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A Diagnostic Test for Danish Middle School Arithmetics

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UNIVERSITY OF COPENHAGEN
DEPARTMENT OF SCIENCE EDUCATION

Master's Thesis

Derya Diana Cosan

A DIAGNOSTIC TEST FOR DANISH MIDDLE SCHOOL ARITHMETICS

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Submitted on: May 31st, 2021

Abstract

This present thesis investigates to what extent arithmetic and algebraic praxeologies are mastered by Danish fifth- and seventh-grade students. The Anthropological Theory of Didactics (ATD) and the notions of praxeologies are the main tools for the examination of this. These are used to develop praxeological reference models for arithmetic and algebra in fifth- and seventh grade. The construction of the models is based on two different textbooks for Danish fifth- and seventh-grade students respectively and a review of the official programmes for these grades. The identified types of task related to arithmetic and algebra are among other things related to fractions, decimals, solving of first-degree equations etc. These praxeological reference models illustrates what arithmetic and algebraic praxeologies Danish fifth- and seventh-grade students are supposed to learn.

For the examination of the extent to which these praxeologies related to arithmetic and algebra are mastered by the students, a diagnostic test for fifth and seventh grade respectively is developed. The tests are developed in relation to the praxeological models.

The diagnostic test is performed by 86 Danish fifth grade students and 78 seventh grade students in Nkøbing municipality.

The results from the diagnostic tests show that the students, both in fifth and seventh grade, master almost all types of task to different extents, while similarities and difference between the classes at the same grade are identified.

It is observed that some students possess some misconceptions in types of task related to arithmetic and algebra, while other students have lack of knowledge. The most prominent misconceptions among these Danish fifth- and seventh-grade students are related to fractions and calculation of these, decimals and the magnitudes of fractions and decimals. Furthermore, it is observed that the majority of the students have an operational approach to the equal sign instead of a relational approach.

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1 Introduction

Among Danish high school students, there is a high failure rate and low algebraic skills (Ministry of Education, 2019). Students' low algebraic skills have an impact on their success in exams. In the summer exam in 2019, it was observed that 31.2% of the students at Danish upper secondary school STX failed the mathematics exam at B-level (Ministry of Education, 2019). In this exam, for example, students were introduced to the following algebraic exercise:

$$\text{Reduce the expression: } (a+b)^2 - b \cdot (2a+b)$$

According to Grønbæk, Winsløw, & Jessen (2019), this is an exercise which could appear in ninth grade exam, why it can be expected already in ninth grade that the students are able to solve this type of task. Data from the exam for Danish ninth grade students show that only approximately 60% of students master simple operations with fractions.

According to Siegler, Duncan, Davis-Kean, Duckworth, Claessens, Engel, Susperreguy & Chen (2012), students' algebra performance and overall mathematics achievements in high school can be predicted by their knowledge of fractions at age 10. Specifically, Siegler et al. (2012) found that students' skills in fractions in fifth and sixth grade are a strong predictor of success in mathematics in high school, after controlling for other factors such as gender, social background, etc. According to Siegler et al. (2012), one of the reasons for this is that if students do not master and understand fractions, they will have difficulties answering completely basic algebraic equations. Arithmetic with fractions contains the basic arithmetic principles that must subsequently be used in algebra, including arithmetic with letters, in high school.

10-year-olds' knowledge of fractions as a predictor of their algebra knowledge and mathematical achievement at age 16 (Siegler et al., 2012), is the motivating phenomenon for the present thesis. Furthermore, if misconceptions and lack of knowledge related to arithmetic and algebra can be identified and resolved early in primary and lower secondary school, it could result in students with higher mastery of arithmetic and algebra later on.

The purpose of this thesis is therefore to identify the lack of knowledge and misconceptions regarding to arithmetic and algebra among Danish fifth- and seventh-grade students. In order to identify and examine the knowledge gaps and misconceptions, diagnostic tests were developed. In the present thesis, two diagnostic tests, for fifth and seventh grade respectively, have been developed to identify: 1. The misconceptions related to arithmetic and algebra among fifth- and seventh-grade students, and 2. how many and which, if the test is not anonymous, of the students have low or no mastery of arithmetic and algebra, in order to pay more attention to the development of these students.

By explicitly identifying these misconceptions and knowledge gaps related to arithmetic and algebra, it allows for the possibility to address these issues in early primary and lower secondary school, and thus prevents low or no mastery of algebra for future high school students.

1.1 The structure of the thesis

First, the thesis will present the theoretical framework i.e., the Anthropological Theory of Didactics, which is used throughout the whole thesis. This will be followed by a review of the research regarding the understanding of arithmetic and algebra and the relation between the two. Subsequently, misconceptions related to arithmetic and algebra among students will be outlined in relation to previous research. Thus, section 2 is constructed.

Hereafter, in section 3, the three research questions will be presented. Based on these three research questions, the thesis is divided into three parts, where each part examines a research question. Section 4 outlines the context through which the three research questions are examined. Section 5 and 6 present the methodology and results, respectively, for the first research question. The examination of the first research question is based on the official programmes for Danish 4.-6. Grade students, but mainly the textbooks in mathematics for Danish fifth- and seventh-grade students respectively. Specifically, the textbooks utilized are Gyldendal MULTI 5 i-bog and Gyldendal MULTI 7 i-bog and Alinea Matematrix, (specifically Matematrix 5 and Matematrix 7). Based on these, the answer for the first research question is praxeological reference models related to arithmetic and algebra in fifth and seventh grade is developed.

Section 7 and 8 present the methodology and results for the second research question. This research question is examined by developing a diagnostic test for arithmetic and algebra in fifth and seventh grade. While section 7 describes the development of the diagnostic test, section 8 outlines the results obtained the diagnostic test.

Sections 9 and 10 address the third research question, which is answered on the basis of the results from research question 2.

Finally, section 10, the discussion, evaluates the methods used throughout the thesis and the results obtained. This section is followed by a conclusion and suggestions for future research in section 11.

2 Theoretical framework

The theoretical and methodological frameworks used in this thesis are based on the Anthropological Theory of Didactics (ATD), introduced by Yves Chevallard. This didactical theory, which is used in research in mathematics education, offers the possibility to construct an epistemological reference model for a particular mathematical knowledge. ATD is chosen as the theoretical framework, rather than the Theory of Didactical Situations (TDS), because ATD concentrates on describing, explaining, and analyzing school algebra, whereas Brousseau's (founded in the early 1970s) TDS is concerned with designing and investigating teaching sequences (Winsløw, 2007). Furthermore, another important aspect is that school algebra has been central to the development of ATD (Bosch, 2015).

The Anthropological Theory of Didactics "offers a general epistemological model of mathematical knowledge where mathematics is seen as a human activity of study of types of problems" (Barbé, Bosch, Espinoza, & Gascón, 2005, p. 236). Hence, the anthropological approach is used to investigate and analyze human knowledge and activity, mathematical or otherwise, within the notion of a praxeology. (Chevallard, 2019).

2.1 The Anthropological Theory of Didactics

The starting point in the Anthropological Theory of Didactics is that human activity can be described in two inseparable blocks, the *praxis*- and *logos* blocks respectively (Barbé et al., 2005), where Chevallard (2006) points out that "human praxeologies are open to change, adaptation, and improvement" (p. 23).

The praxis, or practical, block (or the know-how) consists of types of problems or problematic tasks to be solved and investigated, and by the techniques utilized to solve these types of tasks (Barbé et al., 2015). The *type of tasks* T and a *technique* τ for solving task $t \in T$ together order a pair denoted by $\Pi = [T / \tau]$ and is called the praxis block (Chevallard, 2019). Furthermore, the anthropological approach posits that each way of working, investigating, or solving a task needs the existence of a technique (Barbé et al., 2005).

The second block of a praxeology is the logos, or knowledge block, whose aim is to offer the mathematical discourse required to justify and interpret the praxis block (Barbé et al., 2005).

Concretely, the logos block contains the levels *technology* θ and *theory* Θ, where the first-mentioned are utilized to give an explanation, justification, and description of the used techniques whereas the theoretical level refers to the formal justification of the technologies

(Barbé et al., 2005, p. 237). The theoretical level contains general models, notions and basic assumptions which validate the technological discourse and organize the praxeological elements (Bosch, 2015, p.54). Thus, the technology θ aims to make the technique τ understandable and explain and describe what and why it is as it is (Chevallard, 2019), while the theory level Θ is a mathematical discourse used to control, justify, and make a technology (or a lot of technologies) θ understandable (Chevallard, 2019). To specify, the theoretical level contains a general justification of the practice and could for instance be models, assumptions, and concepts, where it applies that it is often implicit for the students (Bosch, 2015, p.54). The technology and theory levels together constitute the logos block denoted by $\Lambda = [\theta/ \Theta]$.

Hence, these two inseparable blocks, the praxis and logos blocks, are fundamental parts of the anthropological model of mathematical activity. These two blocks form a mathematical praxeology (also denoted mathematical organizations or mathematical praxeologies) (Barbé et al., 2005) written in the form $\Pi \oplus \Lambda = [T/ \tau] \oplus [\theta/ \Theta] = [T/\tau/\theta/ \Theta]$ (Chevallard, 2019).

According to ATD, the reason for the interrelated connection between the praxis and logos blocks within a praxeology $[T/\tau/\theta/ \Theta]$ is that “no human action can exist without being, at least partially, “explained”, made “intelligible”, “justified”, “accounted for”, in whatever styles of “reasoning” such an explanation or justification may be cast” (Chevallard, 2006, p.23).

2.1.1 How is ATD and praxeologies used in the research

In analyzing, teaching, or learning situations of mathematical content, it applies that in the interpretation of the mathematics, some questions arise related to these contents. These questions could be “what is elementary algebra or geometry, or statistics? How is it interpreted in a given educational institution? Why is it taught? How is it related to other content?” (Ruiz-Munzón, Bosch, & Gascón, 2013, p. 3). These questions, and the like, can be examined on the basis of ATD and epistemological reference model (EMR) for the different mathematical domains involved in teaching and learning processes (Ruiz- Munzón et al., 2013, p.3), where these models are formulated in terms of and consisting of praxeologies. These models are always temporary and often need to be modified.

According to ATD, a praxeological reference model (PRM) has some specific features where the empirical data used for developing the model is from the school mathematics and the different institutions, such that the school, policymakers etc., which is a part of the transformation of the mathematical object or body of knowledge. A PRM contains “concrete activities that can be considered as the raison d’être of the mathematical content involved in terms of problems to be solved or questions to be addressed, as well as the way it evolves to give rise to new problematic questions" (Ruiz- Munzón et al., 2013, p.3).

ATD and PRM can be used in many varied circumstances. In the following, research that has used ATD and PRM in practice will be presented.

2.1.2 ATD and PRM in practice

The didactical theory, ATD, and reference models have been used in many circumstances among other Putra (2018) and Wijayanti & Winsløw (2017).

For instance, Putra (2018) has, in his PhD Thesis, used ATD and reference models for rational numbers to study and compare Indonesian and Danish pre-service teachers' mathematical and didactical knowledge of rational numbers.

Wijayanti & Winsløw (2017) have used ATD and demonstrated how the notion of praxeological reference model allows us to analyze the mathematical content in textbooks. According to Wijayanti & Winsløw (2017) this approach enables us to make an objective and detailed praxeological reference model "which could contribute to "common measures" for both comparative and historical studies of how a sector or theme appears in mathematics textbooks" (p. 327), where the teachers are able to use this model for comparing different types of task from the textbooks with types of task that occur in national examinations.

2.2 What is algebra? Relation between algebra and arithmetic

In the following, algebra will be elucidated from different perspectives.

2.2.1 Algebra as generalized arithmetic

Arithmetic has an essential role in the development of algebra, where research shows that algebra develops in close interaction with arithmetic. Students have a close relation to arithmetic and therefore it appears as a reference point for the students' later work with algebra (Bolea, Bosch, & Gascón, 2004). Specifically, this means that algebra in school occurs as "algebraic language", which refers to "a way of expressing the general properties of arithmetic operations" (Bolea et al., 2004, p.126) and algebra is therefore understood as generalized arithmetic. This prevailing view of algebra as generalized arithmetic is seen in types of school mathematical tasks such as: 1. Writing numerical expressions with symbols that describe and generalize arithmetic calculation techniques (Bolea et al., 2004, p. 126), 2. Manipulating algebraic expressions by simplifying or transforming it. Furthermore, letters in the expressions represent unknown numbers, meaning that tasks about solving equations are understood as equalities between algebraic expressions., 3. A type of task is solving word problems by using equations. It consists of translating a verbal problem by letting the unknown quantities and the numerical values get a name (Bolea et al., 2004).

Beyond the view of algebra as a generalized arithmetic, it is also possible to consider algebra as a process of algebraization.

2.2.2 Algebra as a process of algebraization

From an ATD point of view school algebra is considered as a *process of algebraization* of already existing mathematical praxeologies, which assumes that algebra does not occur as a content of its own in the same way as other mathematical praxeologies, like arithmetic, statistics, or geometry in school (Ruiz-Munzón et al., 2013). In this view, algebra is considered as a general mathematical modelling tool for any school mathematical praxeologies i.e., it is regarded as a tool for modelling mathematical systems (Bolea et al., 2004) which affects all sectors of mathematics (Ruiz-Munzón et al., 2013).

Ruiz-Munzón et al. (2013) points out that “algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge” (p. 4), which emphasizes the crucial role of algebra as a tool to consider theoretical questions that occur in different school mathematical domains, such as arithmetic and geometry, in which these theoretical questions cannot be solved within these mathematical domains (Ruiz-Munzón et al., 2013). Work with patterns or sequences is an example of this, in which a building principle is presented, and one must make a prediction before determining the rule or general law that characterizes it (Ruiz-Munzón et al., 2013). From this, another important property of algebra, universal arithmetic, appears which contains “the possibility of using it to study relationships independently of the nature of the related objects, leading to generalised solutions of a whole type of problems, instead of a single answer to isolated problems, as is the case in arithmetic” (Bosch, 2015, p.61).

Algebra, as a modelling tool, has the property of giving "answers to questions related to the scope, reliability and justification of the mathematical activity which is carried out in the initial system" (Bolea et al., 2004, p.127), where it should further apply that the algebraic model has the possibility to give a description, generalisation and justification of problem-solving processes, where it can collect techniques and problems that tend to be unrelated at first (Bolea et al., 2004).

Furthermore, by using the algebraic modelling, it is possible to get an expanded and "a progressive transformation of the initially system" (Bolea et al., 2004, p.127) with further addition of new kinds of problems and techniques for solving these problems and new links to other fields. Additionally, expressions in the algebraic modelling process contains letters, which does not only include numbers but magnitudes too.

2.2.3 The algebraization process

Ruiz et al. (2013) presents an epistemological reference model (ERM) for elementary algebra, which starts with arithmetical praxeologies and their connection with techniques built around a set of *calculation programmes* (CP). A CP is a sequence of arithmetic operations that can be performed “step by step” on an initial set of numbers or quantities and gives as a result a final number of quantity (Ruiz-Munzón et al., 2013). Through the production and use of the calculation programmes, it is possible to solve the corpus of problems of classic elementary arithmetic. But the use of CP will cause some technical limitations and theoretical questions about the reason of the obtained results, inclusive a justification and interpretation of it, and the relation between different kinds of problems and techniques will appear (Ruiz-Munzón et al., 2013). Because of these questions, an expansion of the initial system occurs where modelling is applied and leads into three different stages of the “algebraization” process (Ruiz-Munzón et al., 2013).

The first stage of the algebraization process concerns making the structure of the CP explicit, where it is essential to look at a CP as a whole and not only as a process (Ruiz et al., 2013). In this stage, it is not required to use letters for indicating the different numbers and quantities that occur in a CP, but it is necessary to pay attention to the hierarchy of arithmetic operations and the bracket rules. In this stage new techniques as creating and simplifying algebraic expressions were constructed and "this stage requires the operation of "simplifying" and "transposing" equivalent terms, but not the operation of "cancelling""(Ruiz-Munzón et al., 2013), where the notion of algebraic expressions is considered as the symbolic model of a CP. In continuation of these new techniques, a new theoretical block was developed to justify these new techniques.

The second stage of the algebraization process happens when it is necessary to consider the relation between variables of a CP. Specifically, this stage presents a new mathematical object including the consideration of equations and the accompanying techniques for solving these equations (Ruiz-Munzón et al., 2013). This stage furthermore contains the solution of equations with one unknown and one parameter, where problems in this stage "are modelled with CP with two arguments and the solutions are given as a relationship between the arguments involved" (Ruiz-Munzón et al., 2013, p. 5). The problem will contain solution of a one-variable equation if one of the arguments get a concrete numerical value. In this second stage of modelling, there exists a clear distinction between "parameters" and "variables" such that they cannot substitute each other (Ruiz, Bosch, & Gascón, 2007).

The third, and last, stage of the algebraization process occurs when "the number of arguments of the CP is not limited and the distinction between unknowns and parameters is eliminated" (Ruiz-Munzón et al., 2013, p.5), where the roles of "parameters" and "variables" are exchangeable (Ruiz et al., 2007). This stage is furthermore characterized with the fact that the new mathematical organizations include the production, transformation, and interpretation of formulas.

These stages will be illustrated by an example in Ruiz-Munzón et al., (2013, p.6), which is the following task:

"Think of a number, multiply it by 4, add 10, divide the result by 2 and subtract the initial number"

This task can be represented by a CP illustrated by $P(n) = \frac{4n+10}{2} - n$. The first stage of the algebraization process begins with the problem of solving $P(n) = 7$, where the solving starts with simplifying the algebraic expression of $P(n)$ and subsequently find the equivalence $P(n) \equiv n + 5$ i.e., $P(n) = \frac{4n+10}{2} - n = n + 5$. This simplification and the equivalence expression $P(n) = n + 5$ easily show that for $n = 2$ it applies that $P(n) = 7$ (Ruiz-Munzón et al., 2013).

The second stage of the algebraization process will begin with the solving of the problem in the form $P(n) = 3n - 7$. Problems in that form are possible to solve in the first stage, but it is more complex than using techniques from the second stage.

For finding the solution of that problem it can be done by solving the equation $n + 5 = 3n - 7$. For instance, if the task was *"Think of a number, multiply by 4, add another number… divide the result by 2 and subtract the initial number"* (Ruiz-Munzón et al., 2013, p. 6), it can be represented by a CP illustrated by $P(n, a) = \frac{4n+a}{2} - n$, with a as a parameter. With a CP as $P(n, a) = \frac{4n+a}{2} - n$ and with the problem $P(n, a) = 2n - a$, this can be solved by the same type of techniques and theory as before, where the situation now is solving the equation $n + \frac{a}{2} = 2n - a$ and get $n = \frac{3a}{2}$. The solution is a relation between n and a (Ruiz-Munzón et al., 2013) .

The third stage of the algebraization process is achieved when the CP includes more than two arguments such that $P(n, a, b) = \frac{4n+a}{2} - b$. A solution for a problem related to this CP needs new techniques for describing the obtained relation (Ruiz-Munzón et al., 2013).

There exist different reasons for making an epistemological reference model for elementary algebra explicit. This model can be used as a tool to analyze, examine, and describe what kind of algebra is taught and learnt in various educational systems, what elements are absent and what elements can be integrated in any teaching process (Bosch, 2015).

In conclusion, although algebra is understood as generalized arithmetic and as a modelling tool in two separate ways this is not the case. Algebra involves arithmetic with letter, which is an abstract and generalized arithmetic in which letters stand for numbers. The fact that letters represent numbers is exactly what models do, so letters and symbolic language are used as a kind of model for arithmetic. Therefore, there is not much difference between algebra as a generalized arithmetic and algebra as a modeling tool. The crucial difference is that algebra as a modeling tool is a more general modeling tool which do not only model numbers (which is the case in algebra as generalized arithmetic). So, algebra as a modeling tool is more comprehensive and where letters for instance in mathematics, and other scientific subjects, are used for other abstracts magnitudes than numbers.

2.3 Four conceptions of school algebra and variables

Usiskin (1988) presents four conceptions of school algebra and the uses of variables. He points out that school algebra is about the use and understand of "letters" and their operations, where the letters refer to variables, but since variables contains many aspects, Usiskin (1988) points out that it is not adequate to define algebra as the study of variables. Variables can occur in many ways. Variables can occur in formulas such as $A = LW$, where A, L and W stand for quantities area, length and width and can be considered as known. Variables in equations such as $40 = 5x$ can occur, where x is an unknown, and for an identity such that $\sin(x) = \cos(x) \cdot \tan(x)$, x represents an argument of a function (Usiskin, 1988).

In general, many students think that variables are letters that stand for numbers, but this is not always the case. For instance, in geometry variables are used as representing points; for example, by representing the points A, B and C in a triangle (Usiskin, 1988). It is noted in Usiskin (1988) that students believe that a variable is always a letter, which is confirmed by many educators who emphasizes that students think that $3 + x = 7$ and $3 + \triangle = 7$ are algebra, while expressions like $3 + \_\_\_\_ = 7$ are not algebra. Four conceptions of algebra, presented by Usiskin (1988), will be reviewed in this section.

The first conception of algebra he presents is algebra as generalized arithmetic, which has been presented earlier. Usiskin (1988) posits that in this conception, variables are seen as pattern generalizers. This can be illustrated with the example $3 + 5.7 = 5.7 + 3$ which can be generalized to $a + b = b + a$ (Usiskin, 1988). In this conception, the key instructions for the students are *translate* and *generalize* and furthermore unknowns do not exist in this conception of algebra since the aim of this conception is to generalize known relationships among numbers (Usiskin, 1988).

The second conception of algebra presented in Usiskin (1988) is algebra as a study of procedures for solving certain kinds of problems. Specifically, a problem could be "When 3 is added to 5 times a certain number, the sum is 40. Find the number." (Usiskin, 1988, p. 9), which translated to algebraic language will be $5x + 3 = 40$, and then solve it by add -3 on each side and finally divide with 5 on each side. In this conception, the variables involved are called unknowns or constants and the key instructions in this conception are to *simplify* and *solve*.

The third conception of algebra according to Usiskin (1988) is algebra as the study of relationships among quantities for instance the area formula for rectangle $A = LW$. In this example, the relationship between three quantities is illustrated. In such a conception of algebra, the involved variables vary. In this conception a variable is understood as an argument or as a parameter, where the first mentioned means that a variable "stand for a domain value of a function" (Usiskin, 1988, p.10) while a variable as a parameter means that the variable stands for a number such as the other numbers depend on this variable. Furthermore, only in this conception the notions of dependent and independent variables occur.

According to Usiskin (1988), the fourth and last conception of algebra is algebra as the study of structures. This conception understand algebra "as the study of structures by the properties we ascribe to operations on real numbers and polynomials" (Usiskin, 1988, p. 11). Look at the problem "Factor $3x^2 + 4ax - 132a^2$" (Usiskin, 1988, p.11). In this problem, the variable is not understood as one of the earlier presented ways since the variable is not a function or relation, it is not an argument and there is no equation that must be solved so the variable is not considered as an unknown too. Furthermore, there is not any arithmetic pattern to generalize, so the variable in the above-mentioned problem is not the same as one of the earlier presented conceptions of variables. The answer to the problem is $(3x + 22a)(x - 6a)$. The students can check whether it is correct by multiplying the binomials which is exactly the same way that the students have get the answer in the beginning. Usiskin (1988) points out that although "it is silly to check by repeating the process used to get the answer" (p.11), the students work with the variables as marks on a paper. In this fourth conception of algebra, a variable is much more than an arbitrary symbol and in this conception the variable are used as arbitrary marks on paper.

2.4 Didactics of arithmetics with focus on students' misconceptions

In the following section, central misconceptions, and challenges in arithmetic and algebra will be identified and examined based on previous research and literature.

Before the detailed presentation of the central misconceptions students may have in arithmetic and algebra, it is important to outline what the term misconception explicitly involves. In Durkin & Rittle-Johnson (2014) a misconception is defined as: "a label for synthetic concepts that do not match the accepted view and that form as students attempt to integrate existing knowledge with new information, before deeper conceptual change occurs" (p.22).

From a more ATD point of view, a misconception is a well-established praxeology in the students, that does not match the accepted praxeologies in mathematics. Based on the notions in ATD, is it possible to distinguish between technical and theoretical misconception. A technical misconception is the use of a wrong technique, which could be the common misconception related to addition or subtraction of fractions, which involves adding or subtracting the numerators and denominators separately i.e.

$$\tau_1^*: \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c+c}$$

$$\tau_2^*: \frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

Where τ_n^*, for $n \in \mathbb{N}$, denote a wrong technique. These are examples of techniques that does not conform to the existing institutional techniques about addition and subtraction of fractions with like or different denominators.

A more theoretical misconception could be a wrong definition of a notion. For instance, a theoretical misconception related to fraction could be to consider $\frac{a}{b}$ as two separate numbers above each other, instead of considering it as a number, which illustrates a wrong theoretical definition of the notion of fractions.

2.4.1 Students' misconceptions of the equal sign (=)

This section examines research dealing with the use and understanding of the equal sign amongst students. Understanding the equal sign is crucial to students' understanding of algebra, which is why it is important for students to have a relational understanding of the equal sign early on.

The equal sign is a relational symbol, which emphasize that two sides of an equation are equal and interchangeable (Byrd, McNeil, Chesney, & Matthews, 2015), but the dominant understanding of the equals sign among students is an operational approach, which includes the following understandings: "do something signal" (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007, p. 223), "the answer is" (Bush & Karp, 2013, p. 620) or "add the number" (Alibali et al., 2007, p. 227). Almost all students at all grades have developed an insufficient understanding of the equal sign, which shows that the notion of 'equal' is complex and difficult for students to understand (Alibali et al., 2007).

Based on some research, the following section will present different types of task which specifically show the students' operational and relational understanding of the equal sign and what consequences this understanding has for the students' ability in algebraic problems.

Falkner et al. (1999) (in Alibali et al., 2007) found that the students' solutions to the problem $8 + 4 = \boxed{} + 5$ clearly illustrates the students' understanding of the equal sign. The students' solutions were either 12 or 17, which were dominant answers among first- through sixth-grade students in Falkner et al. (1999), and these solutions "are consistent with a view of the equal sign as announcing the result of an arithmetic operation" (Alibali et al., 2007, p. 223). According to Stephens, Knuth, Blanton, Islera, Gardiner, Marum (2013), students typically answer in one of three ways: First, students may understand the equal sign as 'the answer comes next' concept and therefore answer that the missing number is 12. Secondly, some students may use the concept "use all numbers" and then let the missing number be 17. Third, some students may use the concept to 'extend the problem' and write 12 in the space of the missing number, but then add another equal sign in the following way: $8 + 4 = 12 + 5 = 17$. These three inaccurate ways of answering the above-mentioned question illustrate an operational view of the equal sign, which can cause problems.

In cases of solving elementary school arithmetic problems such as: $3 + 8 = \square$ (Alibali et al., 2007, p. 223), it is sufficient to use an operational understanding of the equal sign, but in later grades when the problems and equations get more complex, such as: $3x + 5 = 11$ or $2x - 3 = 4x + 5$ (Alibali et al., 2007), the operational understanding will not be useful, and the understanding of the equal sign as a symbol which represents a relationship between quantities rather that a signal to perform arithmetic operations, will be necessary. Hence, it is a necessity for students to develop a relational understanding of the equal sign for later work with algebraic problems.

Steinberg, Sleeman, and Ktorza (1990) in Alibali et al. (2007) found that although many eighth- and ninth-grade students could solve and make transformations in equation solving, some problems occur in a type of task where the students have to determine whether two equations were equivalent, where some transformations had been applied to the equation. These equations $2x + 6 = 10$ and $2x + 6 - 8 = 10 - 8$ could be an example of this, since these tasks require an understanding that the transformation preserves the equivalence relation expressed in the first equation (Alibali et al., 2007). The students with a relational understanding of the equal sign will, in such a problem, recognize that "the transformation performed on the second equation of each pair preserved the quantitative relationship expressed in the first equation of each pair" (Alibali et al., 2007, p. 226) and then answer that the solution of the equations is the same. Similar type of task is demonstrated in Byrd, McNeil, Chesney & Matthews (2015).

Elementary school students understand $+, -$ and $=$ as actions to be performed and Ginsburg (1977) in Kieran (1981) found that for the problem $3 + 4 = \square$ some students answer that "the equal sign means what is adds up to" (Kieran, 1981, s.318) while another says "3 and 5 make 8"(Kieran, 1981, s. 318), and for problem like $\square = 3 + 4$ students would answer that ""blank equals 3 plus 4", but then add: "It's backwards! Do you read backwards?"" (Kieran, 1981, p.318), which would make them read the problem as $4 + 3 = \square$. These observations show that a consequence of students' understanding of equalities as actions is that they found it "difficult to read arithmetic sentences that do not reflect the order of his calculations" (Kieran, 1981, p. 318). A further consequence of this is that students have problems reading sentences like $3 = 3$ since they believe that the answer must be after the equal sign (Kieran, 1981).

A study by Behr, Erlwanger, and Nicols (1976) in Kieran (1981) shows that students in grades 1 to 6 do not change their understanding of the equal sign as the change to upper grades, where Behr et al. (1976) in Kieran (1981) found that sixth-grade students understand the equal sign as a "do something signal". Another understanding of the equal sign presented by Behr et al. (1976) is that “After “=” should be your answer. It’s the end, not another problem” (Kieran, 1981, p. 319) and these understanding of the equal sign shows that the sign is understood as an operator and hence as an action to perform, and not as a relational symbol.

Stacey and MacGregor (1997a) in Bush & Karp (2013) point out that the students’ misconception about the equal sign can be observed in students' work when they link many unequal problems with an equal sign. This is also noted in Kieran (1981) who identifies that there exists a misconception of the equal sign which occurs during students' written work. This misunderstanding is expressed in instances where students use the equal sign between each calculation, which is why false equalities occur; for example, in the following word problem example: ““In an existing forest 425 new trees were planted. A few years later, the 217 oldest trees were cut. The forest then contains 1063 trees. How many trees were there before the new trees were planted?”” (Kieran, 1981, p.320), where the students’ calculation was expressed in the following way $1063 + 217 = 1280 - 425 = 1063$. This calculation shows that the difficulty for the students is at the symbolic level, where the operations are written in the order in which the students think about the problem (Kieran, 1981).

It is noted in Welder (2012) that a correct interpretation of the equal sign thus a relational view, is central for learning algebra. The reason is that a relational view of the equal sign is crucial to manipulate and solve equations, to understand that the two sides of an equation are equivalent and that it is always possible to change an equation with another equivalent equation. Furthermore, algebraic reasoning is rooted in an understanding of the equality and "appropriate use of the equal sign for expressing generalizations" (Welder, 2012, p. 258). Pointed by Carpenter, Franke & Levi (2003) in Welder (2012), one of the biggest factors that can occur and impact the students learning of algebra, is the limited understanding of the equal sign they have.

In conclusion, an operational view of the equal sign will create limitations and act as a barrier when students encounter algebra, as this approach will make it difficult for students to work with equations and for instance isolate variables. This limitation, in turn, will possibly not be present in students with a relational view.

2.4.2 Students' misconceptions of decimals

This section looks at research dealing with the misconceptions of the decimal numbers, which is a major topic that has been researched and discussed in many parts of the world in many years. A crucial task concerning decimals has been the comparison of the magnitudes of two or more decimals, and this type of task has been used as a basis for studies about misconceptions of decimals (Stacey, Helme, & Steinle, 2001).

Students' knowledge about the natural numbers becomes a barrier for them in the learning of decimal numbers, as students may have a tendency to use the properties of natural numbers on decimal numbers.

Misconception about decimals is formed by the transition from the perception of numbers as natural numbers to the perception of numbers that contain natural, rational, and real numbers. Durkin & Rittle-Johnson (2014) point out that three common misconceptions about decimals are defined by Resnick, Nesher, Leonard, Magone, Omanson, & Irit Peled (1989) and Irwin (2001). These misconceptions and others also exist in later research, which will be presented as well.

The three common misconceptions about decimals presented by Resnick et al. (1989) and Irwin (2001) in Durkin & Rittle-Johnson (2014) are 1) The whole number misconception, 2) The role of zero misconception, and 3) The fraction misconception.

The whole number misconception contains the interpretation of decimals as whole numbers, where the properties of the whole numbers were used on the decimals. A concrete task which examines this is when students must evaluate which of the following decimals 0.25 and 0.7 is largest. By thinking of decimals as they contain all the properties of whole numbers, students incorrectly used their knowledge about whole numbers to decimals and then the students will express that 0.25 is larger than 0.7 since 25 is larger than 7 (Durkin & Rittle-Johnson, 2014). This misconception appears in Bush & Karp (2013) and Irwin (2001) where they have pointed that longer decimal fractions are larger i.e., in a comparison between the magnitude of 0.456 and 0.47 students answer that 0.456 is larger since it has more digits.

The second misconception presented by Resnick et al. (1989) and Irwin (2001) in Durkin & Rittle-Johnson (2014) is the role of zero misconception. This misconception contains the role of zero, which is about students "overgeneralization of a particular property of whole numbers to decimals" (Durkin & Rittle-Johnson, 2014). To specify, it means that students ignore the zero when it is placed in the tenths place which means that they consider 0.07 and 0.7 as the same value, again by using their knowledge about whole numbers an incorrect way on decimal numbers. A type of task that examines this misconception in Durkin & Rittle-Johnson (2014) is "Circle all the numbers that are worth the same amount as 0.51: 0.5100; 0.051; 0.510; 51" (p. 25). Furthermore, students think that when a zero is added to the end of a decimal, it becomes 10 times larger i.e., students think that 0.320 is larger than 0.32 since a zero is added (Durkin & Rittle-Johnson, 2014).

The third misconception is related to students understanding of fractions. A misconception pointed out in Durkin & Rittle-Johnson (2014) is that students use their knowledge about fractions on decimals. Students know from their prior knowledge of fractions that the longer the denominator is in a fraction, the smaller is the value of the fraction. By using this knowledge in an incorrect way, students believe that the longer the decimal is, the smaller is the value of it since they contain smaller parts, which is the case for fractions. Thus, the students will believe that 0.784 is lower than 0.3 since 0.784 is thousandths while 0.3 is tenths. Hence, as with the whole numbers, students use their knowledge about fractions in an incorrect way regarding to decimals.

A type of task examined in Durkin & Rittle-Johnson (2014), which shows that all these three misconceptions can occur, is a task concerning the choice between a correct decimal on number line i.e., "What number tells about where the slash is on the number line? 0.76 0.3 0.08 0.401 with slash at 0.76" (Durkin & Rittle-Johnson, 2014, p.25). If the students answer is 0.401, it shows that the whole number misconception occurs, as the students indicate that since 401 is largest then 0.401 should be largest among the four decimals. If the students identify that 0.08 is largest, the role of zero misconception occurs, since the students consider 0.08 to be the same as 0.8 and then conclude that 0.8 is largest. Finally, if the students indicate that 0.3 is largest, the third above-mentioned misconception occur, since the students use the properties for fractions on the decimal and then conclude since 0.3 is the decimal with shortest decimals then it must be the one with largest value among the other three decimals (Durkin & Rittle-Johnson, 2014).

A misconception, which exists but is less prominent among students, is that longer decimal fractions are smaller (Resnick et al., 1989) and (Irwin, 2001). To specify, this means that when students must examine whether 1.35 or 1.2 is smallest, they choose 1.35 since it has a longer decimal than 1.2.

2.4.3 Students' misconceptions of fractions

Another important concept which has an influence on students' success in mathematics, concretely in algebra, is fractions (Barbieri, Rodrigues, Dyson, & Jordan, 2020). In Barbieri et al. (2020) it is pointed out that students who begin in seventh grade without any knowledge of fractions will get many mathematical difficulties why it is important to master fractions. In the following section research and studies about students' difficulties and misconceptions in fractions will be elucidated.

As mentioned in section 2.1, Chevallard (2006) posits that human praxeologies are open to change, and this will be useful in the transition from whole numbers to fractions. The students' needs to change their mathematical praxeologies when move from natural numbers to rational numbers, since it is not possible to use the properties of natural numbers directly on rational numbers. When the students transfer their knowledge of whole numbers to fractions, this can provide problems.

Whole numbers have the property that they are represented linearly, where any number is followed by exactly one number greater than the preceding number. Furthermore, for whole numbers, a number only represents one magnitude. This knowledge can give students difficulties and conflicts in the learning of fractions, since, for fractions it applies that it can be represented in many ways; for instance, $\frac{1}{5}$ and $\frac{2}{10}$, etc. (Barbieri et al., 2020). Furthermore, in determining the magnitude of a fraction, it is necessary to consider the numerator and the denominator at the same time, and not separately as whole numbers, which students may tend to do. That students consider the numerator and the denominator separately as whole numbers, can be illustrated in a type of task, where they have to determine the magnitudes of the fractions $\frac{1}{4}$ and $\frac{1}{2}$. By considering the numerator and denominator separately as whole numbers, students will tend to answer that $\frac{1}{4}$ is greater than $\frac{1}{2}$ since 4 is greater than 2 (Barbieri et al., 2020).

Ashlock (2006), in Bush & Karp (2013), has identified some misconceptions where few of these will be illustrated in the following. A type of task where students must simplify fraction to lowest terms, the students tend to simplify only the numerator and not the denominator for instance they simplify $\frac{5}{10}$ to $\frac{1}{10}$.

In addition and subtraction of fractions Ashlock (2006) in Bush & Karp (2013) found that students add or subtract the numerators and add or subtract the denominator separately, which could be due to overgeneralization of whole number knowledge to fractions. According to Namkung, Fuchs & Koziol (2018), the utilization of whole-number concepts into the understanding of fractions could lead to substantial misconceptions about fractions since fundamental differences exist between whole numbers and fractions. The difference is first the symbolic representation which is different for whole numbers and fractions. While a whole number is one number, a fraction is a number represented by two numerals and a fraction bar (Namkung, Fuchs & Koziol, 2018). With whole numbers it is possible for students to "counting on" for placing numbers in order, while no discrete number precedes a fraction and there exist infinite quantities of fractions between any two fractions (Namkung, Fuchs & Koziol, 2018). So, the misconceptions which occur according to Namkung, Fuchs & Koziol (2018) are that students consider the numerator and denominator in a fraction as independent numbers. This is possible to observe, as mentioned, in tasks about comparing the magnitudes of fractions or ordering fractions, where these tasks are solved by the students with their knowledge about whole numbers (Namkung, Fuchs & Koziol (2018). So, by using the whole-number knowledge the students order the fractions $\frac{1}{2}, \frac{1}{8}$ and $\frac{1}{12}$ as $\frac{1}{2} < \frac{1}{8} < \frac{1}{12}$ because $2 < 8 < 12$ (Namkung, Fuchs & Koziol (2018). Another identified misconception, by Ashlock (2006) in Bush & Karp (2013), with addition and subtraction of fractions is that students identify a common denominator but fail to change the whole fraction into an equivalent form.

With multiplication and division of fractions some misconceptions are identified too. According to Ashlock (2006), in Bush & Karp (2013), students tend to multiply two fractions by cross-multiplying as they are used to from addition and subtraction of fractions. When dividing two fractions, it seems that the students divide the numerators and then divide the denominators, which illustrates how the students overgeneralizing whole number operations. When changing a whole number into a fraction, students may do it wrongly by for instance change the whole number 6 into $\frac{6}{6}$ instead of $\frac{6}{1}$ as described in Ashlock (2006) in Bush & Karp (2013).

Pointed by Deringöl (2019) one of the main reasons for why the students consider fraction and fraction operations as difficult is that students memorize formulas and algorithms without any understanding of fractions. Some findings, outlined in Deringöl (2019), suggest that "an early and hasty transition to representation of fractions in the classroom with abstract symbols without dependence on student experience and a basic conceptual framework leads to misconceptions", why the presentation of fractions in the teaching has an impact on the misconceptions students developed later.

2.4.4 Students' misconceptions in calculation of arithmetic expressions and negative numbers

In the following section, students' misconceptions in calculation of arithmetic expressions and negative numbers will be elucidated.

Understanding the correct use of integers and order of operations with whole numbers, fractions, and decimals are important skills for students to succeed in algebra, but many misconceptions in relation to this are identified in the literature and research.

In Bush & Karp (2013), much research and studies about the use of negative numbers are presented. One of these is Ashlock (2006), in Bush & Karp (2013), who has identified some misconception involving integers. The students know to subtract when they take the sum of a positive and negative integer, but they do not know which sign the resulting answer should have. Furthermore, another misconception identified in Ashlock (2006) in Bush & Karp (2013) is that students have the understanding that a sum of two negative numbers is a positive number, which according to Ashlock (2006), may be because multiplication and division of two negative numbers is positive, so the students master a technique but use it in the wrong way.

A study by Kloosterman (2012) in Khalid & Embong (2020) shows that a quarter of 13-year-old students have problems with addition of positive and negative numbers in a correct way, where half of these students have difficulties with division of integers too. Such difficulties and misconceptions have been investigated and documented globally and one of these is Khalid & Embong (2020).

Khalid & Embong (2020) has investigated the misconceptions that year 7 students in public Malaysian schools have when they solve tasks involving addition, subtraction, multiplication, and division with integers. In this study, Khalid & Embong (2020) found that some students make mistakes in subtraction of negative numbers such as $-6-(-2)=-(6+2)=-8$, where this type of mistake, according to Khalid & Embong (2020), is due to a lack of, or poor knowledge of, parentheses where students are not able to remove parentheses correctly.

In a task where the students must calculate $-6-(-2)$, some students answer that $-6-(-2)=6-2=4$. According to Khalid & Embong (2020), this mistake is due to fact that the students still treat the integers as whole numbers, which indicates that they ignore the negative signs and operations and calculate the expression as whole numbers. This is exactly the case for the calculation for $-6-(-2)$ where students ignore the negative signs and transform the expression to $6-2=4$. Another incorrect technique the students use is that they add the negative sign to the answer they have produced in their calculation.

A type of mistake, which a majority of the Malaysian students made, is what Khalid & Embong (2020) called rule-mix. Many students learn and remember rules as negative and negative becomes positive, and this rule is used by students in another situation where it is not appropriate. Although this rule is used for multiplication and division, some students use this rule in addition and subtraction. For the tasks $-2-6$ and $-6-(-2)$ some Malaysian students has the answers $-2-6=8$ and $-6-(-2)=8$ and Khalid & Embong (2020) point out that those students who answer $-2-6=8$ and $-6-(-2)=8$ have argued that "negative meets with negative becomes positive" (p. 4), which illustrate how students use the negative and negative becomes positive rule incorrectly. So, the students master the technique that negative multiplied with negative becomes positive, but by using this technique in addition and subtraction, this illustrates a technical misconception.

In their study, Linchevski and Livneh (1999) in Bush & Karp (2013) asked 53 twelve-year-old students to solve these five tasks:

1. $5 + 6 \times 10 =$

2. $17 - 3 \times 5 =$

3. $8 \times (5 + 7) =$

4. $27 - 5 + 3 =$

5. $24 \div 3 \times 2 =$

In question 1 and 2, the majority of students began with addition instead of multiplication. Questions 3 was answered correct by all students which illustrates, that they know to calculate and simplify the parentheses before other operations. On the other hand, Booth (1988) in Bush & Karp found that the students do not use the parentheses before other operations, but instead think that the operations have to performed from left to right. Both questions 4 and 5 were answered correct by the majority of students, but they notice in an interview that they must add before subtraction in questions 4 and multiply before division in questions 5. These existing misconceptions can be a barrier, hindering students in their learning of and achievement in algebra.

3 Research Questions

Based on the Anthropological Theory of Didactics and the previously presented problematic types of task belonging to arithmetic and algebra i.e., calculation of arithmetic expressions and negative integers, decimals, fractions and the equal sign, this thesis will examine the misconceptions in arithmetic and algebra that may occur among Danish fifth- and seventh-grade students. This will be elucidated through the following three research questions:

> RQ1: What arithmetic and algebraic praxeologies are Danish fifth- and seventh-grade students supposed to learn?
>
> RQ2: To what extent are these praxeologies actually mastered by the students?
>
> RQ3: How can teachers be usefully informed of answers to RQ2?

The first research questions will be examined and answered by constructing a praxeological reference model for the topics of arithmetic and algebra for Danish fifth- and seventh-grade students respectively. The construction of the models will be based on an analysis of two different textbooks for Danish fifth- and seventh-grade students respectively and a review of the official programmes (Official programme, 2019) for these two grades. Although a praxeological reference model is a model with all praxeologies, the developed praxeological reference models in the thesis will mainly be based on the praxis block and supplemented with some technological and theoretical elements.

The second question will be answered and investigated by developing a diagnostic test, which is done by using the developed praxeological reference models. Since the praxeological reference models explicitly declare what the students should master in the different grades, these models illustrate what the test is specifically about. The answer to RQ2 is thus about developing a method to answer this question, which a diagnostic test can fulfill. Analysis of the test answers from students in fifth and seventh grade will also be used as an answer to RQ2.

The third research question will be answered in continuation of the results from the second research question, where the feedback for the teachers of the fifth and seventh grades will be based on the results from RQ2. All three research questions are therefore interrelated.

4 The context

To investigate to what extent the arithmetic and algebraic praxeologies are mastered by Danish fifth- and seventh-grade students, a collaboration with Nkøbing municipality took place. Notice that the two schools involved in the collaboration will be called School A and School B from Nkøbing municipality.

The main purpose for the collaboration was to develop a diagnostic tool, which consists of two parts: a diagnostic test for the students in fifth and seventh grade, which examine the students' knowledge (and lack of knowledge) and misconception in arithmetic and algebra, and a guide for the teachers, which contain information and results from the tests.

The praxeological models are based on the official programmes (Official programme, 2019) for Danish fifth- and seventh-grade students and the textbooks that are used in fifth and seventh grades in Nkøbing municipality. Concretely, the textbooks are Gyldendal MULTI 5 i-bog and Gyldendal MULTI 7 i-bog and Alinea Matematrix, (specifically Matematrix 5 and Matematrix 7).

The developed diagnostic tests were performed as a pilot test on two Danish fifth-grade students and five Danish seventh-grade student from School A in Nkøbing municipality. The students in the pilot test and in the official test have 60 minutes for solving the tests.

The pilot test was performed at the students' home and not in the schools due to restrictions during COVID-19. Therefore, there may be uncertainties about the results of the pilot test, since it can happen that the students have for instance used a calculator in the test, even though this is not allowed (details in Appendix 1).

The final tests were performed on 86 Danish fifth-grade students and 78 Danish seventh-grade student from School B in Nkøbing municipality. The students and classes for the pilot and final test are selected by the municipal chief consultant for mathematics and science in Nkøbing, which is why I have not had an impact on this selection.

Before performing the tests, it was necessary to inform the teachers about the project (Appendix 2) and inform the parents and ask for their consent.

Finally, a description of the test was made (Appendix 1) which the teachers used to inform the students about the test. Notice that the students were informed that the test was anonymous. This was said, to illustrate that the teachers are not interested in judging the students' individually, but they are instead interested in finding out what the whole class has challenges with.

5 Methodology for RQ1

In the following section, the methodology behind the first research question: what arithmetic and algebraic praxeologies are Danish fifth-and seventh-grade students supposed to learn, will be elucidated. Before the answering and examination of this question, an explanation of how arithmetic and algebra is defined in the thesis will be clarified. In the thesis, arithmetic and algebra is defined as calculation with numbers and calculation with letters respectively, which for instance involves solving of equations. In addition, some choices have been made in the development of the praxeological models. First, not everything in arithmetic and algebra will be incorporated in the praxeological models for instance the notion of percentage. It is of course, important in applications, but because the percentage concept is a variation of fractions, it was chosen to be excluded. Furthermore, the focus will be on tasks related to arithmetic and algebra where algebraic and arithmetical techniques are enough for solving the tasks and where instrumented techniques are not required. The reason for why the focus is not on instrumental techniques are that the non-instrumental techniques are the only relevant ones in the preparation for algebra.

The answer to the first research question will specifically be praxeological reference models (PRM) that illustrates praxeologies on arithmetic and algebra, which Danish fifth- and seventh-grade students may master. This section will explain how the PRMs are developed and why it is developed as it is. These PRMs are developed and constructed along with the analysis of the material and are therefore not independent from these materials. By constructing the models in that way, the analysis will be utterly explicit and as pointed in Wijayanti & Winsløw (2017) the analysis should "be reproducible in the sense that the same analysis would be made by other researchers who have familiarized themselves with the model" (p. 315).

In order to develop these PRMs, it is central to take the official programmes (Official programme, 2019) for Danish fifth- and seventh-grade students, respectively, as a starting point whereby it is possible to see what requirements there are for these students when it comes to contents and competences in different mathematical domains such as arithmetic, algebra, geometry, etc.

The reason why the development of PRMs starts with looking at the official programmes (Official programme, 2019) for Danish fourth to sixth- and seventh to ninth-grade students, is that these are the official guideline for what students are expected to master in different grades. Compared to textbooks, the official programmes (Official programme, 2019) are the material all teachers are required to use, which is why these are the first material to look at for the examination of what arithmetic and algebraic praxeologies are Danish fifth- and seventh-grade students supposed to learn.

Furthermore, it will be relevant to look at official tests for the different grades as an expression of what is required to master in these grades. Such official tests as exam sets are only available for Danish nineth grade students. In these final exam sets it is possible to find what Danish ninth-grade students must master at the end of ninth grade and therefore these sets are not useful in the development of the PRMs for fifth- and seventh-grade students since it is not possible to see in the final exam sets what belongs to Danish fifth and seventh grades.

Since the official programme (Official programme, 2019) is relatively imprecise because it applies for fourth to sixth- and seventh to ninth-grade students, and since there are no official tests for Danish fifth- and seventh-grade students, such as the final exam for Danish nineth-grade students, the most important source for answering RQ1 will be the textbooks for the development of PRMs.

Concretely, the textbooks are Gyldendal MULTI 5 i-bog and Gyldendal MULTI 7 i-bog and Alinea Matematrix, (specifically Matematrix 5 and Matematrix 7). These textbooks have a central role in the formation of the PRMs, since they are the teaching material for the classes, who will be tested. Based on these textbooks, it is therefore possible to specify typical types of task, associated techniques, technological and theoretical elements, related to arithmetic and algebra, the Danish fifth- and seventh-grade students from Nkøbing municipality must master in these two grades, and to test students' actual knowledge about the content they are taught.

The official programmes (Official programme, 2019) in mathematics for Danish fifth- and seventh-grade students do not have explicit contents goals for each grade in arithmetic and algebra, as is the case in Japan for instance (Winsløw, 2018). Therefore, it is central to identify parts of the textbooks which correspond to the topics of arithmetic and algebra.

This was done by examining the contents and the table of contents for the chapters, that contained the following headlines: 1. Calculation of arithmetic expressions involving integers and two or more operations, 2. Calculation with negative numbers, 3. Decimal numbers and calculation with these, determination of magnitude, 4. Fractions and calculation with these, equivalence of fractions, determination of the magnitude of fractions, 5. Conversions between decimals and fractions, 6. Equation solving, and 7. Reduction. This was done for both textbooks for both grades.

The remaining chapters in the books were read, but not as thoroughly as the chapters corresponding to arithmetic and algebra, to ensure that all relevant types of task and associated techniques were identified.

According to ATD, a praxeological reference model always starts with the types of task. Therefore, the first step in analyzing the textbooks was to identify all types of task, both in examples and exercises, corresponding to arithmetic and algebra. After this identification, the next step was to analyze all examples and exercises since these will give explicit information about the required techniques students have to master for solving the tasks. All task were categorized depending on according to the techniques that are required to solve them; so that when a certain task requires a new technique, a new type of task will be added to the model. In that way, all types of task and techniques were identified explicitly. The developed praxeological reference models are both a result of the analysis of the textbooks and in the same way it is a tool of the analysis in the sense that all the tasks related to arithmetic and algebra in the textbooks could be examined on the basis of the models.

In the development of these models, the theoretical elements will not be excluded, but there will be a more peripheral and less systematic attention to these elements.

In conclusion, the praxeological reference models, for the topics of arithmetic and algebra, are mainly developed from these two above-mentioned textbooks and the main purpose for the development of the PRMs is to design the diagnostic tests.

The following section will show examples of how the official programmes (Official programme, 2019) and the textbooks were analyzed to produce the PRM.

5.1 How are the PRMs for Danish fifth- and seventh-grade students developed?

5.1.1 Official programmes for Danish fifth- and seventh-grade students

In the following, parts of the official programmes (Official programme, 2019), dealing with arithmetic and algebra, for Danish fifth and seventh grades will be clarified.

The official programme (Official programme, 2019) for fourth to sixth grade in mathematics, says the following about arithmetic and algebra:

"The area of skills and knowledge focuses on the 2nd stage [which means fourth to sixth grade class] on the development of calculation methods with rational numbers. (…) Gradually, the focus is expanded to include calculations that consist of several steps, and to composite calculations that include several types of calculations. The work includes knowledge about order of arithmetic operations. (…) Throughout the stage, calculation strategies related to simple fractions, decimal numbers, percentages, and simple negative numbers are further developed. (…) In the stage, the teaching focuses on simple equation solving. (…) which consists of problems and calculation from everyday life, among other things that can be described with equations" (Official programme, 2019, p. 22-23)

Although the official programme (Official programme, 2019) shows that Danish fourth to sixth-grade students must, among other things, master fractions, decimals, percentages, simple negative numbers, the hierarchy of arithmetic operations, and simple equation solving, many unclear formulations exist. It is seen in the official programme (Official programme, 2019) that "The area of skills and knowledge focuses on the 2nd stage [which means fourth to sixth grade class] on the development of calculation methods with rational numbers" i.e., the students must be able to calculate with rational numbers, but it is not clear whether the students, for instance, must be able to add two fractions or multiply two decimal numbers.

The official programme (Official programme, 2019) uses concepts as "simple negative numbers" and "simple equation solving", but what does 'simple' actually mean and which type of equations does the official programme refer to? These formulations do not indicate anything about whether it is first degree equation with integer coefficients or clarify what simple negative numbers really are.

Furthermore, the official programme emphasizes:

"Gradually, the focus is expanded to include calculations that consist of several steps, and to composite calculations that include several types of calculations"

This shows that calculations must consist of several steps, but what is meant by 'several steps' is not clear.

Furthermore, "composite calculation" is not explained either, but are clarified by including" several types of calculations".

The same case occurs for the official programme for Danish seventh to ninth grade in mathematics, which says the following about arithmetic and algebra:

"First in the stage, the work from the 2nd stage [means 4.-6. class] with rational numbers is continued. In the teaching, the emphasis is placed on the close relationship between fractions, decimals, and percentages and on the use of these in both theoretical and practical contexts. (…) In the stage the work continues from the 2nd stage regarding the students' development of methods for calculations with whole numbers, fractions, decimal numbers and percentages. (…) From the beginning of the stage, the students work on representing algebraic expressions geometrically and describing properties of geometric figures using algebra. (…) The teaching also aims to enable students to reduce algebraic expressions when appropriate." (Official programme, 2019, p. 29-31).

Danish seventh- to ninth-grade students should master the same mathematical content as the Danish fourth- to sixth-grade students, and further how to solve algebraic expressions and reduction of expressions. In addition to these elements, no other formulations have been given of what the students should specifically learn and master in the different grades. Therefore, the official programme mainly specifies the mathematical domains and sectors the students should meet, but it is too general and without any precise content goals and any specific types of task for each grade.

Another central reason why the official programme is not sufficient to answering RQ1 is that the official programme applies to three years at a time rather than for a single grade or year group.

These deficiencies can be met by turning the attention to the textbooks, where a great variety and a large collection of examples and exercises occurs.

5.1.2 Mathematics textbooks for Danish fifth- and seventh-grade students

In the following, the most important material used to develop the praxeological reference models, namely the textbooks, will be presented and an explanation of how they are used for the development will be central in this section.

The used materials are the textbooks Gyldendal MULTI 5 i-bog (denoted M5) Alinea Matematrix 5 (denoted AM5) for Danish fifth-grade students and Gyldendal MULTI 7 i-bog (denoted M7) and Alinea Matematrix 7 (denoted AM7) for Danish seventh-grade students.

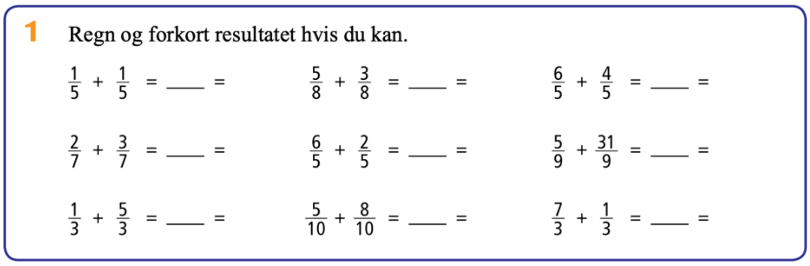
Before the selection of task related to arithmetic and algebra a definition of tasks related to arithmetic and algebra is necessary. As mentioned in section 5, arithmetic and algebra is defined as calculation with numbers and calculation with letters respectively, which for instance involves solving of equations, calculation with fractions etc.

Since in the construction of the praxeological models it is not necessary to include all the material from the textbooks, for instance with the notion of percentage as mentioned in section 5, the following section will present which choices that have been made in the development of PRMs.

5.1.3 Danish fifth-grade students

As mentioned, for the development of the PRMs, an analysis of all examples and exercises occurs. In a detailed reading of AM5 and M5, types of task related to arithmetic and algebra were identified. For the fifth-grade students, types of task related to fractions, decimals and negative integers were identified.

Specifically, nine types of task related to fractions, including arithmetic operations with fractions and determination of magnitudes of fractions, two types of task related to decimals and two types of task related to negative integers were identified in the textbooks. The explanation of each type of task is followed by the technique and an example of how it is illustrated in the textbooks, in Table 1. Some examples of the nine tasks related to fractions, identified in the textbooks could be in AM5 p. 71-72:



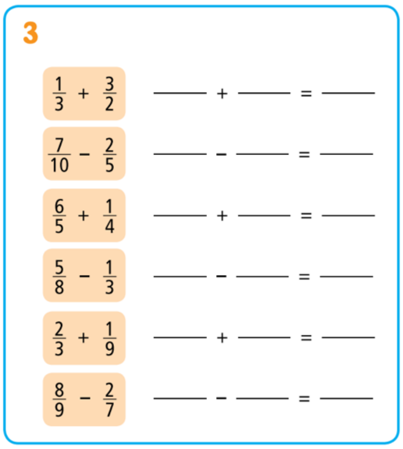
1 Regn og forkort resultatet hvis du kan.

$\frac{1}{5} + \frac{1}{5} = \_\_\_ =$ $\quad \frac{5}{8} + \frac{3}{8} = \_\_\_ =$ $\quad \frac{6}{5} + \frac{4}{5} = \_\_\_ =$

$\frac{2}{7} + \frac{3}{7} = \_\_\_ =$ $\quad \frac{6}{5} + \frac{2}{5} = \_\_\_ =$ $\quad \frac{5}{9} + \frac{31}{9} = \_\_\_ =$

$\frac{1}{3} + \frac{5}{3} = \_\_\_ =$ $\quad \frac{5}{10} + \frac{8}{10} = \_\_\_ =$ $\quad \frac{7}{3} + \frac{1}{3} = \_\_\_ =$

Figure 1: Addition of fractions with like denominator, AM5 p. 71-72



3

$\frac{1}{3} + \frac{3}{2}$ $\quad \_\_\_ + \_\_\_ = \_\_\_$

$\frac{7}{10} - \frac{2}{5}$ $\quad \_\_\_ - \_\_\_ = \_\_\_$

$\frac{6}{5} + \frac{1}{4}$ $\quad \_\_\_ + \_\_\_ = \_\_\_$

$\frac{5}{8} - \frac{1}{3}$ $\quad \_\_\_ - \_\_\_ = \_\_\_$

$\frac{2}{3} + \frac{1}{9}$ $\quad \_\_\_ + \_\_\_ = \_\_\_$

$\frac{8}{9} - \frac{2}{7}$ $\quad \_\_\_ - \_\_\_ = \_\_\_$

Figure 2: Addition/subtraction of fractions with different denominators, AM5 p. 71-72

These tasks all belong to addition and subtraction of fractions but are different types of task and therefore require different techniques. Figure 1 shows a type of task about addition of fractions with like denominators, which requires the technique $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$, denoted as τ_8 in Table 1 and τ_{10} i Table 2. Subtraction of fractions with like denominators are identified too, where the required technique is $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$, denoted as τ_{10} i Table 1 and τ_{12} i Table 2.
Although addition and subtraction of fractions with like denominators can be considered as the same type of task, but with different operators, it is decided to consider these as two different types of task, since they require two different techniques.
In Figure 2 task about addition and subtraction of fractions with different denominators are identified. These are categorized as separate types of task since they require another different technique than the above-mentioned technique for addition and subtraction of fractions with like denominators. For addition of fractions with different denominators the following technique is relevant: $\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}$. This technique is illustrated as τ_9 in Table 1 and as τ_{11} in Table 2.
Subtraction of fractions with like denominators is identified too, where the necessary technique is $\frac{a}{b}-\frac{c}{d}=\frac{ad-cb}{bd}$, denoted as τ_{11} in Table 1 and τ_{13} in Table 2.

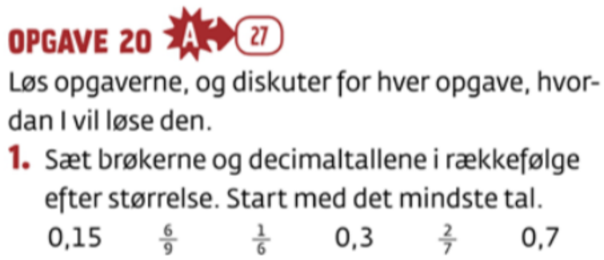
The following task was found in the textbook M5 p. 55:

Figure 3: The magnitude of unit fractions, M5 p.55

Many variations of this task exist in the textbooks. In the analysis of the textbooks three different types of task related to the magnitude of fractions were identified, why three different techniques are emphasized. In the above-mentioned task (Figure 3) students must assess which of the unit fractions $\frac{1}{4}$ and $\frac{1}{5}$ is the lowest fraction. Since the fractions are unit fractions, the techniques related to this type of task is that the fraction with lowest denominator is largest i.e., $\frac{1}{5}$ is the lowest one. This technique is illustrated in Table 1 as τ_{18} and as τ_{24} in Table 2. The other two types of task related to the magnitude of the fraction are fractions with like numerators and different denominators and fractions with like denominators and different numerators. For these types of task to different techniques appears. For fractions such that $\frac{2}{5}$ and $\frac{2}{10}$ with like numerators and different denominator, the fraction with lowest denominator is largest as illustrated in τ_{17} in Table 1 and τ_{23} in Table 2. For the magnitude of fractions with like denominators and different numerators such as $\frac{9}{12}$ and $\frac{10}{12}$, which is a type of task different from the other two, the related technique is that the fraction with the highest numerator is largest. This technique is illustrated in τ_{16} in Table 1 and τ_{22} in Table 2.

So, although all the tasks are about the magnitude of fractions, different types of task are identified in the textbooks since they require different techniques, which is emphasized in the PRMs (in Table 1 and Table 2).

As with the case of fractions, task related to the magnitude of decimals were identified too, for instance in M5 p.56:


OPGAVE 20 A 27

Løs opgaverne, og diskuter for hver opgave, hvordan I vil løse den.

1. Sæt brøkerne og decimaltallene i rækkefølge efter størrelse. Start med det mindste tal.

0,15 $\frac{6}{9}$ $\frac{1}{6}$ 0,3 $\frac{2}{7}$ 0,7

Figure 4: The magnitude of decimals, M5 p.56

The identified technique for the type of task related to the magnitude of decimal numbers is a lexicographical ordering. This involves that the parts on the left side of the decimal point are compared with each other, if it is not possible to assess the value of the decimal number from the number on the right side of the decimal point. For instance, the magnitude for the decimals 0.15, 0.3 and 0.7 can be determined by looking at the left side of the decimal point and asses the decimals one by one. For instance, 7 is larger than 3 and 1, and 3 is larger than 1, which gives that the order of the magnitudes of the decimals is 0.7, 0.3 and 0.15.

This technique is illustrated in in τ_{21} in Table 1 and τ_{27} in Table 2

On page 20 in AM5, it is possible to see tasks as below and similar:

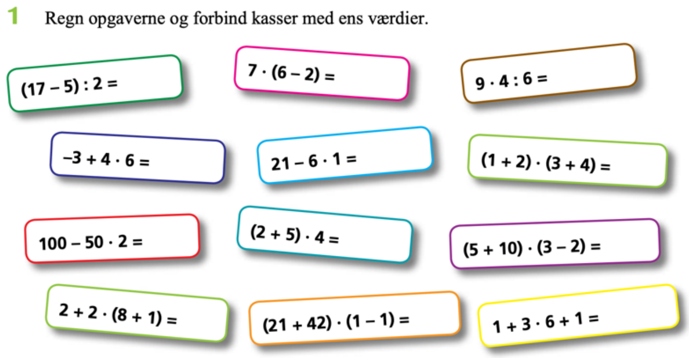


Figure 5: Calculation of arithmetic expressions with two or more operations, AM5 p. 20

Although these tasks at first sight look different, they are categorized as one type of task in the PRMs. This is because these tasks are all about calculation of arithmetic expressions involving integers and two or more operations, with or without brackets. Based on this, these tasks can be solved with the same technique which is illustrated in in $\tau_1$and τ_2 in Table 1 and τ_1 and $\tau_2$in Table 2. All these tasks can be solved by first calculate the brackets in the expression which is followed by the calculation of all other arithmetic operations. If no brackets appear, which is the case for some of the above-mentioned tasks, the techniques involve to multiplicate and divide before addition and subtraction. The decision of why the tasks (Figure 5) are categorized as one type of task can also be supported by noticing that the textbook AM5 has grouped the tasks and presented them together, suggesting that they belong to the same type of task. Considering these tasks as one type of task is also supported by the following technological element in AM5 p. 18:

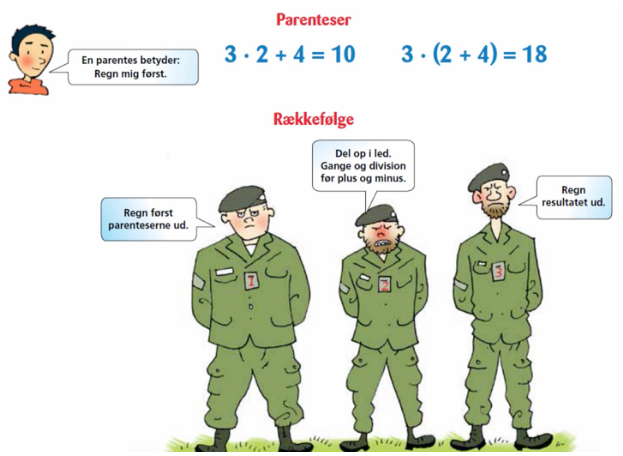


Figure 6: Order of arithmetic operations from AM5 p. 18

This thus confirms that these tasks belong to the same type of task with the same technique, since the textbook explains the technique for these tasks as being brackets first, subsequently multiplication and division before addition and subtraction.

Based on these tasks and the like, it is clear, that the students must be able to calculate arithmetic expressions, involving integers and two or more operations, both with or without brackets and that they need to know the order of the arithmetic operations.

A case, where an important decision was made for the PRMs was for the task in M5 on page 62:

OPGAVE 6

1. Skriv som brøk.

a. 0,25 **b.** 0,10 **c.** 0,75

d. 0,12 **e.** 0,08 **f.** 0,50

Figure 7: Conversion from a decimal to a fraction, M5 p. 62

This task is about conversion a decimal to a fraction. Depending on which decimal must be converted to a fraction, the task can be solved with two different techniques. In the case where the conversion of 0.25 or 0.50 to a fraction is required, the conversion will happen directly since some conversions like 0.25 and 0.50 are typically memorized by the students.

But if the student must convert 0.12 to a fraction another technique is necessary. This technique involves writing the decimal in the form $\frac{a}{10}$ or $\frac{a}{100}$, where a is the decimal, and reduce it or find an equivalent fraction. So, for 0.12 will be written in the form $\frac{12}{100}$ and by reducing $0.12 = \frac{3}{25}$.

So, in the type of tasks where conversion of a decimal to a fraction is necessary, there exist two different techniques, which is illustrated in the PRMs as τ_{15a} and τ_{15b} in Table 1. The same type of tasks is identified in the textbooks for seventh grade, where the techniques are illustrated in the PRM as τ_{21a} and τ_{21b} in Table 2. The same case occurs for types of task where a conversion from fractions to decimals occur.

Furthermore, tasks related to fractions and decimals on a number line were prominent in the textbooks too. For instance, on p. 65 in M5:

OPGAVE 12

1. Tegn en tallinje på 8 cm. Afsæt $\frac{1}{4}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{16}$, $\frac{1}{2}$ og $\frac{12}{16}$.

2. Tegn en tallinje på 12 cm. Afsæt $\frac{1}{6}$, $\frac{2}{3}$, $\frac{3}{8}$, $\frac{3}{12}$, $\frac{17}{24}$ og $\frac{2}{4}$.

Figure 8: Fractions on a number line, M5 p.65

In the official programme for Danish seventh to ninth grade, it is pointed out that:

"From the beginning of the stage, the students work on representing algebraic expressions geometrically and describing properties of geometric figures using algebra" (Official programme, 2019, p. 30)

This illustrates that geometry is used to present algebraic contexts and vice versa, which is also the case with tasks related to the number line. Placing a fraction or a decimal number on a number line can be understood in a more geometric way, in which these numbers are transferred to points on a line. Thus, tasks related to fractions and decimals on a number line are not types of task that examine arithmetic techniques, but these say more about the students' theoretical knowledge of fractions. Some students will even solve tasks related to fractions on a number line by rewriting this fraction to decimal numbers and then placing it on the number line. That geometry is used to present algebraic context is not only applied for the number line, but also for other algebraic relation. For instance, the algebraic relation: $a \cdot (b + c) = ab + ac$, can geometrically be described with the following:

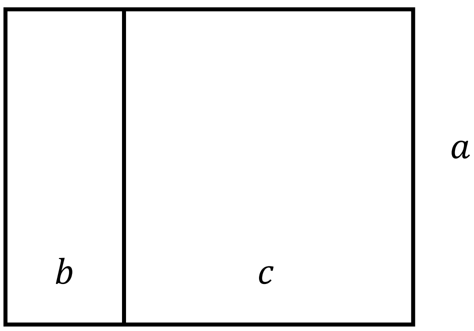


Figure 9: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ geometrically described

For instance, in Figure 8 exercise 1, the students must draw a number line which is eight cm. Afterwards, the students have to place $\frac{3}{8}$ on the number line. The number line with eight cm will be divided into eight equal parts with a fixed distance between each part. For placing $\frac{3}{8}$ on this number line it is central to see $\frac{3}{8}$ as $3 \cdot \frac{1}{8}$. Multiplication of a natural number with a fraction is the same as adding the fraction $\frac{1}{8}$ together with itself 3 times. By using this, $\frac{3}{8}$ is placed after three out of eight equal parts of the number line i.e., $\frac{3}{8}$ is placed as illustrated in Figure 10:

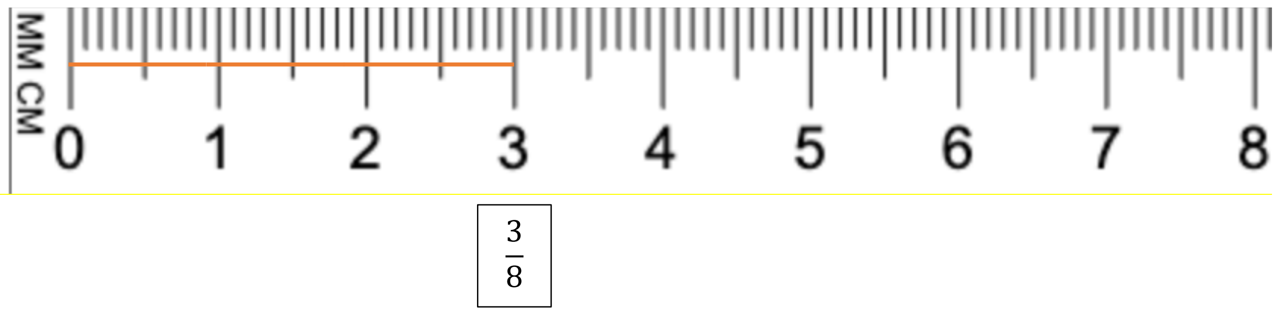
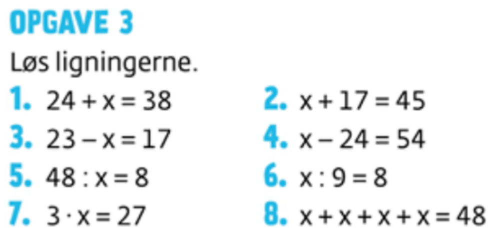


Figure 10: Example of a fraction on a number line

Finally, tasks about solving a first-degree equation and reduction tasks were prominent in both textbooks, as in the following examples in M5 p.137:



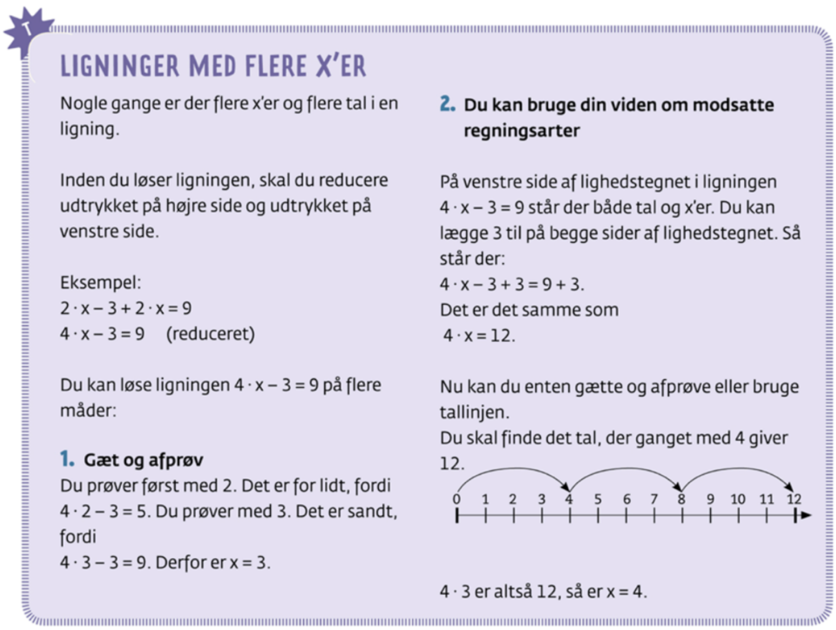
OPGAVE 3

Løs ligningerne.

1. $24 + x = 38$
2. $x + 17 = 45$
3. $23 - x = 17$
4. $x - 24 = 54$
5. $48 : x = 8$
6. $x : 9 = 8$
7. $3 \cdot x = 27$
8. $x + x + x + x = 48$

Figure 11: First-degree equations, M5 p.137

For tasks related to solving a first-degree equation, two techniques were identified as shown in Table 1 and Table 2. These tasks can be solved by the students either with a guess-and-try-method or by using opposite arithmetic operators as presented in M5 p. 140:



LIGNINGER MED FLERE X'ER

Nogle gange er der flere x'er og flere tal i en ligning.

Inden du løser ligningen, skal du reducere udtrykket på højre side og udtrykket på venstre side.

Eksempel:
$2 \cdot x - 3 + 2 \cdot x = 9$
$4 \cdot x - 3 = 9$ (reduceret)

Du kan løse ligningen $4 \cdot x - 3 = 9$ på flere måder:

1. Gæt og afprøv
Du prøver først med 2. Det er for lidt, fordi $4 \cdot 2 - 3 = 5$. Du prøver med 3. Det er sandt, fordi $4 \cdot 3 - 3 = 9$. Derfor er $x = 3$.

2. Du kan bruge din viden om modsatte regningsarter

På venstre side af lighedstegnet i ligningen $4 \cdot x - 3 = 9$ står der både tal og x'er. Du kan lægge 3 til på begge sider af lighedstegnet. Så står der:
$4 \cdot x - 3 + 3 = 9 + 3$.
Det er det samme som
$4 \cdot x = 12$.

Nu kan du enten gætte og afprøve eller bruge tallinjen.
Du skal finde det tal, der ganget med 4 giver 12.

$4 \cdot 3$ er altså 12, så er $x = 4$.

Figure 12: How to solve a first-degree equation in M5 p.140

The use of opposite arithmetic operators involves isolating the unknown (e.g., the variable x)

by using an opposite operator such as the additive inverse or multiplicative inverse (reciprocal). The additive inverse is the number with opposite sign so the additive inverse for 4 is -4 and 2 is the additive inverse for -2. The multiplicative inverse or reciprocal is the number which is written as the bottom of a fraction with 1 as the numerator i.e., $\frac{1}{3}$ is the reciprocal of 3, while the reciprocal for a fraction is the opposite fraction i.e., $\frac{6}{5}$ is the reciprocal of $\frac{5}{6}$.

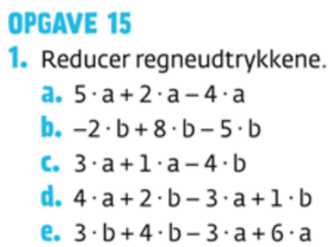
For solving an equation, the reciprocal is used if a number multiplies or divided the unknown. For instance, the problem $4x - 6 = 5$ will be solved by using the additive invers for -6 which gives $4x = 5 + 6$. The next step is to use the opposite operation to solve the unknown by multiplying the reciprocal of 4 on both side of the equal sign which gives:

$$4x = 11 \Rightarrow \frac{4x}{4} = \frac{11}{4} \Rightarrow x = \frac{11}{4}$$

So, by using the opposite operators it is possible to solve the equation for the variable x.

The guess-and-try techniques involves trying with different values of the unknown until both side of the equal sign is equal.

Finally, Danish fifth-grade students are supposed to learn about reduction of arithmetic expression, which is illustrated in M5 p.141:


OPGAVE 15

1. Reducer regneudtrykkene.

a. $5 \cdot a + 2 \cdot a - 4 \cdot a$

b. $-2 \cdot b + 8 \cdot b - 5 \cdot b$

c. $3 \cdot a + 1 \cdot a - 4 \cdot b$

d. $4 \cdot a + 2 \cdot b - 3 \cdot a + 1 \cdot b$

e. $3 \cdot b + 4 \cdot b - 3 \cdot a + 6 \cdot a$

Figure 13: Reduction of arithmetic expression M5 p.141

The type of task related to reduction of arithmetic expression is present in PRM for Danish fifth- and seventh-grade students. The difference occurs when seventh-grade students work with reduction of arithmetic expression up to three variables and fifth-grade students work with reduction of arithmetic expression up to two variables. In addition to this difference, the used technique in both cases is similar, namely simplify by collecting and reduce same kind of terms, which is τ_{24} in Table 1 and τ_{30} in Table 2.

This technique is also presented on p. 140 in M5:

Figure 14: Simplifying by collecting and reduce in M5 p. 140

All the above-mentioned decisions and choices made for the PRM for Danish fifth-grade students also apply for the PRM for Danish seventh-grade students, but for the seventh-grade students, additional types of task were identified, which the following section will illustrate.

5.1.4 Danish seventh-grade students

In a detailed reading of AM7 and M7, types of task related to arithmetic and algebra were identified. For the seventh-grade students the same types of task as for the fifth-grade students were identified. In this section, types of task which only belongs to the Danish seventh-grade students will be presented, as the other types of task are already examined.

Concretely, following types of task were furthermore identified in AM7 and M7:

multiplication of decimals, multiplication of negative integers, multiplication of a negative and positive integers, multiplication and division of two fractions, division of an integer with a fraction, division of a fraction with an integer and reduction tasks up to 3 variables.

The explanation of each types of task is followed by the technique and an example of how it is illustrated in the textbooks, in Table 2. Some examples of these types of tasks will be illustrated in the following.

For multiplication of negative integers and multiplication of a negative and positive integers, the following types of tasks and like were identified in M7 on p. 30:

OPGAVE 6

Udregn:

A $6 \cdot (-2)$

B $(-6) \cdot (-2)$

C $(-6) \cdot (\frac{1}{2})$

D $(\frac{1}{2}) \cdot (-6)$

Figure 15: Multiplication of negative and positive integers, M7 p. 30

For such tasks it was decided to consider them as different types of task i.e., multiplication of negative integers and multiplication of a negative and positive integers, since two different techniques are required. As illustrated in Table 2, for tasks related to multiplication of two negative integers as $(-6) \cdot (-2)$, the technique is to multiply the integers without the negative sign, since negative sign multiplied with negative sign gives a positive sign. The technique for this type of task is illustrated in Table 2 as τ_4.

For type of task related to multiplication of a negative and a positive integer as $6 \cdot (-2)$, the technique is to multiply the integers and add a negative sign, since it applies that two like signs gives a positive sign while two unlike signs make a negative sign. The technique for this type of task is illustrated in Table 2 as τ_5.

In the analysis of the textbooks, types of task related to multiplication of two fractions, division of two fractions, division of an integer with a fraction and division of a fraction with an integer were identified.

On p. 31 in AM7, it is possible to see tasks as below and similar:

5 Multiplikation	**6** Division
a $\frac{1}{5} \cdot \frac{4}{6}$ = ______	**a** $\frac{4}{6} : \frac{5}{9}$ = ______
b $\frac{4}{5} \cdot \frac{2}{3}$ = ______	**b** $\frac{3}{4} : \frac{7}{8}$ = ______
c $\frac{1}{2} \cdot \frac{1}{2}$ = ______	**c** $\frac{1}{6} : \frac{9}{10}$ = ______
d $\frac{4}{6} \cdot \frac{3}{6}$ = ______	**d** $\frac{3}{5} : \frac{4}{6}$ = ______
e $\frac{1}{4} \cdot \frac{1}{4}$ = ______	**e** $\frac{7}{9} : \frac{1}{2}$ = ______
f $\frac{7}{8} \cdot \frac{1}{8}$ = ______	**f** $\frac{1}{4} : \frac{1}{2}$ = ______
g $\frac{9}{12} \cdot \frac{3}{4}$ = ______	**g** $\frac{5}{6} : \frac{3}{5}$ = ______
h $\frac{5}{9} \cdot \frac{3}{6}$ = ______	**h** $\frac{12}{17} : \frac{6}{25}$ = ______

Figure 16: Multiplication and division of fractions, AM7 p. 31

The first type of task (exercise 5 in Figure 16) is about multiplication of fractions. As illustrated in Table 2 the corresponding technique τ_{14} for this type of task is to multiply numerator of the first fraction with the numerator of the second fraction and the same for the denominators.

For the second type of task (exercise 6 in Figure 16), the first step, in solving it, is to find the reciprocal (i.e., reverse the numerator and the denominator) of the second fraction. The second step is to multiply the two numerators and the two denominators. Hence, division of two fractions require the technique: $\frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{a\cdot d}{b\cdot c}$, as illustrated in τ_{15} in Table 2.

On p. 151 in M7,


OPGAVE 6

Beregn

A $31 : \frac{2}{3}$ **B** $\frac{1}{4} : 5$ **C** $16 : \frac{2}{3}$

Figure 17: Division between fractions and integers, M7 p. 151

it is observed that Danish seventh-grade students may solve tasks related to division of an integer with a fraction and division of a fraction with an integer. Although both tasks are related to integers and divisions, two different techniques are required to solve these, why they are illustrated as two separate types of task in Table 2. For division of an integer with a fraction, τ_{18} in Table 2 is required to solve it. Specifically, the first step is to treat the integer as a fraction and therefore place it over the denominator 1. This gives the type of task division of two fractions, which can be solved by the earlier-mentioned technique i.e., τ_{15} in Table 2. So, for tasks related to division of an integer with a fraction the following technique are required (where k is an integer):

$$\tau_{18}: k : \frac{a}{b}=\frac{k}{1}:\frac{a}{b}=\frac{k}{1}\cdot\frac{b}{a}=\frac{k\cdot b}{a}$$

For the type of task related to division of fraction with an integer, in the same way, the first step is to treat the integer as a fraction and then place it over the denominator 1. This gives division of two fractions which can be solved by the earlier-mentioned technique i.e., τ_{15} in Table 2. So, for tasks related to division of a fraction with an integer the following technique (τ_{19} in Table 2) are required (where k is an integer):

$$\tau_{19}: \frac{a}{b} : k=\frac{a}{b}:\frac{k}{1}=\frac{a}{b}\cdot\frac{1}{k}=\frac{a}{kb}$$

Finally, compared with fifth-grade students, who is presented to tasks related to reduction of expression with two variable, seventh-grade students are presented to tasks related to reduction of expression up to three variables, such as in AM7 on p.15:

1

Reducer mest muligt:

a $5p + 6q - 4r + 2p - 3q + 7r - 3p$

b $7r + 4 \cdot 2r - 5p + 8q + 4 \cdot 2p - 6q$

2

Reducer mest muligt:

a $4x + y + 5x + 3y - 6x - 2y$

b $12y - 4x + 5x - 15y - 11x + 4y$

Figure 18: Reduction up to three variables, AM7 p. 15

As emphasized, the PRMs consist of types of task and techniques Danish fifth- and seven-grade students must master, but some technological and theoretical element, which students must master, are possible to find in the official programme and the textbooks, but in a limited extent. The students should therefore not only master the identified techniques but must according to the official programme for fourth to sixth grade also:

“It is central in the work with equations that the students develop their understanding of the fact that the equals sign means that the expressions on the left and the right side of it have (or should have) the same value (as opposed to an understanding that the equals sign is a signal to calculate).” (Official programme, 2019, p.23)

This shows that the students must have a relational understanding of the equal sign, since lack of this understanding can have an impact on solving equations.

Furthermore, in fifth and seventh grade, the students must master the technological and theoretical elements related to the mentioned types of task and related techniques. Although it is not explicit in the official programmes for both grades, it is for instance illustrated in the textbooks that the students must understand the distributive law. For instance, on p. 57 in M7:

OPGAVE 10

$a \cdot (b + c) = a \cdot b + a \cdot c$

A Efterprøv sammenhængen med fem taleksempler.

B Skriv en forklaring af sammenhængen med egne ord.

Figure 19: Distributive law in M7 p. 57

This task illustrates that it is expected that seventh-grade students are able to explain the distributive law $a \cdot (b + c) = a \cdot b + a \cdot c$, and such tasks which require an explain is considered as a technological task. This task and the like show that students are not only supposed to learn the corresponding technique for each type of task (as illustrated in Table 1 and Table 2), but should furthermore explain and describe in some tasks, why the theoretical elements are required from the students too.

6 Results for RQ1

6.1 Praxeological reference model fifth and seventh grade

Based on the official programme for Danish fourth- to sixth-grade students and the textbooks AM5 and M5, it is possible to develop a praxeological reference model for arithmetic and algebra in fifth grade. In the same way, the praxeological reference model for arithmetic and algebra for Danish seventh-grade students are developed and based on the official programme for Danish seventh- to ninth-grade students, and the textbooks AM7 and M7. As mentioned in section 3, although the PRMs only show the praxis blocks, the logos blocks i.e., the technological and theoretical elements, will be discussed later, so Table 1 and Table 2 are not a complete praxeological reference model.

A praxeological analysis of the textbooks has been made and types of task related to arithmetic and algebra were identified. The identified types of task are listed in the first column of the models (Table 1 and Table 2), and the identified techniques in the second column of the model (Table 1 and Table 2). In the third column examples from the textbooks AM5 and M5 for the PRM for fifth grade and AM7 and M7 for the PRM for seventh grade are given.

Types of task	Techniques	Examples
Calculation of arithmetic expressions involving integers and two or more operations	τ_1: Brackets before all other arithmetic operations τ_2: Multiplication and division before addition and subtraction	AM5, p. 20 Calculate $(5+10)\cdot(3-2)$ $(5+10)\cdot(3-2) =$ $15\cdot 1 = 15$ AM5, p. 20 $21-6\cdot 1 = 15$
Multiplication of a decimal with an integer	τ_3: Multiply normally as integers without the decimal points. Then, add the decimal point in the final answer with the same number of decimals as in the decimal number.	AM5, p.49 Calculate $7.5\cdot 6$ $75\cdot 6 = 450$ Then $7.5\cdot 6 = 45.0$
Addition with negative integers	τ_4: $a+(-b) = a-b$ τ_5: $(-a)+(-b) = -(a+b)$	M5, p. 19 $5+(-3) = 5-3 = 2$ M5, p. 20 $(-8)+(-5) = -(8+5) = -13$
Subtraction with negative integers	τ_6: $a-(-b) = a+b$ τ_7: $(-a)-(-b) = -a+b$	M5, p. 20 $14-(-3) = 14+3 = 17$ M5, p. 20 $(-7)-(-7) = -7+7 = 0$
Addition of fractions with like denominator	τ_8: $\frac{a}{c}+\frac{b}{c} = \frac{a+b}{c}$	AM5, p. 72 $\frac{1}{5}+\frac{1}{5} = \frac{1+1}{5} = \frac{2}{5}$

Addition of fractions with different denominator	$\tau_9: \frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}$	M5, p. 62 $$\frac{3}{4}+\frac{1}{8}=\frac{3\cdot 8+1\cdot 4}{4\cdot 8}=\frac{24+4}{32}=\frac{28}{32}$$
Subtraction of fractions with like denominator	$\tau_{10}: \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$	M5, p. 59 $$\frac{4}{8}-\frac{2}{8}=\frac{4-2}{8}=\frac{2}{8}$$
Subtraction of fractions with different denominator	$\tau_{11}: \frac{a}{b}-\frac{c}{d}=\frac{ad-cb}{bd}$	AM5, p. 72 $$\frac{5}{7}-\frac{2}{5}=\frac{5\cdot 5-2\cdot 7}{7\cdot 5}=\frac{25-14}{35}=\frac{11}{35}$$
Equivalence of fractions	$\tau_{12}: \frac{a}{b}=\frac{k\cdot a}{k\cdot b}$ for $k\in\mathbb{N}$	AM5, p. 71 $$\frac{2}{5}=\frac{2\cdot 2}{2\cdot 5}=\frac{4}{10}$$
Multiplication of a fraction with an integer	$\tau_{13}: \frac{a}{b}\cdot k=\frac{ak}{b}$ for $k\in\mathbb{Z}$	AM5, p. 72 $$3\cdot\frac{2}{3}=\frac{3\cdot 2}{3}=\frac{6}{3}$$
Convert a fraction to a decimal	τ_{14a}: Some conversions from fraction to a decimal are memorized such that $$\frac{1}{4}=0.25$$ $$\frac{1}{2}=0.5$$ and $$\frac{3}{4}=0.75$$ τ_{14b}: Write the fraction in the form $\frac{a}{10}$ or $\frac{a}{100}$ and then convert it to a decimal.	M5, p. 56 Convert $\frac{2}{5}$ to a decimal. Write the fraction in the form $\frac{a}{10}$ and then convert it to a decimal $$\frac{2}{5}=\frac{2\cdot 2}{5\cdot 2}=\frac{4}{10}=0.4$$
Convert a decimal to a fraction	τ_{15a}: Some conversions from decimals to fractions are memorized such that: $$0.25=\frac{1}{4}$$ $$0.50=\frac{1}{2}$$ and $$0.75=\frac{3}{4}$$ τ_{15b}: Write the decimal in the form $\frac{a}{10}$ or $\frac{a}{100}$ and reduce it or find an equivalent fraction.	M5, p.56 Convert 0.25 to a fraction. It is memorized such that: $$0.25=\frac{1}{4}$$ M5, p.62 Convert 0.12 to a fraction. Write the decimal 0.12 in the form $\frac{a}{100}$ and the reduce with 4 $$0.12=\frac{12}{100}=\frac{3}{25}$$
Examine which fraction with like denominators and different numerators is largest	τ_{16}: The fraction with highest numerator is largest i.e., if $a<b \Rightarrow \frac{a}{c}<\frac{b}{c}$	M5, p.54 $\frac{9}{12}$ and $\frac{10}{12}$ Since $9<10$ then $\frac{9}{12}<\frac{10}{12}$

Examine which fraction with like numerators and different denominators is largest	τ_{17}: The fraction with lowest denominator is largest i.e., if $b < c \Rightarrow \frac{a}{b} > \frac{a}{c}$	M5, p.55 $\frac{2}{5}$ and $\frac{2}{10}$ Since 5$<$ 10 then $\frac{2}{5} > \frac{2}{10}$
Examine which unit fraction $\frac{1}{a}$ and $\frac{1}{b}$ is largest	τ_{18}: The fraction with lowest denominator is largest i.e., if $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$	M5, p. 65 $\frac{1}{2}$ and $\frac{1}{5}$ Since 2$<$ 5 then $\frac{1}{2} > \frac{1}{5}$
Placing fractions and decimals on a number line (the relationship between number symbols and their value understood as points on a number line)	τ_{19}: If placing a fraction on a number line, convert the fraction to a decimal and place the decimal on the number line τ_{20}: If placing a fraction on a number line, the fraction can be changed to another equivalent fraction and then place it on the number line.	M5, p.65 Draw a number line which is 12 cm. Draw the fraction $\frac{2}{4}$ on the number line. Convert $\frac{2}{4}$ to 0.5 and then place it on the number line, which is 12 cm. M5, p.65 Draw a number line which is 8 cm. Draw the fraction $\frac{1}{4}$ on the number line. Convert $\frac{1}{4}$ to $\frac{2}{8}$ and then it is easier to place on the number line with 8 cm.
Examine which decimal is largest	τ_{21}: Lexicographical ordering	M5, p. 56 Examine whether 0.15 and 0.3 is largest.
Solve a first-degree equation	τ_{22}: Addition, subtraction, multiplication and division on both side of the equal sign (use opposite arithmetic operators) τ_{23}: Guess and try method	M5, p.137 Example 1 $24 + x = 38$ $x = 38 - 24 = 14$ Example 2 $3 \cdot x = 27$ $x = \frac{27}{3} = 9$
Reduction tasks with one variable	τ_{24}: Simplify by collecting and reduce same kind of terms	M5, p.141 Reduce $5a + 2a - 4a$ $5a + 2a - 4a = 3a$

Table 1: Praxeological reference model fifth grade

As mentioned, the third column provides examples from the textbooks, but in AM7 and M7 it was not possible to find examples related to the examination of the magnitude of decimals. But since it is in the curriculum for Danish fifth-grade students, it is expected that seventh-grade students master this type of task. Because of that, the example of τ_{25} in Table 2 is an example from the curriculum for fifth-grade students namely M5.

Types of task	Techniques	Examples
Calculation of arithmetic expressions involving integers and two or more operations	τ_1: Brackets before all other arithmetic operations τ_2: Multiplication and division before addition and subtraction	M7, p. 60 $37 \cdot 3 + (4 \cdot 10) = 37 \cdot 3 + 40 = 111 + 40 = 151$
Multiplication of a decimal with an integer and multiplication of decimals	τ_3: Multiply normally as integers without the decimal points. Then, add the decimal point in the final answer with the sum of the number of decimals.	M7, p. 60 Calculate $3.5 \cdot 9$ $35 \cdot 9 = 315$ Then $3.5 \cdot 9 = 31.5$ M7, p. 58 Calculate $3.5 \cdot 3.5$ $35 \cdot 35 = 1225$ Then $3.5 \cdot 3.5 = 12.25$
Multiplication of negative integers	τ_4: $(-a) \cdot (-b) = ab$	M7, p. 30 $(-6) \cdot (-2) = 6 \cdot 2 = 12$
Multiplication of a negative and positive integers	τ_5: $(-a) \cdot b = -ab$	M7, p. 30 $6 \cdot (-2) = -12$
Addition with negative integers	τ_6: $a + (-b) = a - b$ τ_7: $(-a) + (-b) = -(a + b)$	M7, p. 29 $5 + (-24) = 5 - 24 = -19$ M7, p. 29 $(-12) + (-26) = -(12 + 26) = -38$
Subtraction with negative integers	τ_8: $a - (-b) = a + b$ τ_9: $(-a) - (-b) = -a + b$	M7, p. 21 $1 - (-4) = 1 + 4 = 5$ M7, p. 29 $(-25) - (-8) = -25 + 8 = -17$
Addition of fractions with like denominator	$\tau_{10}: \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	AM7, p. 31 $\frac{4}{8} + \frac{3}{8} = \frac{7}{8}$
Addition of fractions with different denominator	$\tau_{11}: \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$	AM7, p. 31 $\frac{3}{7} + \frac{5}{9} = \frac{3 \cdot 9 + 5 \cdot 7}{7 \cdot 9} = \frac{27+35}{63} = \frac{62}{63}$
Subtraction of fractions with like denominator	$\tau_{12}: \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$	AM7, p. 31 $\frac{8}{12} - \frac{3}{12} = \frac{8-3}{12} = \frac{5}{12}$
Subtraction of fractions with different denominator	$\tau_{13}: \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$	AM7, p. 31 $\frac{7}{8} - \frac{2}{3} = \frac{7 \cdot 3 - 2 \cdot 8}{8 \cdot 3} = \frac{21-16}{24} = \frac{5}{24}$
Multiplication of two fractions	$\tau_{14}: \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$	AM7, p. 31 $\frac{4}{5} \cdot \frac{2}{3} = \frac{6}{15}$
Division of two fractions	$\tau_{15}: \frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	AM7, p. 31 $\frac{4}{6} : \frac{5}{9} = \frac{4}{6} \cdot \frac{9}{5} = \frac{36}{30}$
Equivalence of fractions	$\tau_{16}: \frac{a}{b} = \frac{k \cdot a}{k \cdot b}$	AM7, p. 31

	for $k \in \mathbb{N}$	$$\frac{15}{20} = \frac{5 \cdot 3}{5 \cdot 4}$$
Multiplication of a fraction with an integer	$\tau_{17}: \frac{a}{b} \cdot k = \frac{ak}{b}$ for $k \in \mathbb{Z}$	M7, p. 150 $$\frac{2}{3} \cdot 15 = \frac{2 \cdot 15}{3} = \frac{30}{3}$$
Division of an integer with a fraction	$\tau_{18}: k : \frac{a}{b} = \frac{k}{1} : \frac{a}{b} = \frac{k}{1} \cdot \frac{b}{a} = \frac{k \cdot b}{a}$	M7, p. 151 $$31 : \frac{2}{3} = \frac{31}{1} : \frac{2}{3} = \frac{31}{1} \cdot \frac{3}{2} = \frac{31 \cdot 3}{1 \cdot 2} = \frac{93}{2}$$
Division of a fraction with an integer	$$\tau_{19}: \frac{a}{b} : k = \frac{a}{b} : \frac{k}{1} = \frac{a}{b} \cdot \frac{1}{k} = \frac{a}{kb}$$	M7, p. 151 $$\frac{1}{4} : 5 = \frac{1}{4} : \frac{5}{1} = \frac{1}{4} \cdot \frac{1}{5} = \frac{1 \cdot 1}{4 \cdot 5} = \frac{1}{20}$$
Convert a fraction to a decimal	τ_{20a}: Some conversions from fraction to a decimal are memorized such that $$\frac{1}{4} = 0.25$$ $$\frac{1}{2} = 0.5$$ and $$\frac{3}{4} = 0.75$$ τ_{20b}: Write the fraction in the form $\frac{a}{10}$ or $\frac{a}{100}$ and then convert it to a decimal.	M7, p. 150 Convert $\frac{8}{20}$ to a decimal. Write the fraction in the form $\frac{a}{100}$ and then convert it to a decimal. $$\frac{8}{20} = \frac{8 \cdot 5}{20 \cdot 5} = \frac{40}{100} = 0.4$$
Convert a decimal to a fraction	τ_{21a}: Some conversions from decimals to fractions are memorized such that: $$0.25 = \frac{1}{4}$$ $$0.50 = \frac{1}{2}$$ and $$0.75 = \frac{3}{4}$$ τ_{21b}: Write the decimal in the form $\frac{a}{10}$ or $\frac{a}{100}$ and reduce it or find an equivalent fraction.	M7, p. 138 Convert 0.75 to a fraction $$0.75 = \frac{3}{4}$$ M7, p. 138 Convert 0.44 to a fraction. Write the decimal 0.44 in the form $\frac{a}{100}$ and the reduce with 4 $$0.44 = \frac{44}{100} = \frac{11}{25}$$
Examine which fraction with like denominators and different numerators is largest	τ_{22}: The fraction with highest numerator is largest i.e., if $a < b \Rightarrow \frac{a}{c} < \frac{b}{c}$	AM7, p. 42 $\frac{5}{12}$ and $\frac{3}{12}$ Since 3< 5 then $\frac{3}{12} < \frac{5}{12}$
Examine which fraction with like numerators and different denominators is largest	τ_{23}: The fraction with lowest denominator is largest i.e., if $b < c \Rightarrow \frac{a}{b} > \frac{a}{c}$	AM7, p. 42 $\frac{6}{18}$ and $\frac{6}{24}$ Since 18< 24 then $\frac{6}{18} > \frac{6}{24}$
Examine which unit fraction $\frac{1}{a}$ and $\frac{1}{b}$ is largest	τ_{24}: The fraction with lowest denominator is largest i.e., if $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$	AM7, p. 42 $\frac{1}{4}$ and $\frac{1}{9}$ Since 4< 9 then $\frac{1}{4} > \frac{1}{9}$
Placing fractions and decimals	τ_{25}: If placing a fraction on a number line, convert the fraction to a decimal	M7, p.136 Draw a number line which is 20 cm. Draw the fraction $\frac{1}{2}$ on the number line.

on a number line (the relationship between number symbols and their value understood as points on a number line)	and place the decimal on the number line τ_{26}: If placing a fraction on a number line, the fraction can be changed to another equivalent fraction and then place it on the number line.	Convert $\frac{1}{2}$ to 0.5 and then place it on the number line, which is 20 cm. M7, p.136 Draw a number line which is 12 cm. Draw the fraction $\frac{1}{4}$ on the number line. Convert $\frac{1}{4}$ to $\frac{3}{12}$ and then it is easier to place on the number line with 12 cm.
Examine which decimal is largest	τ_{27}: Lexicographical ordering	M5, p. 56 Examine whether 0.15 and 0.3 is largest.
Solve a first-degree equation	τ_{28}: Addition, subtraction, multiplication and division on both side of the equal sign (use opposite arithmetic operators) τ_{29}: Guess and try method	AM7, p. 50 Solve $9x - 15 + 2x + 28 = 46$ $9x - 15 + 2x + 28 = 46 \Rightarrow$ $9x + 2x = 46 + 15 - 28 \Rightarrow$ $11x = 33 \Rightarrow$ $x = \frac{33}{11} = 3$
Reduction tasks up to 3 variables	τ_{30}: Simplify by collecting and reduce same kind of terms	AM7, p. 15 Reduce $3q + 4p + 5r + 3 \cdot 3p + 4 \cdot 4q + 5 \cdot 5r$ $3q + 4p + 5r + 3 \cdot 3p + 4 \cdot 4q + 5 \cdot 5r$ $= 3q + 4p + 5r + 9p + 16q + 25r$ $= 12q + 20p + 30r$

Table 2: Praxeological reference model seventh grade

Table 1 and Table 2 constitute a model of the arithmetic and algebraic praxis blocks that Danish fifth- and seventh-grade students are supposed to learn. Additionally, and in relation to these praxis blocks, there are theoretical elements the students must master, which are only partially visible through the examples. As mentioned before, the official programmes state that the student should have a relational understanding of the equal sign since this can have an impact on later equation solving. Furthermore, the students should master the technological and the theoretical aspects related to the listed praxis blocks, such as the distributive law.

Table 1 and Table 2 are not very detailed or complete at the theoretical level, as the PRMs are mainly built around the praxis blocks. However, we do test also theoretical aspects in the diagnostic tests, where students are asked to provide explanations and justifications in some items. Indeed, how and why questions are asked in the diagnostic tests to make the theoretical aspect more explicit.

7 Methodology for RQ2

In the following section, the methodology for RQ2 is presented. To examine in what extent the identified praxeologies in Table 1 and Table 2 are mastered by the students, we prepared a diagnostic test. How the test was developed will now be presented.

Table 1 and Table 2 are models which show the praxis the students in Danish fifth and seventh grade respectively are expected to master. They were therefore used to develop the test, which aimed to detect students' technique and technology related to a particular type of task (Brekke, 2002).

The diagnostic test contains items which all correspond to a type of task and detect the corresponding technique stated in the PRMs. The test also provided an opportunity to assess how prevalent a particular incorrect arithmetical or algebraic technique is among the students. After testing students' praxeologies in arithmetic and algebra, we aimed to inform the teachers about their students' difficulties. According to Brekke (2002), by informing the teachers in that way, the teachers will then have the possibility to make a better plan for the teaching to overcome the misconceptions and shortcomings their students have.

In order to develop the tests, it was essential to first identify the misconceptions students have in arithmetical and algebraic tasks according to previous research. This research was presented in section 2.4 where students' misconception about types of task belonging to arithmetic and algebra are identified.

As mentioned earlier in section 2.4, a misconception is defined either as a wrong technique or a wrong theoretical knowledge about mathematical notions.

7.1 How is the test developed?

In the construction of the diagnostic test, the developed PRMs were the main tool. The model presents concrete types of task and techniques, so the first step in the construction of the test was to produce items belonging to these specific types of task. Hereafter, items for the test for fifth- and seventh-grade students are referred to as follows: (5^{th} gr., I_n) and (7^{th} gr., I_n) respectively, where I_n is the item number.

For instance, for the type of task 'Addition of fractions with like denominator', items like the following were produced:

(7^{th} gr.,I_5)

Write a number in the empty boxes so the calculation is correct

$$\frac{2}{7} + \frac{\square}{\square} = \frac{5}{7}$$

(5^{th} gr., I_{20})

Calculate:

$$\frac{1}{5} + \frac{3}{5} = \frac{\square}{\square}$$

These items have in common that they all require the same technique such that $\tau_8 : \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ in PRM for fifth grade and τ_{10} in PRM for seventh grade, as illustrated in Table 1 and 2. In the same way items corresponding to all techniques that are contained in PRM for fifth grade and seventh grade were constructed, which results in two tests containing items which examine a specific technique from the PRMs.

As mentioned above, the tasks are not limited to only items that examine the techniques from PRMs, but also technological and theoretical elements are tested. These technological and theoretical items also relate to the PRMs because they test students' capacity to explain and justify techniques and to give a justification of the technologies. Theoretical elements are most often relevant to several practice blocks and make different practice blocks cohesive. For example, items exist in the tests where the students must use the technique of equivalent fractions to solve a specific item, but in the same way equivalent fractions is also used in addition with fractions. For instance:

(7^{th} gr.,I_{86}):

Expand the fractions

$$\frac{1 \cdot \square}{4 \cdot \square} = \frac{\square}{20}$$

By which number do you expand?

Explain your answer in the box

This item is not directly about using the technique of equivalent fractions, but the theory of equivalent fractions is used to justify several techniques and to link the techniques together.
Items that examine the technological or theoretical level can be used to gain a deeper understanding of how the students perceive and solve a specific type of task.
For instance, a theoretical item from the test is (7^{th} gr.,I_{81}):
What has gone wrong in this transformation?

$$7x - 7 = 13 - 3x$$
$$10x - 7 = 13$$
$$10x = 20$$
$$x = 10$$

In this item the theoretical level is clear since the students must give an explanation of what has gone wrong in the transformation.
A more technological item is (5^{th} gr.,I_{75}):
Which number is largest? (circle your answer)

0.362	0.37

In this technological item the students must examine which number is largest by using the technique about lexicographical ordering and then explain the used technique.

As illustrated in Table 1 and Table 2, two techniques are identified in the type of task of solving a first-degree equation: the use of opposite arithmetic operator (see Figure 12), and guess-and-try method. Items that examine the technological and theoretical elements can also help identifying what technique the students' use. This can be illustrated by looking at (7^{th} gr.,I_{69}) and (7^{th} gr.,I_{79}):

I_{69}:

Solve the equation

$$2x = 16 + 2$$

I_{79}:

Write the solution in the box.

Explain in the box how you found the solution to the equation

$$3x + 2 = 8$$

In item 69 it is not possible to see whether a seventh-grade students have used τ_{28} or τ_{29} from PRM in Table 2. This will become more apparent in item 79 where the student must explain the solution of the task.

Furthermore, in the design of the tests, items have been made to capture some of the difficulties that are documented in the research literature related to arithmetic and algebra. From the literature, as mentioned in section 2.4.1, it can be observed that students may have misconceptions about the equal sign. And since the understanding of the equal sign as relational is also mentioned in the official programme for Danish fourth to sixth grade, items that examine students' understanding of the equals sign were incorporated in both tests. Falkner et al. (1999) used (in Alibali et al., 2007) the task $8 + 4 = \square + 5$ for the examination of students' understanding of the equal sign. Inspired by this, item 91 in the 7^{th} grade test is:

(7^{th} gr.,I_{91})

Write a number in the box so the calculation is correct

$$8 - 2 = \square + 5$$

Explain how you found the solution to the answer:

One decision made in the development of both tests was to place all tasks randomly, to avoid that items which belong to the same type of tasks are placed one after the other. This is done, for instance for (5th gr.,I_{40}) and (5th gr.,I_{49}) to avoid that students compare the two similar items and being affected.

So, (5th gr.,I_{40}) and (5th gr.,I_{49}) is:

I_{40}:

Write the numbers in order with the smallest first

0.27 0.267 0.1200 0.05

I_{49}:

Write the numbers in order with the smallest first

0.1798 0.18 0.2 0.09

These items belong to the same type of task, namely 'Examine which decimal is largest'. If these items were placed one after the other, students may want to compare these items. Students may become aware of some stuff in one item and can get affected by this in the other item. By placing similar items one after the other, the students may become better at solving one type of item and this is not the purpose of the diagnostic test. The purpose is to investigate whether the students can solve these items individually without being affected by the other items.

On the other hand, there are tasks that are deliberately placed one after the other. This is because, an example is needed at the beginning and to avoid giving an example each time, it is decided to place them together. This applies for instance for item 41, item 42, and item 43 in the final test for seventh-grade students (Appendix 6).

Finally, items that examine techniques come first in the test, whereas the theoretical questions come later.

This is due firstly to the fact that according to ATD the praxis block is a prerequisite for mastering the theoretical block. It will therefore not make sense to provide the students with items that examine theoretical blocks if the students do not master the technical items. Therefore, it is most appropriate that the technical items are placed first in the tests and are followed by items belonging to the theoretical blocks, as it is not possible to explain techniques you do not master.

Another argument for placing the theoretical questions later is to avoid a situation in which students become stuck in a theoretical question.

7.2 The pilot test

As mentioned in section 4. 'Context', a pilot version of the test was given to two fifth-grade students and five seventh-grade students from School A. This pilot test is a way to prepare the final test, as the answers from the pilot test can answer the following questions:

1. How were the amounts of items? Did most students get through all the items?
2. Were there any items the student misunderstood? This question can be identified, for instance, by looking at items that many students have answered incorrectly. These items can tell us whether the students have had incorrect answers in these items because they did not know how to solve it, or maybe the items were poorly formulated.
3. The number of correct answers to type of task and item can help decide whether some items should be added or removed.
4. Finally, the formulations in the theoretical oriented items can be revised if it turns out that many have misunderstood their formulation.

Based on these questions the test was revised (see section 7.2.1).

For the pilot tests for fifth and seventh grade, a table relating types of task from the PRM with items was developed (Appendix 3). This model has firstly been useful in the development and revision of the final tests for fifth and seventh grade, since it illustrates how much a type of task appears in both tests. If the items in the tests belong to the same type of task with the same technique some of these items are excluded and filtered.

For both pilot tests, an excel sheet has been developed to record the results (Appendix 4a & Appendix 4b). The columns contain the student numbers, while the rows have item numbers.

If the student answers an item correctly, this is noted with a 1, and if the student answers incorrectly, it is noted with a 0. Finally, the sum for each student can be determined, which shows how many items the student has done correctly. The sum of the whole class for each item shows how many students have solved the item correctly. This can be used to examine which tasks were difficult and easy and thus have an impact on the revision of the pilot test, and in final test, it is also relevant as basis for the feedback for teachers.

7.2.1 The pilot test for fifth-grade students

The following section will explain the changes that have been made to the fifth-grade test based on the pilot test. Notice that the item numbers referred to in this section are item numbers from the pilot test which do not match the item numbers of the final test (Appendix 5 & Appendix 6), since changes have occurred.

The Excel sheet for fifth grade (see Appendix 4a) shows that both students get through all tasks, which may mean that the number of items in the test was not too high. Furthermore, the Excel sheet (Appendix 4a) shows that both students have mistakes in item 26, 30, 36, 62, 64, 66, 68, 69, 80, 83 and 85, which is why an extra look was taken at these items. In the following, some of these items will be discussed and it will be examined whether the students' mistakes in these items are due to the items being poorly formulated or whether it is due to students' difficulties with solving these items.

Item 30 and item 66 are about subtraction of fractions with different denominators. This type of task is also found in item 55, where one of two students has performed it correctly. According to the praxeological reference model for fifth grade, subtraction of fractions with different denominators is a type of task fifth-grade students should master. Based on this and since item 55 was solved by one student out of two, item 30 and item 66 were retained in the final test, since it appeared that it was not the formulation of the item that was problematic.

Item 36 (in the pilot test) is changed from the pilot test to the final test. This item contained several parts, which have been separated into several items such as 34, 35, 36 and 37 in the final test (see Appendix 5).

Item 62 and item 69 about addition and subtraction with negative integers both students answered incorrectly, but since the students should master calculation with negative integers (according to the PRMs Table 1), this was not removed from the final test.

There has also been changes in items about reduction of expressions. These items were formulated in two different ways to investigate whether the students know the concept of ‘reduce’.

(5^{th} gr.,I_{64}) was formulated as follows:
Write the expression $7a - 3a$ in a simpler way here: $\boxed{}$

While e.g., (5^{th} gr.,I_{71}) (this also applies to I_{80} and I_{83}) is formulated in the following way:
Reduce

$$8b + 5b = \boxed{}$$

From the results of the pilot test (Appendix 4a) it is possible to see that item 64 was solved incorrectly by both students, but whereas both students solved item 71 correctly. This may possibly mean that they did not understand the term "Write the expression $7a - 3a$ in a simpler way here", but that they mastered the term 'reduce'. Therefore, item 64 from the pilot test is changed in the final test (see the change in item 64 in Appendix 5).
Finally, it was observed that there were quite a few items about equivalence of fractions, and many of these were of the same type. Therefore, some of these were removed in the final test.
For instance, item 22 (from the pilot test):
Write two new fractions that are of the same value as the first fraction

$$\frac{1}{5} \quad \boxed{} \quad \boxed{}$$

In the pilot test there exits four items of this type of task about equivalence of fractions and both fifth-grade students in the pilot test answered these items correctly. Therefore, two out of these four items, including item 12 and item 32, were removed, since it is not necessary to test the same in exactly the same way. The number of correct answers to a type of task and item had thus an influence on the revision of the test.

7.2.2 The pilot test for seventh-grade students

Notice that the item numbers referred to in this section are item numbers from the pilot test which do not match the item number of the final test (Appendix 6) since changes have occurred. The Excel sheet for seventh grade (Appendix 4b) shows that four out of five students get through all tasks, why it is conceivable that the number of items in the test was not too high.

The changes that were made for the fifth-grade test in items related to reduction of expressions were also done for the seventh-grade test, for the same reason.

In the pilot test for seventh-grade students, 39 out of 105 items were answered correctly by all students, which is why I have chosen to look at these to assess whether some could be removed, as there is no reason to test the same several times.

The three types of tasks that were conspicuous were 1) equivalence of fractions, 2) examination of the magnitude of a decimal number, and 3) solving of a first-degree equation.

14 out of 25 equivalence of fractions items, were answered correctly by all the students. Based on this, it was decided that some of these items could be replaced with items related to addition and subtraction of fractions with like and different denominator. These new items were added in the tests since it is noticed from the item & task table (Appendix 3) for the pilot test that there were few items of these types of task.

Furthermore, the items related to the examination of the magnitude of a decimal and solving a first-degree equation was also reduced.

(7^{th} gr.,I_{66}) in the pilot test is:

Write a number in the empty boxes so the calculation is correct

$$\frac{\boxed{}}{\boxed{}} : \frac{1}{5} = \frac{3}{4}$$

None of the students had this item correct.

It is obvious that it is not the division sign, i.e., : which is the reason for the students' mistakes since other items in the pilot test involve this sign too and were answered correct by the students.

In the final test, several variations of this item have been added, since one item possibly cannot detect whether it is specifically this item that is complicated or whether it is generally this type of task that students have problems with.

To examine this, several variations of item 66 are added to the final test to investigate whether it is only item 66 which is complicated or whether the students generally have no technique for the division of fractions. It could also be that the sample in the pilot test is too small, so that with a larger sample one gets different answer.

More generally, the pilot test showed that a fairly reasonable level had been achieved, where everyone could answer some items (some more than others). In this loose sense, the test is neither too difficult nor too easy.

7.3 The relation between the PRMs and the final test

Based on the pilot tests and a table relating types of tasks from the PRM with items for the pilot test (Appendix 3), a revision of the test has taken place and the final test has been developed.

As mentioned earlier, the final tests are designed such that each item in the test, which belongs to a type of task, has a one-to-one correspondence to a specific technique. This means, that a correct answer to an item in the test diagnose mastery or non-mastery of the technique in question.

Therefore, Table 3 has been made to illustrate the relation between types of task from the praxeological models and the items in the final tests for fifth- and seventh-grade students. This table illustrates how much a type of task appears in both tests.

Table 3 can be used as a tool to answer RQ3 about how teachers can be usefully informed of the answers from the tests for Danish fifth- and seventh-grade students.

Types of task	Item (I) in test for fifth grade	Item (I) in test for seventh grade
T_1: Calculation of arithmetic expressions involving integers and two or more operations	$I_1, I_9, I_{12}, I_{27}, I_{28}, I_{44}$	I_1
T_2: Multiplication of a decimal with an integer	I_{11}, I_{15}, I_{26}	I_2, I_{34}
T_3: Multiplication of decimals		I_{11}, I_{29}, I_{93}
T_4: Multiplication of negative integers		I_3
T_5: Multiplication of a negative and positive integers		I_4
T_6: Addition with negative integers	I_{65}, I_{70}	I_{12}, I_{36}
T_7: Subtraction with negative integers	I_{62}, I_{68}	I_{30}, I_{35}
T_8: Addition of fractions with like denominator	$I_2, I_{20}, I_{25}, I_{29}$	I_5, I_{26}, I_{85}
T_9: Addition of fractions with different denominator	I_{45}, I_{56}, I_{59}	I_{17}, I_{37}
T_{10}: Subtraction of fractions with like denominator	$I_{10}, I_{14}, I_{21}, I_{30}, I_{43}$	I_{13}, I_{19}, I_{22}

T_{11}: Subtraction of fractions with different denominator	I_{31}, I_{55}, I_{66}	I_{28}, I_{31}
T_{12}: Multiplication of two fractions		I_{14}, I_{15}, I_{38}
T_{13}: Division of two fractions		$I_{32}, I_{39}, I_{64}, I_{70}, I_{92}$
T_{14}: Equivalence of fractions	$I_6, I_7, I_8, I_{13}, I_{22}, I_{23}, I_{42}, I_{47}, I_{60}, I_{73}$ $I_{76}, I_{77}, I_{78}, I_{81}, I_{82}$	$I_6, I_{16}, I_{41}, I_{42}, I_{43}, I_{48}, I_{54}, I_{66}, I_{71}$ $I_{75}, I_{78}, I_{83}, I_{86}$
T_{15}: Multiplication of a fraction with an integer	I_3, I_{16}, I_{24}	I_{46}, I_{50}
T_{16}: Division of an integer with a fraction		I_{33}
T_{17}: Division of a fraction with an integer		I_{49},
T_{18}: Convert a fraction to a decimal	$I_{17}, I_{32}, I_{41}, I_{46}, I_{67}$	$I_7, I_{18}, I_{21}, I_{44}$
T_{19}: Convert a decimal to a fraction	$I_{19}, I_{33}, I_{54}, I_{61}$,	$I_8, I_{45}, I_{67}, I_{72}$
T_{20}: Examine which fraction with like denominators and different numerators is largest	I_{18}, I_{39},	I_{20}, I_{58}
T_{21}: Examine which fraction with like numerators and different denominators is largest	I_{48}, I_{74}	I_{59}, I_{76}
T_{22}: Examine which unit fraction $\frac{1}{a}$ and $\frac{1}{b}$ is largest	I_5, I_{79}	$I_9, I_{62}, I_{80}, I_{82}$
T_{23}: Placing fractions and decimals on a number line (the relationship between number symbols and their value understood as points on a number line)	$I_{34}, I_{35}, I_{36}, I_{37}, I_{50}, I_{51}, I_{57}$	$I_{23}, I_{24}, I_{25}, I_{51}, I_{52}, I_{60}$
T_{24}: Examine which decimal is largest	$I_{38}, I_{40}, I_{49}, I_{53}, I_{75}$	$I_{47}, I_{56}, I_{68}, I_{77}$
T_{25}: Solve a first-degree equation	$I_4, I_{52}, I_{58}, I_{63}, I_{69}, I_{71}, I_{86}$	$I_{40}, I_{53}, I_{55}, I_{57}, I_{61}, I_{63}, I_{69}, I_{79}, I_{81}$
T_{26}: Reduction tasks with one variable	$I_{64}, I_{72}, I_{80}, I_{83}$	I_{27}
T_{27}: Reduction tasks up to 2 variables		I_{10}, I_{74}
T_{28}: Reduction tasks up to 3 variables		I_{65}, I_{73}
T_{29}: Equal sign	I_{84}	$I_{87}, I_{88}, I_{89}, I_{90}, I_{91}$
T_{30}:Distributive law	I_{85}	I_{84}

Table 3: Relation between types of tasks and the items in the diagnostic tests

8 Results for RQ2

To examine to what extent, the praxeologies in Table 1 and Table 2 are mastered by the students, diagnostic tests were developed. The following sections will present the results from the diagnostic tests and then provide an answer of RQ2 namely: To what extent are these praxeologies actually mastered by the students?

The diagnostic tests were performed in School B in Nkøbing municipality. Due to anonymity, the involved classes in the tests will be denoted as 5.Cn and 7.Cn (for n=1,…,4), where C refers to class. A total of 86 fifth-grade students and 78 seventh-grade students performed the test. Not all students get through the entire test, which will be taken into consideration in the presentation of the results. In the following, a student will be denoted with S_n and an item will be denoted as I_n (where n denoted the student (S) and item (I) number) i.e., a reference to the test answers from a student will be denoted as (5.Cn, I_n, S_n) and (7.Cn, I_n, S_n).

All the items in the following section can be found in Appendix 5 (final test for fifth grade) and Appendix 6 (final test for seventh grade)

Before the presentation of how the results for both final tests are coded, notice that I_{86} (5$^{\text{th}}$ gr.) and I_{80} (7th gr.) are not included in the final tests due to technical problems. In addition, two identical tasks have been added by mistake in the final test for fifth grade (it applies I_{28} and I_{44}). An incorrectly answered item gives 0 point while a correctly answered item gives 1 point, but there are a few exceptions.

In items which require an explanation, 1 point is given for a correct answer and explanation, 0.5 point for either a correct answer or explanation and 0 points if neither the answer nor explanation is correct.

In items that contain the number line, where four different fractions must be placed on the number line, 0.25 point is given for each correctly placed fraction (a total of 1 point can be obtained).

In addition, in items where two answers are given (e.g. I_{23} and I_{27} in 5$^{\text{th}}$ gr.), 0.5 point is given for each correct answer (a total of 1 point can be obtained). Furthermore, I_{87} (7$^{\text{th}}$ grade) and I_{84} (in 5$^{\text{th}}$ grade) is separated into two parts I_{87a} & I_{87b} and I_{84a} & I_{84b} respectively. Each item gives 1 point.

The following sections will present the results for the final test for fifth- and seventh-grade students. The sections will in some cases presents the results for each class but will also provide a picture of how fifth- and seventh-grade students generally master different types of task in School B.

8.1 Fifth-grade students praxeologies

Table 4 shows the results from the final test for fifth-grade students. The first column shows the types of task, while the other four columns show how many points the class has received per item (converted to percent). In a class with e.g., 23 students, an item can get a maximum of 23 points.

From Table 4, it is observed that different classes master different types of task belonging to arithmetic and algebra in fifth grade, which may be due to the big variation in what the teachers emphasize in their teaching, since the official programme for Danish fourth to sixth grade (Official programme, 2019) is quite unclear about which praxis blocks that should be taught. Not all the results in Table 4, will be presented one by one, but instead, only some types of misconceptions and lack of knowledge among the students will be illustrated in the following.

Types of task	5.C1	5.C2	5.C3	5.C4
T_1: Calculation of arithmetic expressions involving integers and two or more operations	I_1:56,2% I_9:85,95% I_{12}:91,3% I_{27}:52,1% I_{28}:34,8% I_{44}:39,1%	I_1: 31,57% I_9: 78,94% I_{12}: 78,94% I_{27}: 23,68% I_{28}: 21% I_{44}: 15,79%	I_1:73,9% I_9:100% I_{12}: 86,95% I_{27}: 45,65% I_{28}: 47,82% I_{44}: 43,47%	I_1:42,85% I_9:100% I_{12}:76,2% I_{27}:33,33% I_{28}:28,5% I_{44}:28,5%
T_2: Multiplication of a decimal with an integer	I_{11}: 65,2% I_{15}: 43,48% I_{26}: 43,48%	I_{11}: 57,89% I_{15}: 52,63% I_{26}: 47,36%	I_{11}: 34,78% I_{15}: 21,73% I_{26}: 30,43%	I_{11}: 76,2% I_{15}: 52,4% I_{26}: 42,8%
T_3: Multiplication of decimals				
T_4: Multiplication of negative integers				
T_5: Multiplication of a negative and positive integers				
T_6: Addition with negative integers	I_{65}: 60,87% I_{70}: 39,13%	I_{65}:68,42 % I_{70}: 73,78 %	I_{65}: 78,26% I_{70}: 39,13 %	I_{65}: 66,66% I_{70}: 71,4 %
T_7: Subtraction with negative integers	I_{62}: 13% I_{68}: 26%	I_{62}:52,6 % I_{68}: 36,84%	I_{62}: 0% I_{68}: 8,7%	I_{62}: 52,3% I_{68}: 23,8 %
T_8: Addition of fractions with like denominator	I_2:39,13% I_{20}:43,48% I_{25}:43,48% I_{29}:39,13%	I_2: 21% I_{20}: 15,79% I_{25}: 15,79% I_{29}: 21%	I_2:39,13% I_{20}: 47,82% I_{25}: 52,17 % I_{29}:47,82%	I_2:28,5% I_{20}:33,33% I_{25}:38% I_{29}:33,33%
T_9: Addition of fractions with different denominator	I_{45}:8,7% I_{56}:17,39% I_{59}:8,7%	I_{45}: 0% I_{56}: 0% I_{59}: 0%	I_{45}:0% I_{56}:0% I_{59}:0%	I_{45}: 4,7% I_{56}:9,5% I_{59}: 0%
T_{10}: Subtraction of fractions with like denominator	I_{10}:39,13% I_{14}:47,82% I_{21}: 43,48% I_{30}:39,13% I_{43}:43,48	I_{10}: 15,79% I_{14}: 15,70% I_{21}: 15,79% I_{30}: 21% I_{43}: 15,79%	I_{10}:30,43% I_{14}:30,43% I_{21}: 56,5% I_{30}:30,43% I_{43}:43,47%	I_{10}:42,8% I_{14}:42,8% I_{21}: 42,8% I_{30}:38% I_{43}:33,33%
T_{11}: Subtraction of fractions with different denominator	I_{31}:8,7% I_{55}:0% I_{66}:0%	I_{31}: 0% I_{55}: 0% I_{66}: 0%	I_{31}:0% I_{55}:0% I_{66}:0%	I_{31}:14,2% I_{55}:4,7% I_{66}:4,7%
T_{12}: Multiplication of two fractions				
T_{13}: Division of two fractions				

T_{14}: Equivalence of fractions	I_6:34,78% I_7:34,78% I_8:39,13% I_{13}:34,78% I_{22}:52,17% I_{23}:47,82% I_{42}:52,17% I_{47}:41,3% I_{60}:52,17% I_{73}: 13% I_{76}:39,13% I_{77}:56,52% I_{78}:30,43% I_{81}:39,13% I_{82}:8,7%	I_6:63,15% I_7: 36,84% I_8:36,84% I_{13}:31,57% I_{22}:57,89% I_{23}:63,15% I_{42}:52,63% I_{47}:47,37% I_{60}:42,1% I_{73}: 21% I_{76}:47,37% I_{77}:42,1% I_{78}:10,5% I_{81}:42,1% I_{82}:15,79%	I_6:30,43% I_7:30,43% I_8:39,13% I_{13}:30,435% I_{22}:39,135% I_{23}:41,3% I_{42}:34,78% I_{47}:26% I_{60}:34,78% I_{73}: 10,87% I_{76}:39,13% I_{77}:65,21% I_{78}:4,3% I_{81}:52,1% I_{82}:2,1%	I_6:66,66% I_7:66,66% I_8:80,9% I_{13}:57,1% I_{22}:71,4% I_{23}:66,66% I_{42}:66,66% I_{47}:54,76% I_{60}:61,9% I_{73}:30,95% I_{76}:42,8% I_{77}:52,38% I_{78}:14,28% I_{81}:42,85% I_{82}:21,42%
T_{15}: Multiplication of a fraction with an integer	I_3:39,13% I_{16}: 26% I_{24}: 26%	I_3: 0% I_{16}: 5,26% I_{24}: 0%	I_3:8,7% I_{16}:0% I_{24}:13%	I_3:19% I_{16}:14,2% I_{24}: 28,5%
T_{16}: Division of an integer with a fraction				
T_{17}: Division of a fraction with an integer				
T_{18}: Convert a fraction to a decimal	I_{17}:43,47% I_{32}:47,82% I_{41}:47,82% I_{46}:52,17% I_{67}:30,43%	I_{17}: 52,63% I_{32}: 31,57% I_{41}: 31,58% I_{46}: 42,1% I_{67}: 26,32%	I_{17}:26% I_{32}:43,47% I_{41}:39,13% I_{46}:43,47% I_{67}:26%	I_{17}:57,14% I_{32}:52,38% I_{41}:47,62% I_{46}:47,62% I_{67}:19%
T_{19}: Convert a decimal to a fraction	I_{19}:65,2% I_{33}:21,74% I_{54}:60,87% I_{61}: 47,82 %	I_{19}: 68,42% I_{33}: 57,89% I_{54}: 57,89% I_{61}: 57,89%	I_{19}:39,13% I_{33}:17,4% I_{54}:47,8% I_{61}:39,13%	I_{19}:71,4% I_{33}: 52,38% I_{54}:75,19% I_{61}:52,38%
T_{20}: Examine which fraction with like denominators and different numerators is largest	I_{18}:78,26% I_{39}:69,56%	I_{18}: 73,68% I_{39}: 78,94%	I_{18}:47,82% I_{39}:65,21%	I_{18}:80,9% I_{39}:71,4%
T_{21}: Examine which fraction with like numerators and different denominators is largest	I_{48}:86,96% I_{74}:56,52%	I_{48}: 73,68% I_{74}: 21%	I_{48}:65,21% I_{74}:43,47%	I_{48}:66,66% I_{74}:45,23%
T_{22}: Examine which unit fraction $\frac{1}{a}$ and $\frac{1}{b}$ is largest	I_5:91,3% I_{79}:89,1%	I_5: 68,42% I_{79}: 31,57%	I_5:73,9% I_{79}:52,17%	I_5:85,71% I_{79}:45,23%
T_{23}: Placing fractions and decimals on a number line (the relationship between number symbols and their value understood as points on a number line)	I_{34}:30,43% I_{35}:30,43% I_{36}:17,39% I_{37}:25% I_{50}:23,9% I_{51}:73,9% I_{57}:65,2%	I_{34}: 19,73% I_{35}: 21% I_{36}: 19,73% I_{37}: 17,1% I_{50}: 18,4% I_{51}: 78,9% I_{57}: 68,42%	I_{34}:17,39% I_{35}:21,7% I_{36}:16,3% I_{37}:22,8% I_{50}:23,9% I_{51}:65,2% I_{57}:56,5%	I_{34}:33,33% I_{35}:22,62% I_{36}:25% I_{37}:33,33% I_{50}:28,57% I_{51}:80,9% I_{57}:66,66%
T_{24}: Examine which decimal is largest	I_{38}:78,26% I_{40}:73,9% I_{49}:52,17% I_{53}:56,52% I_{75}:52,17%	I_{38}: 57,89% I_{40}: 21% I_{49}: 15,79% I_{53}: 52,63% I_{75}: 36,84%	I_{38}:56,5% I_{40}:21,7% I_{49}:21,7% I_{53}:56,5% I_{75}:45,65%	I_{38}:71,42% I_{40}:19% I_{49}:23,8% I_{53}:61,9% I_{75}:52,4%
T_{25}: Solve a first-degree equation	I_4: 86,95%	I_4: 84,21%	I_4: 86,95%	I_4:85,7%

	I_{52}:78,26% I_{58}:34,78% I_{63}:21,74% I_{69}:34,78% I_{71}:17,39% I_{86}:	I_{52}: 78,94% I_{58}: 31,57% I_{63}: 36,84% I_{69}:26,31% I_{71}:10,52% I_{86}:	I_{52}:96,95% I_{58}:30,4% I_{63}:21,7% I_{69}:26% I_{71}:17,4% I_{86}:	I_{52}:85,7% I_{58}:28,5% I_{63}:47,6% I_{69}: 42,85% I_{71}:23,8% I_{86}:
T_{26}: Reduction tasks with one variable	I_{64}:4,34% I_{72}:4,34% I_{80}:4,34% I_{83}:4,34%	I_{64}:5,26% I_{72}:10,52% I_{80}:0% I_{83}:0%	I_{64}:21,7% I_{72}:17,4% I_{80}:0% I_{83}: 0%	I_{64}:47,62% I_{72}:33,33% I_{80}:0% I_{83}:9,5%
T_{27}: Reduction tasks up to 2 variables				
T_{28}: Reduction tasks up to 3 variables				
T_{29}: Equal sign	I_{84a}: 86,95% I_{84b}: 17,39	I_{84a}:73,68% I_{84b}: 15,8%	I_{84a}:78,26% I_{84b}: 0	I_{84a}:80,9% I_{84b}: 4,7%
T_{30}:Distributive law	I_{85}:56,52%	I_{85}:36,84%	I_{85}:47,8%	I_{85}:38%

Table 4: Results from the final test for fifth grade students

For the analysis of the results in Table 4, some decisions have been made. It has been decided that an item where less than half of the class master it, is considered as low mastery, but with a few exceptions (in T_1), which will be described later. Furthermore, if a very large gap exists among the items in a type of task, this will be addressed.

The analysis is done by first looking at the columns in Table 4 for identifying where the classes individually have low mastery. The next step was to look at and select some of the types of task the classes have in common and discuss a possible reason for the low mastery.

The types of tasks $T_3, T_4, T_5, T_{12}, T_{13}, T_{16}, T_{17}, T_{27}$ and T_{28} are in Table 4, but the rows are empty since these types of task only exist in the final tests for seventh-grade students, but since Table 3 is made for both grades together with items for each type of task it is not possible to change the number of the types of task.

8.1.1 Fifth-grade students' mastery of fractions

8.1.1.1 Addition and subtraction of fractions

In Table 4 it is observed that for types of task related to fractions, there exists some types of task which fifth-grade students (in all classes) show low mastery of, since the mastery percentage for the majority of the items is below 50%. Addition and subtraction of fractions with like and different denominators are types of task the students show a low mastery about, but the following will be based on addition and subtraction of fraction with like denominators. This is because very few students (applies to all four classes) (see Table 4) show mastery of addition and subtraction of fractions with different denominators and therefore it is conceivable that this is not types of task the students have been taught, although it is a types of task, they should master according to the PRM for fifth-grade student (see Table 1).

On the other hand, the textbook Alinea Matematrix (Matematrix 4) for fourth grade shows that addition and subtraction of fractions with like denominators is presented and taught in fourth grade. Therefore, a higher degree of these types of task, than Table 4 shows, is expected.

In addition and subtraction of fractions with like denominators, a very dominant technical misconception is observed. For addition of fractions with like denominators for instance I_2 and I_{29} typical wrong answers are:

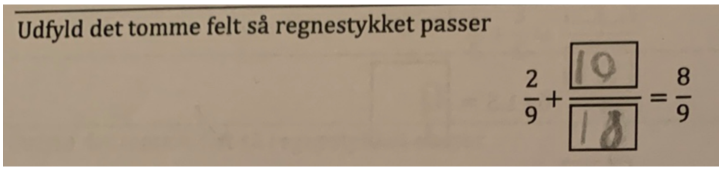


Figure 20: Student answer (5.C2, I_2, S_7)

Figure 21: Student answer (5.C4, I_{29}, S_{15})

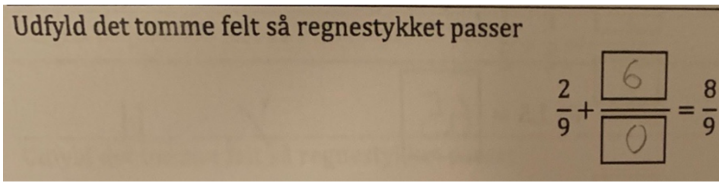


Figure 22: Student answer (5.C1, I_2, S_{21})

In all these answers it is possible to see that the students have worked with the numerator and denominator as two separated parts. In such items, it is possible to observe that a technical misconception exists among the students. Instead of only add the numerators of the two fractions, the students add the numerators and denominators of the two fractions separately, which shows that they possibly consider the fractions as two separate whole numbers.

This technical misconception among the students is exactly a misconception that has been identified from previous research (see section 2.4.3 e.g., Ashlock (2006) in Bush & Karp (2013) identified the same misconception).

The same technical misconception exists in subtraction of fractions with like denominators. Some answers from the students are:

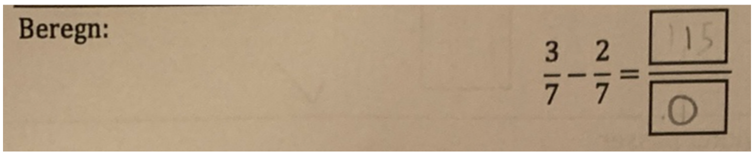


Figure 23: Student answer (5.C2, I_{10}, S_{15})

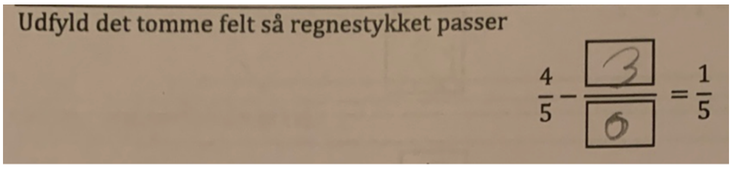


Figure 24: Student answer (5.C3, I_{21}, S_1)

In the same way, the technical misconception involve that the students subtract the numerators and the denominators separately as two whole numbers. This could be because the students mix different calculation rules of fractions. For instance, the misconception may occur because the students use the technique for multiplication of fractions and do the same with addition and subtraction of fractions with like denominators. Furthermore, the answers (Figure 23 and Figure 24) show that they do not master the theoretical knowledge that a fraction is a number with a denominator $\neq 0$.

8.1.1.2 The number line and the magnitude of fractions

In items in the type of task related to placing fractions on a number line, a misconception is observed. For instance, in I_{34} an example of how the fraction $\frac{1}{2}$ is placed on the number line is given and the students should place three other fractions as $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$.

The most prevalent answer for this item among the students was:

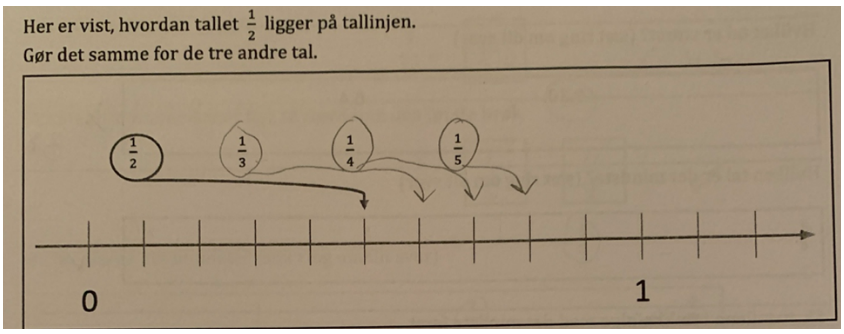


Figure 25: Student answer (5.C2, I_{34}, S_6)

The students probably consider the fraction as containing two separate parts and as whole numbers, where they may think that since 2 comes before 3 as whole numbers, it should be the same for the fractions $\frac{1}{2}$ and $\frac{1}{3}$. The teachers will be informed about this misconception, and it will be recommended that the students should be taught more thoroughly what the meaning of simple fraction (of type $\frac{1}{x}$) is.

Although there is a high mastery of the type of task related to the examination of the magnitude of fractions among the fifth-grade students (T_{20}, T_{21} and T_{22} in Table 4), there are still some misconceptions the teachers have to be informed about. These misconceptions are most prevalent in 5th gr.,I_{74} and 5th gr.,I_{79}. Some answers from the students for these items are:

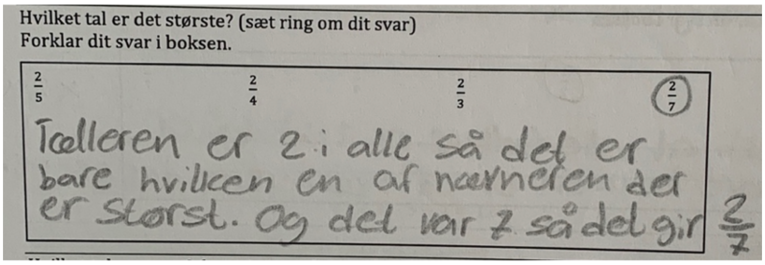


Figure 26: Student answer (5.C3, I_{74} , S_{23})

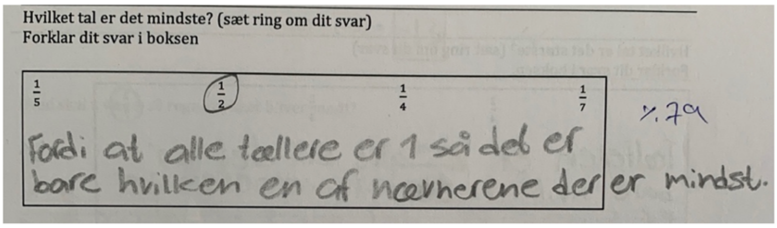


Figure 27: Student answer (5.C3, I_{79} , S_{23})

When examine which unit fraction or which fraction with like numerator and different denominator is largest or lowest, the students mostly compare the denominators without taking in consideration the whole fraction. Therefore, the typical answers for I_{74} is that "The numerator is 2 in all so it is just which one of the denominators are the largest one. And it was 7 so it gives $\frac{2}{7}$" (Figure 26), where the student explains that 7 is larger than 3,4,5 and therefor $\frac{2}{7}$ is largest. A prevalent answer for I_{79} is: "Because all numerators are 1, it is just which one of the denominators is the smallest" (Figure 27). Exactly in the same way as before, the student notice that 2 is lower than 4, 5 and 7 and therefor $\frac{1}{2}$ is the lowest fraction.

It is important to notice that some students give correct answers and explanations in I_{74} and I_{79}, but answers like Figure 26 and Figure 27 are still very prominent types of answers.

8.1.1.3 Equivalence of fractions

In continuation of the notion of fractions, some technical misconceptions and lack of knowledge are observed in T_{14} and T_{15}. In T_{14} about equivalence of fractions, a big variation among the items exists, but in general a low mastery of this type of task is dominant among these fifth-grade students.

Most of the items which belongs to equivalence of fractions are not answered by the students, and since lack of an answer gives 0 point, it is possible in Table 4 to observe a low percent of mastery of equivalence of fractions. In T_{14} there exists different type of items and in some of these, it is possible to observe the technical- and theoretical misconception the students have about equivalence of fraction.

A theoretical misconception about equivalence of fractions can be observed in for instance I_{76} and I_{81} and some comments will be provided for I_{78}.

I_{76}, I_{81} and I_{78} test whether the students understand the notions of 'Expanding' and 'Simplifying'.

Very common and repeated answer from the students for items about 'Expanding' are:

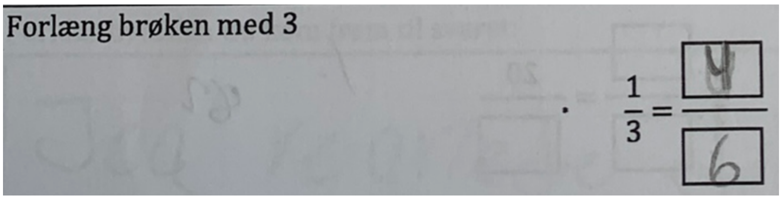


Figure 28: Student answer (5.C3, I_{76}, S_{20})

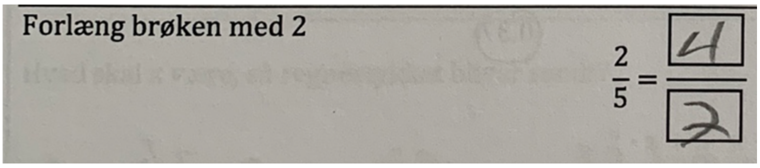


Figure 29: Student answer (5.C4, I_{81}, S_{19})

This shows that the students possibly understand the notion of 'Expand' as 'add' since many of the students have added 3 or 2 to the numerator and denominator in I_{76} and I_{81} respectively. This misconception could be described as a technical misconception since the students use a wrong technique. Instead of using $\tau_{12}: \frac{a}{b} = \frac{k \cdot a}{k \cdot b}$ for $k \in \mathbb{N}$ the students do the following:

$$\tau_{12}^{*}: \frac{a}{b} = \frac{k+a}{k+b}$$

But this misconception is a more theoretical misconception since the students have a wrong definition of the notion ‘Expand’.

In addition, as it is illustrated in Table 4, for all classes it is observed that I_{76} and I_{81}, where the students must ‘Expand’, have got a higher correctness than I_{78}, where the students have to ‘Simplify’. This may indicate that there are a greater understand of the notion of ‘Expand’ rather than ‘Simplify’ among these students.

In multiplication of a fraction with an integer (T_{15}) a very common technical misconception among the students is to multiply both the numerator and denominator with the integer instead of only multiply the numerator with the integer. Some answers from the students are:

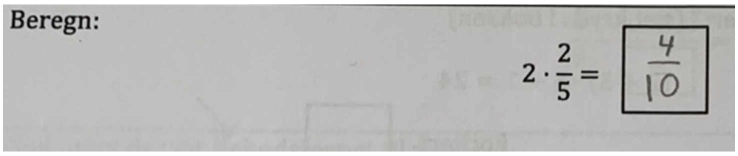


Figure 30: Student answer (5.C1, I_3, S_2)

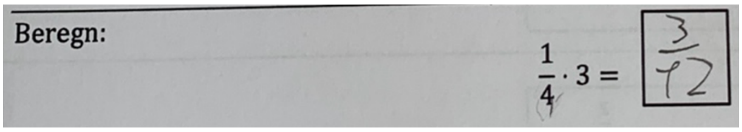


Figure 31: Student answer (5.C3, I_{16}, S_{20})

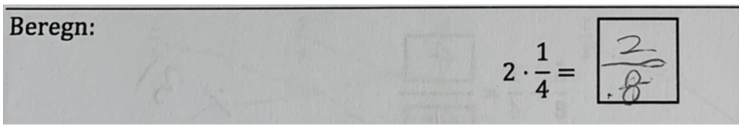


Figure 32: Student answer (5.C3, I_{24}, S_{21})

It is possible to see that instead of using τ_{13} in Table 1 the students apply a wrong technique denoted as τ_{13}^* i.e.

$$\tau_{13}^* = \frac{a}{b} \cdot k = \frac{a \cdot k}{b \cdot k}$$

From Table 4 it is observed that 5.C4 shows most mastery of equivalence of fractions compared with the three other classes, while 5.C1 and 5.C3 are the classes with lowest mastery (almost half as many as in 5.C4). And 5.C2, 5.C3 and 5.C4 show very low or almost no mastery of multiplication of a fraction with an integer. The mastery is higher in 5.C1 (almost 40%) but it is still not sufficient since multiplication of an integer with a fraction is an important type of task to master for later work with more complex algebraic expressions. This shows that all the teachers can be advised to focus more on multiplication of an integer with a fraction in their teaching, while the teachers primarily from 5.C1 and 5.C3 should also focus on equivalence of fractions.

8.1.2 Fifth-grade students' mastery of calculation of arithmetic expressions

Calculation of arithmetic expressions involving integers and two or more operations (T_1) is a very broad type of task which contains items with different operations and order of these. Depending on the operation and the order of the operations, some huge differences can be observed in Table 4. In Table 4 it is possible to observe a high correctness for I_9 in all four classes (between 80-100%), while there exists almost half (or lower) as much correctness in for instance I_1 and I_{28}. Since there is a big variation between these items belonging to the same type of task, an extra attention will be laid on these and a possible reason for the high difference among the items in the same type of task will be presented.

5th gr.,I_9 in the final test is:
What is the solution to the calculation: $3 \cdot 2 + 5$ $\square$

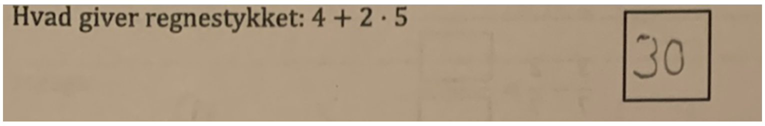
While 5th gr.,I_1 and I_{28} are:

I_1
What is the solution to the calculation: $4 + 2 \cdot 5$ $\square$

I_{28}
What is the solution to the calculation: $3 - 2 + 5 \cdot 2$ $\square$

It is possible to see that the students master it when the expression is written as in I_9. In I_9 the student will possibly solve this item by calculation from left to right, and since the multiplication is the first operation in the calculation, no problems occur. But the technical misconception becomes much more apparent in I_1 and I_{28}, where the multiplication is the last operation in the expression. If the student master the technique belonging to T_1 (see τ_2 in Table 1), there would possible not be such a big difference between the correctness of I_9 and I_1 & I_{28}, but in Table 4 it is possible to observe a big difference which may indicate that the students do not master the techniques belonging to T_1.

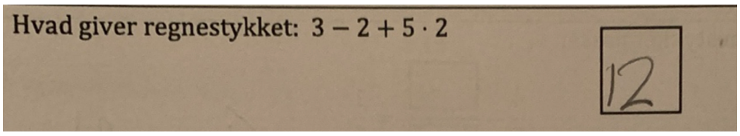
In addition to the fact that there is a significant difference in the correctness of these items according to Table 4, it is also important to show what type of mistakes belonging to T_1 the students actually made. Some examples from the students are:


Hvad giver regnestykket: $4 + 2 \cdot 5$

30

Figure 33: Student answer (5.C2, I_1, S_8)

And

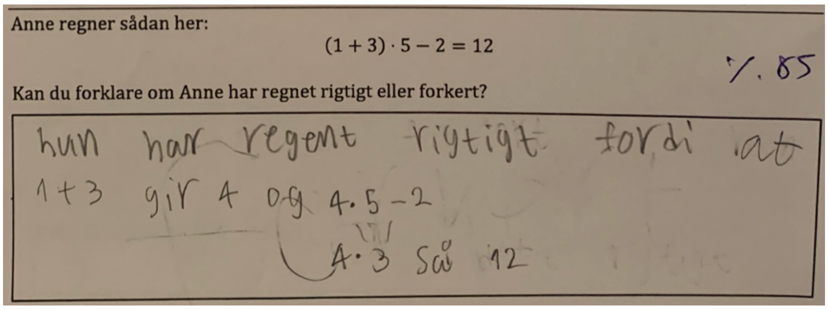

Hvad giver regnestykket: $3 - 2 + 5 \cdot 2$

12

Figure 34: Student answer (5.C3, I_{28}, S_1)

Although there exist students who master the technique belonging to T_1, a very big part of the students had a misconception about the technique belonging T_1. The students who have answered I_1 incorrectly have given the answer 30, while a very common incorrect answer for I_{28} among the students is 12. This indicates that in both cases most of the students calculate the expression from left to right without taking into consideration the operations and the order of these.

Another interesting observation in Table 4 is T_{30} about the distributive law. There are students who master the distributive law, but a common wrong use of the distributive law can be observed in the following answer from a student:


Anne regner sådan her:

$(1 + 3) \cdot 5 - 2 = 12$

I. 85

Kan du forklare om Anne har regnet rigtigt eller forkert?

hun har regent rigtigt fordi at
1+3 gir 4 og 4·5−2
4·3 så 12

Figure 35: Student answer (5.C4, I_{85}, S_{18})

The question is that the students should assess whether Anna has calculated the expression $(1 + 3) \cdot 5 - 2 = 12$ correct and furthermore explain their answers. A student says that "Anna has calculated the expression correct because $1 + 3$ is 4 and $4 \cdot 5 - 2 = 4 \cdot 3 = 12$" (Figure 35). Such an explanation is a prominent answer among the fifth-grade students. Instead of the use of the distributive law, the student calculates the parenthesis and $5 - 2$ separately and then multiply these two values. By working with this item in that way, it may indicate that there exists a lack of knowledge about the use of parenthesis. Furthermore, by answering as the student in Figure 35, a lack of theoretical understanding of the notion of the distributive law can be observed.

So, in conclusion, the results from the final tests were presented to discuss in what extent the fifth-grade students, who performed the diagnostic test, master the praxeologies belonging to arithmetic and algebra (see Table 1). As it can be observed in Table 4, the students master all types of task but in different extent (which both applies for the students individually and to the whole class), but in above section some central misconception made by the students belonging to some of the types of task have been presented.

8.1.3 Solving a first-degree equation and the equal sign

All four classes show a mastery of the solution of a first-degree equation (about 80-90%) when the item is like:

5th gr.,I_4:

Write a number in the empty box so the calculation is correct

$$36 - \square = 29$$

But when the item involves the variable x as the case in I_{63} and I_{69}:

5th gr.,I_{63}:

What should x be so that the calculation becomes true?

$$2x = 10$$

$$\square$$

Or

5th gr.,I_{69}:

What should x be so that the calculation becomes true?

$$x - 36 = 48$$

$$\square$$

the correctness for these items among the Danish fifth-grade students is about 20-35% (and 40-50% in 5.C4). The teachers will be informed about this, by telling them that the student master to solve a first-degree equation without the variable x. It is conceivable that the variable x has not been presented to the students yet, although according to the praxeological reference model for fifth grade (Table 1) it should, but the teachers will still get some feedback for this anyway.

Sure, eventually, if the student understands the connection between the variable x and the empty box, they will possibly have fewer problems in solving a first-degree equation.

In continuation of this, about 15% in 5.C1 and 5.C2 and 0% in 5.C3 and 5.C4 do have a relational understanding of the equal sign i.e., the students understand the equal sign as indicating a relationship between the two sides of the equal sign, while most of the students have an operational understanding of the equal sign. The operational understandings of the equal sign are expressed in the students answers in the following ways:

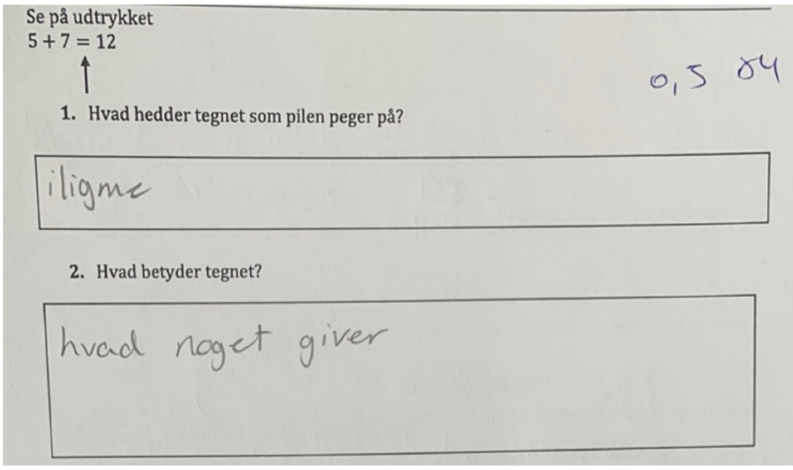


Figure 36: Student answer (5.C4, I_{84a} & I_{84b}, S_{10})

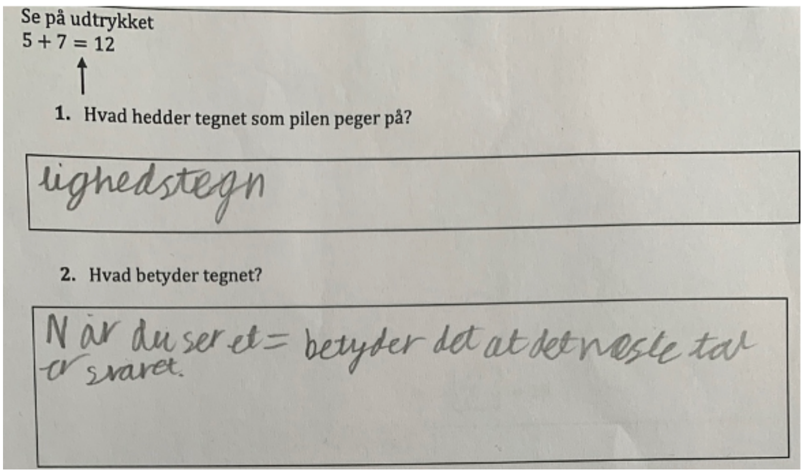


Figure 37: Student answer (5.C1, I_{84a} & I_{84b}, S_6)

Thus, a majority of the students have an operational approach to the equal sign since they give answers such that "When you see a = it means that the next number is the answer" (Figure 37) and "What something gives" (Figure 36). This is known from the research as mentioned in section 2.4.1. A relational understanding of the equal sign is therefore necessary, as mentioned in section 2.4.1, in the solution of algebraic items such that the solution of a first-degree equation why the teachers could be advised to focus on this in their teaching.

The following will present more essential differences between the four different classes, which the teachers will be informed about too, but notice that the teachers do not get information about how the specific other classes have coped with the test, but they get information about how their own class coped, compared with the other classes.

5.C2 and 5.C4 show greatest mastery in addition of negative integers compared with 5.C1 and 5.C3. Furthermore, although the mastery of subtraction with negative integers is most high in 5.C2 and 5.C4 compared to the other two classes (according to Table 4) it is still very low. Therefore, especially the teachers of 5.C1 and 5.C3 will be advised to revisit addition and subtraction with negative integers, but the teachers of 5.C2 and 5.C4 might fruitfully do that too, since the mastery of these types of tasks is also relatively low. The teachers will furthermore be informed about that page 12 in Gyldendal MULTI 5 i-bog (denoted M5) will be very usable in the teaching of the negative integers.

8.1.4 The magnitude of decimals

The last misconceptions the teachers for all classes could be informed about is related to examination of which decimal is largest (T_{24} in Table 4). When the students should order decimals from the lowest to the highest (I_{40} and I_{49}) a very prevalent misconception is observed among all the classes. Notice that 5.C1 is the class who show the highest mastery of this type of task, but the prevalent misconception also exists in this class. The misconception can be observed in the following answers from the students:

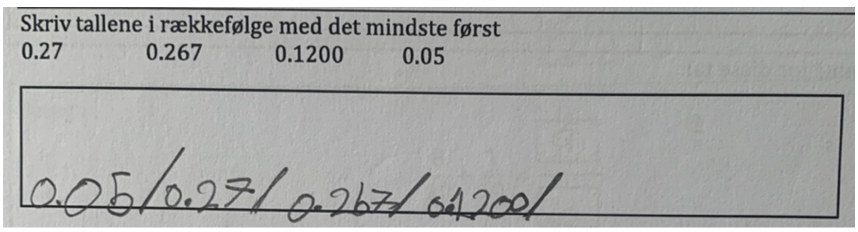


Figure 38: Student answer (5.C3, I_{40} , S_1)

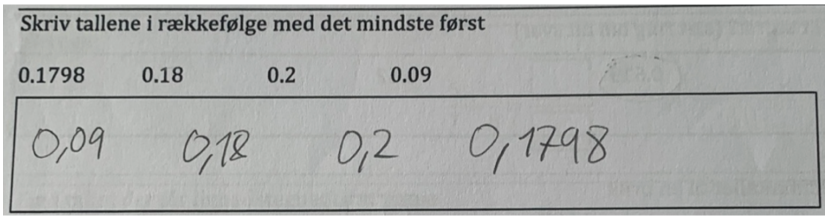


Figure 39: Student answer (5.C1, I_{49} , S_5)

As it can be seen the observed misconceptions among fifth-grade students is to confuse the decimals with whole numbers and therefore, the students sometimes consider the number with most decimals, which is also discussed in section 2.4.2, as the largest one. This is not always the case, since the student write that 0.18 is larger than 0.2, but the misconception is still prominent. For instance, in Figure 38 it is possible to observe that the student think that '267 is larger that 27' or '1200 is larger that 267' and therefor '0.267 should be larger than 0.27' and '0.1200 should be larger than 0.267'.

This misconception is more obvious in the following student answer, which has chosen 0.362 to be larger than 0.37:

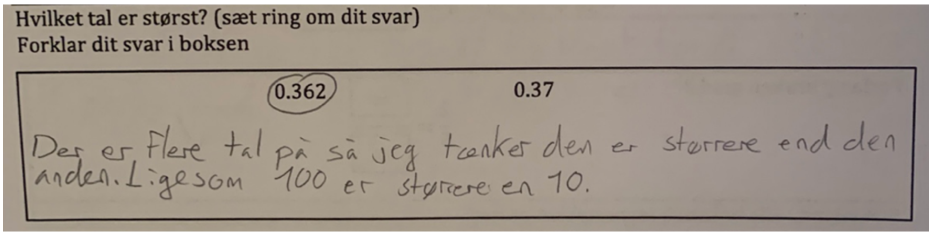


Figure 40: Student answer (5.C1, I_{75} , S_{21})

Where the student's answer is: "There are more numbers on, so I think it is bigger than the other. Like 100 is bigger than 10". This confirms that this prevalent misconception is because of a technical misconception or lack of knowledge about the lexicographical ordering.

8.2 Seventh-grade students praxeologies

The diagnostic test performed by 78 seventh-grade students contain 30 types of tasks. The following will try to give a picture of what the seventh-grade students have low and high mastery about, and further give some comments about what the individual classes do not master.

The results from the diagnostic test, performed by four Danish seventh grade classes, are visible in Table 5. These results will be used to answer to what extent, the praxeologies from Table 2 are mastered by seventh-grade students. As the case with Table 4, Table 5 contains five columns, where the first contains the types of task that occurs in the diagnostic test, while the other four columns show number of points each class has received per item (converted to percent). These classes are denoted with 7.C1, 7.C2, 7.C3 and 7.C4.

As with the case of Table 4, it is possible to observe from Table 5 that there are manifest differences in how well the classes master certain different types of task related to arithmetic and algebra. This variation probably exists because the teachers have taught and focused on different parts of arithmetic and algebra, since the concrete praxis blocks the teachers should teach about are not clarified in the official programme for Danish for seventh to ninth grades. The existing variation among the classes will be further clarified in the next section. Not all the results from Table 5 will be presented, but some interesting observations and types of task where a high rate of mistakes occurs, will be analyzed.

As with the case in section 8.1, some decisions are made for the presentation and analysis of the results. It has been decided pragmatically that from 70% and upwards the concrete type of task will be considered as mastered by the class, and if there is a large gap among the different items in a type of task, this will be addressed.

There are a total of 93 items in the diagnostic test for seventh grade and it turns out that many students have not had enough time to complete the entire test. Since a wrong answer and an unanswered item give 0 point, it is decided that the last items in a type of task does not have as crucial importance as the first items. To specify, this means that if a type of task consists of for instance three items and the first two items show a high mastery of this type of task, while the last item shows half as high mastery, the last item will not be taking into consideration, as it is conceivable that many have not reached all the item.

The analysis of Table 5 is done by first looking at each type of task and assess whether the classes show a low mastery of the type of task. By doing this, some common types of task and differences among the classes were identified. In the following section these common types of task and differences will be presented.

Types of task	7.C1	7.C2	7.C3	7.C4
T_1: Calculation of arithmetic expressions involving integers and two or more operations	I_1:86,4%	I_1:76,2%	I_1:41%	I_1:38,9%
T_2: Multiplication of a decimal with an integer	I_2:100% I_{34}:81,8%	I_2:85,7% I_{34}:76,2%	I_2:76,5% I_{34}:70,6%	I_2: 88,9% I_{34}:55,6%
T_3: Multiplication of decimals	I_{11}:13,6% I_{29}:13,6	I_{11}:52,4% I_{29}:42,8%	I_{11}:11,7% I_{29}:17,7%	I_{11}: 5,6% I_{29}:16,7%

	I_{93}:0%	I_{93}:4,8%	I_{93}:0%	I_{93}:5,6%
T_4: Multiplication of negative integers	I_3:72,7%	I_3:71,4%	I_3:58,8%	I_3:38,9%
T_5: Multiplication of a negative and positive integers	I_4:77,2%	I_4:71,4%	I_4:70,6	I_4:22,22%
T_6: Addition with negative integers	I_{12}:77,2% I_{36}:50%	I_{12}:80,9% I_{36}:28,6%	I_{12}:94,1% I_{36}:58,8%	I_{12}:77,8% I_{36}: 38,9%
T_7: Subtraction with negative integers	I_{30}:36,4% I_{35}:54,5%	I_{30}:33,33% I_{35}:14,3%	I_{30}:35,3% I_{35}:41,2%	I_{30}:22,2% I_{35}:11,1%
T_8: Addition of fractions with like denominator	I_5: 68,2% I_{26}:81,8% I_{85}:13,6%	I_5:90,5% I_{26}:80,9 I_{85}:30,9%	I_5:70,6% I_{26}:88,2% I_{85}:20,6%	I_5:88,9% I_{26}:94,4% I_{85}:36,1%
T_9: Addition of fractions with different denominator	I_{17}:31,8% I_{37}:22,7%	I_{17}:61,9% I_{37}:14,3%	I_{17}:0% I_{37}:0%	I_{17}:33,3% I_{37}:0%
T_{10}: Subtraction of fractions with like denominator	I_{13}:81,8 I_{19}:63,6% I_{22}:72,7%	I_{13}:61,9% I_{19}:85,7% I_{22}:90,5%	I_{13}:64,7% I_{19}:64,7% I_{22}:88,2%	I_{13}:72,2% I_{19}:66,7% I_{22}:94,4%
T_{11}: Subtraction of fractions with different denominator	I_{28}:36,4% I_{31}:36,4%	I_{28}:57,1% I_{31}:52,4%	I_{28}:17,7% I_{31}:17,7%	I_{28}:44,4% I_{31}:33,33%
T_{12}: Multiplication of two fractions	I_{14}:36,4% I_{15}:68,2% I_{38}:68,2%	I_{14}: 42,9% I_{15}:57,1% I_{38}:71,4%	I_{14}:70,6% I_{15}:88,2% I_{38}:70,6	I_{14}:44,4% I_{15}:61,1% I_{38}:55,6%
T_{13}: Division of two fractions	I_{32}:0% I_{39}:0% I_{64}:13,6% I_{70}: 4,5% I_{92}: 0%	I_{32}:14,3% I_{39}:28,5% I_{64}:33,33% I_{70}:14,3% I_{92}:4,7%	I_{32}:0% I_{39}:0% I_{64}:11,8% I_{70}:11,8% I_{92}:0%	I_{32}:11,1% I_{39}:5,6% I_{64}:11,1% I_{70}:11,1% I_{92}:8,3%
T_{14}: Equivalence of fractions	I_6:72,7% I_{16}:68,2% I_{41}:72,7% I_{42}:72,7% I_{43}: 72,7% I_{48}:77,2% I_{54}:40,9% I_{66}:40,9% I_{71}:54,5% I_{75}:45,4% I_{78}:27,2% I_{83}:40,9% I_{86}: 20,5%	I_6:80,9% I_{16}:66,66% I_{41}:80,9% I_{42}:80,9% I_{43}:76,2% I_{48}:85,7% I_{54}:76,2% I_{66}:33,33% I_{71}:57,1% I_{75}:38% I_{78}:28,6% I_{83}:35,7% I_{86}:21,4%	I_6:58,8% I_{16}:88,2% I_{41}:76,5% I_{42}:64,7% I_{43}:64,7% I_{48}:82,4% I_{54}:35,3% I_{66}:35,3% I_{71}:58,8% I_{75}: 50% I_{78}:17,7% I_{83}: 17,7% I_{86}:17,7%	I_6:83,3% I_{16}:83,3% I_{41}:50% I_{42}:55,5% I_{43}:44,4% I_{48}:66,6% I_{54}:33,3% I_{66}:33,3% I_{71}:55,5% I_{75}:36,1% I_{78}:11,1% I_{83}:22,2% I_{86}:27,8%
T_{15}: Multiplication of a fraction with an integer	I_{46}:50% I_{50}:45,5%	I_{46}:42,8% I_{50}:52,4%	I_{46}: 70,6% I_{50}:64,7%	I_{46}:50% I_{50}:50%
T_{16}: Division of an integer with a fraction	I_{33}:13,6%	I_{33}:14,3%	I_{33}:0%	I_{33}:0%
T_{17}: Division of a fraction with an integer	I_{49}:18,2%	I_{49}:28,6%	I_{49}:35,3%	I_{49}:27,8%
T_{18}: Convert a fraction to a decimal	I_7:63,6% I_{18}:59% I_{21}:59% I_{44}:40,9%	I_7: 80,9% I_{18}:80,9% I_{21}:80,9% I_{44}: 52,4%	I_7: 76,5% I_{18}:76,5% I_{21}:82,4% I_{44}:41,2%	I_7:77,8% I_{18}:77,8% I_{21}:66,7% I_{44}:27,8%
T_{19}: Convert a decimal to a fraction	I_8:63,6%	I_8:80,9%	I_8: 88,2%	I_8:66,7%

	I_{45}:72,7% I_{67}: 77,2% I_{72}:45,5%	I_{45}:80,9% I_{67}:61,9% I_{72}:38%	I_{45}: 82,4% I_{67}:53% I_{72}:41,2%	I_{45}:72,2% I_{67}:50% I_{72}:33,3%
T_{20}: Examine which fraction with like denominators and different numerators is largest	I_{20}:95,5% I_{58}:86,4%	I_{20}:85,7% I_{58}:76,2%	I_{20}: 82,4% I_{58}:58,8%	I_{20}:83,3% I_{58}:61,1%
T_{21}: Examine which fraction with like numerators and different denominators is largest	I_{59}:86,4% I_{76}:59%	I_{59}:66,66% I_{76}:45,2%	I_{59}:76,5% I_{76}:41,2%	I_{59}:61,1% I_{76}:27,8%
T_{22}: Examine which unit fraction $\frac{1}{a}$ and $\frac{1}{b}$ is largest	I_9:90,9% I_{62}:54,5% I_{80}: I_{82}:43,2%	I_9:90,5% I_{62}:52,4% I_{80}: I_{82}:40,5%	I_9:88,2% I_{62}:35,3% I_{80}: I_{82}:32,4%	I_9: 94,4%% I_{62}:0% I_{80}: I_{82}:47,2%
T_{23}: Placing fractions and decimals on a number line (the relationship between number symbols and their value understood as points on a number line)	I_{23}:48,8% I_{24}:53,4% I_{25}:45,5% I_{51}: 52,3% I_{52}:77,3% I_{60}:42%	I_{23}:77,4% I_{24}:78,5% I_{25}:61,9% I_{51}:61,9% I_{52}:76,2% I_{60}:57,1%	I_{23}: 47% I_{24}:47,4% I_{25}:41,2% I_{51}:44,1% I_{52}:70,6% I_{60}:30,9%	I_{23}:32% I_{24}:32% I_{25}:27,8% I_{51}:30,5% I_{52}:55,5% I_{60}:27,8%
T_{24}: Examine which decimal is largest	I_{47}:72,7% I_{56}:86,4% I_{68}:81,8% I_{77}:54,5%	I_{47}:76,2% I_{56}:76,2% I_{68}:66,66% I_{77}:47,6%	I_{47}:58,8% I_{56}:88,2% I_{68}:70,6% I_{77}:55,9%	I_{47}:59% I_{56}:72,2% I_{68}:55,5% I_{77}:36,1%
T_{25}: Solve a first-degree equation	I_{40}:63,6% I_{53}:68,2% I_{55}: 81,8% I_{57}:27,3% I_{61}:68,2% I_{63}:13,6% I_{69}: 59% I_{79}: 56,8% I_{81}:27,3%	I_{40}:42,8% I_{53}: 42,8 % I_{55}:71,4% I_{57}:47,6% I_{61}:57,1% I_{63}:33,33% I_{69}:61,9% I_{79}:52,4% I_{81}:35,7%	I_{40}:47% I_{53}:76,5% I_{55}: 76,5% I_{57}:11,8% I_{61}:70,6 I_{63}:23,5 I_{69}:53% I_{79}:47% I_{81}:11,8%	I_{40}:27,8% I_{53}:55,5% I_{55}:61,1% I_{57}:22,2% I_{61}:44,4% I_{63}:16,7% I_{69}:38,9% I_{79}:27,8% I_{81}:0%
T_{26}: Reduction tasks with one variable	I_{27}:59%	I_{27}:61,9%	I_{27}:35,3%	I_{27}:11,1%
T_{27}: Reduction tasks up to 2 variables	I_{10}:50% I_{74}:18,2%	I_{10}:61,9 I_{74}:33,33%	I_{10}:41,2% I_{74}:11,8%	I_{10}:0% I_{74}:5,5%
T_{28}: Reduction tasks up to 3 variables	I_{65}: 9% I_{73}: 13,6%	I_{65}:19% I_{73}:57,1%	I_{65}:5,9% I_{73}:17,7	I_{65}:0% I_{73}:0%
T_{29}: Equal sign	I_{87a}:50% I_{87b}:31,8% I_{88}:40,9% I_{89}:4,5% I_{90}:4,5% I_{91}:20,5%	I_{87a}:42,8% I_{87b}:38% I_{88}:35,7% I_{89}:23,8% I_{90}:23,8% I_{91}:16,66%	I_{87a}:41,2% I_{87b}:11,8% I_{88}:20,6% I_{89}:8,8% I_{90}:0 I_{91}:11,8%	I_{87a}:66,7% I_{87b}:22,2% I_{88}:41,7% I_{89}: 8,3% I_{90}:11,1% I_{91}:27,8%
T_{30}:Distributive law	I_{84}:25%	I_{84}:23,8%	I_{84}:17,7%	I_{84}:8,3%

Table 5: Results from the final test for seventh grade students

8.2.1 Seventh-grade students' mastery of fractions

8.2.1.1 Addition and subtraction of fractions

Compared with the students, who had performed the diagnostic test for fifth grade (see section 8.1), it is observed in Table 5 that among the students in seventh grade, there exists a high mastery (between 70-90%) of the items for the types of task related to addition and subtraction of fractions with like denominators, except in I_{85} (see Appendix 6). This can have many reasons, for instance:

1. that the students do not understand the formulation of the item since it was different compared with the other two items, so they skipped I_{85}, 2. that they did not manage to complete the item due to time pressure or 3. because they give a wrong answer to the item. All these reasons resulted in 0 point. By a detailed examination of all the test answers from the 76 seventh-grade students who participate in the test, it turns out that 56 of these have not answered this item. In many cases it seems like it is the time pressure which is the reason for why I_{85} shows a low mastery in Table 5, and more generally many of the last items are not answered by the students. Therefore, the low percentage for I_{85} in all classes does not necessarily mean that they do not master the types of tasks related to addition of fractions with like denominators, since the other two items in T_8 show the opposite.

On the other hand, in Table 5 it is possible to observe a low mastery of types of task related to addition and subtraction of fractions with different denominators. These types of task are prominent in the praxeological reference model for fifth grade (see Table 1), so the students should have been presented to these types of task much earlier and therefore should have a higher mastery of T_9 and T_{11} than observed in Table 5.

Although a low mastery of addition and subtraction of fractions with different denominators occurs among all classes, it is especially the results for 7.C3 which are conspicuous. While there are 7, 13 and 6 students respectively in 7.C1, 7.C2 and 7.C4 who show a mastery of for instance I_{17}, there are no students who master this in 7.C4. In the same way, while there are 8, 12, and 8 students respectively in 7.C1, 7.C2 and 7.C4 which show a mastery of for instance I_{28}, this item is only mastered by 3 students in 7.C3.

In the test answers, it is observed that the students approach I_{17} and I_{37} in three different ways. A large portion of the students have not answered the items, which is not due to the time pressure, since the items come early in the test and the students have also answered the subsequent items.

Two typical answers to items I_{17} and I_{37} are observed:

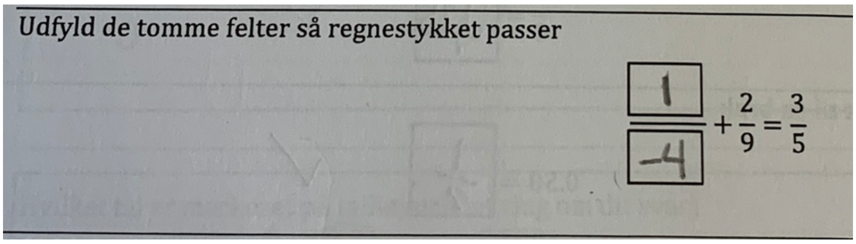


Figure 41: Student answer (7.C3, I_{37}, S_4)

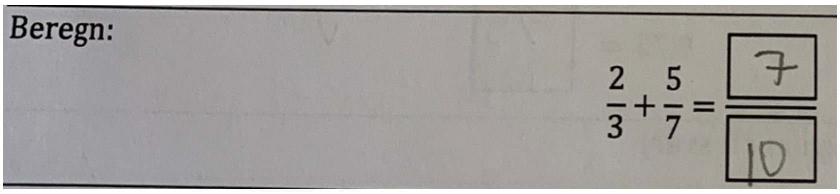


Figure 42: Student answer (7.C4, I_{17}, S_5)

The technical misconception that occurs in these students' answers, was described in section 2.4.3 about addition and subtraction of like denominators. The students consider the fraction as to separate numbers since the students add the numerators and denominators of the two fractions separately. These items should be solved by either using $\tau_{11}: \frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$ or $\tau_{10}: \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ in Table 2, or by using equivalence of fractions.

The same technical misconception about subtraction of fractions with different denominators is seen here:

Figure 43: Student answer (7.C1, I_{28}, S_5)

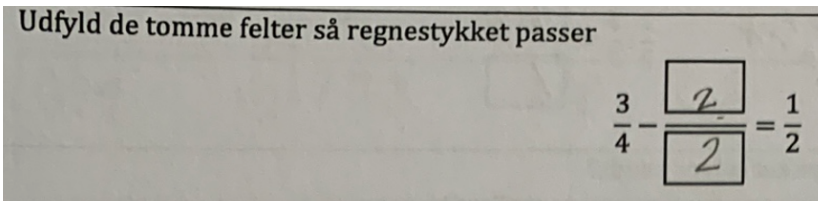


Figure 44: Student answer (7.C2, I_{31}, S_1)

These answers and like show that the students have the same technical misconception about subtraction of fractions with different denominators as with addition of two fractions with different denominators.

These technical misconception or lack of knowledge about addition and subtraction of fractions with different denominators may suggest that some seventh-grade students could have a low mastery of addition and subtraction of fractions with different denominators.

8.2.1.2 Multiplication and division with fractions

T_{12} and T_{13}, about multiplication and division of two fractions respectively, are types of task which only exist in the test for seventh-grade students since it is not a part of the praxeological reference model for Danish fifth-grade students (see Table 1).

Although 7.C3 is the class with the lowest mastery of addition and subtraction of fractions with different denominator, they are in turn the class that has the highest mastery of multiplication of two fractions. The results for I_{14} about multiplication of two fractions are quite conspicuous (See Table 5). I_{14} is about multiplication of two fractions, while I_{15} is a varied version of I_{14}. It was expected that I_{14}would be performed better than I_{15}, since I_{14} is written as two fractions multiplied with each other, while I_{15} is more complicated since it is written as a fraction multiplied with an unknown fraction, nevertheless students in 7.C1, 7.C2 and 7.C4 all show a low mastery of I_{14} and have a higher success rate in I_{15} than I_{14}:

(7th gr.,I_{14})

Calculate:

$$\frac{1}{3} \cdot \frac{2}{7} = \frac{\square}{\square}$$

(7th gr.,I_{15})

Write a number in the empty boxes so the calculation is correct

$$\frac{2}{5} \cdot \frac{\square}{\square} = \frac{6}{25}$$

It is difficult to point the reasons behind this. The only repeated answer from the students for I_{14} is to add the numerator and denominator separately instead of multiplying, so a technical misconception occurs, as seen here:

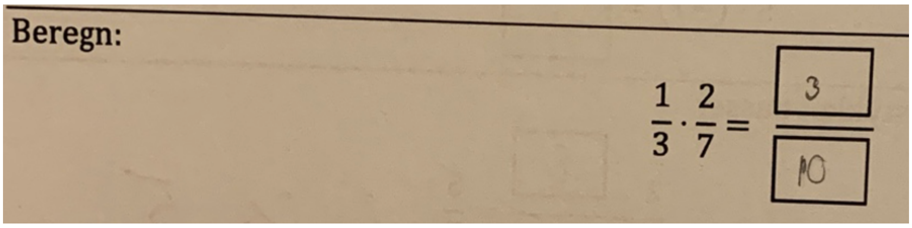


Figure 45: Student answer (7.C1, I_{14}, S_{20})

From this student answer it is possible to see that the student master some techniques about types of task related to fractions, but it seems like the students mix the techniques and show some technical misconception. Instead of multiplying the numerator and denominator of the fractions separately, the students add them.

A possible reason for this technical misconception (and in fact other technical challenges) may be due to the fact that the teachers skip the theory and mostly focus on the calculation methods. This means that the teachers may have tried to teach the students some calculation methods, like algorithms, but neglected to give the students a sense and understanding of why these calculation methods are correct. When the students memorize a lot of techniques, without understanding, and when the students solving an item by using a wrong method, it may indicate that the students know something, but use it incorrectly. This must indicate that the teaching has been technique-focused, where a lot of training of techniques has been central in these students' teaching, without theoretical review.

In addition to this, many students do not answer the item. In general, the students' answers for I_{14} show that multiplication of two fractions is probably not a type of task students have technical or theoretical misconceptions about, but a lack of knowledge about multiplication of two fractions possibly exist among some students.

By considering the first three items in T_{13} (division of two fractions), a low mastery of division of two fractions exists among the seventh-grade students. Table 5 shows that the five items belonging to T_{13} are only mastered by very few students and there exists a pattern which shows that many students are not taught this, as they cannot solve the items.

For division of two fractions, the results observed in Table 5 show that most of the students either have a technical misconception or a lack of knowledge about division of fractions. This will be assessed on the basis of the answers from the students that have performed the test.

Division of two fractions has been tested differently in each item and it is expected that I_{39} wil have the highest success rate since I_{39} is division of two fractions while the other items have some variations.

I_{39}: Calculate:

$$\frac{1}{2} : \frac{5}{7} = \frac{\square}{\square}$$

Some examples of the variations are:

(7^{th} gr.,I_{32})

Write a number in the empty boxes so the calculation is correct

$$\frac{2}{5} : \frac{\square}{\square} = \frac{6}{10}$$

(7^{th} gr.,I_{64})

Calculate:

$$\frac{\square}{\square} : \frac{3}{4} = \frac{8}{15}$$

If students are not able to solve I_{39}, it probably shows a lack of knowledge about division of two fractions, and this lack of knowledge is an obstacle in the remaining items, since they cannot be solved without a mastery of the knowledge related to division of two fractions and the belonging techniques.

I_{39} is solved correctly by 7 out of 78 students, who performed the test for seventh grade, while the remaining items in T_{13} are variations of I_{39}. I_{64} is solved correctly by 14/78 students, I_{32} is solved by 5/78 and I_{70} is solved by 8/78 students, while I_{92} is one of the last items that many students did not get through due to time pressure. Many students have skipped the items related to division of two fractions, while few shows technical misconceptions such as:

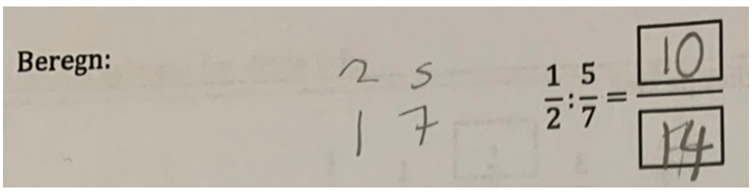


Figure 46: Student answer (7.C4, I_{39}, S_{12})

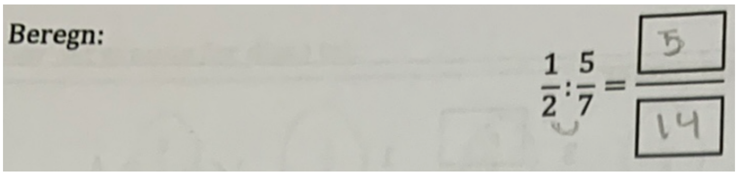


Figure 47: Student answer (7.C2, I_{39}, S_{11})

These two answers from two students shows different misconceptions. In the first one (Figure 46), the student possibly knows that the denominator and numerator may have to be switched in one of the fractions, but the student shows incorrect use of a technique, as he / she switch the numerator's and denominator's place in the first fraction rather than the second and then multiplies the fractions incorrectly. This thus shows use of a wrong technique.

In the second answer (Figure 47), it is possible to observe that the student directly multiplies the fractions without switching the numerator and denominator in the last fraction.

This shows that students have a knowledge and a technique about that multiplication is involved in the division of fractions, but this technique is deficient and is used incorrectly by the students. As mentioned earlier, this may be due to the teachers' dominating focus on techniques, rather than the theoretical knowledge, which could be why the students mix the different techniques.

In general, the answers from the students for this type of task show there is mostly a lack of knowledge and understanding of what division of fractions are and very few have wrong techniques for solving items related to division of two fractions. This lack of knowledge could be because the teacher for the classes has not taught this type of task, which is why the teachers will be informed about their students' misconceptions and lack of knowledge within this type of tasks.

8.2.1.3 Multiplication and division with a fraction and an integer

In Table 4, it is observed that the fifth-grade students who have performed the diagnostic have a low mastery of multiplication of a fraction with an integer. The seventh-grade students show a higher mastery of this type of task compared to the fifth-grade students, but it is still not a high mastery for all classes. 7.C3 shows the highest mastery of T_{15}, about multiplication of a fraction with an integer, while only half of the students in 7.C1, 7.C2 and 7.C4 show a mastery of T_{15}. There is no pattern of what type of mistakes the students have made in items belonging to T_{15}. The students who got 0 points for the items belonging to T_{15}, have in the vast majority not instances made the items (this applies to all four classes) and they thus skipped the item. This possibly illustrates that the students who have received 0 points in items for T_{15} may simply have a lack of knowledge about multiplication of a fraction with an integer, since the students have skipped these items rather than answering the item incorrectly.

A few students answered I_{46} and I_{50} as follows:

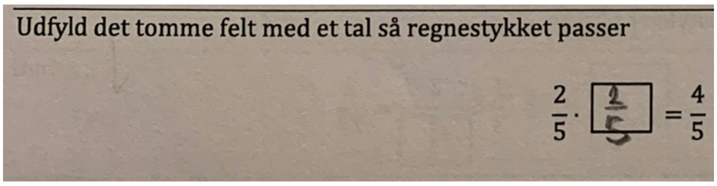


Figure 48: Student answer (7.C4, I_{46}, S_{16})

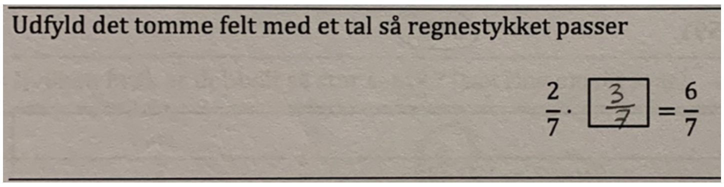


Figure 49: Student answer (7.C3, I_{50}, S_{11})

These students approach items related to multiplication of a fraction with an integer in a wrong way. In both cases it is the same type of technical misconception which is prominent. From these student answers it is possible to see that the students use their knowledge about techniques related to addition of fractions with like denominators to items related to multiplication of fractions with an integer. If these two items were about addition of fractions with like denominators, the students' solution would be correct, but the mistake is that the students multiply two fractions as it were addition of fractions with like denominator. From these answers, it can thus be observed that the students have a technique that is remembered and used incorrectly.

As mentioned in section 8.2.1.2, this may indicate that the teaching for these seventh-grade students has been very oriented and focused on practicing techniques without giving the students a theoretical understanding, resulting in that the students mastering a lot of techniques but use them incorrectly. For instance, if the students had a feeling and understanding about that by multiplying two fractions, the result will be smaller, they would not perform the mistake in Figure 48 or Figure 49.

The results in Table 5 for the types of task related to division of an integer with and fraction (T_{16}) and division of a fraction with an integer (T_{17}) show that the students have a lack of knowledge related to these types of task. The type of task T_{16} were tested in I_{33} and T_{17} were tested in I_{49}. Table 5 shows that only about 14% students in 7.C1 and 7.C2 probably master division of an integer with a fraction, while the other two classes have no mastery of this type of task. By a thorough review of all the answers from the students, it is observed that out of 78 students, only 6 students give a correct answer to I_{33} while 44 students do not answer the item at all. This possibly illustrates that over half of the seventh-grade students do not master division of an integer with a fraction.

In addition, 22 out of 78 seventh-grade students have answered as follows:

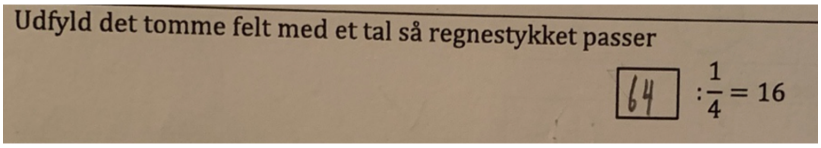


Figure 50: Student answer (7.C3, I_{33}, S_{10})

According to the praxeological reference model for Danish seventh-grade students (Table 2), T_{16} is a type of task the students should master, but it is only 6 out of 78 students which show a mastery of these. The remaining students show either a lack of knowledge about T_{16} (applies for 44 out of 78 students) or a misconception of I_{33} (applies for 22 out of 78 students) since the students consider $\frac{1}{4}$ as 4 and then consider 64 as the answer as $64: 4 = 16$.

A higher mastery of division of a fraction with an integer (T_{17}) exists compared with T_{16}, but this mastery is still very low among the students. Out of 78 students, 21 students have provided a correct answer to I_{49} while 47 students have not answered the item. Again, as mentioned earlier, the lack of answers from so many students could indicate a lack of knowledge about this item instead of lack of time, since I_{49} is not one of the last items.

8.2.1.4 Magnitude of fractions

The last interesting observation of the results in Table 5 about fractions is types of task related to assessing the magnitude of the fractions

At T_{20}, T_{21} and T_{22} a high mastery of all these is observed, but the results for I_{62} show half as much mastery. This result is quite conspicuous and could be because I_{62} is formulated more complicated than for instance I_9. To give the teachers a concrete explanation of the test results and what misconceptions and lack of knowledge that has occurred about I_{62}, I_9 and I_{62} will be presented and the difference between these two items will possibly illustrate the big difference between the results.

7th gr.,I_9:

Which number is the largest? (circle your answer)

$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{7}$

7th gr.,I_{62}:

Which fraction is twice as large as $\frac{1}{4}$? (circle your answer)

$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{2}$

These items are both about examination of which unit fractions is largest, but for I_{62} the situation becomes more complicated. In I_{62} the student should assess which fraction that is twice as large as $\frac{1}{4}$ and many students give the answer 2/8, which illustrates that they double the denominator and the numerator separately rather than considering the fraction as a number.

Therefore, the results shows that it is possibly easier to solve items related to the magnitudes of the fractions when the fractions are given explicit as I_9 while more complicated items as I_{62} can provide some mistakes among the students.

On the other hand, the teachers will be informed about the large difference in the results for I_9 and I_{62} and furthermore make it clear for the teachers that this difference may be due to students' lack of knowledge about a fraction as a number rather than considering a fraction as two separate parts.

8.2.2 Seventh-grade students' mastery of first-degree equations

From Table 5 it is observed that there is a big variation between the results of the classes for each item related to solving a first-degree equation. In the following, it will be described which variations of the items that could be the causes of such different results. Furthermore, based on

these results, it will be explained whether the seventh-grade students, who performed the test, master the type of task related to solving a first-degree equation.
The big variation between the results for each item may be due to the fact that for T_{25} there are different items that are formulated in different ways, some more complicated than others.
Some of the classes show a greater mastery of T_{25} than others, but overall a low mastery of T_{25} occurs, except in I_{55}. I_{55} is also the only item where the unknown variable is not denoted by x but instead by a box. As in the case of fifth-grade students, the students also show greater mastery when the item is formulated as follows:
(7th gr.,I_{55})
Write a number in the empty box so the calculation is correct

$$56 - \boxed{} = 44$$

The high mastery of this item may either be due to the presence of a box like the unknown rather than a variable x, or it may be due to the item being an easier first-degree equation compared to the other items. Another central reason for the low mastery of items, where x is involved instead of, for instance, a box, could be that the variable x is new for the students and therefore they have not solved many items with letter calculation yet. This is surprising since both textbooks Gyldendal MULTI 7 and Alinea Matematrix 7, both contain chapters (Chapter 4 and Chapter 3, respectively), where letter calculation and equation solution are central.
Overall I_{69} is solved correctly by 42/78 of the students, i.e., a little over half. There should be a higher mastery of this item as it is less complicated than e.g., I_{57}. In I_{69} it is expected to add the right side and divide by 2 on both sides, while I_{57} requires more steps and greater mastery. From Table 5 we generally see a higher mastery of I_{69} than I_{57}, but this is still low.
For I_{69} the mastery is 59% in 7.C1, 62% in 7.C2, 53% in 7.C3 and 39% in 7.C4 This mastery decreases significant in I_{57}, where the mastery is 27% in 7.C1, 48% in 7.C2, 12% in 7.C3 and 22% in 7.C4. I_{57} is the following item:
Solve the equation

$$7x - 7 = 13 - 3x$$

$$\boxed{}$$

I_{69} examines whether seventh-grade students master a technique for solving a first-degree equation. Since there are two techniques for solving a first-degree equation (see τ_{28} and τ_{29} in Table 2), I_{79} has been developed to illustrate which technique the students mainly use to solve items on the form as in I_{69}. These items are:

(7^{th} gr.,I_{69})

Solve the equation

$$2x = 16 + 2$$

(7^{th} gr.,I_{79})

Write the solution in the box.

Explain in the box how you found the solution to the equation

$$3x + 2 = 8$$

In the answers from the students, it is possible to observe that both techniques (τ_{28} and τ_{29}) from Table 2 occur for solving a first-degree equation.

For instance:

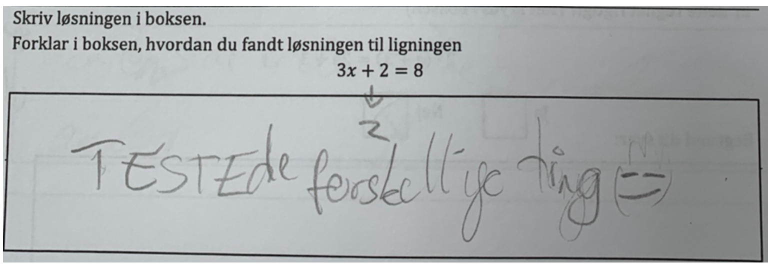


Figure 51: Student answer (7.C4, I_{79}, S_1)

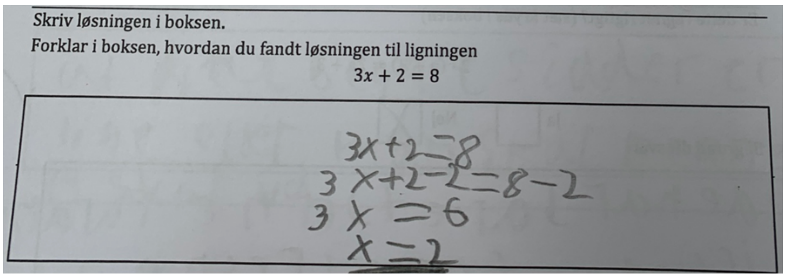


Figure 52: Student answer (7.C1, I_{79}, S_6)

While the first answer (Figure 51) from the student is: "I tested different things" thus solving by guessing and substitution, the second student (Figure 52) use opposite arithmetic operators to solve the first-degree equation. Although both techniques are available among Danish seventh-grade students, the most prominent technique is τ_{28} in Table 2 i.e., the students use opposite arithmetic operators to solve the first-degree equation.

Furthermore, I_{57} and I_{81} are the same items:

(7th gr.,I_{57}):

Solve the equation

$$7x - 7 = 13 - 3x$$

(7th gr.,I_{81})

What has gone wrong in this transformation?

$$7x - 7 = 13 - 3x$$
$$10x - 7 = 13$$
$$10x = 20$$
$$x = 10$$

but while I_{57} examines whether the students master a technique for solving a first-degree equation, I_{81} have a more technological approach, where it is possible to see whether the students master to explain the solution of a first-degree equation.

For two of the classes (7.C1 and 7.C3) it is the same number of students who solve I_{57} and I_{81}, while in the other two classes there are multiple who solve I_{57} rather than I_{81}. This may be because the students did not manage to complete the item due to time pressure, while another explanation may be that the students master a technique to solve I_{57} but have a lack of technological knowledge for I_{81}.

8.2.3 Reduction tasks

Finally, it is observed in the results in Table 5 that the seventh-grade students have a low mastery of reductions tasks. Specifically, it is observed that 7.C1 and 7.C2 show a higher mastery of reduction tasks than 7.C3 and 7.C4, but still very low mastery. 7.C1 and 7.C2 show a mastery of reduction tasks when the task is written on the form as in 7th gr., I_{27}:

Write the expression as short as possible:

$$8 - 2 \cdot 3 + 8\text{b} = \boxed{}$$

But this mastery only exists among 2/18 students in 7.C4 and 6/17 students in 7.C3.

In general, all teachers will be informed about that their students have a medium mastery when the reduction tasks are written as: "write the expression as short as possible", whereas the students show almost no mastery of items related to reduction when the task contains the notion "reduce" (e.g., I_{65} see Appendix 6). Again, as noted earlier, the items containing the word ‘reduce’ is placed as one of the last in the test, which could have an impact on the low result, as it can be expected that many did not manage to complete the item due to time pressure. But on the other hand, it is seen that items that come after I_{65} belonging to other types of task show a high mastery, which could indicate that it is not due to time pressure but due to lack of knowledge that there is a low mastery of reduction tasks. Overall, there is a variation of the results for the four classes, which the teacher will be informed about.

Overall, Table 5 shows that seventh-grade students have a varying mastery of the type of task related to solving a first-degree equation depending on how the item is formulated.

So, in conclusion, this section has presented the results from the final test for seventh-grade students in School B and has explained in what extent they master the praxeologies belonging to arithmetic and algebra (see Table 2). From Table 5 it is possible to observe that the students, who have performed the test, master almost all types of task to different extents, and it is possible to consider similarities and difference between the classes. From the explanation in the above section, it is possible to conclude that some students possess some misconceptions related to arithmetic and algebra while other students have a lack of knowledge in some parts of tasks related to arithmetic and algebra.

9 Methodology for RQ3

The following section will explain how the teachers of the different classes in the test can be usefully informed of the answers to RQ2, since the purpose of the test is to help teachers identify where there is potential for improvements in their classes.

The following sections present how feedback for the teachers is designed. The teachers for the individual classes will be informed about both positive and negative results of the diagnostic test. In the next sections, it is only the negative results which are presented, but in Appendix 7 and Appendix 8 (Feedback for the teachers) it is possible to see feedback of the positive results given to the teachers. So, feedback for the teachers will includes an overview that highlights what each class has a high and low mastery of compared to other classes, but in the following only some types of task with low mastery will be addressed. With this comparison of the classes, it is made clear to the teachers that the types of task his/her entire class does not master, are mastered by the other three classes, which is a confirmation of that the types of task are not superficial, but that it is actually types of task that must be mastered in this grade, which in turn are mastered in the other classes.

Furthermore, the feedback is based on the classes individually and their individual performance, which is why this feedback also includes a comment on the strengths and weaknesses of each class (even if they are common to all classes). Finally, in the feedback for the teachers, it will be made clear how many of the students in their class show almost no or no mastery of some types of task related to arithmetic and algebra, since it can offer the teachers an insight into the level for the whole class and not just some of the students.

The teachers will be informed about four to five types of task their own class should focus more on. Although the previous sections explain several types of task the fifth- and seventh-grade students have misconception about, the feedback for the teachers will only be based on four to five types of task, because it is more important to choose four to five types of task and elaborate on these, rather than choosing more where the feedback becomes more superficial.

In continuation of this, the class's misconceptions are made clear, and advice is given on what the teachers can work on to improve students' knowledge.

Feedback for the teachers, to a large extent, contains examples of items that the students have misconceptions about, which the teachers can take into consideration in their teaching.

Examples of misconceptions will be added to make the feedback clearer and more concrete, rather than informing the teachers of a lot of abstract types of task which will be difficult to understand. To ensure anonymity, student answers will not be added to the feedback if it is supposed that the teacher can decide, on the basis of this answer, which student the answer belongs to.

Furthermore, in some cases the teachers will be informed about where in the textbooks the different types of task are located, so that it becomes clearer for the teachers which types of task and in which extent, according to the textbooks, the students should master these different types of tasks.

The next two sections (section 10.1 and 10.2) contain the aspects the teachers will be informed about, but the concrete feedback for the teachers is available in Appendix 7 and Appendix 8.

10 Results for RQ3

The following section presents what the teachers were informed about, based on the results in section 8.1 and 8.2, respectively. General information were given to all teachers in fifth and seventh grade, respectively, and then, all teachers also receive feedback regarding the level of their class.

10.1 Information for teachers of fifth grade students

The following section will give an explanation of how feedback, and which feedback, is given to the teachers of fifth-grade students. The feedback is based on the following types of task: 1. Addition of fractions with like denominators (T_8), 2. Subtraction of fractions with like denominators (T_{10}), 3. Equivalence of fractions (T_{14}) and 4. The calculation of arithmetic expressions involving integers and two or more operations (T_1).

The teachers of all four classes were informed about the results belonging to the types of task such as addition and subtraction of fractions with like denominators. Although there exist other types of task related to fractions where misconceptions exist among the students, the types of task related to addition and subtraction of fractions with like denominators were chosen.

This is because it is the same misconception about fractions that exists among all classes, so if the misconception related to addition and subtraction of fractions with like denominators can be solved and addressed in all classes, this may have an impact on other types of task related to fractions. The most prominent technical misconception among all four classes related to addition and subtraction of fractions with like denominators, involves that the students, typically add and subtract the numerator and denominator separately.

In items related to the number line, it was furthermore possible to observe some lack of theoretical knowledge. The students, in some cases, consider the fraction as a whole number when placing $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ on a number line. They placed it as whole numbers on a number line by placing $\frac{1}{2}$ followed by $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$.

From Table 4 it is observed that there are half as many who master addition and subtraction of fractions with like denominators in 5.C2, compared to the remaining three classes. Although this difference exists, it is worth noticing that there are still not many who master this type of task in the other classes.

For instance, I_2, about addition of fractions with like denominators, are mastered by 40% of the students in 5.C1 and 5.C3, while this mastery is about 21% in 5.C 2 and 29% in 5.C4. Teachers for all classes were also informed about this difference and the low mastery.

In the same way, among the classes a low mastery of subtraction of fractions with like denominators exists. Although 5.C2 is again the class with the lowest mastery, all teachers were informed about the existing misconception in relation to these types of tasks.

The teachers were informed that the students consider a fraction as divided into two separate parts and use the operations addition and subtractions separately in the numerator and denominator, instead of considering the fraction as a number. Addressing this misconception is central to the learning of other types of task related to fractions.

The third type of task which teachers were informed about is T_{14} about equivalence of fractions. This type of task is very central in the textbooks, Gyldendal MULTI 5 and Alinea Matematrix 5, compared with, for instance, multiplication of a fraction with an integer.

For instance, on page 54 in Multi 5 and on page 68 in Alinea Matematrix 5, there is a thorough explanation of the equivalence of fractions, while multiplication of a fraction with an integer mostly occurs as items without thorough explanations.

The fact that equivalence of fraction has such a central role in the textbooks illustrates the importance of this type of task. Equivalence of fractions is not only important in items about 'expanding' and 'simplifying' fractions but is furthermore a central theoretical element in the learning and performance of addition and subtractions with fractions with different denominators. Because of this importance, the teachers will be informed about the type of task related to equivalence of fractions. Furthermore, it is possible, from Table 4, to observe that 5.C4 shows most mastery of equivalence of fractions compared with the other three classes, while 5.C1 and 5.C3 are the classes with lowest mastery (almost half as many as in 5.C4).

Finally, the teachers were informed about the results and misconception related to the calculation of arithmetic expressions involving integers and two or more operations.

This was chosen since research (mentioned on section 2.4.4) shows that a misconception or lack of knowledge about calculation of arithmetic expressions involving integers and two or more operations can be a barrier and hinder students in learning and doing algebra. Because of the importance of this type of task, the teachers were informed about the results and misconceptions belonging to this type of task. The teachers were further informed that the results for this type of task are very varying for all items (between 20-100% mastery), which may, as previously mentioned, have been due to how the arithmetic expressions are written. For instance,

I_1:

What is the solution to the calculation: $4 + 2 \cdot 5$ $\boxed{}$

I_9:

What is the solution to the calculation: $3 \cdot 2 + 5$ $\boxed{}$

This shows that depending on what the first operation is in the arithmetic expressions, the result for the item is varying. Based on the test answers from the students, the teachers were informed that there is an overall higher success rate among their students by solving I_9 rather than I_1, which must indicate that most of the students tend to calculate from left to right without taking the operations into consideration. The mastery of this type of task is significantly lower in 5.C2 compared to the other three classes which show a fairly similar mastery. Common to all four classes, as observed in Table 4, is that there is a higher correctness for I_9 (between 80-100%), while there exists almost half (or lower) as much correctness for I_1 and I_{28} (see Appendix 5).

The teacher for the individual classes will furthermore receive information about the number of students who show very little or no mastery. Based on the student answers, it has been observed that the following number of students have only solved (and shown mastery of) approximately between 10-15 items (which is the same as approximately 12-18% of the whole test). It is 3 students in 5.C1, 2 students in 5.C2, 5-6 students in 5.C3 and 1 student i 5.C4.

In addition, the results show that the performance of 5.C1 and 5.C4 classes in percentage is higher than the classes 5.C2 and 5.C3. This performance is assessed by looking at the number of points the whole class can get in total and how many they have received. There is a total of 85 items in the test for fifth-grade students and since 23 students from 5.C1 have performed the test, the class can get a total of $85 \cdot 23 = 1955$ point.

The results shows that the total performance of the class is 842,25 point, which gives that the total performance for 5.C1 is $\frac{842{,}25}{1955} \cdot 100 = 43{,}1\%$.

The teachers were informed about the performance of the whole class compared to other classes and most importantly that each class has some students who have a very low overall performance in arithmetic and algebra. This is especially prominent in 5.C3.

10.2 Information for teachers of seventh-grade students

The following section will explain how feedback, and which type of feedback were given to the teachers of seventh-grade students. Notice that the section presents the feedback about the negative results, but the teachers will also receive feedback about the positive results (see Appendix 7).

The feedback is based on the following types of tasks:

1. Addition of fractions with different denominators (T_9), 2. Subtraction of fractions with different denominators (T_{11}), 3. Multiplication of two fractions (T_{12}), 4. Division of two fractions (T_{13}) and 5. Solving a first-degree equation (T_{25}).

Although there are many types of task that are worth addressing and informing the teachers about, it was decided that types of task related to fractions and first-degree equation receive most attention. This is because fractions and first-degree equations are the two types of task related to arithmetic and algebra, that are most dominant in the textbooks, treated by entire chapters, compared with for instance, multiplication of positive and negative integers, which is only visible on few pages.

Not all types of task related to fractions will be addressed in the following. Although division of an integer with a fraction (T_{16}) and vice versa (T_{17}) show a low mastery among all classes, this is not focused on, because compared with the other types of task related to fractions, T_{16} and T_{17} is contained in a lesser degree in both textbooks. While the other types of task related to fractions contain both examples and explanations and a lot of tasks and items, T_{16} and T_{17} occur in a much lesser extent in the textbooks with very few examples. And since these types of task are less prominent than the above-mentioned types of task related to fractions (T_9, T_{11}, T_{12} and T_{13}), the low results for T_{16} and T_{17} may be due to the teachers' lesser focus on these in their teaching.

Types of task related to addition and subtraction of fractions with different denominators have a low mastery in all classes, but these types of task are mastered mostly by 7.C2, while 7.C1 and

7.C4 have a roughly equal low mastery. The most conspicuous is 7.C3. While the remaining three classes show either lack of knowledge or a misconception related to addition and subtraction of fractions with different denominators, 7.C3 show no knowledge and mastery of this. While there are 7,13, and 6 students respectively in 7.C1, 7.C2 and 7.C4 who show a mastery of for instance I_{17} about addition of fractions with different denominators, there are no students who master this in 7.C4. In the same way, while there are 8, 12, 8 students respectively in 7.C1, 7.C2 and 7.C4 which show a mastery of for instance I_{28} about subtraction of fractions with different denominators, this item is only mastered by 3 students in 7.C3.

The teachers for all classes will be informed about this low mastery, since the mastery is low in all classes, but especially the teacher of 7.C3 will be informed that there is no mastery in that class in contrast to the other classes, which should be worked on, since T_9 (about addition of fractions with different denominators) and T_{11} (about subtraction of fractions with different denominator) have been part of the curriculum since fifth grade.

This lack of knowledge in 7.C3 may possibly be due to the teacher's focus on other types of task, such as multiplication of two fractions (T_{12}), where 7.C3 shows the highest mastery.

For this type of task, 7.C3 is the only class with a high mastery, while the remaining three classes have a varying mastery depending on items. While there are twice as many in 7.C3 who can solve I_{14}, there are half as many in the other classes.

I_{14}: Calculate

$$\frac{1}{3} \cdot \frac{2}{7} = \frac{\boxed{}}{\boxed{}}$$

Division of two fractions shows surprising results. On page 146 in the textbook Multi 7, it is possible to find a review of all the techniques belonging to calculation with fractions, where division of fractions is also included. Taking into consideration that arithmetic with fractions has such a crucial place in the textbooks for seventh grade, it is surprising that so few master this. Out of 78 students who have performed the test, 5 students have solved I_{32} correctly, where 2 students are from 7.C4 and 3 from 7.C2.

I_{32}:

Write a number in the empty boxes so the calculation is correct

$$\frac{2}{5} : \frac{\boxed{}}{\boxed{}} = \frac{6}{10}$$

As mentioned in section 8.2.1.1 and 8.2.1.2, misconceptions related to addition, subtraction, multiplication, and division with two fractions, occur among the students in seventh grade. The teachers were informed about the most dominant technical misconceptions that exist within fractions, as these can provide an overview of what the teachers could change their focus to in their teaching. Within types of task related to fractions, it is seen that the students have some techniques, but use these incorrectly (see section 8.2.1.2 for further explanation), which may be due to the great attention to techniques in their teaching, which is why the teacher will be recommended to have a greater focus on the theoretical aspects and knowledge.

Finally, the teachers were informed about the results and misconception related to solving first-degree equations (T_{25}). As mentioned in section 8.1.2, some varying results occur for items related to T_{25}, so the mastery of T_{25} in all classes is dependent on the items.
Based on the test answers from the students, the teachers were also informed about that there is a higher success rate among their students in solving for instance I_{55} rather than I_{69}.
I_{55}:
Write a number in the empty box so the calculation is correct

$$56 - \boxed{} = 44$$

I_{69}:
Solve the equation

$$2x = 16 + 2$$

$$\boxed{}$$

Although there are varying results for the classes, 7.C4 is the class with the lowest mastery of solving a first-degree equation, while 7.C1, 7.C2 and 7.C3 have roughly the same mastery. The classes will be informed about this.
From these results, it is decided that the teachers for the four classes were informed about this low mastery so their teaching can focus more on solving a first-degree equation. This type of task becomes especially important for students in upper secondary school.

In addition to the teachers are informed about their students' mastery on carefully selected types of task, the number of students who show very little or no mastery will also be addressed in the feedback to the teachers.
Based on the student answers, it can be observed how many students have only solved (and shown mastery) approximately between 11-17 items out of 93, which roughly corresponds to the students having solved approximately 12-18% of the test.
It is 2 students i 7.C1, 3 students in 7.C2, 2 students in 7.C3 and 2 students in 7.C4, and further one student at the limit in 7.C1 and 7.C3. Based on the results, it can further be observed that the overall performance (in percentage) for 7.C1, 7.C2 and 7.C3 is higher than 7.C4.
While the first three classes have an overall performance of 49%, 54% and 45% respectively, this overall performance is 39% in 7.C4.
So, the teachers for the individual classes could be informed about their class's overall performance, but more importantly, the teachers were told that there are actually 2-3 students in their class who have a very low mastery (almost none) within types of task related to arithmetic and algebra.

Misconceptions observed in the classes give the impression that the focus has been very much on techniques. The students have developed a lot of technical misconceptions, in almost all types of tasks, so the students' low mastery is not necessarily due to lack of technique or lack of training of technique in the class, but perhaps lack of theoretical understanding.
The teachers were not told to practice the techniques more because if the students master a lot of incorrect techniques, it may be because the way these techniques are practiced has not focused on understanding based on clear technology and reasoning. Therefore, teachers were advised to focus on understanding and meaning of the used techniques, rather than mere skill-drill.

11 Discussion

In the following section, there will be a discussion of the presented and analyzed results from sections 8.1 and 8.2. This discussion aims to address the tendencies that occur in the results of the diagnostic test and further in what extent these results can be representative of the reality. In continuation of this, the challenges and uncertainties that may occur in the analysis of the results will also be addressed.

Furthermore, the identified misconceptions among fifth- and seventh-grade students will be compared with the misconceptions identified in previous research (section 2.4). This will confirm that the misconceptions identified in this project are also prevalent misconceptions in other countries.

The methods used for the development of the praxeological models and both tests will be discussed, including both weaknesses and strengths.

11.1 A discussion of the results

11.1.1 What do the results tell us?

The obtained results from the diagnostic tests performed by some Danish fifth- and seventh-grade students in Nkøbing municipality showed several misconceptions in types of task related to arithmetic and algebra. Some of these misconceptions were related to calculation of fractions, calculation of arithmetic expressions involving integers and two or more operations and solving a first-degree equation. In the section about the theoretical framework, some typical misconceptions related to types of task about arithmetic and algebra among students in primary and secondary school were identified and presented. In the following the findings of the analysis of the diagnostic tests will be discussed, among other things, in relation to the presented misconceptions in the theoretical framework (section 2.4).

The results of the diagnostic tests performed by Danish 86 fifth- and 78 seventh-grade students, show that the students master types of task related to arithmetic and algebra to different extent. While some students master almost all techniques, it is also possible to find some students who master almost nothing. The reason for the low mastery could be that the students just struggle with arithmetic and algebra in general but based on test answers from a total of 86 fifth-grade students and 78 seventh-grade students, there is no doubt that some crucial misconceptions related to arithmetic and algebra among the students exist.

This diagnostic test can, among other things, identify how many and also who, if the test was not anonymous, need more attention.

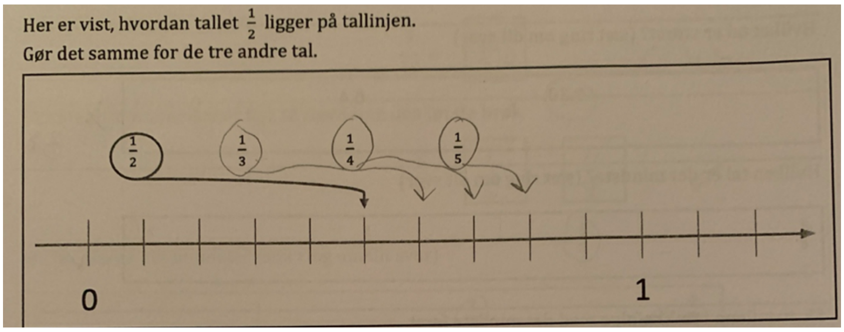
It is worth to consider the students who have the lowest results, as this is not a single student, but a total of 21 students (a total of fifth- and seventh-grade students), which is worrying. If the teaching continues as usual (before the test is performed), it will end up that these 12/86 fifth- and 9/78 seventh-grade students do not learn anything and will be behind.

As mentioned in section 1 the introduction, Siegler et al. (2012) found that 10-year-olds' students' knowledge of fractions would predict their algebra knowledge and mathematical achievement at age 16. Based on this, the low or almost no mastery of types of task related to arithmetic and algebra among these 12/86 fifth- and 9/78 seventh-grade students could be a predictor for their success in the educational system.

These students, if everything continues as usual, will also be the students who not only do poorly in high school, but also other places where mathematics is required. This test is not about identifying students who need special and extra individual teaching. Instead, it is about identifying students who have not gotten much out of the teaching in the different grade levels as they should (like their classmates) and make it clear that if the teaching and the class continues as they usually do, this will affect the students with the lowest mastery, and it will probably not be possible for them catch up on the missing.

These mistakes made by the students in the tests are not just isolated calculations mistake but are related to more serious issues. A very prominent technical misconception among the fifth- and seventh-grade students is related to addition and subtraction of fractions, where the students use a wrong technique by add or subtract the numerator and denominator of the fractions separately. This identified misconception among the students, is exactly a misconception similarly with what Ashlock (2006) in Bush & Karp (2013) have found. Ashlock (2006) in Bush & Karp (2013) posits that this misconception could be due to overgeneralization of whole numbers to fractions, which could be the case with these students too.

While some students use wrong techniques in their calculation of fractions, for instance addition and subtraction, other theoretical lack of knowledge were found among the students too. Items related to the number line makes it possible to observe whether the students have a sense of the order of magnitude of fractions which they should master in connection with unit fractions. These items can illustrate whether the students have an understanding of the concept of fractions in relation to the numbers the students are already familiar with, i.e., whole numbers. For instance, when the students order the unit fractions as the whole numbers, this could illustrate that they do not have the feeling and understanding of the order and magnitude about unit fractions. In an item where the students must place some unit fractions on a number line, this lack of theoretical knowledge where observed. For instance, in the following student answer:



This answer shows that the students do not have a sense of the order of the unit fractions, but instead order these fractions as whole numbers.

Based on these observations, it is worth to notice that presentations of fractions in middle school probably should not directly involve a lot of calculation with fractions. Instead, the focus should be on giving students an understanding of fractions as numbers and magnitudes before calculation of these. The fact that the students master this understanding will also give them a better starting point when calculating fractions.

When students master fractions as numbers and magnitudes and do not use their knowledge of whole numbers directly, this may also have a positive effect on, for example, adding fractions. For instance, for the item $\frac{1}{2}+\frac{1}{2}$ if the students consider these fractions as numbers and magnitudes, they will probably be able to see that $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$ does not make any sense, since they end the calculation with a result smaller than 1, but without this knowledge, the result $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$ will probably not bother them. So, the fact that the students answer $\frac{2}{4}$ for the calculation $\frac{1}{2}+\frac{1}{2}$ could illustrate that they do not master a sense of the magnitude of a fraction, as they should in fifth- and seventh grade.

Among the seventh-grade students, misconceptions related to decimals occur. It was found (see section 8.1.4) that some students consider the decimal 0.362 as larger than 0.37, and the argument from the student is: "There are more numbers on, so I think it is bigger than the other.

Like 100 is bigger than 10" (Figure 40). This misconception is consistent with the whole number misconception presented by Resnick et al. (1989) and Irwin (2001) in Durkin & Rittle-Johnson (2014) (see section 2.4.2). As presented in section 2.4.2, students use their knowledge about the natural numbers, which becomes a barrier in the learning of decimal numbers, since the students use the properties of natural numbers on decimal numbers.

That the students consider 0.362 as larger than 0.37, can thus indicate a lack of understanding of decimals as being a part of a whole and that the students use their whole number knowledge when examining the magnitude of a decimal. As with the case of fractions, it is central that the presentation of decimal numbers in middle school focuses on the magnitudes of decimal numbers compared to the students' existing knowledge of whole numbers. In order to strengthen the students' knowledge of decimals and fractions, it is central to focus on the relationship between these, as they both have in common that they both represent the relationship of part by whole.

This confirm that the existing misconceptions in previous research and literature, also exist among some Danish students. Resnick et al. (1989) and Irwin (2001) in Durkin & Rittle-Johnson (2014) found other misconceptions related to decimals, such as: 2) The role of zero misconception, and 3) The fraction misconception, which was not identified among Danish fifth- and seventh-grade students, which could be due to the size of the sample of the students who performed the diagnostic test.

It is also possible to observe that within the same grade levels, differences may exist among the classes. While some students master one type of task, other students master another. This does not necessarily mean that students have misconceptions or lack of knowledge, but that they simply have not been taught about it. This could be an argument for the varying results across the classes, but the most solid argument for the variation of the students' mastery could either be that the techniques have been incorrectly handled or not mastered at all.

Based on the results of the diagnostic test, a new question arises: why do these misconceptions actually exist?

There could be many reasons for this, but a very essential reason could be a lesser focus on technological and theoretical discourse in the teaching. Based on the results, it is possible to observe a greater focus on techniques rather than theories among the students. This is observed among the students when they mix different techniques. For instance, with addition of two fractions, it is observed that the students add the denominator and numerator separately. This misconception could originate in students' technique of multiplying fractions. In multiplication of two fraction, the students multiply the numerator and the denominator separately and by using this technique in addition of fractions, the students thus show a wrong use of technique and thus a misconception.

This could indicate that there is no lack of techniques among the students, but the great focus on techniques without connecting it to a theoretical starting point, is what limits the students, which could give arise to misconception.

As presented in section 2.4.1 in the theoretical framework, students generally have an operational understanding of the equal sign, which is also found among the Danish fifth- and seventh-grade students who performed the diagnostic test. As mentioned in section 8.1.3, the most common explanation from the students to the question about what the equal sign is, was for instance "When you see a = it means that the next number is the answer" (5th gr.,I_{84b}, S_6) in Figure 37 and "What something gives" (5th gr.,I_{84b}, S_{10}) in Figure 36. These answers are consistent with the findings in the theoretical framework, where in previous research for instance in Bush & Karp (2013) a student gives the following explanation about the equal sign: "the answer is" (p. 620). This well-known approach to the equal sign is thus also existing among some Danish fifth- and seventh-grade students. This shows that the Danish fifth- and seventh-grade students in the diagnostic tests mainly master an operational approach of the equal sign. This limited and wrong approach to the equal sign will according to Carpenter, Franke & Levi (2003) in Welder (2012) impact the students learning of algebra, which could be one of the reasons for the fifth- and seventh-grade students limited mastery of solving tasks related to first degree equation.

It is observed in the results that I_{69} (Appendix 6) is solved by half of the seventh-grade students who have performed the diagnostic test. It cannot be concluded with certainty that the low mastery in solving a first-degree equation among the students is due to an operational understanding of the equal sign, but the students' approach to the equal sign could have an impact in solving equations as I_{69} (Appendix 6). An operational approach to the equal sign will according to Alibali et al., (2007) (see section 2.4.1) be sufficient in solving items such that $3 + 8 = \_\_\_\_$. In more complex equations, as in the case in the diagnostic test for seventh grade for instance I_{57} : Solve the equation

$$7x - 7 = 13 - 3x$$

$$\boxed{}$$

This approach will provide issues.
The findings from the diagnostic test in seventh grade show in total that 21/78 (27% of the seventh-grade students) students have a relational approach to the equal sign. This low mastery of a relational understanding of the equal sign could be one reason for the low mastery in solving first degree equations. The low mastery can of course also be due to the low mastery of the techniques needed to solve a first-degree equation.
On the other hand, the results show that among all fifth-grade students there are 8/86 (9.3% of the fifth-grade students) students with a relational approach to the equal sign. This could indicate that although the mastery of the equal sign as relational is low in seventh grade, it is higher than the mastery the fifth-grade students, why it is appropriate to expect that the mastery of the equal sign as relational is something students develops through the years. Therefore, it can be said that fifth-grade students develop a relational approach later, to some extent at least, but the big question is: does it develop fast enough? There is, of course, an improvement from fifth to seventh grade, but the big question is whether it is going fast enough too and whether there is a sufficient development. There are still seventh-grade students who have a low mastery in many types of task so the big question is whether the development is fast enough and whether these students will be prepared for high school with this development.
The fact that students gain a higher level of mastery of types of task related to arithmetic and algebra over the years is also observed in other types of task. While there is a low mastery of addition and subtraction of fractions with like denominators among fifth-grade students, this is mastered by far more in seventh grade. The same applies, for instance, to multiplication of a decimal with an integer. These differences must indicate that the students become better and master types of task related to arithmetic and algebra to a greater extent over the years. So, it can be expected that types of task where seventh grade students show a low mastery about for instance addition and subtraction of fractions with different denominators, will change over the years and students will probably show a higher mastery of these later in lower secondary school.
In the examination of what fifth- and seventh-grade students master in types of task related to arithmetic and algebra; it is interesting to observe the smaller number of students who master almost nothing. There are few items with a mastery of 100% among some seventh-grade classes, but otherwise there are items where 80% of the class answers correctly.

These items are considered as high level of mastery in the class, but this can be problematic. That 80% respond correctly to an item means that 20% do not show a mastery. Considering 80% to be a high level of mastery may emphasize insufficient minimal goals. For instance, for calculation of arithmetic expression involving integers and two or more operations, the class with the highest mastery has a mastery of 86.5% for I_1.

I_1:

What is the solution to the calculation:

$$32 + 3 \cdot (7 - 5) = \boxed{}$$

In seventh grade, this should be an item which almost 100% of the class should master, but with a mastery of 86.5%, we still consider it as high mastery. Although there are many students at a middle level, there is a large group of students who master almost nothing. In their test answers, it has been observed that they have answered almost nothing. This can of course be laziness, but these empty answers must indicate that it is probably not misconception that is dominant among the students who show a very low level of mastery, but instead lack of knowledge.

11.1.2 Representativity of the results

At School B in Nkøbing, it has been found that 14% of all fifth-grade students who performed the test and 11.5% of the seventh-grade students show almost no mastery of types of task related to arithmetic and algebra. The schools in Nkøbing municipality are some of the highest performing schools in the final examination for ninth grade (Beck, 2019) and six schools from Nkøbing municipality is among the top 100 schools in Denmark (Sjöberg, 2019). It is therefore conceivable that among fifth- and seventh-grade students in more challenging schools in other municipalities in Denmark there will be a higher proportion of students with low or no mastery of types of task related to arithmetic and algebra.

The identified misconceptions among fifth and seventh grade students in Nkøbing municipality, will possibly also be found at other schools, and it will even be possible to find more misconceptions in other schools in Denmark. As mentioned earlier, not all of the aforementioned misconceptions presented in the theoretical framework occur among the students from Nkøbing municipality, but it is conceivable that these misconceptions presented in the theoretical framework will show up when testing other students at other school.

Therefore, if the diagnostic test were performed by fifth- and seventh-grade students in schools, not placed in Nkøbing municipality, other results are expected. As the sample only consists of students from Nkøbing municipality, the results found among the students in Nkøbing municipality will not be representative of all students in Denmark. It is therefore not possible to conclude that the found misconceptions related to arithmetic and algebra among fifth- and seventh-grade students from Nkøbing municipality applies for all fifth- and seventh-grade students in Denmark. Instead, the results can provide an insight into the fact that the misconceptions that exist in other research that has been carried out in other countries are also found among some Danish fifth- and seventh-grade students. The results found will therefore not be able to answer RQ2 completely, as the sample does not consist of all Danish fifth- and seventh-grade students, but just a few students.

11.1.3 Challenges in analysis of the results

In the analysis of the results of the diagnostic tests performed by fifth- and seventh-grade students, some uncertainties occur. To distinguish between lack of time and lack of knowledge has been one of the biggest challenges in analyzing the results.

If the last items in the test were not answered by the student, it was considered as lack of time, but if an item that occurs early in the test was not answered, it was considered as lack of knowledge. Although this decision was made at the beginning of the analysis of the results, it is not possible to say with certainty that it is correct.

To address this uncertainty, one option could be to reduce the number of items in the test or to offer more time for solving the items. The uncertainty about lack of time can be avoided if the students were given infinite time, but this will practically not be possible, and the students' concentration will probably not allow this either.

Furthermore, in the analysis of the results, an item has been awarded 0 point for both incorrectly answered and unanswered items, which can affect the overall picture of the individual student or class. However, by assigning 0 point to these two types of answers, it was possible to observe the sum of each student or class.

In the analysis of the results, it has sometimes been difficult to assess which technique the student has used in her/his solution of the item, if the item could be solved in several ways or if the student has solved the task correctly, but with a wrong technique. This is not possible to observe in items where only the students' answer is available without explanations.

In the review of the results, an interesting approach would be to focus more on the correlation of individual items. For instance, it was observed in the diagnostic test for seventh grade, that students give the answer $\frac{3}{10}$ for item 14:

I_{14}:

Calculate:

$$\frac{1}{3} \cdot \frac{2}{7} = \frac{\square}{\square}$$

Based on this answer, it was concluded that the students have a technical misconception, but to be able to conclude this, one has to investigate whether the students use this technical misconception consistently in items related to the same type of task.

In Deringöl (2019) it is identified that for assessing whether a wrong answer to an item is a misconception or a mistake, it is important to consider the frequency and consistency of that mistake (p. 29). This could be done in the analysis of the results for each grade by considering whether a student makes a type of mistake frequently and consistent in items related to the same type of task or whether they only make the mistake in for instance one item related to the same type of task.

Although individual items also provide a picture of the class' mastery of types of task related to arithmetic and algebra, another methodological change, which could offer another view, is to involve more statistical analyses. By doing this, it could be possible to assess different correlations between the items or it could be possible to say more about the individual student or class. So, a more thorough investigation of the correlations between items related to the same types of task could conceivably provide a more accurate picture of whether the student is making a single mistake in the items, or whether the student has a technical misconception. It could for instance be relevant to examine whether the students who answer $\frac{3}{10}$ in I_{14}, provide the answer $\frac{4}{20}$ for item 15:

I_{15}:

Write a number in the empty box so the calculation is correct

$$\frac{2}{5} \cdot \frac{\square}{\square} = \frac{6}{25}$$

11.2 Comments and changes of the methodology

Before the final test, a pilot test was performed by two fifth- and five seventh-grade students form School A. The students for the pilot test show a higher mastery of types of task related to arithmetic and algebra and got through all the items in the pilot test. Among other things based on this, the number of items was not changed. In the results of the final test, it was possible to observe many students who do not complete the entire test. It should also be noted that the pilot test was performed in other circumstances, which could lead to better results. For instance, the students were chosen by the school, which is why it is conceivable that those with the highest mastery are chosen. Furthermore, the fifth- and seventh-grade students performed the pilot test at home (due to Covid-19 and restrictions), which is why it is possible that the students used a calculator or received help at home, even though this was not allowed.

It thus seems clear that the pilot test had serious limitations.

Given the large difference between the schools in Nkøbing municipality and other schools that may have lower performing students, it is worth thinking about whether this diagnostic test can be used among other students at the same grade level.

Although the items in the diagnostic test are items fifth- and seventh-grade students should master according to the praxeological models for fifth and seventh grade (Table 1 and Table 2), one should, in the testing of other lower performing schools, be aware that the test should not be more difficult than it is and possibly consider focusing on a smaller number of types of task where the most important ones could be selected.

The number of items in the test may also have an importance on the performance of the test in other municipalities on lower performing students. If students in other municipalities are significantly weaker, the number of items needs to be considered, as many items in a test can result in that not all students get through all types of task. Thus, a real picture will not be obtained of what the students master and what misconceptions they have.

It will not be possible to assess whether items are unanswered due to lack of time or lack of knowledge, and this problem could be solved by a small number of items. On the other hand, one can of course argue that many items could be necessary to make sure that all students have items to solve in 60 minutes, but with a dominance of lower performing students, the most realistic will be a smaller number of items.

In the diagnostic tests, there are items which examine the students' theoretical knowledge, as it is a central part of the students' learning. If the focus was only on techniques, we would end up educating some students who know a lot of arithmetic techniques and have no knowledge about the mathematical field.

In the development of the diagnostic tests, it was decided to place items related to the theoretical aspects as the last items. This can be criticized in the sense that slow students do not get through these items at all because of lack of time and it would therefore not be possible to get an insight into their theoretical knowledge related to arithmetic and algebra.

But on the other hand, the placement of the theoretical items is a conscious choice, as the expectation is that if students cannot solve technical items related to arithmetic and algebra, it is also not certain that they will be able to perform the theoretical items.

12 Conclusion

The present thesis intended first to investigate what arithmetic and algebraic praxeologies Danish fifth- and seventh-grade students are supposed to learn. This examination was based on the Anthropological Theory of Didactics, concretely, a praxeological analysis mainly of two mathematics textbooks used by fifth- and seventh-grade students in the schools where the tests were trialed. In this praxeological analysis of the textbooks, 30 types of tasks related to arithmetic and algebra in seventh grade and 21 types of tasks related to arithmetic and algebra in fifth grade, were identified. Furthermore, the related techniques were presented too. Based on these types of task and techniques, diagnostic tests were developed for the purpose of examining whether fifth- and seventh-grade students master these types of task and related techniques. The results of the diagnostic tests, which were performed by 86 fifth-grade students and 78 seventh-grade students, provide insight into what some fifth- and seventh-grade students in Denmark master in terms of types of task related to arithmetic and algebra, and which misconception occurs most prominently among these students. The most prominent misconceptions related to arithmetic and algebra was related to types of task about fractions. A prominent misconception, which was consistent with what other research found, is that the students consider a fraction as a whole number, which was prominent in items about the magnitudes of fractions. Furthermore, technical misconceptions were identified too, for instance, in types of task related to addition, subtraction, and multiplication of fractions. In addition of fractions, both with like and different denominators, it was observed that the students add and subtract the numerators and denominators separately. Furthermore, in the type of task about the examination of the magnitude of a decimal, an identified prominent misconception is that fifth- and seventh-grade students confuse the ordering of decimals with whole numbers, which gives that the students sometimes consider the number with most decimals as the largest one.

Based on these, and more results, the teachers for the classes which perform the test, get a feedback about which types of task their students master and which types of task that require more attention and more focus in the teaching.

Although many misconceptions were identified, a further study would focus on the relation between items in the same type of task to examine whether the student has a misconception, which apply to frequent and consistent types of mistakes, or whether the student performs a mistake once.

Since the diagnostic tests are performed by students from a school in Nkøbing municipality, which is among the highest performing schools in the final examination for ninth grade (Beck, 2019), and among the top 100 schools in Denmark (Sjöberg, 2019), it is not possible to say something general from these results. These results are preliminary results. Therefore, the tests must be performed by students from other schools in other municipalities, to obtain a more representative insight into how Danish fifth and seventh grade students master types of task related to arithmetic and algebra. From the current obtained results, it is only possible to conclude how the students from Nkøbing municipality master types of task related to arithmetic and algebra. Since the results from Nkøbing municipality show a lack of theoretical knowledge and techniques related to arithmetic and algebra among fifth- and seventh-grade students, it is expected to identify more misconception among fifth- and seventh-grade students in other middle schools in other municipalities. Thus, by expanding the diagnostic test to other municipalities, it could be interesting to further research, whether the identified misconceptions also appear in other schools and whether other misconceptions related to arithmetic and algebra exist. An advantage of performing the diagnostic tests several times is that the diagnostic tests could improve in the process.

The development of these diagnostic tests for fifth-and seventh grade is central in the sense that on the basis of these tests it will be possible to identify misconceptions and lack of theoretical knowledge early so as to avoid the well-known transition problems between primary and lower secondary school and high school.

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Appendices

Appendix 1: Information for teachers and students

Information til lærerne og eleverne, der deltager i projektet.

Testen, der skal udføres i henholdsvis 5. og 7. klasse, har en varighed på 60 minutter. Lærerne udleverer et samlet opgavesæt til eleverne, som de skal løse individuelt. Efter 60 minutter afleverer eleverne opgavesættet tilbage med deres besvarelser. Der må ikke afleveres andet end opgavesættet (fx et kladdepapir med mellemregninger). Elever, der færdiggør testen, inden de 60 minutter er gået, afleverer opgavesættet til lærerne og afslutter testen. Elever, der ikke færdiggør testen, inden de 60 minutter er gået, afleverer dét de har nået indenfor de 60 minutter til lærerne og afslutter dermed testen.

Lærerne skal informere eleverne om følgende:

1. Alle svar i testen skal skrives i en boks. Mellemregninger fx på et kladdepapir skal ikke afleveres, når prøven er slut.
2. Testen udføres anonymt, hvorfor der ikke skal skrives navn på.
3. Testen har til formål at oplyse læreren om, hvad hele klassen kan finde ud af, og hvad klassen har udfordringer med.
4. Det er vigtigt, at eleverne gør sig umage i testen, så læreren ikke bruger tid i timen på ting, som klassen i forvejen kan finde ud af.
5. Læreren skal informere eleverne om, at det ikke er ham/hende, der retter testen, men at testen bliver rettet af den projektansvarlige. Den projektansvarlige vil dernæst fortælle læreren, hvilke udfordringer klassen har, og ikke enkelte elever.
6. Læreren skal gøre det tydeligt, at han/hun ikke er interesseret i at bedømme eleverne enkeltvis.
7. Hvis eleverne ikke kan svare eller er i tvivl, om de kan svare, må de springe opgaven over.
8. For løsning af testen må lommeregner og mobiltelefoner ikke benyttes.

Appendix 2: Information for the teachers

Diagnostisk værktøj for matematiklærere

Diagnostiske værktøjer er generelt en mangelvare for matematiklærere. Der findes tests, der hjælper lærerne med at registrere matematiske færdigheder, som givne elever har udfordringer med. Disse tests er dog ikke udformet til at hjælpe læreren med at finde ud af, hvad de pågældende udfordringer bunder i: Hvilke specifikke fejl eller mangler elevernes svar beror på.

Følgende projekt har til formål at udvikle et diagnostisk værktøj, der består af to dele: et opgavesæt til eleverne (en test) og en guide til lærerne, der indeholder information om hvad eleverne har lært, og hvad de ikke har tilstrækkelig viden om inden for aritmetik og algebra. Konkret vil testen og de dertilhørende resultater udstyre lærere med et værktøj, der kan benyttes til at forstå elevernes korrekte og ukorrekte tanker i besvarelsen af opgaverne. Dette værktøj vil eksplicit illustrere, hvad eleverne kan, og hvilke matematisk teoretiske udfordringer, eleverne har indenfor aritmetik og algebra.

Det er veldokumenteret, at udfordringer i aritmetik og algebra i folkeskolen har en afgørende betydning for elevernes generelle forståelse og mestring af matematik, hvilket bl.a. kommer til udtryk i gymnasiet. Derfor fokuserer dette diagnostiske værktøj i første omgang på at støtte lærerne i at kortlægge elevernes udfordringer på netop disse to emneområder.

Appendix 3: Relation between types of task and the items in the pilot test

Types of task	Item (I) fifth grade	Item (I) seventh grade
T_1: Order of operations (eller multiplication before addition)	$I_1, I_7, I_{10}, I_{26}, I_{27}, I_{43}$	$I_1,$
T_2: Multiplication of a decimal with an integer	$I_9, I_{14}, I_{25},$	I_2, I_{29}
T_3: Multiplication of decimals		I_{11}, I_{22}
T_4: Multiplication of negative numbers		$I_3,$
T_5: Multiplication of a negative and positive number		$I_4,$
T_6: Addition with negative numbers	I_{65}, I_{69}	I_{12}
T_7: Subtraction with negative numbers	I_{62}, I_{67}	I_{23}, I_{31}
T_8: Addition of fractions with like denominator	$I_2, I_{19}, I_{24}, I_{28}$	I_5, I_{98}
T_9: Addition of fractions with different denominator	I_{44}, I_{56}, I_{59}	I_{32}
T_{10}: Subtraction of fractions with like denominator	$I_8, I_{13}, I_{20}, I_{29}, I_{42}$	I_{13}
T_{11}: Subtraction of fractions with different denominator	I_{30}, I_{55}, I_{66}	I_{24}
T_{12}: Multiplication of two fractions		$I_{14}, I_{15}, I_{25}, I_{33}, I_{34}$
T_{13}: Division of fractions		$I_{26}, I_{35}, I_{41}, I_{65}, I_{66}$
T_{14}: Equivalence of fractions	$I_6, I_{11}, I_{12}, I_{21}, I_{22}, I_{31}, I_{32}, I_{41}$ $I_{72}, I_{75}, I_{76}, I_{77}, I_{79}, I_{81}, I_{82}, I_8$	$I_6, I_{16}, I_{28}, I_{37}, I_{38}, I_{44},$ $I_{45}, I_{53}, I_{67}, I_{68}, I_{74}, I_{75},$ $I_{76}, I_{77}, I_{78}, I_{80}, I_{81}, I_{84},$ $I_{85}, I_{86}, I_{87}, I_{91}, I_{96},$ I_{99}, I_{105}
T_{15}: Multiplication of a fraction with an integer	$I_3, I_{15}, I_{23},$	I_{42}, I_{47}
T_{16}: Division of an integer with a fraction		I_{27}
T_{17}: Division of a fraction with an integer	$I_{33},$	I_{46}
T_{18}: Convert a fraction to a decimal	$I_{16}, I_{34}, I_{40}, I_{45},$	$I_7, I_{17}, I_{19}, I_{39}$
T_{19}: Convert a decimal to a fraction	$I_{18}, I_{35}, I_{53}, I_{61}$	$I_8, I_{40}, I_{69}, I_{70}$
T_{20}: Examine which fraction with like denominators and different numerators is largest	I_{17}, I_{36}, I_{38}	I_{18}, I_{20}, I_{58}

T_{21}: Examine which fraction with like numerators and different denominators is largest	I_{36}, I_{47}, I_{73}	I_{20}, I_{59}, I_{89}
T_{22}: Examine which unit fraction $\frac{1}{a}$ and $\frac{1}{b}$ is largest	I_5, I_{36}, I_{78}	$I_9, I_{20}, I_{63}, I_{93}, I_{95}$
T_{23}: Examine which decimal is largest	$I_{37}, I_{39}, I_{48}, I_{49}, I_{50}, I_{52}, I_{57}, I_{7}$	$I_{48}, I_{49}, I_{50}, I_{51}, I_{52}, I_{61}, I$
T_{24}: Solve a first-degree equation	$I_4, I_{51}, I_{58}, I_{63}, I_{68}, I_{70}$	$I_{36}, I_{54}, I_{55}, I_{56}, I_{57}, I_{64}, I$ I_{92}, I_{94}
T_{25}: Reduction tasks up to 2 variables	$I_{64}, I_{71}, I_{80}, I_{83}$	I_{10}, I_{21}, I_{83}
T_{26}: Reduction tasks up to 3 variables		I_{73}, I_{79}
T_{27}: Equality sign	I_{85}, I_{87}	$I_{100}, I_{101}, I_{102}, I_{103}, I_{104}$
T_{28}:Distributive law	I_{86}	I_{97}

Appendix 4a: Excel sheet fifth grade

5th grade	S1	S2	Sum
Item			
1	1	0	1
2	1	1	2
3	0	1	1
4	1	1	2
5	1	1	2
6	0	1	1
7	0	1	1
8	0	1	1
9	1	1	2
10	1	1	2
11	0	1	1
12	1	1	2
13	1	1	2
14	1	1	2
15	0	1	1
16	0	1	1
17	1	0	1
18	1	1	2
19	1	1	2
20	1	1	2
21	0	1	1
22	1	1	2
23	0	1	1
24	1	1	2
25	1	1	2
26	0	0	0
27	0	1	1
28	1	1	2
29	1	1	2
30	0	0	0
31	0	1	1
32	1	1	2
33	0	1	1
34	1	1	2
35	1	1	1
36	0	0	0
37	1	1	1
38	1	1	1
39	1	0	1
40	1	1	2
41	1	0	1
42	1	1	2
43	0	1	1
44	1	1	2
45	1	1	2
46	1	1	2
47	1	1	2
48	1	1	2
49	1	0	1
50	1	1	2
51	1	1	2
52	1	1	2
53	1	1	2
54	0	1	1
55	0	1	1
56	1	0	1
57	1	0	1
58	0	1	1
59	1	0	1
60	0	1	1
61	1	1	2
62	0	0	0
63	1	1	2
64	0	0	0
65	1	1	2
66	0	0	0
67	1	0	1
68	0	0	0
69	0	0	0
70	1	0	1
71	1	1	2
72	1	0	1
73	1	0	1
74	1	1	2
75	1	1	2
76	1	1	2
77	0	1	1
78	1	1	2
79	1	0	1
80	0	0	0
81	1	1	2
82	1	0	1
83	0	0	0
84	1	1	2
85	0	0	0
86	1	1	2
87	1	1	2
Sum	58	62	

Appendix 4b: Excel sheet seventh grade

7th grade	S1	S2	S3	S4	S5	Sum
Item						
1	1	1	1	0	0	3
2	1	1	1	1	0	4
3	1	1	1	1	1	5
4	1	1	1	1	1	5
5	1	1	1	1	1	5
6	1	1	1	1	1	5
7	1	1	1	1	1	5
8	1	1	1	1	1	5
9	1	1	1	1	1	5
10	0	1	1	1	1	4
11	0	1	0	1	1	3
12	1	1	1	1	1	5
13	0	1	1	1	1	4
14	1	1	0	1	1	4
15	1	1	0	1	1	4
16	1	1	1	1	1	5
17	1	1	1	1	1	5
18	1	1	0	1	1	4
19	1	1	1	1	1	5
20	1	1	1	1	1	5
21	0	1	0	1	1	3
22	0	1	1	0	1	3
23	1	1	0	0	0	2
24	1	1	1	0	0	3
25	1	1	0	1	1	4
26	1	1	1	0	1	4
27	1	1	0	0	0	2
28	1	1	1	1	1	5
29	1	1	1	1	1	5
30	1	0	0	1	1	3
31	1	1	0	1	1	4
32	0	1	0	0	0	1
33	1	1	0	1	0	3
34	1	1	1	1	1	5
35	1	1	1	0	1	4
36	1	1	1	1	1	5
37	1	1	1	1	1	5
38	1	1	1	1	1	5
39	1	1	0	1	1	4
40	1	1	1	1	1	5
41	0	1	1	0	1	3
42	1	1	1	0	1	4
43	1	1	1	0	1	4
44	1	1	1	1	1	5
45	1	1	1	1	1	5
46	1	1	0	1	0	3
47	1	1	0	0	1	3
48	1	1	1	1	1	5
49	0	1	1	1	1	4
50	1	1	1	1	1	5
51	1	1	1	1	1	5
52	1	1	1	1	1	5
53	1	1	1	1	1	5
54	1	1	1	1	1	5
55	1	1	1	1	1	5
56	1	1	0	1	1	4
57	1	1	0	1	1	4
58	1	1	1	1	1	5
59	1	1	1	1	1	5
60	1	1	0	0	0	2
61	1	1	1	1	1	5
62	1	1	1	1	1	5
63	1	0	1	1	0	3
64	1	1	0	0	1	3
65	1	1	1	0	1	4
66	0	0	0	0	0	0
67	1	1	0	1	1	4
68	1	1	1	1	1	5
69	1	1	1	0	1	4
70	1	1	1	0	1	4
71	1	1	1	1	1	5
72	1	1	1	1	1	5
73	0	1	1	1	1	4
74	1	1	1	1	1	5
75	1	1	1	1	1	5
76	1	1	1	1	1	5
77	1	1	1	1	1	5
78	1	1	1	1	1	5
79	0	1	0	0	0	1
80	0	1	0	1	1	3
81	0	1	1	0	1	3
82	0	1	1	1	1	4
83	0	1	1	1	1	4
84	0	1	1	1	1	4
85	0	1	1	1	1	4
86	0	1	1	1	1	4
87	0	1	1	1	1	4
88	0	1	1	1	1	4
89	0	1	1	1	1	4
90	0	1	1	1	1	4
91	0	1	0	1	0	2
92	0	1	0	1	1	3
93	0	1	1	1	1	4
94	0	1	0	0	1	2
95	0	1	0	1	1	3
96	0	1	0	1	1	3
97	0	1	0	1	0	2
98	0	0	1	1	1	3
99	0	1	0	0	1	2
100	0	0	0	1	0	1
101	0	1	1	1	1	4
102	0	1	0	0	1	2
103	0	1	0	0	1	2
104	0	1	1	0	1	3
105	0	1	0	0	1	2
Sum	68	100	71	79	90	

Appendix 5: Final test for fifth grade

Test 5th grade

All answers in this test must be written in a box.

Item 1

What is the solution to the calculation: $4 + 2 \cdot 5$ $\boxed{}$

Item 2

Write a number in the empty boxes so the calculation is correct

$$\frac{2}{9} + \frac{\boxed{}}{\boxed{}} = \frac{8}{9}$$

Item 3

Calculate:

$$2 \cdot \frac{2}{5} = \boxed{}$$

Item 4

Write a number in the empty box so the calculation is correct

$$36 - \boxed{} = 29$$

Item 5

Which number is the largest? (circle your answer)

$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{7}$

Item 6

Two of the fractions in the box are equal. (circle the fractions)

For example

$\frac{2}{5}$	$\frac{1}{6}$	$\frac{6}{15}$	$\frac{2}{6}$

Do the same for this box

$\frac{3}{5}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{6}{10}$

Item 7

Do the same for this box

$\frac{1}{4}$	$\frac{4}{14}$	$\frac{2}{6}$	$\frac{2}{7}$

Item 8
Do the same for this box

$\frac{3}{4}$	$\frac{1}{5}$	$\frac{3}{12}$	$\frac{9}{12}$

Item 9
What is the solution to the calculation: $3 \cdot 2 + 5$ $\square$

Item 10
Calculate:

$$\frac{3}{7} - \frac{2}{7} = \frac{\square}{\square}$$

Item 11
Calculate:

$$6 \cdot 1.5 = \square$$

Item 12
Is the calculation right or wrong? (tick the box)

$$(2 + 3) \cdot 5 - 1 = 24$$

Right $\square$ Wrong $\square$

Item 13
Write a number in the box so the equal sign is correct

$$\frac{3}{7} = \frac{9}{\square}$$

Item 14
Calculate:

$$\frac{3}{4} - \frac{1}{4} = \frac{\square}{\square}$$

Item 15
Calculate:

$$0.7 \cdot 3 = \square$$

Item 16

Calculate:

$$\frac{1}{4} \cdot 3 = \boxed{}$$

Item17

Convert the fraction to a decimal number

$$\frac{13}{100} = \boxed{}$$

Item 18

Which number is the largest? (circle your answer)

$\frac{3}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{6}$

Item 19

Convert the decimal number to a fraction

$$0.75 = \boxed{}$$

Item 20

Calculate:

$$\frac{1}{5} + \frac{3}{5} = \frac{\boxed{}}{\boxed{}}$$

Item 21

Write a number in the empty boxes so the calculation is correct

$$\frac{4}{5} - \frac{\boxed{}}{\boxed{}} = \frac{1}{5}$$

Item 22

Write a number in the box so the equal sign is correct

$$\frac{2}{5} = \frac{\boxed{}}{15}$$

Item 23

Write two new fractions that are of the same value as the first fraction

$$\frac{1}{5} \quad \boxed{} \quad \boxed{}$$

Item 24
Calculate:

$$2 \cdot \frac{1}{4} = \square$$

Item 25
Calculate:

$$\frac{2}{7} + \frac{4}{7} = \frac{\square}{\square}$$

Item 26
Calculate:

$$2 \cdot 2.7 = \square$$

Item 27
In some of the calculations below, a parenthesis is missing.
Write the parentheses where they are missing.

$$2 \cdot 4 + 3 = 14$$

$$2 \cdot 4 + 3 = 11$$

$$20 - 2 \cdot 5 + 3 = 4$$

Item 28
What is the solution to the calculation: $3 - 2 + 5 \cdot 2$ $\square$

Item 29
Calculate

$$\frac{8}{10} + \frac{1}{10} = \frac{\square}{\square}$$

Item 30
Write a number in the empty boxes so the calculation is correct

$$\frac{5}{6} - \frac{\square}{\square} = \frac{2}{6}$$

Item 31
Calculate:

$$\frac{5}{8}-\frac{1}{4}=\frac{\square}{\square}$$

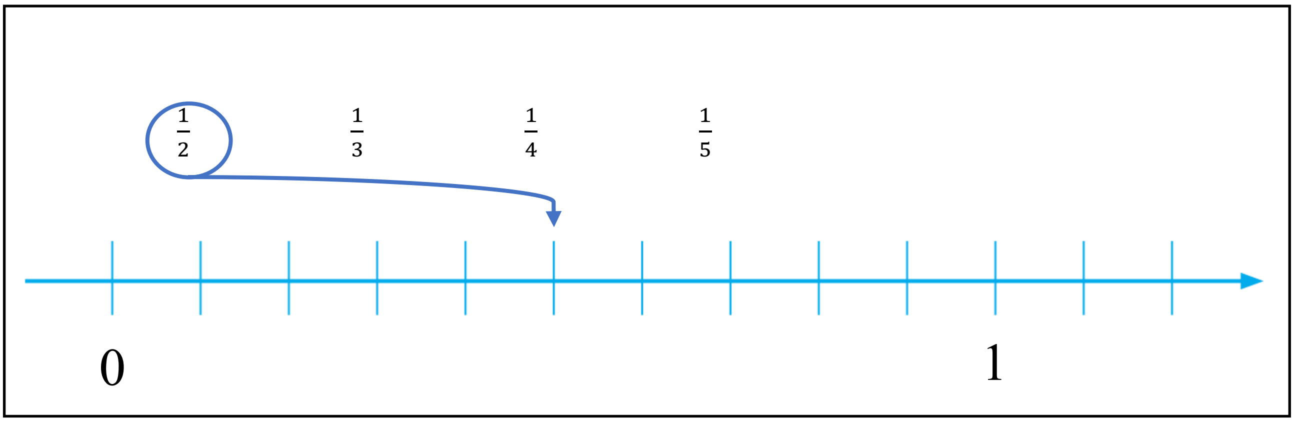
Item 32
Convert the fraction to a decimal number

$$\frac{1}{4}=\square$$

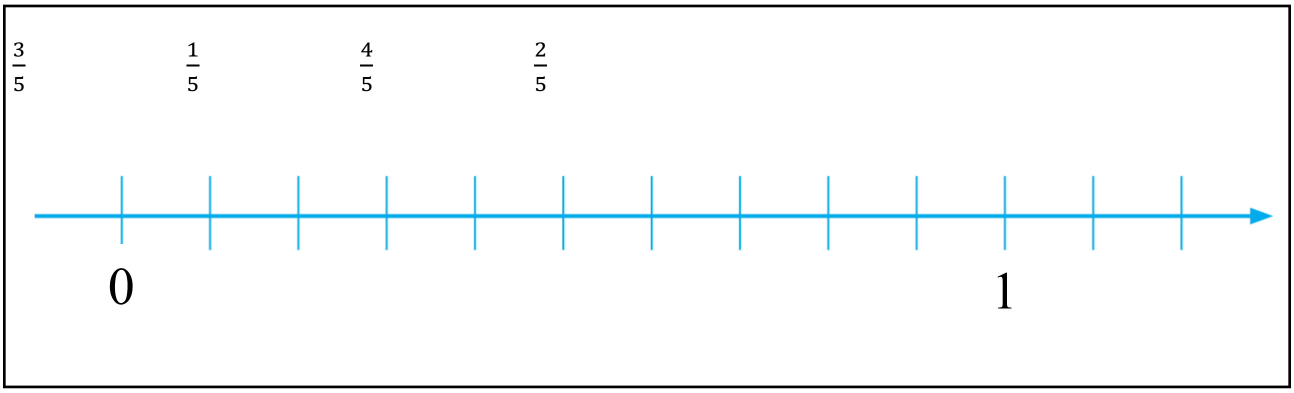
Item 33
Convert the decimal number to a fraction

$$0.36=\square$$

Item 34
In the following it is shown where the number $\frac{1}{2}$ is placed on the number line.
Do the same for the other three numbers



Item 35
Do the same for these numbers:



Item 36

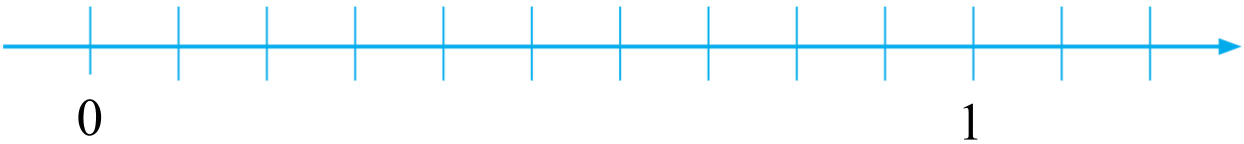
Do the same for these numbers:

$\frac{2}{5}$ $\frac{2}{4}$ $\frac{2}{6}$ $\frac{2}{3}$

Item 37

Do the same for these numbers:

$\frac{1}{5}$ $\frac{1}{4}$ $\frac{3}{4}$ $\frac{1}{2}$



Item 38

Which number is the largest? (circle your answer)

0.30 0.4

Item 39

Which number is the smallest? (circle your answer)

$\frac{3}{6}$ $\frac{1}{6}$ $\frac{5}{6}$ $\frac{2}{6}$

Item 40

Write the numbers in order with the smallest first

0.27 0.267 0.1200 0.05

Item 41

Convert the fraction to a decimal number

$$\frac{3}{4} = \boxed{}$$

Item 42

Write a number in the box so the equal sign is correct

$$\frac{\boxed{}}{5} = \frac{3}{15}$$

Item 43

Calculate:

$$\frac{2}{3} - \frac{1}{3} = \frac{\boxed{}}{\boxed{}}$$

Item 44

What is the solution to the calculation: $3 - 2 + 5 \cdot 2$

$$\boxed{}$$

Item 45

Calculate:

$$\frac{1}{5} + \frac{1}{2} = \frac{\boxed{}}{\boxed{}}$$

Item 46

Convert the fraction to a decimal number

$$\frac{7}{10} = \boxed{}$$

Item 47

Write two new fractions that are of the same value as the first fraction

$$\frac{2}{3} \quad \boxed{} \quad \boxed{}$$

Item 48

Which number is the smallest? (circle your answer)

$\frac{2}{5}$	$\frac{2}{4}$	$\frac{2}{3}$	$\frac{2}{7}$

Item 49

Write the numbers in order with the smallest first

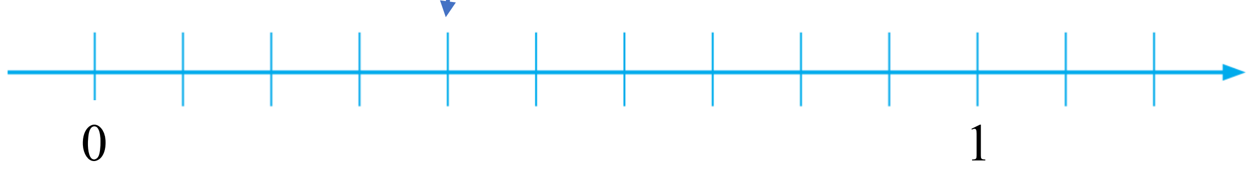
0.1798 0.18 0.2 0.09

Item 50

In the following it is shown where the number 0.4 is placed on the number line.

Do the same for the other two numbers

0.4 0.250 0.05



Item 51

Which number is marked on the number line (circle your answer)

0.7 0.09 0.590

Item 52

Write a number in the empty box so the calculation is correct

$$45 - \boxed{} = 32$$

Opgave 53

Hvilket tal er størst? (sæt ring om dit svar)

0.519	0.62

Item 54

Convert the decimal number to a fraction

$$0.50 = \boxed{}$$

Item 55

Calculate:

$$\frac{3}{4} - \frac{2}{5} = \frac{\boxed{}}{\boxed{}}$$

Item 56

Calculate:

$$\frac{8}{10} + \frac{1}{2} = \frac{\boxed{}}{\boxed{}}$$

Item 57

Which number is marked on the number line (circle your answer)

0.25   0.200   0.0250

Item 58

Write a number in the empty box so the calculation is correct

$$\boxed{} : 4 = 4$$

Item 59

Calculate:

$$\frac{5}{7} + \frac{1}{8} = \frac{\boxed{}}{\boxed{}}$$

Opgave 60
Write a number in the box so the equal sign is correct

$$\frac{1}{6} = \frac{\boxed{}}{24}$$

Item 61
Convert the decimal number to a fraction

$$0.20 = \boxed{}$$

Item 62
Calculate:

$$7 - (-9) = \boxed{}$$

Item 63
What should x be so that the calculation becomes true?

$$2x = 10 \quad \boxed{}$$

Item 64
Reduce

$$7a - 3a = \boxed{}$$

Item 65
Calculate:

$$6 + (-5) = \boxed{}$$

Item 66
Calculate:

$$\frac{2}{3} - \frac{1}{2} = \frac{\boxed{}}{\boxed{}}$$

Item 67
Convert the fraction to a decimal number

$$\frac{3}{5} = \boxed{}$$

Item 68
Calculate:

$$\boxed{}$$

$$(-18) - (-5) =$$

Item 69
What should x be so that the calculation becomes true?

$$x - 36 = 48$$

$$\boxed{}$$

Item70
Calculate:

$$(-2) + (-3) = \boxed{}$$

Item 71
What should x be so that the calculation becomes true?

$$2 \cdot x + 5 \cdot x - 3 = 11$$

$$\boxed{}$$

Item 72
Reduce

$$8b + 5b = \boxed{}$$

Item 73
Write a number in the empty boxes so the calculation is correct

$$\frac{1 \cdot \boxed{}}{4 \cdot \boxed{}} = \frac{\boxed{}}{20}$$

Which number did you write in all the boxes? Why?

Item 74
Which number is the largest? (circle your answer)

$\frac{2}{5}$	$\frac{2}{4}$	$\frac{2}{3}$	$\frac{2}{7}$

Explain your answer in the box

Item 75

Which number is the largest? (circle your answer)

0.362	0.37

Explain your answer in the box

Item 76

Expand the fraction with 3

$$\frac{1}{3} = \frac{\boxed{}}{\boxed{}}$$

Item 77

Why are $\frac{1}{2}$ og $\frac{2}{4}$ equal?

Explain your answer in the box

Item 78

Simplify the fraction as much as possible

$$\frac{15}{25} = \frac{\boxed{}}{\boxed{}}$$

Item 79

Which number is the smallest? (circle your answer)

$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{7}$

Explain your answer in the box

Item 80
Reduce

$$-3b + 7 + 5b - 4 = \boxed{}$$

Item 81
Expand the fraction with 2

$$\frac{2}{5} = \frac{\boxed{}}{\boxed{}}$$

Item 82
Write a number in the empty boxes so the calculation is correct:

$$\frac{5 \cdot \boxed{}}{7 \cdot \boxed{}} = \frac{20}{\boxed{}}$$

Which number did you write in all the boxes? Why?

Item 83
Reduce

$$16a - 3a + 7 = \boxed{}$$

Item 84
Look at the expression
$5 + 7 = 12$

1. What is the name of the sign the arrow points at?

2. What does the sign mean?

Item 85
Anne calculates like this:

$$(1 + 3) \cdot 5 - 2 = 12$$

Can you explain whether Anne has calculated the expression right or wrong?

Item 86
Write a number in the empty box so the calculation is correct

$$8 + 4 = \boxed{} + 5$$

Explain how you found the answer:

Appendix 6: Final test for seventh grade

Test 7th grade

All answers in this test must be written in a box.

Item 1

What is the solution to the calculation:

$$32 + 3 \cdot (7 - 5) = \boxed{}$$

Item 2

Calculate:

$$3 \cdot 1.5 = \boxed{}$$

Item 3

Calculate:

$$(-2) \cdot (-5) = \boxed{}$$

Item 4

Calculate:

$$5 \cdot (-7) = \boxed{}$$

Item 5

Write a number in the empty boxes so the calculation is correct

$$\frac{2}{7} + \frac{\boxed{}}{\boxed{}} = \frac{5}{7}$$

Item 6

Write a number in the empty box so the calculation is correct

$$\frac{2}{5} = \frac{\boxed{}}{15}$$

Item 7

Convert the fraction to a decimal number

$$\frac{13}{100} = \boxed{}$$

Item 8
Convert the decimal number to a fraction

$$0.75 = \boxed{}$$

Item 9
Which number is the largest? (circle your answer)

$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{7}$

Item 10
Write the expression as short as possible:

$$7b + a - 3b - 5 = \boxed{}$$

Item 11
Calculate:

$$4.07 \cdot 5.1 = \boxed{}$$

Item 12
Calculate:

$$17 + (-5) = \boxed{}$$

Item 13
Write a number in the empty box so the calculation is correct

$$\frac{\boxed{}}{9} - \frac{2}{9} = \frac{5}{9}$$

Item 14
Calculate:

$$\frac{1}{3} \cdot \frac{2}{7} = \frac{\boxed{}}{\boxed{}}$$

Item 15
Write a number in the empty box so the calculation is correct

$$\frac{2}{5} \cdot \frac{\boxed{}}{\boxed{}} = \frac{6}{25}$$

Item16

Write a number in the box so the equal sign is correct

$$\frac{3}{7} = \frac{9}{\boxed{}}$$

Item 17

Calculate:

$$\frac{2}{3} + \frac{5}{7} = \frac{\boxed{}}{\boxed{}}$$

Item 18

Convert the fraction to a decimal number

$$\frac{7}{10} = \boxed{}$$

Item 19

Write a number in the empty boxes so the calculation is correct

$$\frac{4}{5} - \frac{\boxed{}}{\boxed{}} = \frac{1}{5}$$

Item 20

Which number is the largest? (circle your answer)

$\frac{3}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{6}$

Item 21

Convert the fraction to a decimal number

$$\frac{3}{4} = \boxed{}$$

Item 22

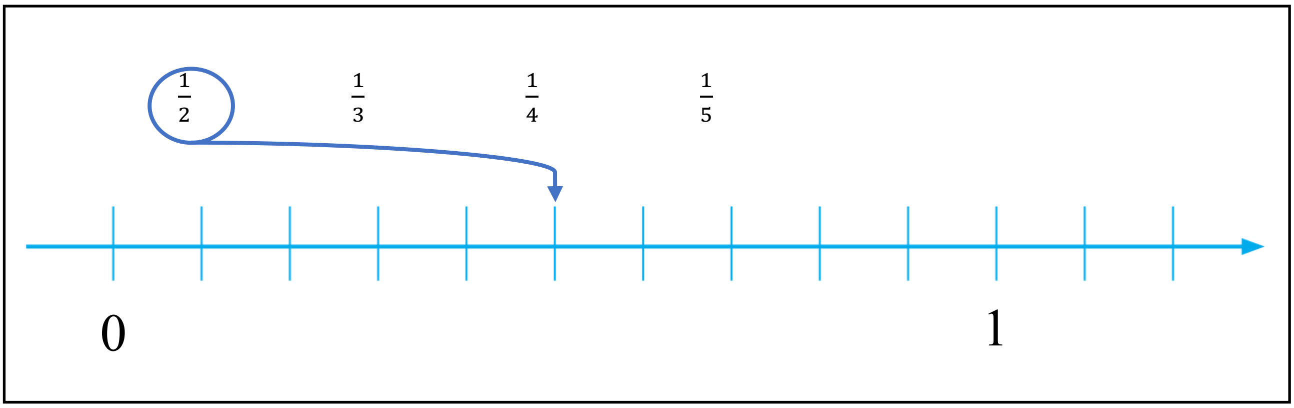
Calculate:

$$\frac{5}{8} - \frac{3}{8} = \frac{\boxed{}}{\boxed{}}$$

Item 23

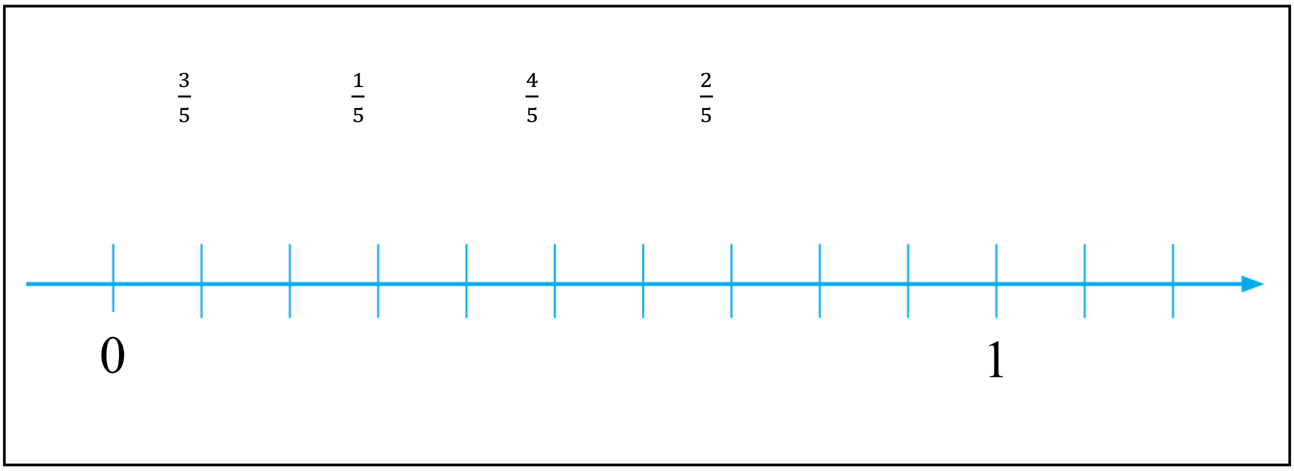
In the following it is shown how the number $\frac{1}{2}$ is placed on the number line.

Do the same for the other three numbers



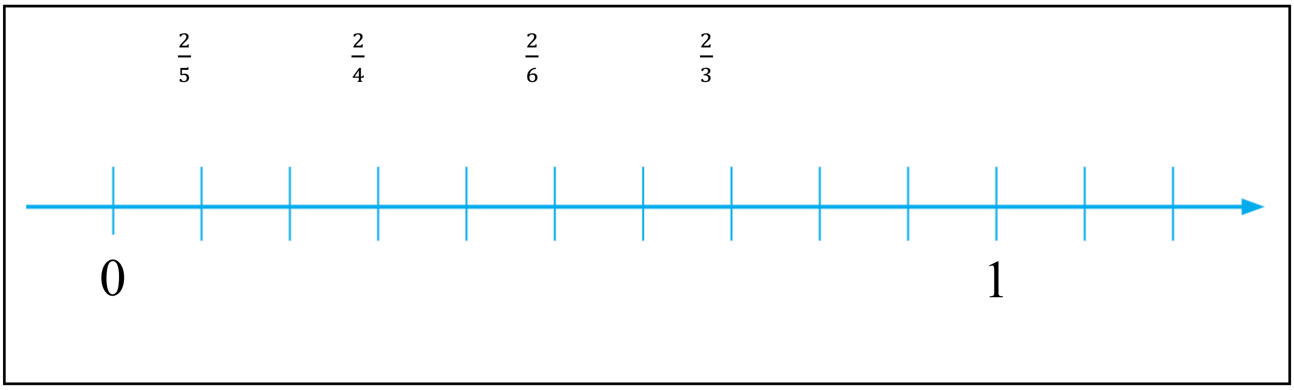
Item 24

Do the same for these numbers:



Item 25

Do the same for these numbers:



Item 26
Calculate:

$$\frac{3}{9}+\frac{2}{9}=\frac{\square}{\square}$$

Item 27
Write the expression as short as possible:

$$8-2\cdot 3+8\text{b}=\square$$

Item 28
Calculate:

$$\frac{7}{8}-\frac{3}{4}=\frac{\square}{\square}$$

Item 29
Calculate:

$$3.2\cdot 1.53=\square$$

Item 30
Calculate:

$$3-(-7)=\square$$

Item 31
Write a number in the empty boxes so the calculation is correct

$$\frac{3}{4}-\frac{\square}{\square}=\frac{1}{2}$$

Item 32
Write a number in the empty boxes so the calculation is correct

$$\frac{2}{5}:\frac{\square}{\square}=\frac{6}{10}$$

Item 33
Write a number in the empty box so the calculation is correct

$$\square:\frac{1}{4}=16$$

Item 34

Calculate:

$$7.5 \cdot 3 = \boxed{}$$

Item 35

Calculate:

$$(-4) - (-8) = \boxed{}$$

Item 36

Calculate:

$$(-12) + (-18) = \boxed{}$$

Item 37

Write a number in the empty boxes so the calculation is correct

$$\frac{\boxed{}}{\boxed{}} + \frac{2}{9} = \frac{3}{5}$$

Item 38

Write a number in the empty box so the calculation is correct

$$\frac{1}{3} \cdot \frac{2}{\boxed{}} = \frac{2}{27}$$

Item 39

Calculate:

$$\frac{1}{2} : \frac{5}{7} = \frac{\boxed{}}{\boxed{}}$$

Item 40

What should x be, so the calculation is correct?

$$x \cdot 5 = 1$$

$$\boxed{}$$

Item 41

Two of the fractions in the box are equal. (circle the fractions)

For example

$\frac{2}{5}$	$\frac{1}{6}$	$\frac{6}{15}$	$\frac{2}{6}$

Do the same for this box

$\frac{3}{5}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{6}{10}$

Item 42

Two of the fractions in the box are equal. (circle the fractions)

$\frac{1}{4}$	$\frac{4}{14}$	$\frac{2}{6}$	$\frac{2}{7}$

Item 43

Two of the fractions in the box are equal. (circle the fractions)

$\frac{3}{4}$	$\frac{1}{5}$	$\frac{3}{12}$	$\frac{9}{12}$

Item 44

Convert the fraction to a decimal number

$$\frac{2}{8} = \square$$

Item 45

Convert the decimal number to a fraction

$$0.50 = \square$$

Item 46

Write a number in the empty box so the calculation is correct

$$\frac{2}{5} \cdot \square = \frac{4}{5}$$

Item 47

Write the numbers in order with the smallest first

0.1798 0.18 0.2 0.09

Item 48

Write a number in the empty box so the calculation is correct

$$\frac{\square}{5} = \frac{3}{15}$$

Item 49

Write a number in the empty box so the calculation is correct

$$\frac{1}{5} : \square = \frac{1}{20}$$

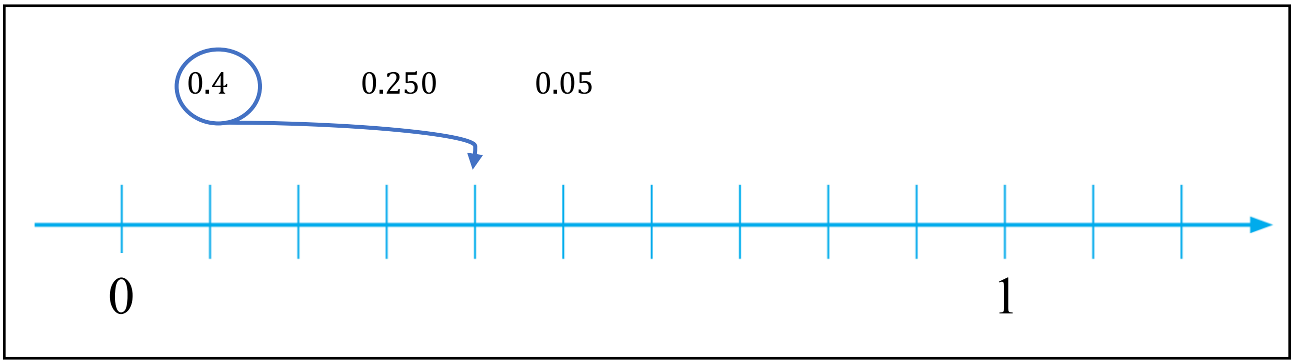
Item 50

Write a number in the empty box so the calculation is correct

$$\frac{2}{7} \cdot \square = \frac{6}{7}$$

Item 51

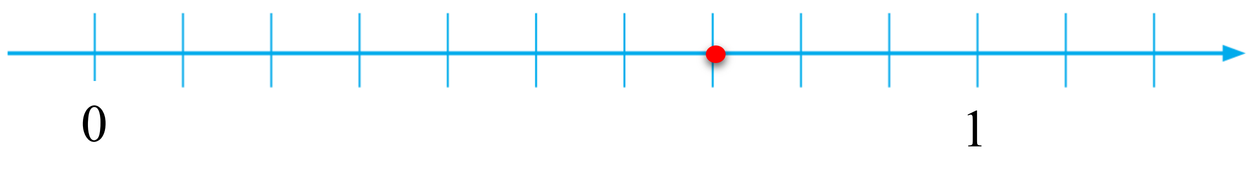
In the following it is shown how the number 0.4 is placed on the number line.
Do the same for the other two numbers



Item 52

Which number is marked on the number line (circle your answer)

0.09 0.7 0.590



Item 53

What should x be, so the calculation is correct?

$$x \cdot \frac{3}{4} = \frac{15}{4}$$

Item 54

Simplify the fraction as much as possible

$$\frac{21}{27} = \frac{\square}{\square}$$

Item 55

Write a number in the empty box so the calculation is correct

$$56 - \square = 44$$

Item 56

Which number is the largest? (circle your answer)

0.591	0.62

Item 57

Solve the equation

$$7x - 7 = 13 - 3x$$

Item 58

Which number is the smallest? (circle your answer)

$\frac{3}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{6}$

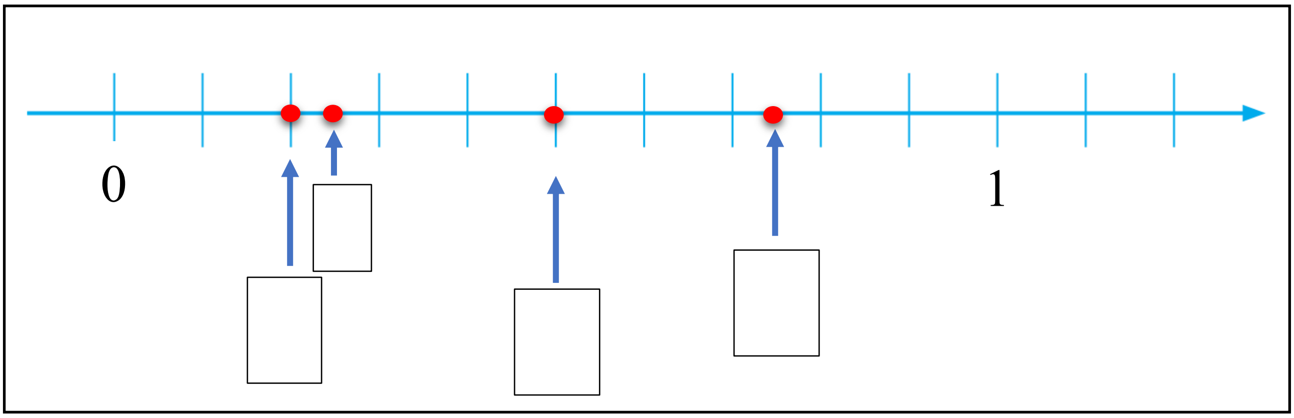
Item 59

Which number is the smallest? (circle your answer)

$\frac{2}{5}$	$\frac{2}{4}$	$\frac{2}{3}$	$\frac{2}{7}$

Item 60

Which fractions should be written under the arrows? (write the fraction in the box)



Item 61

Write a number in the empty box so the calculation is correct

$$\boxed{} : 4 = 4$$

Item 62

Which fraction is twice as large as $\frac{1}{4}$? (circle your answer)

$\frac{1}{8}$ $\frac{2}{8}$ $\frac{1}{2}$

Item 63

What is p?

$$\frac{3p}{5} = 6$$

$$\boxed{}$$

Item 64

Calculate:

$$\frac{\boxed{}}{\boxed{}} : \frac{3}{4} = \frac{8}{15}$$

Item 65

Reduce $7b + 2a - 5c - 4b - c = \boxed{}$

Item 66

Write a number in the empty boxes so the calculation is correct

$$\frac{5 \cdot \square}{7 \cdot \square} = \frac{15}{\square}$$

Item 67

Convert the decimal number to a fraction

$$0.20 = \square$$

Item 68

Which number is the largest? (circle your answer)

0.30	0.4

Item 69

Solve the equation

$$2x = 16 + 2$$

$$\square$$

Item 70

Write a number in the empty boxes so the calculation is correct

$$\frac{\square}{\square} : \frac{1}{5} = \frac{3}{4}$$

Item 71

Expand the fraction with 3

$$\frac{4}{7} = \frac{\square}{\square}$$

Item72

Convert the decimal number to a fraction

$$0.36 = \square$$

Item 73

Reduce

$$\frac{3p + 3q + 3r}{3} = \boxed{}$$

Item 74

Reduce

$$4x - 2y + 3x + 5y - 2x = \boxed{}$$

Item 75

Write two new fractions that are of the same value as the first fraction

$$\frac{4}{7} \quad \boxed{} \quad \boxed{}$$

Item 76

Which number is the largest? (circle your answer)

$\frac{2}{5}$ $\frac{2}{4}$ $\frac{2}{3}$ $\frac{2}{7}$

Why?

Item 77

Which number is the largest? (circle your answer)

0.37 0.362

Explain your answer in the box

Item 78

Write a fraction that is half as big as $\frac{3}{5}$ $\boxed{}$

Item 79

Write the solution in the box.

Explain in the box, how you found the solution to the equation

$$3x + 2 = 8$$

Item 80

Write a fraction which is greater than $\frac{1}{5}$ and less than $\frac{1}{2}$

Item 81

What has gone wrong in this transformation?

$$7x - 7 = 13 - 3x$$
$$10x - 7 = 13$$
$$10x = 20$$
$$x = 10$$

Item 82

Which number is the smallest? (circle your answer)

$\frac{1}{7}$ $\frac{1}{5}$ $\frac{1}{2}$ $\frac{1}{4}$

Why?

Item 83

Why are $\frac{1}{5}$ and $\frac{2}{10}$ equally big?

Item 84

Is the expression calculated correctly? (tick the box)

$$2 \cdot (a + 3) = 2a + 3$$

Yes ☐ No ☐

Justify your answer:

Item 85

Which fraction should be added to 4/7 to get 1?
Explain your answer in the box

Item 86

Expand the fraction

$$\frac{1 \cdot \boxed{}}{4 \cdot \boxed{}} = \frac{\boxed{}}{20}$$

What number did you expand with?
Explain your answer in the box

Item 87

Look at the expression

$5 + 7 = 12$

$\uparrow$

1. What is the name of the sign the arrow points at?

2. What does the sign mean?

Item 88

Write a number in the empty box so the calculation is correct

$$35 + \boxed{} = 45 + 5$$

Explain how you found the answer:

Item 89

Is the number which should be inserted in the box, the same number in these two equations?

Explain your answer in the box

$$3 \cdot \boxed{} + 17 = 41$$

$$3 \cdot \boxed{} + 17 - 5 = 41 - 5$$

Item 90

Do these two equations have the same solution?

Explain your answer in the box

$$5 \cdot m - 7 = 43$$

$$5 \cdot m - 7 + 5 = 43 + 5$$

Item 91

Write a number in the empty box so the calculation is correct

$$8 - 2 = \boxed{} + 5$$

Explain how you found the answer:

Item 92

Write a number in the empty boxes so the calculation is correct:

$$\frac{\boxed{}}{\boxed{}} : \frac{4}{6} = \frac{9}{10}$$

Explain how you found the answer. If you cannot solve the item, then explain what you think was difficult.

Item 93
Calculate $1.75 \cdot 3.7$
Explain how you found the answer.
How did you put the comma in the result?

Appendix 7: Feedback for teachers in fifth grade

Kære matematiklærer i 5.A

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 5.A. Der informeres om hvilke fejltyper eleverne i 5.A hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 5.klasser.

5.A viser en god beherskelse af:

1. Opgaver omhandlende brøkers størrelse. Konkret betyder dette, at en stor andel af eleverne i 5.A er i stand til at vurdere hvilke brøker der har større værdi end andre. For eksempel kan mellem 80-90% af eleverne i 5.A vurdere følgende brøkers indbyrdes størrelse:

   Hvilket tal er det største? (sæt ring om dit svar)

$\frac{3}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{6}$

   Hvilken tal er det mindste? (sæt ring om dit svar)

$\frac{2}{5}$	$\frac{2}{4}$	$\frac{2}{3}$	$\frac{2}{7}$

   De elever der ikke er i stand til at løse opgaver af denne type, har oftest en misforståelse om brøker. Konkret vil disse elever fx mene at $\frac{1}{7}$ er større end $\frac{1}{2}$ fordi 7 er større end 2.

2. Sammenlignet med de resterende tre 5-klasser, er 5.A den klasse der klarer sig bedst i opgaver om at vurdere forskellige decimaltals størrelse. Fx er ca. 80% af 5.A i stand til at løse: Hvilket tal er størst? (sæt ring om dit svar)

0.30	0.4

3. 5.A er den klasse, hvor flest elever er i stand til at løse opgaver om multiplikation af en brøk med et heltal (ca. 40%). Selvom 5.A er den klasse der viser størst beherskelse af denne opgave, er det kun under halvdelen af klassen, hvorfor dette bør tages op i undervisningen.

5.A viser en lav beherskelse af:

1. Ækvivalens af brøker. Mens fx 35% af 5.A besvarer opgave 6 nedenfor korrekt, er der dobbelt så mange der behersker dette i to andre 5.klasser.

   Opgave 6: To er brøkerne I boksen er lige store. (Sæt ring om disse)

$\frac{3}{5}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{6}{10}$

2. 5.A har en lav beherskelse af addition og subtraktion med brøker med samme nævner (cirka 40% behersker dette), mens næsten ingen (kun 4 elever) kan løse opgaver om addition og subtraktion af brøker med forskellige nævnere.

   Typisk fejl der begås i denne type opgave er:

   $$\frac{1}{5}+\frac{3}{5}=\frac{4}{10}$$

   $$\frac{5}{8}-\frac{1}{4}=\frac{4}{4}$$

   Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

3. 5.A (ca. 87% af klasse) er i stand til at løse ligninger af første grad når opgaven er på formen: Skriv et tal i boksen så regnestykket passer $36 - \boxed{} = 29$ men opgaver på formen: Hvad skal x være, så regnestykket bliver sandt?

   $$2x = 10$$

   kan kun løses af ca. 22% af klassen.

4. I opgaver relateret til regnearternes hierarki er det observeret at 86% af 5.A svarer korrekt på opgave 9 nedenfor, mens det kun er 56% der svarer korrekt på opgave 1. Dette må betyde at de fleste elever regner direkte fra venstre mod højre uden at tage højde for regnearternes hierarki.

   Opgave 1: Hvad er løsningen til beregningen: $4 + 2 \cdot 5 =$

   Opgave 9: Hvad er løsningen til beregningen: $3 \cdot 2 + 5 =$

5. Der findes 3 elever i 5.A som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Kære matematiklærer i 5.B

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 5.B. Der informeres om hvilke fejltyper eleverne i 5.B hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 5.klasser.

5.B viser en god beherskelse af:

1. 5.B er en af de klasser der viser størst forståelse (ca. 70%) af addition med negative heltal. Fx er der 74% der kan løse opgaven 70: Beregn $(-2) + (-3) =$
2. Ca. 74-80% af klassen kan løse opgaver af denne type:

   Hvilket tal er det største? (sæt ring om dit svar)

$\frac{3}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{6}$

   Hvilken tal er det mindste? (sæt ring om dit svar)

$\frac{2}{5}$	$\frac{2}{4}$	$\frac{2}{3}$	$\frac{2}{7}$

5.B viser en lav beherskelse af:

1. 5.B har en lav beherskelse af addition og subtraktion med brøker med samme nævner (cirka 15-20% behersker dette), mens næsten ingen elever kan løse opgaver om addition og subtraktion af brøker med forskellige nævnere.

   Typisk fejl der begås i denne type opgave er:

   $$\frac{1}{5} + \frac{3}{5} = \frac{4}{10}$$

   $$\frac{5}{8} - \frac{1}{4} = \frac{4}{4}$$

   Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

2. Sammenlignet med de andre 5.klasser, er der dobbelt så mange (ca. 63%) der kan løse opgave 6 nedenfor i 5.B. Selvom det er højere end de andre klasser er det stadig en lav beherskelse, da der ved andre opgaver om ækvivalens af brøker kun er omkring ca. 36% der kan løse denne.

   Opgave 6

   To er brøkerne I boksen er lige store. (Sæt ring om disse

$\frac{3}{5}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{6}{10}$

3. 5.B (ca. 85% af klasse) er i stand til at løse ligninger af første grad når opgaven er på formen: Skriv et tal i boksen så regnestykket passer $36 - \boxed{} = 29$ men opgaver på formen: Hvad skal x være, så regnestykket bliver sandt?

$$2x = 10$$

Kan kun løses af ca. 37% af klassen.

4. 5.B er den klasse der har klarest dårligt i opgaver der undersøger regnearternes hierarki. For opgave 1: Hvad er løsningen til dette $4 + 2 \cdot 5$, er det kun 32% der svarer korrekt på dette, hvorimod der er 74% i en af de andre 5.klasser.
   Tværtimod i opgave 9: Hvad er løsningen til beregningen: $3 \cdot 2 + 5 =$, der handler om regnearternes hierarki er der 79% der svarer korrekt. Dette kan betyde at de fleste regner direkte fra venstre mod højre, uden at tage højde for operationerne.
5. Der findes 2 elever i 5.B som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Kære matematiklærer i 5.C

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 5.C. Der informeres om hvilke fejltyper eleverne i 5.C hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 5.klasser.

5.C viser en god beherskelse af:

1. 5.C er den klasse der viser højst score (ca. 80%) i opgave 65 om addition af negative heltal fx: Opgave 65: Beregn $6 + (-5) =$
2. 5.C er den klasse der viser højst beherskelse i opgaver om regnearternes hierarki fx

Opgave 1: Hvad er løsningen til beregningen: $4 + 2 \cdot 5 =$

Opgave 9: Hvad er løsningen til beregningen: $3 \cdot 2 + 5 =$

Opgave 1 er løst af 74% og opgave 9 er løst af 100% af klassen. Altså 5.C er den klasse med højest beherskelse af opgaver på denne type.

5.C viser en lav beherskelse af:

1. Selvom 5.C er den klasse der viser højst beherskelse med opgaver om addition og subtraktion af brøker med samme nævner, ligger denne beherskelse på højst 50% afhængigt af opgave. Addition og subtraktion af brøker med forskellige nævnere er der ingen i 5.C der behersker, sammenlignet med andre 5.klasse, hvor der er enkelte elever der behersker dette.

   Typisk fejl der begås i denne type opgave er:

$$\frac{1}{5} + \frac{3}{5} = \frac{4}{10}$$

$$\frac{5}{8} - \frac{1}{4} = \frac{4}{4}$$

   Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

2. Ved ækvivalens af brøker ses at 5.C er en af de klasser der behersker det mindst. Fx er det kun 30% af eleverne i 5.C der kan løse denne opgave korrekt:

   Opgave 6

   To af brøkerne i boksen er lige store. (sæt ring om disse)

$\frac{3}{5}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{6}{10}$

Opgave 7

Gør det samme for denne boks.

$\frac{1}{4}$	$\frac{4}{14}$	$\frac{2}{6}$	$\frac{2}{7}$

3. Der findes 6 elever i 5.C som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Kære matematiklærer i 5.D

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 5.D. Der informeres om hvilke fejltyper eleverne i 5.D hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 5.klasser.

5.D viser en god beherskelse af:

1. Der er dobbelt så mange elever i 5.D der viser en beherskelse af ækvivalens af brøker sammenlignet med andre klasser. Fx er der 80% der har løst opgave 8 nedenfor korrekt, mens dette ligger på ca. 40% i de andre 5.klasser.

   Opgave 8: To af brøkerne i boksen er lige store (sæt ring om disse)

$\frac{3}{4}$	$\frac{1}{5}$	$\frac{3}{12}$	$\frac{9}{12}$

2. Blandt alle 5.klasser, er 5.D den klasse der viser højest beherskelse i opgaver om at konvertere en brøk til et decimaltal og et decimaltal til en brøk. Selvom det er den højeste beherskelse blandt alle 5.klasser, er beherskelse af konvertering fra brøk til decimaltal på 57% mens konverteringen af decimaltal til brøk er ca. 70%.

5.D viser en lav beherskelse af:

1. I opgaver relateret til regneartemes hierarki er det observeret at 100% af 5.D svarer korrekt på opgave 9 nedenfor, mens det kun er 43% der svarer korrekt på opgave 1 nedenfor. Dette betyder at eleverne regner direkte fra venstre mod højre uden at tage højde for regneartemes hierarki.

   Opgave 1: Hvad er løsningen til beregningen: $4 + 2 \cdot 5 =$

   Opgave 9: Hvad er løsningen til beregningen: $3 \cdot 2 + 5 =$

2. For addition og subtraktion af brøker med ens nævnere er det mellem 30-40% af eleverne i 5.D der viser en beherskelse af dette. Til gengæld er der ingen der svarer korrekt på opgaver om addition og subtraktion af brøker med forskellige nævnere.

Typisk fejl der begås i denne type opgave er:

$$\frac{1}{5}+\frac{3}{5}=\frac{4}{10}$$

$$\frac{5}{8}-\frac{1}{4}=\frac{4}{4}$$

Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

3. Der findes 1 elev i 5.D som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Appendix 8: Feedback for teachers in seventh grade

Kære matematiklærer i 7.A

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 7.A. Der informeres om hvilke fejltyper eleverne i 7.A hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 7.klasser.

7.A viser en god beherskelse af

1. 7.A er den klasse der viser størst beherskelse af opgave 1 nedenfor, om regneartернes hierarki. I 7.A forekommer beherskelsen på 86%

   Opgave 1: Hvad er løsningen til udregningen:

   $$32 + 3 \cdot (7 - 5) = \boxed{}$$

7.A viser en lav beherskelse af:

1. 7.A er den klasse der klarer det mindst godt i opgaver om addition af brøker med ens nævnere. Fx er der 68% der har løst opgave 5 nedenfor korrekt, mens dette er på 90% i andre 7.klasser.

   Opgave 5: Udfyld de tomme felter så regnestykket passer

   $$\frac{2}{7} + \frac{\boxed{}}{\boxed{}} = \frac{5}{7}$$

2. I opgaver om subtraktion af brøker med ens nævnere er der for eksempel 81% der har løst opgave 13 nedenfor rigtig. Dette viser at 7.A har en større beherskelse af subtraktion af brøker med ens nævnere sammenlignet med addition af brøker med ens nævner. Både addition og subtraktion af brøker med ens nævner har været en del af pensum siden 5.klasse, hvorfor flere bør beherske dette.

   Opgave 13: Udfyld det tomme felt så regnestykket passer:

   $$\frac{\boxed{}}{9} - \frac{2}{9} = \frac{5}{9}$$

3. Opgaver om addition og subtraktion af brøker med forskellige nævnere er der meget lav beherskelse i 7.A. Kun 32% af klassen har besvaret opgave 17 nedenfor korrekt, mens

kun 36% af klassen har besvaret opgave nedenfor 28 korrekt. Generelt forekommer der en lav beherskelse af addition og subtraktion af brøker med forskellige nævnere i 7.A.

Opgave 17: Beregn $\frac{2}{3}+\frac{5}{7}$

Opgave 28: Beregn $\frac{7}{8}-\frac{3}{4}$

Typisk fejl der begås i denne type opgave er:

$$\frac{2}{3}+\frac{5}{7}=\frac{7}{10}$$

$$\frac{7}{8}-\frac{3}{4}=\frac{4}{4}$$

Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

4. For opgaver om multiplikation og division af brøker forekommer der en lav beherskelse i 7.A. I opgave 14: Beregn $\frac{1}{3}\cdot\frac{2}{7}$, er der kun 36% af klassen der behersker dette, sammenlignet med en af de andre 7.klasser, hvor beherskelsen er på 70%.

   For division af brøker, fx opgave 39: Beregn $\frac{1}{2}:\frac{5}{7}$, er der ingen i 7.A der har løst denne korrekt, hvorimod der er 6 elever i en af de andre 7.klasser, som kan løse denne opgave.

5. 7.A er den klasse der viser den højeste beherskelse (ca. 63%) af opgave 40 nedenfor, om løsning af første gradsligning, blandt alle fire 7.klasser.

   Opgave 40: Hvad skal x være så regnestykket passer?

$$x\cdot 5=1$$

$$\boxed{}$$

   7.A viser den største beherskelse i opgaver om løsning af første gradsligning når opgaven er på formen som i opgave 55:

   Skriv et tal i det tomme felt boksen regnestykket passer:

$$56-\boxed{}=44$$

Til gengæld viser 7.A en lav beherskelse i løsningen af mere komplicerede første gradsligninger fx opgave 57: løs ligningen

$$7x - 7 = 13 - 3x$$

Mens 27% af 7.A kan løse opgave 57, er der fx 48% i en anden klasse.

6. Der findes 2 elever i 7.A (og en ved grænsen) som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Kære matematiklærer i 7.B

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 7.B. Der informeres om hvilke fejltyper eleverne i 7.B hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 7.klasser.

7.B viser en god beherskelse af:

1. 7.B viser en god beherskelse af addition af brøker med ens nævner. Fx er der 91% af 7.B der har løst opgave 5 nedenfor korrekt.

   Opgave 5: Udfyld de tomme felter så regnestykket passer

   $$\frac{2}{7}+\frac{\boxed{}}{\boxed{}}=\frac{5}{7}$$

2. 7.B er den klasse der viser højst beherskelse af addition med brøker med forskellige nævnere. Mens mellem 0-30% af eleverne i den andre 7.klasser løser opgave 17 nedenfor korrekt, er beherskelsen på 62% i 7.B.

   Opgave 17: Beregn $\frac{2}{3}+\frac{5}{7}$

   7.B er også den klasse der viser højst beherskelse af subtraktion med brøker med forskellige nævnere. Opgave 28: Beregn $\frac{7}{8}-\frac{3}{4}$, er løst korrekt af 57% af 7.B, mens denne beherskelse er mellem 17-44% i de andre 7.klasser. Men bemærk at 57% beherskelse stadig er lavt, da subtraktion og addition af brøker med forskellige nævner har været en del af pensum for 7.klasse siden 5.klasse.

   Typisk fejl der begås i denne type opgave er:

   $$\frac{2}{3}+\frac{5}{7}=\frac{7}{10}$$

   $$\frac{7}{8}-\frac{3}{4}=\frac{4}{4}$$

   Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

7.B viser en lav beherskelse af:

1. For opgaver om multiplikation og division af brøker forekommer der en lav beherskelse i 7.B. I opgave 14: Beregn $\frac{1}{3} \cdot \frac{2}{7}$, er der kun 43% af klassen der behersker dette, sammenlignet med en af de andre 7.klasser, hvor beherskelsen er på 70%.

   For division af brøker, fx opgave 39: Beregn $\frac{1}{2} : \frac{5}{7}$, er der 29% i 7.B der har løst denne korrekt. 7.B er den klasse med den højeste beherskelse af opgave 39 selvom dette stadig er meget lavt.

2. 7.B er den klasse der viser den højeste beherskelse (ca. 63%) af opgave 40 nedenfor, om løsning af første gradsligning, blandt alle fire 7.klasser.

   Opgave 40: Hvad skal x være så regnestykket passer?

   $$x \cdot 5 = 1$$

   $$\boxed{}$$

   7.B viser en beherskelse på 71% i opgaver om løsning af første gradsligning når opgaven er på formen som i opgave 55:

   Skriv et tal i det tomme felt boksen regnestykket passer:

   $$56 - \boxed{} = 44$$

   Selvom dette er en høj beherskelse, er dette en så simpel opgave at mange flere i 7.B bør kunne løse den.

   I mere komplicerede første gradsligninger fx opgave 57:

   løs ligningen

   $$7x - 7 = 13 - 3x$$

   er det 48% af 7.B der kan løse opgave 57. Selvom dette stadig er en lav beherskelse, er det den højeste beherskelse blandt alle 7.klasser.

3. Der findes 3 elever i 7.B som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

   Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Kære matematiklærer i 7.C

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 7.C. Der informeres om hvilke fejltyper eleverne i 7.C hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 7.klasser.

7.C viser en god beherskelse af:

1. den klasse der viser bedst beherskelse af multiplikation af brøker. Fx er opgave 14: Beregn $\frac{1}{3} \cdot \frac{2}{7}$, løst korrekt af 71% af klassen, mens dette ligger mellem 35-45% i de andre 7.klasser.
2. Af opgaver om addition med negative heltal. Fx i opgave 12: Beregn $17 + (-5)$, er beherskelsen på 94% i 7.C.
3. For opgaver om addition af brøker med samme nævner er beherskelsen mellem 70-88% i 7.B. Dette er en høj beherskelse, men da denne opgavetype har været fremtrædende siden 5.klasse, bør der være en højere beherskelse (i hvert fald i den opgave hvor beherskelsen er 70%)

7.C viser en lav beherskelse af:

1. Sammenlignet med de tre andre 7.klasser, er der ingen elever i 7.C der kan løse opgaver med addition af brøker med forskellige nævner. Fx opgave 17 nedenfor, er beherskelsen på 0% i 7.C, mens den fx er 62% i en af de andre 7.klasser.

   Opgave 17: Beregn $\frac{2}{3} + \frac{5}{7}$

   Ligeledes er 7.C der klasse, der viser lavest beherskelse af subtraktion af brøker med samme nævner. Fx er opgave 28: Beregn $\frac{7}{8} - \frac{3}{4}$, løst af 18% af eleverne i 7.C, mens denne beherskelse er på 57% i en af de andre 7.klasser.

   Typisk fejl der begås i denne type opgave er:

   $$\frac{2}{3} + \frac{5}{7} = \frac{7}{10}$$

   $$\frac{7}{8} - \frac{3}{4} = \frac{4}{4}$$

   Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

2. I opgaver om division af to brøker er der kun 2 elever der kan løse 2/5 opgaver om division af brøker. Ellers er der ingen i 7.C der kan regne opgaver om division af brøker.

Disse to elever har løst opgave 64 og 70 nedenfor korrekt.

Opgave 64: Beregn

$$\frac{\square}{\square} : \frac{3}{4} = \frac{8}{15}$$

Opgave 70: Udfyld de tomme felter så regnestykket passer

$$\frac{\square}{\square} : \frac{1}{5} = \frac{3}{4}$$

Hvorimod en mere almindelig opgave om division af brøker fx opgave 39: Beregn $\frac{1}{2} : \frac{5}{7}$, er der ingen elever i 7.C der kan løse.

3. For opgaver om løsning af en første gradsligning er resultaterne forskellige. For opgave 53 og opgave 61 nedenfor, ses det at den 7.C har den højeste beherskelse.

   7.C viser en beherskelse på 77% i opgaver om løsning af første gradsligning når opgaven er på formen som i opgave 55:
   Skriv et tal i det tomme felt boksen regnestykket passer:

$$56 - \square = 44$$

   Selvom dette er en høj beherskelse, er dette en så simpel opgave at mange flere i 7.C bør kunne løse den.
   På den anden side er 7.C også den klasse der har mindst beherskelse af mere komplicerede opgaver om løsningen af første gradsligning. Dette gælder fx opgave 57, hvor kun 12% viste en beherskelse af denne, sammenlignet med de andre klasser, hvor beherskelsen er på 48% i en af klasserne.

Opgave 53

Hvad skal x være så udtrykket passer

$$x \cdot \frac{3}{4} = \frac{15}{4} \quad \boxed{}$$

Opgave 57: Løs ligningen: $7x - 7 = 13 - 3x$

4. Der findes 2 elever i 7.C (og en ved grænsen) som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

   Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.

Kære matematiklærer i 7.D

I det følgende informeres om resultaterne for den diagnostiske test foretaget i 7.D. Der informeres om hvilke fejltyper eleverne i 7.D hovedsageligt har lavet og hvordan de har klaret testen sammenlignet med de andre 7.klasser.

7.D viser en god beherskelse af:

1. Addition og subtraktion af brøker med ens nævnere. 7.D den klasse med højest beherskelse i disse opgaver. Fx i opgave 26: Beregn: $\frac{3}{9}+\frac{2}{9}$, er der 94% beherskelse i 7.D. Ved subtraktion af brøker med ens nævnere er beherskelsen er beherskelsen 94% i fx opgave 22: Beregn $\frac{5}{8}-\frac{3}{8}$, mens beherskelsen er 66% i opgave 19.

   Opgave 19: Udfyld de tomme felter så regnestykket passer

   $$\frac{4}{5}-\frac{\boxed{}}{\boxed{}}=\frac{1}{5}$$

   Altså er der en varierende beherskelse i 7.D i opgaver om subtraktion af brøker med ens nævnere.

7.D viser en lav beherskelse af:

1. 7.D er den klasse der viser den laveste beherskelse af opgaver om regneartternes hierarki. Opgave 1: Hvad er løsningen til regnestykket: $32+3\cdot(7-5)$, er løst korrekt af 39% af 7.D, mens denne beherskelse er på 86% i en af de andre 7.klasse.

2. I opgaver om multiplikation af negative heltal, og multiplikation af negative og positive heltal, er 7.D den klasse der behersker det mindst. Fx opgave 3: Beregn $(-2)\cdot(-5)$, er det 39% der svarer korrekt på denne opgave, mens der er 73% i en af den andre 7.klasser. Ligeledes med opgave 4:Beregn $5\cdot(-7)$, er det 23% af 7.D der kan svare på dette, mens der er 77% i en af de andre 7.klasser. I denne opgavetype forekommer en meget stor variation mellem 7.D og de andre 7.klasser.

3. Opgaver om addition og subtraktion af brøker med forskellige nævnere er der meget lav beherskelse i 7.D. Kun 33% af klassen har besvaret opgave 17 nedenfor korrekt, mens kun 44% af klassen har besvaret opgave nedenfor 28 korrekt. Generelt forekommer der en lav beherskelse af addition og subtraktion af brøker med forskellige nævnere i 7.D.

   Opgave 17: Beregn $\frac{2}{3}+\frac{5}{7}$

   Opgave 28: Beregn $\frac{7}{8}-\frac{3}{4}$

   Typisk fejl der begås i denne type opgave er:

$$\frac{2}{3}+\frac{5}{7}=\frac{7}{10}$$

$$\frac{7}{8}-\frac{3}{4}=\frac{4}{4}$$

   Det vil sige, eleverne adderer og subtraherer nævner og tæller separat.

4. For opgaver om multiplikation og division af brøker forekommer der en lav beherskelse i 7.D. I opgave 14: Beregn $\frac{1}{3}\cdot\frac{2}{7}$, er der kun 44% af klassen der behersker dette, sammenlignet med en af de andre 7.klasser, hvor beherskelsen er på 70%.

   For division af brøker, fx opgave 39: Beregn $\frac{1}{2}:\frac{5}{7}$, er der 6% i 7.D der har løst denne korrekt, sammenlignet med en af de andre 7.klasser, hvor beherskelen er op 29%..

5. For opgaver om løsning af en første gradsligning er resultaterne forskellige. For opgave 53 og opgave 55 nedenfor, ses det at den 7.D viser en højere beherskelse end andre opgaver om løsning af en første gradsligning.

   7.C viser beherskelse på 61% i opgaver om løsning af første gradsligning når opgaven er på formen som i opgave 55: Skriv et tal i det tomme felt boksen regnestykket passer:

$$56-\boxed{}=44$$

   Denne beherskelse er højere i en anden 7.klasse, hvor beherskelsen er på 81%.

   På den anden side er 7.D også en af de klasser der har mindst beherskelse af mere komplicerede opgaver om løsningen af første gradsligning. Dette gælder fx opgave 57 nedenfor, hvor kun 22% viste en beherskelse af denne, sammenlignet med de andre klasser, hvor beherskelsen er på 48% i en af klasserne.

Opgave 53

Hvad skal x være så udtrykket passer $x \cdot \frac{3}{4} = \frac{15}{4}$

Opgave 57: Løs ligningen: $7x - 7 = 13 - 3x$

6. Yderligere er det observeret i 7.D, at 28% har besvaret opgave 44 nedenfor korrekt, mens dette ligger på 64% i anden 7.klasse
   Der findes 2 elever i 7.D som næsten ingen beherskelse har i opgaver om aritmetik og algebra (mellem 12-18% af hele testen er løst korrekt).

Ovenstående feedback fokuserer på tekniske enkeltheder. Forskning viser dog at det ikke er nok at arbejde med disse gennem træning, men at det også er vigtigt at undervisningen skaber indsigt i hvorfor teknikken virker. Fx sker fejl i regning med brøker typisk fordi elever blander regler sammen, som de bare har trænet men ikke forstået betydningen og gyldigheden af. Så der skal både arbejdes med at træne og forstå regneregler mv., hvis eleverne skal ende med at kunne bruge dem når det er relevant.