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A diagnostic study on functions

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Speciale – Matematik

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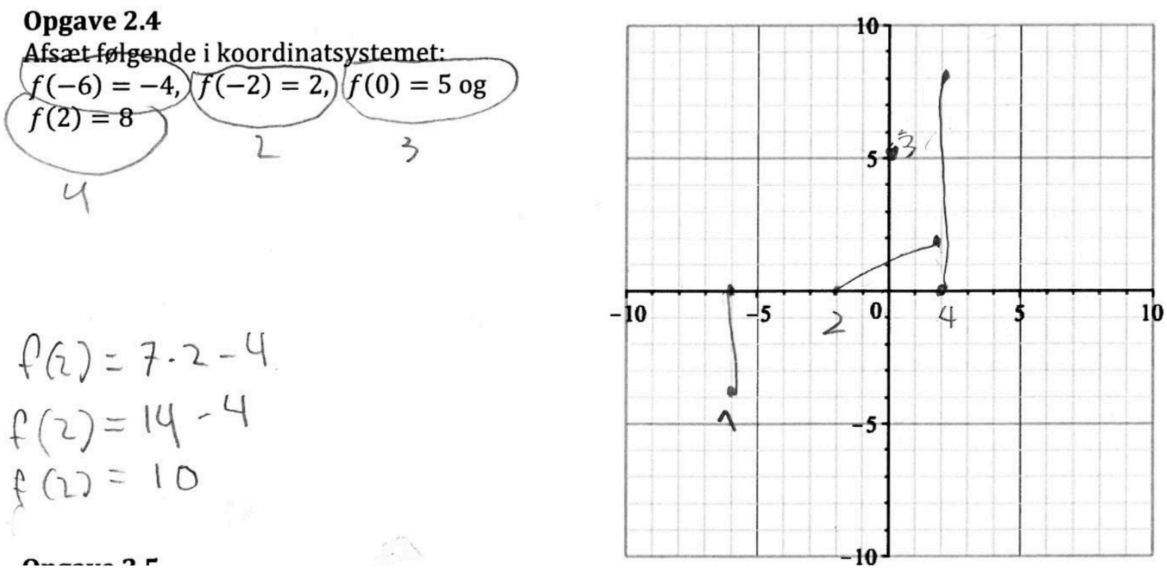

Master’s thesis

By Nicole Jonasen

A diagnostic study on functions

A study on first year high school students’ challenges with various aspects of functions

Supervisors: Mikkel Willum Johansen and Britta Eyrich Jessen

Submitted on: September 2nd 2019

Abstract

The purpose of this master's thesis is to examine high school students' understanding of functions. The examination of the students' understanding of the concept of function is done on the base of Tall and Vinner's theory about concept images and concept definitions. This master's thesis will present a diagnostic questionnaire, which is developed on the base of diagnostic teaching and testing and cognitive theory. The diagnostic questionnaire will examine why the students are incapable of drawing linear functions. The results of the questionnaire are based on 52 first year high school students' answers. Among other things the answers to the questionnaire revealed that students have a narrow concept definition of the concept of functions and that almost a third of the students perceived that a function is an algebraic expression. The results of the questionnaire showed that there is a relation between being incapable of drawing linear functions by hand and not understanding the conceptual blend of the Cartesian plane. Furthermore the results of the questionnaire also showed that there is a relation between not understanding and using the $f(x)$-notation correct and not being able to draw linear functions by hand. The relations were tested through statistical testing of several hypotheses. In addition, the answers to the questionnaire also showed that the students' usages of CAS-tools do not have an influence on their abilities to draw linear functions by hand. These above relations suggest a change in the approach to teaching the $f(x)$-notation. Furthermore this thesis also suggests that a greater focus on teaching the students conceptual metaphors, conceptual blends and metonymy of mathematics could maybe improve the students' understanding of various aspects of functions.

Acknowledgement

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I would also like to give great thanks to the high school and the teachers who were willing to lend me their classes. Without you the research would not have been possible. Thanks to the students for being open-minded and agreeing to let me use their answers of the questionnaire for my research.

Lastly I will also like to give thanks to my family and friends for being supportive and helpful during the writing of my thesis. Thanks to my parents and brother for always being there when things seemed too overwhelming. Thanks to my boyfriend Niels for; his unconditional love, being supportive, optimistic and cheerful and putting up with my thesis-emotions. Thanks to Luisa and Niels for proofreading my thesis. Finally thanks to my fellow thesis student Vibeke who I shared the thesis office with. You have been a great support both emotionally and professionally if I needed to discuss aspects of my research.

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Introduction

The new high school reform in Denmark contains some ambitious goals for mathematics. One of the goals is that almost all high school students must have mathematics at least at level B (Winsløv, 2017). In April 2019 the students being at their second year of high school are the first to be 'forced' to have mathematics at level B. They had their midterms and the results of their midterms were discouraging. 34% of the students would have failed if their midterms were their final exam (Romme-Mølby, 2019). In May/June 2019 the first students that were 'forced' to have mathematics at level B took their written exams, and the outcome was that the percentage that one needs to have correct to pass got adjusted. It got reduced to 21% (Svensson, 2019).

With this evolvement of mathematics in high school I believe that it is important to investigate aspects of students' understating of mathematical concepts. Function is one of the basic concepts of mathematics and much time as well as attention has been spent on the concept of function, but it still remains a difficult concept. Functions are amazing in their variety of interpretation and representations (Sajka, 2003). Though translation between representations of functions is something that is difficult for the students (Leinhardt et al., 1990). However the translating of representations is considered a fundamental process that is leading to mathematical understanding and successful problem-solving (Elia et al., 2008). This means that being able to translate between a function's different representations is important. Since there have been spent a lot of time and attention to the concept of function and it is still a difficult concept, this could indicate that there is a need for a new approach to find out why the concept of function is difficult. One such approach could be 'diagnostic teaching and testing' which is an approach that has been used in Norway to diagnose the students' mathematical knowledge (Brekke, 1994). Another approach could be the theory of cognition, which is a theory used in my bachelor project with interesting results. Theory of cognition used together with the theory of diagnostic teaching, could be a new approach to enlighten the difficulties students experience when translating between different representations that functions have. Furthermore there is a need for a theory that is handling students understanding of mathematical concepts, since it is the students' understanding that is in question. On the grounds of these considerations it has led me to the following problem statement.

Problem statement

In this thesis I will examine high school students' understanding of the mathematical concept function. Furthermore I will on the base of diagnostic teaching together with cognitive theory develop a questionnaire. The questionnaire will examine high school students' conception of functions and their mastering of translating from a function's algebraic expression to its graphical representation. More specifically, the questionnaire will examine four hypotheses. The four hypotheses are developed on the grounds of cognitive theory, experience from high school teachers and from the results of the pilot test. The four hypotheses are:

> H1: There is a relation between understanding the conceptual blend of the Cartesian plane and being able to draw linear functions in the Cartesian plane by hand correct.
>
> H2: There is a relation between being able to understand and use the $f(x)$-notation correct and being able to draw linear functions in the Cartesian plane by hand correct.
>
> H3: If the students mostly use CAS-tools to draw functions then they are not able to draw linear functions by hand correct.
>
> H4: There is a relation between being able to tell what the algebraic expression for a linear function is, based on a graphical representation of the function and being able to draw linear functions by hand correct.

Theory

In this section I will present the theories that creates the theoretical framework for this thesis. The theories are chosen so that I am able to answer the thesis' problem statement. To be able to examine the students' understanding of the mathematical concept function, a theory, which is dealing with understanding concepts in general, is needed. Therefore, I will in this section present David Tall and Schlomo Vinner's theory about how students percept concrete mathematical concepts through concept images. In this section I will also present what diagnostic teaching and diagnostic testing is. This is needed to be able to understand how to develop a diagnostic test. To be able to answer hypotheses H1 and H2, especially, there is a need for understanding conceptual blends, conceptual metaphors and metonymy. On the grounds of this I will therefore in this section present George Lakoff and Rafael E. Núñez' theory of how mathematics is based on conceptual metaphors. Lastly, I will present what the literature and high school teachers find that the students have problems with when working with functions. The knowledge of students' problems on a specific subject is necessary to be able to create a diagnostic test. The literature review was done on the bases of three main publications: Brekke (2002), Niss and Jankvist (2016) and Tall and Vinner (1981). My supervisors handed the main publications to me. From the main publications I discovered the usage of the following keywords: 'misconceptions of functions', 'understanding functions', 'conception of functions', 'representations of functions'. This lead me to searched REX for publications related to the keywords.

Concept image and concept definition

Mathematical results are built on proofs. The constructions of mathematical proofs are often based on logical deductions and formal definitions. However, the human brain is not a purely logical entity. The human brain is not as logical thinking as the field of mathematics assumes and sometimes the situation that humans attempt to think logically can lead to mistakes (Tall & Vinner, 1981, p. 151). Because of this situation Tall and Vinner (1981) find it important that we distinguish between formal mathematical concepts and the way students perceive mathematical concepts. To make this distinction Tall and Vinner (1981) use the terms concept image and concept definition. They define the term concept image:

”We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.” (Tall & Vinner, 1981, p. 152)

They deem that a concept image consists of all the associations one does when thinking of a concept. A mental picture could be visual representations of a concept e.g. a graph for a specific function and/or the symbols $y = f(x)$. Associated properties could be that someone thinks that a function is something that should always be defined by means of algebraic expressions (Vinner, 1983, p. 293). This means that if a student, when thinking about the concept function, thinks about graphs, equations, drawing lines, coordinate system etc. Then all these associations would constitute the concept image to the concept function for this particular student. The concept image is individual since the cognitive structures in humans are different. A concept image is constructed over the years, and it is constructed on the base of different experiences that a student does. The concept image will consequently be altered as the student meets new stimuli and matures (Tall & Vinner, 1981, p. 152). The second term that Tall and Vinner (1981) uses is concept definition.

”We shall regard the concept definition to be a form of words used to specify that concept.” (Tall & Vinner, 1981, p. 152)

The concept definition is the wording that is used to specify a concept. The concept definition can be a personal reconstruction of the student’s way of understating the concept. Hence the concept definition would be the student’s description of its evoked concept image. As a result of the student’s concept image changing over the years the student’s concept definition will also change over the years. Thus the personal concept definition can de different from the formal concept definition. The formal definition is the definition that is accepted by the mathematical community (Tall & Vinner, 1981, p. 151).
Vinner and Dreyfus (1989) did an empirical study of 271 college students’ and 36 junior high school teachers’ concept images of a mathematical function. For this study, a questionnaire

was designed to examine the participants' concept image of the concept function. One of the questions was: "*What is a function in your opinion*?" (Vinner & Dreyfus, 1989, p. 359). The students' definitions were categorised in six categories: 'Correspondence', 'Dependence relation', 'Rule', 'Operation', 'Formula' and 'Representation'. The categories were a refinement categorisation of Vinner's (1983) categorisation. One of the participants of the study wrote this answer to the question:

"It is an equation expressing certain relation between two objects." (Vinner & Dreyfus, 1989, p. 360).

The participant defined a function as something that has an equation. That student's answer is not the formal definition of a function. The formal definition is:

"Let A and B be non-empty sets. A relation f between A and B is called [...] *a function from A to B (and we write $f: A \rightarrow B$), if the following two restrictions are true:*

1. *For all $x \in A$ there is a $y \in B$, such that xfy*
2. *If xfy_1 and xfy_2 (or $(x, y_1) \in f$ and $(x, y_2) \in f$), then $y_1 = y_2$"* (Lützen, 2012, p. 92, my translation).

This is an example of an individual's concept definition that is different from the formal definition of the concept.

When a concept image is being evolved it does not have to be coherent all the time. The reason for this is that some sensory input will excite certain neuronal pathways and inhibit others. Thusly parts of the concept image can be evoked and sensory input will develop the evoked part of the concept image. This part of the concept image might no longer be coherent with other parts of the concept image. It is in this setting that a conflict can occur. A student can have contradictory concept images without being aware of if. The reason of this is that it is only possible to experience the dissonance if both concept images are evoked simultaneously (Tall & Vinner, 1981, pp. 153). A student can work with the graphical representation of a piecewise defined function without knowing the function's algebraic expression. The student can perceive the graph as a function while simultaneously have a concept definition: 'that a

function only has one algebraic expression'. This conflict will only be uncovered if the student is asked to write the function's algebraic expression. The student's conflict will be uncovered because the student will evoke both concept images at the same time (Tall & Vinner, 1981, pp. 153).
For this potential situation Tall and Vinner (1981) use the term potential conflict factor.

"We shall call part of the concept image or concept definition which may conflict with another part of the concept image or concept definition a potential conflict factor." (Tall & Vinner, 1981, p. 153).

The potential conflict factor will create a cognitive conflict when both parts of the concept image are evoked simultaneously but this situation might not occur. If the parts of the concept image are evoked in such a way that a cognitive conflict occurs then Tall and Vinner (1981) denotes this a cognitive conflict factor. According to Tall and Vinner (1981) a more serious type of potential conflict factor is when a student's concept image is in conflict with the formal concept definition. This type of conflict factor can seriously impede the students leaning of formal theory. This type of conflict factor can only be uncovered if the student creates a new concept image from the formal definition, that then is needed to be evoked simultaneously with the existing concept image (Tall & Vinner, 1981, p. 154).

Students rely on their concept images. This means that if they are presented to a definition of a concept that they have already constructed a concept image for, then one of the three following situations can happen.

1. The student will alter its concept image, such that the concept image will contain both concepts. The old and the new.
2. The student will preserve the existing concept image and adopt the new concept image, but the student will also forget the new concept image again.
3. Both concept images will be preserved. If the student is asked to define the concept then the student will use the formal concept definition but in all other situations the student will use its original concept image (Vinner, 1983, p. 294).

The students rely so much on their concept image that they only in one out of three situations will alter their concept image.

Diagnostic tests and teaching

The idea about diagnostic tests and diagnostic teaching is founded on theory about constructivism. A constructivist view on teaching is that a student's learning is based on the student's actions and experience. These two components are the base for learning. However, the reflections that a student does, based on the experience, is a crucial factor for developing the target knowledge (Brekke, 2002, p. 3).

To be able to develop good diagnostic items one needs to have an overview of the misconceptions that are linked to the different mathematical concepts (Brekke, 2000, p. 6). For Brekke (2002) incomplete thoughts connected to a concept is called misconceptions. Leinhardt et al. (1990) define a misconception as incorrect features of a student's knowledge that are repeatable and explicit. Brekke (2002) points out that it is important to understand the difference between a student's mistake and the misconceptions one has. A mistake can be random because of the lack of attention, concentration or that they did not read the task thorough. The misconceptions are not random. Behind the misconceptions the students have a specific way of thinking that they use consistently (Brekke, 2002, pp. 10). Leinhardt et al. (1990) and Brekke's (2002) way of defining misconceptions are different but when Brekke points out the differences between a mistake and a misconception it becomes clear that the basic idea of misconceptions is the same. It is this definition of misconception that I use in my thesis.

Diagnostic teaching can according to Brekke (2002) be divided in four phases:

1. Identify misconceptions and partly developed concepts that the students have.
2. Plan the teaching such that possible misconceptions and partly developed concepts can be enhanced. One can call this a cognitive conflict.
3. Solve the cognitive conflict through discussions and reflections.
4. Use the expanded (or new) concept in other contexts. (Brekke, 2002, p. 19)

To identify the misconceptions that the students have, diagnostic teaching uses a diagnostic test that contains diagnostic tasks. A diagnostic test will preferable contain types of tasks that the students have not worked with a lot. In some of the tasks the students would be encouraged to show how they came to their answer (Brekke, 2002, p. 16). In a diagnostic test, one would try not to ask questions where the student could answer correct even though the student has misconceptions. An example of a task that will not give information is a task such as: *'Circle which number is larges of 0,23, 0,62 and 0,42'*.

This would not be a good diagnostic task since, if the students have a misconception that 'decimals is a pair of integers' then the students would circle '0,62' and be correct. The same type of task but with the numbers '0,62, 0,4 and 0,236' would be a good diagnostic task (Brekke, 2002, pp. 16), since it would reveal the misconception 'that decimal numbers are a pair of integers' if the student answers '0,236'. A good diagnostic task when working with the concept of functions could be a task as the one seen in figure 1.

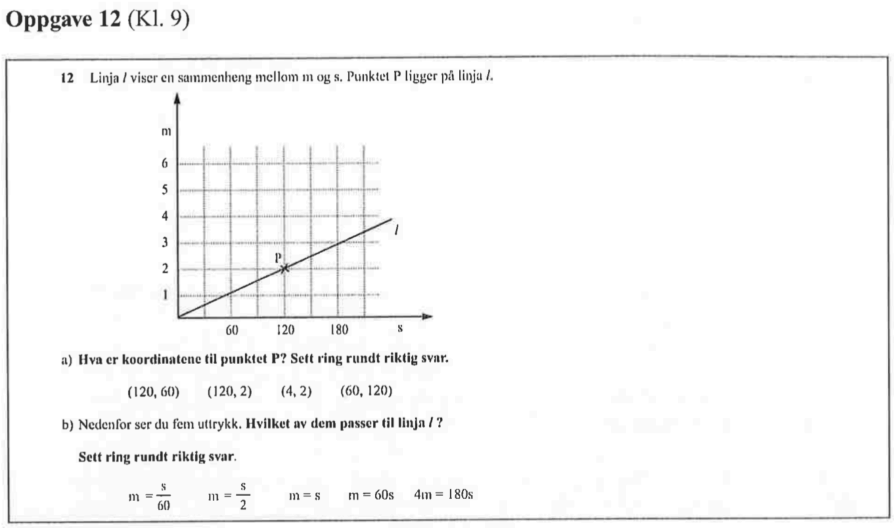


Figure 1: Task 12 from Gjone (1997, p. 19)

The task in figure 1 is a task made for ninth grade students. The students are asked to circle the right answer to what coordinates the point *P* has. The different answer possibilities will all expose different misconceptions. The answer '(4,2)' will for example reveal that the student just counted the number of squares and does not understand the labelling on the axis.

Cognitive semantics

Metaphors are a part of our everyday language we operate with them without being aware of it. 'She didn't defend her opinion' and 'He shot down all my arguments' these are both expressions that use metaphors. 'Defend' and 'shot' are both words used in a context with war. Thus an argument is conceptualized as 'Argument is war' (Lakoff & Johnson, 1980, chap. 1). These are expressions that we use in our everyday language, but few people think about how this

mechanism works and why we use metaphors. Cognitive linguistic and philosopher George Lakoff and cognitive scientist Rafael E. Núñez have studied the role of metaphors in mathematics. They do not believe that we only use metaphors to enrichen our language. There are a lot of examples of expression such as the ones given above. The expressions do not make sense if one thinks about it; arguments cannot be shot because it is not an object, but we understand the expression metaphorically.
There has been a study done by Thibodeau and Boroditsky (2011) on how metaphors affect the way that we reason about complex issues. The study consisted of two groups where both groups was handed a text about crime in the fictional city Addison. The difference in the two texts that the two groups were handed was; in one of them there was an embedded metaphor that 'crime is a beast' and in the other there was an imbedded metaphor that 'crime is a virus'. This was the only difference between the two texts. After reading the text, the groups were asked to answer two questions:

"1) In your opinion what does Addison need to do to reduce crime? 2) Please underline the part of the report that was most influential in your decision." (Thibodeau & Boroditsky, 2011, p. 3)

The outcome of this study was that; the group that read the text where crime was described like a monster proposed that Addison should fight crime by hiring more police, build jails, law enforcement and capturing and locking up the criminals. The other group that had read the report where crime was described like a virus was more likely to propose that Addison should investigate the root causes of the issue and instituting social reforms (Thibodeau & Boroditsky, 2011, p. 5). This gives us an idea of what great effect metaphors can have on our understanding of concepts and thereby how important they are in relation to understanding concepts and solving issues.

The most starling result in cognitive science is that most of the thoughts that we have are unconscious, which makes them inaccessible to our conscious. Most of our everyday thoughts happen so fast and on such a low level that it is not accessible to us. This is also the case with mathematics (Lakoff & Núñez, 2000, chap. 2).

A conceptual metaphor is inference-preserving. Núñez (2009) explains it in this way:

"These conceptual metaphors, which are inference-preserving cross-domain mappings, are cognitive mechanisms that allow us to project the inferential structure from a source domain which usually is grounded in some form of basic bodily experience, into another one, target domain, usually more abstract" (Núñez, 2009, p. 73).

Conceptual metaphors are mappings between two domains: a source domain and a target domain. According to Lakoff and Núñez (2000) one needs to create a mapping from a source domain, which is grounded in everyday experience, to a target domain, which is abstract, to be able to understand something abstract. It is this mapping that makes it possible for us to understand abstract concepts such as affection or time (Núñez, 2009, pp. 73). Lakoff and Núñez (2000) believe that all mathematics is based on this mapping mechanism.

We can understand arithmetic from everyday experiences. The experiences we humans have are the base of the source domain. The experiences that we do, could be by operating with collections of objects and the target domain is arithmetic, this mapping would be inference-preserving. The mapping can be expressed in this way:

Arithmetic is object collection

Source domain Object collations		*Target Domain* Arithmetic
Collections of objects of the same size	→	Numbers
The size of the collection	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
The smallest collection	→	The unit (one)
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtracting

(Lakoff & Núñez, 2000, p. 55).

To be able to go beyond counting we need the capacity to be able to form correspondences across conceptual domains and this sort of mapping is called a conceptual blend (Núñez, 2009, pp. 77).

"*Conceptual metaphor and conceptual blending are among the most basic everyday cognitive mechanisms that take us beyond minimal early abilities and simple counting to the primary arithmetic of natural numbers*."(Núñez, 2009, p. 78).

An example of a conceptual blend that most are familiar with could be combining numbers with lines to make 'numbers lines' (Núñez, 2009, pp. 77). The union of geometry and algebra is call analytic geometry. This is a field that much of mathematic depend on. Analytic geometry rests on the concept of the Cartesian plane. The Cartesian plane is a conceptual blend of two number lines and the Euclidean plane, with two lines perpendicular to each other (Lakoff & Núñez, 2000, pp. 284). The conceptual blend is mapped in this way:

The Cartesian plane blend

Conceptual domain 1 Number lines		*Conceptual domain 2* The Euclidean plane with line X perpendicular to line Y
Number line x	↔	Line X
Number line y	↔	Line Y
Number m on number line x	↔	Line M parallel to line Y
Number n on number line y	↔	Line N parallel to line X
The ordered pair of numbers (m, n)	↔	The point where M intersects N
The ordered pair of numbers (0,0)	↔	The point where X intersects Y
A function $y = f(x)$; that is, a set of ordered pairs (x, y)	↔	A curve with each point being the intersection of two lines, one parallel to X and one parallel to Y
An equation linking x and y; that is, a set of ordered pairs (x, y)	↔	A figure with each point being the intersection of two lines, one parallel to X and one parallel to Y
The solution to two simultaneous equations is variables x and y	↔	The intersection points of two figures in the plane

(Lakoff & Núñez, 2000, p. 285)

As mentioned analytic geometry rests on the Cartesian plane and functions is a part of analytic geometry. Thus understanding the Cartesian plane is an important part of understanding functions.

Lakoff and Núñez (2000, pp. 74) use the term conceptual metonymy. If we consider the sentence “When the pizza boy comes, give him a good tip.” The conceptual frame of the sentence is ‘Ordering pizza for delivery’. In this frame the pizza delivery boy has a role, namely delivering pizza for the costumer. The costumer that says: “When the pizza boy comes, give him a good tip.” does not know which individual will be delivering the pizza. But we need to conceptualize and talk about the individual that is bringing the pizza. The ‘pizza delivery boy’ comes to stand metonymically for the individual that delivers the pizza that particular day. Conceptual metonymy is also a part of mathematics it allows us to go from concrete arithmetic to general algebraic thinking. When writing $x + 3 = 8$, x is our notation for a ‘role’, just like the pizza delivery boy. x is a number and $x + 3 = 8$ says that whatever number x happens to be, adding three to it will yield eight. It is this mechanism that makes the discipline of algebra possible (Lakoff & Núñez, 2000, pp. 74).

Student problems in their work with functions

The concept of function is an important part of high school mathematics since it is a central part of modern mathematics. The concept of function is a key concept when working together with other disciplines (da: fag), since functions are often used when describing and working with phenomena in the world. A strong understanding of the concept of function is crucial for further education where mathematics is a big part of the education (Danish Ministry of Education, 2019, pp. 15).

“*The concept of function is* [….] *hard to learn, and students often just think about functions as symbol manipulations or procedures…*” (Danish Ministry of Education, 2019, p. 15, my translation)

It is well known that functions and the concept of function are difficult for students. As preparation for the investigation of students’ problems when working with functions, I spoke with six experienced high school teachers. Read more about this in the method section below. The experienced high school teachers I spoke with experienced, that students have problems when working with functions. Two mentioned that the students have trouble using the $f(x)$-notation correct. Four of the teachers mentioned that the students have a narrow understanding of the concept of function. They experienced that many students understand a function as its algebraic expression and that they do not see the domain as a part of the defining characteristics of a function. The students perceive functions as machines and believe that only continuous functions are functions, or they believe that if a function is discontinuous then it is more than one function. Further, two of the experienced high school teachers mentioned that the students have a hard time connecting a function's algebraic expression with the graphic representation (Personal correspondences). These are just some of the examples that the high school teachers gave me, but it showed that the teachers experienced that high school students have problems when working with functions.

From my literature review I found that there are multiple aspects of functions that the students find hard to grasp. I found that the problems that students have when working with functions could be divided into two themes:

Translating between different representations of functions.

Understanding what a function is.

When translating between different representations of functions the students have the most problems when translating from one representation to the algebraic expression of a function (Rønningstad, 2009; Gjone, 1997; Leinhardt et al., 1990; Niss & Jankist, 2016, 2017). When logically analysing tasks where students have to translate between graphs and algebraic expression, it would suggest that translating from graphs to their algebraic expression would be harder since it requires pattern recognition. Translating the other way around requires a series of steps; generating ordered pairs, placing them in the Cartesian plane and connecting the dots (Leinhardt et al., 1990, p. 35). There is also empirical work that supports the assertion that it is more difficult to translate from graphical representation to an algebraic expression. The results of a National Assessment of Educational Progress showed that only 5% of 17-year-olds high-school students could generate an equation when given a graph of a straight line with indicated intercepts $(-3,0)$ and $(0,5)$. The assessment also showed that only 18% of the 17-year-old high school students could generate the correct graph when given a linear equation (Leinhardt et al., 1990, p. 35). In a Norwegian assessment where 471 ninth-grade students participated, it was found that when given a graph of a linear function, only 19,7% could connect the right equation to the graph (Gjone, 1997, p. 60). The task in the Norwegian assessment was in a multiple-choice form, where the students had five possible answers where only one is correct. Hence they didn't have to deduce the equation themselves. As part of her a master's thesis, Rønningstad (2009) found that 49,3% of the participating students, which were students from first and second year of high school, was able to find which of the given four possible algebraic expressions that fitted the graph. She also found that a large part of the students that chose a wrong answer, chose one that indicated that they believed that the constant b in $f(x) = ax + b$ is where the graph intersects the x-axis. Further Rønningstad (2009) found that 32,2% could draw the linear function $y = x$ and 36,0% could draw the function $y = 5$.

Students' definition of functions is something that has been researched substantially. Vinner(1983), Vinner and Dreyfus (1989), Leinhardt et al. (1990) and Niss and Jankivst (2016) have researched students' perception of functions. There a multiple examples on students' perception of functions and most of the research shows that a lot of students have a narrow or an incorrect perception of what a function is (Vinner, 1983, pp. 299; Vinner & Dreyfus, 1989, pp. 359; Leinhardt et al., 1990, pp. 30; Niss & Jankivst, 2016, chap. 3).
As an example Vinner and Dreyfus (1989) found that the students' view of functions could be categorised in the following six categories:

1. Correspondence: A function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set.
2. Dependence relation: A function is a dependence relation between two variables.
3. Rule: A function is a rule.
4. Operation: A function is an operation or manipulation.
5. Formula: A function is a formula, an algebraic expression, or an equation.
6. Representation: The function is identified, in a possibly meaningless way, with one of its graphical or symbolic representations. (Vinner & Dreyfus, 1989, p. 359-360)

Take the perception 'a function is a formula'. This is a very narrow perception of the concept of function. This definition excludes a group of functions that does not have an algebraic expression as a representation. Also the perception categorised as 'representation' is also a narrow perception. This perception could perhaps leave no room for functions that does not have a graphical representation.

Method

In this section I will present the choices I have made and the methods I have used, to be able to examine the problem statement and the four hypotheses. This section is important to be able to guarantee the reliability and validity of this thesis. I will present my method for researching what high school teachers experience students having problems with, when working with functions. I will present how the questionnaire for the pilot test was developed and implemented. Further, I will describe the final questionnaire, how it was implemented and what purposes the different tasks have. Lastly, I will present my coding of the empirical data.

Preparatory investigation

In the brainstorming phase of the project I wrote emails to six experienced high school teachers. I asked the teachers if they had experienced some general problems that high school students have when working with functions. This was to get a feeling of what problems Danish high school students have when working with functions. The high school teachers I contacted are teachers from different high schools. Both current and former high school teachers were contacted and one of them was a former consultant in the field of high school mathematics (da: fagkonsulent). I talked to those particular high school teachers since they were persons that were either familiar to one of my supervisors or I.

Pilot test

I developed a questionnaire, which worked as a pilot-testing instrument. The tasks in the questionnaire for the pilot test were inspired by: Gjone (1997), Rønningstad (2009), Vinner (1983) as well as Detektionstest one and three (appendix A) and the answers that the experienced high school teachers had given in the preliminary investigation. The questionnaire was implemented in a first year high school class at a high school (STX[1]) in the vicinity of Copenhagen. The class had mathematics at level B. I came in contact with the high school since I knew a person working at the high school. 21 students participated in the questionnaire. The results of the pilot test made me change the focus of the questionnaire away from only testing

[1] STX is a type of high school that does not specialise in a particular field, but aim to prepare students for a wide range of further education.

if the students could translate between the different representations that functions have. The results of the pilot test showed that many of the students had a hard time drawing graphs of functions when given the algebraic expression. I decided in the new questionnaire to focus more on why the students had trouble with translating from an algebraic expression of a function to the graphical representation. Thus, by combining the results of the pilot test with the theory presented in the theory section it lead me to form hypotheses that would be interesting to explore in the new and final questionnaire. The questionnaire for the pilot test can be seen in the appendix B.

The questionnaire

To be able to explore the students' understanding of the concept of function and to be able to test my hypothesis I developed a questionnaire. Based on the literature, the preparatory investigation and the pilot test, I felt confident that I understood the problem area sufficiently well to proceed with a quantitative investigation in the form of a questionnaire, rather than performing further qualitative instigations. Hence I chose a questionnaire since I found it the best way to explore and test my hypotheses and problem statement. Further the hypotheses are hypotheses that have not been explored greatly. Therefore I wanted to be able to test the hypotheses for a larger group of students. Hence testing a larger group of students would make the results of testing the hypotheses more reliable. The questionnaire consisted of two parts. The questionnaire can be seen in appendix C. The questionnaire was developed on the grounds of the theories presented in the theory section and some aspects of the questionnaire were inspired by Gjone (1997), Rønningstad (2009), Vinner (1983), Detektionstest one and three (Appendix A) and the results of the pilot test.

Presentation of the tasks

This section will contain a presentation of some of the tasks from the questionnaire. In table 1 the reader can get an overview of the different tasks and their purpose.

Task	Purpose
1.1	Examine the students understanding of the concept of function
1.2 b), 1.2 c), 1.2 d), 1.2 e)	Examine if the students can draw linear functions
1.3	Examine what the students use to draw graphs
2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.8, 2.9	Examine if the students understand the conceptual blend of Cartesian plane
2.4, 2.5 a), 2.7, 2.10	Examine if the students understand the $f(x)$-notation
1.2 b), 1.2 c), 1.2 d), 1.2 e), 2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.8, 2.9	Testing hypothesis H1
1.2 b), 1.2 c), 1.2 d), 1.2 e), 2.4, 2.5 a), 2.7, 2.10	Testing hypothesis H2
1.2 b), 1.2 c), 1.2 d), 1.2 e), 1.3	Testing hypothesis H3
1.2 b), 1.2 c), 1.2 d), 1.2 e), 2.11	Testing hypothesis H4

Table 1: An overview of the tasks' purpose

In the following section I will explain and elaborate the purposes of a selection of tasks from the questionnaire.

Task 1.1 consisted of two tasks. Task 1.1 is shown in figure 2.

Opgave 1.1

Giv et eller flere eksempler på funktioner:

Beskriv med dine egne ord hvad en funktion er:

Figure 2: Task 1.1 of the final questionnaire

The students were asked to give examples of functions. This task was intended to 'warm up' the students. It might have been a too abstract question to begin with, if the students were asked to describe what a function is.

In task 1.2 a) the students were asked to 'Explain in your own words why the point (3,1) is on the graph for $f(x) = 2x - 5$ and why the point (7,4) is not on the graph for the function $f(x) = 2x - 5$'. Task 1.2 a) can be seen in figure 3.

Opgave 1.2
a) Forklar med dine egne ord hvorfor punktet (3,1) ligger på grafen for $f(x) = 2x - 5$ og hvorfor punktet (7,4) ikke ligger på grafen for $f(x) = 2x - 5$.

____

____

____

Figure 3: Task 1.2 a) of the final questionnaire

Task 1.2 a) was intended to gain insight so as to the students' preferred way of giving an argument. I expected that some of the students would give a graphic argument and other would give an algebraic argument.

The tasks: 2.2, 2.3 and 2.4 are of the same type. The students were given a quadratic coordinate system with axis running from -10 to 10 vertically and horizontally and then they were asked to plot the given points in the coordinate system. The representations of the points are different in the three tasks. As seen in figure 4 the points in task 2.2 were presented using coordinate notation. In figure 5 it is showed that the points in task 2.3 were presented using a table and in figure 6 it can be seen that the points in task 2.4 was presented using $f(x)$-notation.

Opgave 2.2
Afsæt følgende i koordinatsystemet:
$A(4,1)$, $B(-3,-8)$ og $C(-6,7)$

Figure 4: Task 2.2 of the final questionnaire

Opgave 2.3
Afsæt følgende i koordinatsystemet:

x	-6	-4	1	6
$f(x)$	-5	2	4	0

Figure 5: Task 2.3 of the final questionnaire

Opgave 2.4
Afsæt følgende i koordinatsystemet:
$f(-6) = -4,\ f(-2) = 2,\ f(0) = 5$ og
$f(2) = 8$

Figure 6: Task 2.4 of the final questionnaire

I chose three different notation forms to make sure that I tested the students' ability to plot points in the Cartesian plane and not their ability to understand notation.

Task 2.6, 2.7 and 2.8 are of the same type. The students are asked to find the algebraic expression that fits the information they are given. The representation of the given information is different in the three tasks. In task 2.6, figure 7, the students are given a table that shows the relation between x and $f(x)$. In task 2.7, figure 8 the students are given information using $f(x_1) = y_1$ notation. In task 2.8 the students are presented with coordinate pairs that show the relation between x and $f(x)$. Task 2.8 can be seen in figure 9.

Opgave 2.6
Følgende tabel viser sammenhængen mellem x og $f(x)$

x	1	3	4	7	9
$f(x)$	5	11	14	23	29

Hvilke regneforskrifter passer til tabellen? (sæt kryds)

☐ $f(x) = x + 4$
☐ $f(x) - 3x - 2 = 0$
☐ $f(x) = 5x$
☐ $f(x) = 3x + 2$

Figure 7: Task 2.6 of the final questionnaire

Opgave 2.7
Følgende x-værdier og tilhørende funktionsværdier viser sammenhængen mellem x og $f(x)$
$f(1) = 1,\quad f(3) = 9,\quad f(4) = 13,\quad f(7) = 25,\quad f(9) = 33$
Hvilke regneforskrifter passer til x-værdierne og de tilhørende funktionsværdier? (sæt kryds)

☐ $f(x) - 4x + 3 = 0$
☐ $f(x) = x$
☐ $f(x) = 4x - 3$
☐ $f(x) = x + 6$

Figure 8: Task 2.7 of the final questionnaire

Opgave 2.8
Følgende koordinatpar viser sammenhængen mellem x og $f(x)$:
(1,3), (3,7), (4,9), (7,15), (9,19)
Hvilke regneforskrifter passer til koordinatparrene? (sæt kryds)

☐ $f(x) = 2x + 1$
☐ $f(x) = x + 2$
☐ $f(x) = 3x$
☐ $f(x) - 2x - 1 = 0$

Figure 9: Task 2.8 of the final questionnaire

I chose three different notation forms to make sure, that I tested the students' ability to connect a series of points with an algebraic expression and not their ability to understand notation.

In task 2.9 the students are presented with a graph of a function. The students are asked to use the graph to fill out the table that they are given. The table is only partly filled out in advance. Task 2.9 can be seen in figure 10.

Opgave 2.9
Til højre er givet en graf for en funktion.
Brug grafen til at udfylde hele tabellen.

x		-2	2	6	
$f(x)$	-3				4

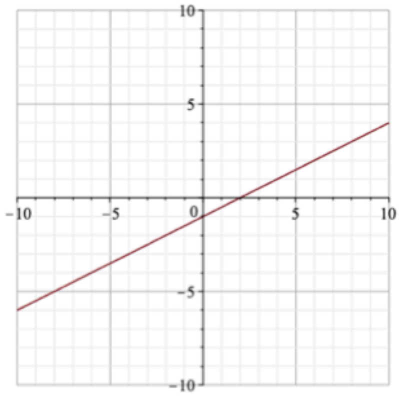


Figure 10: Task 2.9 of the questionnaire

I chose to use table notations since I believe that the students are most familiar with this notation. The choice to use table notations was discussed with my two supervisors.

Task 2.11 serves two purposes. Firstly, its purpose was to test hypothesis H4. Secondly its purpose was to explore in cooperation with task 2.12 if the students can just remember the definition, for what role a and b has for a linear function, without understanding the definition.

Implementation of the questionnaire

The questionnaire was implemented in two first year high school classes at a high school in the vicinity of Copenhagen. This was the same high school as where the pilot test took place. The high school is to my knowledge an average high school, which is important as it allows the results of the questionnaire to be generalised. I approached a mathematics teacher at the high school through an acquaintance who worked at the high school. The teacher agreed to let me borrow two of his classes. Hence the two classes had the same teacher in mathematics.

One of the classes has mathematics at level B and the other has mathematics at level A. The questionnaire were carried out in the two classes back-to-back meaning that they would not have the time to affect the answers.
When I implemented the questionnaire in the classes I made a short introduction to the students. The introduction can be seen in appendix D. In the introduction I asked the students to be open-minded and answer the questionnaire as best as they could. I also assured them that their answers for the questionnaire would not influence their grades. The students were instructed to first do part one of the questionnaire and they were instructed to do it chronologically and not go back and edit their answers. They were also instructed to leave part one of the questionnaire on the front of their tables when they were done. I then continuously collected part one of the questionnaire. I collected part one of the questionnaire so that the students would not be biased by what they had to do in the second part of the questionnaire. The students were also told not to use CAS-tools or other aids. They were also instructed to use the margin of the questionnaire if they needed to do intermediate calculations. The students were also informed that there were tasks on both sides of the paper. The students had approximately 45 minutes to finish, and many of the students were done in less than 45 minutes.

Coding

The students that wanted to participate were asked to fill out and sign a deceleration of informed consent and since some of the students were underage, their parents were asked to fill out and sign a deceleration on their behalf. The deceleration templates can be seen in appendix E. The declaration of informed consent and the students' answers of the questionnaire have been stored on a safe and logged server provided by the university of Copenhagen, fulfilling the requirements of the GDPR. Further the University of Copenhagen have approved a data processing agreement allowing me to work with the data.

There were 58 students participating. 28 of the students were from the first year class that have mathematics at level B. The remaining 30 of the students were also at their first year and they have mathematics at level A. 53 of the students gave their consent, meaning I could use their results for my analysis. The questionnaires of the remaining five students were destroyed.

One student's missing answers gave me a strong feeling that he/she was not aware that there was a page two in the first part of the questionnaire. Thus that student's answers are contaminated and not representative. For that reason that student's answers were left out of the results and analysis. Consequently, 52 students' answers of the questionnaire constitute the results and the analysis.
When evaluating the answers of the questionnaires, I discussed answers I was unsure how to approach with a fellow thesis student who is also an experienced high school teacher.
When executing the statistical tests and doing the qualitative and quantitative analysis I have looked at all of the students as one sample. However I have a variable that register which class the participants are a part of.

Quantitative coding and testing

I have used the statistical program SPSS to: code my results, do the statistical tests and create tables of the results. I got guidance form Data Science Laboratory of Copenhagen University to which statistical tests I could use to test my hypotheses and what method I should use to explore other possible significant connections.
The quantitative method is based on numerical data (Hansen & Andersen, 2009, chap. 7) therefore I converted the results to numbers. For some of the tasks I did precoding (da: prækodning) for other tasks where I didn't have sufficient sense of how the answers would spread out I used postcoding (da: postkodning) (Hansen & Andersen, 2009, chap. 7).

Due to the extensive amount of variables and their coding, thorough explanations of this would be tiresome and unnecessary for the reader. Thus, for deeper insights into my coding see appendix F. Here the reader can find answers to questions regarding how I coded my variables and the frequency of each code within each variable.

I created and coded new variables. I created a variable called 'Drawn functions'. For this variable I coded 'Have drawn all functions correct', given value 1, if the students had drawn task 1.2 b) to 1.2 e) correct. I coded 'Have drawn one or more functions wrong', given the value 0, if one or more of task 1.2 b) to 1.2 e) was not correct or if one or more of the tasks had missing answers. I chose that a missing answer could be evaluated as 'not drawn correct' when coding

this new variable. I found this reasonable since with a missing answer, the student did not try to draw the function. Hence the student probably did not know how to draw it.
Another new variable I created was 'Understanding the conceptual blend of the Cartesian plane'. This variable was coded as 'Understand', given the value 1, if the students had done the following:

Plotted both points correct in task 2.1

Plotted all point correct in either task 2.2, 2.3 or 2.4

Choose at least one correct answer in task 2.6, 2.7 or 2.8

Filled the table correct in task 2.9

The variable was coded as 'Do not understand', given the value 0, if the student did not fulfil the above demands.
I created a new variable called 'Understanding the $f(x)$-notation'. This variable was coded as 'understand', given the value 1, if the student had done the following:

Plotted all points correct in task 2.4

Calculated correct and used the notation correct in task 2.5 a)

Choose at least one correct answer in task 2.7

Chose the correct answer in task 2.10

If the students did not fulfil the above demands then the variable was coded 'Do not understand', given the value 0.

To explore if there were significant connections that I did not foresee I made a correlation matrix. Finding the correlations was done in an explorative way. I am aware that these correlations only suggest relationship between variables and it cannot uncover if one variable is the cause of the other variable (Hansen & Andersen, 2009, chap. 7). Further I am aware that since this was done in an explorative way with 19 different variables my chance for getting at least one false positive is considerable (Johansen, 2019, pp. 9).
When examining correlations it is important to both look at the correlation coefficient and the significance level, since one could have a weak correlation but the p-value could be 0,000 (Bryman, 2012, pp. 349). Thus, in my results both the correlation coefficient and the p-value are listed. For my correlations I had the significance level at 95%.

Qualitative coding

For analysing and coding the qualitative data I have used the program NVivo 12. I have directly translated the Questionnaires' answers, where I have only corrected the students' misspelling.

For task 1.1, where the students describe in their own words what a function is, I used Vinner's (1983) and Vinner and Dreyfus' (1989) categories of students' concept images as an inspiration to my categorisation. The students' answers have been categorised in five categories. Some of the students might have used words that could categorise them in two of the categories. If this was the case I have evaluated which category the answer would fit best. This evaluation was based on what the student had first written, since that would be the students' first intuition, or if the student used words like 'but the most important thing is' or such expressions. The categories can be seen in the analysis section.

Results and Analysis

This section will present and analyse the quantitative data. The results are based on a quantitative data processing and statistical analysis. Firstly I will introduce the data. Thereafter I will present the statistical testing of my hypothesis and analysis about these. Further I will present some of the other significant connections that I did not foresee. Lastly I will present and carry out an analysis of the qualitative data that were extracted from the questionnaires.

Quantitative results and analysis

Introduction to the data

In this section tables of frequencies of some of the tasks will be presented. Frequency tables of all the tasks and variables can bee seen in the appendix G. The valid percentage is sometimes different from the percentage; this is the case if there is some missing cases i.e. missing answers. In this case the valid percentage is the percentage of the percentage that has answered the task. I will in this section regard a student's missing answer as a wrong answer.

In the questionnaire the students were asked to draw four different functions. The results of my questionnaire showed that 57,7% of the students were not able to draw all functions correctly. Meaning only 42,2% students was able to draw all four linear functions correct. The frequencies can be seen in table 2.

Drawn functions

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have drawn one or more functions wrong	30	57,7	57,7	57,7
	Have drawn all functions correct	22	42,3	42,3	100,0
	Total	52	100,0	100,0	

Table 2

In table 3 to 6 it can be seen that out of the four functions, function $f(x) = 2x - 5$ was the function that the largest part of the students could draw. 78,8% of the students succeeded in drawing that function correct. The students did not succeed in drawing the other thee functions correct almost equally unsatisfactory. 36,5% was not able to draw $f(x) = 3$ correct, 34,6% was not able to draw $f(x) = 4x$ correct and 36,5% was not able to draw $f(x) = -x + 6$ correct. There were 11,5% of the students that did not draw or tried to draw the function $f(x) = 3$, this could indicate that the students found this function the most complex.

Task 1.2 b) Draw the function f(x)=2x-5

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	10	19,2	19,6	19,6
	Drawn correct	41	78,8	80,4	100,0
	Total	51	98,1	100,0	
Missing	- 1	1	1,9		
Total		52	100,0		

Table 3

Task 1.2 c) Draw the function f(x)=-x+6

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	15	28,8	31,3	31,3
	Drawn correct	33	63,5	68,8	100,0
	Total	48	92,3	100,0	
Missing	- 1	4	7,7		
Total		52	100,0		

Table 4

Task 1.2 d) Draw the function f(x)=4x

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	14	26,9	29,2	29,2
	Drawn correct	34	65,4	70,8	100,0
	Total	48	92,3	100,0	
Missing	- 1	4	7,7		
Total		52	100,0		

Table 5

Task 1.2 e) Draw the function f(x)=3

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	13	25,0	28,3	28,3
	Drawn correct	33	63,5	71,7	100,0
	Total	46	88,5	100,0	
Missing	- 1	6	11,5		
Total		52	100,0		

Table 6

After drawing the functions the students got a question about what they normally use when drawing graphs. In table 7 it is shown what the different answers frequencies are.

Task 1.3 What do you normally use when drawing graphs?

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Draws with paper and pencil	14	26,9	26,9	26,9
	Gets a CAS-tool to draw it	26	50,0	50,0	76,9
	Draw with paper and pencil and gets a CAS-tool to draw it	12	23,1	23,1	100,0
	Total	52	100,0	100,0	

Table 7

My results showed that 50,0% of the students normally use CAS-tools when drawing graphs. Also my results showed that 23,1% both use CAS-tool and paper and pencil. It is only 26,9% of the students that normally use paper and pencil to draw graphs. It needs to be noted that many of the students wrote as a comment for this task, that the teacher often decide what they should use to draw graphs. The students also commented that their choice of tool depended on the difficulty of the function.

In the questionnaire, the students were asked to place numbers on a number line. As can be seen in table 8 the majority of the students can place the numbers on the number line. 90,4% of the students succeeded in placing the numbers.

Task 2.1 Place the numbers on the number line

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Only placed -2 correct	5	9,6	9,6	9,6
	Placed both correct	47	90,4	90,4	100,0
	Total	52	100,0	100,0	

Table 8

It is easy to see from table 9 to 11 that the students overall struggled the most with task 2.4. 38,5% of the students was not able to plot any of the points correct in task 2.4. Also 17,3% of the students did not write anything in task 2.4, which could indicate that these students do not feel familiar with this type of notation.
My results showed that 86,5% of the students succeed to plot all points correct in task 2.2. The results of task 2.2 can be seen in table 9. Task 2.2 is the one of task 2.2., 2.3, and 2.4 that

the largest part of the students succeeded with. This could indicate that the students are most familiar with plotting points given by coordinate notation. In task 2.3 it was only 61,5% of the students that succeeded to plot all the point correct, this can be seen in table 10. As mentioned the students seemed to find task 2.4 particularly hard since only 46,2% of the students succeeded in plotting all points correct.

Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Haven't plotted any points correct	2	3,8	3,8	3,8
	Have only plotted one point correct	1	1,9	1,9	5,8
	Have only plotted two points correct	4	7,7	7,7	13,5
	Have plotted all points correct	45	86,5	86,5	100,0
	Total	52	100,0	100,0	

Table 9

Task 2.3 Plot the points in the Cartesian plane (Table)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Haven't plotted any points correct	2	3,8	4,0	4,0
	Have only plotted two points correct	3	5,8	6,0	10,0
	Have only plotted three points correct	13	25,0	26,0	36,0
	Have plotted all points correct	32	61,5	64,0	100,0
	Total	50	96,2	100,0	
Missing	- 1	2	3,8		
Total		52	100,0		

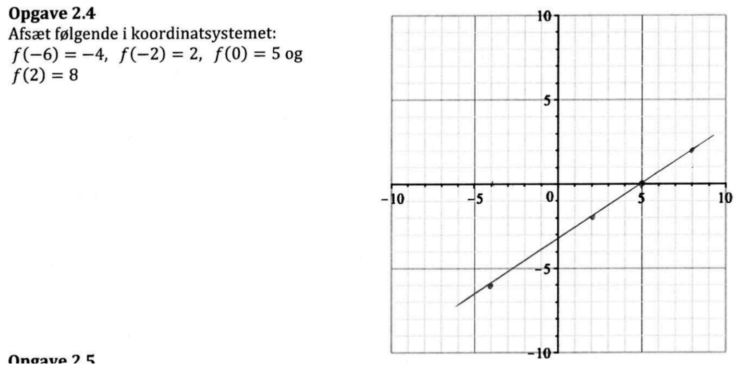
Table 10

Task 2.4 Plot the points in the Cartesian plane (f(x1)=y1 notation)

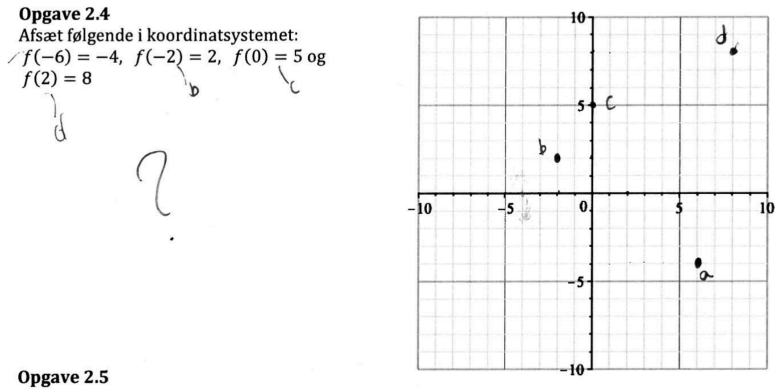
		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Haven't plotted any points correct	11	21,2	25,6	25,6
	Have only plotted one point correct	4	7,7	9,3	34,9
	Have only plotted two points correct	2	3,8	4,7	39,5
	Have only plotted three points correct	2	3,8	4,7	44,2
	Have plotted all points correct	24	46,2	55,8	100,0
	Total	43	82,7	100,0	
Missing	- 1	9	17,3		
Total		52	100,0		

Table 11

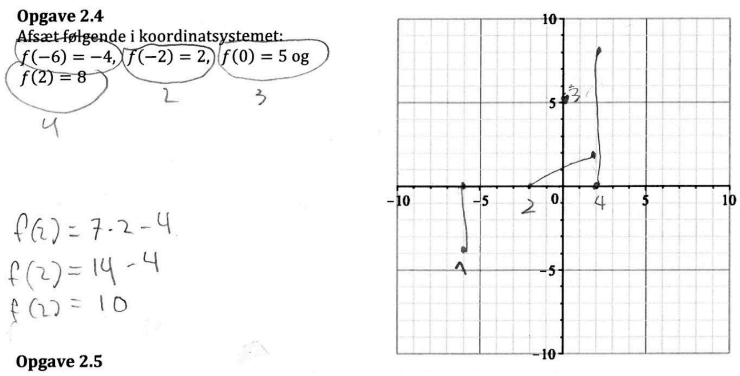
My results showed that some of the students, that could not plot the points correct in task 2.4, swapped the x-values with the functional values completely. An example of this can be seen in example 1. Some other students did not have a pattern in their way of plotting the points in task 2.4. An example of this is example 2 and 3.



Example 1: Questionnaire 231's answer to task 2.4



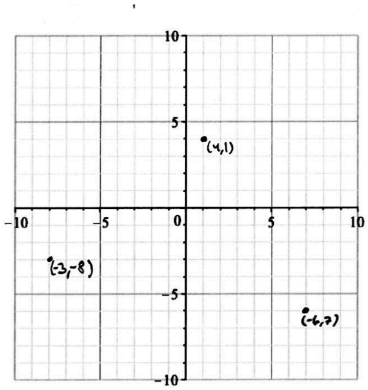
Example 2: Questionnaire 252's answer to task 2.4



Example 3: Questionnaire 150's answer to task 2.4

Only 3,8% of the students could not plot any points correct in task 2.2. In task 2.3, 7,6% of the students were not able to plot any points correct. The students that have not plotted any points correct in task 2.2 have swapped the x-values and the y-values. An example of this can be seen in example 4. The students that have not plotted all the points or some of them correct in task 2.3 tend to swap the x-values and the functional values. There are also cases where the students are not consistent with their mistakes. An example of the swapping and not being consistent can be seen in example 5.

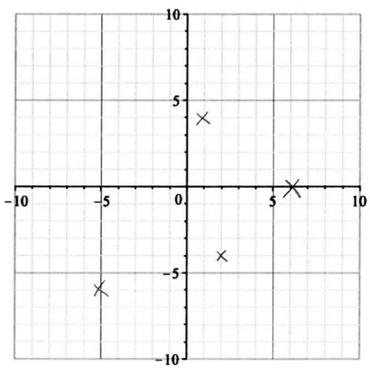
Opgave 2.2
Afsæt følgende i koordinatsystemet:
$A(4,1)$, $B(-3,-8)$ og $C(-6,7)$



Example 4: Questionnaire 90's answer to task 2.2

Opgave 2.3
Afsæt følgende i koordinatsystemet:

x	-6	-4	1	6
$f(x)$	-5	2	4	0



Example 5: Questionnaire 180's answer to task 2.3

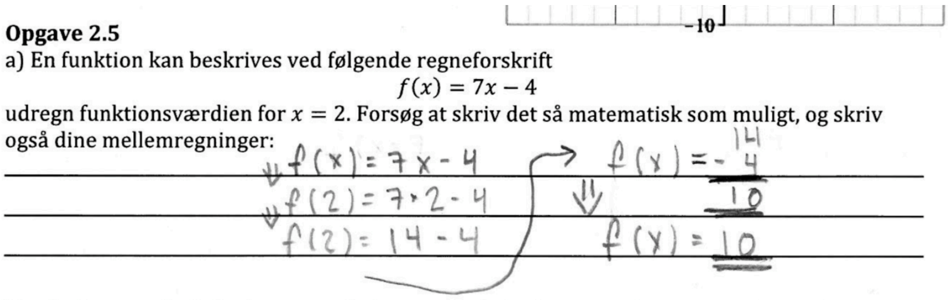
The subsequent task the students had to do were task 2.5 a). As seen in table 12 only 5,7% of the students gave an incorrect answer. My results showed that only 50,0% of the students were able to both give the correct answer and use the $f(x)$-notation correct. This result of the questionnaire indicates that a large group of the students do not understand the $f(x)$-notation fully. They might try to use the notation but they do not fully know the correct way to use it.

Task 2.5 a) Calculate the function value for x=2 for the function f(x) =7x-4

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Incorrect answer	1	1,9	2,0	2,0
	Correct answer but incorrect use of the notation	23	44,2	46,0	48,0
	Correct answer and correct use of the notation	26	50,0	52,0	100,0
	Total	50	96,2	100,0	
Missing	- 1	2	3,8		
Total		52	100,0		

Table 12

Some examples where it becomes clear that the student does not fully understand the $f(x)$-notation is example 6 and example 7. In example 6 it can be seen that the student tries to use the $f(x)$-notation when writing $f(2) = 7 \cdot 2 - 4$ and $f(2) = 14 - 4$. Then in the next step the student stops writing $f(2) = \cdots$ and thereby the students stops replacing 2 with x. Instead the student writes $f(x) = 10$. The student further concludes that $f(x) = 10$, which is incorrect since $f(x) = 7x - 4$. In my evaluations of task 2.5 a) I focused on notation and therefore denoted that this student did not use the notation correct.

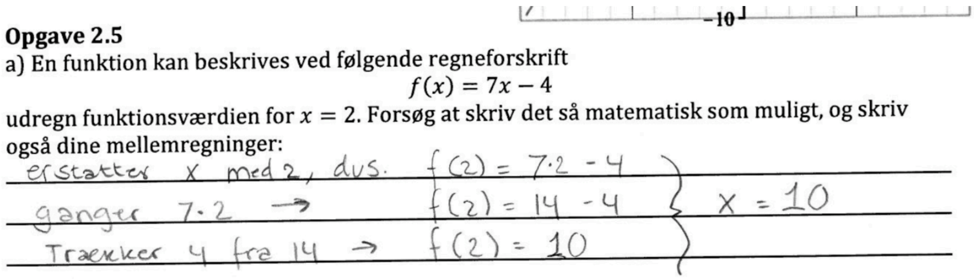

-10
Opgave 2.5
a) En funktion kan beskrives ved følgende regneforskrift
$$f(x) = 7x - 4$$
udregn funktionsværdien for $x = 2$. Forsøg at skriv det så matematisk som muligt, og skriv også dine mellemregninger:

$f(x) = 7x - 4$
$f(2) = 7 \cdot 2 - 4$
$f(2) = 14 - 4$

$f(x) = \overset{14}{-4}$
10
$f(x) = 10$

Example 6: Questionnaire 51's answer to task 2.5 a)

Another student ended up writing $x = 10$ as the answer to task 2.5 a), it can be seen in example 7. One can see that the student is using the right notation when writing $f(2) = 7 \cdot 2 - 4$, $f(2) = 14 - 4$ and $f(2) = 10$ and the result $f(2) = 10$ is correct. But the student concludes that the previous calculations leads to $x = 10$, which is not correct.



Opgave 2.5

a) En funktion kan beskrives ved følgende regneforskrift

$$f(x) = 7x - 4$$

udregn funktionsværdien for $x = 2$. Forsøg at skriv det så matematisk som muligt, og skriv også dine mellemregninger:

erstatter x med 2, dvs. $f(2) = 7 \cdot 2 - 4$

ganger $7 \cdot 2$ → $f(2) = 14 - 4$ $x = 10$

Traekker 4 fra 14 → $f(2) = 10$

Example 7: Questionnaire 21's answer to task 2.5 a)

These are some of the examples of students calculating correct but not using the notation correct.

In the questionnaire the students were asked to find which algebraic expression that fits the information they got. There were tree tasks of this form but the information the students got were represented in different ways. The results of the questionnaire can be seen in table 13 to 15.

Task 2.6 Which algebraic expression fits the table

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have answered one or more wrong possible answers	6	11,5	11,8	11,8
	Have answered one possible answer correct and one wrong	1	1,9	2,0	13,7
	Have answered one of the correct possible answers	36	69,2	70,6	84,3
	Have answered both of the correct possible answers	8	15,4	15,7	100,0
	Total	51	98,1	100,0	
Missing	-1,0	1	1,9		
Total		52	100,0		

Table 13

Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have answered one or more wrong possible answers	3	5,8	5,9	5,9
	Have answered one of the correct possible answers	42	80,8	82,4	88,2
	Have answered both of the correct possible answers	6	11,5	11,8	100,0
	Total	51	98,1	100,0	
Missing	-1,0	1	1,9		
Total		52	100,0		

Table 14

Task 2.8 Which algebraic expression fits the coordiate pairs

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have answered one or more wrong possible answers	5	9,6	10,0	10,0
	Have answered one possible answer correct and one wrong	1	1,9	2,0	12,0
	Have answered one of the correct possible answers	37	71,2	74,0	86,0
	Have answered both of the correct possible answers	7	13,5	14,0	100,0
	Total	50	96,2	100,0	
Missing	-1,0	2	3,8		
Total		52	100,0		

Table 15

In all three tasks the largest part of the students have only provided one correct answer. In task 2.6 it is 69,2% of the students, in task 2.7 it is 80,8% of the students and in task 2.8 it is 71,2% of the students. It seems from the results that the students find task 2.6 and task 2.8 equally hard since 13,4% failed to do both of them correct. Also, I will note that this task was of multiple-choice form with four answer possibilities, where two of the answers are correct. Thus if the students only guessed and chose to mark one answer this gives them a 50% chance of answering correct.

As shown in table 16, 5,8% of the students have not answered anything for task 2.9, and it is only 57,7% of the students that have completed the table correct. Also my results showed that the students struggle the most with finding the correct x-values from the graph. 23,1% of the students has written all the functional values correct but they have errors when finding and writing the x-values.

Task 2.9 Fill out the table from the graph

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Errors in both categories but some points are correct	3	5,8	6,1	6,1
	Have written all the f(x) values correct, but there are errors in the x-values	12	23,1	24,5	30,6
	Have written all the x-values correct but there are errors in the f(x) values	4	7,7	8,2	38,8
	Everything is correct	30	57,7	61,2	100,0
	Total	49	94,2	100,0	
Missing	- 1	3	5,8		
Total		52	100,0		

Table 16

In table 17 it can be seen that 23,1% answered task 2.10 incorrect. This indicates that these 23,1% have trouble with understanding the $f(x)$-notation.

Task 2.10 The point f(3)=7 is on the graph for the function. Which graph fulfil this demand?

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Answered incorrect	10	19,2	20,0	20,0
	Answered correct	40	76,9	80,0	100,0
	Total	50	96,2	100,0	
Missing	- 1	2	3,8		
Total		52	100,0		

Table 17

The results of my questionnaire shows that 23,1% of the student have a hard time translating from graphical representation to the algebraic expression, since they do not answer correct in task 2.11. The results of task 2.11 can be seen in table 18.

Task 2.11 Which algebraic expression fits the graph

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Answered incorrect	9	17,3	18,4	18,4
	Answered correct	40	76,9	81,6	100,0
	Total	49	94,2	100,0	
Missing	- 1	3	5,8		
Total		52	100,0		

Table 18

A great part of the tasks in the questionnaire are tasks where the students are given one representation of a function e.g. an algebraic expression, and they are asked to translate it into another representation of a function e.g. a graph. The above results also show that there is a part of the students that does not master translating between different representations of functions.

Testing the hypothesis

After having presented the main results of my empirical study I will in this section analyse whether my four initial hypotheses are falsified or corroborated by the empirical data. I will remind the reader that the four hypotheses that I am testing are:

> H1: There is a relation between understanding the conceptual blend of the Cartesian plane and being able to draw linear functions in the Cartesian plane by hand correct.
>
> H2: There is a relation between being able to understand and use the $f(x)$-notation correct and being able to draw linear functions in the Cartesian plane by hand correct.
>
> H3: If the students mostly use CAS-tools to draw functions then they are not able to draw linear functions by hand correct.
>
> H4: There is a relation between being able to tell what the algebraic expression for a linear function is, based on a graphical representation of the function and being able to draw linear functions by hand correct.

Hypothesis H1

When testing hypothesis H1 I did a χ^2-test at a 95% significant level. As seen in table 19 and 20 we have an χ^2-value at 4,690 and a p-value at 0,030 which means that I cannot discard my hypothesis. So it seems from my data that there is a relation between understanding the conceptual blend of the Cartesian plane and being able to draw linear functions by hand correctly.

Understanding the conceptual blend of the Cartesian plane * Drawn functions Crosstabulation

			Drawn functions		
			Have drawn one or more functions wrong	Have drawn all functions correct	Total
Understanding the conceptual blend of the Cartesian plane	Do not understand	Count	20	8	28
		Expected Count	16,2	11,8	28,0
		% of Total	38,5%	15,4%	53,8%
	Understand	Count	10	14	24
		Expected Count	13,8	10,2	24,0
		% of Total	19,2%	26,9%	46,2%
Total		Count	30	22	52
		Expected Count	30,0	22,0	52,0
		% of Total	57,7%	42,3%	100,0%

Table 19

Chi-Square Tests

	Value	df	Asymptotic Significance (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-Square	4,690[a]	1	,030		
Continuity Correction[b]	3,550	1	,060		
Likelihood Ratio	4,747	1	,029		
Fisher's Exact Test				,048	,029
Linear-by-Linear Association	4,600	1	,032		
N of Valid Cases	52				

a. 0 cells (0,0%) have expected count less than 5. The minimum expected count is 10,15.

b. Computed only for a 2x2 table

Table 20

Also it can be seen from table 19 that 53,8% does not understand the conceptual blend of the Cartesian plane, which is a large part of the participating students. It is problematic that 53,8% of the students does not understand the conceptual blend of the Cartesian plane when my results showed, that there is a connection between understanding the Cartesian plane blend and being able to draw linear functions by hand correct. This means that 53,8% the students might not be able to draw any linear functions by hand correct, since they do not understand the conceptual blend of the Cartesian plane. The results of testing this hypothesis could lead to the question ‘Can the students in general draw function by hand correct if they do not understand the conceptual blend of the Cartesian plane?’. I will return to educational implications of this result in the discussion section below.

Hypothesis H2

Testing hypothesis H2 was done using an χ^2-test at a 95% significance level. My results showed that 73,1% does not understand the $f(x)$-notation. This can be seen in table 21. The results of the χ^2-test is showed in table 22. The χ^2-test gave a p-value at 0,010. Hence I cannot discard my hypothesis. From the test it seems that there is a relation between understanding the $f(x)$-notation and being able to draw linear functions by hand correct.

Understanding the f(x)-notation * Drawn functions Crosstabulation

			Drawn functions		
			Have drawn one or more functions wrong	Have drawn all functions correct	Total
Understanding the f(x)-notation	Do not understand	Count	26	12	38
		Expected Count	21,9	16,1	38,0
		% of Total	50,0%	23,1%	73,1%
	Understand	Count	4	10	14
		Expected Count	8,1	5,9	14,0
		% of Total	7,7%	19,2%	26,9%
Total		Count	30	22	52
		Expected Count	30,0	22,0	52,0
		% of Total	57,7%	42,3%	100,0%

Table 21

Chi-Square Tests

	Value	df	Asymptotic Significance (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-Square	6,656[a]	1	,010		
Continuity Correction[b]	5,124	1	,024		
Likelihood Ratio	6,702	1	,010		
Fisher's Exact Test				,013	,012
Linear-by-Linear Association	6,528	1	,011		
N of Valid Cases	52				

a. 0 cells (0,0%) have expected count less than 5. The minimum expected count is 5,92.

b. Computed only for a 2x2 table

Table 22

This test indicates that there should be a greater focus on making sure that the students understands the notation. One could argue that understanding the notation is a part of understanding the concept. I will return to this in the discussion section below.

Hypothesis H3

Hypothesis H3 was also tested with an χ^2-test at a 95% significance level. In table 24 it can be seen that the χ^2-value is 1,970 and the p-value is 0,373. Thus at a 95% significance level I must discard hypothesis H3. Concluding that the results of my questionnaire showed that there is no relation between students mostly using CAS-tools and them not being able to draw the linear functions by hand correct.

Task 1.3 What do you normally use when drawing graphs? * Drawn functions Crosstabulation

			Drawn functions		
			Have drawn one or more functions wrong	Have drawn all functions correct	Total
Task 1.3 What do you normally use when drawing graphs?	Draws with paper and pencil	Count	7	7	14
		Expected Count	8,1	5,9	14,0
		% of Total	13,5%	13,5%	26,9%
	Gets a CAS-tool to draw it	Count	14	12	26
		Expected Count	15,0	11,0	26,0
		% of Total	26,9%	23,1%	50,0%
	Draw with paper and pencil and gets a CAS-tool to draw it	Count	9	3	12
		Expected Count	6,9	5,1	12,0
		% of Total	17,3%	5,8%	23,1%
Total		Count	30	22	52
		Expected Count	30,0	22,0	52,0
		% of Total	57,7%	42,3%	100,0%

Table 23

Chi-Square Tests

	Value	df	Asymptotic Significance (2-sided)
Pearson Chi-Square	1,970[a]	2	,373
Likelihood Ratio	2,058	2	,357
Linear-by-Linear Association	1,830	1	,176
N of Valid Cases	52		

a. 0 cells (0,0%) have expected count less than 5. The minimum expected count is 5,08.

Table 24

This result is interesting since there is heavy discussion on CAS-tools and that they inhibit the students' learning. Further discussion of this result will be presented in the discussion section.

Hypothesis H4

Testing hypothesis H4 I tried doing an χ^2-test. However there were two cases that had expected count five or less. Hence I cannot do an χ^2-test. So instead to test hypothesis H4 I carried out a correlation test. The results of the test can be seen in table 25. The result of the test was that the correlation between the two variables "being able to draw functions correct" and "doing task 2.11" is 0,216 and the p-value is at 0,136.

Correlations

		Drawn functions	Task 2.11 Which algebraic expression fits the graph
Drawn functions	Pearson Correlation	1	,216
	Sig. (2-tailed)		,136
	N	52	49
Task 2.11 Which algebraic expression fits the graph	Pearson Correlation	,216	1
	Sig. (2-tailed)	,136	
	N	49	49

Table 25

At a 95% significance level I must discard my hypothesis. Hence it seems from my results that there is no relation between being able to tell what the algebraic expression for a linear function is based on a graphical representation and being able to draw linear functions by hand correct. Concluding that hypothesis H4 is falsified. Noting that the correlation coefficient is less than 0,8, hence even if there were a significant correlation, it would not have been a strong relation between the two variables.

Correlations

This section will present some of the correlations I found that I did not foresee. The entire correlation matrix can be seen in appendix H.

Testing my hypothesis H4 showed that I needed to discard it. In the exploration of correlations I found that task 2.11[2] and 1.2 b)[3] correlates. The correlation test is based on 49 respondents. The correlations coefficient between these two variables is 0,319 and it has a p-value equal to 0,025, which means that at a significance level at 95% they correlate significantly. The correlation coefficient is below 0,8. Hence there is not a strong relationship between the two variables. It is however interesting that these two correlates. The two functions that are being addressed in the two tasks are of the form $f(x) = ax - b$, which could be an explanation to why they correlate. If a student knows how the function $f(x) = 2x - 5$ is drawn then it seems likely that the same student would be able to tell that the function in task 2.11 has the algebraic expression $f(x) = 4x - 8$. One can use the same knowledge to both draw and find the algebraic expression. Namely that b is where the function intersects the second axis and that a represents "how many steps you have to go upwards when you step one to the right". Based on some of the tasks where the students have to write answers in their own words, this seems like a plausible explanation to why the two correlates. The students' written answers will be analysed in the section about qualitative results and analysis.

When drawing the functions the students were asked to draw a function of this form: $f(x) = b$ or more precisely the students had to draw $f(x) = 3$. The function $f(x) = 3$ stick out from the other functions since it is a constant function. Hence there is no x present on the right hand side of the equality sign. The constant function also stuck out from the others based on the correlation matrix. The three functions $f(x) = 2x - 5$, $f(x) = -x + 6$ and $f(x) = 4x$ all correlated with each other, albeit not strongly but significantly and $f(x) = 3$ did not correlate with any of them. This could indicate that the students are more familiar with functions of the form as the three that correlates and not as familiar with constant functions.

[2] Task 2.11 is the task where the students need to look at the graphical representation and choose which algebraic expression fits the graph.
[3] Task 1.2 b) is where the students are asked to draw the function $f(x) = 2x - 5$.

Qualitative results and analysis

Graphical or algebraic argument

For task 1.2 a)[4] 21 of the students gave an explanation from a graphical view, 12 gave an explanation from an algebraic view and three explains that it is because of the two points formula. The remaining students either have not answered the question or their answers were not satisfying because they did not make sense or were irrelevant in the context. These were answers such as: "*Because it doesn't – I don't know why*" (Questionnaire 171, my translation), "*I don't know, it is written in my notes*" (Questionnaire 201, my translation) or "*The point* $(3,1)$ *lies on* $(0,1)$*, the point* $(7,4)$ *lies on* $(4,8)$" (Questionnaire 70, my translation).

As mentioned 52 students' answers is the base of the results of the questionnaire. Hence it is only 69,2% of the students that answered task 1.2 a) with a satisfying argument. 40,4% of the students gave a graphical argument and only 23,1% gave an algebraic argument. This could imply that a great part of the students, namely 40,4% are more visual learners. It could also imply that the students that gave a graphical argument know the graphical procedures better than the algebraic. The students' use of graphical arguments could also be a result of me using the word 'graph' in the task. The wording of the student's graphical argument gives insight to what procedure they might have used to draw the functions that were presented in the tasks after. One students wrote:

"$f(x) = 2x - 5$ *is a linear function. You start at* -5 *on the vertical axis, say* 1 *across and* 2 *up <-> $2x$. Then one can see that the point* $(3,1)$ *lies on the line, but not* $(7,4)$." (Questionnaire 161, my translation)

Here the wording gives us a chance to follow the student's way of thinking. Questionnaire 161 starts at $(0, -5)$ and then from that point moves one to the right and two upwards and then probably draws a line. This could indicate that this is the procedure the student used to draw

[4] Task 1.2 a) is the task where the students was asked to explain in their own words why the point $(3,1)$ lies on the graph of the function $f(x) = 2x - 5$ and why the point $(7,4)$ doesn't lie on the graph of the function $f(x) = 2x - 5$.

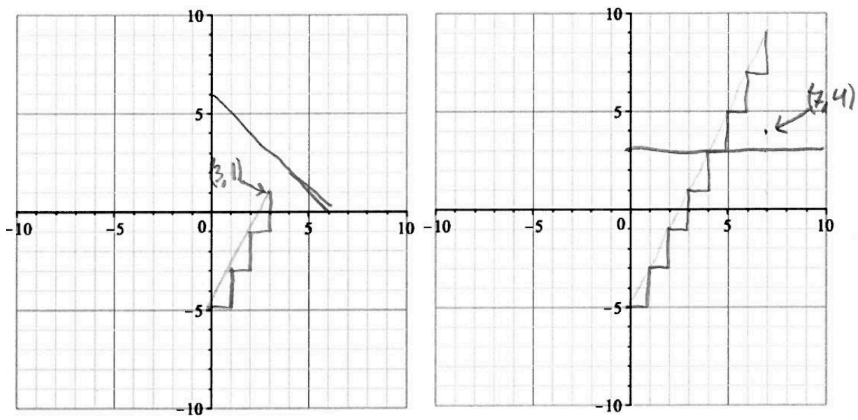
the function $f(x) = 2x - 5$. This example is not unique, as there are other students that also argued like this. The following are examples of this:

“b tells where the line hits the y-axis, while a tells how much the line tilts when you go one step out on the x-axis. This fits with $(3{,}1)$ *lying on the graph.”* (Questionnaire 170, my translation)

“We start at -5 *on the y-axis, and go one to the right and two up, then we draw a line and we can see that the point* 3,1 *lies on the graph and* 7,4 *doesn’t.”* (Questionnaire 200, my translation)

“If one uses the constants to move in the coordinate system one ends at $(3{,}1)$ *(shown in the 2. (c) task below). Same in the next task. Shown in task (e) below.”* (Questionnaire 230, my translation)

Questionnaire 230 used the following two drawings as a part of the argument.



Example 8: Questionnaire 230's drawings to task 1.2 c) and 1.2 e)

From these drawings and from the wording in the quotes it is quite clear that students use the same procedure to give an argument and to draw the functions. These are just some examples of students that gave a graphical argument. The students that are giving a graphical argument all use the same procedure which is: they first locate where the line intersects the y-axis and afterwards move one to the right and count how many they would have to go up, according to a and then draw a line.

The procedure that the students that gave a graphical argument used, can also be used to tell what algebraic expression fits a graph. As I mentioned in the previous section this could be a plausible explanation to why task 2.11 and 1.2b) has a significant correlation.

The other large category was the students that gave an algebraic argument. These were arguments such as:

"By inserting the x and y values -> $1 = 2 \cdot 3 - 5$*, one can see that the right hand side is the same as the left hand side. The point* $(7,4)$ *does not lie on the graph since the right hand side is not equal to the left hand side* $4 \neq 2 \cdot 7 - 5$*."* (Questionnaire 101, my translation)

"*Because* $3 \cdot 2 - 5 = 1$ *and* $2 \cdot 7 - 5 \neq 4$"(Questionnaire 141, my translation)

"If one inserts the x-coordinates on x's placing in the algebraic expression it has to be equal to the y-coordinate. If one inserts 3 in the algebraic expression one would get the result 1, which is the y-coordinate. If one inserts 7 in the algebraic expression the result would be 9 and not 4." (Questionnaire 190, my translation)

These are just some examples of students that gave an algebraic argument. It is clear that they only use the algebraic expression to give an argument. From a formalistic view one could argue that the algebraic argument is more precise than the graphical argument. Although the students that gave a graphical argument did not give a wrong answer, their way of arguing just have limitations. This will be discussed more in the discussion section.

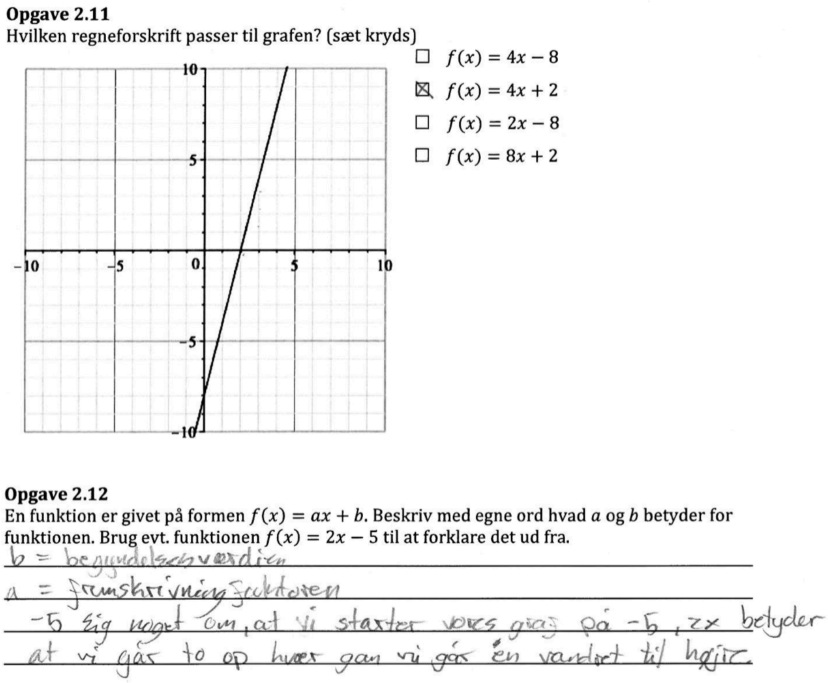
Task 2.11 and 2.12

23,1% of the students either did not answer task 2.11[5] correct or did not give an answer. I wanted to analyse the answers of the students that answered 2.11 wrong and also answered

[5] In task 2.11 the students were asked to find the algebraic expression, from four answer possibilities, when given a graph.

task 2.12[6]. There were only 13,5% students that both answered 2.11 wrong and answered something in task 2.12. The remaining 9,6% that answered 2.11 wrong did not give an answer to task 2.12. Thus these students were not interesting for this context.

What is interesting is that the answers that 14,5% of the students' answered in task 2.12 indicate that the students know the 'formal' explanation to what role a and b has for the function. Although when connecting their answer in 2.12 with their answers in task 2.11 it becomes clearer that they have a misconception of what role constants a and b have. An example of this can be seen in example 9.


Opgave 2.11
Hvilken regneforskrift passer til grafen? (sæt kryds)

- ☐ $f(x) = 4x - 8$
- ☒ $f(x) = 4x + 2$
- ☐ $f(x) = 2x - 8$
- ☐ $f(x) = 8x + 2$

Opgave 2.12
En funktion er givet på formen $f(x) = ax + b$. Beskriv med egne ord hvad a og b betyder for funktionen. Brug evt. funktionen $f(x) = 2x - 5$ til at forklare det ud fra.

b = begyndelsesværdien
a = fremskrivningsfaktoren
-5 sig noget om, at vi starter vores graf på -5, 2x betyder at vi går to op hver gang vi går én vandret til højre.

Example 9: Questionnaire 161's answers to task 2.11 and 2.12

The student from example 9 writes as an answer to task 2.12:

"*b= starting value*

a= projection factor (da: fremskrivningsfaktor)

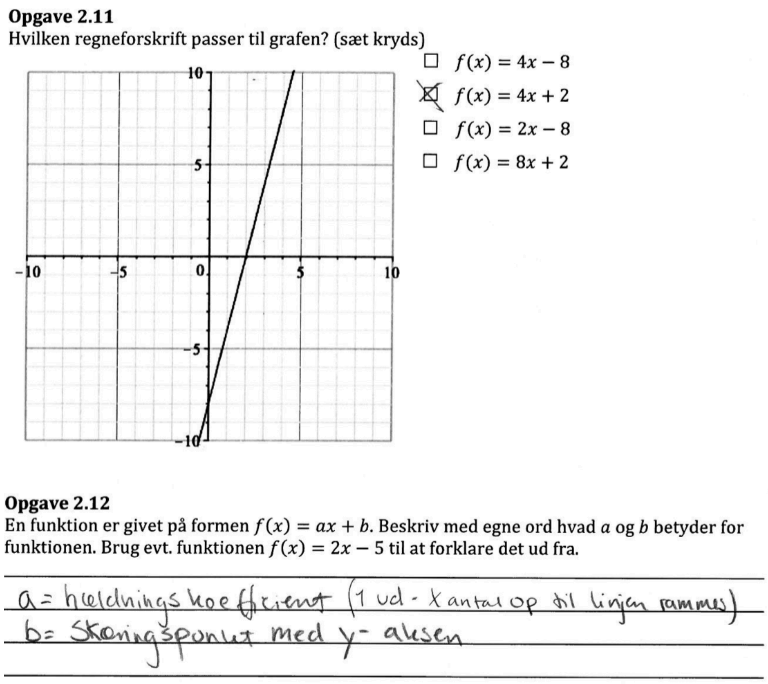
---

[6] In task 2.12 the students were asked to explain in their own words what role a and b has for the function $f(x) = ax + b$, and that they could use the function $f(x) = 2x - 5$ in their explanation.

-5 *says something about, that we start our graph at* -5, $2x$ *means that we go two upwards every time we go one horizontal to the right"* (Questionnaire 161, my translation)

Reading task 2.12 and more importantly reading what the student wrote as an answer when using the example, evaluation of this task would probably give the student a checkmark for doing it right. However what task 2.11 uncovers is that it seems that the student knows what role a has but not what role b has. The answers for task 2.11 and task 2.12 also uncovers that what the student believes is the starting value is where the graph intersects the x-axis, hence a misconception. If the student were only asked to do task 2.12 the misconception of the constant b would not have been uncovered since this student have learned the 'formal' words for what role a and b has. The Questionnaire 161 uses the word 'projection factor' for the constant a. 'Projection factor' is a word that is used when working with exponential functions. Thus it is incorrect to use the word 'projection factor' in the context of linear functions.

In example 10 there is shown another student's answers of the two tasks.


Opgave 2.11
Hvilken regneforskrift passer til grafen? (sæt kryds)

- ☐ $f(x) = 4x - 8$
- ☒ $f(x) = 4x + 2$
- ☐ $f(x) = 2x - 8$
- ☐ $f(x) = 8x + 2$

Opgave 2.12
En funktion er givet på formen $f(x) = ax + b$. Beskriv med egne ord hvad a og b betyder for funktionen. Brug evt. funktionen $f(x) = 2x - 5$ til at forklare det ud fra.

a= hældnings koefficient (1 ud - x antal op til linjen rammes)
b= skæringspunkt med y- aksen

Example 10: Questionnaire 180's answers to task 2.11 and 2.12

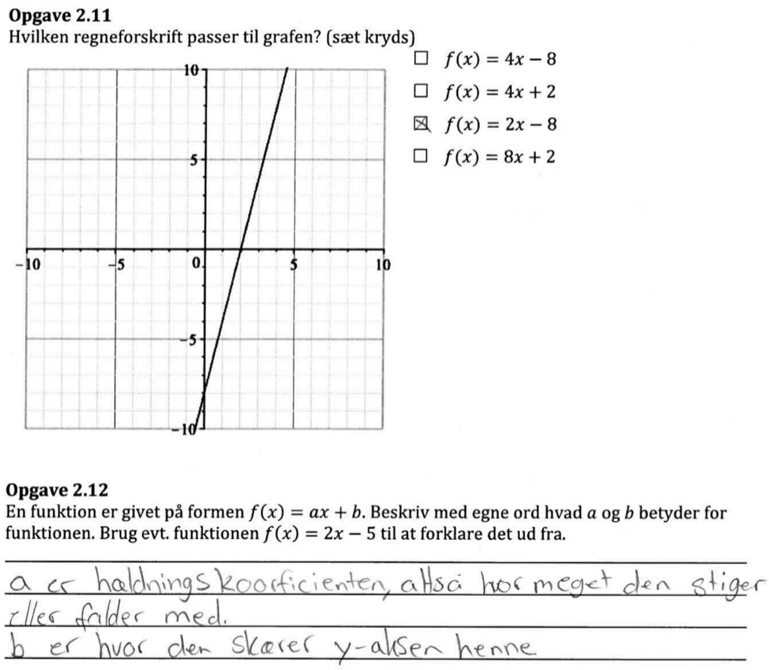
The student wrote as an answer to task 2.12:

"a=slope (1 out – x times up until the line is hit)
b= the intersection with the y-axis" (Questionnaire 180, my translation)

Questionnaire 180 writes that the constant b is the point where the line intersects the y-axis. Although this is not consistent with what the student have answered in task 2.11. From comparing the answers to task 2.11 and 2.12 it seems that Questionnaire 180 knows what the slope is, but he/she does not know which axis that is referred to as the y-axis. From the answer to task 2.11 it seems that b is the intersection with the x-axis. If the student had only answered task 2.12 it would not have been uncovered that Questionnaire 180 has a misconception of which axis is which.

The student shown in example 11 have answered that the function that is drawn in the Cartesian plane in task 2.11 is $f(x) = 2x - 8$. In task 2.12 the student from example 11 wrote:

"a is the slope, how much it increases or decreases.
b is where it intersects the y-axis" (Questionnaire 231, my translation)


Opgave 2.11
Hvilken regneforskrift passer til grafen? (sæt kryds)

- ☐ $f(x) = 4x - 8$
- ☐ $f(x) = 4x + 2$
- ☒ $f(x) = 2x - 8$
- ☐ $f(x) = 8x + 2$

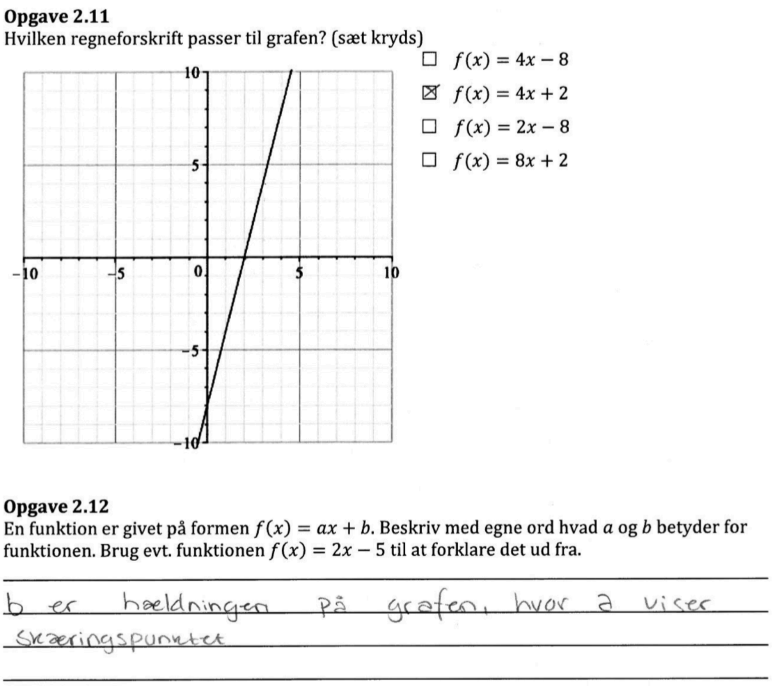
Opgave 2.12
En funktion er givet på formen $f(x) = ax + b$. Beskriv med egne ord hvad a og b betyder for funktionen. Brug evt. funktionen $f(x) = 2x - 5$ til at forklare det ud fra.

a er hældningskoefficienten, altså hvor meget den stiger
eller falder med.
b er hvor den skærer y-aksen henne

Example 11: Questionnaire 231's answers to task 2.11 and 2.12

From the answers in task 2.12 Questionnaire 231 would convince the reader that he/she knows what role the constants a and b has for a liner function. If one compares the two tasks 2.11 and 2.12 one would get the impression that Questionnaire 231 knows what role the constant b has, but the student has a misconception about the constant a.
From what the student have answered in task 2.11 it could indicate that the student has a misconception that the slope, the constant a, is where the function intersects the x-axis.

Two other students' answers to task 2.12 do not quite correlate with what they have answered in task 2.11. As seen in example 12 this student answered to task 2.11 that the function that is drawn is $f(x) = 4x + 2$.



Opgave 2.11
Hvilken regneforskrift passer til grafen? (sæt kryds)

- ☐ $f(x) = 4x - 8$
- ☒ $f(x) = 4x + 2$
- ☐ $f(x) = 2x - 8$
- ☐ $f(x) = 8x + 2$

Opgave 2.12
En funktion er givet på formen $f(x) = ax + b$. Beskriv med egne ord hvad a og b betyder for funktionen. Brug evt. funktionen $f(x) = 2x - 5$ til at forklare det ud fra.

b er hældningen på grafen, hvor a viser skæringspunktet

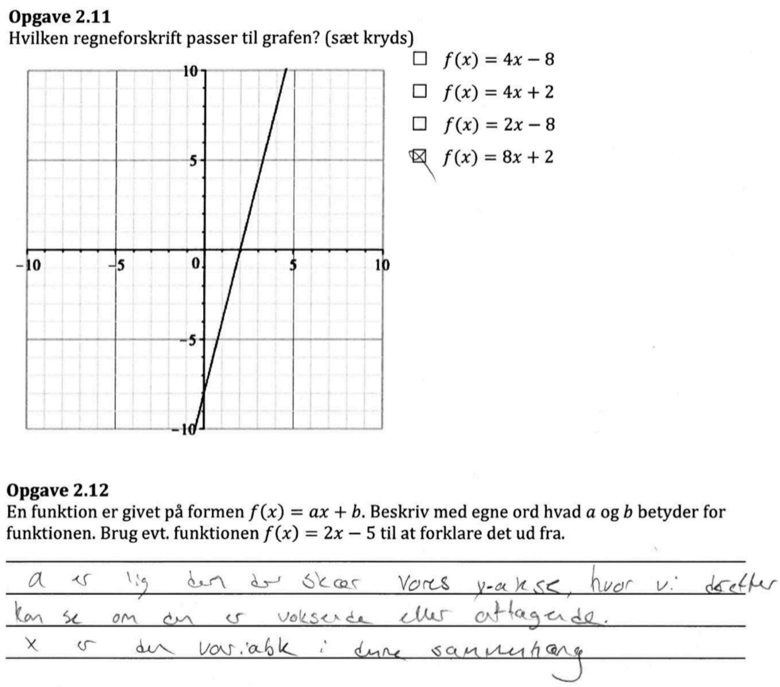
Example 12: Questionnaire 21's answers to task 2.11 and 2.12

At the same time the student wrote to task 2.12 that:
"*b is the slope of the graph, where a shows the intersection*" (Questionnaire 21, my translation)

However none of the answer possibilities in task 2.11 would fit the way this student have misunderstood the roles of the constants a and b. For Questionnaire 21 there might be missing an answer possibility in task 2.11 such as $f(x) = -8x + 4$. The explanation to Questionnaire 21's answer in task 2.11 could be that he/she would have wanted to answer something else, but decided to just choose one of the answer possibilities. In Questionnaire 21's case task 2.12 alone uncovers a misconception about the roles of the constants a and b.

The second student where the answers from task 2.11 and 2.12 do not quite correlate is Questionnaire 250, which can be seen in example 13. To task 2.12 the student wrote:

"a is equal to where it intersects our y-axis, and from there we can see if it is increasing or decreasing.
x is the variable in this interrelation" (Questionnaire 250, my translation)


Opgave 2.11
Hvilken regneforskrift passer til grafen? (sæt kryds)

- ☐ $f(x) = 4x - 8$
- ☐ $f(x) = 4x + 2$
- ☐ $f(x) = 2x - 8$
- ☒ $f(x) = 8x + 2$

Opgave 2.12
En funktion er givet på formen $f(x) = ax + b$. Beskriv med egne ord hvad a og b betyder for funktionen. Brug evt. funktionen $f(x) = 2x - 5$ til at forklare det ud fra.

a er lig den der skær vores y-akse, hvor vi derefter
kan se om den er voksende eller aftagende.
x er den variabel i denne sammenhæng

Example 13: Questionnaire 250's answers to task 2.11 and 2.12

Questionnaire 250 does not explain what role the constant b has for the linear function. He/she only explains what role the constant a has. This student has the misconception that a is the intersection with the y-axis. From this student's misconception then the function $f(x) = 8x + 2$ form the answer possibilities, is the one that comes closest to what the student have answered in task 2.12.

What is interesting about this is that Questionnaire 250 answer to task 2.12 raises the question on what the student really understands as the roles of the constant a. If the constant a is the intersection then a should be -8. However the student also writes that we can see from a if the function is increasing or decreasing. The student might believe that one can see from the constant a where it intersect, that it is at 8, whether or not this is positive or negative is maybe secondary, because whether or not a is positive or negative indicates if the function is increasing or decreasing.

The examples above shows that some students can remember definitions without really knowing what they say and what they mean. These two tasks together reveal misconceptions that these students have, but it is necessary that the students answer both tasks to get a clear picture of what misconceptions the students have.

Concept definition

In the questionnaire the students were asked to describe in their own words what a function is. This is an abstract question, however, all 52 participating students gave some sort of an answer to the question.
The five categories, that the answers have been categorised in, are:

A function is a tool (13,5%)
A function is an algebraic expression (32,7%)
A function is a model based on data (13,5%)
A function is a graph (21,2%)
A function is interrelation (da: sammenhæng) (19,2%)

13,5% of the students perceives a function as a tool. They consider a function as a 'thing' that gives answers and some of the students consider the function as a machine. The students in this category have used words such as "input", "output" and "tool" some examples of this category are:

"A function is a tool (mathematical) to express the interrelation, between two variables, x and y or in other words the dependent and independent."[7] (Questionnaire 211, my translation)

"There comes an input 'x' in a function ('machine') and an output 'y'."[8] (Questionnaire 190, my translation)

"A function is a sort of mathematical tool that, given constants can put two variables in interrelation. Especially the degree of explanation gives an idea about how good a function reflects the interrelation between to axes."[9] (Questionnaire 230, my translation)

"A function is a mathematical tool to describe or predict the reality. Totally simply said a function is maybe also just a mathematical machine, that gets some inputs and brings some outputs." [10] (Questionnaire 180, my translation)

The teacher that these students have for mathematics, told me that he uses the metaphor that 'a function is a machine' in his teaching. From research I know that this metaphor is also used in some teaching books e.g. Clausen et al. (2018, p. 14). These two aspects combined could be an explanation to why some of the students define a function as a machine. Looking at Questionnaire 230's answer, the student writes that: "*the degree of explanation gives an idea about how good* […] *the interrelation between to axes*". The degree of explanation is a concept the students use when working with datasets and modelling. Questionnaire 230's answer indicates that this student does not understand what information the degree of explanation gives or the answer indicates that the student is not certain about what the axes are.

[7] The quote in Danish: "En funktion er et redskab (matematisk) til at udtrykke sammenhængen mellem to variabler, x og y eller med andre ord den afhængige og uafhængige"
[8] The quote in Danish: "Der kommer et input 'x' ind i en funktion ('maskine') og et output 'y'.
[9] The quote in Danish: "En funktion er en form for matematisk værktøj, som givet konstanter, kan sætte 2 variable i en sammenhæng. Særligt forklaringsgraden giver et billede af hvor godt en funktion afspejler sammenhængen mellem to akser."
[10] The quote in Danish: "En funktion er et matematisk redskab til at beskrive eller forudse virkeligheden. Helt simpelt er en funktion måske også bare en matematisk maskine der får nogle input og kommer med nogle output."

32,7% of the students perceive that a function is an algebraic expression and this is the largest category. The students in this group have used words such as “formula”, “calculation”, “equation”, “recipe” and “rule” (da: forskrift) some examples of this grouping are:

“A function is a small calculation one can solve to find the interrelation between 2 factors, e.g. how much food a man of 70 kg should eat on a daily basis.”[11] (Questionnaire 130, my translation)

“A function is an equation that can be set up on a graph, which contains a x and a y.”[12] (Questionnaire 161, my translation)

“A function is a rule that has a specific function in a coordinate system. x and y are included, though it is not always clear, sometimes it can be ‘implicit’. One can input values in the functional rule (da: funktionsforskrift) *and form this get a specific line in the coordinate system, where one can read points and values.”*[13] (Questionnaire 151, my translation)

“*I would describe it as some sort of recipe of a e.g. line.*”[14] (Questionnaire 30, my translation)

“A function is a formula that describes a constant and a slope for a line in a coordinate system.”[15] (Questionnaire 120, my translation)

The most important quality for the students in this category is that the function either has or is an algebraic expression. Thinking that a function either has or is an algebraic expression narrows the students’ concept detention of functions. The functions that high school students

[11] The quote in Danish: ”En funktion er et lille regnestykke man kan løse, for at finde sammenhængen mellem 2 faktorer, f.eks. hvor meget mad skal en mand på 70 kg spise dagligt.”
[12] The quote in Danish: ”En funktion er en ligning som kan stilles op på en graf, som indeholder et x og et y.”
[13] The quote in Danish: ”En funktion er en forskrift som har en bestemt funktion i et koordinatsystem. Der indgår x og y, dog ikke altid helt klart, det kan godt være ’implicit’. Man kan indsætte værdier i funktionsforskriften og dermed få en bestemt linje i koordinatsystemet, som man kan aflæse punkter og værdier på.”
[14] The quote in Danish: ”Jeg ville beskrive det som en slags opskrift på en f.eks. linje.”
[15] The quote in Danish: ”En funktion er en formel som beskriver en konstant og en hældning for en linje i et koordinatsystem.”

meet in the teaching are often functions that have an algebraic expression, which then could lead the students to make the assumption that functions always has an algebraic expression.

13,5% of the students thinks of a functions as a model. It is something that is based on data. The students in this category have used words such as “model”, “dataset”, “data” and “development” (da: udvikling) some examples of this category are:

“A function is a model that gives us permission to generalise and predict or generalise and look back at different issues as well as explaining a mathematical development.” [16] (Questionnaire 111, my translation)

“A function is a positive or negative development.”[17] (Questionnaire 241, my translation)

“By using functions one can calculate different data and it can especially be used when one wants to look a head in time, e.g. in connection to a production and when you want to look back in time.”[18] (Questionnaire 91, my translation)

“A function can be used to create an overview of a given dataset.”[19] (Questionnaire 101, my translation)

Some of the students say directly that the function is a model, others use wordings such as “one can use the function” and then describes situations that is usually in connection with modelling. Most of the students describe the function in connection with a dataset, something that is connected to the real world. It seems that the students of this category does not think that functions are abstract.

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[16] The quote in Danish: ”En funktion er en model der giver os lov til at generaliser og forudsige eller generalisere og tilbage se diverse problemstillinger samt forklare en matematisk udvikling.”
[17] The quote in Danish: ”En funktion er en positiv eller negativ udvikling.”
[18] The quote in Danish: ”Ved at bruge funktioner kan man udregne forskellige dataer og det kan især bruges når man vil kigge frem i tiden fx i forhold til en produktion og når man vil kigge tilbage i tiden.”
[19] The quote in Danish: ”En funktion kan bruges til at skabe et overblik over et givent datasæt.”

21,2% of the students understands the function as a graph. This category is the second largest. In this category the students have primarily used the words "graph" and "line" some examples of this grouping are:

"A function is a graphical interrelation between some x-values and y-values. All functions consist of a dependent (y) and an independent (x) value. Functions can have constants such as e.g. a and b."[20] (Questionnaire 71, my translation)

"A function is a way to define a graph. Shortly said the graph shows what we have according to the function."[21] (Questionnaire 61, my translation)

"A line in a coordinate system, that changes according to a rule."[22] (Questionnaire 41, my translation)

"A function is a graph in a coordinate system it can show how much something will increase or the relation between two functions, e.g. how much something increased in 2019 and how much it increased in 2002 and then one can compare them."[23] (Questionnaire 261, my translation)

"A function is some sort of a line that moves. Raises or falls in different ways that you then can read in a coordinate system." [24](Questionnaire 251, my translation)

The students in this category have all firstly explained a function as a graph. Indicating that this is the most important quality that a function has. One could think that these students might be more visual learners than the other students hence this is why the graph quality is so

[20] The quote in Danish: " En funktion er en grafisk sammenhæng mellem nogle x-værdier og y-værdier. Alle funktioner består af en afhængig (y) og en uafhængig (x) værdi. Funktioner kan have konstanter som fx a og b."
[21] The quote in Danish: "En funktion er en måde at definere en graf på. Kort sagt viser grafen det vi har i forhold til funktionen."
[22] The quote in Danish: "En linje i et koordinatsystem, som ændre sig efter en forskrift."
[23] The quote in Danish: "En funktion er en graf i et koordinatsystem den kan vise hvor meget noget stiger, eller forholdet mellem 2 funktioner, fx hvor meget noget steg i 2019 og hvor meget det steg i 2002 og så kan man sammenligne dem."
[24] The quote in Danish: " En funktion er en form for linje som bevæger sig. Stiger eller falder på forskellige måder som så kan aflæses i et koordinatsystem."

important to them. The answers of the students in this category could also indicate that they are most familiar with functions that have a graphical representation or they simply have not met a function that could not be represented as a graph.

19,2% of the students thinks of a function as an interrelation. Students in this category have used words such as “interrelation” and “relation” some examples of these grouping are:

“A function describes a interrelation between two variables, two factors.”[25] (Questionnaire 100, my translation)

“Can describe the relation between two factors.” [26](Questionnaire 150, my translation)

“A function shows an interrelation.” [27](Questionnaire 191, my translation)

“A function is a description of an interrelation between a number of numbers. It is a recipe on how one can calculate an interrelation.”[28] (Questionnaire 220, my translation)

“It is a interrelation between x and y.” [29](Questionnaire 20, my translation)

The word interrelation is a word that is often used in the Danish textbooks for mathematics when working with the concept of functions e.g. Clausen et al. (2018), Grøn et al. (2017), Carstensen et al. (2017). The students’ using of the word interrelation could originate from the textbooks. These students’ explanation is probably the one that comes closest to the formal definition that the students have met. It is not clear from these students’ answers whether or not they know what is meant by ‘interrelation’. There is a chance that the students have just learned this ‘definition’ of a function and does not know what it means.

[25] The quote in Danish: ”En funktion beskriver sammenhængen mellem to variabler altså to faktorer.”
[26] The quote in Danish: ”Kan beskrive forholdene mellem to faktorer.”
[27] The quote in Danish: ”En funktion viser en sammenhæng.”
[28] The quote in Danish: ”En funktion er en beskrivelse af en sammenhæng mellem en række tal. Det er en opskrift på hvordan man kan regne en sammenhæng ud.”
[29] The quote in Danish: ”Det er en sammenhæng mellem x og y.”

Discussion

In this section I will firstly present a discussion of the two main results of this study. Then I will also discuss of some of the other quantitative results. I will also discuss some of the qualitative results that were presented in the last section. The discussions will involve possible explanations for the results and educational implications they can have. I will further in the discussions include already established research that has relevance to the discussion.

Two main results

The two main results of my thesis are the corroborated hypotheses H1 and H2. These results are especially interesting since they are new results and possibly important to new ideas on how to teach mathematics.

Hypothesis H1

The result of testing hypothesis H1 showed that there is a relation between understanding the conceptual blend of the Cartesian plane and being able to draw linear functions in the Cartesian plane by hand correct. There have been done a similar discovery by Schoenfeld et al. (1993). The study by Schoenfeld et al. (1993) revealed that a particular student's misconceptions about algebraic and graphical representations could be traced back to what Schoenfeld et al. (1993) calls a missing 'Cartesian connection'. A missing 'Cartesian connection' is when the student is not able to connect the two-dimensional graphic world with the algebraic world (Schoenfeldt, 1993, pp. 108). Recalling that the conceptual blend of the Cartesian plane is a central concept in analytic geometry and that analytic geometry is a union of algebra and geometry. Thus, the missing 'Cartesian connection' could be 'translated' to not understanding the conceptual blend of Cartesian plane. Hence the results by Schoenfeld et al. (1993) and mine support each other. The results from Schoenfeld et al. (1993) were based on the study of one student where my findings indicates that the relation in hypothesis H1 is not only true for one student but it can be generalised to more students.

Considering the relation between understanding the conceptual blend of the Cartesian plane and being able to draw linear function by hand correct. It indicates that there should be a larger focus on making sure that the students understand the conceptual blend of the Cartesian plane. The Cartesian plane is an important aspect when working with functions.
The questionnaire only explored if hypothesis H1 is true for linear functions. One could ask if this relation is also true for functions in general. It could be interesting to explore if there is a relation between understanding the conceptual blend of the Cartesian blend and being able to draw functions in general. From the results of Shoenfeld et al. (1993) it seems plausible that there could be a connection between understanding the conceptual blend of the Cartesian plane and being able to do other translations between graphical and algebraic representations. E.g. translating all types of functions algebraic representation to their graphical representation. This could potentially be a field of study that should be investigated further.

The results of testing hypothesis H1 also evokes a discussion on, if there should be a greater focus on making sure that the students understands the conceptual blends that constitutes the mathematics they are taught. Since hypothesis H1 was corroborated one might wonder if there could be a relation between understanding other conceptual blends and being able to do other aspects of mathematics.
As an example the unit circle is also a conceptual blend. It is a blend of 'A circle in the Euclidian plane with center and radius' and 'The Cartesian plane, with x-axis, y-axis, and origin at (0,0)' (Lakoff & Núñez, 2000, p. 388). In mathematics at level B the students meets the trigonometric functions sine and cosine: "*Cosine and sine are implemented as coordinates for the direction vector to a given point on the unit circle….*" (Danish Ministry of education, 2019, p. 18, my translation). Thus the students need to understand the unit circle. It could be that not understanding the conceptual blend of the unit circle could be connected to not being able to understand direction vector. This is by all means just speculation that could be explored further.

As mentioned in the theory section the conceptual blend of the Cartesian plane is a blend between two conceptual metaphors. According to Lakoff & Núñez (2000) mathematics is built on conceptual metaphors since humans conceptualize abstract concepts in concrete terms. There is a risk that the students that fail to understand the conceptual blend of the Cartesian

plane, might be failing at understanding one of the two conceptual metaphors that the conceptual blend consists of. Thus making it important to teach and make sure that the students understand each of the conceptual metaphors. Conceptual metaphors are a part of high school mathematics. Conceptual metaphors are used in textbooks (Jonasen & Johansen, in preparation) and they are also used in these two classes' teaching. Since the teacher told me that he uses the metaphor that the function is a machine in his teaching of the classes. Considering that mathematics is based on conceptual metaphors according to Lakoff and Núñez (2000) and understating them might have a relation to some mathematical 'skills'. This could give an implication that there should be a greater focus on making sure that the students understand conceptual metaphors that are present in the mathematics they are taught.
The results of testing hypothesis H1 could indicate that there should be focus on analysing the different conceptual metaphors and conceptual blends that appears in high school mathematics. Such that there can be evolved series of tasks that can examine the students understanding of conceptual metaphors and conceptual blends. However even more importantly is the question of what educational initiative could be made, to make sure that the students understand the conceptual metaphors and conceptual blends? One way of doing this could be to have more explicit approach in teaching of what the conceptual metaphors and conceptual blends consist of and what effect they have on our understanding of abstract concepts. A more explicit-reflective framework for teaching is supported by Abd-El-Khalick (2012). When helping precollege students develop informed conceptions of the Nature of Science Abd-El-Khalick (2012) found that it was necessary to engage the students with inquiry. However it was not sufficient to teach the students to develop informed conceptions of Nature of Science. Abd-El-Khalick (2012) found that:

"…an explicit-reflective framework is needed to achieve the goal of improving understandings about NOS [Nature of science] *among science teachers and students."* (Abd-El-Khalick, 2012, p. 2090)

The label 'reflective' has for Abd-El-Kahlick (2012) instructional implications in the form that there should be designed structured opportunities to help the students examine and reflect on their experiences. For Abd-El-Khalick (2012) the label 'explicit' has curricular implications and he believes that specific Nature of Science learning outcomes should explicitly be a part of the curriculum (Abd-El-Kahlick, 2012, p. 2091). It might not be necessary that the learning about the conceptual metaphors and conceptual blends are a part of the curricular. However the practical way to implement the explicit-reflective framework is out of this thesis range. Though having a more explicit approach to teaching the students about the conceptual blends and the conceptual metaphors that mathematics are formed by could maybe be beneficial for the students' mathematical understanding in general. The explicit-reflective framework has tentatively by Johansen and Kjeldsen (2018) been transferred to the teaching of mathematics. Thus it is possible to use the explicit-reflective framework in the context of teaching mathematics. Further more diagnostic teaching also uses discussion and reflections as a part of solving a cognitive conflict (Brekke, 2002, p. 19). Thus reflections are used in mathematical teaching before.

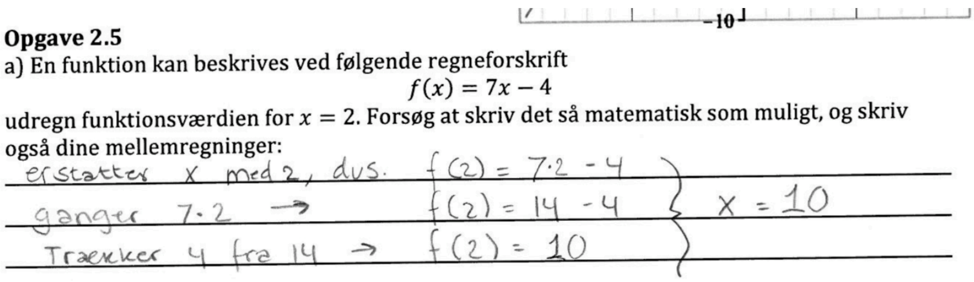
What I did not explore in the questionnaire was the students' ability to construct the axes in the coordinate system. In the questionnaire, the students were given axes that were scaled. However the axes are a part of the conceptual blend of the Cartesian plane, namely the number lines X and Y. There is evidence that the construction of axes requires a sophisticated set of skills and knowledge (Leinhardt et al., 1990). Thus exploring the students' ability to construct the axes would have been beneficial for the knowledge of the students understanding the conceptual blend of the Cartesian plane.

Understanding the concept of function, meaning understanding all aspect of the concept of function, also includes understanding the conceptual blend of the Cartesian plane. The Cartesian plane is a significant part of high school students' meeting with functions, since many of the functions they work with have graphical representations. Also, the results of testing hypothesis H1 indicate the importance of making sure that the students understand the conceptual blend of the Cartesian plane.

Hypothesis H2

The results of testing hypothesis H2 showed that there is a relation between being able to understand the $f(x)$-notation and being able to draw linear functions in the Cartesian plane by hand correct. The results of the questionnaire also showed that 73,1% of the students does not fully understand the $f(x)$-notation. Since hypothesis H2 were corroborated and it was found that 73,1% of the students does not fully understand the $f(x)$-notation. This indicates that there should be a greater focus on teaching and making sure that the students know what the $f(x)$-notation means and know how to use it correctly. As mentioned in the theory section, two of the experienced high school teachers also experienced that the students does not know how to use the $f(x)$-notation correct. Thus it is not unique that there are students in these two classes that does not fully understand the $f(x)$-notation.

When I coded the students' answers to task 2.5 a) I was persistent, such that if they did not use the $f(x)$-notation correct all the way through task 2.5 a) then I coded them as not using the $f(x)$-notation correct. For example Questionnaire 21 in example 14 used the notation correct almost all throughout and also writes the correct answer with correct notation, when writing $f(2) = 10$. However I coded Questionnaire 21's answer as not using the notation correct since he/she concludes that $x = 10$. It is clear from Questionnaire 21's answer that he/she did not believe that $f(2) = 10$ was the answer and he/she therefore concludes that $x = 10$.


Opgave 2.5
a) En funktion kan beskrives ved følgende regneforskrift
$$f(x) = 7x - 4$$
udregn funktionsværdien for $x = 2$. Forsøg at skriv det så matematisk som muligt, og skriv også dine mellemregninger:

erstatter x med 2, dvs. $f(2) = 7 \cdot 2 - 4$
ganger $7 \cdot 2$ → $f(2) = 14 - 4$
Traekker 4 fra 14 → $f(2) = 10$
$x = 10$

Example 14: Questionnaire 21's answer to task 2.5 a)

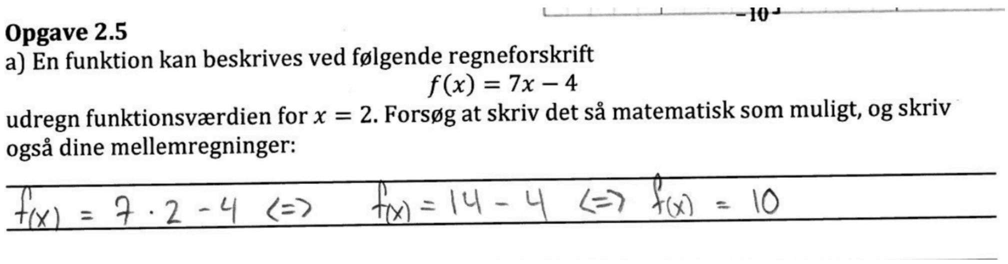
In general one might not evaluate the notation as persistently as I did in my coding. However the corroboration of hypothesis H2 raises a speculation on, that maybe we are doing the students a disservice by not correcting them when they do not use the $f(x)$-notation correct. In other disciplines, would a teacher look trough such mistakes? It is hard to really compare this type of task with another task in another discipline and sometimes there could be greater problems that needed focusing. Hence a teacher would not correct a student's misuse of notation. Then again, what defines what problems are greater than others? A misuse of notation might seem like a small error that does not need focus. However since hypothesis H2 was cor-

roborated it indicates that the misuse of notation is a significant error and therefore should be corrected.
Looking at example 14 the student uses the notation correct, hence one might think that it is not necessary to correct the student. However Questionnaire 21's answer could be an example of a student using the notation properly without even knowing why and what it means. My results showed that 73,1% of the students, does not understand the notation, but some of them still try to use it. Example 14 and the results of the study show that the students know that they are supposed to use the notation, but it seems that they do not really know how and why. This could be a result of the following observations that Shoenfeld (1988) has made:

"*Mathematics curricula have been chopped into small pieces, which focus on the mastery of algorithmic procedures as isolated skills. Most textbooks present "problems" that can be solved without thinking about the underlying mathematics, but blindly applying the procedures that have just been studied*." (Schoenfeld, 1988, p. 163)

If the students are just using procedures to answer the tasks and using the notation that goes with the procedures, then they will not learn the correct notation and why the notation is important. This means that they might not learn, that when they are asked to find the functional value, then the answer $x = 10$ is not correct. If the students are just using procedures to solve the problems and not focusing on the notation, then an answer such as $x = 10$ can appear.
The students learn to solve equations in primary school (Danish Ministry of children and education, 2019, pp. 27). In primary school they are asked to find x in equations, this could be an explanation to why Questionnaire 21 in example 14 concludes that $x = 10$.

In the case of the $f(x)$-notation, x is a metonymy that can be replaced by a number which will then, depending on f, have a corresponding value. Meaning that in the expression $f(x) = 7x - 4$, if x is replaced with 2 on the left hand side of the equality sign then it should also be replaced with 2 on the right hand side, or the other way around. Questionnaire 261 in example 15 has not replaced 2 with x on both sides of the equality sign. Hence Questionnaire 261 has not metonymically replaced 2 with x.



Opgave 2.5

a) En funktion kan beskrives ved følgende regneforskrift

$$f(x) = 7x - 4$$

udregn funktionsværdien for $x = 2$. Forsøg at skriv det så matematisk som muligt, og skriv også dine mellemregninger:

$f(x) = 7 \cdot 2 - 4 \Leftrightarrow f(x) = 14 - 4 \Leftrightarrow f(x) = 10$

Example 15: Questionnaire 261's answer to task 2.5 a)

In a study done by Sajka (2003) it was found that the student being interviewed treated the symbols $f(x)$, $f(y)$ and $f(b)$ as three different names of the same function. The student from Sajka's (2003) interview clearly did not understand the metonymic role that x, y and b have. The student in Sajka's (2003) interview saw the variables as being the indicator of the function's name. Thus for the student in Sajka's (2003) interview f did not determine the name of the function, instead the variables x, y and b determined the name of the function.

It could have been interesting in the questionnaire to further test the participating students' understanding of metonymies for example by asking if $f(x)$, $f(y)$ and $f(b)$ are different functions.

The students' ability to try to use the $f(x)$-notation but not using it correct could be a result of what Sajka (2003) says:

"What 'we usually write' and do in mathematics lessons is very important for the student. It is more important than thinking about the meaning of the symbol." (Sajka, 2003, p. 247)

If the students are only focused on what 'we usually write' rather than thinking about what the meaning of what we write is, then it could lead to an answer such as the one that Questionnaire 21 gave in example 14, namely $x = 10$. Meaning that, for Questionnaire 21, what the students are usually asked to find is x, hence the answer must be x equal to something, combined with that when working with functions they usually use $f(x)$-notation.

Focus on notation or symbolism in general is not new in the field of researching mathematical education. Different symbols are a part of mathematics. Even from the beginning of learning mathematics the children meet different symbols and notations, these are symbols such as

$+, -, =$ etc. Ginsburg (1997) found that many children do not understand what the mathematical symbols refer to. Since 73,1% of the students do not understand the $f(x)$-notation fully it indicates that the students find it difficult to fully understand the $f(x)$-notation. The $f(x)$-notation can be difficult to understand because one needs flexibility to understand the notation, since $f(x)$ both represent the name of the function and it also represents the value of the function f. The way to interpret the notation depends on the context, which can confuse a non-advanced student (Sajka, 2003, p. 230). If the students do not have the flexibility to understand the $f(x)$-notation and if they are not explicitly told that the meaning depends on the context, then how should they understand the notation completely? The way to interpret the notation in different contexts might be something that is just embedded in teachers' knowledge and something that they therefore are not aware of. When learning a mathematical concept the representation and the notation is a part of it. A study done by Johansen and Misfeldt (2018) found that for mathematicians thinking and working mathematical means interacting with the notation. Symbolic representations can suggest new moves or ideas for the mathematician and different stages of representation can open new possible venues for investigation (Johansen & Misfeldt, 2018). One could imagine that it would also suggest new moves for the students when working with different stages of representations. Further it is possible that working and thinking mathematical also means for the students, to interact with the notation. To fully understand a concept would then also mean to fully understand the different representations, being able to translate between them and to fully understand the notation of the concept.

If the students are not able to follow the $f(x)$-notation fully, this could indicate that there should be a greater focus on teaching and making sure that the students understands the fundamental metonymy of algebra. Moreover there should maybe be a greater focus on explaining the cognitive functions of metonymies. Just like the case with conceptual metaphors, a more explicit-reflective approach, to the role of metonymies could maybe be beneficial for the students' understanding of the $f(x)$-notation.

Quantitative results and statistically tests

My results showed that a 57,7% of the students was not able to draw all functions correct. The result that students find it difficult to draw graphs is not ground breaking. A study done by the National Assessment of Educational Progress revealed that only 18% of 17-year-old students were able to produce the correct graph corresponding to a linear equation. This was when the students were given a ruler and a piece of paper with labelled axes (Leinhardt et al., 1990, p. 35). The results of my questionnaire showed that even though more than one fifth of the students could not draw the graph of $f(x) = 2x - 5$ correct. Comparing my results with the results of National Assessment of Educational Progress it shows that the two high school classes that participated in my study are better at drawing linear functions. However results of National Assessment of Educational Progress are not recent findings. Thus it is possible, that the 17-year-old students today all together could have become better at mastering the skill of drawing linear functions.

As mentioned, since 11,5% of the students did not try to draw the constant function $f(x) = 3$ this could indicate, that they found this function the most difficult to draw. This result is supported by a study done by Markovits, Eylon and Bruckheimer (Leinhardt et al., 1990, pp. 35). They found that translation between graph and algebraic equation where constant functions were involved was exceptionally difficult. They believed that this translation is probably harder because of the 'missing' variables. Markovits, Eylon and Bruckheimer's (Leinhardt et al., 1990, pp. 35) findings were also supported by a study done by Zaslavsky (Leinhardt et al., 1990, pp. 35). My explorative work done with correlation also supports that the constant function stands out from the other functions, since the other three functions that the students had to draw correlated. However the constant function did not correlate with any of the other functions.
In the teaching guidelines for STX A/B/C (Danish Ministry of Education, 2019, p. 23) it is noted that as a part of the minimum requirement, high school students should be able to draw functions by hand. Hence a minimum of 21,1% of the students does not fulfil this requirement. It is not specified in the teaching guidelines which graphs they should be able to draw. However since linear function is a subject they learn about in primary school (Danish Ministry of children and education, 2019, pp. 27) it would make sense to expect that the students posses

this skill. When reading the primary schools goal of knowledge (da: vidensmål) it is not transparent what the students should learn about linear functions. It is expected that they learn to use graphs, but it does not say that they should be capable of drawing them by hand (Danish Ministry of children and education, 2019, pp. 27). Previous research has pointed out that high school teachers have a feeling, that first year high school students do not master skills they should have learned in primary school (Jensen, et al., 2015, pp. 51). The result, that the participating students are not able to draw all four functions correct could be a result of this. The result that a minimum of 21,1% of students does not fulfil the minimum requirements given by the Danish Ministry of Education (2019) is an important problem. These results should evoke a discussion on, that either the teachers in high schools need to have a larger focus on evolving these skills or there should be a better collaboration between high schools and primary schools, such that it is calcified what the students learn in primary school. Thus what they need to learn in high school.

Hypothesis H3

On the grounds of the empirical data and statistical testing I had to discard my hypothesis H3. Hence there is no relation between students mostly using CAS-tools and them not being able to draw linear functions by hand correct. There is an on going heavy discussion on what CAS-tools do to the students learning and the term black boxing is a part of this discussion. An explanation of CAS-induced learning difficulties is that central concepts in mathematics are being black boxed (Jankvist & Misfeldt, 2015). When students are using CAS-tools then both the calculations and procedures are performed by the tool, hence they are being black boxed. That makes the use of CAS-tools problematic since both calculations and procedures performed by the students, heavily influence the students' concept development (Jankvist et al. 2019).
It is some high school teachers' experience that it is hard to be idealistic to teach the students the mathematics in question, when they can get by just as good when learning commands for CAS-tools that can solve the task (Jensen et al., 2015, pp. 36). Hence the students learn to solve the tasks instrumental with the support of CAS-tools, without being challenged with the target knowledge (Blomhøj, 2016, chap. 5).

The arguments that CAS-tools should inhibit the students learning are a bit different from the results of testing my hypothesis H3. As mentioned for task 1.3 some of the students commented that often the teacher decides what tool they should use when drawing a graphs. This could be a reason for why my result is different from other research on CAS-tools. In task 1.3 it is not clear if the students should answer what tool they have used most frequently when they are drawing all types of graphs. The students could possibly have answered what tool they would prefer to use to draw graphs. It could also be that they have answered what tool they would use for drawing graphs like the ones in task 1.2 b), 1.2 c), 1.2 d) and 1.2 e). It could be interesting to ask the students what they have used most frequently when drawing graphs in primary and high school combined. This is not explicit in the question and could be altered. If the students have been schooled in only using CAS-tools this could mean that they do not know how to do it without CAS-tools.

What cannot be seen from testing the hypothesis H3 is whether or not the students that have answered that they normally use CAS-tools. They are also students that are very skilful in mathematics. This is important information since, as Jankvist et al. (2019) writes:

> *"Of course, if a student is familiar with the traditional techniques behind these CAS procedures, then everything in the garden is lovely."* (Jankvist et al., 2019, p. 68)

So if the students are able to draw the graphs in the questionnaire and they normally use CAS-tools to draw them then there are no problems. The result of testing hypothesis H3 indicated that these students' usages of CAS-tools to draw graphs are not problematic. Whether or not this is also the case for other types of mathematical tasks cannot be answered from the result of testing hypothesis H3.

Qualitative discussion

Graphical and algebraic argument

As answers to task 1.2 a) I expected that some of the students would give a graphical argument and that some would give an algebraic argument, which was also the outcome. 40,4% of the students gave a graphical argument. As mentioned, this way of arguing is not incorrect it just have limitations. The students will not be able to give a graphical argument if they get a polynomial of degree 17 or some other functions they have not met before, since they cannot know how it is evolved graphically. The students would then need to draw it, and it would still be a vague argument. If the same type of task as task 1.2 a) was asked, but the points in question was very close, maybe so close that one cannot see it clearly from a graph, then this way of giving an argument also have limitations. An example of this could be the points $(1, \sqrt{2})$ and $(1,1.4142135623)$. Further if the points in question were not a part of the quadratic coordinate system they were given, then they could not give a graphical argument.

Since it is a large part of the students that gives a graphical argument, which have some limitations, it could indicate that it would be beneficial to the students to engage in a discussion on arguments and limitations. More explicit, a discussions on the different ways of giving a mathematical argument, as some arguments seem more convincing and different ways of arguing have different limitations.

40,4% of the students gave, as mentioned, a graphical argument, this could indicate that they are more visual learners. This result could give some inspiration on how to approach new subjects in the classes. If a large group of the class are more visual learners then this could, if it is possible, be a better approach in the introduction of new subjects. The graphical argument could also indicate that these students are more familiar with the graphical procedures rather than the algebraic. Hence this could give the teacher knowledge on what processes are needed to have more focus on. In task 1.2 a) I used the word 'graph' and this could lead the students to use a graphical argument rather than an algebraic. However if the word 'graph' lead the students to use a graphical argument then it indicates, that the students do not completely understand the connection between a function's graph and its set of points (da: punktmængde).

Concept definition

I found five categories for the students' concept definition of the concept function. The largest category was 'A function is an algebraic expression'. An algebraic expression is an associated property that a function can have. As mentioned 32,7% of the students perceive functions in this manner. Perceiving a function as an algebraic expression is not a special case for these two high school classes that my results are based on. Four of the experienced high school teachers said that in their experience students have a narrow understanding of the concept of function. Part of this narrow understanding is that they view a function as an algebraic expression. Vinner (1983) found that some 10- and 11-graders had a concept image that a function is formula. A similar result was also true for some junior high school teachers and some college students (Vinner & Dreyfus, 1989). Having a narrow concept definition is problematic since it can be a potential conflict factor.

It is important to get an insight to what the students' concept image and concept definition are since it can give us a better understanding of the students and knowing why they act the way they do. Further an insight to the students' concept image can also suggest some improvements to the teaching (Vinner, 1983).
Vinner and Dreyfus (1989) did a categorization of concept images. As mentioned, the categories were; correspondence, dependence relation, rule, operation, formula and representation. Some of these categories are the same as what I found. Both the category formula and representations are one to one categories to 'A function is an algebraic expression' and 'A function is a graph', respectively.
As mentioned in the method section I found inspiration for the categorization from both Vinner (1983) and Vinner and Dreyfus (1989). However I found that especially because there is not a one to one correspondence from Danish to English a full adoption of Vinner and Dreyfus's (1989) categories would not be accurate. The wordings that the students use are the insight we get to what their concept image and concept definition are and some Danish words contain other meanings than a translation to English would. Also there is a difference between what is taught at what level in different countries and the way that concepts are being taught are also different from country to country. It is even different from teacher to teacher.

As mentioned in the theory section when a student is writing what he or she believes is a function it is not certain that they write their concept image. There can be aspect of the concept image that he or she is not fully aware of and therefore cannot formulate it in words. Also as Vinner (1983, p. 294) noted there is a chance that some of the students would write what they would think is the closest to the formal definition, even though they do not think of a function in that way. This could be the case for the students that were categorised as 'A function is a interrelation'. I tried to avoid this situation by writing in the questionnaire and telling them in the introduction that I wanted them to use their own words. Although, I cannot be sure if the students did not use their own words or if they did not describe their full concept image.

The students' concept image and concept definition is important since they can have a narrow definition of the concept. Hence excluding functions that are in fact functions. 32,7% of the students found that a function is an algebraic expression, which is a narrow definition of a function. This definition leaves no space for functions that does not have an algebraic expression as its representations. Some of the students in this category wrote, that a function is an equation. Hence for these students there is probably not a difference between equations and a calculation formula (da: regneforskrift). However according to the teaching guidelines for STX A/B/C (Danish Ministry of Education, 2019, p. 13) it should be clarified through activities in the teaching what the difference between a calculation formula and an equation is. Also the category 'the function is a graph' which 21,2% of the students belonged to, making it the second largest category. This concept definition is also narrow. A graph is a visual representation of the function. Further the students that define a function in this way might not believe that functions that do not have a graphical representation are functions at all.

The students had concept definitions that would cover more than one category; this is also a situation that has been seen before. In an interview with a student, Sajka (2003) found that the particular student's concept image was $Function = formular + graph$ which for my categories would have been part of two different categories. The reason for dividing the concept definitions into categories is to get an overview of the students' concept definitions. However if there is a need for an in depth analysis of a particular student's concept image then this division would not give the right picture.

Some of the students used the metaphor that a function is a machine, these students' answers were categorised as: ' The function is a tool'. Some of the teachers that I corresponded with also experienced that the students understand functions as a machine. Defining a function as a machine sets some implicit boundaries that one might not be aware of. If the function is perceived as a machine, then the function is discrete (da: diskret) since a machine does not construct indiscrete objects. Further a machine cannot 'produce' in the opposite direction, hence the function cannot work in the opposite direction, which implicitly indicates that the invers function does not exist.

As mentioned in the theory section, the students rely a great deal on their concept image. Using a metaphor such as 'the function is a machine' and what indirect implications it has could affect the students understanding of the concept. Since the students rely on their concept image so greatly and they have a hard time modifying it, then a narrow concept definition could prevent them to fully understand the concept in question.

Final remarks

My study is only at the size of 52 students and they are from a specific high school and have a specific teacher. This means that my study cannot be generalized to every high school and high school student. However it can give some indications of some tendencies that could also be in other high schools classes. Further the findings do evoke some new questions and some tendencies that could lead to further study. The importance of understanding the conceptual metaphors, conceptual blends and metonymy that is a part of mathematics, could be an important new focus. In this way my study could work as a pilot study where one could make some adjustments of the questionnaire that the students was given. Two things that I personally would like to adjust is the formulation of task 1.3. Where I would have wished that it was more explicit that I wanted to know what method they have used the most, meaning what they are being schooled in. To hopefully avoid having to exclude a student's answer I would like to alter the questionnaire such that on page one of the questionnaire I would have liked to write 'FLIP!' so I could be more certain that all the students saw that page.

For further study it could be interesting to explore what the effects of an explicit-reflective approach to leaning high school students the conceptual metaphors, conceptual blends and metonymy could have on the students' understanding of various aspects of the concept of function or mathematics in general.

Conclusion

Many high school students find it difficult to work with functions. Especially, it is difficult for them to translate between the different representations a function may have. This is supported by several studies. The results of the pilot test showed that especially the translation from a function's algebraic expression to its graphical representation is difficult for the students. This is also the result from the final questionnaire. 57,7% of the students failed at drawing all four linear graphs correct, meaning that they fail to live up to the minimum requirement for high school students stated by the Danish Ministry of Education. Hence there is a need for a greater focus on what the students have learned in primary school and what they need to learn in high school.

Through the work of this thesis I have found that students have a narrow perception of the concept of function. The results of my study suggest the some Danish high school students understand functions as: algebraic expressions, graphs, tools, interrelations or models. The result that students have a narrow perception of the concept of function is also supported by the literature.

The qualitative results of the questionnaire showed that students have different approaches to giving a mathematical argument. Different ways of giving a mathematical argument have different limitations. Hence it could be beneficial for the students to engage in a discussion of different arguments and their limitations.

Testing hypothesis H1 resulted in a corroboration of the hypothesis. Hence is seems from my results that there is a relation between understanding the conceptual blend of the Cartesian plane and being able to draw linear functions in the Cartesian plane by hand correct.

Testing hypothesis H2 resulted in a corroboration of the hypothesis. Hence is seems from my results that there is relation between being able to understand and use the $f(x)$-notation correct and being able to draw linear functions in the Cartesian plane by hand correct.

Testing hypothesis H3 resulted in it being disproved. Hence in my study there is no relation between students mostly using CAS-tools and them not being able to draw linear functions by hand correct. The result of testing hypothesis H3 is not supported by the literature since CAS-tools are often blamed for the students' difficulties.

Hypothesis H4 was falsified. Meaning that in my study there is no relation between being able to tell what the algebraic expression for a linear function is, based on a graphical representation of the function and to being able to draw linear functions by hand correct.
On the basis of the empirical observations from my study I will suggest that there should be a greater focus on learning the students conceptual metaphors, conceptual blends, notation and metonymy. I would suggest a more explicit approach could be beneficial. Further teaching more explicitly could be achieved by using an explicit-reflective framework.

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Appendix A

Detektionstest 1

1. Giv et andet udtryk for $1\frac{1}{2}$:
2. Giv et andet udtryk for $3\frac{2}{3}$:
3. Betyder a^2 det samme som $2a$? Ja: _ Nej: _
4. Er $3a = a3$? Ja: _ Nej: _
5. Betyder $4b$ det samme som $4+b$? Ja: _ Nej: _
6. Hvad er $\frac{a}{b}\cdot\frac{b}{a}$? (Hvor hverken a eller b er 0.)
7. Er tallet $-a$ positivt eller negativt, eller kan det ikke afgøres? Positivt: _ Negativt: _
   Kan ikke afgøres: _
8. Hvor meget er 110 % af 85?
9. En vare koster inklusive 25 % moms 150 kr. Hvor stor en procentdel af de 150 kr. udgør momsen?
10. Hvad er $\frac{3}{\sqrt{2}}\cdot\frac{\sqrt{2}}{3}$?
11. Hvilket tal er størst: $\frac{5}{9}$ eller 0,6?
12. Hvad er $\frac{c/d}{c}$?
13. Hvilket tal er størst: $\frac{13}{3}$ eller $\frac{13}{4}$?
14. Hvad er $0\cdot x$?
15. Hvad er $0+x$?
16. Hvad er $\frac{a^5}{a^5}$? (Hvor a ikke er 0.)
17. Findes der nogen værdier af a, således at $a^2 = 2a$? Ja: _ Nej: _
18. Findes der nogen værdier af b, således at $4b = 4+b$? Ja: _ Nej: _
19. Om tallene k og s ved vi, at $\frac{4}{5}k = s$. Isolér k:
20. Hvad er løsningen/løsningerne til ligningen $3x - x = 2x$?
21. Hvad kan du sige om de to tal *c* og *d* når: $7c + 22 = 109$ og $7d + 22 = 109$?
22. Hvad kan du med ord sige om sammenhængen mellem x og y når $y = x + 5$?
23. Er $f(x) = x^2 - x$ og $g(x) = x(x-1)$ lig hinanden eller er de forskellige?
    Lig hinanden: _ Forskellige: _
24. Hvis $f(x) = 5$, er det så en funktion? Ja: _ Nej: _
25. Er $x = 0$ en løsning til ligningen $3x - x = 2x$? Ja: _ Nej: _
26. Afrund $148,72 + 51,351$ til et helt tal:
27. Hvilke af følgende brøker er lig hinanden: $\frac{1}{4}, \frac{4}{16}, \frac{4}{12}, \frac{2}{8}$
28. Opskriv $\frac{3}{20}$ som decimaltal:

29. Hvilken er sødest: En blanding af 2 teskefulde sukker og 6 teskefulde citronsaft eller en blanding af 8 teskefulde sukker og 24 teskefulde citronsaft?
Den første: _ Den anden: _ Lige søde: _
30. Hvilket tal er størst: $0,32$ eller $0,315$?
31. Hvis P er antal professorer og S er antal studerende, hvad udtrykker følgende ligning da om sammenhængen mellem antallet af professorer og antallet af studerende: $6 \cdot P = S$?
32. Hvornår er de to udtryk $a+b-c$ og $a-b+c$ lig hinanden?
33. Løs ligningen: $(x-3)(x-5)=0$.
34. Bestem n når: $4+n-2+5=11+3+5$.
35. Løs ligningen: $3x+20=x+64$.
36. Løs ligningen: $-6x=24$.
37. For hvilke x gælder: $38x+72=38x$?
38. Er $\frac{1}{10}$ og $0,1$ det samme? Ja: _ Nej: _
39. Er $\frac{1}{4}$ og $0,4$ det samme? Ja: _ Nej: _
40. Ligger $\frac{9}{11}$ mellem 1,0 og 1,2? Ja: _ Nej: _
41. Hvor mange decimaltal er der mellem $\frac{2}{7}$ og $\frac{3}{7}$?
42. Hvor mange brøker er der mellem 0,65 og 0,66?
43. Opskriv $\frac{1/2}{1/4}$ og $\frac{2,1}{4,1}$ som sædvanlige brøker:
44. Hvad er arealet af et rektangel med siderne s og $\frac{1}{s}$ (med $s>0$)? Hvad er omkredsen?
45. I et koordinatsystem er punkterne med koordinaterne $(2,-7)$ og $(7,-2)$ endepunkterne af et linjestykke. Hvad er koordinaterne for linjestykkets midtpunkt?
46. I et koordinatsystem går der en ret linje gennem punkterne $(0,0)$ og $(1,3)$. Opskriv en forskrift for denne linje.

47. Hvilke af figurerne A, B, C, D, E og F forestiller funktioner?

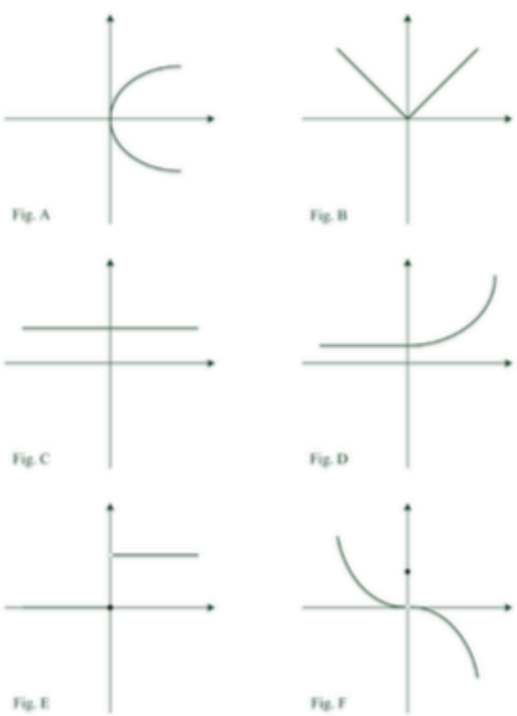


Fig. A: En funktion: _ Ikke en funktion: _
Fig. B: En funktion: _ Ikke en funktion: _
Fig. C: En funktion: _ Ikke en funktion: _
Fig. D: En funktion: _ Ikke en funktion: _
Fig. E: En funktion: _ Ikke en funktion: _
Fig. F: En funktion: _ Ikke en funktion: _

Hvis du har svaret 'Ikke en funktion' til en eller flere af figurerne, forklar da hvorfor der ikke er tale om en funktion.

48. Hvilke hele tal er der mellem -2 og $3,5$?

49. Findes der et største tal a som opfylder: $2 \leq a < 4$? Ja: _ Nej: _

50. Hvis $a = b$ er så $b = a$? Ja: _ Nej: _

51. Hvis $a < b$ er så $b < a$? Ja: _ Nej: _

52. Hvis $a < b$ er så $b > a$? Ja: _ Nej: _

53. Er $\frac{a-1}{b+1} = \frac{a}{b} - 1$? Ja: _ Nej: _

54. Hvad er $\frac{a \cdot (-1)}{b \cdot (-1)}$?

55. Er $\frac{a+3}{a+4} = \frac{3}{4}$? Ja: _ Nej: _

56. Er $\frac{a-1}{b-1} = \frac{a}{b}$? Ja: _ Nej: _

57. Hvad er $\frac{3x-1}{-3x+1}$?

Detektionstest 3

14 Spørgsmål fra Professoren

Her er 14 spørgsmål fra professoren. Det er meget vigtigt for vores undersøgelse, at du svarer på alle spørgsmålene, også hvis der skulle være nogle du ikke synes du kan gøre noget ved. På forhånd stor tak for hjælpen!

Spørgsmål 1

Hans kan gå fra Roskilde Station til Roskilde Domkirke på 6 minutter. Grethe skal bruge 8 minutter. Hvor lang tid tager det, hvis de følges ad? Begrund dit svar.

Spørgsmål 2

Du er ved at lave din egen dressing til en salat. Her er en opskrift på 100 milliliter (ml) dressing.

Salatolie	60 ml
Eddike	30 ml
Soyasauce	10 ml

Hvor mange ml salatolie skal du bruge for at lave 150 ml af denne dressing? Begrund dit svar.

Spørgsmål 3

Søren vil sætte sine sparepenge i banken. Banken ***Tæsk*** tilbyder 0,25 % i rente hvert kvartal, hvis han lader pengene stå i 2 år. Banken ***Bank*** tilbyder 1 % i årlig rente, hvis han lader pengene stå i 2 år. Er det ligegyldigt hvilken bank Søren vælger, eller har han fordel af at vælge den ene frem for den anden? Begrund dit svar.

Spørgsmål 4

På en bestemt skole er der 6 gange så mange elever som lærere. Opskriv en formel der udtrykker sammenhængen mellem antallet, E, af elever og antallet, L, af lærere. Begrund dit svar.

Spørgsmål 5
Se på billedet ovenfor. Hvor høj er den forreste bygning cirka? Begrund dit svar.

Spørgsmål 6
Et oliefelt indeholder 100 millioner tønder olie. Ali siger, at hvis man hvert år udvinder 1 million tønder olie, slipper olien op efter 100 år. Aya siger, at hvis man hvert år udvinder 1 % af den olie, der er tilbage, slipper olien aldrig op. Hvem har ret og hvorfor?

Spørgsmål 7
På en ret stejl bakke i Athen findes en vej op, der er ca. 4 km lang. Rikke, som er i god form, kan bestige bakken med en gennemsnitsfart på 3 km i timen, og gå ned igen med den dobbelte fart. Hvad er Rikkes gennemsnitsfart for den samlede tur? Begrund dit svar.

Spørgsmål 8

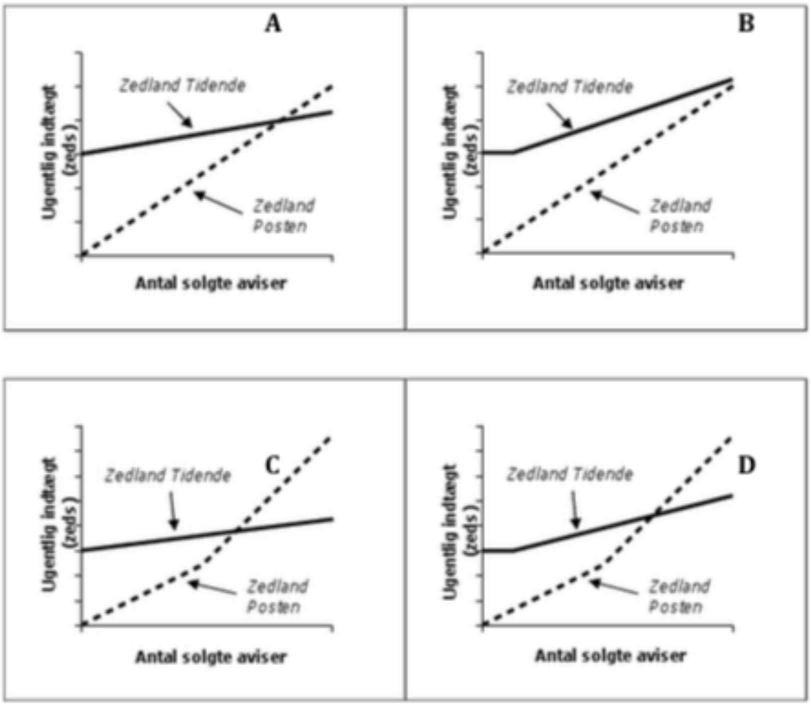
Et pizzeria serverer to runde frokostpizzaer af samme slags og tykkelse, men i forskellig størrelse. Den mindste har en diameter på 30 cm og koster 30 kr. Den største har en diameter på 40 cm og koster 40 kr. Hvilken pizza giver mest for pengene? Vis, hvordan du kom frem til dit resultat.

Spørgsmål 9

I landet *Zedland* er der to aviser, der søger sælgere. Annoncerne nedenfor viser, hvordan de betaler deres sælgere.

ZEDLAND POSTEN	**ZEDLAND TIDENDE**
BRUG FOR EKSTRA PENGE? SÆLG VORES AVISER Du vil blive betalt: 0,20 zeds pr. avis for de første 240 aviser, du sælger på en uge, plus 0,40 zeds for hver ekstra avis, du sælger.	*GODT BETALT JOB, DER IKKE TAGER LANG TID!* Sælg *Zedland Tidende* og tjen 60 zeds om ugen, plus ekstra 0,05 zeds pr. avis du sælger.

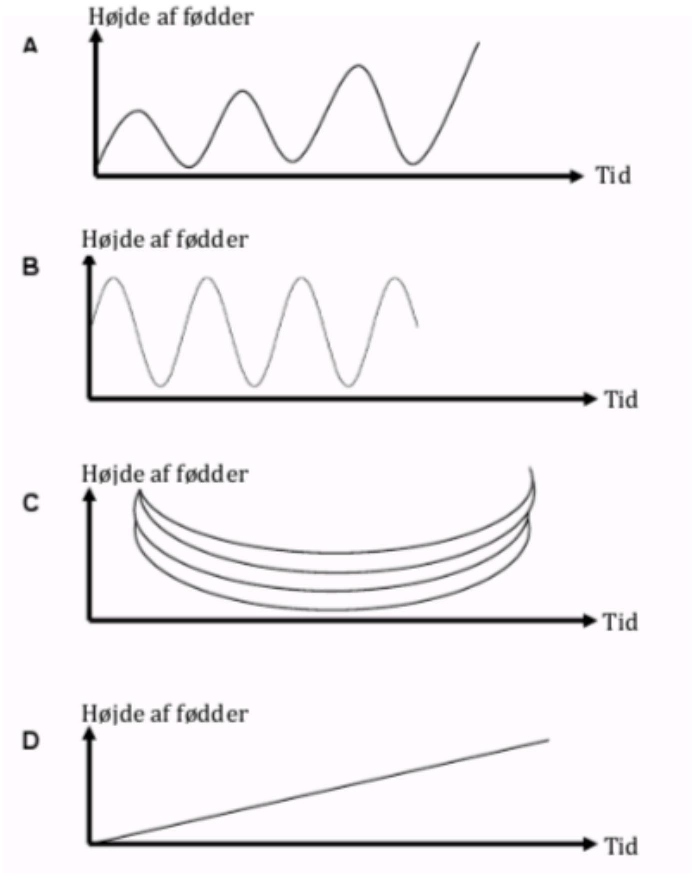
John beslutter sig for at søge en stilling som avissælger. Han skal vælge mellem *Zedland Posten* og *Zedland Tidende*. Hvilken af de følgende grafer (A, B, C eller D) er en korrekt fremstilling af, hvordan de to aviser betaler deres sælgere? Begrund dit svar.



Spørgsmål 10

Mohammed sidder på en gynge. Han begynder at gynge. Han forsøger at komme så højt op som muligt.

Hvilken af følgende grafer (A, B, C eller D) afbilder bedst højden af hans fødder over jorden mens han gynger? Begrund dit svar.



Spørgsmål 11

En træterning med alle sider lig 2 cm vejer 4,8 gram. Hvad vejer en træterning, hvor alle siderne er 4 cm? Begrund dit svar.

Spørgsmål 12

Af helbredsmæssige årsager bør folk begrænse deres anstrengelser, fx under udøvelse af sport, for ikke at overskride en bestemt hjertefrekvens (antal hjerteslag pr. minut). Før i tiden var sammenhængen mellem en persons anbefalede maksimale hjertefrekvens og personens alder (målt i år) beskrevet ved følgende formel (hvor der ses væk fra enheder):

Anbefalet maksimale hjertefrekvens = 220 – alder.

Nyere forskning viste at denne formel burde ændres en smule. Den nye formel er som følger:

Anbefalet maksimale hjertefrekvens = 208 – (0,7 × alder).

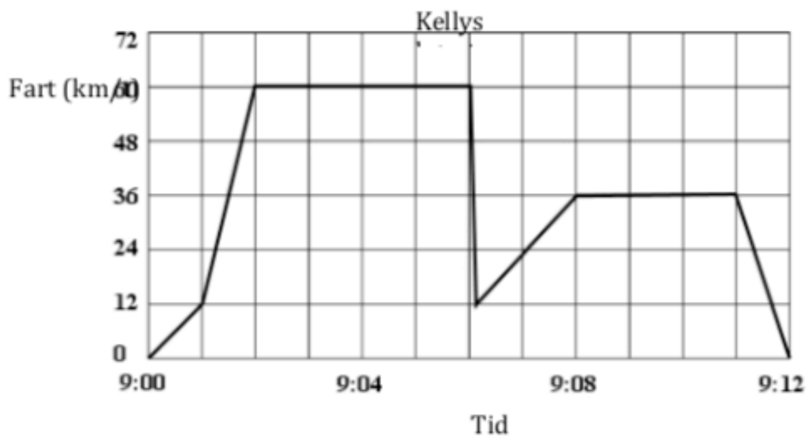
En avisartikel skrev: "Et resultat af at benytte den nye formel i stedet for den gamle er, at det anbefalede maksimale antal hjerteslag per minut for yngre mennesker nedsættes en smule, mens det for ældre mennesker forhøjes en smule."

Avisens påstand er korrekt. Fra hvilken alder og frem forhøjes den anbefalede maksimale hjertefrekvens ved overgang til den nye formel? Begrund dit svar.

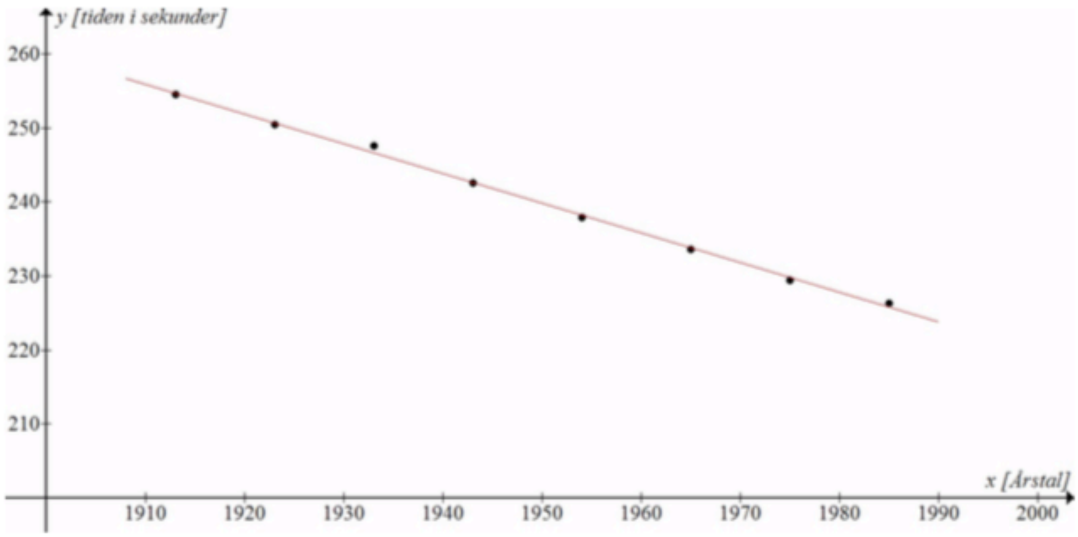
Spørgsmål 13

Kelly kørte en tur i sin bil. Pludselig løb en kat ud foran bilen. Kelly bremsede hårdt op og undgik at ramme katten. Lettere rystet besluttede Kelly sig for at køre hjem igen. Diagrammet nedenfor viser en forenklet gengivelse af bilens fart i løbet af turen.

Hvad var klokken, da Kelly bremsede hårdt op for at undgå at ramme katten? Begrund dit svar.



Opgave 14
På nedenstående figur er udvalgte verdensrekordtider for at løbe en mil indtegnet for perioden 1913-1985.
Man bemærker, at punkterne tilnærmelsesvis ligger på en ret linje, og derfor har man lavet en lineær model, der beskriver verdensrekordtiden (y) som funktion af årstallet (x).
Benyt figuren til at vurdere, hvor god modellen er, og indenfor hvilken periode modellen kan benyttes. Begrund dit svar



Appendix B

Pilot questionnaire

Navn:____ **Klasse:**__________ **Dato:**_________

Test om funktioner

Opgave 1

Giv et eller flere eksempler på funktioner:

____

____

____

____

Beskriv med dine egne ord hvad en funktion er:

____

____

____

____

____

Beskriv med dine egne ord hvad relationen mellem en funktion og en graf er:

____

____

____

____

____

Navn:______________________________________ **Klasse:**__________ **Dato:**__________

Opgave 2
På en lille ø ude i Stillehavet er der 4 gange så mange edderkopper som der er sommerfugle. Opskriv en formel der udtrykker sammenhængen mellem antallet, E, af edderkopper og antallet, S, af sommerfugle.

____

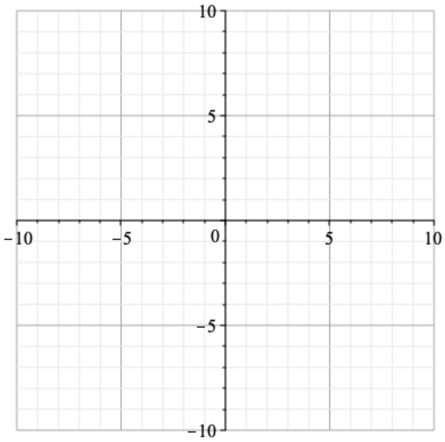
Opgave 3
Tabellen viser sammenhængen mellem x og $f(x)$

x	1	3	4	7	9
$f(x)$	5	11	14	23	29

Hvilke funktionsudtryk passer til tabellen? (sæt kryds)

☐ $f(x) = x + 4$
☐ $f(x) - 3x - 2 = 0$
☐ $f(x) = 5x$
☐ $f(x) = 3x + 2$

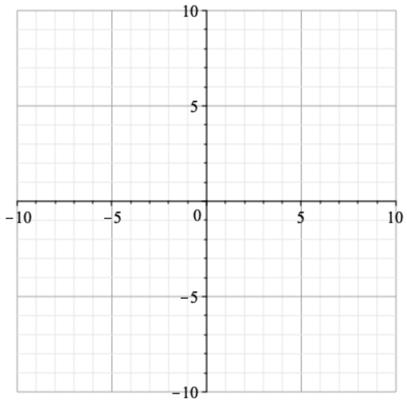
Opgave 4
Afsæt følgende punkter i koordinatsystemet:
$A(4,1)$, $B(-3,-8)$ og $C(-6,7)$



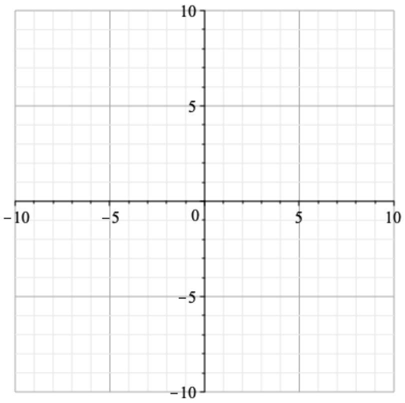
Navn:____ **Klasse:**__________ **Dato:**__________

Opgave 5
Her er fire funktionsudtryk, tegn deres grafer:

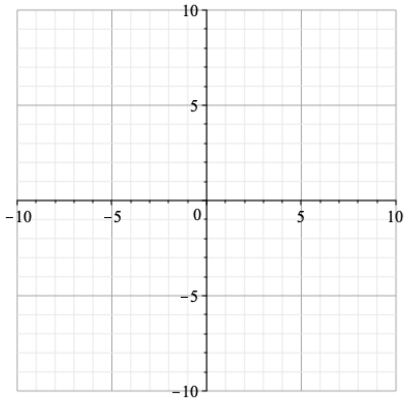
a) $f(x) = 3$



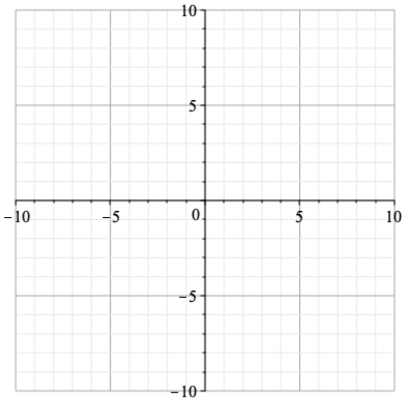
b) $2x - f(x) = -3$



c) $f(x) = 3x$



d) $f(x) = -x + 2$



e) Forklar med dine egne ord hvordan grafen for $f(x) = -x + 2$ kan tegnes i et koordinatsystem.

____

____

____

f) Forklar med dine egne ord hvorfor punktet $(5, -3)$ ligger på grafen for $f(x) = -x + 2$ og hvorfor punktet $(8,2)$ ikke ligger på grafen for $f(x) = -x + 2$.

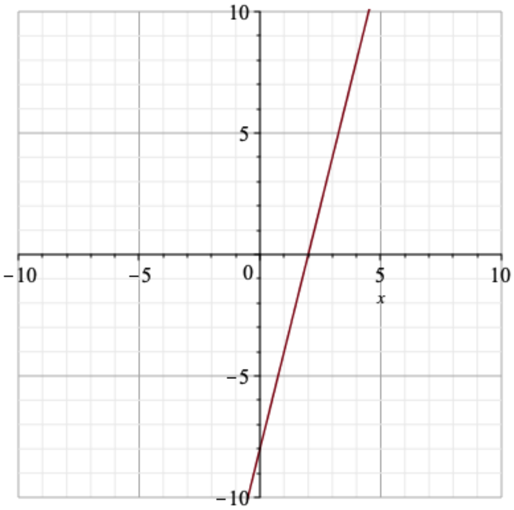
____

____

____

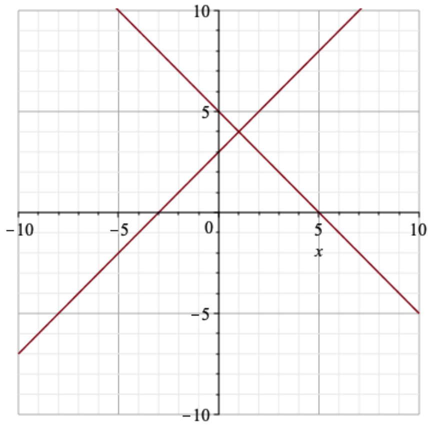
Navn:____ **Klasse:**__________ **Dato:**__________

Opgave 6
Hvilken regneforskrift passer til grafen? (sæt kryds)



☐ $f(x) = 4x - 8$
☐ $f(x) = 4x + 2$
☐ $f(x) = x - 8$
☐ $f(x) = 8x + 2$

Opgave 7
Se på tegningen, hvilke af følgende udsagn er korrekte: (sæt kryds)



☐ Funktionerne skærer hinanden i (1,4)

☐ Funktionerne har punktet (1,4) tilfælles

☐ Funktionerne skærer hinanden i (4,1)

☐ Funktionerne har punktet (4,1) tilfælles

Navn:______________________________________ **Klasse:**__________ **Dato:**__________

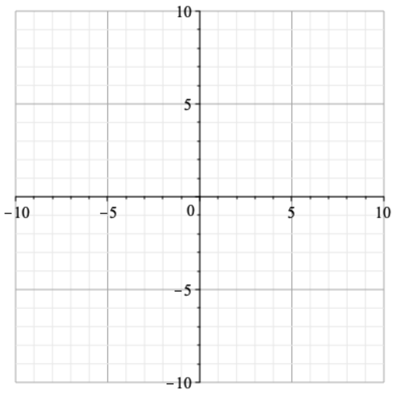
Opgave 8
Vi kender følgende funktionsværdier for en funktion $f(x)$:

$$f(1) = 2, \quad f(2) = 3, \quad f(3) = 4, \quad f(5) = 6$$

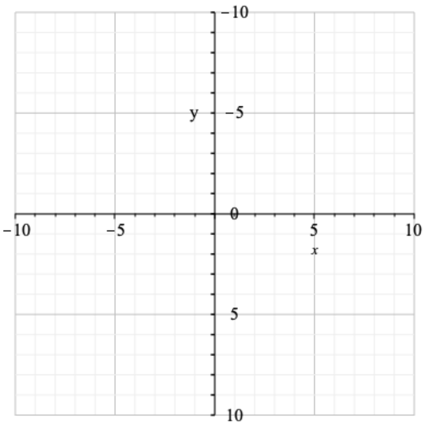
Hvis vi kræver at $f(x)$ skal være en funktion, hvilke af følgende funktionsværdier er det så muligt at $f(x)$ tager, udover dem vi kender: (sæt kryds)

☐ $f(5) = 3$
☐ $f(4) = 10$
☐ $f(4) = 3$
☐ $f(4) = 5$

Opgave 9
a) Tegn i koordinat systemet en mulig graf for en funktion som er voksende i intervallet [1;6]



b) Nu har vi vendt y-aksen om (den lodrette akse), dvs. -10 er i toppen af koordinatsystemet og 10 er i bunden af koordinatsystemet. Tegn nu en mulig graf for en funktion som er voksende i intervallet [1;6]



Appendix C

Final questionnaire

Navn:____ **Klasse:__________** **Dato:__________**

Undersøgelse af funktionsforståelse

Del 1

Opgave 1.1

Giv et eller flere eksempler på funktioner:

____

____

____

____

Beskriv med dine egne ord hvad en funktion er:

____

____

____

____

____

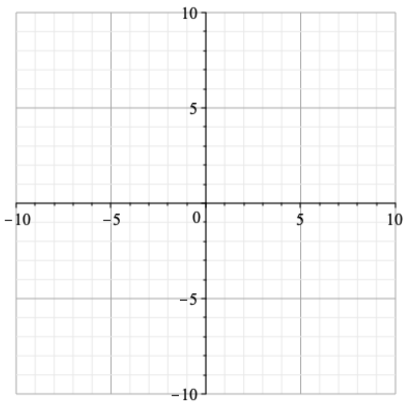
Navn:______________________________ **Klasse:**________ **Dato:**________

Opgave 1.2

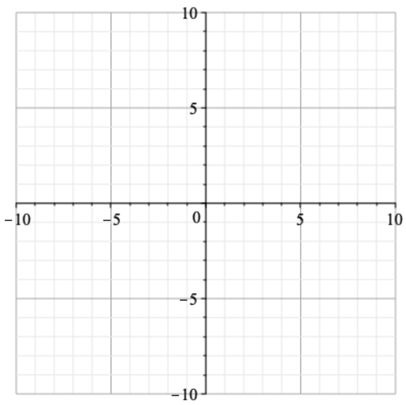
a) Forklar med dine egne ord hvorfor punktet (3,1) ligger på grafen for $f(x) = 2x - 5$ og hvorfor punktet (7,4) ikke ligger på grafen for $f(x) = 2x - 5$.

Tegn graferne for de fire regneforskrifter:

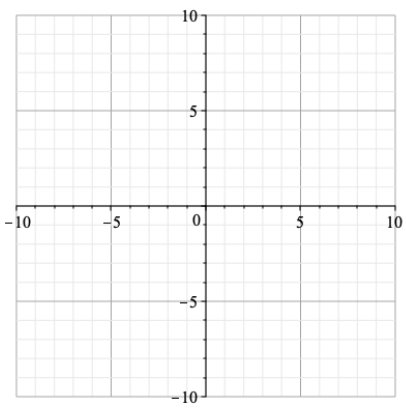
b) $f(x) = 2x - 5$



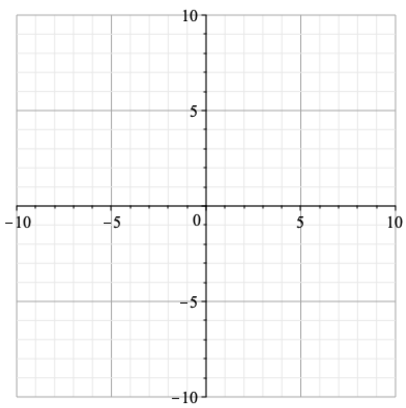
c) $f(x) = -x + 6$



d) $f(x) = 4x$



e) $f(x) = 3$



Opgave 1.3

Når du normalt bliver bedt om at tegne grafen for en funktion, hvordan gør du så? (sæt kryds)

☐ Tegne den med papir og blyant

☐ Få Ti-Nspire/ Geo-Gebra /Maple/ Wolframalpha/ lommeregner til at tegne den

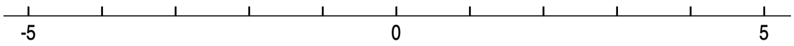
☐ Andet______________________(Hvis du sætter kryds ved 'andet', så skriv hvordan)

Navn:____ **Klasse:**__________ **Dato:**__________

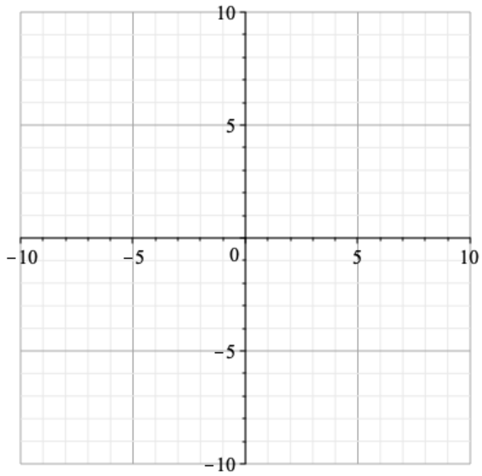
Undersøgelse af funktionsforståelse

Del 2

Opgave 2.1
Afsæt tallene: -2 og 4,3 på tallinjen

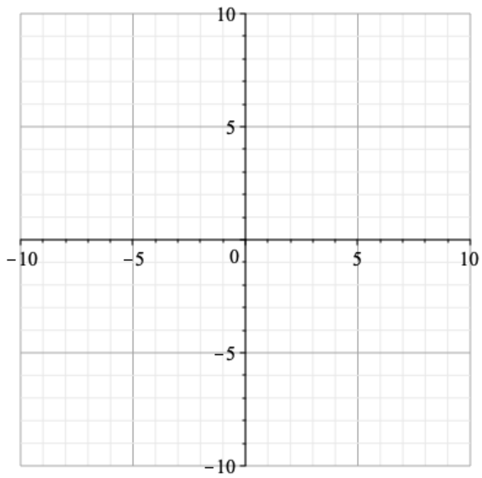


Opgave 2.2
Afsæt følgende i koordinatsystemet:
$A(4,1)$, $B(-3,-8)$ og $C(-6,7)$



Opgave 2.3
Afsæt følgende i koordinatsystemet:

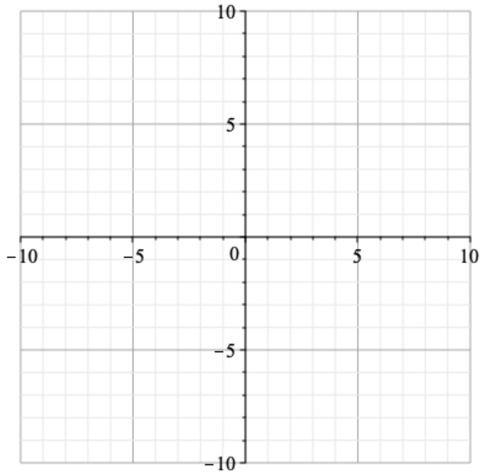
x	-6	-4	1	6
$f(x)$	-5	2	4	0



Navn:____ **Klasse:**__________ **Dato:**__________

Opgave 2.4

Afsæt følgende i koordinatsystemet:
$f(-6) = -4$, $f(-2) = 2$, $f(0) = 5$ og $f(2) = 8$



Opgave 2.5

a) En funktion kan beskrives ved følgende regneforskrift

$$f(x) = 7x - 4$$

udregn funktionsværdien for $x = 2$. Forsøg at skriv det så matematisk som muligt, og skriv også dine mellemregninger:

____

____

____

b) Når du normalt skal udregne en funktionsværdi, hvilken metode benytter du dig så mest af? (sæt kryds)

☐ Udregner det med lommeregner/Ti-Nspire/ Geo-Gebra /Maple/ Wolframalpha

☐ Udregner det i hovedet

☐ Udregner det i hovedet men benytter papir og blyant som støtte, til fx mellemregninger

☐ Andet____
(Hvis du sætter kryds ved 'andet', så skriv hvordan)

Opgave 2.6

Følgende tabel viser sammenhængen mellem x og $f(x)$

x	1	3	4	7	9
$f(x)$	5	11	14	23	29

Hvilke regneforskrifter passer til tabellen? (sæt kryds)

☐ $f(x) = x + 4$
☐ $f(x) - 3x - 2 = 0$
☐ $f(x) = 5x$
☐ $f(x) = 3x + 2$

Navn:____ **Klasse:**__________ **Dato:**__________

Opgave 2.7

Følgende x-værdier og tilhørende funktionsværdier viser sammenhængen mellem x og $f(x)$

$$f(1) = 1, \quad f(3) = 9, \quad f(4) = 13, \quad f(7) = 25, \quad f(9) = 33$$

Hvilke regneforskrifter passer til x-værdierne og de tilhørende funktionsværdier? (sæt kryds)

☐ $f(x) - 4x + 3 = 0$

☐ $f(x) = x$

☐ $f(x) = 4x - 3$

☐ $f(x) = x + 6$

Opgave 2.8

Følgende koordinatpar viser sammenhængen mellem x og $f(x)$:

$$(1,3), \quad (3,7), \quad (4,9), \quad (7,15), \quad (9,19)$$

Hvilke regneforskrifter passer til koordinatparrene? (sæt kryds)

☐ $f(x) = 2x + 1$

☐ $f(x) = x + 2$

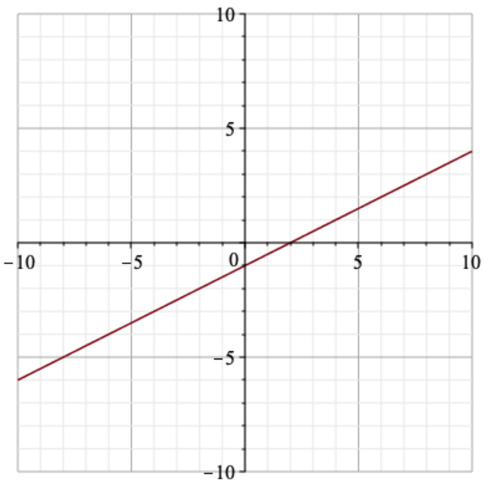
☐ $f(x) = 3x$

☐ $f(x) - 2x - 1 = 0$

Opgave 2.9

Til højre er givet en graf for en funktion.
Brug grafen til at udfylde hele tabellen.

x		-2	2	6	
$f(x)$	-3				4

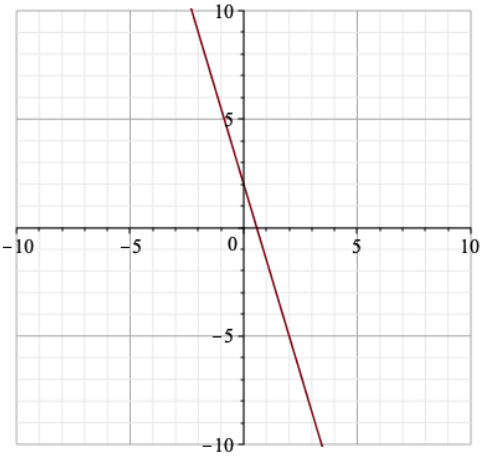


Navn:____ **Klasse:**__________ **Dato:**__________

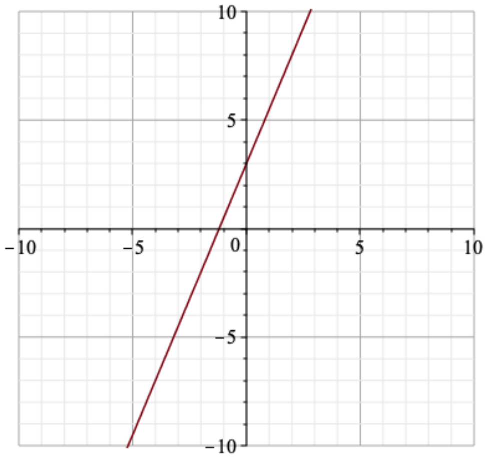
Opgave 2.10
Vi ved om en funktion af punktet $f(3) = -7$ ligger på grafen for funktionen. Hvilken af følgende grafer gælder dette for? (sæt kryds)

☐ Graf A ☐ Graf B ☐ Graf C ☐ Graf D

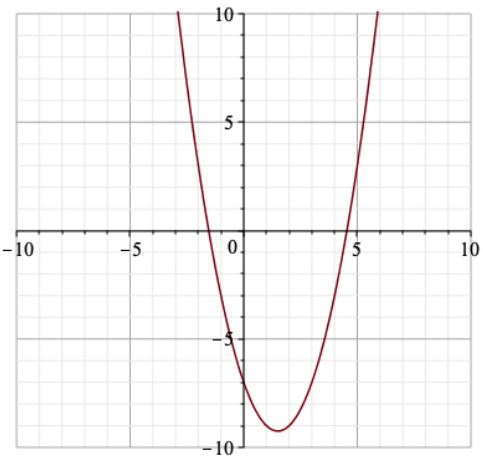
Graf A



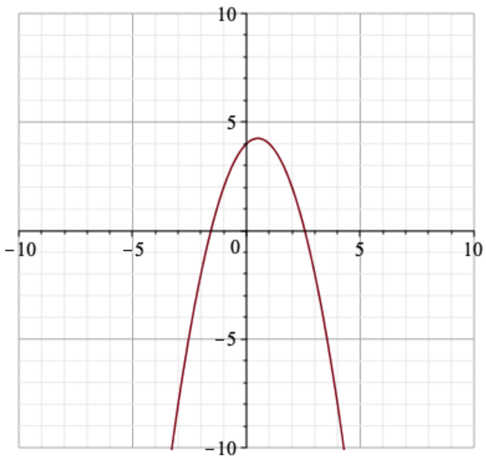
Graf B



Graf C



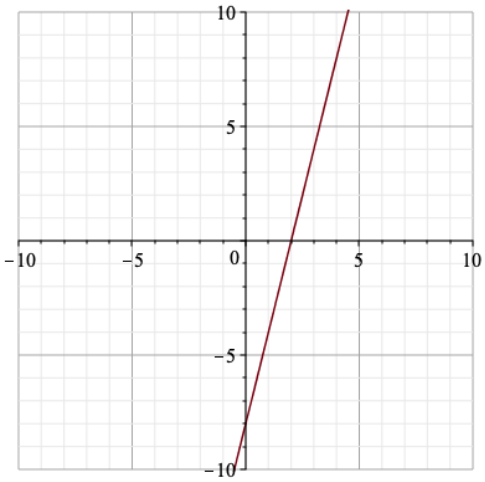
Graf D



Navn:____ **Klasse:**____________ **Dato:**___________

Opgave 2.11

Hvilken regneforskrift passer til grafen? (sæt kryds)



- ☐ $f(x) = 4x - 8$
- ☐ $f(x) = 4x + 2$
- ☐ $f(x) = 2x - 8$
- ☐ $f(x) = 8x + 2$

Opgave 2.12

En funktion er givet på formen $f(x) = ax + b$. Beskriv med egne ord hvad a og b betyder for funktionen. Brug evt. funktionen $f(x) = 2x - 5$ til at forklare det ud fra.

____

____

____

____

Appendix D

Introduction to the questionnaire

Hej jeg hedder Nicole og er specialestuderende hos Københavns universitet.
Jeg har jo fået lov til lige at låne jer og i skal besvare en lille undersøgelse. Det er ikke en undersøgelse som kommer til at tælle med i jeres karaktere, det er bare en undersøgelse for min skyld, til mit speciale. Så jeg vil gerne bede jer om at være meget åbne, og besvare den så godt i med jeres egne ord. Undersøgelse består af to dele. Første del der skal i lave opgaverne i kronologisk rækkefølge. Jeg vil gerne have at i laver opgave 1.1 først, og når i har besvaret den, så gå videre til de næste opgaver, og gå ikke tilbage og ret i jeres svar. Når i har lavet første del så lig den frem på bordet, så vil jeg samle dem ind løbende.
Hvis i har brug for at lave mellemregninger eller lignende, så bare gør det på det udleveret papir (ude i siden), i må ikke bruge Ti-Nspire eller lommeregner, eller andre hjælpemidler.
Vær opmærksom på at der er opgaver på begge sider af papiret.
Er der nogen spørgsmål?

Appendix E

Declaration of informed consent template

KØBENHAVNS UNIVERSITET
DET NATUR- OG BIOVIDENSKABELIGE FAKULTET

SAMTYKKEERKLÆRING

Vedrørende deltagelse i undersøgelse af funktionsbegrebet

I forbindelse med undervisningen vil specialestuderende Nicole Jonasen fra Københavns Universitet give eleverne i klassen en test, der kan sige noget om elevernes forståelse af centrale matematiske begreber.

Undersøgelsen vil indgå som en del i Nicole Jonasens speciale og evt. i efterfølgende forskningspublikationer. I specialet og evt. forskningspublikationer vil testresultaterne blive anonymiseret, og kan/vil derfor ikke føres tilbage til specifikke elever.

Jeg bekræfter, at jeg er blevet underrettet om formålet med undersøgelsen og at Nicole Jonasen har besvaret eventuelle spørgsmål, som jeg havde om, hvordan min deltagelse bruges.

Jeg er gjort bekendt med, at min deltagelse er frivilligt. Jeg har mulighed for at trække min deltagelse tilbage, hvorefter al indsamlet materiale om mig destrueres.

Sæt kryds:
☐ Jeg giver hermed min samtykke til, at jeg deltager i undersøgelsen ved at besvare en kort test om funktioner.

☐ Jeg giver hermed min samtykke til, at jeg evt. kan opringes til uddybbende interview.

____
NAVN (BLOKBOGSTAVER)

____ ________
UNDERSKRIFT DATO

____
TELEFONNUMMER

INSTITUT FOR NATURFAGENES DIDAKTIK
ØSTER VOLDGADE 3

SPECIALESTUDERENDE
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JVB346@ALUMNI.KU.DK

VEJLEDER
MIKKEL W JOHANSEN
DIR 28 72 84 41
mwj@ind.ku.dk
REF: MWJ

KØBENHAVNS UNIVERSITET
DET NATUR- OG BIOVIDENSKABELIGE FAKULTET

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Jeg bekræfter, at jeg er blevet underrettet om formålet med undersøgelsen og at Nicole Jonasen har besvaret eventuelle spørgsmål, som jeg havde om, hvordan mit barns deltagelse bruges.

Jeg er gjort bekendt med, at mit barns deltagelse er frivilligt. Jeg har mulighed for at trække mit barns deltagelse tilbage, hvorefter al indsamlet materiale om barnet destrueres.

Sæt kryds:
☐ Jeg giver hermed min samtykke til, at mit barn deltager i undersøgelsen ved at besvare en kort test om funktioner.

☐ Jeg giver hermed min samtykke til, at mit barn evt. kan opringes til uddybbende interview.

____
BARNETS NAVN (BLOKBOGSTAVER)

____
FORÆLDRE/VÆRGES NAVN (BLOKBOGSTAVER)

____ ________
UNDERSKRIFT (FORÆLDRE/VÆRGE) DATO

____
BARNETS TELEFONNUMMER

INSTITUT FOR NATURFAGENES DIDAKTIK
ØSTER VOLDGADE 3

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VEJLEDER
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Appendix F

Codebook

Codebook

Klasse

		Value	Count	Percent
Standard Attributes	Position	2		
	Label	Class		
	Type	Numeric		
	Format	F1		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Level B mathematics	25	48,1%
	1	Level A mathematics	27	51,9%

@3

		Value	Count	Percent
Standard Attributes	Position	3		
	Label	Task 1.2 b) Draw the function f(x) =2x-5		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Drawn incorrect	10	19,2%
	1	Drawn correct	41	78,8%
Missing Values	-1		1	1,9%

@4

		Value	Count	Percent
Standard Attributes	Position	4		
	Label	Task 1.2 c) Draw the function f(x) = - x + 6		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Drawn incorrect	15	28,8%
	1	Drawn correct	33	63,5%
Missing Values	- 1		4	7,7%

@5

		Value	Count	Percent
Standard Attributes	Position	5		
	Label	Task 1.2 d) Draw the function f(x) = 4 x		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Drawn incorrect	14	26,9%
	1	Drawn correct	34	65,4%
Missing Values	- 1		4	7,7%

@6

		Value	Count	Percent
Standard Attributes	Position	6		
	Label	Task 1.2 e) Draw the function f(x) = 3		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Drawn incorrect	13	25,0%
	1	Drawn correct	33	63,5%
Missing Values	-1		6	11,5%

@7

		Value	Count	Percent
Standard Attributes	Position	7		
	Label	Task 1.3 What do you normally use when drawing graphs?		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	1	Draws with paper and pencil	14	26,9%
	2	Gets a CAS-tool to draw it	26	50,0%
	3	Other	0	0,0%
	4	Draw with paper and pencil and gets a CAS-tool to draw it	12	23,1%

@8

		Value	Count	Percent
Standard Attributes	Position	8		
	Label	Task 2.1 Place the numbers on the number line		
	Type	Numeric		
	Format	F1		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Haven't placed any numbers correct	0	0,0%
	1	Only placed -2 correct	5	9,6%
	2	Only placed 4,3 correct	0	0,0%
	3	Placed both correct	47	90,4%

@9

		Value	Count	Percent
Standard Attributes	Position	9		
	Label	Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)		
	Type	Numeric		
	Format	F1		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Haven't plotted any points correct	2	3,8%
	1	Have only plotted one point correct	1	1,9%

@9

		Value	Count	Percent
	2	Have only plotted two points correct	4	7,7%
	3	Have plotted all points correct	45	86,5%

@10

		Value	Count	Percent
Standard Attributes	Position	10		
	Label	Task 2.3 Plot the points in the Cartesian plane (Table)		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Haven't plotted any points correct	2	3,8%
	1	Have only plotted one point correct	0	0,0%
	2	Have only plotted two points correct	3	5,8%
	3	Have only plotted three points correct	13	25,0%
	4	Have plotted all points correct	32	61,5%
Missing Values	-1		2	3,8%

@11

		Value	Count	Percent
Standard Attributes	Position	11		
	Label	Task 2.4 Plot the points in the Cartesian plane (f(x1) =y1 notation)		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Haven't plotted any points correct	11	21,2%
	1	Have only plotted one point correct	4	7,7%
	2	Have only plotted two points correct	2	3,8%
	3	Have only plotted three points correct	2	3,8%
	4	Have plotted all points correct	24	46,2%
Missing Values	-1		9	17,3%

@12

		Value	Count	Percent
Standard Attributes	Position	12		
	Label	Task 2.5 a) Calculate the function value for x=2 for the function f(x) =7x-4		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Incorrect answer	1	1,9%
	1	Correct answer but incorrect use of the notation	23	44,2%
	2	Correct answer and correct use of the notation	26	50,0%
Missing Values	-1		2	3,8%

@13

		Value	Count	Percent
Standard Attributes	Position	13		
	Label	Task 2.5 b) What do you normally use when calculating functional values?		
	Type	Numeric		
	Format	F1		
	Measurement	Nominal		
	Role	Input		
Valid Values	1	Calculate with CAS-tools	11	21,2%
	2	Mental calculation	3	5,8%
	3	Mental calculation but uses paper and pencil as support	22	42,3%
	4	Other	1	1,9%
	5	Calculate with CAS-tools and mental calculation	5	9,6%
	6	Calculate with CAS-tools and mental calculation but uses paper and pencil as support	10	19,2%

@14

		Value	Count	Percent
Standard Attributes	Position	14		
	Label	Task 2.6 Which algebraic expression fits the table		
	Type	Numeric		
	Format	F3.1		
	Measurement	Nominal		
	Role	Input		
Valid Values	,0	Have answered one or more wrong possible answers	6	11,5%
	,5	Have answered one possible answer correct and one wrong	1	1,9%
	1,0	Have answered one of the correct possible answers	36	69,2%
	2,0	Have answered both of the correct possible answers	8	15,4%
Missing Values	-1,0		1	1,9%

@15

		Value	Count	Percent
Standard Attributes	Position	15		
	Label	Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values		
	Type	Numeric		
	Format	F2.1		
	Measurement	Nominal		
	Role	Input		
Valid Values	,0	Have answered one or more wrong possible answers	3	5,8%
	,5	Have answered one possible answer correct and one wrong	0	0,0%
	1,0	Have answered one of the correct possible answers	42	80,8%
	2,0	Have answered both of the correct possible answers	6	11,5%
Missing Values	-1,0		1	1,9%

@16

		Value	Count	Percent
Standard Attributes	Position	16		
	Label	Task 2.8 Which algebraic expression fits the coordiate pairs		
	Type	Numeric		
	Format	F3.1		
	Measurement	Nominal		
	Role	Input		
Valid Values	,0	Have answered one or more wrong possible answers	5	9,6%
	,5	Have answered one possible answer correct and one wrong	1	1,9%
	1,0	Have answered one of the correct possible answers	37	71,2%
	2,0	Have answered both of the correct possible answers	7	13,5%
Missing Values	-1,0		2	3,8%

@17

		Value	Count	Percent
Standard Attributes	Position	17		
	Label	Task 2.9 Fill out the table from the graph		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	All wrong	0	0,0%
	1	Errors in both categories but some points are correct	3	5,8%
	2	Have written all the f(x) values correct, but there are errors in the x-values	12	23,1%
	3	Have written all the x-values correct but there are errors in the f(x) values	4	7,7%
	4	Everything is correct	30	57,7%
Missing Values	-1		3	5,8%

@18

		Value	Count	Percent
Standard Attributes	Position	18		
	Label	Task 2.10 The point f (3)=7 is on the graph for the function. Which graph fulfil this demand?		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Answered incorrect	10	19,2%
	1	Answered correct	40	76,9%
Missing Values	- 1		2	3,8%

@19

		Value	Count	Percent
Standard Attributes	Position	19		
	Label	Task 2.11 Which algebraic expression fits the graph		
	Type	Numeric		
	Format	F2		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Answered incorrect	9	17,3%
	1	Answered correct	40	76,9%
Missing Values	- 1		3	5,8%

Tegnet_alle_funktioner_rigtigt

		Value	Count	Percent
Standard Attributes	Position	20		
	Label	Drawn functions		
	Type	Numeric		
	Format	F8		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Have drawn one or more functions wrong	30	57,7%
	1	Have drawn all functions correct	22	42,3%

Konceptualiseringen

		Value	Count	Percent
Standard Attributes	Position	21		
	Label	Understandin g the conceptual blend of the Cartesian plane		
	Type	Numeric		
	Format	F8		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Do not understand	28	53,8%
	1	Understand	24	46,2%

Notationen_fx

		Value	Count	Percent
Standard Attributes	Position	22		
	Label	Understanding the f(x)-notation		
	Type	Numeric		
	Format	F8		
	Measurement	Nominal		
	Role	Input		
Valid Values	0	Do not understand	38	73,1%
	1	Understand	14	26,9%

Appendix G

Frequencies tables

Frequency Table

Class

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Level B mathematics	25	48,1	48,1	48,1
	Level A mathematics	27	51,9	51,9	100,0
	Total	52	100,0	100,0	

Task 1.2 b) Draw the function f(x)=2x-5

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	10	19,2	19,6	19,6
	Drawn correct	41	78,8	80,4	100,0
	Total	51	98,1	100,0	
Missing	-1	1	1,9		
Total		52	100,0		

Task 1.2 c) Draw the function f(x)=-x+6

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	15	28,8	31,3	31,3
	Drawn correct	33	63,5	68,8	100,0
	Total	48	92,3	100,0	
Missing	-1	4	7,7		
Total		52	100,0		

Task 1.2 d) Draw the function f(x)=4x

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	14	26,9	29,2	29,2
	Drawn correct	34	65,4	70,8	100,0
	Total	48	92,3	100,0	
Missing	-1	4	7,7		
Total		52	100,0		

Task 1.2 e) Draw the function f(x)=3

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Drawn incorrect	13	25,0	28,3	28,3
	Drawn correct	33	63,5	71,7	100,0
	Total	46	88,5	100,0	
Missing	- 1	6	11,5		
Total		52	100,0		

Task 1.3 What do you normally use when drawing graphs?

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Draws with paper and pencil	14	26,9	26,9	26,9
	Gets a CAS-tool to draw it	26	50,0	50,0	76,9
	Draw with paper and pencil and gets a CAS-tool to draw it	12	23,1	23,1	100,0
	Total	52	100,0	100,0	

Task 2.1 Place the numbers on the number line

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Only placed -2 correct	5	9,6	9,6	9,6
	Placed both correct	47	90,4	90,4	100,0
	Total	52	100,0	100,0	

Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Haven't plotted any points correct	2	3,8	3,8	3,8
	Have only plotted one point correct	1	1,9	1,9	5,8
	Have only plotted two points correct	4	7,7	7,7	13,5
	Have plotted all points correct	45	86,5	86,5	100,0
	Total	52	100,0	100,0	

Task 2.3 Plot the points in the Cartesian plane (Table)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Haven't plotted any points correct	2	3,8	4,0	4,0
	Have only plotted two points correct	3	5,8	6,0	10,0
	Have only plotted three points correct	13	25,0	26,0	36,0
	Have plotted all points correct	32	61,5	64,0	100,0
	Total	50	96,2	100,0	
Missing	- 1	2	3,8		
Total		52	100,0		

Task 2.4 Plot the points in the Cartesian plane (f(x1)=y1 notation)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Haven't plotted any points correct	11	21,2	25,6	25,6
	Have only plotted one point correct	4	7,7	9,3	34,9
	Have only plotted two points correct	2	3,8	4,7	39,5
	Have only plotted three points correct	2	3,8	4,7	44,2
	Have plotted all points correct	24	46,2	55,8	100,0
	Total	43	82,7	100,0	
Missing	- 1	9	17,3		
Total		52	100,0		

Task 2.5 a) Calculate the function value for x=2 for the function f(x) =7x-4

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Incorrect answer	1	1,9	2,0	2,0
	Correct answer but incorrect use of the notation	23	44,2	46,0	48,0
	Correct answer and correct use of the notation	26	50,0	52,0	100,0
	Total	50	96,2	100,0	
Missing	- 1	2	3,8		
Total		52	100,0		

Task 2.5 b) What do you normally use when calculating functional values?

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Calculate with CAS-tools	11	21,2	21,2	21,2
	Mental calculation	3	5,8	5,8	26,9
	Mental calculation but uses paper and pencil as support	22	42,3	42,3	69,2
	Other	1	1,9	1,9	71,2
	Calculate with CAS-tools and mental calculation	5	9,6	9,6	80,8
	Calculate with CAS-tools and mental calculation but uses paper and pencil as support	10	19,2	19,2	100,0
	Total	52	100,0	100,0	

Task 2.6 Which algebraic expression fits the table

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have answered one or more wrong possible answers	6	11,5	11,8	11,8
	Have answered one possible answer correct and one wrong	1	1,9	2,0	13,7
	Have answered one of the correct possible answers	36	69,2	70,6	84,3
	Have answered both of the correct possible answers	8	15,4	15,7	100,0
	Total	51	98,1	100,0	
Missing	-1,0	1	1,9		
Total		52	100,0		

Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have answered one or more wrong possible answers	3	5,8	5,9	5,9
	Have answered one of the correct possible answers	42	80,8	82,4	88,2
	Have answered both of the correct possible answers	6	11,5	11,8	100,0
	Total	51	98,1	100,0	
Missing	-1,0	1	1,9		
Total		52	100,0		

Task 2.8 Which algebraic expression fits the coordiate pairs

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have answered one or more wrong possible answers	5	9,6	10,0	10,0
	Have answered one possible answer correct and one wrong	1	1,9	2,0	12,0
	Have answered one of the correct possible answers	37	71,2	74,0	86,0
	Have answered both of the correct possible answers	7	13,5	14,0	100,0
	Total	50	96,2	100,0	
Missing	-1,0	2	3,8		
Total		52	100,0		

Task 2.9 Fill out the table from the graph

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Errors in both categories but some points are correct	3	5,8	6,1	6,1
	Have written all the f(x) values correct, but there are errors in the x-values	12	23,1	24,5	30,6
	Have written all the x-values correct but there are errors in the f(x) values	4	7,7	8,2	38,8
	Everything is correct	30	57,7	61,2	100,0
	Total	49	94,2	100,0	
Missing	- 1	3	5,8		
Total		52	100,0		

Task 2.10 The point f(3)=7 is on the graph for the function. Which graph fulfil this demand?

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Answered incorrect	10	19,2	20,0	20,0
	Answered correct	40	76,9	80,0	100,0
	Total	50	96,2	100,0	
Missing	- 1	2	3,8		
Total		52	100,0		

Task 2.11 Which algebraic expression fits the graph

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Answered incorrect	9	17,3	18,4	18,4
	Answered correct	40	76,9	81,6	100,0
	Total	49	94,2	100,0	
Missing	- 1	3	5,8		
Total		52	100,0		

Drawn functions

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Have drawn one or more functions wrong	30	57,7	57,7	57,7
	Have drawn all functions correct	22	42,3	42,3	100,0
	Total	52	100,0	100,0	

Understanding the conceptual blend of the Cartesian plane

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Do not understand	28	53,8	53,8	53,8
	Understand	24	46,2	46,2	100,0
	Total	52	100,0	100,0	

Understanding the f(x)-notation

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Do not understand	38	73,1	73,1	73,1
	Understand	14	26,9	26,9	100,0
	Total	52	100,0	100,0	

Appendix H

Correlation matrix

Correlations

Correlations

		Class	Task 1.2 b) Draw the function f(x) =2x-5	Task 1.2 c) Draw the function f(x)=-x+6	Task 1.2 d) Draw the function f(x) =4x	Task 1.2 e) Draw the function f(x)=3	Task 1.3 What do you normally use when drawing graphs?
Class	Pearson Correlation	1	,207	,315*	,183	,408**	-,150
	Sig. (2-tailed)		,144	,029	,212	,005	,288
	N	52	51	48	48	46	52
Task 1.2 b) Draw the function f(x)=2x-5	Pearson Correlation	,207	1	,486**	,384**	,187	-,275
	Sig. (2-tailed)	,144		,000	,007	,213	,051
	N	51	51	48	48	46	51
Task 1.2 c) Draw the function f(x)=-x+6	Pearson Correlation	,315*	,486**	1	,457**	,249	-,133
	Sig. (2-tailed)	,029	,000		,001	,095	,366
	N	48	48	48	48	46	48
Task 1.2 d) Draw the function f(x)=4x	Pearson Correlation	,183	,384**	,457**	1	,067	-,141
	Sig. (2-tailed)	,212	,007	,001		,659	,338
	N	48	48	48	48	46	48
Task 1.2 e) Draw the function f(x)=3	Pearson Correlation	,408**	,187	,249	,067	1	-,046
	Sig. (2-tailed)	,005	,213	,095	,659		,759
	N	46	46	46	46	46	46

Correlations

		Task 2.1 Place the numbers on the number line	Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Task 2.3 Plot the points in the Cartesian plane (Table)	Task 2.4 Plot the points in the Cartesian plane (f(x1) =y1 notation)	Task 2.5 a) Calculate the function value for x=2 for the function f (x)=7x-4	Task 2.5 b) What do you normally use when calculating functional values?
Class	Pearson Correlation	,078	,301*	,306*	,203	,261	-,119
	Sig. (2-tailed)	,583	,030	,031	,192	,067	,400
	N	52	52	50	43	50	52
Task 1.2 b) Draw the function f(x)=2x-5	Pearson Correlation	,335*	,047	,273	,024	,010	,004
	Sig. (2-tailed)	,016	,741	,057	,879	,948	,978
	N	51	51	49	42	49	51
Task 1.2 c) Draw the function f(x)=-x+6	Pearson Correlation	-,041	,081	,274	,135	,247	-,026
	Sig. (2-tailed)	,784	,583	,062	,405	,097	,858
	N	48	48	47	40	46	48
Task 1.2 d) Draw the function f(x)=4x	Pearson Correlation	-,193	-,033	,156	-,139	,200	,072
	Sig. (2-tailed)	,188	,823	,296	,393	,182	,627
	N	48	48	47	40	46	48
Task 1.2 e) Draw the function f(x)=3	Pearson Correlation	,149	,384**	,137	,101	,211	-,196
	Sig. (2-tailed)	,323	,008	,371	,534	,170	,191
	N	46	46	45	40	44	46

Correlations

		Task 2.6 Which algebraic expression fits the table	Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Task 2.8 Which algebraic expression fits the coordiate pairs	Task 2.9 Fill out the table from the graph	Task 2.10 The point f(3)=7 is on the graph for the function. Which graph fulfil this demand?	Task 2.11 Which algebraic expression fits the graph
Class	Pearson Correlation	,164	,133	,261	,105	,220	,082
	Sig. (2-tailed)	,250	,351	,067	,471	,124	,576
	N	51	51	50	49	50	49
Task 1.2 b) Draw the function f(x)=2x-5	Pearson Correlation	,028	,190	,030	,181	,125	,319*
	Sig. (2-tailed)	,846	,185	,834	,219	,387	,025
	N	50	50	50	48	50	49
Task 1.2 c) Draw the function f(x)=-x+6	Pearson Correlation	,247	,312*	,132	,158	,131	,024
	Sig. (2-tailed)	,094	,033	,375	,294	,381	,872
	N	47	47	47	46	47	46
Task 1.2 d) Draw the function f(x)=4x	Pearson Correlation	,165	,204	,041	,013	,156	,044
	Sig. (2-tailed)	,268	,170	,785	,934	,295	,773
	N	47	47	47	46	47	46
Task 1.2 e) Draw the function f(x)=3	Pearson Correlation	,215	,318*	,370*	,221	,294*	,289
	Sig. (2-tailed)	,157	,033	,012	,149	,050	,057
	N	45	45	45	44	45	44

Correlations

		Class	Task 1.2 b) Draw the function f(x) =2x-5	Task 1.2 c) Draw the function f(x)=-x+6	Task 1.2 d) Draw the function f(x) =4x	Task 1.2 e) Draw the function f(x)=3	Task 1.3 What do you normally use when drawing graphs?
Task 1.3 What do you normally use when drawing graphs?	Pearson Correlation	-,150	-,275	-,133	-,141	-,046	1
	Sig. (2-tailed)	,288	,051	,366	,338	,759	
	N	52	51	48	48	46	52
Task 2.1 Place the numbers on the number line	Pearson Correlation	,078	,335*	-,041	-,193	,149	-,002
	Sig. (2-tailed)	,583	,016	,784	,188	,323	,987
	N	52	51	48	48	46	52
Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Pearson Correlation	,301*	,047	,081	-,033	,384**	-,126
	Sig. (2-tailed)	,030	,741	,583	,823	,008	,375
	N	52	51	48	48	46	52
Task 2.3 Plot the points in the Cartesian plane (Table)	Pearson Correlation	,306*	,273	,274	,156	,137	-,037
	Sig. (2-tailed)	,031	,057	,062	,296	,371	,799
	N	50	49	47	47	45	50
Task 2.4 Plot the points in the Cartesian plane (f (x1)=y1 notation)	Pearson Correlation	,203	,024	,135	-,139	,101	-,201
	Sig. (2-tailed)	,192	,879	,405	,393	,534	,196
	N	43	42	40	40	40	43

Correlations

		Task 2.1 Place the numbers on the number line	Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Task 2.3 Plot the points in the Cartesian plane (Table)	Task 2.4 Plot the points in the Cartesian plane (f(x1) =y1 notation)	Task 2.5 a) Calculate the function value for x=2 for the function f (x)=7x-4	Task 2.5 b) What do you normally use when calculating functional values?
Task 1.3 What do you normally use when drawing graphs?	Pearson Correlation	-,002	-,126	-,037	-,201	,051	,249
	Sig. (2-tailed)	,987	,375	,799	,196	,723	,075
	N	52	52	50	43	50	52
Task 2.1 Place the numbers on the number line	Pearson Correlation	1	,375**	,311*	-,126	,062	,097
	Sig. (2-tailed)		,006	,028	,420	,669	,496
	N	52	52	50	43	50	52
Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Pearson Correlation	,375**	1	,559**	,271	,193	-,106
	Sig. (2-tailed)	,006		,000	,079	,178	,455
	N	52	52	50	43	50	52
Task 2.3 Plot the points in the Cartesian plane (Table)	Pearson Correlation	,311*	,559**	1	,046	,204	,064
	Sig. (2-tailed)	,028	,000		,775	,164	,659
	N	50	50	50	41	48	50
Task 2.4 Plot the points in the Cartesian plane (f (x1)=y1 notation)	Pearson Correlation	-,126	,271	,046	1	-,079	-,110
	Sig. (2-tailed)	,420	,079	,775		,617	,484
	N	43	43	41	43	42	43

Correlations

		Task 2.6 Which algebraic expression fits the table	Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Task 2.8 Which algebraic expression fits the coordiate pairs	Task 2.9 Fill out the table from the graph	Task 2.10 The point f(3)=7 is on the graph for the function. Which graph fulfil this demand?	Task 2.11 Which algebraic expression fits the graph
Task 1.3 What do you normally use when drawing graphs?	Pearson Correlation	,075	-,116	-,031	-,232	-,019	-,315*
	Sig. (2-tailed)	,599	,416	,831	,108	,897	,028
	N	51	51	50	49	50	49
Task 2.1 Place the numbers on the number line	Pearson Correlation	,018	,047	,155	-,249	,000	,362*
	Sig. (2-tailed)	,898	,745	,282	,084	1,000	,010
	N	51	51	50	49	50	49
Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Pearson Correlation	,075	,049	,200	-,089	,044	,445**
	Sig. (2-tailed)	,603	,731	,164	,542	,761	,001
	N	51	51	50	49	50	49
Task 2.3 Plot the points in the Cartesian plane (Table)	Pearson Correlation	,141	,118	,225	-,182	,168	,399**
	Sig. (2-tailed)	,333	,420	,124	,221	,254	,005
	N	49	49	48	47	48	48
Task 2.4 Plot the points in the Cartesian plane (f (x1)=y1 notation)	Pearson Correlation	,022	,092	,159	,267	,474**	-,016
	Sig. (2-tailed)	,888	,562	,322	,087	,002	,922
	N	42	42	41	42	41	40

Correlations

		Class	Task 1.2 b) Draw the function f(x) =2x-5	Task 1.2 c) Draw the function f(x)=-x+6	Task 1.2 d) Draw the function f(x) =4x	Task 1.2 e) Draw the function f(x)=3	Task 1.3 What do you normally use when drawing graphs?
Task 2.5 a) Calculate the function value for x=2 for the function f(x)=7x-4	Pearson Correlation	,261	,010	,247	,200	,211	,051
	Sig. (2-tailed)	,067	,948	,097	,182	,170	,723
	N	50	49	46	46	44	50
Task 2.5 b) What do you normally use when calculating functional values?	Pearson Correlation	-,119	,004	-,026	,072	-,196	,249
	Sig. (2-tailed)	,400	,978	,858	,627	,191	,075
	N	52	51	48	48	46	52
Task 2.6 Which algebraic expression fits the table	Pearson Correlation	,164	,028	,247	,165	,215	,075
	Sig. (2-tailed)	,250	,846	,094	,268	,157	,599
	N	51	50	47	47	45	51
Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Pearson Correlation	,133	,190	,312*	,204	,318*	-,116
	Sig. (2-tailed)	,351	,185	,033	,170	,033	,416
	N	51	50	47	47	45	51
Task 2.8 Which algebraic expression fits the coordiate pairs	Pearson Correlation	,261	,030	,132	,041	,370*	-,031
	Sig. (2-tailed)	,067	,834	,375	,785	,012	,831
	N	50	50	47	47	45	50

Correlations

		Task 2.1 Place the numbers on the number line	Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Task 2.3 Plot the points in the Cartesian plane (Table)	Task 2.4 Plot the points in the Cartesian plane (f(x1) =y1 notation)	Task 2.5 a) Calculate the function value for x=2 for the function f (x)=7x-4	Task 2.5 b) What do you normally use when calculating functional values?
Task 2.5 a) Calculate the function value for x=2 for the function f(x)=7x-4	Pearson Correlation	,062	,193	,204	-,079	1	-,064
	Sig. (2-tailed)	,669	,178	,164	,617		,660
	N	50	50	48	42	50	50
Task 2.5 b) What do you normally use when calculating functional values?	Pearson Correlation	,097	-,106	,064	-,110	-,064	1
	Sig. (2-tailed)	,496	,455	,659	,484	,660	
	N	52	52	50	43	50	52
Task 2.6 Which algebraic expression fits the table	Pearson Correlation	,018	,075	,141	,022	,053	,247
	Sig. (2-tailed)	,898	,603	,333	,888	,718	,080
	N	51	51	49	42	49	51
Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Pearson Correlation	,047	,049	,118	,092	,042	,192
	Sig. (2-tailed)	,745	,731	,420	,562	,775	,177
	N	51	51	49	42	49	51
Task 2.8 Which algebraic expression fits the coordiate pairs	Pearson Correlation	,155	,200	,225	,159	,093	,109
	Sig. (2-tailed)	,282	,164	,124	,322	,528	,450
	N	50	50	48	41	48	50

Correlations

		Task 2.6 Which algebraic expression fits the table	Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Task 2.8 Which algebraic expression fits the coordiate pairs	Task 2.9 Fill out the table from the graph	Task 2.10 The point f(3)=7 is on the graph for the function. Which graph fulfil this demand?	Task 2.11 Which algebraic expression fits the graph
Task 2.5 a) Calculate the function value for x=2 for the function f(x)=7x-4	Pearson Correlation	,053	,042	,093	-,008	,017	,360*
	Sig. (2-tailed)	,718	,775	,528	,957	,907	,013
	N	49	49	48	47	48	47
Task 2.5 b) What do you normally use when calculating functional values?	Pearson Correlation	,247	,192	,109	-,264	,139	,028
	Sig. (2-tailed)	,080	,177	,450	,067	,337	,848
	N	51	51	50	49	50	49
Task 2.6 Which algebraic expression fits the table	Pearson Correlation	1	,617**	,687**	,302*	,156	,149
	Sig. (2-tailed)		,000	,000	,035	,284	,308
	N	51	51	49	49	49	49
Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Pearson Correlation	,617**	1	,768**	,405**	,303*	,098
	Sig. (2-tailed)	,000		,000	,004	,034	,504
	N	51	51	49	49	49	49
Task 2.8 Which algebraic expression fits the coordiate pairs	Pearson Correlation	,687**	,768**	1	,364*	,222	,176
	Sig. (2-tailed)	,000	,000		,012	,125	,232
	N	49	49	50	47	49	48

Correlations

		Class	Task 1.2 b) Draw the function f(x) =2x-5	Task 1.2 c) Draw the function f(x)=-x+6	Task 1.2 d) Draw the function f(x) =4x	Task 1.2 e) Draw the function f(x)=3	Task 1.3 What do you normally use when drawing graphs?
Task 2.9 Fill out the table from the graph	Pearson Correlation	,105	,181	,158	,013	,221	-,232
	Sig. (2-tailed)	,471	,219	,294	,934	,149	,108
	N	49	48	46	46	44	49
Task 2.10 The point f(3) =7 is on the graph for the function. Which graph fulfil this demand?	Pearson Correlation	,220	,125	,131	,156	,294*	-,019
	Sig. (2-tailed)	,124	,387	,381	,295	,050	,897
	N	50	50	47	47	45	50
Task 2.11 Which algebraic expression fits the graph	Pearson Correlation	,082	,319*	,024	,044	,289	-,315*
	Sig. (2-tailed)	,576	,025	,872	,773	,057	,028
	N	49	49	46	46	44	49

Correlations

		Task 2.1 Place the numbers on the number line	Task 2.2 Plot the points in the Cartesian plane (Coordinate pairs)	Task 2.3 Plot the points in the Cartesian plane (Table)	Task 2.4 Plot the points in the Cartesian plane (f(x1) =y1 notation)	Task 2.5 a) Calculate the function value for x=2 for the function f (x)=7x-4	Task 2.5 b) What do you normally use when calculating functional values?
Task 2.9 Fill out the table from the graph	Pearson Correlation	-,249	-,089	-,182	,267	-,008	-,264
	Sig. (2-tailed)	,084	,542	,221	,087	,957	,067
	N	49	49	47	42	47	49
Task 2.10 The point f(3) =7 is on the graph for the function. Which graph fulfil this demand?	Pearson Correlation	,000	,044	,168	,474**	,017	,139
	Sig. (2-tailed)	1,000	,761	,254	,002	,907	,337
	N	50	50	48	41	48	50
Task 2.11 Which algebraic expression fits the graph	Pearson Correlation	,362*	,445**	,399**	-,016	,360*	,028
	Sig. (2-tailed)	,010	,001	,005	,922	,013	,848
	N	49	49	48	40	47	49

Correlations

		Task 2.6 Which algebraic expression fits the table	Task 2.7 Which algebraic expression fits the x-values and their associated fuctional values	Task 2.8 Which algebraic expression fits the coordiate pairs	Task 2.9 Fill out the table from the graph	Task 2.10 The point f(3)=7 is on the graph for the function. Which graph fulfil this demand?	Task 2.11 Which algebraic expression fits the graph
Task 2.9 Fill out the table from the graph	Pearson Correlation	,302*	,405**	,364*	1	,281	-,143
	Sig. (2-tailed)	,035	,004	,012		,056	,338
	N	49	49	47	49	47	47
Task 2.10 The point f(3) =7 is on the graph for the function. Which graph fulfil this demand?	Pearson Correlation	,156	,303*	,222	,281	1	,043
	Sig. (2-tailed)	,284	,034	,125	,056		,773
	N	49	49	49	47	50	48
Task 2.11 Which algebraic expression fits the graph	Pearson Correlation	,149	,098	,176	-,143	,043	1
	Sig. (2-tailed)	,308	,504	,232	,338	,773	
	N	49	49	48	47	48	49

*. Correlation is significant at the 0.05 level (2-tailed).

**. Correlation is significant at the 0.01 level (2-tailed).