case he must have considered an interval where the function is increasing. The same would also happen for the unit circle setting: the direction of the rotation of an angle would “go up” but it would also “go down”, depending on the quadrant. The student tried to give an explanation by changing from the symbolic representation of an equation. We suspected that he would also try to work similarly by changing representation for the next question, and possibly trying to calculate separately the two sides of the equation with the help of the unit circle. So, we intervened in order to take his focus from the unit circle. I asked him directly if it is possible to move something from the input of a function. We are uncertain if this hint guided the student into answering no, or if it was something that he was already aware of. He said that it cannot happen because then the result would change, adding that “you have to save the sine off the π”. We cannot know what the student meant with this last phrase and as mentioned before, we are not sure if we hinted the answer to the student. Even though he seems to have known that if a part of the input is “moved” outside of the input (or in other words if the input changes), then the output would also change, and thus perceiving the difference between a function’s input and output, he did not realize that x was the input of sine in the first question.

Student 5:

> Student 5: I am not sure if it is sin times x or $\sin(x)$. I do not know if there is supposed to be a parenthesis.
>
> Interviewer: Yes there is supposed to be.
>
> Student 5: I think it is untrue, but I cannot explain why. That is a whole number, $\sin x$, so dividing it by x, does not make it sine.
>
> Interviewer: And what about the other question?
>
> Student 5: That is the same as the one before, $\sin(-\frac{\pi}{2})$ is a number plus something else, and if it is in the parenthesis, all really is the same.

Student 5 immediately asked about the unknown to her notation, which seemed to be an obstacle which she overcame by a clarification in the function notation. She then explained that $\sin x$ is a "whole number", meaning that she could not take x from the input and divide it with the x in the denominator. She then related the two questions, by stating that "$\sin(-\frac{\pi}{2})$ is a number". Hence, according to the student, "$\sin(-\frac{\pi}{2}) + \pi$" is a number plus π, whereas "$\sin(-\frac{\pi}{2} + \pi)$" "is the same", probably referring to the fact that π is a part of the input of sine, just as $-\frac{\pi}{2}$ is.

Student 6:

> Student 6: In the eighth question, I would say yes, because those two x can delete each other.
>
> Interviewer: And then it is sine of what?
>
> Student 6: Then it is just sine …, just like an equation we can say, it just stands alone. About question nine, I would say it is true, because … or no, it is not correct, because here we have our π without a parenthesis, but here it is inside, so we calculate it with the expression here.
>
> Interviewer: So, if here [Question 8], it was like this[$\frac{\sin(x)}{x}$], would it be correct or not?
>
> Student 6: Eh.. No, I think not, because now x is in the parenthesis.

Here, the parenthesis is the determining factor for this student to view something as an input of the function sine, or not. Without the use of parenthesis, he said that the two x can "delete each other", most likely referring to division, and so, he must have perceived $\sin x$ as a multiplication between sin and x, as other students before him. When he was asked about the lack of input, his answer was that sine "just stands alone". Hence, the existence of an input does not seem necessary to Student 6. His words "just like an equation", imply that he was considering of sin = sin, perhaps thinking that an equation can exist without the use of variables, like 2 = 2. However, this misconception disappeared when there was a parenthesis in $\sin(x)$.

In the next question, where the parenthesis exists, the student immediately distinguished between the input and the output of the function sine and the different results he would get depending whether $+\pi$ was a part of the input or not. So, when he was asked again about Question 8, but this time after having a parenthesis around sine's input, he said it is not true, with his only explanation being that x was now in the parenthesis. He did not find it necessary to explain it further, because he probably thought that it was apparent that what is inside the parenthesis cannot go outside, or be divided with something outside of the parenthesis etc.

Student 7:

> Interviewer: I see you answered "yes", in $\frac{\sin x}{x}$ = sin, but "no" in $\sin(-\frac{\pi}{2} + \pi) = \sin(-\frac{\pi}{2}) + \pi$. Is there something different?
>
> Student 7: Yes, because the $+\pi$ is out of the parenthesis and that usually you put that plus, after [the parenthesis].
>
> Interviewer: So if $\frac{\sin x}{x}$ = sin, was written like $\frac{\sin(x)}{x}$ = sin, is it a yes or a no?
>
> Student 7: I would say no as well, maybe..
>
> Interviewer: Because of the parenthesis or something else?
>
> Student 7: Not because of the parenthesis. I think I would actually say yes.
>
> Interviewer: Ok, so it is a yes for both.[$\frac{\sin x}{x}$ = sin and $\frac{\sin(x)}{x}$ = sin].

Student 7 was immediately asked to compare his two answers, as he answered "yes" in the first one, but "no" in the second one. Instead of finding the difference between the two questions, it seems that he compared the two sides of the equality "$\sin(-\frac{\pi}{2} + \pi) = \sin(-\frac{\pi}{2}) + \pi$". He also mentioned that he usually encounters it in the form of the right-hand side of the equation, with the "$+\pi$" not being a part of the input of sine. His impression seems accurate, because it is not often that a student has to make calculations inside the input of the function.

Then, I tried to be more direct in order to find out if it was the parenthesis that made the student answer differently in the two questions. However, contrary to what we expected, the student replied that it was not because of the parenthesis that his answers were different, and that he would also agree with "$\frac{\sin(x)}{x}$ = sin". He did not explain why. Overall, it does not seem that Student 7 considered $\sin(x)$ as a function, where x is the input and $\sin(x)$ is the output.

Student 8:

Student 8: $\frac{\sin x}{x}$ = sin is also true, x's go out together.

Interviewer: And then what do we have?

Student 8: It is equal to sine.

Interviewer: What is sine?

Student 8: Sine is a posing line in a circle, so if we have this triangle, this would be the adjacent and this would be the opposite side. So, this sine is when we try to calculate this here ["this" refers to the angle which he drew-see figure 7.19].

Interviewer: So this is what it is [showed the leg that he marked on the right-angled triangle].

Student 8: Yes.

Student 8: $\sin(-\frac{\pi}{2}+\pi) = \sin(-\frac{\pi}{2})+\pi$, it is not the same as the one above, I think, because up here [$\sin(-\frac{\pi}{2}+\pi)$], we multiply sine with $\frac{\pi}{2}+\pi$, but here [$\sin(-\frac{\pi}{2})+\pi$], it is like sine of something $+\pi$, so I would definitely say they are not the same.

Student 8 is the only one who directly answered what $\sin x$ is. According to him, $\frac{\sin x}{x}$ equals sin, which he defined with respect to the unit circle setting. In figure (7.19) we can see the geometrical representation that he constructed in order to give the definition of $\sin x$. He did not refer to the unit circle, but to any circle, and then he drew a right-angled triangle, where the leg opposite from the angle x would be equal to $\sin x$. However, he neglected to mark the angle as "x", so the question whether he had connected x to an angle is created. So, Student 8 considered $\sin x$ to be the vertical distance from a point of the unit circle to the x-axis. Then, he marked that leg of the triangle, and added that "sine is when we try to calculate this here". We assume that he was referring to finding the length of the leg of the triangle, corresponding it to the y-axis, in the unit circle setting. Moreover, the student answered that "$\frac{\sin x}{x}$ = sin", because he was under the assumption that there is a multiplication between sin and x, and he did not consider $\sin x$ to be a function. We have reached to this conclusion, firstly, because he said that "the x's go together", implying that he divided the x in $\sin x$ with the x in the denominator, and secondly, because of the last part of this excerpt, where he said that in "$\sin(-\frac{\pi}{2} + \pi)$", he multiplied sine with $\frac{\pi}{2} + \pi$. Hence, his answer "yes" referring to "$\frac{\sin x}{x}$ = sin", does not seem to be due to the lack of parenthesis, but due to other notation problems. This can be considered as a didactical obstacle, where the old knowledge of "hiding" the multiplication sign between numbers, for example 2·x can be written as 2x, contradicts a newer piece of knowledge about functions, where f(x), or $\sin(x)$ represents a different notion. What is remarkable though, is that even though he wanted to multiply, he considered "$-\frac{\pi}{2} + \pi$" as an entity, multiplying sine with all of "$-\frac{\pi}{2} + \pi$", and not separating it, implying that he might have perceived it as the input of sine.

Regarding "$\sin(-\frac{\pi}{2}) + \pi$", Student 8 changed strategy. He did not mention again the multiplication, but he said "sine of something plus π". This can be translated as an attempt to consider $\sin(-\frac{\pi}{2})$ as the output of a sine, and adding π, as something external and separate. Trying to understand why he distinguished between the two sides of the equation

$\sin(-\frac{\pi}{2}+\pi) = \sin(-\frac{\pi}{2}) + \pi$, and the reason why he did not consider them both as multiplications, we assume that this was due to an additional calculation in the input in the left-hand side of the equation. Considering the remark of Student 7 ("usually you put that plus, after"), we suppose that the reason why the student handled differently the two sides of the equation was that he had not previously encountered an input like this one, where calculations inside the input were required.

> Interviewer: What if you have $\frac{\sin(x)}{x}$ = sin? Is it a yes or a no?
>
> Student 8: It is true.
>
> Interviewer: So if this [$\frac{\sin(x)}{x}$ = sin] is true, what is the difference between Questions 8 and 9? Because you said "no" to Question 9. Should they maybe be both true or both false?
>
> Student 8: Yes, that is actually also true.
>
> Interviewer: So, which of the two is it?
>
> Student 8: Then I think it is a yes, even though it is a lot of yes, which is suspicious.

Our former assumption of the problem not lying in the lack of the parenthesis (but in the function notation) for the student's answer on "$\frac{\sin x}{x} = \sin$" is confirmed, when he once more replied that "$\frac{\sin(x)}{x}$ = sin" holds. Later in the dialogue, I tried to make the student compare Questions 8 and 9, aiming for him to see the difference between them, and recognize x in $\sin x$, as the input of a function, which cannot be deleted or divided with another x. However, Student 8 did not seem to have connected sine to being a function, as he concluded that he could divide the x of the nominator with the x of the denominator.

It should be mentioned that in Class Y, the teacher had presented the definition of a function, and she had explained that for one input there must only be one output. Hence, the students had the opportunity to examine if $\sin x$ is a function. Moreover, while using Nspire, the students had to name different functions as f1(x) = $\sin(x)$, f2(x) = $\sin(x-2)$, etc., equating the function of sine to f(x), a symbolism which is the usual notation for a function. So, it is curious how Student 8 did not seem to view sin x as a function which cannot be partially divided by another x. It seems that due to the old knowledge of handling multiplication, the student found it challenging to distinguish between multiplication and function notation. Our estimation about his confusion is that he was considering $\frac{\sin(x)}{x}$ to be similar to something like $\frac{\alpha\,(3-1)}{2} = \frac{\alpha\,(2)}{2} = \alpha$. To sum up, the student's final answer for both questions was "yes", but he did not explain why. His suspicions about the many "yes" also refers to Question 7.

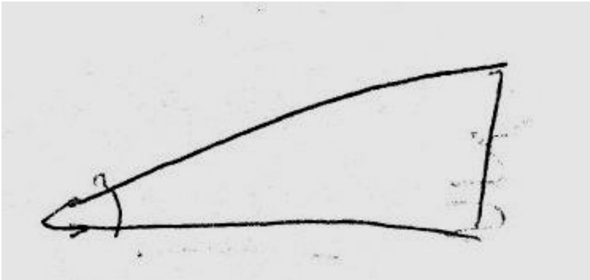


Figure 7.19: Definition of sine by Student 8.

Student 9:

Student 9: I know that $\frac{\sin x}{x}$ = sin is yes, because sine times x divided by x, then you can eliminate the x's.

Interviewer: What about here $\frac{\sin(x)}{x}$ = sin?

Student 9: But then sine does not have a value, if you just write sine, does it?

Interviewer: So is it a yes or a no?

Student 9: What about if you write sine without the parenthesis and then it has a value? [she probably means sinx=sinx]

Interviewer: Well you decide, is it a yes or a no?

Student 9: I would say no.

Student 9 also had a problem with the function notation, misinterpreting the lack of parenthesis for multiplication. When the parenthesis was added, she realized that "sin" did not have an input. However, this did not constitute a problem for her in the first situation where the parenthesis was not there. An assumption we make from her answer is that she meant to multiply both sides of the equation with x, resulting in $\sin x = \sin x$. So in that case, she did not consider it a problem to start from $\sin x = \sin x$ and then divide both sides by x. This shows that she saw a difference between $\sin(x)$ and $\sin x$, where we cannot know what she considered $\sin x$ to represent and why she saw a difference between the two. It could perhaps be, that as long as there was a lack of parenthesis, she considered $\sin x$ to be a multiplication, as for example 2·x = x·2. Similarly to this example, she could have thought that this can also happen for: 2·sin = sin·2, or, x·sin = sin·x. In the end, she chose to answer "no", without giving further explanation on which the difference is between the two and why she chose to answer "no" after all.

> Student9: This one is a no too, because if you have sine, then first you have to say $\frac{\pi}{2} + \pi$ and then it is the sine value of that, but if you say $\sin(-\frac{\pi}{2})$ and then you $+\ \pi$, then it does not give you the same answer.

Here, the student seems to have understood that sine is a function, and she seemed to be able to distinguish between the input and the output of a function. When she said "it is the sine value of that", she implied that "$\frac{\pi}{2} + \pi$" is the input, whereas in the right-hand side of the equation, the input is different, and so, the result would also be different. The student stayed in the symbolic representation of the equalities, and she did not use any trigonometric setting to explain her answers.

Student 10:

> Student 10: What is this x here? [$\frac{\sin x}{x}$ = sin]
>
> Interviewer: [I put a parenthesis so that it is sin(x)]
>
> Student 10: Ah ok! It is wrong because it is.. you cannot.. this just cannot go out with each other, because it is not times x, it is sin(x), it is a function.
>
> Interviewer: Yes.

> Student 10: [She writes "no" to the Question 9]. Here in Question 9 we have some things in the parenthesis, whereas there, the plus is outside.

Student 10 was the only student who called $\sin x$ "a function". At first, she asked what x was in $\sin x$, also being confused with the function notation. When I put the parenthesis, the student explained that it is not possible to divide with x, since the x of the nominator was a part of the input. She explicitly said that $\sin x$ does not stand for "sin times x", but that it is a function, making her the only student who used the term "function". Regarding the second equation, it was clear to her that the equality was wrong, since the "$+\pi$" in the left-hand side was a part of the input, whereas in the right-hand side it was not.

Student 11:

> Student 11: $\frac{\sin x}{x}$ = sin is true, because if you can move this x, it is times..
>
> Interviewer: It is not times… what if I put a parenthesis here?
>
> Student 11: Oh, I am not sure about that, we did not have about that, but I think you can move it [x] over to the other side [to sin] like this [so that $\sin(x) = \sin x$], but I am not sure.
>
> Interviewer: So is it correct or not?
>
> Student 11: I would say correct.
>
> Student 11: $\sin(-\frac{\pi}{2} + \pi) = \sin(-\frac{\pi}{2}) + \pi$ is not correct.
>
> Interviewer: What is the difference between those two?
>
> Student 11: You have to multiply this one and this one with this [she meant multiply sin with $-\frac{\pi}{2}$ and then multiply sin with π], but in $\sin(-\frac{\pi}{2}) + \pi$, you only multiply sin with $-\frac{\pi}{2}$, and π is next to it.

With or without the parenthesis, Student 11 decided that the equation holds. She seemed to perceive $\sin x$ as a multiplication. Even after I told her that there is no multiplication, and I wrote the equation again with a parenthesis, the student seemed uncertain and she recommended that we multiplied both sides of the equation with x ("move x over to the other side”) and have $\sin(x) = \sin x$, which is surprisingly true as an equation, but by accident. We can observe the didactical obstacle which this student had in distinguishing between the multiplication and the function notation.

Then, the student answered “no” to the second equation, and she was asked to find the difference between the two questions. In comparison to Student 8, where he only perceived the left-hand side of the equation $\sin(-\frac{\pi}{2} + \pi) = \sin(-\frac{\pi}{2}) + \pi$ as a multiplication between “sin” and “$-\frac{\pi}{2} + \pi$”, Student 11 perceived both sides as a multiplication. Nevertheless, she detected the difference between the two sides to be that in the left-hand side the multiplication was between “sin” and “$-\frac{\pi}{2} + \pi$”, whereas in the other side the multiplication was only between “sin” and “$-\frac{\pi}{2}$”, adding that “π is next to it”. Therefore, even though we see a confusion with the function notation and even though she did not explain why she answered “no” to the second equation, we understand that she could have had a perception of the input of sine, but she had not probably realized that she was referring to the input of a function. This can be seen because she distinguished between the “$+\pi$” being inside the sine’s input, whereas in the right-hand side of the equation, “next to it”. However, she might have been considering something similar to “$\alpha(-2) + 2 \neq \alpha(-2 + 2)$”, and that could be a reason why she saw a difference between the two sides of the equation.

Now, adding to the results which are related to students’ views of sine and cosine, we will now analyze a photo which was taken during the second hour of Trigonometry of Class X. The teacher had not yet introduced the function setting. At that point where the photo was taken, the students had to calculate different values of sine and cosine by the use of the unit circle. As we can see, the student had found two outputs for each input of cosine and sine. It is not certain whether the student did not have the prerequisite knowledge of the notion of function, or whether he had not connected the fact that sine and cosine are functions.

On the image’s first line, the student seems to have corresponded 90° to $\frac{\pi}{2}$, and then he wrote 0, probably wanting to express that $\cos(\frac{\pi}{2}) = 0$. It seems that he first tried to convert from degrees to radians and then calculate the cosine value of that angle, which is also what he did for the third line. He first converted $-90°$ to $\frac{3\pi}{2}$ and then he wrote -1, probably meaning that $\sin(-90°) = -1$. The student seems to have converted radians to degrees on the second line. We can understand this from the "°" he put after the first 0. However, π rads does not correspond to 0°. The student does not handle sine and cosine as functions: it seems that he converted from degrees to radians and conversely, writing the result where the output of the function should be written. Also, it seems that he had not realized that even though the input of the trigonometric function sine or cosine is an angle, which can either be in degrees or radians, the same does not apply for the output, which is not angle.

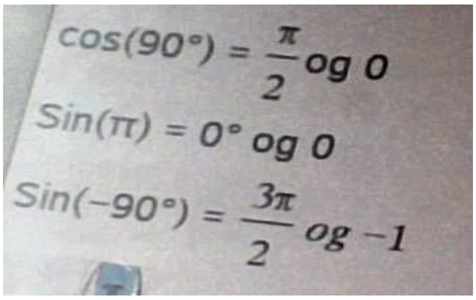


Figure 7.20: Student's screen: The sine and cosine function giving two outputs.

8. Discussion:

According to our results, we can see that the unit circle setting was the most chosen one among the students who took the interviews. For instance, five out of six students who participated in the interviews, used the unit circle setting to explain the identity $\cos(x + 2\pi) = \cos(x)$, and one, the function setting. Regarding finding the difference between $\cos(\frac{\pi}{2}) = 0$ and solving $\cos(x) = 0$, those students who used a trigonometric setting as a part of their explanation, used the unit circle setting. Also, all six students used the unit circle to explain the identity $\sin^2 x + \cos^2 x = 1$. This is different to Weber's results in 2005, where the great majority of the students had not chosen the unit circle to work with (see p. 22), but the triangle setting, when they explained the trigonometric identity $\sin^2 x + \cos^2 x = 1$. In particular, in the first group, none of the students of that research chose the unit circle to explain the identity $\sin^2 x + \cos^2 x = 1$. Also, from the fifteen out of the forty students of the second group who gave a valid explanation, only two used the unit circle. However, this was not the case for our students, whose choice for the unit circle almost seemed as a one-way street: we believe that they chose the unit circle, not only because they were influenced by their class's didactical contract, since it was the setting they had worked most within the classroom, but also because they did not want to work with the function setting. Student 1 told us that the graph of cosine confuses him, whereas the unit circle provided a relation between sine and cosine. Student 2 also mentioned that the function setting was harder for him, whereas the unit circle "made more sense".

From what we have analyzed, there are two possible reasons why the majority of the students did not choose the function setting. Firstly, agreeing with Brown (2005), it was due to the lack of prerequisite knowledge. Indeed, some students did not have the necessary knowledge about interpreting graphs of functions, something which prevented them from using the function setting. For example, Students 7 and 8 could not find where $\cos(\pi)$ is on the cosine graph. Also, when Student 10 saw the cosine graph, she said "is this line cosine?", revealing how unfamiliar the function setting was to her. Moreover, Student 6 was under the impression that when a graph of a function decreases, then the function is negative and when it increases, it is positive. Secondly, using Douady's terminology (see section 3.1), we detect an insufficient translation from the unit circle to the function setting. During this incomplete translation, some information gets lost, resulting in the creation of imperfect links between the two settings. For example, even though Student 2 used both settings, he referred to the range of cosine as $(3,-3)$, when he used the graph setting. We assume that he would not have said the same, if he was working on the unit circle, as he had mostly worked with the unit circle during the lessons, and so, he would have probably realized that since the radius of the unit circle is 1, the range of cosine cannot be $(3,-3)$. However, we cannot neglect the possibility that this was not due to the transition between the settings, but due to a gap in his knowledge regarding both settings. In other words, it is possible, that even if we assume that he knew that the radius of the unit circle is 1, he might not have connected this to the range of cosine. Student 7 on the other hand, did not even try to transfer from the unit circle to the function setting. He stated that he could not work with the graph of cosine, since he had learnt how to work with the unit circle in class. The imperfect link at this case, could perhaps be better described as a non-existing link between the two settings.

Overall, only Student 9 seemed comfortable both with the unit circle and the function setting. She had explained the identity $\sin^2 x + \cos^2 x = 1$ during class using the unit circle, and the identity $\cos(x + 2\pi) = \cos(x)$ during the interview using the graph of cosine. She was also the only student who had attended lower secondary school in another European country. We do not know though, how this could affect her answers or whether this was a coincidence. In her case, the imperfect link lied between the triangle and the other two trigonometric settings. In particular, she had strictly connected the degrees to the triangle setting, and as a result, she could not explain the existence of negative angles measured in degrees. This constitutes a didactical obstacle, where the old knowledge of the angles being strictly positive, contradicts the new knowledge. This also agrees with Akkoc (2008) and Moore (2013), because the difficulty in transitioning between the settings appears to be due to a problematic connection between degrees and radians. Furthermore, agreeing with Bressoud, we observe that the remains from the triangle trigonometry, become an obstacle in transitioning to unit circle trigonometry.

We cannot not present the setting preference of Student 11, who showed a strong preference for the unit circle. She explained the identity $\cos(-x) = \cos(x)$ with respect to the unit circle setting, even though the question asked her to use the graph setting. What is remarkable though, is that when she was asked what an angle is, she started by: “We use the unit circle and we can see the angles”. It was also Student 6, that when he defined what an angle is, he asked if he had to use the unit circle as a part of his explanation, showing that he could also connect the notion of angle to the unit circle setting. Had he chosen the unit circle setting, he might have chosen radians instead of degrees. This is perhaps the reason why he asked which setting he must use, right after he mentioned the degrees. In other words, it could be that he had connected degrees to the triangle Trigonometry, and radians to the unit circle Trigonometry. However, it is not possible to say whether the link between these two settings was well established for the student, as he only explained what an angle is, with respect to the triangle setting.

The analysis of the students’ answers in relation to whether the equality $8\pi = 2\pi$ is correct or false, showed that not all students based their answer in relation to a trigonometric setting. However, those who did, did it in terms of the unit circle setting, or the use of a circle. In particular, out of the eight students who had time to answer the yes/no questions, three replied that the equality does not hold. Students 8 and 9 argued that $\pi = 3.14$ and thus, it cannot be that $8\pi = 2\pi$. Student 10 did not explain his answer. Student 6 originally supported the equality, claiming that π stands for how many times one goes around the unit circle, and not connecting it to 3.14. Then he changed his mind about his original answer, when he was told that $\pi = 3.14$. In the end, he concluded that his answer would depend on whether he thought about the unit circle or not. This is related to Kupkova’s results in 2008, where the majority of the students admitted that they saw no relation between π which they see in the unit circle setting, or as an input of trigonometric functions, to the number 3.14. Student 11 claimed that $8\pi = 2\pi$ holds and it does not hold at the same time, supporting her opinion with the same argument as Student 6. She marked both “yes” and “no” on the questionnaire. Two students (Students 2 and 7) said that they did not know, and only one student (Student 1) thought that the equality is true. He did not have time though, to explain his answer. In general, we can see that only Student 1 answered that the equality holds, probably considering that 2π and 8π lie on the same place on the unit circle and not recognizing that $\pi = 3.14$. Overall, we can see that the difficulty in transferring from $\pi = 3.14$ (old knowledge), to π in the context of the unit circle (new knowledge), that is, as an input of sine and cosine and as a number which is used to indicate lengths of arcs, constitutes this as a didactical obstacle.

Changing the focus to students' misconceptions related to the properties of the trigonometric functions sine and cosine, it was found that there was a problem determining their range. As already presented, Student 2 wrote that the range of cosine is $(3,-3)$. Moreover Students 1 and 10, and some students whose writings we have obtained through photos during the lessons, had identified cosine and sine to be the x-axis and the y-axis respectively in the unit circle setting, neglecting that the range of sine and cosine is $[-1,1]$. Let us consider this scenario for a moment, where an angle $\theta = 45°$ is formed in the unit circle by the line $y = x$. Then by definition we have that the point A where the angle cuts the unit circle is $A(\cos 45°, \sin 45°)$. The range of sine and cosine must lie between -1 and 1. Now, when someone claims that $\cos\theta$ and $\sin\theta$ are the x-axis and the y-axis respectively, he implies that their values are not necessarily described by the coordinates of the point A, but from other possible points on the line $y = x$. This contradicts firstly the way $\cos\theta$ and $\sin\theta$ were defined in the unit circle setting, and secondly, even if that were true, then how would one pick which point of the $y = x$ would determine the values of $\cos\theta$ and $\sin\theta$? Taking it a step further, could it be more than one point, contradicting the uniqueness of the outputs $\cos\theta$ and $\sin\theta$? At this point, we should mention that in Class Y, the teacher had briefly explained about the uniqueness of the output of a function. However, the students of both classes had just started to learn about functions, so it is possible that they would not realize that having more than one output, is problematic. Returning to the students' results, it is clear from the unit circle constructions of Students 8 and 9, that they had not made this identification.

Having as the main criteria the distinguishing between the input and the output of the functions sine and cosine, as well as the fact that for every input of a function there exists a unique output, we tried to gain an understanding on students' perceptions on sine and cosine as functions. What we found, was that six out of ten students who participated in the interviews, perceived sine as a function. From those six, one student (Student 2) was not confused with the lack of parenthesis in $\sin x$, referring to the input of sine as a box, or as a whole number which could not be separated. Two other students (Students 5 and 10) confronted the different notation, asking what $\sin x$ stood for, and in particular, Student 10 said that $\sin x$ is a function, and not a multiplication between sin and x. She was the only student who used the word "function". The other three students (Students 1, 6 and 9) found the parenthesis to be the deciding factor for distinguishing between the input and the output of a function. Without the parenthesis, those three students handled $\sin x$ as a multiplication between sin and x. In particular, Student 9 was puzzled over sine not having an input and she then tried to prove the equality $\frac{\sin x}{x} = x$ through algebraic manipulations. Student 6 stated that $\frac{\sin x}{x} = \sin$, where "sin stands alone just like an equation". The lack of input did not seem to worry him. Only when I put the parenthesis, the student said that this equation was not true. We cannot consider his answer wrong, as this was clearly a problem of function notation. Our results agree with Sajka (2003), who supports that the importance of notation is big and if the students do not understand the symbolic notation or confuse it with other already known rules of algebra, this will lead to not solving mathematical problems in new situations. Here, we saw that for the above students, the notation was the determining factor for distinguishing between the input and output of the trigonometric function.

Regarding the rest of the students, one out of four (Student 3) did not provide enough data so that we can have a certain understanding whether he perceived sine as a trigonometric function. Student 7 replied that the equality $\frac{\sin x}{x} = \sin$ holds, in both cases where the parenthesis is or is not there, giving the impression that he did not recognize sine as a function, with x being the input and $\sin x$ the output. We cannot know if he considered it as a multiplication instead. Now, the reason why he said that the equality $\sin(-\frac{\pi}{2}+\pi) = \sin\left(-\frac{\pi}{2}\right)+\pi$ does not hold, is because he had not encountered the left-hand side of this equality before. So this student decided that the equality does not hold, not based on algebraic manipulations, nor on his knowledge about functions, but based on his experience on what he had encountered more often. Student 11 handled $\sin x$ as a multiplication, even though she was told that it is not. Even though I put a parenthesis afterwards, the student relied on old techniques on performing algebraic calculations and "moved" the x from the denominator to the right-hand side, getting $\sin(x) = \sin x$ as a result. She then handled the equality $\sin(-\frac{\pi}{2}+\pi) = \sin\left(-\frac{\pi}{2}\right)+\pi$ in a similar way, where she applied the distributive law, by multiplying sin with $\frac{-\pi}{2}$ and then with $+\pi$, whereas on the right-hand side she multiplied sin with $\frac{-\pi}{2}$, and then she added π. This reveals a didactical obstacle, where the student had relied on old techniques about algebraic manipulations like $\alpha(-2)+2 \neq \alpha(-2+2)$, she had overgeneralized them, and applied them in this case, neglecting the new knowledge about the notation of a function's input.

The last of those four students was Student 8, who was the only student to have answered what $\sin x$ is. According to Student 8, sine is a side, which is opposite of an angle, in a triangle which lies in a circle, most definitely meaning in the unit circle. He did not seem to perceive it as a function. Neither in his drawing, or when he spoke, did he include an input in sine, amplifying our assumption. Also, he handled $\sin x$ as a multiplication, revealing a didactical obstacle, where the old knowledge about hiding the multiplication sign conflicts the new knowledge about function notation. We can see this because he used the phrase "the x's go out together" and he admitted that in the equality $\sin(-\frac{\pi}{2}+\pi) = \sin\left(-\frac{\pi}{2}\right)+\pi$, he multiplied sine with $-\frac{\pi}{2}+\pi$, making us wonder if he treated $(-\frac{\pi}{2}+\pi)$ as an input of the function sine, or as a parenthesis, like Student 11. For the right-hand side, Student 8 said that it is not a multiplication, but "sine of something". We cannot be sure what he meant and whether in this case, he considered sine to be a function with $-\frac{\pi}{2}$ being the input. The answer why he treated the left from the right-hand side of the equality in a different way, could be found in what Student 7 said: it is more often that one sees the right-hand side of this equality. Hence the problem here, could be that $\sin(-\frac{\pi}{2}+\pi)$ was unfamiliar, compared to what the students had encountered before. Overall, it was not the parenthesis itself which confused Student 8. His confusion was due to other notation problems. We believe that it was the combination of the trigonometric function sine, together with the fact that there were required calculations inside the input of sine. Those results are also consistent with Sajka's (2003).

Now, regarding the students' perceptions on sine and cosine as functions, we find figure 7.20 rather important. If the content of this photo was true, then by definition, cosine and sine would not be functions, because in the photo, we see two outputs for each input. Moreover, one output is measured in degrees, revealing that this student had not realized that the range of sine and cosine must be a real number. We assume that the problem lied in the transition from angles to real numbers as inputs. It could perhaps be that this student thought that just as the input of a trigonometric function can be both in radians, as a real number, and in degrees, this could also occur with the output, which can be in degrees and radians as well, neglecting the fact that the output should be a real number.

Finally, summing up the most important information we could gather regarding students' perceptions of the input of sine and cosine, we will first refer to their perceptions of an angle as quale, quantum or/and relation, and whether they connected angles to a trigonometric setting. Out of the four students who were asked what an angle is, two (Students 8 and 9) described an angle in terms of its quality, one student (Student 6) both as a quality and a quantity and one student (Student 11) in terms of relation. Student 11 also mentioned radians in her attempt to show the connection between the arc and the subtended angle in the figure 7.15 of the unit circle which she had drawn. In general, most students linked the notion of angle to a circle, or, specifically, to the unit circle. For example, in Question 7, where the students had to decide whether the equality $-390° = 30$ was correct, those students who based their explanation on a trigonometric setting, and did not only stay in the algebraic calculations, connected the notion of angle to the unit circle setting, or to a circle. It should also be mentioned at this point, that as expected, out of the four students who had time to answer Question 7, and who based their answer on the fact that a circle corresponds to 360°, three miscalculated the sum of $-390° + 360°$ to be $30°$. We assume that this happened because students usually perform algebraic calculations with a CAS-tool or a calculator. Now, continuing about the connection of angles to trigonometric settings, we found that students' perceptions varied. Students 6 and 9 had connected degrees to the triangle setting and radians to the unit circle setting. In particular, Student 6 supported the equality 8π = 2π, referring to 8π being on the same place on the unit circle as 2π. At the same time, he handled the equality "$-390° = 30°$" differently, not using the unit circle, or a circle, to see if the equality holds, revealing that he had not connected degrees to the unit circle setting. As we can see, this problem in the transition between the triangle and unit circle setting agrees with Akkoc (2008) and Moore (2013), who argued that it could be due to problems in connecting degrees and radians. On the other hand, Students 8 and 10 had not made this distinguishing. Even though at first Student 10 wondered how something negative can be equal to something positive, she later thought of using the unit circle. Finally, from figure 7.20, we can observe that the student who wrote what is in that photo, had connected both radians and degrees to the unit circle, which we know he was using when the photo was taken.

9. Conclusion:

The aim of this thesis was to gain an insight on students' perceptions of the trigonometric functions sine and cosine, as well as which trigonometric setting they primarily choose to work with, and how they transfer between these settings.

According to our results, most students chose the unit circle setting. The choice for some students was double: Not only did they choose to explain a trigonometric identity that they were given by the use of the unit circle, but afterwards, when it was required that they used the graph of cosine to describe another trigonometric identity, they insisted on doing so, again, by using the unit circle. The reason why most students did not choose the function setting was due to lack of prerequisite knowledge on interpreting graphs of functions, and due to problems in the transition from the triangle or unit circle to the function setting. Overall, only one student was comfortable using both settings to explain trigonometric identities. None of the students used the triangle setting to support their explanations, but this is not something which cannot be interpreted, as none of the two teachers mentioned this setting during the lessons.

The first misconception we observed, is the identification which some students had made between the functions sine and cosine to the y-axis and the x-axis respectively, resulting in problematic perceptions regarding the range of the functions sine and cosine. Overall, it seemed that the students faced several challenges into handling sine and cosine as trigonometric functions. Some of the students did not seem to understand the importance of the existence of an input in a function, neglecting to mention it in their explanations and geometric constructions (for example Student 10 constructed the unit circle, marking the x-axis as "sin", and Student 6 described sine as something which "stands alone"). Furthermore, one student wrote "$\sin -(x)$", where the "$-$" was not a part of the input, nor the output, revealing a confusion regarding the input and the output of a function. Another student had written two outputs for each input of sine and cosine, contradicting the fact that sine and cosine are functions. However, it should be mentioned that at that point, the notion of function had only recently been introduced to the students, and thus, it is understandable that it was not yet well-established.

Another misconception was created due to the function notation $\sin x$, where some students perceived this as a multiplication between sin and x. Here, the existence or lack of the parenthesis in $\sin(x)$ played an important role in the recognition of x as the input of the function for some students. For some others though, this was not the determining factor, and their perception of recognizing $\sin(x)$ as a multiplication did not change. Weber (2008) had suggested that students should treat a trigonometric function like the square root of a number. It would be interesting to see whether students' misconception regarding $\sin(x)$ could be overcome, by comparing the input x, to x as the radicand, and the output $\sin(x)$, to $\sqrt{x}$.

The last misconception which we observed, and which is connected to perceiving sine and cosine as trigonometric functions, was from the same student who had written two outputs for each input. In addition to having two outputs for each input, the student had one of the outputs measured in degrees. The confusion in handling radians and degrees could be because the student had not realized that radians and degrees are used to measure angles. Therefore, this could imply that the student had not connected the input of sine and cosine to angles. Secondly, we understand that the student was not aware that the output of sine and cosine must be a real number.

Furthermore, from the four students who were asked to describe what an angle is, only one (Student 11) connected the notion of angle to the input of trigonometric functions. This does not necessarily mean that if we asked the students whether the input of sine or cosine is an angle, they would have denied it. However, the fact that they did not immediately make this connection, could be a sign of a partial perception of the input of trigonometric functions, as it could imply that they had not connected it to angles. This relates to Dejarnette (2014), who argues that those students who perceive the relation between an angle and a trigonometric function, can use angles as inputs of those functions. Indeed, we do not think that it was a coincidence, that the only student (Student 11) who explained angles with respect to lengths of arcs on the unit circle, was the only student who mentioned that angles can be used as inputs of trigonometric functions, as one can observe the relation between the length of an arc x, and $\sin x$ or $\cos x$. Also, if we carefully observe this student's answers, we will notice that almost all of her answers show a perception of angles as lengths of arcs in the unit circle, also corresponding them to radians. She was also the only student who had not substituted x in $\cos(-x) = \cos(x)$ in her effort to explain this identity, handling x as any length of an arc which subtends the angle x. Moreover, she had perceived the equality $8\pi = 2\pi$, as an equality which does not hold, being able to combine the old knowledge that $\pi = 3.14$, to the new knowledge of the unit circle, and the function setting. In summary, it seems that this student's perception of sine and cosine was broader and more complete in comparison to other students, who had fragmented perceptions of sine and cosine. Overall, she seemed to connect the input of sine or cosine, to an angle in the unit circle, and to the length of the arc which subtends it. However, the student had a difficulty with handling the function notation, perceiving it as a multiplication.

To sum up, it is important that the students have the prerequisite knowledge of translating graphs of functions, before they handle graphs of trigonometric functions in the function setting. Also, taking as an example this of Student 11, we conclude that it is important for students to have a good understanding of the concept of radians. We believe that it is beneficial that they first get introduced to radians through the unit circle, and not through a conversion formula between radians and degrees, which could create a problematic connection between the triangle and the unit circle setting. This does not mean that the students should not learn this conversion formula. However, in our opinion, they should learn it after they have gotten familiar with the unit circle and measuring angles in radians, as well as having connected radians to lengths of arcs in the unit circle. This way, there can be better links which connect the triangle to the unit circle and the function setting, in order to facilitate the transition between them, and as a result, to perceive sine and cosine in all three settings.

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