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A Study on Teacher Knowledge Employing Hypothetical Teacher Tasks

Based on the Principles on the Anthropological Theory of Didactics

Camilla Margrethe Mattsson

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UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

Master's Thesis

Camilla Margrethe Mattsson

A Study on Teacher Knowledge Employing Hypothetical Teacher Tasks

Based on the Principles of the Anthropological Theory of Didactics

Supervisor: Carl Winsløw

Submitted: August 8, 2016

Name: Camilla Margrethe Mattsson

Title: A Study on Teacher Knowledge Employing Hypothetical Teacher Tasks – Based on the Principles of the Anthropological Theory of Didactics

Department: Department of Science Education, University of Copenhagen

Supervisor: Carl Winsløw

Time Frame: February – August 2016

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This thesis focus on teacher knowledge related to the theme of derivative functions in Danish high schools. First, it is clarified what the notion of *teacher knowledge* entail, within the Anthropological Theory of Didactics (ATD) and what principles of research this programme advocates. Secondly, a method involving the use of hypothetical teacher tasks (HTTs) for accessing and assessing teacher knowledge, which builds on the principles of ATD, is investigated. For this purpose, a subject matter didactical analysis of the theme of functions derivatives is performed and the mathematical theme, as it exists in Danish High schools, is investigated. Together, these analyses constitute the *reference model* of the study, upon which, five HTTs are designed and presented, along with an *a priori* analysis of each task. These HTTs are employed in an empirical study, where the teacher knowledge of five teacher students and four high school teachers, related to the theme of derivative functions, is investigated. The data from the empirical study showed that the participants' different teaching experience was not generally reflected in their performances. The capacity of the study does not allow for any conclusions as to *why* the participants' various teaching experience is not reflected in their answers to the HTTs.

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Contents

1 Introduction

The thesis was initially inspired by a large problematic not within the scope of a thesis as such. The official requirements in Denmark and many other countries, to teach, or more precise, to become tenured in upper secondary school[1] is to have completed a Master's degree with two disciplines, a major and a minor (in disciplines taught in upper secondary school) which furthermore fulfils certain requirements settled by the Ministry of education. In addition to this, upper secondary schools have to administer the teachers with courses on pedagogy within the first year of hiring (Pædagogikum, n.d.).

My initial question was:

Do these requirements produce teachers?

The official statement is that the education we receive at the universities provide the professional competence, while the pedagogy courses provide teaching competencies (Sådan bliver du gymnasielærer, n.d.). The pedagogy courses have received much criticism over the years and latest in a report by Jessen, Holm & Winsløw (2015) on the role of secondary mathematics and its needs for development. Jessen et al. report how teachers experience discontinuity between their university education and the pedagogy courses. In addition, teachers are expressing a need for tools to translate their subject matter knowledge into inspiring and motivating teaching on a suitable level. In all, 40-50 % of the teachers[2] expressed that they did not feel properly prepared for teaching in regards to their pedagogical and didactical skills. In their survey, a group of mathematic teachers directly pointed to a need for a separate teacher education at the universities (Jessen et al., 2015).

To improve the education of teachers is not a simple matter though. The initial question regarding whether or not the requirements to become a teacher produce teachers, is in reality many-fold and involve questions such as 'what do a teachers need to know in order to perform successfully?', 'how can such knowledge be developed?' and 'is this in line with the way teachers are being educated?'. Whereas the answer to the latter appears to be 'no' it is also clear that an improvement of mathematics teacher education depends on the answers to the first two questions. While these questions are not new, they have not been answered in full either.

[1] The terms *upper secondary school* and *high school* will be used interchangeably throughout the thesis. They are both referring to the part of the school system, which in Danish is called *gymnasium* and encompass the 10th, 11th and 12th grade.
[2] Only 37 % of the teachers in the survey answered the questionnaire in full (Jessen et al. 2015).

1.1 Knowledge for Teaching

The present section will outline various studies and researches for the purpose, of presenting the basis literature from which the thesis takes its departure. The presentation will incorporate the earlier work of Felix Klein, tracing up to the research of Hill, Ball and Schilling, and finally presenting a contribution by Durand-Guerrier, Winsløw and Yoshida.

Felix Klein, a German mathematician and didactician, raised questions of the kind presented above already in 1932. In his book *Elementary Mathematics – from an advanced standpoint* (1932) Klein describes the consequences of the 'state of affairs', namely that no alliance existed between school and university:

> When, after finishing his course of study [at university], he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching. (p. 1)

Klein also describes how he noticed the attention towards appropriate training of teachers began to rise and described this as a 'new phenomenon'. Klein sought to help abolish the discontinuity in transitioning from being a student at university to becoming a teacher in high school through lectures tending to the needs of the prospective teachers (Klein, 1932). In his opinion, the teacher should know his field to the extent of being able to follow its development and he should 'stand above' his subject*.* The latter referring to the ability of seeing the connection between the 'versions of mathematics'[3] taught in high school and the mathematics taught at university (Winsløw, 2013).

Klein thus, directly and through his lectures, addressed the questions and problematics that constituted the initial motivation for this thesis. Despite of this, today, more than sixty years after Klein wrote Elementary Mathematics, researchers have yet to establish a theoretical consensus regarding what teachers need to know and how they learn it. This is not to say that teacher knowledge and all it entails has not been investigated. Since Klein's days, a lot of effort has been placed in trying to answer the questions surrounding this widely recognized discontinuity. For a long time, the optimizing of teacher education centred on an expansion of the *mathematics* presented to prospective teachers during their studies. In 1999, Cooney wrote, "Formerly our conception of teacher knowledge consisted primarily of understanding what teachers knew about mathematics" (Cooney, 1999, p. 163). He also

[3] Winsløw speaks of this in terms of the connection between praxeological organisations taught in school and praxeological organisations taught at university. These are concepts to be presented later.

stated that the complexity of the issue regarding teacher knowledge was becoming more recognized along with the fact that "mathematical knowledge does not alone translate into better teaching" (Cooney, 1999, p. 163). A surprising finding of Eisenberg in 1977 was an initial contributor to this development (Cooney, 1999). In the study performed by Eisenberg, no correlation was found between teachers' mathematical subject-matter knowledge and students' achievements (Bromme, 1994). As a reaction to these studies, researchers started to investigate and model other areas of knowledge possibly crucial in the teacher's practice. The teachers' mathematical knowledge would although not be undermined completely as it has been determined that the mathematical knowledge in fact plays a key role in teaching, which is also acknowledged by Bromme (1994). Studies like the Eisenberg study simply suggest a deep complexity of the teacher's practice and moreover, that many other factors bear key roles in correlation between teachers' mathematical knowledge and students' learning outcome.

Over the years, studies in the field of mathematical educational research have divided into several branches, constituting different programmes of research. Among these is the classroom-level educational research, searching to uncover the influence of teachers' classroom teaching-behaviour, especially including pedagogical methods, on students' learning, while the *educational production function studies* comprise another research programme, focused on the influence of resources held by schools, students and teachers, i.e. teachers' salaries, student families' socioeconomic status and schools' material resources. Within this programme, a particular focus on teachers' characteristics also developed; some studies mapped teacher characteristics based on educational training, courses taken and teaching experience, while other studies examined teachers' results in various mathematical competence tests (Hill, Rowan & Ball, 2005). According to Hill and colleagues (2005) the problem in this research programme "remains [the] imprecise definition and indirect measurement of teachers' intellectual resources and, by extension, the misspecification of the causal processes linking teacher knowledge to student learning" (Hill et al., 2005, p. 375). Adding that, "Effectiveness in teaching resides not simply in the knowledge a teacher has accrued but how this knowledge is used in classrooms" (Hill et al., 2005, p. 376).

A third research programme takes a different approach, investigating the *mathematical knowledge for teaching* held by the teachers. This programme initiated by Shulman and colleagues, differentiates between mathematical knowledge that any educated person can hold and the mathematical knowledge that teachers should hold; it reframed the study of teacher knowledge and it was largely embraced by the research community (Ball, Thames & Phelps, 2008). In a 1986 article, Shulman points to the fact that the cognitive psychology concerned with learning, had focused its research primarily on the student's point of view and Shulman expressed a need to be asking questions about how teachers learn. He centralized questions such as "How does the successful college student transform his or her expertise in the subject matter into a form that high school students can comprehend?" (Shulman, 1986, p. 8). Shulman proposed three areas of content knowledge for teaching: 1) subject matter content knowledge, 2) pedagogical content knowledge and 3) curricular knowledge; among which pedagogical content knowledge (PCK) has received the most attention. This type of content knowledge is defined as a knowledge that succeed subject matter knowledge, namely knowledge on subject matter *for* teaching which includes "in a word, the ways of representing and formulating the subject that makes it comprehensible to others" (Shulman, 1986, p. 9). Shulman further points to the results of the aforementioned emphasis on *student learning* within the research community, as key components of PCK, as knowledge regarding what makes some tasks difficult while others easy is an essential part of PCK (Shulman, 1986). The concept of PCK has subsequently been taken up and developed by many researchers, along with strategies in regards to measuring and comparing teachers' knowledge in this area.

Among these were Bromme, whom in a 1994 article presented a topology of areas of knowledge necessary in the teacher's practice (Bromme, 1994). This topology is an extension of Shulman's areas of knowledge for teaching and proposes different but interconnected fields of knowledge, constituting in all, the teacher's professional knowledge. In particular, Bromme distinguishes between content knowledge about mathematics as a discipline and school mathematics knowledge while also adding philosophy of school mathematics, which refers to the teacher's view on the epistemological foundation of mathematics (Bromme, 1994).

In a 2008 paper, however, Hill, Ball and Schilling asserts that the existence of the area of knowledge we call pedagogical content knowledge, have been assumed in the field of research and they report that the scholarly evidence concerning what this knowledge really is and how it affects students' outcome is still lacking. They also stress that methods of measuring pedagogical content knowledge has yet to be developed. In the 2008 article, Hill et al. presents an area of teacher knowledge called *knowledge of content and student,* and its relation to Shulmans pedagogical content knowledge is defined and delimited, with the aim of developing a large-scale method of measurement (Hill et al., 2008). Figure 1 below, shows the map of *mathematical knowledge for teaching* constructed by Hill and colleagues. Upon this, the researchers constructed multiple task items, presenting various teaching situations to a group of respondents consisting of a large-scale sample of teachers. A central question in this study was whether *knowledge of content and student* could be identified through the items and thus, said to exist. The study concluded that teachers "do seem to hold *knowledge of content and students*" (Hill et al., 2008, p. 395), but that, despite a thorough conceptualization and distinction from other areas of knowledge, it was found difficult to measure (Hill et al., 2008).

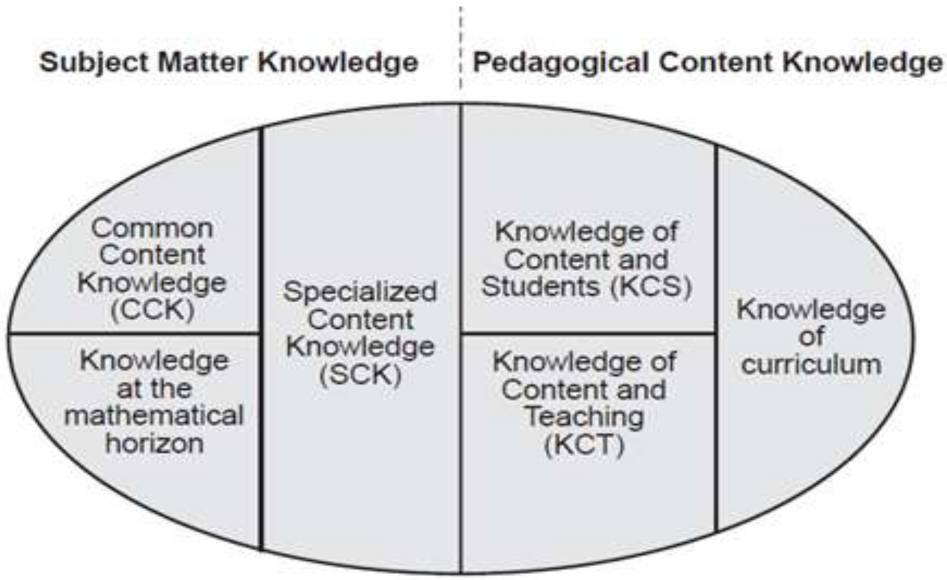


Figure 1: Ball and colleagues map of mathematical knowledge for teaching (Hill et al., 2008, p. 377)

Durand-Guerrier, Winsløw and Yoshida (2010) also brings up the questions "what does a mathematics teacher need to know, and how should preservice education prepare future teachers?" (Durand-Guerrier et al., 2010, p. 1) pointing to the fact that these questions remain unanswered. With quite a different approach to the solution than Hill and colleagues, they however also point to the root of the problem being a lack of methods for modelling and thus for describing and assessing teacher knowledge (Durand-Guerrier et al., 2010). Durand-Guerrier and colleagues propose a method for assessing mathematics teacher knowledge based on the concept of modelling mathematical activity within the Anthropological Theory of didactics.

It appears that no agreement or consensus presents itself in the literature on the subject regarding theoretical models or methods related to the previous stated questions and that this has created what Durand-Guerrier and colleagues calls a "black hole" (Durand-Guerrier et al., 2010, p. 2).

1.2 Aim and Structure of the Thesis

Based on the initial question and the above outline of the *status quo* within the research field, this thesis will explore the method to access and asses teacher knowledge, proposed by Durand-Guerrier, Winsløw and Yoshida (2010), namely that of *hypothetical teacher tasks*. Specifically, the method will be explored related to the theme of derivative functions in Danish high schools. The first part of thesis is guided by the following questions:

How is mathematical knowledge for teaching, perceived within the framework of the Anthropological Theory of Didactics?

How can one measure teacher knowledge, based on the principles of the Anthropological Theory of Didactics? Specifically what method are Durand-Guerrier and colleagues suggesting?

These questions are answered in chapter 2, which constitute the thesis' theoretical basis. Upon this, the thesis' research questions are presented in chapter 3, followed by an outline of the study's methodology in chapter 4.

The second part of the thesis aims to serve as a basis for answering the thesis' Research Question 1. This includes a subject matter didactical analysis of the chosen theme of differential calculus, which in particular entails, exploring how this theme can be perceived within the framework of the Anthropological Theory of Didactics (chapter 5) and upon this an analysis of five hypothetical teacher tasks, designed by the author (chapter 6).

In the third part of the thesis the results from an empirical study is analysed, which aims to serve as a basis to answer Research Question 2. The purpose of the empirical study is two-fold: the hypothetical teacher tasks was answered by 9 participants, four high school teachers and five university students, to investigate firstly, the participants' teacher knowledge related to the theme and secondly, the potential of the designed hypothetical teacher tasks, in particular; if the tasks conveyed the participants varying teaching experience (chapter 7).

The results of the empirical study is discussed in chapter 8, particularly including considerations concerning the data collecting methods and the characteristics of the designed hypothetical teacher tasks. The thesis' conclusion is presented in chapter 9.

2 Theory

In this chapter, the theoretical basis of the thesis is presented. The theoretical basis is comprised of the Anthropological Theory of Didactics and includes the key concepts of mathematical and didactical organisations, which are to be presented, in detail, in section 2.2 and 2.3 respectively, with the aim of determining how *mathematical knowledge for teaching,* is perceived within the Anthropological Theory of didactics. Upon this, the method for accessing and assessing teacher knowledge, employed in this thesis, is elaborated in the final section of the chapter (2.4).

2.1 The Anthropological Theory of Didactics

The Anthropological Theory of Didactics (henceforth abbreviated: ATD) is a research programme, initiated by the French didactician Yves Chevallard, for analysing and evolving mathematics education (Holm & Pelger, 2015). ATD constitutes a branch in the *epistemological programme* (Barbé, Bosch, Espinoza & Gascón, 2005), which originates from the work of Guy Brousseau and the research paradigm developed in the 1970s based on the Theory of Didactic Situations (Bosch & Gascón, 2006).

The central object of ATD is the learning and teaching of mathematics relative to the institutions in which, these processes take place (Bosch, Chevallard & Gascón, 2005). A fundamental part of the epistemological programme is the conviction that didactics research must incorporate *epistemological reference models* to be used as a mean to avoid "Spontaneous conceptions of mathematical knowledge that researchers could assume" and thus being subject to the institution of interest (Bosch & Gascón, 2006, p. 61). The notion of reference models highly relates to the concept of "didactic transposition", which will be explained thoroughly in the next subsection (2.1.1).

ATD proposes a use of epistemological models as tools to describe mathematical knowledge (Bosch, Chevallard & Gascón, 2005). This is based on the central idea of ATD for studying the phenomena of teaching and learning, namely the idea that one can model all human activity related to different types of tasks i.e. mow the lawn, set the alarm and measure your heartrate. Accordingly, it is possible to model mathematical knowledge by perceiving mathematical activity as a human activity, in which certain types of tasks or problems are being studied (Bosch & Gascón, 2006). Still, these tasks are to be construed as embedded in a context, an institution, which the researcher must incorporate in his analysis. This is what the concept of reference models entails, and as mentioned, these are related to the process of didactic transposition; indeed, reference models are actually justified by this phenomenon (Bosch & Gascón, 2006).

2.1.1 Didactic transposition

The theory of *didactic transposition* concerns the 'moving' of knowledge between institutions and between the actors in the didactic process (Bosch & Gascón, 2006). A key aspect related to the didactic transposition is the recognition that the *knowledge to be taught* in schools is a product of a process, taking place *outside* school, originating from the institutions in which the mathematical knowledge is produced. The didactic transposition entails an *adaption* of the so-called *scholarly knowledge* i.e. the knowledge as it is produced by mathematicians to the relevant teaching institution. This is a process of selection, delimiting, reorganising and a redefining of knowledge and thus enable the teaching of this, but simultaneously creating various limitations. For example, the phenomenon of *monumentalistic* education, where the adapting of knowledge has resulted in a removal of the motivation and justification of the knowledge (Bosch & Gascón, 2006).

The actors in the part of the transposition, taking place outside school, are collectively called the *noosphere* and includes politicians, teachers and professionals within the discipline. The result is the *knowledge to be taught* (Bosch & Gascón, 2006). The didactic transposition furthermore include the *knowledge actually taught* and the *knowledge learned* as figure 2 illustrates. The last two 'steps' in the figure reflects the role of the teacher and the student, respectively (Bosch & Gascón, 2006).

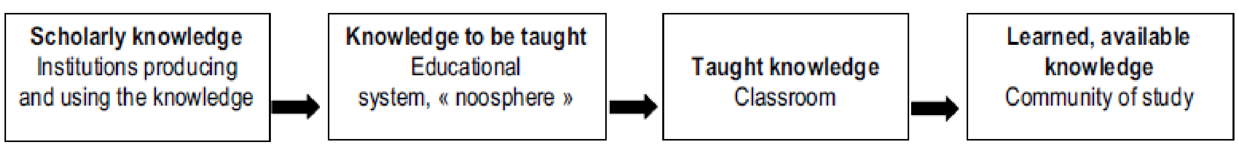


Figure 2: The didactic transposition process (Bosch et al., 2005, p. 1257, edited).

The selected *knowledge to be taught*, communicated to the teachers through curricula, available textbooks and official exams, creates conditions and constrains in the teacher's praxis as well as a portion of freedom and there will naturally be a difference between the knowledge to be taught and the knowledge actually taught (Barbé et al., 2005). The last step of the transposition concerns the actual teaching situation taking place in the classroom (Bosch & Gascón, 2006).

For the didactician, acknowledging the didactic transposition and the need to study it in order to understand what is going on in concrete teaching situations, means to incorporate it when studying mathematics education and mathematical activity and thus the field of research widens extensively (Bosch & Gascón, 2006). To meet this objective, the reference model is an important tool – as Bosch and Gascón writes (2006):

> When looking at this new empirical object that includes all steps from scholarly mathematics to taught and learnt mathematics, we need to elaborate our own 'reference' model of the corresponding body of mathematical knowledge (p. 57).

This elaboration enables the researcher to capture in full the limits and restrictions within a teaching institution and to capture why something is done in a certain way and not another and thus "contributes to explain, in a more comprehensive way, what teachers and students do when they teach, study and learn mathematics" (Bosch & Gascón, 2006, p. 53).

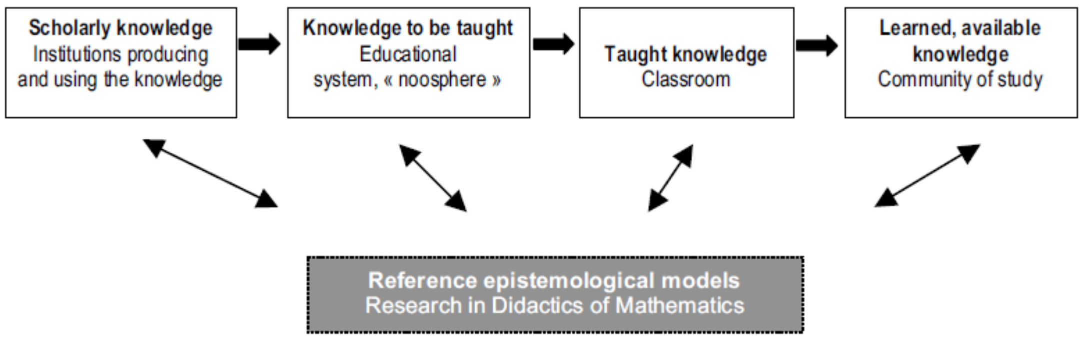


Figure 3: The reference praxeological model incorporates every level of the didactic transposition (Bosch et al., 2005, p. 1257).

There exists no general or widespread standard reference model for the bodies of mathematical knowledge that are taught in secondary school and thus it is the researcher's job to develop and validate these. The tool of praxeological reference models proposed by ATD was in fact, introduced in order to manage this new empirical object (Bosch & Gascón, 2006).

2.2 Mathematical Organisation

> Mathematics teaching and learning situations are characterized by the construction and sharing of *practice* and *knowledge* of a mathematical kind. (Miyakawa & Winsløw, 2013, p. 4)

Within ATD such practice and knowledge are – in the most elementary version and in one word - called a *praxeology*, which is described by Chevallard (2006) as "The basic unit into which one can analyze human action at large" (From Bosch & Gascón, 2006, p. 59). In the following, the focus will be to describe what a mathematical praxeology entails, but as stated in the section 2.1, the notion of a praxeology can be applicated to all human activity.

A mathematical praxeology takes as its base a *type of task* (denoted T) (Barbé et al., 2005). For example, consider the mathematical task:

t: Let $f(x) = x^2 + 7x + 18$ and determine $f'(x)$

Such a task belongs to a more general class of types of tasks on the form:

$t \in T$: Determine $f'(x)$ when given the algebraic expression for $f(x)$

For every type of task, there belongs a *technique* (denoted τ) or possibly multiple techniques, which is used in order to solve the task (Durand-Guerrier et al., 2010). For example, the mathematical technique associated with the type of task T comprises of algebraic manipulations combined with a certain algorithmic procedures.

Types of tasks and corresponding techniques constitutes the *praxis* or *practice block* of a praxeology. However, as expressed by Chevallard (2006) "No human action can exist without being, at least partially, 'explained', made 'intelligible', 'justified', 'accounted for', in whatever style of 'reasoning'" (Bosch & Gascón, 2006, p. 59). Hence, there will always exist some sort of justification related to the methods used and thus a practice block always relates to some *logos* or *knowledge block*. According to ATD, such a knowledge block is likewise comprised of two parts, called *technology* (denoted θ) and *theory* (denoted Θ), both integrating the purpose, explanation and justification of the practical block (Bosch & Gascón, 2006).

The technology part encompass "The important characteristic of human activity to allow for coherent discourse about tasks and techniques" (Durand-Guerrier et al., 2010, p. 4). Thus, the technology embodies our description of the tasks and techniques. This imply, that once you set out to describe in full length a technique employed, you are necessarily operating on a technological level since it will necessarily be done so through a certain discourse surrounding the particular task and the tools to solve it. The theory is the incorporation and organisation of the discourses surrounding the techniques we use during the study and solving of mathematical problems, such that it forms a coherent net of explanations and justifications for our actions (Durand-Guerrier et al., 2010). In short: "Praxis […] entails logos which in turn backs up praxis" as stated by Chevallard (Bosch & Gascón, 2006, 59). However, in their 2010 article Durand-Guerrier and colleagues adds that a praxis in some cases "may exist independently of the techno-theoretical block" (Durand-Guerrier et al., 2010, p. 5). This relates to the assertion that human activity can be performed without any accompanying description or justification (technology) and further, some tasks and related techniques are associated with technological elements but is not justified further on a theoretical level (Durand-Guerrier et al., 2010). A praxeology is thus comprised of four elements: a type of task, a technique, technology and theory forming the family $(T, \tau, \theta, \Theta)$.

Praxeologies often appear in coherent families. Such collections of praxeologies combines to form *mathematical organisations* (henceforth abbreviated: MO) (Miyakawa & Winsløw, 2013). A MO can assemble in different manners. In order to provide a strong and precise tool in varying situations, ATD differentiates between *punctual* (a praxeology), *local, regional* and *global* MOs to describe increasingly complicated situations (Bosch & Gascón, 2006). Figure 4 illustrates the four types of organisations. The punctual organisation builds upon a single type of task with an associated technique. Increasing the collection of tasks solvable, with various techniques that can be explained and justified with reference to the same technology and theory, creates a local organisation $(T_i, \tau_i, \theta, \Theta)$. A collection of multiple punctual praxeologies all sharing the same theory, forms a regional organisation $(T_i, \tau_i, \theta_i, \Theta)$ and lastly, a collection of local and regional organisations all sharing the same theory, create a global MO (Durand-Guerrier et al., 2010). The local, regional and global organisations corresponds to mathematical themes, sectors and domains, respectively (Bosch & Gascón, 2006).

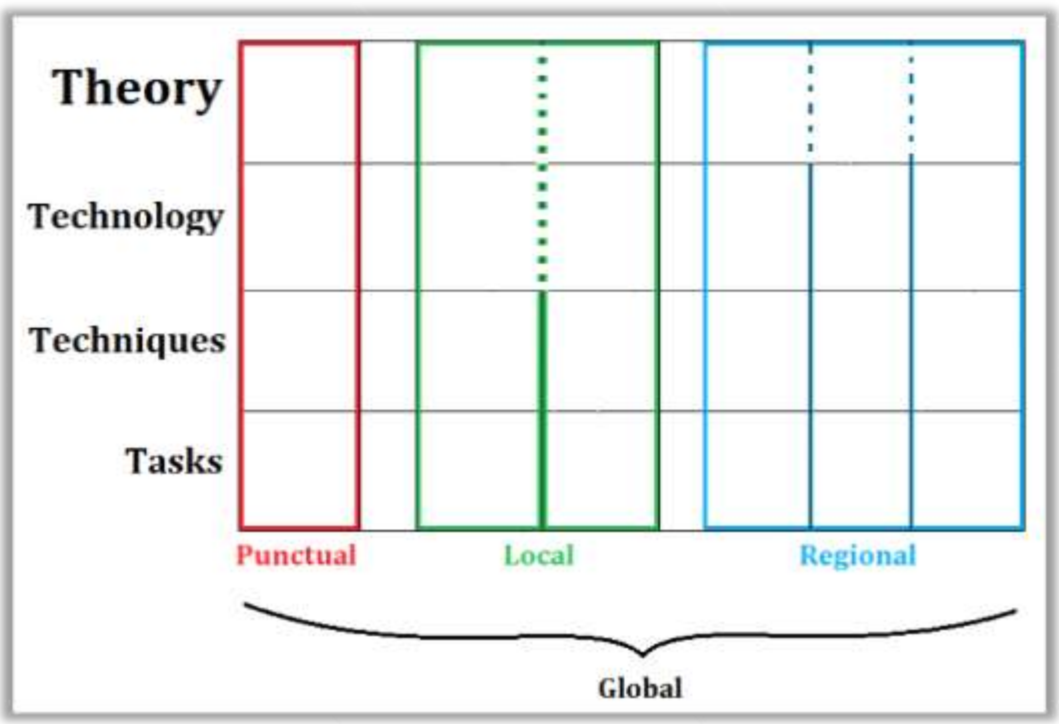


Figure 4: Illustrating punctual, local, regional and global organisations.

For example, task T given above, creates a punctual praxeology belonging to the theme *differentiation*, which belongs to the sector *differential calculus*, which in turn is a part of the domain *mathematical analysis*. A description of a regional mathematical organisation can constitute a reference model for the researcher (Durand-Guerrier et al., 2010).

2.3 Didactical Organisation

As it is the case with knowledge in a general sense, mathematical praxis and knowledge in the form of MOs are not absolute and rigid entities. Knowledge will always be a product of certain processes to which, it is related (Bosch & Gascón, 2006). So what creates and shapes the mathematical organisations created in the classroom?

The answer is the *process of study*, which in turn is to be modelled and understood as *didactical praxeologies*, which combines to forms didactical organisations (henceforth abbreviated: DO)(Bosch & Gascón, 2006). A DO models the teacher's activity based on the teacher's tasks (didactical types of tasks are denoted T^*) and is often directly linked to MOs; indeed, a DO can be regarded as the 'answer' to the question "How does one establish a MO [for students]" (Durand-Guerrier et al., 2010, p. 5). Such a relation between DOs and MOs is illustrated with the following example. Consider the task:

T^*: Plan a teaching session on the determination of functions monotonicity properties, using its derivative function.

This didactical task serves as a base for a local DO. However, this DO explicitly relates to a local MO build upon types of tasks such as:

T: Determine the monotonicity of f given the algebraic expression of f'.

The DO will in fact depend on such a corresponding MO. Furthermore, in a concrete teaching situation, for example the practical execution of an answer to T^*, the MO created will naturally depend on the DO, as the taught knowledge transpose to the learned knowledge. This will possibly affect the teacher to modify the DO, for example if the DO has created 'misconceptions' and hence, the MO created affects the DO.

Though didactical praxeologies are of growing interest alongside the interest in teacher's role in the didactic process (Barbé et al., 2005), the experience with DOs and modelling of teaching activity according to the principles of ATD is not extensive in the literature (Durand-Guerrier et al., 2010). Therefore, an exact conceptualization of DOs related to MOs is not widespread. Durand-Guerrier and colleagues (2010) propose the following model:

> A local DO consists of a family of punctual DOs, which in a teaching activity will be enacted consecutively in time […] Some of the task types (defining the punctual DOs) relate directly to a MO, for instance a DO task type may be to construct a question for students that will enable them to work on the MO. The teacher employs, to solve the

task of a given punctual DO, a technique which is at least potentially *explained* by the overarching technology; the latter will then also refer to the MO in case the task type is related to it. (p. 6)

A teacher's activity during a teaching session should thus, be considered as creating a local DO. This will consist of many individual tasks with corresponding techniques but will be united in a local DO through the overall goal of the session which, combined with the technology of a possible related MO, constitutes the technology.

***

Based on the concepts of MOs (section 2.2) and DOs (section 2.3), mathematical teacher knowledge is thus perceived as technology and theory belonging to MOs and DOs within the framework of ATD (Miyakawa & Winsløw, 2013).

2.4 Hypothetical Teacher Tasks

To access and assess teacher knowledge and to do this precisely, Durand-Guerrier and colleagues (2010) suggest using an *operational epistemological model*, a model based on the principles of ATD, intended to model DOs related to specific MOs (Durand-Guerrier et al., 2010).

The model proposed is an activity-oriented model based on *hypothetical teacher tasks* (henceforth abbreviated: HTTs). The idea is to formulate tasks that are meaningful for the teacher, but simplistic, as the tasks are 'removed' from the conditions or constrains associated with the actual teaching practice. Furthermore, many of the teachers' DOs are *not* related to or dependent on mathematical didactical knowledge, but simply based on tasks regarding pedagogy and organisation e.g. management of time during a lesson. Therefore, since it is the goal to access and assess in particular the teachers' DOs related to specific MOs, it is necessary to create a situation by means of the task that minimizes the involvement of DOs related to pedagogical or organisational tasks. In an ordinary teaching situation, a teacher's answer to a student's question will, for example depend on time, the immediate goal of the session etc. The name *hypothetical* stems from this removal of the task from a 'real life' setting and into a frame with less and optimally, no factors in play concerning pedagogy and organisation However, the tasks should maintain a certain characteristic of relevance for the teachers (Durand-Guerrier et al., 2010). Furthermore, the assessment of the answers to the HTTs, entail a construction and employment of reference praxeologies of the DOs as well as MOs related to the HTTs in accordance with the principles of ATD.

3 Research Questions

Upon the theory presented in chapter 2, the thesis' research questions will be presented. The method proposed by Durand-Guerrier and colleagues, presented in section 2.4, to access and assess mathematics teachers' knowledge, namely that of HTTs, will be explored in this thesis with the main aim of answering the following two questions.

Research question 1:

> *Based on a subject matter analysis of the theme of differential calculus in Danish high school, how can one model non-trivial teacher knowledge related to this theme in terms of HTT's?*

This method employed for this research question is elaborated in the next section; however answering this question ultimately produces *actual* HTTs. These HTTs will be utilized in an empirical study seeking to answer the following research question.

Research question 2:

> *Do the participants' answers to the HTT reflect their different amounts of teaching experience? In what way?*

The formulation 'non-trivial teacher knowledge' in research question one, needs some elaboration. In this context, the meaning of this formulation is considered as two-fold. On the one hand, it refers to mathematical knowledge (i.e. techno-theoretical components of MOs) associated with tasks which are non-typical in the transposition of the theme of differential calculus to Danish high schools and thus, it refers to knowledge related to mathematical tasks which are not commonly taught or widespread in Danish high schools. Simultaneously it refers to didactical knowledge (i.e. techno-theoretical components of DOs) associated with didactical tasks which relates to some MO. Meaning that the term non-trivial teacher knowledge excludes knowledge related to didactical tasks which could be relevant to pose to any teacher.

4 Methodology

In this chapter, I will outline the methodology employed to answer the thesis' research questions.

4.1 Research Question 1

As is explicit in Research Question 1, a subject matter didactical analysis of the theme of differential calculus is the first step. This will be presented in chapter 5. It aims firstly, at providing an insight into the possible structure of a mathematical organisation constituted by this theme. Secondly, it aims at analysing the knowledge block associated with such an organisation. This analysis builds largely on the presentation of the theme given in an introductory analysis book by Lindstrøm (2006) and seeks to uncover the more implicit aspects of the theory's inherent mathematical objects as well as the interconnections between the theory's various definitions and theorems. Lastly, the subject matter analysis seeks to explore the transposition of the subject matter to Danish high schools. In this respect, it is of particular importance to identify the elements that are *not* transposed to high school but 'left at university', the possible consequences related to the *rationale* of the subject matter and the justification of the techniques associated with 'typical tasks' of high school curriculum.

Based on the above analysis, five HTTs was designed. These are presented in chapter 6 along with an *a priori* analysis of each subtask. The design process necessarily included a selection of some tasks, while others were abandoned; a rather brief literature study, outlined in the introduction of chapter 6, compelled to select task that were placed in a graphical setting. Furthermore, as explained in section 2.4, the goal when using HTTs is to access and assess the teachers' DOs related to *specific* MOs. Hence, the designed tasks all relates to specific punctual MOs and the *a priori* analysis shall make this relation explicit. Regarding the inherent mathematical tasks, the analysis will uncover, which mathematical techniques are necessitated by the tasks, and identify the technology (θ) and theory (Θ) associated with these techniques.

The techno-theoretical components related to the mathematical techniques are thus identified, while those related to the didactical tasks and techniques are not. This difference in treatment originates from the current absence of established and widely acknowledged theoretical ground within which the activities finds its explanations and justifications. The impossibility of creating complete reference models related to didactical tasks is thus an expression of the scarcity of theoretical models within the teaching profession (Durand-Guerrier et al. 2010). The choice of one technique over another does not necessarily reflect some entrenched theoretical knowlegde, but could just as well reflect the respondent own personal beliefs about teaching (Miyakawa & Winsløw, 2013).

'Standard answers' to the HTTs was developed upon the *a priori analysis*, comprised mainly of the key didactical and mathematical techniques identified. The tasks together with the established model of associated technique, technology and theory constitutes the thesis' answer to Research Question 1.

4.2 Research Question 2

To answer research question 2 the HTTs was answered by, nine participants, which naturally divided into two groups of respondents. Below is a description of the groups, followed by an outline of the method for collecting data.

4.2.1 The Respondents

The study involved nine divided into two groups. The first group consisted of five university students (identified as S5-S9) for which, mathematics was their minor subject and they were all at the stage in their education of finishing their mathematical studies and thus, they have accumulated the mathematics knowledge that is required to teach in secondary school. The students were enrolled in a course called "Mathematics in a Teaching Context" offered at the University of Copenhagen, and therefore it is asserted that these participants have an interest in the teaching of mathematics. However, none of the participant in this group have any teaching experience in secondary school. The second group consisted of four in-service high school teachers (identified as T1-T4). This group is more diverse internally. One of the teachers was still a university student, though with six full years of teaching experience in secondary school (T2). One of the teachers had studied mathematics as minor subject (T4), while the remaining three teachers had studied mathematics as a major subject. Lastly, their specific teaching experiences varied in terms of which levels, i.e. some had much experience with teaching A-level mathematics while one had never taught A-level mathematics and furthermore, their teaching experience varied related to the specific theme of differential calculus.

All the participants had, for this purpose, one important thing in common: they had all followed courses at university covering the theme in focus. At the University of Copenhagen, the theme of differential calculus is treated in the first semester, primarily in the course "Introduction to the Mathematical Sciences" (treating differentiability of functions of one variable) (Introduktion til de matematiske fag, 2016). This particular course is also included in the university's course description for students taking mathematics as a minor

(Sidefag i matematik, n.d.). In a more general context, in the guidelines settled by the Ministry of Education and Research for the universities providing teacher educations, the theme of Calculus is included and related to which, it is stated: "The candidate must have solid knowledge of the following mathematical themes" (Retningslinjer for universitets uddannelser, 2006). Thus, the participants' mathematical education may vary, but they have all attended courses at university covering the theme on which the HTTs centre.

4.2.2 Collecting data

Circumstances regarding accessibility of the participants meant that the method for collecting data varied between the two groups.

The teachers worked with the HTTs individually and answered the tasks in writing within a timeframe of fifty minutes while the researcher (author) was present, sitting across from the participant. These meetings was also audio recorded. The purpose of the researcher being present was to encourage and provide a more natural scene for the teachers to "think out loud" when working with the tasks and thus, to ensure (in a higher degree) access to the teachers' technology and theory.

The university students on the other hand, was accessible collectively, in an extraordinary teaching session in the course "Mathematics in a teaching context". Under these circumstances, it was chosen to place the participants in pairs, to create an environment, which encouraged "thinking out loud", more exactly, to encourage the student participants to share their thoughts, regarding the tasks and their solutions, with each other. Time showed that the number of university students participating, was limited to five and thus, they were placed in groups of two and three. The students were asked to consider each task individually and give a preliminary answer and first then, discuss the task with the co-student(s), to ensure an insight into each participant's ability to mobilize appropriate techniques. The students' conversations was audio recorded and they were also given 50 minutes (exceeding the official timeframe of the session by 5 minutes and two of the participants did not have the opportunity to stay longer, which meant that their timeframe was limited to 45 minutes).

4.2.3 The A Posteriori Analysis

To answer Research Question 2, the participants' responses to the HTTs was analysed *a posteriori* (chapter 9). This entailed specifically, an identification of the specific mathematical and didactical techniques activated by the participants and, based on the former; identifying the participants' technology and theory. The *a priori* analysis of the HTTs served as a reference model in the identification process. For the purpose of creating an initial overview of the results, the analysis of the participants' answers was compared to the 'standard answers' developed in the *a priori* analysis of the HTTs and based on this comparison, each answer was given points varying between 0-3 in the following way:

0 points:	The participant did not answer or provided a wrong answer.
1 point:	The participant's answer included one or few correct elements.
2 points:	The participant's answer included multiple correct elements.
3 points:	The participant's answer covered all *a priori* identified aspects and possibly additional relevant elements.

It is stressed, that the purpose of the points was to provide an overview of the participants' performances on the HTTs and is quite superficial. Furthermore, since the standard answers are constructed through an analysis of the tasks performed by the author and in cooperation with the thesis supervisor, some subjectivity is consequently related to the distribution of points; indeed, the coding of the participants' answers bear the same subjectivity.

Lastly, in order to provide a full answer to Research Question 2, considerations regarding the character of the HTTs, the data collecting methods and various limitations of the study, is discussed.

5 Subject Matter – Didactical Analysis

In the presentation in section 2.2, it was emphasized how all human activity can be described as organisations consisting of a coherent family of praxeologies and that this holds for mathematical activity as well. Such organisations can vary in size and complexity. The aim of this chapter is to explore the mathematical organisation, which encapsulate the theme of differential calculus and in particular, to explore the theory that unifies such an organisation and justifies the techniques associated with the generating tasks. Furthermore, the goal of the chapter is to clarify how such an organisation is transposed to high school, i.e. what is the *knowledge to be taught*.

5.1 Algebraic and Topological Organisations in Analysis

The aim of this section is to clarify how an epistemological reference model of differential calculus, would present itself and to some extent, its relation to other sections of analysis taught in secondary school. A 2015 article by Winsløw will constitute the starting point.

Inspired by the work of Barbé et al. (2005) regarding the restrictions of the teacher when teaching limits in Spanish high schools, Winsløw proposed in a 2015 article, six local organisations, encompassing the themes of limits, derived functions and integrals. In their article, Barbé and colleagues propose a reference model on the subject of limits consisting of two separate but connected local MOs; an *algebraic* MO called the "Algebra of Limits" and a *topological* MO called the "Topology of Limits" (Barbé et al., 2005). Based on these results, Winsløw suggests that the same structure is detectible when considering other elements of calculus, pointing to the integral and the derivative function. Furthermore, Winsløw stresses how a connection exists between these three pairs of local MOs – each pair constituting a reginal MO.

Figure 5 illustrates the simplified reference model proposed by Winsløw consisting of six local MOs. For an elaborate discussion of MO_1 and MO_2, regarding limits of functions see Barbé et al. (2005) and for a more in depth discussion of MO_5 and MO_6 see Winsløw (2015). In this context, mainly MO_3 an MO_4 will be of interest. However, the names of the six organisation given in this scheme will be preserved throughout, in order to make referencing clear. The main structure, as it also appears in the figure below, is made up of a division, but as the arrows in the middle column suggests, the MOs are connected.

Object	Existence/"topology"	Computation/"algebra"
Limit of function f at point $a \in [-\infty, \infty]$	**MO$_2$** T_{21}: Does $\lim_{x \to a} f(x)$ exist? T_{22}: Justify rules and properties $\rightarrow$	**MO$_1$** T_{11}: Find $\lim_{x \to a} f(x)$. THEORY BLOCK
Derivative of function f	**MO$_4$** T_{41}: Does f' exist? Where? T_{42}: Justify rules and properties $\rightarrow$	**MO$_3$** T_{32}: Find f'. THEORY BLOCK
Integral of function f on interval $[a,b] \subseteq [-\infty, \infty]$	**MO$_6$** T_{61}: Does $\int_a^b f(x)\,dx$ exist? T_{62}: Justify rules and properties $\rightarrow$	**MO$_5$** T_{51}: Find $\int_a^b f(x)\,dx$. THEORY BLOCK

Figure 5: Six local MOs constituting a simplified reference praxeological model (Winsløw, 2015, p. 203).

MO$_4$ concerns the topology of the derivative of a function f. Its base is comprised of types of tasks considering the existence of the derivative as well as tasks seeking to justify the differentiation rules and properties of the derivative function (Winsløw, 2015). In this thesis, MO$_4$ is considered as based on five *major* types of tasks:

$\mathcal{T}_{4.1}$: What is the derivative of f in a point $a \in D_f$?

$\mathcal{T}_{4.2}$: What is the derivative function f'?

$\mathcal{T}_{4.3}$: For a given f does f' exists? Where?

$\mathcal{T}_{4.4}$: Justify the properties of the derivative function.

$\mathcal{T}_{4.5}$: Justify the differentiation rules (e.g. $f\left(g(x)\right)' = f'(g(x)) * g'(x)$).

An example of a type of task regarding the general properties of the derivative is $T \in \mathcal{T}_{4.4}$:

T: Show that if $f'(x) = 0$ for all $x \in D_f$ then f is constant on D_f.

These types of tasks together with techniques for solving them constitutes the practice block of MO$_4$ and are unified by a knowledge block with its primary element being the definition of the derivative. However, some of the properties and results associated with differential calculus is in addition to the definition of the derivative also dependent on elements from other local organisations (Winsløw, 2015). First of all the definition of the derivative itself is strongly and explicitly related to the notion of a limit. MO_4, as figure 5 illustrates, is 'built' on MO_2, "The Topology of Limits" – in fact, a central type of task in MO_2 concerns the existence of limits and hence a type of task in this organisation is to determine whether the derivative exists in a given point for a given function. The corresponding algebraic organisation MO_3, "The Algebra of the Derivative Function", bases on *major* types of task regarding the determination of f' when existence is given (Winsløw, 2015):

$\mathcal{T}_{3.1}$: Given the algebraic expression of f, determine $f'(x)$.

Which often appear in the extended version of:

$\mathcal{T}_{3.2}$: Given the algebraic expression of f and a point $a \in D_f$, determine $f'(a)$.

For which the basic technique is comprised of algebraic manipulations. The technology entails a discourse about the correct way of computing derivatives, largely through suitable use of the differentiation rules. Within MO_3, these rules also constitute the theory i.e., the differentiation rules justifies the *praxis* of MO_3. Additionally, $\mathcal{T}_{3.1}$ also exist in the following extended version:

$\mathcal{T}_{3.3}$: Given the algebraic expression of f, determine the monotonicity of f.

For which, the technique is justified by the differentiation rules as well as the 'rules' concerning the properties of the derivative. The differentiation rules and the properties of the derivative are in turn, justified in the practice block of MO_4 and hence, the practice block of MO_4 provides the justification of the knowledge block of MO_3. The following type of task also belongs to MO_3:

$\mathcal{T}_{3.4}$: Given the graphical representation of f and a point $a \in D_f$, determine $f'(a)$.

This closely relates to $\mathcal{T}_{3.2}$, but is associated with techniques that are not algebraic in nature, but comprises of reading and interpreting graphs.

It should be stressed that that the reference model in figure 5 above, proposed by Winsløw (2015), is not asserted to be comprehensive or to be considered as constituting a reference model of all the subjects of analysis, taught in high school. The model is simplistic, however for the purpose of identifying primary challenges in the transposition of elements in the domain; the model is also sufficient. What the model does not encompass is, for example the local organisation generated by tasks concerning optimization (Winsløw, 2015). In section 5.3, we will return to the primary challenges identified by Winsløw regarding this transposition.

5.2 The 'Scholarly' Knowledge

In this section, the local organisations, MO_3 and MO_4, as they were defined in the former section, is explored further; in particular the knowledge blocks of the two MOs. However, since the knowledge block of MO_3 is justified in the practice block of MO_4, it could also be said, that the focus of this section is the entirety of MO_4. This section thus explores the main elements of the knowledge block of MO_4, as well as the results justified in the practice block. The analysis of the theory presented in the following, aims in particular at making explicit some of the hidden assumptions and consequences, and at providing an overall picture of the theoretical landscape surrounding the derivative and thereby contribute to the reference model of the subject.

Large parts of the presentation in this chapter bases on an introductory analysis book by Lindstrøm (2006), which is widely used at the University of Copenhagen. The book is written for teaching purposes, and therefore the content has been subjected to a didactic transposition process and hence, it is not 'scholarly knowledge' in a pure form. However, since it is a book addressed to university students, it is asserted that nothing is 'left out'. When investigating the book, only one exception to this assertion was found, which could be interpreted as stemming from an expectation from the author of the book, namely that the university students prior to attending university have been taught differential calculus (some version of it) in high school.

5.2.1 The Definition of the Derivative

The central element in the theoretical level of MO_4 is the rigorous definition of the derivative of a function. Preceding this, however, is a theory, which generates a whole other MO, namely MO_2 – the topology of limits. We shall see in detail, what is meant by MO_4 being "directly derived" from MO_2 (Winsløw, 2015, p. 200). Furthermore, the derivative function bears with it many additional underlying concepts such as the concept of a function, continuity and the concept of the real numbers, which are crucial for the rigorous definition, which in turn is central for the acknowledgement of any theory in today's mathematical realm. Through an examination of Lindstrøm (2006), a landscape of results and definitions appeared surrounding the definition of the derivative of a function f at a point $a \in D_f$. Figure 6 illustrates this landscape and shows how the various mathematical definitions and theorems appears interconnected. Not all the definitions and results in the figure are a part of the local

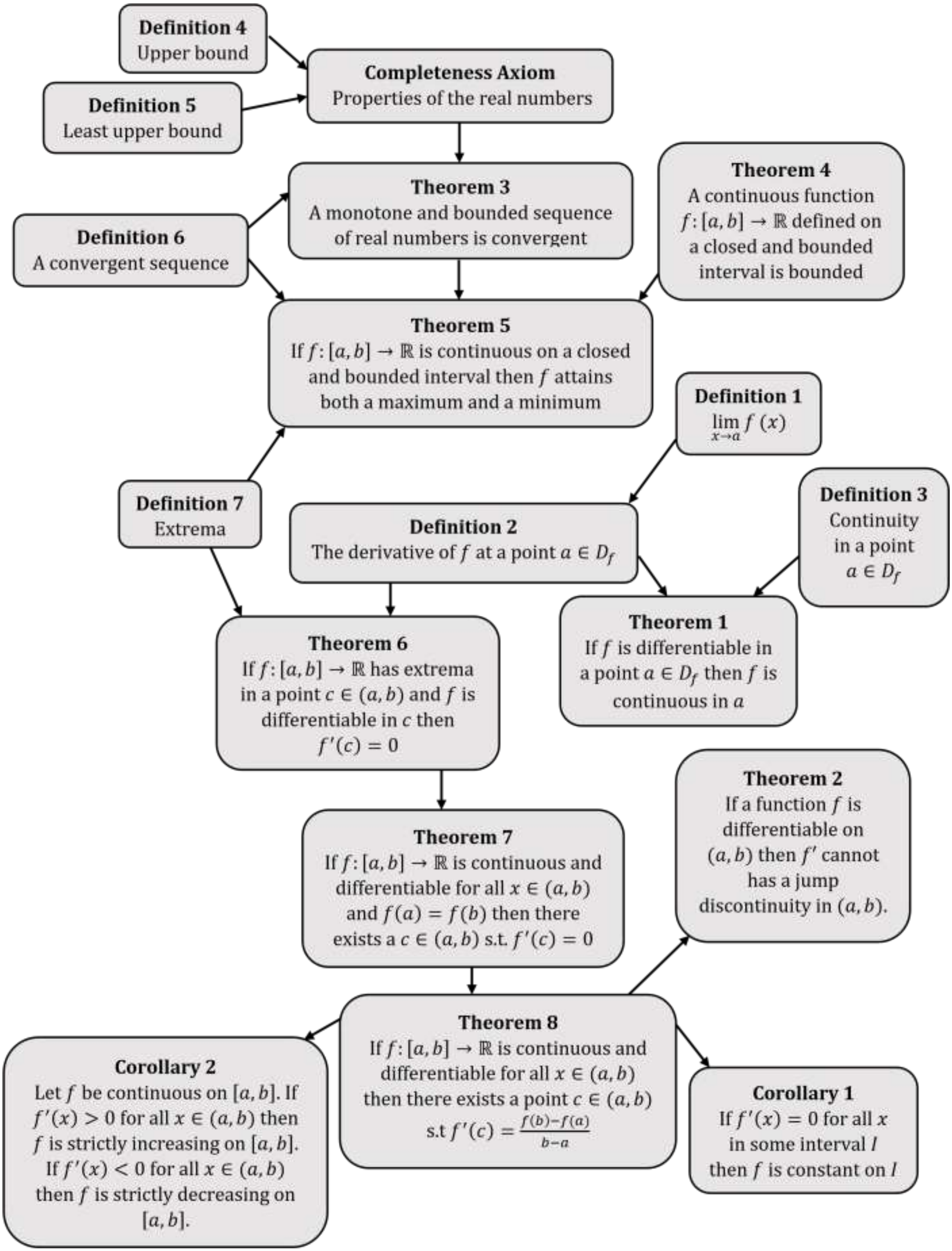


Figure 6: A landscape of definitions and theorems related to the derivative function.

organisation MO_4; some of the definitions generates other MOs. The elements, which belongs to the MOs presented in section 5.1, are labelled accordingly; the rules for differentiation is not included in figure 6, these are the subject of subsection 5.2.5. In the following, the concept of the derivative function as well as its rigorous definition, as it is stated in Lindstrøm (2006) (my own translation from Norwegian[4]) will constitute the starting point of the didactical analysis. From the definition of the derivative, we will move forward as well as back and zoom in to uncover results and concepts both preceding, and derived from, the definition of the derivative. The proofs for the results are included in the analysis as these convey the exact way in which the results are dependent upon each other. On a general note, let us first establish four ways in which the concept of the derivative can be approached. According to Zandieh (1997):

> The concept of derivative can be seen (a) graphically as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification, (b) verbally as the instantaneous rate of change, (c) physically as speed or velocity, and (d) symbolically as the limit of the difference quotient. (p. 65)

An addition, or a variation of (a) could be the interpretation of the derivative at a point $a \in D_f$ as the "limit" of the slope of the secant lines through $(a, f(a))$ and $\big(x, f(x)\big)$ as $x \to a$ (see figure 7).

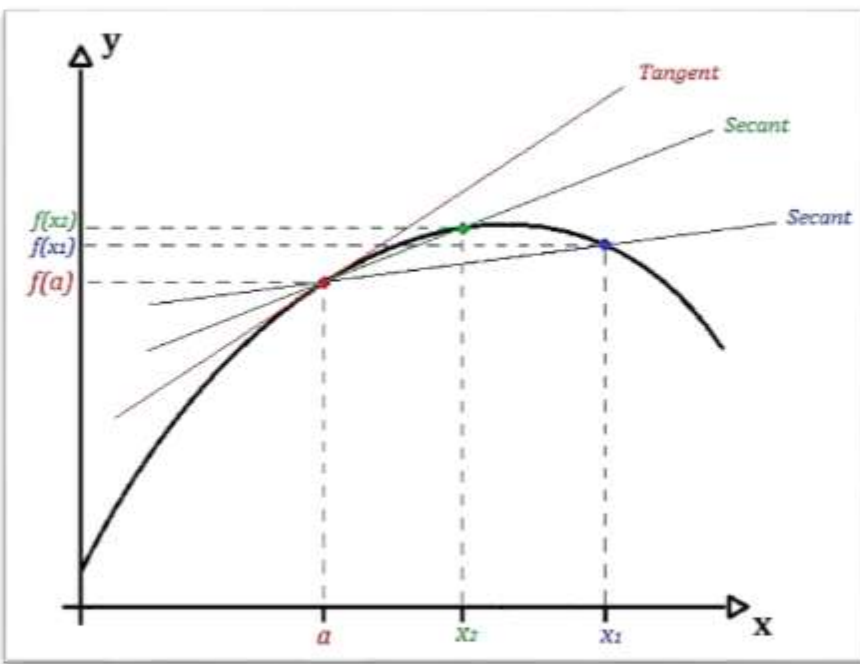


Figure 7: The derivative at ***a*** *is the "limit" of the slope of the secants.*

[4] All citations from Lindstrøm (2006) are translated from Norwegian by the author.

Leaving (b) and (c) for now, let us consider (d): The derivative considered symbolically as the limit of the difference quotient. In this description, two aspects of the derivative appear explicitly; *limit* and *difference quotient*. The difference quotient is the average rate of change of the dependent variable in respect to the independent variable over a given interval $[a, x]$. In symbols, we write:

$$\frac{f(x) - f(a)}{x - a} \qquad \text{or} \qquad \frac{f(x+h) - f(x)}{h}.$$

Where $x, a \in D_f$ and $|h|$ is the distance between x and a. (Zandieh, 1997). Furthermore, the concept of a limit is obviously inherent and, in fact central, in the definition of the derivative of f at a point $a \in D_f$. To make sense of the definition of the derivative, thus means, making sense of the concept of limits. The definition states (Lindstrøm, 2006, p. 231, own translation):

Definition 1 Assume that f is defined in a neighborhood around a point a. We say that $f(x)$ has limit b when x approaches a if the following holds. For any number $\varepsilon > 0$ (regardless of how small) there exists a number $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all x if $0 < |x - a| < \delta$. In symbols, we write:

$$\lim_{x \to a} f(x) = b$$

Note that the definition does not state whether x approaches a from *above* or *below.* In fact the definition involves both $a < x < a + \delta$ and $a - \delta < x < a$ and only when the limit from above and below are the same, do we say that the limit exists:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = b$$

Furthermore, the definition does not require f to be defined in a. From the requirement, $0 < |x - a|$ it is evident that only x near a is of importance, and that $x = a$ is not considered. Thus, a function does not need to be defined in a to have a limit for x approaching a. If f however, *is* defined in a and if f is continuous, the limit for x approaching a will always be $f(a)$ as we shall see in subsection 5.2.2. We can now state the definition of the derivative in a point $a \in D_f$ (Lindstrøm, 2006, p. 254, my own translation).

Definition 2 Assume that f is defined in a neighborhood around a point a (hence there exists an interval $(a - c, a + c)$ s.t. $f(x)$ is defined for all x in this interval). If the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, we call f differentiable in a. We write

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and call this *the derivative of f at the point a.*

Notice how two thing are going on. Firstly, definition 2 presents a (potential) property belonging to a function f and a point $a \in D_f$, namely the existence of the limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Secondly, if f holds this property, the definition ties an *object* to f and the point a, namely the number $f'(a)$. Due to the basic ontological difference, it is important to acknowledge the first part of the definition and not simply state that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Since the latter assumes existence of the limit, which consequently entails the false statement, that all functions defined in on interval around a are differentiable in a. It should thus be recognized how the definition of the derivative of f in a point $a \in D_f$ is far from trivial or self-evident as we are in reality dealing with the (potential) limit of a *new* function g, which is undefined at a, namely the difference quotient:

$$\lim_{x \to a} g(x), \text{when } g(x) = \frac{f(x) - f(a)}{x - a}.$$

For $x = a$, this quotient function g will have denominator equal to zero and hence be undefined at a and the limit will never simply be $g(a)$. Notice however, that the existence of the limit is not dependent on whether g is defined in a cf. Def. 1. On the contrary, though; the existence of the derivative of f in a point a, is clearly dependent on f being defined in a. Considering, the concept of the derivative at a point as suggested by Zandieh (1997), namely as the slope of the tangent line to a curve at a point, it is clear *why* f need to be defined in a, as f only has a tangent in a if it is defined in a. Let us briefly consider $f(x) = e^x$. For $a = 0$ the relevant difference quotient is:

$$g(x) = \frac{e^x - 1}{x}$$

That the limit exists for x approaching zero is not trivial at all, as the expression appears to approach '0/0' for $x \to 0$. A way to prove that the limit exists for x approaching 0 is by employing the *squeeze theorem* (which will not be proved here, see (Proof of the Squeeze Theorem, n.d.). It states:

The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$ containing c, except possibly for $x = c$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$

If $g(x) = \frac{e^x-1}{x}$ and choosing $f(x) = -|x| + 1$, $h(x) = |x| + 1$ and $(a, b) = (-1,1)$, we have, as figure 8 illustrates:

$$-|x| + 1 \leq \frac{e^x - 1}{x} \leq |x| + 1, \text{for } x \in (-1,1)\backslash\{0\}$$

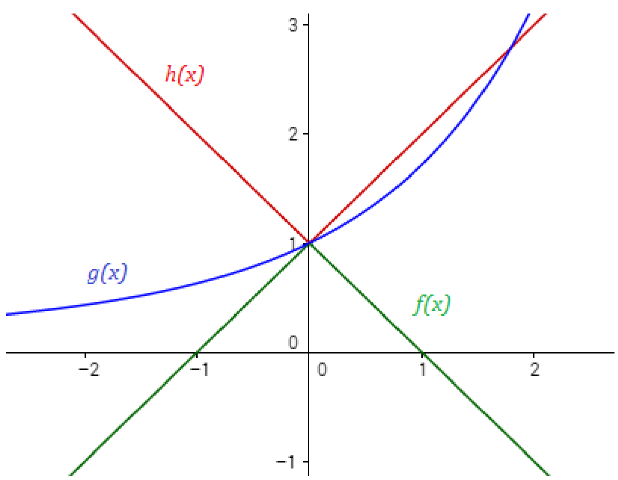


Figure 8: $g(x)$ 'squeezed' between $f(x) = -|x| + 1$ and $h(x) = |x| + 1$.

And the requirements of the theorem is satisfied, yielding:

$$\lim_{x\to 0} -|x| + 1 = \lim_{x\to 0}|x| + 1 = 1, \text{and thus} \lim_{x\to 0} \frac{e^x-1}{x} = 1$$

Notice how g appears to actually attain the value 1 for $x = 0$, which is known to be false. Thus, the graphical setting is not a reliable tool in this respect, as it might lead to the false argument: $\lim_{x \to 0} g(x) = g(0) = 1$.

Another way to write the derivative of f in a point; let us now call this point x_0, is (assuming that f is differentiable in x_0):

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If f is defined on an interval $I = [a, b]$ and the above limit exists for all $x_0 \in (a, b)$ then we call f a *differentiable function* on (a, b) and write (Zandieh, 1997, p. 65):

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \quad x \in (a, b)$$

Notice how f is said to be differentiable on (a, b) and not on the entire domain $[a, b]$. This is a consequence of the definition of limits – f is not defined in a neighborhood around a or b and thus the limit for x approaching a 'from below' and the limit for x approaching b 'from above' is meaningless.

Furthermore, the objects, $f'(x_0)$ and $f'(x)$, that the above formulas define, is of very different nature. The first defines a number and the second defines a function. In Lindstrøm (2006), this 'aspect' of the theory is not presented in detail and the second formula, the derivative *function*, is not defined, or distinguished from the derivative of a function in a point, explicitly. However, it is stated, shortly after a definition corresponding to the present subsections' Definition 2, that "… for the derivative *itself*, there exists multiple notations" (Lindstrøm, 2006, p. 254, my own translation, italics added), pointing to the notations:

$$f'(x), \qquad \frac{df}{dx}(x), \qquad D[f(x)]$$

It is possible that this is 'left out' due to the natural extension from the derivative in a point; in Definition 2, we assign to each x_0 in the interior of D_f (if the limit exists) a number $f'(x_0)$ which is exactly the mechanism associated with functions. Hence, the definition of the derivative *function* $f'(x)$, is a natural consequence of Definition 2.

A relevant aspect of the concept of the derivative should be included in this context. As mentioned above the derivative at a point and the derivative function are two – though inevitably related – very different objects. In fact, while the derivative at a point define an *object*, the derivative function immediately seems to define a *process.* In a general sense, however the concept of a function possess a duality; it can be viewed as a process, taking as input one value and returning another and it can be viewed as a static object – the result of a process (such as derivation). The derivative concept contains multiple such two-sided elements. Firstly, the difference quotient may be thought of as a process of dividing two objects, the result being a ratio and thus an object. Secondly, taking the limit of the ratio (the difference quotient) can be thought of as a dynamic process were x approaches some fixed number (or $\pm\infty$) but simultaneously, it can be thought of as an object, namely the limit itself (Zandieh, 1997). The process/object duality is by Sfard (1991) referred to as an *operational/structural* conception. According to Sfard, the operational understanding precedes the structural understanding, however processes are considered as actions performed on established objects (Sfard, 1991). Hence, when learning the concept of the derivative one needs to transition from an operational understanding to a structural understanding of the difference quotient, of limits and of functions and thus be able to consider these as processes as well as objects (Zandieh, 1997).

5.2.2 Continuity

Another central concept, tightly related to the differentiability of a function is continuity of a function, stated differently: an important property of a differentiable function *is* continuity. Let us consider the definition of continuity (Lindstrøm, 2006, p. 212, own translation).

Definition 3 A function f is contionous in a point $a \in D_f$ if the following holds: for any $\varepsilon > 0$ (regardless of how small) there exists a $\delta > 0$ such that when $x \in D_f$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

As opposed to the definition of limits, the definition of continuity in a point a requires that f is defined in a. Aside from this, the two definitions look very similar. They are connected in the following way. Consider $f: [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. Then the following holds (Lindstrøm, 2006, p. 236):

$$f \text{ is continuous on } [a,b] \Leftrightarrow \lim_{x \to c} f(x) = f(c) \text{ for all } c \in (a,b),$$

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b)$$

The concept of continuity and the concept of limits is thus closely related. This is also reflected by the presence of the concept of continuity within MO_1 and MO_2 (Barbé et al., 2005 and Winsløw, 2015). The following theorem demonstrates how the concept of continuity and differentiability are interrelated (Lindstrøm, 2006, p. 259, own translation).

Theorem 1 If f is differentiable in a point a then f is continuous in a.

Proof Assume f is differentiable in $a \in D_f$. By the definition of continuity in a point, we want to show that $\lim_{x \to a} f(x) = f(a)$. First, we rearrange the expression:

$$\lim_{x \to a} f(x) = f(a) \iff \lim_{x \to a} f(x) - f(a) = 0 \iff \lim_{x \to a}\big(f(x) - f(a)\big) = 0.$$

Using that $\lim_{x \to a} f(a) = f(a)$ and the rules for calculating with limits for the second biimplication (Lindstrøm, 2006, p. 233). From the last equality, we get:

$$\lim_{x \to a}\big(f(x) - f(a)\big) = \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) =$$
$$\lim_{x \to a}\frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0$$

Using, for the second equality, that the limit of $x - a$ exists for $x \to a$ and upon this, the rules for calculating with limits as well as the assumption on f for the third equality. ■

It is worth noticing that if f is not differentiable in a, the limit of the difference quotient would not exist. Consequently, a differentiable function will be continuous and Theorem 1 thus constitutes one of the key properties of differentiable functions, which are to be justified in the practice block of MO_4. The result is however, not a part of the knowledge block of MO_3, as it does not serve as justification for any of the techniques for the tasks belonging to this MO.

In a more general note, continuity is a *necessary condition* but not a *sufficient condition* as there exists functions, which are not differentiable, but are continuous. This means that the reverse does not hold: continuity does not imply differentiability (Lindstrøm, 2006). In fact, there exists functions that are everywhere continuous but nowhere differentiable. Though this might seem counter intuitive, this means that there exists functions with the property that in any interval of the domain (no matter how small) the function will have an 'edge'. Using Zandieh's characterisation (a) given in the introductory of subsection 5.2.1: no matter how much the graph of the function is magnified, no 'line' will appear, and stated formally: the limit of the difference quotient will not exists for any points in the functions domain (i.e. the limit from above and below will be different from each other). The first published example[5] (with proof) of such a function in the history of mathematics was Carl

[5] Other mathematicians before Weierstrass discovered examples of everywhere continuous nowhere differentiable functions but did not publish (Thim, 2003).

Weierstrass' *monster* function illustrated graphically in figure 9, given by the expression (Thim, 2003):

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \text{ where } 0 < a < 1, b \text{ is an odd integer and } ab > 1 + \frac{3\pi}{2}.$$

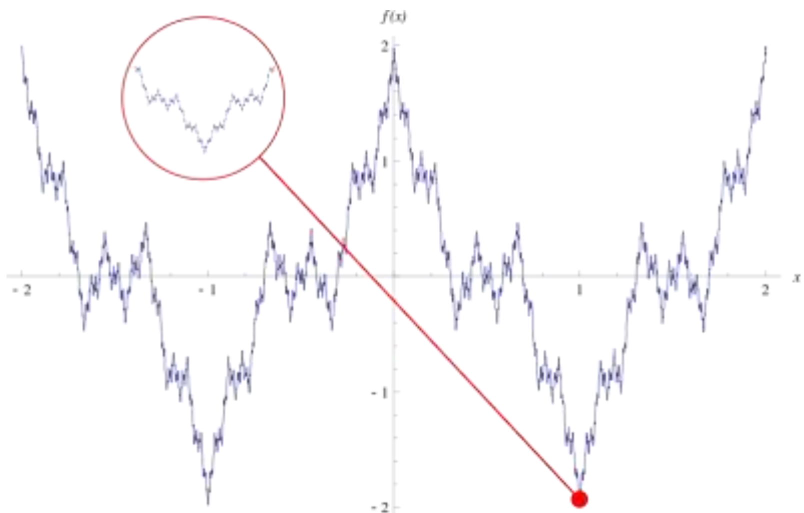


Figure 9: Weierstrass' everywhere continuous nowhere differentiable function (Weierstrass function, n.d.).

Up until this development, it was widely believed that all continuous functions were differentiable, except possibly in a certain limited amount of points (Thim, 2003). An example of the latter type of function is $f(x) = |x|$, which is everywhere continuous but only differentiable for $x \in \mathbb{R}\backslash\{0\}$, since taking the limit of the differential quotient for $x \to 0$ gives:

$$\lim_{x^+ \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x^+ \to 0} \frac{|x|}{x} = 1 \neq -1 = \lim_{x^- \to 0} \frac{|x|}{x} = \lim_{x^- \to 0} \frac{g(x) - g(0)}{x - 0}$$

This result appears in the graph of $|x|$ as an 'edge' at $x = 0$ (figure 10).

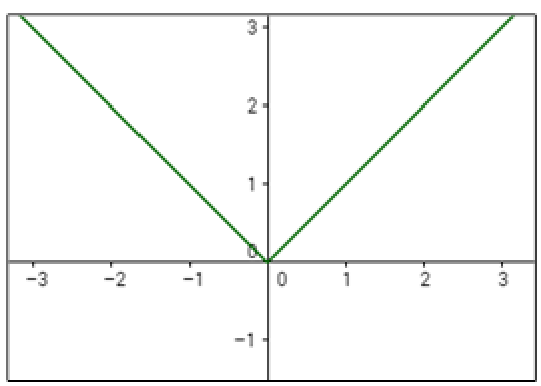


Figure 10: The graph of $|x|$

The property of continuity does not extend to the derivative function, i.e. the derivative function f' need not be continuous. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}^{+}$ given by (The Derivative, n.d.):

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

This function is continuous since $x^2 \sin\left(\frac{1}{x}\right)$ is continuous for all $x \neq 0$ and $\lim_{x\to 0} f(x) = f(0)$: $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$ and thus $\left|x^2 \cdot \sin\left(\frac{1}{x}\right)\right| \leq x^2$, which implies:

$$-(x^2) \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 \text{ for all } x \in \mathbb{R}\backslash\{0\}.$$

Moreover, $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$ and hence, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0 = f(0)$ by the squeeze theorem (see p. 27) and thus f is continuous for all $x \in \mathbb{R}$. By the same argument, it holds:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x\to 0} x \cdot \sin(\frac{1}{x}) = 0.$$

So $f'(0) = 0$ and$f$ has derivative function:

$$f'(x) = \begin{cases} 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), x \neq 0 \\ 0, x = 0 \end{cases}.$$

Using rules for differentiation for f defined on $x \in \mathbb{R}\backslash\{0\}$ (in particular (ii), (v), (II), (V) see subsection 5.2.5). However, the argument below shows that the limit of $f'(x)$ for $x \to 0$ does not exist, and thus f' is not continuous at $x = 0$:

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}(2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)) = \lim_{x\to 0} 2x \cdot \sin\left(\frac{1}{x}\right) - \lim_{x\to 0} \cos\left(\frac{1}{x}\right) =$$
$$0 - \lim_{x\to 0} \cos\left(\frac{1}{x}\right) \text{ Does not exist}$$

Since $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ oscillates between [-1,1].

By the above counterexample, it is clear that a derivative function does not need to be continuous. However, we know that, additionally, a limit is said *not* to exist if $\lim_{x\to a^+} f(x)$ and

$\lim_{x \to a^-} f(x)$ both exists but are not equal. This 'type' of discontinuity is not possible for a derivative function (Derivatives cannot have jump discontinuity, n.d.):

Theorem 2 If a function f is differentiable on (a, b) then f' cannot has a jump discontinuity in (a, b).

Proof Assume f is differentiable on (a, b) and let $c \in (a, b)$. Then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. Assume now that the following limits exists:

$$\lim_{x \to c^+} f'(x) = A \text{ and } \lim_{x \to c^-} f'(x) = B$$

When $x \to c^+$ then $x > c$ and for some $d_1 \in (c, x)$ we have cf. Theorem 8 (to be presented):

$$\frac{f(x) - f(c)}{x - c} = f'(d_1)$$

As $x \to c^+ \Rightarrow d_1 \to c^+$which gives:

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{d_1 \to c^+} f'(d_1) = A$$

When $x \to c^-$ then $x < c$ and for some $d_2 \in (x, c)$ we have cf. Theorem 8:

$$\frac{f(x) - f(c)}{x - c} = f'(d_2)$$

As $x \to c^- \Rightarrow d_2 \to c^-$which gives:

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{d_2 \to c^-} f'(d_2) = B$$

It follows that $A = B$ and f' does not have a jump discontinuity at c. ■

The proof demonstrates the dependence of this result on the Theorem 8: the mean value theorem (to be presented), and thus, as we shall see, a dependence on the properties of the real numbers. Before proceeding to treat other results directly related to the derivative function, we move back, to consider the completeness of the real numbers as this property plays an important role in the theory.

5.2.3 The Property of the Real Numbers

As illustrated in figure 6 (p. 23), the completeness of the real numbers represents a basis for various results regarding the properties of the derivative. This is no surprise since the properties of the real numbers serve as an essential part of the rigorous foundation on which all of analysis rests (Hunter, 2014).

In mathematics, eleven axioms exists describing the properties of the real numbers $\mathbb{R}$. Out of these eleven axioms, only one is specific to the real numbers – the first ten describes the rational numbers $\mathbb{Q}$ completely, while the eleventh axiom describes a property only held by the real numbers (Lindstrøm, 2006). For completeness, the notion of a *least upper bound* is defined prior to stating the completeness axiom (Lindstrøm, 2006, p. 93, own translation).

Definition 4 A nonempty set A has an *upper bound* if there exists an element b s.t. $a \leq b$ for all elements $a \in A$.

Definition 5 A nonempty set A has a *least upper bound* if there exists an element b_0 s.t. $a \leq b_0 < b$ for all elements $a \in A$ and all other upper bounds b.

The completeness axiom states (Lindstrøm, 2006, p. 94, own translation):

The Completeness Axiom Every nonempty set of real numbers having an upper bound has a least upper bound in $\mathbb{R}$.

The theory on the real numbers unifies multiple local organisations within the domain of analysis upon which some of the key results in differential calculus depend (Winsløw, 2015). Such an organisation generated by the theory of the real numbers could rightfully be named MO_0 to acknowledge the primary and preceding role of this MO. The completeness axiom is essential in the net of results, which leads to one of the most important results in analysis, namely the mean value theorem (Theorem 8). Not all of the definitions and theorems, along with their proofs, preceding the mean value theorem will be given here, but is included in figure 6 (p. 23) to illustrate the dependence of MO_4 as well as MO_3 on other local MO's derived from MO_0 within the global organisation of mathematical analysis. Only two result preceding the mean value theorem will be presented here, as these state properties of the derivative function and are to be justified in the practice block of MO_4.

5.2.4 Key Properties of the Derivative Function

In the following, a theorem is presented (Theorem 6) which describes the relationship between f' and the points of extrema of f. It is a result included in the theoretical level of

MO_3, as it in particular justifies the techniques for tasks of type $\mathcal{T}_{3.3}$: given the algebraic expression of f determine the monotonicity of f, and the result itself is to be justified within the practical block of MO_4. However, the result derives from the definition of the derivative together with the definition of *extrema.* The definition states (Lindstrøm, 2006, p. 227, own translation):

Definition 7 A point $a \in D_f$ is called a *maximum point* for the function $f: D_f \to \mathbb{R}$ if $f(a) \geq f(x)$ for all $x \in D_f$. We call a a *minimum point* if $f(a) \leq f(x)$ for all $x \in D_f$. Together we call these points *extrema*.

Theorem 6 states (Lindstrøm, 2006, p. 263, own translation):

Theorem 6 Assume $f: [a, b] \to \mathbb{R}$ attains a maximum or a minimum in an internal point $c \in (a, b)$. If f is differentiable in c then $f'(c) = 0$.

Proof By negating the statement, we are proving that if $f'(c) \neq 0$ then f does not attain a maximum or a minimum in c.

If $f'(c) \neq 0$ then either $f'(c) < 0$ or $f'(c) > 0$. Assume the latter. Since f is differentiable, we thus know that:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$$

Hence, for all $x \in (c - \delta, c + \delta)$ according to the definition of limits, it holds:

$$0 < \frac{f(x) - f(c)}{x - c} \qquad (*)$$

I.e. we know that there exists a $\delta > 0$ s.t. $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon$, for *any given* $\varepsilon > 0$ if $|x - c| < \delta$; and thus, such a δ also exists for $\varepsilon = f'(c)$. Hence, there exists a $\delta > 0$ s.t.:

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < f'(c) \Rightarrow -f'(c) < \frac{f(x)-f(c)}{x-c} - f'(c) \Rightarrow 0 < \frac{f(x)-f(c)}{x-c}$$

For $x > c$ the denominator of $(*)$ will be positive and therefor it must hold that $f(x) > f(c)$ for the whole fraction to be positive. So by definition (cf. def. 7), c is not a maximum. For $x < c$ the denominator will be negative and therefor it must hold that $f(x) < f(c)$ for the fraction to be positive. So by definition c is not a minimum.

Assuming $f'(c) < 0$, the proof is analog. ■

Pausing for a moment – it has been shown that if $f'(c) \neq 0$ then f attains neither a maximum nor a minimum in c and hence if f attain an extreme value in c then $f'(c)$ must be equal to zero. It should be noted that Theorem 6 does not tell us anything about the converse situation i.e. whether f *always* attains a maximum or a minimum in c if $f'(c) = 0$. The short answer is 'no'. Having $f'(c) = 0$ does not necessarily imply that c is an extrema. The function $f(x) = x^3$ illustrates this perfectly, since $f'(0) = 0$ while $f(x) < f(0)$ for $x < 0$ and $f(x) > f(0)$ for $x < 0$ (figure 11). We say that the function has a *point of inflection* in $x = 0$.

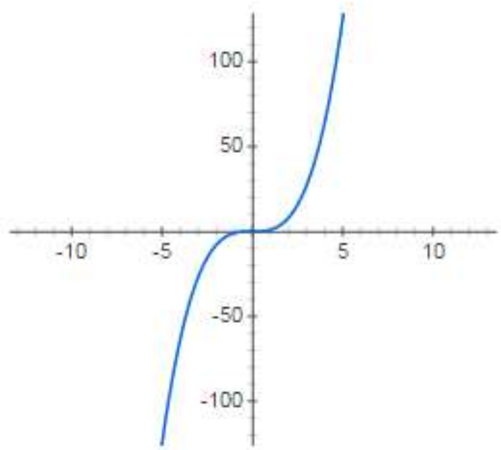


Figure 11: The graph of $f(x) = x^3$

So $f'(c) = 0$ is a necessary but not sufficient condition for c to be an extrema, however $f'(c) = 0$ is a sufficient condition for c to be an extrema *or* an inflection point. This result, in addition to being one of the results included in the theoretical level of MO3; again justifying techniques related to tasks concerning monotonicity, is essential in the net of results illustrated in figure 6 (p. 23) due to its role in proving Theorem 7, *Rolle's Theorem*. This theorem states (Lindstrøm, 2006, p. 263, own translation):

Theorem 7 Assume $f: [a, b] \to \mathbb{R}$ is continuous, and differentiable in all internal points $x \in (a, b)$. Furthermore, assume $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ s.t. $f'(c) = 0$.

Proof Since f is continuous it will attain both a maximum and a minimum in the closed interval $[a, b]$ according to Theorem 5 (see figure). Since $f(a) = f(b)$ there must exists at least one point $c \in (a, b)$ which is a maximum or a minimum. According to Theorem 6 $f'(c) = 0$. ■

The proof is included to make clear the theorem's dependence on Theorem 6, a result derived from the completeness property of the real numbers, and thereby the dependence of the results derived from Rolle's Theorem on the properties of the real numbers. Despite the dependence on MO_0 and thus the profound nature of the result, the statement of the theorem appears rather intuitive, especially considering the graphical interpretation of the theorem (figure 12):

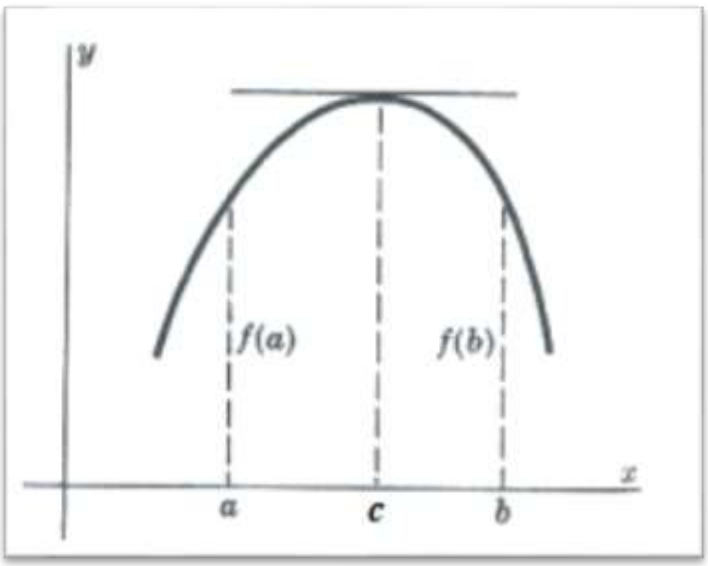


Figure 12: The graphical interpretation of Rolle's Theorem (Miller, n.d.).

Rolle's Theorem is a special case of the following mean value theorem, which is stated here without proof (Lindstrøm, 2006, p. 263, own translation).

Theorem 8 Assume that $f: [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable in all internal points $x \in (a, b)$. Then there exist a point $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The mean value theorem states that for a given function defined on the interval $[a, b]$ there will always exist at least one point $c \in (a, b)$ for which the slope of the associated tangent will be the same as the slope of the straight line connecting the end-points of the curve $(a, f(a))$ and $(b, f(b))$. Figure 13 below, presents this graphical interpretation. Notice that if f is defined on $[a, b]$ then it is also defined on $[a_1, b_1]$ for all a_1, b_1 satisfying $[a_1, b_1] \subseteq [a, b]$. Hence, between any two points in the domain (A and B in the figure), there exists a point for which the associated tangent has the same slope as the line between the between the two points.

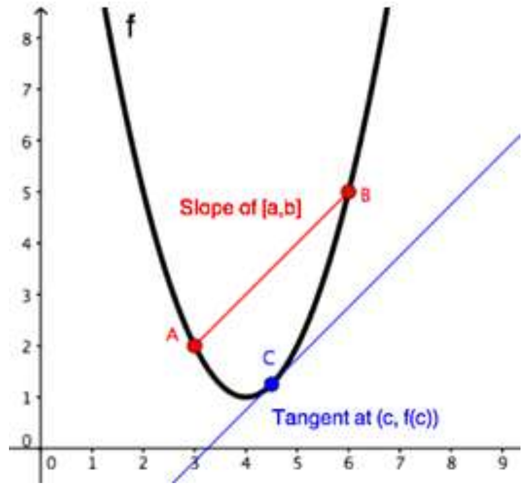


Figure 13: The graphical interpretation of the mean value theorem (Mean Value Theorem Explanation, n.d.)

From Theorem 8 two corollaries are derived, which both belong to the theory of MO_3 and are to be justified within the practice block of MO_4. These corollaries are stated in the following along with their proofs. The latter, to make explicit the reliance on Theorem 8. Corollary 1 states (Lindstrøm, 2006, p. 265, own translation):

Corollary 1 If $f'(x) = 0$ for all x in some interval I then f is constant on I.

Proof Choose a point $a \in I$. For any $x \in I$ different from a we have by Theorem 8

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0$$

for a point c between x and a. Hence $f(x) = f(a)$. Since x was arbitrary, we have shown that $f(x) = f(a)$ for all $x \in I$ and f must be constant. ■

Notice that the corollary states that *only* a constant function has derivative equal to 0 for all x in the domain – not that a constant function has derivative equal to zero. The latter follows immediately from definition 2, since $f(x) = f(a)$ for all points $x \neq a$ in the domain:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{0}{x - a} = \lim_{x \to a} 0 = 0$$

Corollary 2 states (Lindstrøm, 2006, p. 266, own translation):

Corollary 2 Assume f is continuous [and differentiable] on $[a, b]$. If $f'(x) \geq 0$ for all internal points $x \in (a, b)$ then f is increasing on $[a, b]$. If $f'(x) \leq 0$ for all $x \in (a, b)$ then f is decreasing on $[a, b]$. If we instead have strict inequalities (respectively, $f'(x) > 0$ or $f'(x) < 0$ for all internal points x) then the function is *strictly* increasing or *strictly* decreasing.

Proof Assume f is continuous [and differentiable] on $[a,b]$ and that $f'(x) \geq 0$ for all internal points $x \in (a,b)$. Choose $x_1,\ x_2 \in [a,b]$ s.t. $x_1 < x_2$. According to Theorem 8 there exists a $c \in (x_1, x_2)$ s.t.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0$$

Since $x_2 - x_1 > 0$ then $f(x_1) \leq f(x_2)$ and f is increasing.

The three other parts of the corollary are proved completely analogue. ■

The examination if the main properties of the derivative function shows, in particular, that these do not rely on MO_4 alone, but on MO_1 and MO_2 (as the rules for calculating limits, belonging to MO_1 and justified in MO_2 are employed on occasion) and very much so, on various organisations derived from MO_0 as well. These 'various' organisations will not be identified in any detail, in this context, apart from their already established common reliance on the MO_0.

5.2.5 The Differentiation Rules

The rules for differentiation, belonging to the theory of MO_3 are listed in this subsection, though the proofs will not be given here. The rules divide into two groups: (1) the *special* rules for differentiating specific functions and (2) the *general* rules for differentiating. Let $c, a \in \mathbb{R}$, then group (1) comprises of the following rules (Lindstrøm, 2006):

(i) Differentiating a constant function: $(c)' = 0$

(ii) Differentiating a power function: $(x^a)' = ax^{a-1}$

(iii) Differentiating an exponential function: $(a^x)' = a^x \ln(a)$ for $a > 0$.

In particular: $(e^x)' = e^x$ and $(e^{c \cdot x})' = c \cdot e^{kx}$

(iv) Differentiating the natural logarithm: $(\ln(x))' = \frac{1}{x}$

(v) Differentiating sin, cos and tan: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \frac{1}{\cos^2 x}$

Assuming that both f and g are defined and differentiable in a, the general rules comprise of the following (Lindstrøm, 2006):

(I) Differentiating a constant times a function: $\left(c \cdot f(a)\right)' = c \cdot f'(a)$.

(II) Differentiating a sum/difference of functions: $\left(f(a) \pm g(a)\right)' = f'(a) \pm g'(a)$.

(III) Differentiating a product of functions: $\left(f(a) \cdot g(a)\right)' = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

(IV) Differentiating a quotient of functions: $\left(\frac{f(a)}{g(a)}\right)' = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g(a)^2}$, for $g(a) \neq 0$.

Assuming g is defined and differentiable in a and f is defined and differentiable in $g(a)$, then:

(V) Differentiating a composite function: $\left(f\left(g(a)\right)\right)' = f'(g(a)) \cdot g'(a)$.

These rules are to be justified in the practice block of MO4; however, the justification of the majority of the rules depend on theory not included in MO4. One example of a rule derived directly from the definition of the derivative is (i) as we saw on p. 39. An example of a rule, which relies on more than MO4, namely MO2 is (I): Let $g(x) = c \cdot f(x)$

$$\lim_{x \to a} \frac{c \cdot f(x) - c \cdot f(a)}{x - a} = \lim_{x \to a} c \cdot \frac{f(x) - f(a)}{x - a} = c \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = c \cdot f'(a) = g'(x)$$

Since f is assumed differentiable in a and by the *property of limits*. Additionally, the proofs of (II)–(IV) and (iv) requires the use of limit properties why these are also derived (and dependent on) MO2, since the properties of limits are justified within the practice block of this MO (The Derivative, n.d.). Besides the reliance on MO2 multiple of the differentiation rules rely on other mathematical results and techniques not included in MO2 or any other organisation considered previously in this context. For example, one way of justifying the derivative of $(\cos x)' = -\sin x$ is by using the real analytic definitions of these functions, while other ways uses properties of the cosine and sine functions (e.g. the cosine of sum) (Derivative of Cosine Function, n.d.). A way of proving (iv) requires the use of logarithm properties (such as $\frac{\log a}{\log b} = \log a - \log b$) and furthermore depends on the definition of e given in terms of the limit (Proofs of derivatives of ln(x) and e^x, n.d.).

$$e \coloneqq \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Which again, depends on MO_2 to bear any meaning. Furthermore, one way, of proving the special case of (iii), namely $(e^x)' = e^x$ can be proven from (iv) and the use of the (V) (Proofs of derivatives of ln(x) and e^x, n.d.). The proof of (ii) also relies on the natural logarithm together with (V), while the proof of the chain rule, (V), does not rely on the other differentiation rules, but is derived from the definition using an auxiliary function (Lindstrøm, 2006).

Based on this limited outline, it is asserted that the rules in group (1) and (2), creates a somewhat interconnected map like the properties of the derivative do in figure. The rules in group (2) rely mainly on MO_2 and MO_4, while the rules in group (1) rely on a variety of properties and definitions of, and related to, the specific functions.

5.3 The knowledge to be taught

In this section, we will elaborate our epistemological reference model further. As we saw in subsection 2.1.1, such a model must incorporate the *knowledge to be taught*. For this purpose, it will investigate how MO_3 and MO_4 appear on the institutional level of secondary school. (Bosch et al., 2005).

The *knowledge to be taught* is manifested in the official curriculum (Bosch & Gascón, 2006); therefore, this section will first present the part of the official curriculum, settled by the Danish Ministry of Education, which relates to derivative functions (Stx-bekendtgørelsen, 2013). It states:

1. Definition and interpretation of the derivative including growth rate and marginal observations, the derivative function for the elementary functions as well as the calculation rules for differentiating $f + g$, $f - g$, $k * f$, $f * g$, $f \circ g$ and deduction of selected derivatives.
2. Monotonicity properties, extrema and optimizing as well as the relation between these concepts and the derivative.
   (Matematik A – stx, 2013, translated from Danish by the author)

The above is common for level A and level B high school mathematics, except the rule for differentiation of composite functions (the chain rule), which is not included in the B level mathematics curricula (Matematik A – stx, 2013 & Matematik B – stx, 2013). Content related to the derivative is entirely excluded from C level high school mathematics (Matematik C – stx, 2013). Further, in the guidelines provided for the teachers by the ministry, it is stated:

> Work on the concept of derivative imply that the concept of limit is included but it is not intended that this be given a separate treatment. Similarly, the study of the

relationship between monotonicity and local extrema includes involvement of the concept of continuity but it is not intended that this be given a separate treatment. (Matematik – stx, 2013, translated from Danish by the author).

From point 1 and 2 above, as well as the guidance note, it appears that elements from both MO_3 and MO_4 are *to be taught* in secondary school. In the following, it is explored how 1 and 2 manifest itself in the national written-exam tests based on four tests: two tests for B-level students (28th of May 2015 and 22nd of May 2015) and two tests for A-level students (22nd of May 2015 and 28th of May 2015), see Appendix A. Furthermore, a high school B-level mathematics textbook, by Clausen, Schomacker and Tolnø (2011a) as well as the corresponding exercise book (2011b) by the same authors, are examined, to give just one example of the *knowledge to be taught* as manifested in a Danish high school textbook.

The tasks given at the written-exam tests, which relates to the relevant subject matter, reduces to the following types of tasks:

T_1: Given the algebraic expression of f determine f'

$T_{1.1}$: Given the algebraic expression of f and a point $a \in D_f$, determine $f'(a)$.

$T_{1.2}$: Given the algebraic expression of f determine the monotonicity of f.

T_2: Explain $f'(a)$.

T_1 and $T_{1.1}$ belongs to MO_3 corresponding to $\mathcal{T}_{3.1}$ and $\mathcal{T}_{3.2}$, respectively as well as $T_{1.2}$, which ultimately relies on the determination of f' and corresponds to $\mathcal{T}_{3.3}$ while T_2 belongs to MO_4 corresponding to $\mathcal{T}_{4.1}$ (see section 5.1). Thus, the exam question incorporate only a single type of task related to MO_4. Related to this task however, it is relevant to explore what explanations that a high school textbook provide regarding $f'(a)$ and thus what explanations the students are expected to provide.

The high school mathematics textbook by Clausen et al. (2011a) presents the subject in two parts: First, a chapter called *Differential calculus* and later a section of the book is devoted to the *Definition of the derivative*. Part one and two corresponds mainly to the transposition of MO_3 and MO_4, respectively. In the first part, however, the concept of the derivative is introduced and it is initially presented by means of *speed* at a particular point in time. Following, it is identified as the slope of a curve (or the slope of a tangent to the curve) at a given point and 'recipes' are provided to find the derivative in a point (one of the recipes is illustrated figure 14 below), based on a rough definition of the derivative (leaving out considerations on existence).

When a curve is given by $y = f(x)$ the slope a in a point $P(x_0, f(x_0))$ on the curve is determined in the following way: we choose an arbitrary second point $Q(x, f(x))$ on the curve and determine the slope of the secant

$$a_{PQ} = \frac{f(x) - f(x_0)}{x - x_0}.$$

This fraction is rewritten (if possible) such that it is possible to insert x_0 instead of x in the expression for a_{PQ} and thus calculate the value a that a_{PQ} approaches when x approaches x_0. That x approaches x_0, is equivalent to the point Q approaches the point P. The result a is the slope of the curve in the point P. The result a is called the *differential*

Figure 14: "Recipe 1" for finding the derivative (Clausen et al., 2011a, p. 13, translated from Danish by the author)

By far, the most frequent tasks assigned to the students in Clausen et al. (2011b) are of type $\mathcal{T}_{3.1}$, $\mathcal{T}_{3.2}$ and $\mathcal{T}_{3.3}$, all belonging to MO_3. To solve these tasks, the textbook provides the 'recipe' illustrated above and the rules (i) – (iv) including the following special cases of rule (ii) (Clausen, 2011a):

$$* \qquad (x)' = 1 \quad (x^2)' = 2x, \quad (x^3)' = 3x^2, \quad \left(\frac{1}{x}\right)' = \frac{1}{x^2} \quad \text{and} \quad \sqrt{x} = \frac{1}{2\sqrt{x}}$$

Rule (v) is not included, as trigonometric functions are not studied on B-level (Matematik B – stx, 2013). The textbook also provides the rule $(ax + b)' = a$, which is a combination of (i), (I) and (II). The rules (I) – (IV) are presented either in the textbook or in the exercise book, while (V) is not included, as this is not included on B-level (Matematik B – stx, 2013). Furthermore, the textbook presents the properties of the derivative function in corresponding to Theorem 6, Corollary 1 and Corollary 2, the latter stated as: "If $f'(x)$ is positive for all x in an interval, then $f(x)$ is increasing in the interval" and vice versa (Clausen, 2011a, p. 32, translated from Danish by the author). These rules and properties constitute the transposition of the theoretical level of MO_3 (denoted MO_3').

Some rules, here among, the rules cited above $(*)$, derives from the definition of the derivative and rules for calculating limits (e.g. $\lim_{x \to a}\big(f(x) + g(x)\big) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$). Some of the 'proofs' are provided in the book, however without any explicit reference to properties of limits and thus no explanation or justifications of the inherent algebraic manipulations appear, not even in terms of MO_1 as the use of the rules is not explicit. Other rules, such as (iv), are not proved in the book and likewise concerning the properties of the derivative function. The justification of rules such as $(ax+b)' = a$ and $\left(f(x) - g(x)\right)' = f'(x) - g'(x)$, are left to the students; the former, upon numerous examples of the use of the recipes (Clausen et al., 2011a, p. 17) and the latter, with the instruction to proceed as in the proof of $\left(f(x) + g(x)\right)' = f'(x) + g'(x)$ (Clausen et al., 2011b, p. 43). The justification of these rules are part of the practical block of MO_4 but appears in the transposed organisation (MO_4') as almost absent; only to be used by student in few cases and by copying given examples. The types of tasks belonging to MO_4 which are present in MO_4' are $\mathcal{T}_1$ and $\mathcal{T}_2$ (What is $f'(x)$ and $f'(a)$, $a \in D_f$), which brings us to the theoretical level of MO_4.

As we saw in the last section, the definition of the derivative is central in the theory of MO_4 and furthermore, the concept of limit is fundamental to the definition of the derivative. However, the latter concept is given little attention if any; only included in the form of formulations such as "approaching" and "goes toward" and when determining limits, it is done so in a manner, such as "$f'(x_0) = x_0 + x_0 = 2x_0$", given that the slope of the secant was determined as $x + x_0$ (Clausen et al., 2011a, p. 15). In all, MO_1 is completely absent, as the students are not asked to determine limits, except in few cases (determining derivatives) following demonstrated recipes and the theoretical level is not explicit in any way, as pointed out above. In the absence of MO_2, Clausen et al. (2011a) presents the following definition (figure 15):

That a function $f(x)$ is differentiable in x_0 means that the slope of the secant

$$\frac{f(x) - f(x_0)}{x - x_0}$$

Tends toward a specific number a when x tends toward x_0. In that case, this number a is called the derivative of $f(x)$ in x_0 and it is denoted $f'(x_0)$.

Figure 15: Definition of $f'(x_0)$ in a high school textbook (Clausen et al., 2011a, p. 181; translated from Danish by the author).

The definition incorporates the notion of a limit by the formulation "tends toward" ("går mod"). The avoidance of limits consequently means an exclusion of the notion of two-sided limits, which exclude a formal justification of the domain of the derivative on (a, b) for a given differentiable function defined on $[a, b]$. However, an example of this characteristic appear when proving the rule for differentiating the function $f(x) = \sqrt{x}, x \geq 0$ (namely, $\frac{1}{2\sqrt{x}}, x > 0$) and this, with no accompanying explanation (Clausen et al., 2011a, p. 18). Relating to the given definition and the 'recipe' provided for determining derivatives (figure 14 and 15 above), it is stated that "The difference is, that the recipes focus on the determination of the derivative while the definitions also focus on the existence of the derivative" (Clausen et al., 2011a, p. 183, translated from Danish by the author). However, exploring the tasks assigned, relating to this section, these are found to be solely comprised of tasks of type $\mathcal{T}_{3.1} \in MO_3$ (Clausen et al., 2011b, pp. 82-83). The transposition of the theoretical level of MO_4 is highly related to the transposition of MO_2 and thus the transposition of the definition of the derivative suffers from the minimal transposition of MO_2.

A following section in the book elaborates the relation between differentiability and continuity. Continuity is defined as a connectedness-property achieved if "The graph can be drawn without lifting the pencil from the paper" (Clausen et al., 2011a, p. 188, translated from Danish by the author). The meaning of the word continuity is said to be the same as "connected, without jumps" (Clausen et al. 2011a, p. 189, translated from Danish by the author). There are four tasks related to this section, all of the type:

T: Draw the graph of a function with given properties (differentiable in all but one point, continuous in all but one point, continuous but not differentiable in all points).

With corresponding theory (Clausen et al. 2011a):

Θ: A function f is differentiable if its graph is 'smooth', a function is continuous if the graph has no 'jumps' and a function, which is not continuous, is not differentiable.

Where 'smooth' is understood as equivalent to 'no edges'. We saw, however, in the last section that the statement 'a function is continuous if the graph has no jumps', is not exhaustive, as to describe the properties of a continuous functions. In this context, it is also noted that Theorem 2 is not included in any 'version' in the transposed MOs. However, in the absence of the concept of limits, it seems that the concept of continuity is transposed to the extent it is possible.

The primary challenges identified by Winsløw (2015) regarding the transposition of elements of analysis to high school, is that the topological organisations (i.e. MO_2 and MO_4)

are difficult to transpose because of its close relation to the properties of the real numbers, which is not taught in high school. However, the meaning and justification of the algebraic organisations lie within the topological organisations (Winsløw, 2015). This is also pointed out by Barbé et al. whom finds that MO_2 is not taught in any significant manner in Spanish high schools and points to a disappearance of the 'reasons of being' when the topological organisation is not present to justify any of the techniques.

The trace left by MO_3 and MO_4 in the investigated textbook is of the same character proposed by Barbé et al. (2005) regarding limits in Spanish high schools, as also suggested by Winsløw (2015). The uncomplete transposition of the theoretical level of MO_4 in the book can in fact be considered as a consequence of the minimal trace of MO_2. While the trace of the practice block of MO_4 only consists of task types $\mathcal{T}_1$ and $\mathcal{T}_2$; which are to be answered without the rigorous definition of limits and much so by formulations such $f'(a)$ is "The curve's steepness in $P\ [(a, f(a))]$" (Clausen et al. 2011a, p. 13, translated from Danish by the author). MO_3 appears rather complete in the investigated textbook. It follows immediately, that MO_3 dominates the transposition of the regional MO concerning function's derivatives. The majority of the tasks related to relevant sector are of algebraic nature for which the students are provided with algorithms, established through an abundant amount of examples.

6 The Designed Hypothetical Teacher Tasks

This chapter aims to answer Research Question 1, namely the question:

> *Based on a subject matter analysis of the theme of derivative functions in Danish high school, how can one model non-trivial teacher knowledge related to this theme in terms of HTT's?*

The chapter proposes five of such HTTs along with an *a priori* analysis of each subtask. This particularly entails a determination of the various techniques (τ), which are to be mobilized in order to solve the tasks. Preceding this however, a section is devoted to present some considerations concerning the choice of tasks (section 6.1).

All the tasks included in the HTTs either belong to or relate to MO_3 or MO_4 (in one case techniques related to MO_5 are considered relevant). Therefore, it is asserted, that the mathematical content in the tasks is appropriate for the intended participants. 'Appropriate' meaning that it is expected that the participants master the mathematical content, possibly not with a routine approach, but it is expected, based on their mathematical background (see section 4.2.1) that they possess the knowledge associated with the theoretical levels of MO_3 and MO_4 (and MO_5) and are capable of activating techniques related to these organisations. However, after designing the tasks, these were presented to the subject matter adviser of Mathematics in Danish high schools, whom offered an external opinion on the difficulty of the task, specifically, to answer the question: to which extend can one expect teachers to master the mathematics which is included in the tasks? The conclusion was two sided: firstly, it was established that not many teachers (possibly only A-level teachers) would pose mathematical tasks of the type included in the HTTs and therefore, it could not be expected that the participants have experience with providing feedback to such tasks. Secondly, and more important in this context, the teachers *can* (or possibly: *should be able to*) handle [håndtere] the mathematical problems included in the HTTs, without necessarily having experience in teaching such problems (B. Bruun, personal communication, June 23, 2016).

6.1 The Focus on Graphical Representations

The five HTTs contains didactical task all relating to either MO_3 or MO_4 and mathematical tasks belonging to, or special cases of, the types of task which constitute the basis of these MOs. Only HTT 1 focus on a typical task belonging to MO_3 alone, while the remaining, in varying degrees, focus on non-typical tasks related to MO_3 and MO_4.

In section 5.3, we learned that the algebraic organisation, MO_3, constitutes a large part of the *knowledge to be taught* in upper secondary school. The type of task most common is $\mathcal{T}_{3.1}$ and central in the teachers' practice is thus the teaching of the differentiation rules as well as the correct application of these. Therefore, HTT 1 was designed to assess the participants' didactical knowledge related to MO_3, in particular, the task seeks to uncover the participants' didactical techniques when confronted with student difficulties in the application of differentiation rules, which is asserted to be an essential task in the teachers' praxis.

In section 5.3 we also learned that tasks belonging to MO_4 was atypical in upper secondary school as the types of tasks in the transposed MO_4' comprised only of $\mathcal{T}_{4.1}$: what is the derivative of f in a point $a \in D_f$? and $\mathcal{T}_{4.2}$: what is the derivative function f'? and that the transposed theory of MO_4' was comprised of the answers to these tasks, however informally; leaving out the definition of limits. As will become clear, many of the mathematical task included in the HTTs relates to MO_4. This choice stems from an interest in the teachers' mathematical and didactical techniques related to this organisation, which justifies the whole praxis.

The majority of the HTTs are given in a graphical setting or requires translating between the algebraic and graphical setting, for example between function expressions and graphs. Based on a literature study performed prior to the design, this setting for many of the HTTs was chosen. The literature study (though quite modest) revealed an interest, within the field of didactic research, in the significance of the graphical representation when working with and learning about the concept of a function and its derivative (Nemirovsky and Rubin (1992); Thompson (1994); Asiala, Cottrill and Dubinsky (1997); Santos and Thomas (2003); Berry and Nyman (2003); Hähkiöniemi (2006); Abbey (2008); Haciomeroglu, Aspinwall and Presmeg (2010)). Aspinwall, Shaw and Presmeg (1997) reported about the notion of uncontrollable mental images and the possible negative effects related to students' vivid imagery. On the other hand, much research points to visualization as a tool to enhance learning. Berry and Nyman (2003) concluded that "If students can develop the skill of drawing a function graph from its slope graph then their level of conceptual understanding of the derivative and its connections to the concept of the integral will be greatly improved" (Berry & Nyman, 2003, p. 496). While Santos and Thomas (2003) found that fluency between representations have positive effects on students' understanding. Hähkiöniemi (2006) found that the *embodied world* (which includes graphical representation) provided strong tools in the students' learning process. Aspinwall and colleagues (2010) found that synthesizing analytic and visual thinking had positive effect in students understanding of the derivative. However, Abbey (2008) found, that students' graphical knowledge was weak due to students' preference of the algebraic representation and their weakness in graphing functions without an algebraic expression (among other factors).

Based on the literature study it is asserted that working with graphical representations is an important tool and therefore, it is essential that teachers master the derivative function in this setting. Moreover, the graphical setting offers a way to construct tasks, which do not require long calculations nor is associated with specific algorithms for their solutions; they do not require *procedural knowledge* (Rittle-Johnson, Siegler & Alibali, 2001), but on the contrary, requires a *conceptual understanding*. The latter refereeing to "knowledge of concept and an understanding of the principles that govern a domain and the interrelations between units of knowledge in a domain" (Rittle-Johnson et al., 2001, pp. 346-347) and thus, in this context, conceptual understanding is especially understood as knowledge belonging to MO_4. However, it is not to say that no task belonging to MO_4 can be solves through procedural knowledge.

6.2 A Priori Analysis of the Hypothetical Teacher Tasks

This section presents the five HTTs along with an a priori analysis of each task. The HTTs are originally formulated in Danish, see Appendix B. In the following analysis, the exercises appear in an English translation. The tasks included in the HTTs are sometimes denoted with a small letter t and other times, with a capital letter T. The capital letter signify a type of tasks while the small letter signify a concrete task (for example a task in which a specific function is given). Also, reference to the types of tasks generating MO_3 and MO_4 occurs; these are denoted $\mathcal{T}$ as in section 5.1.

6.2.1 HTT 1

HTT 1 focuses on the use of rules for differentiation related to the specific elementary functions: the natural exponential function e^x and the linear function $ax + b$ as well as the general rule for differentiating composite functions. Task 1a of HTT 1 generates a punctual DO related to the mathematical task $t_{1.1} \in \mathcal{T}_{3.1} \subset$ MO_3':

$t_{1.1}$: Given $f(x) = e^{x+1}$ determine the derivative function $f'(x)$.

Since $t_{1.1}$ is of type $\mathcal{T}_{3.1}$, the techniques for $t_{1.1}$ are justified by the technology and theory of MO_3, described in section 5.1. The DO generated in 1a bases on the explicitly given didactical task $t_{1.1}$* ~ MO_3':

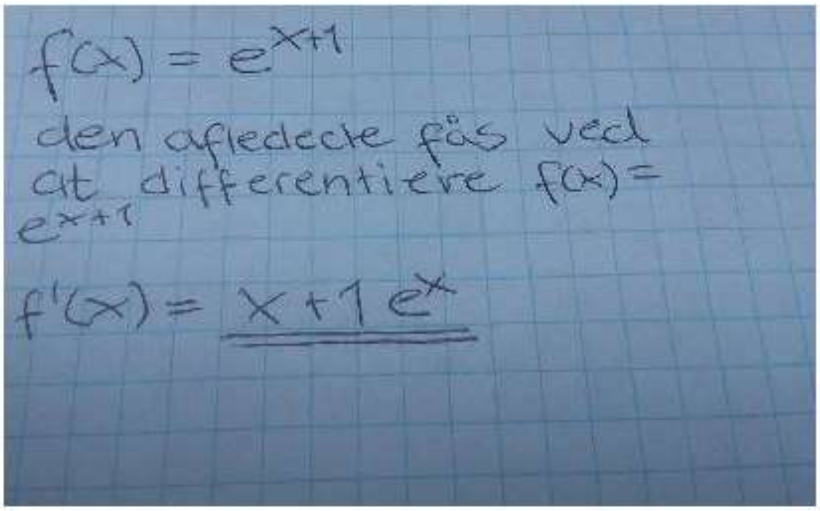
$t_{1.1}$*: Analyse and assess the student's written answer to task $t_{1.1}$.

Which is a didactical task of the type:

$T_{1.1}$*: Analyse and assess the student's written answer to a task of type $\mathcal{T}_{3.1}$.

Exercise 1

In a written assignment, Peter has differentiated the function $f(x) = e^{x+1}$. His work appears in the following picture



a) Analyze and asses Peter's answer.

b) How would you correct the assignment?

c) Propose a new assignment to help uncover whether your student have understood your correction.

Figure 16: HTT 1

In order to respond to $t_{1.1}* \in T_{1.1}*$ a combination of the following techniques can serve as relevant (and possibly in the given order):

$\tau_{1.1-1}$*: Identify that the answer is wrong.

$\tau_{1.1-2}$*: Identify the explicit algebraic manipulations of 'moving' the exponent down in front and subtracting 1 from the original exponent.

$\tau_{1.1-3}$*: Identify the explicit algebraic manipulations as those associated with the rule for differentiating power functions: $(x^a)' = ax^{a-1}$ (ii).

$\tau_{1.1-4}$*: Identify the students' technology: e^{x+1} is a power function.

The latter technique presupposes $\tau_{1.1-1}$* - $\tau_{1.1-3}$*, while exclusively using the first technique $\tau_{1.1-1}$* limits the possibility of identifying the possible misunderstanding or challenges that the student may have. The techniques activated to solve the specific task $t_{1.1}$*, in particular the activation of $\tau_{1.1-1}$* depends on the respondents' ability to mobilize correct mathematical techniques for task $t_{1.1}$. A mobilization of relevant techniques for this task requires a correct answer to the inherent subtask:

$t_{1.1.1}$: Determine what type of function $f(x) = e^{x+1}$ is.

The type of task to which $t_{1.1.1}$ belongs generates a local MO concerning basic function theory, which must precede the teaching and learning of MO_3' (in general preceding all other MOs within the domain of mathematical analysis, assuming f is an elementary function). The technology and theory explaining and justifying the classification of f encompass the difference in notation of variables and constants as well as the knowledge of the characteristic of elementary functions. The technique for solving $t_{1.1}$ is completely dependent upon the answer to $t_{1.1.1}$. The specific technique for task $t_{1.1}$ is:

$\tau_{1.1}$: $(e^{x+1})' = e^{x+1} \cdot (x+1)' = e^{x+1} \cdot 1 = e^{x+1}$.

The didactical techniques $\tau_{1.1-1}$* – $\tau_{1.1-4}$* for solving $t_{1.1}$* can lead to an identification of the student's faulty technology, which is considered as the *primary* problem:

$\theta_{1.1}^{-}$: f as a power function or specifically e is a variable.

Task 1b generates a punctual DO based on the following type of task:

$T_{1.2}$*: Correct in writing your student's work.

The specific didactical task posed in 1b is $t_{1.2}$* ~ MO_3':

$t_{1.2}$*: Correct in writing your student's answer to task $t_{1.1}$.

The techniques are dependent on the respondent's answer to task $t_{1.1}$*, i.e. to which level the respondent recognizes the use of rule (ii) and identifies the student's incorrect technology, $\theta_{1.1}^{-}$. A didactical technique for solving $t_{1.2}$* could be one or a combination of the following techniques:

$\tau_{1.2-1}$*: State that the answer is wrong (in symbols ÷).

$\tau_{1.2-2}$*: Write the correct answer.

$\tau_{1.2-3}$*: Write the correct calculations.

$\tau_{1.2-4}$* State that f is not a power function.

$\tau_{1.2-5}$*: State why f is not a power function (i.e. the difference between a power function and an exponential function).

$\tau_{1.2-6}$*: State that f is a composite function.

$\tau_{1.2-7}$*: State that f is composed of (the exponential) function e^x and (the linear) function $x + 1$.

$\tau_{1.2-8}$*: State that an irrelevant differentiation rule is used.

$\tau_{1.2-9}$*: State the correct differentiation rules associated with $t_{1.1}$.

In this context, no theory on how to provide '*the* correct' written feedback is included. Thus, to create some system and transparency, the assessment of the participants' techniques bases on the following two principles (wherein it is assumed that the participants activated correct techniques for $t_{1.1}$*, in particular, that they identified the student's incorrect technology):

1) The correction shall relate to the analysis and address the primary problem.

2) The correction shall be transferable to other situations where the student might face the same type of challenges.

A combination of the presented techniques, which meets these demand could be $\{\tau_{1.2-4}*, \tau_{1.2-5}*, \tau_{1.2-7}*\}$. Naturally, the following technique is assessed as incorrect:

$\tau_{1.2-10}*^{-}$: State that the answer is correct.

Task 1c also creates a punctual DO based on the didactical task $t_{1.3}* \sim$ MO3':

$t_{1.3}$*: Propose a new task to uncover whether your student understood your correction.

The technique will depend on the respondent's answer to exercise 1b due to the explicit reference to the given correction. For example, if the teacher uses the techniques $\{\tau_{1.2-4}*, \tau_{1.2-5}*\}$ then a technique in 1c could be:

$\tau_{1.3.1}$*: Ask the student to differentiate a sum function made up of an exponential function and a power function (for example $e^x + x^4$).

The use of, for example, technique $\tau_{1.2-3}$ will consequently leave the student with a recipe for differentiating that particular function which means that a technique for 1c could be:

$\tau_{1.3.2}$*: Ask the student to differentiate a composite function similar to f given in $t_{1.1}$ (for example e^{x+6})

However, this will not necessarily test more than the student's ability to follow a recipe. Have the respondent used $\{\tau_{1.2-4} *,\ \tau_{1.2-5} *,\ \tau_{1.2-7} *\}$ a technique for 1c could be:

$\tau_{1.3.3}$*: Ask the student to differentiate e^{x^2}.

This task will uncover if the student can distinguish an exponential function and a power function; indeed this will challenge the student because e^x is raised to a constant. Simultaneously, the task will test the student's ability to use the chain rule correctly. The responses to 1c will be assessed upon its correspondence to both 1a and 1b: does the task address the primary problem identified in 1a and does the task relate to the correction in 1b and hence, can the task be solved with the tools provided in the correction?

HTT 1 is highly related to MO3'. To solve $t_{1.1}* \in T_{1.1}* \sim$ MO3' the respondents must be very familiar with the various rules for differentiating specific functions, not just to know which rules are relevant, but also to recognize the presence of a specific irrelevant rule for differentiation and determine what incorrect technology underlies the technique. 1b and 1c necessitate didactical considerations regarding how one converts an analysis to a written correction, in an effective and meaningful way, and which types of task are appropriate to test a student in this context; meaning for example, which task is in fact testing the intended. The analysis served as a basis for creating the following 'standard answer':

1a. Except a missing parenthesis, Peter's work corresponds with a use of the rule for differentiating power functions: $(x^a)' = ax^{a-1}$. Thus, Peter has treated f as a power function.

1b. Tell the student that an irrelevant rule has been used: f is not a power function (not of the type x^a where the base x is a variable). f is a composite function with inner function $x + 1$ and outer function e^x (the latter is an exponential function: the base e is a constant). I.e. $\{\tau_{1.2-4} *,\ \tau_{1.2-5} *,\ \tau_{1.2-7} *\}$.

1c. Differentiate the $f(x) = e^{x^2}$. This task provides the student with an opportunity to show that he can differentiate a composite function and distinguish between a power function and an exponential functions; this is tested further due to a 'constant on top'.

6.2.2 HTT 2

HTT 2 concerns the algebraic and graphical representation of the derivative of the function $\sqrt{x^2}$. This task was inspired by a study performed by Pino-Fan, Godino, Font and Castro (2012). The specific function was chosen because it offers multiple paths to solution. The expressions $\sqrt{x^2}$ is equivalent to the simpler expression $|x|$ and consequently the task can be solved in two ways: one using techniques justified mainly by the theory of MO_3 and another, using techniques justified by of MO_4. However, activating one technique does not necessarily mean that one is unable to activate another; for the solution of the second task in HTT 2, one must be able to activate both, for a full solution.

Exercise 2

Given $f(x) = \sqrt{x^2}$, $x \in \mathbb{R}$.

a) Determine $f'(x)$ and draw the graph of $f(x)$ and $f'(x)$.

b) One of your students have determined $f'(x)$ via CAS and attained $f'(x) = \frac{x}{\sqrt{x^2}}$.

How do you respond to your student's result?

Figure 17: HTT 2

Task 2a presents three mathematical tasks. The first task is:

$t_{2.1}$: Given $f(x) = \sqrt{x^2}$ for $x \in \mathbb{R}$, determine $f'(x)$.

The first specific technique to solve $t_{2.1}$ is:

$\tau_{2.1-1}$: $\left(\sqrt{x^2}\right)' = \frac{1}{2\sqrt{x^2}} \cdot (x^2)' = \frac{x}{\sqrt{x^2}}, x \neq 0.$

This technique shares technology and theory with $t_{1.1}$ above, specifically applying the elements of the knowledge block of MO_3: (ii) and (V), except the restriction on the domain. The latter signifies that f is not a differentiable function (as it is not differentiable for all x in the domain), why $t_{2.1}$ does not belong to $\mathcal{T}_{3.1}$ and is not a typical task in MO_3'. However, the technique $\tau_{2.1-1}$, can be performed in a rather routine way, as the restriction on the domain can be explained by one of the following technological components, of which the second is part of an algebraic organisation undoubtedly preceding the teaching and learning of MO_3:

$\theta_{2.1-1.1}$: The prerequisites of the chain rule (V).

$\theta_{2.1-1.2}$: A fraction cannot take 0 in its denominator.

The second specific technique to solve $t_{2.1}$ is:

$$\tau_{2.1-2}: \quad \left(\sqrt{x^2}\right)' = (|x|)' = \begin{cases} (-x)', & x < 0 \\ (x)', & x \geq 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}.$$

Specifically justified by the theoretical elements of MO3: (ii) and (I), and the specific technological component:

$$\theta_{2.1-2}: \quad \sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}.$$

Technique $\tau_{2.1-2}$ naturally result in the same domain for f' as technique $\tau_{2.1-1}$, since

$$\lim_{x\to 0^-} \frac{|x|-|0|}{x-0} = -1 \neq 1 = \lim_{x\to 0^+} \frac{|x|-|0|}{x-0}.$$

This explanation signify that the technology explaining $\tau_{2.1-2}$, in particular the restriction of the domain, belongs to MO4 and the theory justifying the determination of the domain of f' is thus the definition of the derivative. Such explanation and justification are however not present in the transposed MO4', but another technological component belonging to MO4' also explains the determined domain of f', namely:

$\theta_{2.1}$: A function is not differentiable in a point if the graph has an 'edge' in that point.

This technology is however, also justified by definition of the derivative. The next mathematical tasks, $t_{2.2}$, asking the respondent to draw the graph of $f(x)$ and task $t_{2.3}$; the graph of $f'(x)$, can be seen as belonging to the same type of task:

$T_{2.2/2.3}$: Draw the graph of a function given its algebraic expression.

Using a technique explained by the following technology, they can in fact, be considered as such:

$\theta_{2.2/2.3-1}$: Graphs are constructed by plotting various coordinates (x_i, y_i) and drawing a curve going through these coordinates.

The above technique is not considered primary in neither MO3 nor MO4; however, this technique is considered as belonging primarily to organisations preceding the teaching and learning of MO4 and MO3, organisation of general function theory, where the theory justifying the above technology concerns theory of functions representations (in particular: algebraic and graphical); and the translation between these.

Upon the construction of the graph of f using a technique explained by $\theta_{2.2/2.3-1}$ an alternative technology to explain the construction f' $(t_{2.3})$ is the following:

$\theta_{2.3-1}$: The graph of f' shows f's progress in slope.

This technological component, though graphical in its nature, is justified by the 'meaning' of the derivative, i.e. the answer to the task $\mathcal{T}2$: what is the derivative function f'? Belonging to MO4 and present in MO4'. Furthermore, if one did not hold $\theta_{2.1-2}$, the work with $t_{2.2}$ is likely to probe the inherent identification: $\sqrt{x^2} = |x|$.

The theory justifying the specific techniques can thus vary: if one draws f' based on the graph of f the theory justifying this will encompass the theory regarding the relationship between a function and its derivative and therefore belong to MO4, while the techniques can also find its justification in the theory on functions and their representations. Task 2b poses a didactical task, $t_{2.1}$* ~ MO3, MO4, generating a punctual DO:

$t_{2.1}$*: How do you, as a teacher, respond to the result $\left(\sqrt{x^2}\right)' = \frac{x}{\sqrt{x^2}}$ attained by a student using a CAS-tool?

In a general sense, this task belongs to the didactical type of task:

$T_{2.1}$*: Respond to a student's mathematical claim.

Didactical techniques for such a task will often focus on getting the student to realize why or why not the mathematical claim holds. Didactical techniques to solve the concrete task $t_{2.1}$* could be a combination of the following, which are aiming at a disclosure of the non-differentiability of f in $x = 0$.:

$\tau_{2.1-1}$*: Ask the student whether $f'(x) = \frac{x}{\sqrt{x^2}}$ is defined for all x.

$\tau_{2.1-1.1}$*: Ask the student what it means that f' is not defined in $x = 0$.

$\tau_{2.1-1.1.1}$*: Ask the student to draw f and determine if it is differentiable in $x = 0$.

Additionally, to address the equality: $\frac{x}{\sqrt{x^2}} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$, and thus in particular $\sqrt{x^2} = |x|$, the following techniques could serve as relevant:

$\tau_{2.1-2}$*: Ask the student to draw f and f'.

$\tau_{2.1-2.1}$*: Ask the student whether $\sqrt{x^2}$ can be expressed any different.

$\tau_{2.1-2.1.1}$*: Ask the student whether $\frac{x}{\sqrt{x^2}}$ can be expressed any different.

However, in order to be able to mobilize these particular techniques, one needs to know first of all, that the student's answer is correct, secondly that the expression is not defined for $x = 0$; what this means and lastly, that the expression is equivalent to a much simpler expression. Thus, to use the didactical techniques $\tau_{2.1-1}$* – $\tau_{2.1-2.1.1}$*, a respondent must be able to mobilize both techniques associated with $t_{2.1}$ posed in 2a and thus techniques belonging to both MO_3 and MO_4. The analysis served as a basis for creating the following 'standard answer':

2a. See $\tau_{2.1-1}$ and $\tau_{2.1-2}$; of which the latter is considered the better solution.

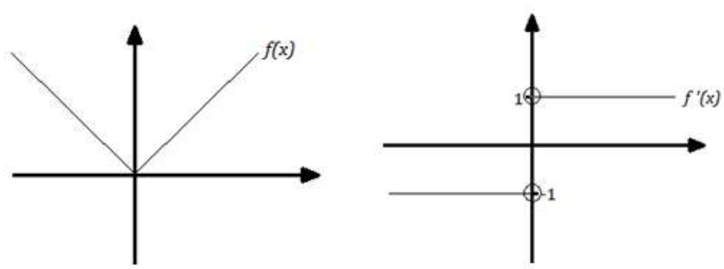


Figure 18: To the left the graph of $f(x) = \sqrt{x^2} = |x|$ and to the right, the graph of its derivative

2b. Firstly, it is clarified whether the student is aware that f is not a differentiable function on all of the interior of D_f, since it not differentiable in $x = 0$. For example via questions such as:

- Is $\frac{x}{\sqrt{x^2}}$ defined for all $x \in D_f$?
- What does it mean that it is not defined in $x = 0$?
- How does the graphical representation of $\sqrt{x^2}$ look like? Is it differentiable in $x = 0$?

It will also be essential that the student consider the meaning of the function expressions, in particular, to realize that $\sqrt{x^2} = |x|$. For example via the questions

- Try to plot f and f' using a CAS-tool.
- Can $\sqrt{x^2}$ and $\frac{x}{\sqrt{x^2}}$ be expressed differently? Why?

6.2.3 HTT 3

HTT 3 concerns the challenges students' face related to the determination of a functions monotonicity given the graphical representation of its derivative. The task is inspired by an exam for A-level students (22nd of May 2015, see Appendix A, p. iii), in which more than 50 percent of the students achieved 0-3 points out of 10 possible points and out of those, more than half achieved 0 points (Ministeriet for børn, undervisning og ligestilling, 2016).The task is thus, considered highly relevant for upper secondary school teachers, as well as pre-service teachers. The didactical task posed in task 3a relates directly to the following type of mathematical task is $T_{3.1} \subset \mathrm{MO}_3$':

$T_{3.1}$: Given the graph of f' determine the monotonicity of f.

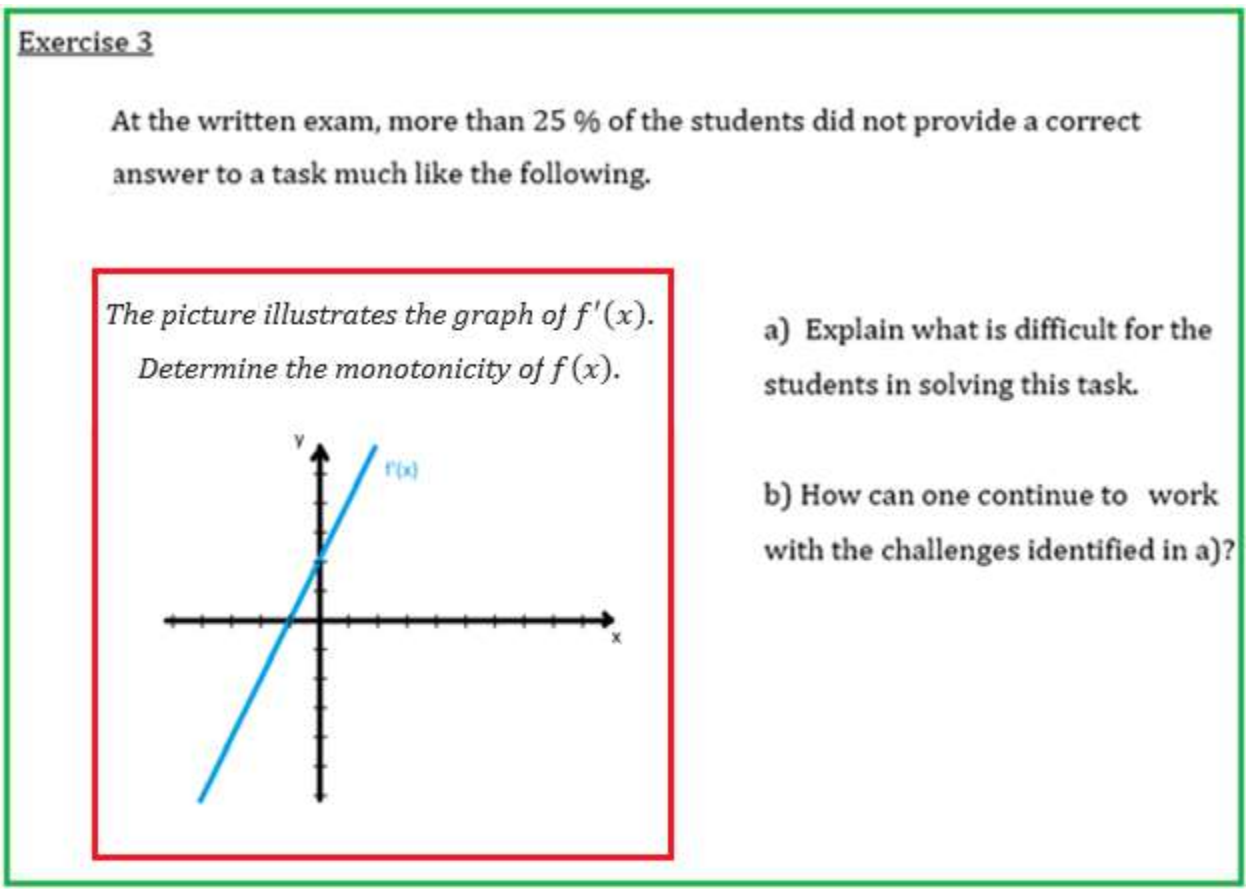


Figure 19: HTT 3

The technological and theoretical components being:

$\theta_{3.1}$: $f'(x) = 0$ when f' intersects the x-axis, f' is positive when $f'$is above the x-axis and f' is negative when f' is below the x-axis.

$\Theta_{3.1}$: Corollary 1, Corollary 2 (section 5.2.4).

We saw in section 5.3 that the latter theoretical component is transposed to MO_3'. The mathematical task $T_{3.1}$ can be seen as belonging to MO_3' as the task is a special case of the task $\mathcal{T}_{3.3}$: given the algebraic expression of f, determine the monotonicity of f. In fact, it is an 'easier' case as f' is given graphically and thus the monotonicity properties of f is derivable directly from the graph. Though this task is asserted to be easier than the typical task $\mathcal{T}_{3.3}$, the technological element, needed to explain the technique comprises of a translation of the prerequisites in Corollary 1 and Corollary 2 from an algebraic setting to a graphical setting. This entails in particular, distinguishing between the notions of *increasing* and *positive* as well as between *decreasing* and *negative*. Furthermore, a confusion between these notions is possibly enforced by students' tendency to assume resemblance between the graphs of a function and its derivative (Nemirovsky and Rubin, 1992). Additionally, this task can challenge students simply because it is not a typical task in MO_3'. As we saw in section 5.3, the tasks posed to students are often algebraic in their nature and are associated with specific algorithms; why the students' might have difficulties activating relevant techniques for $T_{3.1}$. Task 3a generates a punctual DO based on the specific didactical task $T_{3.1}{}^* \sim MO_3$':

$t_{3.1}{}^*$: Explain what is difficult for student in solving $t_{3.1}$.

In responding to such a task, teachers must know the correct techniques for solving the mathematical task, and the techno-theoretical discourse explaining it and therefore, this DO relates to punctual organisation generated by $T_{3.1}$. Some relevant techniques for $t_{3.1}{}^*$ are:

$\tau_{3.1-1}{}^*$: Identify the techniques associated with $T_{3.1}$.

$\tau_{3.1-2}{}^*$: Identify the related technology and theory.

$\tau_{3.1-3}{}^*$: Identify challenges related to the above identifications.

$\tau_{3.1-4}{}^*$: Identify the difference between $T_{3.1}$and other tasks relating monotonicity that students typically find easy/easier.

The two latter techniques, will be based on the participants' own experience with teaching or learning the subject. The task posed in 3a thus sets the stage for own personal conviction and experience with tasks concerning monotonicity as well as the teaching of these. However, it is reasonable to expect that $\tau_{3.1-4}{}^*$ specifically will entail a comparison between $T_{3.1}$ and $\mathcal{T}_{3.3}$ and thereby involve identifying the absence of an algebraic expression and the fact that f' is provided instead of f. Task 3b poses the didactical task generating a punctual DO:

$t_{3.2}$*: How can one continue to work with the challenges identified in 3a?

The participant's techniques to answer $t_{3.2}$* will naturally depend on the participant's respond to $t_{3.1}$*. Relevant possibilities are:

$\tau_{3.2-1}$*: Pose tasks of the same type as $T_{3.1}$.

$\tau_{3.2-2}$*: Ask the students to explain the meaning of f' and its graph.

$\tau_{3.2-3}$*: Pose tasks that involve distinguishing between the notion of an *increasing*, *positive*, *decreasing* and *negative* f'.

$\tau_{3.2-4}$*: Ask the students to draw the graph of f given the graph of f'.

$\tau_{3.2-5}$*: Ask student to draw f and f' when working with the functions algebraic expressions.

$\tau_{3.2-6}$*: Ask the students to explain the theory of MO_3' corresponding to Corollary 1 and Corollary 2 graphically.

A techniques such as $\tau_{3.2-1}$* aims at developing an algorithm for tasks of type $T_{3.1}$, while techniques such as $\tau_{3.2-2}$* and $\tau_{3.2-6}$* aims at developing the students' conceptual knowledge and $\tau_{3.2-5}$* aims at enhancing the inclusion of the graphical setting in teaching, but is not specifically targeted to develop the technology associated with $T_{3.1}$. The choice of technique thus express to some extent the participants' beliefs regarding what the students should learn.

In all, HTT 3 requires knowledge related to MO_3'; however, in a graphical setting. In solving HTT 3, it is paramount that the participants can identify, in particular, the technological component associated with the technique, as this is the key to solving the task and further, one needs to be able to identify the cognitive challenges associated with this component. Solving the didactical task $t_{3.2}$* requires the ability to select tasks in which the specific and necessary knowledge is developed. Based on the analysis of HTT 3, following standard answer was developed:

3a. $T_{3.1}$ is not a typical task since no algebraic expressions are provided and because f' is given instead of f. Students are used to being given an algebraic expression for f when asked to determine monotonicity properties of f. They will in the typical case use the function expression to determine f', determine the solutions to $f'(x) = 0$ and upon this; the signs of f between the zeros. The students might find it difficult to translate this method to a graphical setting and distinguish between the meaning of f' being increasing and f' being positive.

3b. See $\tau_{3.2-1}$* – $\tau_{3.2-6}$*.

6.2.4 HTT 4

HTT 4 presented below, focuses on the relationship between a function and its derivative in a graphical context. HTT 4 was inspired by the work of Haciomeroglu, Aspinwall and Presmeg (2010). The 3 figures in HTT 4 (figure 20) are taken directly from their study (Haciomeroglu et al., 2010, pp. 164-165). In the following the functions presented graphically in figure 1, 2 and 3 will be referred to as f' (though recognizing that this function is not a derivative function), g and h, respectively. Task 4a poses the concrete didactical task $t_{4.1}$* ~ MO4, MO5:

$t_{4.1}$*: Your student presents the graph of a function (figure 1 in HTT 4), which she claims to be a derivative function. What do you say to your student?

Which is of the same type as $T_{2.1}$* in 2b. As the practice of the teacher aims at making the students learn, it is considered implicit that the responds should aim at this in particular (however, it is recognized that this interpretation is not guaranteed). A teacher can take various approaches in responding to the student, one of which could be the following:

$\tau_{4.1-1}$*: Ask your student to explain what it means when f' jumps.

$\tau_{4.1-2}$*: Ask your student to draw the original function f and (based on this) to consider $f'(1)$.

Aiming to uncover the student's argument, the related misconceptions and possibly facilitate a way for the student to realize these misconceptions. However, for such didactical techniques to be meaningful it is asserted that the teacher must be able to assess the student's answers. Task $t_{4.1}$* thus relates to the mathematical tasks:

$t_{4.1}$: Given the graph of a function, in particular f', determine if it is a derivative function.

$T_{4.1.1}$: Can a derivative function have a jump discontinuity?

The latter mathematical task requires a direct activation of a theoretical component belonging to MO4, since the answer to $T_{4.1.1}$ is simply *no*; an answer justified by:

$\Theta_{4.1.1}$: Theorem 2

Exercise 4

Marie has drawn the graph illustrated below (*figure 1*) which she claims to be the graph of a derivative function of some unknown function f.

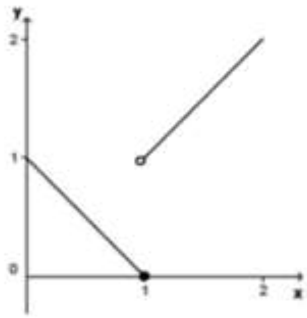


Figure 1

a) How do you respond to her claim?

Marie draws two additional functions, illustrated below (*figure 2* and *figure 3*), which she claims to be antiderivative function to the function presented in (a).

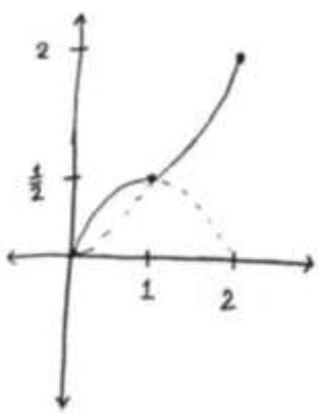


Figure 2

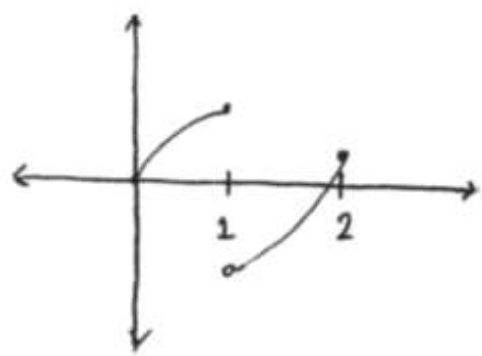


Figure 3

b) Provide Marie with exhaustive feedback.

Figure 20: HTT 4

This theorem was not present in the transposed MO4’, however if one does not hold this knowledge, activating the following mathematical technique might serve as a way to realize that the graph presented by the student is not a derivative:

$\tau_{4.1}$: Reading the graph of f' to identify that the original function cannot be differentiable in $x = 1$.

The technological and theoretical components explaining and justifying $\tau_{4.1}$ belongs to the knowledge block of MO_4':

$\theta_{4.1}$: The graph of f' shows f's progress in slope. A function is not differentiable in a point if the graph has an 'edge' in that point.

$\Theta_{4.1}$: Definition 2.

Also, if one is able to activate $\tau_{4.1}$, a possible didactical technique for $t_{4.1}$* is:

$\tau_{4.1-3}$*: Tell your student that the graph does not represent a derivative function because derivative functions cannot have jump discontinuities.

$\tau_{4.1-4}$*: Tell your student that the presented graph does not represent a derivative function, because the original graph has an 'edge' (and is thus not differentiable) in $x = 1$.

Related to task $T_{4.1.1}$ it is noted, that the correct answer to $T_{4.1.1}$ is considered faulty, if justified by the technological component:

$\theta_{4.1.1^-}$: A derivative function cannot be discontinuous.

We saw a counterexample for this statement in section 5.2.2. Task 4b poses a question encompassing the concrete didactical task:

$t_{4.2}$*: Your student shows you the graphs of g and h and claims that these are antiderivative functions for f'. Provide exhaustive feedback to your student.

This task is also a special version of $T_{2.1}$*. An answer to this task depends on how the respondent have answered in 4a, however, in this context; the task is treated separately from possible answers, given in 4a. To answer this task, respondents must activate techniques for the following mathematical task:

$t_{4.2}$: Given the graph of g and h (figure 2 and 3 in HTT 4), determine if they are differentiable.

$t_{4.3}$: Are g and h, presented graphically in figure 2 and 3, antiderivatives functions of f'?

With corresponding technique:

$\tau_{4.2}$: Reading the graphs of to identify that they are not differentiable functions.

Explained and justified by knowledge components belonging to MO4':

$\theta_{4.2}$: A function is not differentiable in a point if the graph has an 'edge' or is discontinuous, in that point.

$\Theta_{4.2}$: Definition 2

Furthermore, the following technique related to MO6 is relevant:

$\tau_{4.3-2}$: Reading the graphs to identify that h is not an antiderivative of f'.

Explained and justified by:

$\theta_{4.3}$: If $f > 0$ on its entire domain then its antiderivative function is strictly increasing on the entire domain.

$\Theta_{4.3}$: $\int_a^b f(x)\,dx = F(b) - F(a)$ for all $[a, b] \in D_f$ and for an antiderivative F.

Upon these techniques, an answer to $t_{4.2}$* possibly entails showing the student the above mathematical arguments (call these $\tau_{4.2-1}$* and $\tau_{4.2-2}$* corresponding to $\theta_{4.1-1}$ and $\theta_{4.3-2}$, respectively) and by using the following didactical technique, elaborating $\tau_{4.2-1}$*:

$\tau_{4.2-3}$*: Illustrating that none of the function are differentiable in $x = 1$ by using secant lines on the right and left side of $x = 1$; showing that the functions does not have a unique tangent in $x = 1$.

The presented HTT is highly associated with techniques belonging to and MO4'; justified by the definition of the derivative as well Theorem 2. HTT 4 will uncover whether the participants holds the knowledge that is Theorem 2 and if not; whether they are able to activate mathematical techniques associated with MO4' as well as MO6 in order to deal with the student's claim. Notice how it is possible that the relevant techniques are present in the transposed MO6'; however, this has not been investigated, why the techniques are only said to belong to MO6). As a minimum, the task requires activation of mathematical techniques justified by the definition of the derivative. Based on the a priori analysis, the following standard answer was developed:

4a. Ask Marie to explain what it means when f' 'jumps' and ask Marie to draw f as well as to consider what $f'(1)$ is.

4b. Note that Marie has drawn a function with an 'edge' and a discontinuous function. None of these are differentiable in $x = 1$ and therefore figure 1 is not the graph of their derivative. Additionally, figure 3 does not show an antiderivative for the function in figure 1, since the antiderivative of a positive function is increasing. Use the definition of differentiability in a point to explain why a function is not differentiable at an 'edge' and in a point of discontinuity. Illustrate the first with secants to the right and the left of $x = 1$.

6.2.5 HTT 5

HTT 5 continues to focus on MO_4 in the same manner as HTT 4. It is also inspired by the work of Haciomeroglu and colleagues and the figures in HTT 5 (figure 21)are taken directly from their research (Haciomeroglu et al., 2010, pp. 162-163). The first task in HTT 5 centres on the following mathematical type of task:

$T_{5.1}$: Given the graphical representation of f' draw the graph of f with certain properties.

Specifically, the 5a centres on:

$t_{5.1}$: Given the graphical representation of f' (HTT 5a) draw the graph of f with the additional assumption $f(0) = 0$.

The first technique for this task is:

$\tau_{5.1-1}$: Sketching a graph of a function going through (0,0) with decreasingly negative slope on [0,1) (convex); an inflection point in $x = 1$; and with increasingly negative slope on (1,2] (concave).

The explanation and justification of this technique is the same as for $\tau_{3.1}$:

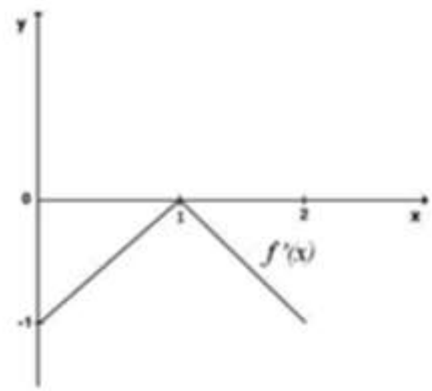
$\theta_{5.1-1}$: $f'(x) = 0$ when f' intersects the x-axis, f' is positive when $f'$is above the x-axis and f' is negative when f' is below the x-axis.

$\Theta_{5.1-1}$: Corollary 1 and Corollary 2 (section 5.2.4)

The convexity and concavity of f on [0,1) and (1,2], respectively, follows immediately, since $f'(x) = 0$ in $x = 1$. The technique is thus explained and justified by MO_3'. Using $\tau_{5.1-1}$, one cannot however, determine the value of the function in the inflection point. The following additional technique for achieves this:

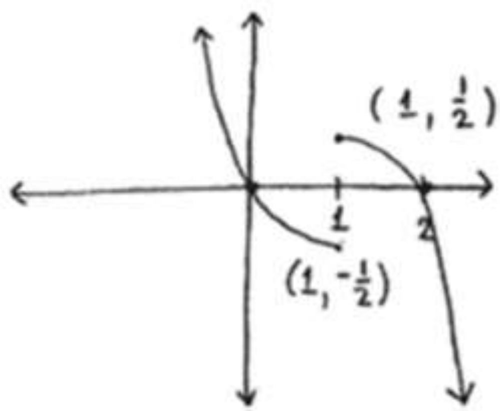
Exercise 5

The picture below show the graphical representation of the function $f'(x)$.



a) Assume $f(0) = 0$ and draw the graph of $f(x)$.

The picture illustrates one of your students' answer to the task proposed in a).



b) How do you respond to your student?

Figure 21: HTT 5

$\tau_{5.1-2}$: Reading the graph to construct the algebraic expression for f', integrating this and using the condition $f(0) = 0$ to determine the integration constant and determining $(1, f(1))$. Using $\tau_{5.1-1}$, indicating the inflection point.

The additional techno-theoretical discourse justifying this technique concerns the extraction of algebraic expression from graphical representations:

$\theta_{5.1-2}$: The graph shows straight lines on (0,1] and (1,2): they can be considered on their natural domain $\mathbb{R}$.

$\Theta_{5.1-2}$: Function theory: The straight line: $y = ax + b$, $a = \frac{y_2 - y_1}{x_2 - x_2}$ and b is intersection with the y-axis, and a function can only attain one function value for each point in the domain.

Furthermore, the integration technique included $\tau_{5.1-2}$ is explained and justified by the knowledge block of MO_5, while the explanation regarding *why* it is relevant to integrate in this context belongs to MO_6 (also MO_5' and MO_6', respectively, see Clausen et al., 2011a, pp. 54-63; as noted earlier, this has not been substantiated, why they are referred to as MO_5 and MO_6). Task 5b poses a didactical task of type $T_{2.1}$* directly related to $t_{5.1}$ in 5b and thus relates directly to MO_3', MO_5 and MO_6.

$t_{5.1}$*: A student presents an answer to $t_{5.1}$ (the graph presented in 5b). What do you say to your student?

The relevant didactical techniques associated with this task requires the ability to activate relevant mathematical techniques for $t_{5.1}$, since it cannot be expected of anyone to provide feedback to a student regarding a task that the person cannot solve. Of course, techniques such as asking the student to explain the result or present arguments as to why the function is in fact the antiderivative of f can be mobilized and through such dialogue, a teacher might be able to realize how the task is solved correctly. In the following however, it is assumed that the participant have answered $t_{5.1}$ correctly. A relevant subtask to $t_{5.1}$* is:

$t_{5.1.1}$*: Analyse and assess the student's answer to $t_{5.1}$.

With corresponding technique:

$\tau_{5.1.1-1}$*: Reading the graph to identify that the function is discontinuous.

$\tau_{5.1.1-2}$*: Reading the graph to identify that the function is not differentiable in $x = 1$.

$\tau_{5.1.1-3}$*: Comparing the student's graph with the correct graph or the graph of f' to see that the student has drawn f on a domain too large.

$\tau_{5.1.1-4}$*: Comparing the student's graph and the correct graph to see that the two graphs have the same progress in slope (apart from in $x = 1$).

$\tau_{5.1.1-5}$*: Identify the intersection on the y-axis of both 'branches' in $y = 0$ (considering the natural extension of the second 'branch').

$\tau_{5.1.1-6}$*: Identifying the arrows as meaningless.

$\tau_{5.1.1-7}$*: Reading the graph to conclude that the student's graph incorrectly attains two values for $x = 1$.

Knowledge belonging to MO4 is necessary for activating techniques $\tau_{5.1.1-1}$* – $\tau_{5.1.1-3}$*, while $\tau_{5.1.1-5}$* requires knowledge related to an organisation of basis function theory. An additional subtask might be:

$t_{5.1.2}$*: Which technique has the student used to get this answer?

Which undeniably relates to the subtask:

$t_{5.1.2.1}$*: How did the student produce the coordinates $(1, -\frac{1}{2})$ and $(1, \frac{1}{2})$?

Solving the latter requires activating the following techniques:

$\tau_{5.1.2.1-1}$*: Identify the use of technique $\tau_{5.1-2}$ to produce the point $(1, -\frac{1}{2})$.

Activating this techniques require that one is able to activate the actual mathematical technique, $\tau_{5.1-2}$. Related to the second coordinate indicated on the students drawing, is the technique:

$\tau_{5.1.2.1-2}$*: Identify the use of the condition $f(0) = 0$ in the determination of the integration constant in the algebraic expression of f defined on (1,2].

This technique however, requires that one holds knowledge regarding the integration constant, which determines the vertical placement of the second 'branch'. It is recognized that this interpretation is only one out of multiple possible interpretations; for example, since the graph drawn by the student is consistent with $k_1 = k_2 = 0$ (k_1 and k_2 being the integration constants in the expression for f on [0,1] and (1,2], respectively), another obvious interpretation could be to say that the student had neglected to include the constants when performing the integration of f'. Upon their analysis of the student's answer, the participants must determine what to say to the student i.e. answer task $t_{5.1}$*. Relevant techniques in this regard could be:

$\tau_{5.1-1}$*: Tell your student that the graph represents a function, which is not differentiable in $x = 1$.

$\tau_{5.1-2}$*: Explain to your student why the function is not differentiable in $x = 1$ (for example by using secant on the left side and right side of $x = 1$).

$\tau_{5.1-3}$*: Tell your student that f' is defined in $x = 1$.

$\tau_{5.1-4}$*: Tell your student what it means for f that f' is defined in $x = 1$.

$\tau_{5.1-5}$*: Tell your student that f is only defined on [0,2].

$\tau_{5.1-6}$*: Tell your student why f is defined on [0,2].

$\tau_{5.1-7}$*: Tell your student that the arrows in the ends of the curve do not have any mathematical meaning.

$\tau_{5.1-8}$*: Tell your student that for each point in the domain of a function there can only be one corresponding function value.

$\tau_{5.1-9}$*: Tell your student that the condition $f(0) = 0$ only applies for the part of f defined for $x = 0$.

All of the above techniques has a counterpart in an approach focusing on dialogue, for example:

$\tau_{5.1-1.1}$*: Ask your student if f is differentiable in $x = 1$.

$\tau_{5.1-4.1}$*: Ask your student to consider how $f'(1) = 0$ while f is not differentiable in $x = 1$.

$\tau_{5.1-9.1}$*: Ask your student if the prerequisite can help determine k_2.

Techno-theoretical components belonging to MO_3', MO_4', MO_5 and MO_6 as well as organisation of basic function theory, are relevant in solving HTT 5. As a minimum, 5a can be solved using techniques justified by MO_3', while 5b requires knowledge related to MO_4' and for a full analysis of the student's answer, techniques of integration is also required. However, HTT 5, as HTT 1, does not only require the participants to use these techniques, but also that they are able to recognize them in a situation where they have been used incorrect.

Based on the a priori analysis the following 'standard answers' was developed:

5a. Method 1: The graph is sketched based on a +/- sign chart constructed by using the graph of f' (which shows that the function is decreasing with inflection point in $x = 1$); Moreover, the derivative is increasing on [0,1] and decreasing on [1,2] which means that the graph of f is convex and concave on these intervals, respectively.

Method 2: Using the provided graph, the algebraic expression of f' is determined:

$$f'(x) = \begin{cases} x-1, & 0 < x \leq 1 \quad := f_1{}' \\ -x+1, & 1 < x < 2 \quad := f_2{}' \end{cases}$$

By integration one gets:

$$f(x) = \begin{cases} \frac{1}{2}x^2 - x + k_1, & 0 \leq x \leq 1 \quad := f_1 \\ -\frac{1}{2}x^2 + x + k_2, & 1 < x \leq 2 \quad := f_2 \end{cases}$$

Using the assumption $f(0) = 0$ one gets:

$$0 = f_1(0) = \tfrac{1}{2}0^2 - 0 + k_1 = k_1, \text{ hence } k_1 = 0$$

The function value for $x = 1$ (the inflection point) can thus be calculated:

$$f_1(1) = \tfrac{1}{2}1^2 - 1 = -\tfrac{1}{2}.$$

Since the function must be continuous, it holds that $f_2(1) = f_1(1)$, we have (this can be left out):

$$f_2(1) = -\tfrac{1}{2}1^2 + 1 + k_2 = -\tfrac{1}{2} = f_1(1), \text{ hence } k_2 = -1\ .$$

Thus, the graph of f must look like the following:

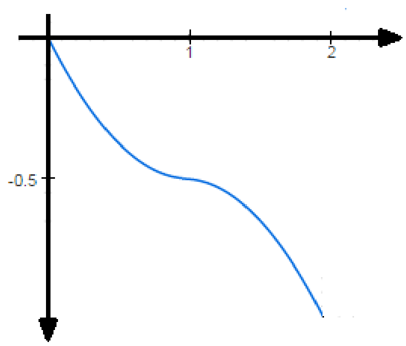


Figure 22: The graph of f as constructed by method 2.

5b. The student likely constructed the algebraic expressions of f' and f since the points $(1,-1/2)$ and $(1,1/2)$ are indicated on the graph. The error occurs when the constants in the two algebraic expressions of f is determined: If one uses $f(0) = 0$ on both expressions and thus overlook that the second expression is defined only for $x \in (1,2]$ then one also attains $k_2 = 0$. The determined expression gives the incorrect 'right branch' of the graph. Moreover, the result is a discontinuous and thus non-differentiable function. The student has furthermore, wrongly drawn a 'function' with two function values in $x = 1$, drawn f on a larger domain than f', and has drawn arrows in the ends of the curve, which has no mathematical meaning .Start by asking if a differentiable function can look like the one drawn – if the student says yes; you point to the discontinuity problem. Following, ask your student to explain the method used and ask if the prerequisite $f(0) = 0$ can help determine k_2 (or less formal: the position of the right branch).

7 The Participants' Performances

In this chapter, the collected empirical data (see App C1 – C5)[6] will be analysed, with the main purpose of creating a basis for answering Research Question 2, namely the question:

> *Do the participants' answers to the HTT reflect their different amounts of teaching experience? In what way?*

Prior to the collection of the empirical data, some of the most central hypotheses, regarding the difference in performances of the two groups, was:

(1) The teachers will activate didactical techniques, which are more appropriate than will the university students.

(2) The university students will, to a greater extent, activate appropriate mathematical techniques related to MO_4 and provide more relevant answers to tasks that belongs to this MO, than will the teachers.

These hypotheses will be addressed throughout the *a posteriori* analysis.

7.1 An overview of the Results

In order to create an overview of the data, the participants' responses are given points, which are collected and presented in a table below (table 1). These points do not directly refer to their techniques in solving the tasks, however they refer implicitly to these, since they are given according to performance, which highly depends on the respondents' activated techniques. The method used to distribute the points was presented in section 4.2.3 and will be discussed further in Chapter 9.

In the table of points, each participant's collected points out of the possible 33 points are presented in percentage (TTL (%) participant). The average collected points among teachers, students and all participants out the possible points are shown in percentage. I.e. all the teachers' collected points out 132 possible points, all the students' collected points out of 165 possible points and all the participants' collected points out of 297 possible points are shown in percentage (in **bold**). All the teachers' collected points on each task out of 12 possible points (TTL T (%) task) and all the students' collected points on each task out of the 15 possible points (TTL S (%) task) are shown in percentage. Furthermore, all the

[6] All the participants' answers to HTT 1 are presented in C1 in the order T1, T2, …, S8, S9. All the participants' answers to HTT 2 are presented in C2 in the order T1, T2, …, S8, S9. And so forth.

participants collected points on each task out of the 27 possible points (TTL (%) task) is shown in percentage.

POINTS (0-3) *Participant/Task*	*1a*	*1b*	*1c*	*2a*	*2b*	*3a*	*3b*	*4a*	*4b*	*5a*	*5b*	*TTL (%) participant*
T1	3	0	1	1	0	3	3	0	0	1	1	39.4
T2	3	2	3	0	0	1	1	0	0	0	0	30.3
T3	1	1	1	3	2	2	2	3	2	0	2	57.6
T4	0	1	1	0	0	2	2	0	0	2	0	24.2
TTL T (%) task	58.3	33.3	50	33.3	16.7	75	66.7	25	16.7	25	25	**37.9**
S5	3	1	1	3	1	1	2	0	1	3	1	51.5
S6	0	1	1	1	1	1	0	0	0	3	1	27.3
S7	3	3	2	3	2	2	1	0	0	3	2	63.6
S8	0	1	1	2	2	2	1	0	1	3	1	42.4
S9	0	0	0	0	0	1	2	0	0	2	1	18.2
TTL S (%) task	40	40	33.3	60	40	46.7	40	0	13.3	93.3	40	**40.6**
TTL (%) task	48.1	37	40.7	48.1	29.6	55.6	51.9	11.1	14.8	63	33.3	**39.4**

Table 1: Scheme of the achieved points for each participant for each task.

The most striking characteristic of the point table is the overall low percentages. In all, the participants only collected 39.4 %, and thus less than half, of the possible points. The teachers collected on average 37.9 % of the possible points while the students on average collected 40.6 % of the possible points and therefore, as groups, the students performed slightly better; however, with the number of participants this difference is vanishing. None of the tasks was answered completely comprehensively by all of the participants. Furthermore, no clear pattern emerges from this table, between the two groups. However, in regards to the two hypotheses of the introduction, the table will not provide this information because the characteristics of the tasks (e.g. which tasks relates/belongs to MO_3 and which relates/belongs to MO_4) do not appear. Furthermore, as we saw in the *a priori* analysis, some tasks could be solved using techniques from either MO_3 or MO_4 or they necessitated to some extent, techniques belonging to both organisations (as well as other organisations). It therefore seems necessary to consider the concrete responses to clarify whether, for example, S7 was given 2 points in task 2b because S7 did not activate the necessary techniques related to MO_4 or because S7 did not include considerations regarding the simplification of the expression $\frac{x}{\sqrt{x^2}}$. In the following section, the participants' responses will be considered in depth, engaging mostly in those, which contain irrelevant techniques or is characterised by a general absence of techniques.

The results presented in the table, will be treated in more detail in the discussion; especially considerations regarding how the two data collecting methods could have had a negative impact on the students' as well as the teachers' performances. In the present context, it should however still be noted that the overall performance of the participants' shows that they were more challenged by the tasks than what was expected (see p. 47).

7.2 Performances on HTT 1

In chapter 6, it became clear that the three didactical tasks included in HTT 1 relates to MO_3'. We shall see in the *a posteriori* analysis below how the hypothesis (1) is not supported by the participants' performances in HTT 1, and we shall see that this HTT proved a challenge for the majority of the participants; in particular to maintain a correspondence between the answers to the three tasks.

About half of the respondents, two teachers and three students (T3, T4, S6, S8 and S9) was not able to mobilize all necessary techniques for solving 1a. T3 and S6 was not able to mobilize the mathematical technique for solving the inherent mathematical task $t_{1.1}$ (differentiate e^{x+1}) (both of these participants received help from an outside source to determine that the student's answer was wrong, and upon this, they carried out their analysis); further T3 and T4 excluded particular aspect of their analysis in 1a and thus came to inexpedient conclusions. S8 and S9 did not mobilize appropriate techniques related to the analysis, to help them identify the student's mathematical technique: S8 reached a correct conclusion with the help of S7, while S9 stated, "The student has not differentiated correctly". Among those who solved the task correctly, a version of techniques $\tau_{1.1-1}{}^*$ – $\tau_{1.1-4}{}^*$ (identified in section 6.2.1) was used. For T3 this was also the case, however, out of the two apparent algebraic manipulations in the student's work, only one was considered; namely that of 'moving' the exponent down in front of the expression. T3 thereby used an insufficient version of technique $\tau_{1.1-2}{}^*$:

$\tau_{1.1-2.1}{}^{*-}$: Identify the explicit algebraic manipulations of 'moving' the exponent down in front.

And upon this, T3 activated:

$\tau_{1.1-3.1}{}^{*-}$: Identify the explicit algebraic manipulations as those associated with the rule for differentiating exponential functions: $(e^{kx})' = ke^{kx}$.

The most interesting performance in 1a was that of T4. In analysing the student's answer to task $t_{1.1}$, T4 makes an assumption, which leads to a wrong analysis. The first observation, which T4 expressed explicitly, was, "He puts the inner function down in front". Combined with the other activated techniques, it seems as if T4 considered f as identified by the student as a composite function:

$\tau_{1.1-1}{}^{*}$	Identify that the answer is wrong.
$\tau_{1.1-2.2}{}^{*-}$:	Identify the explicit algebraic manipulations of 'moving' the *inner function* down in front.
$\tau_{1.1-3.1}{}^{*-}$:	Identify the explicit algebraic manipulations as those associated with the chain rule (V).
$\tau_{1.1-4.1}{}^{*-}$:	Identify the student's technology: e^{x+1} is a composite function.
$\tau_{1.1-5}{}^{*-}$:	Conclude that the student has forgotten the rule for differentiating composite functions.

Illustrated in the picture below is T4's answer to 1a. It states, "He knows that he has to differentiate. And he knows he has to consider the inner and the outer function. He also knows that he has to put the inner function 'down in front'. He forgets the rule for differentiating composite functions".


a) Analysér og vurdér svaret.
Han ved at han skal differentiere.
Og han ved, at han skal kigge på indre og ydre funktion.
Han ved også, at han skal ~~[illegible]~~ den indre funktion "ned foran"
Han glemmer reglen for diff. af sammensat fkt.

Figure 23: T4's answer to exercise 1a.

Thus, T4 did not explicitly consider the possibility that the student have identified f incorrectly i.e. T4 did not acknowledge the inherent task $t_{1.1.1}$: determine what type of function f is, and T4 did not consider the use of any other differentiation rule.

Among those participants who identified the student's use of the rule for differentiating power functions (T1, T2, S5, S7 as well as S6 and S8; though with help), not all pointed to the student's primary problem of identifying f, only T1 and S7 stated this explicitly, though this might be implicit for some of the participants. For example, T2, who in 1a stated, "He has used the calculation rule for how you differentiate a power [function]" and thus, did not explicitly states *why*. However, when solving 1b, T2 stated, "Remember that *e* to the power of anything gives *e* to the power of anything and if 'anything' is a function, you have to use the chain-rule". In this correction, it is apparent that T2 tried to address the student's identification problem. However, T2 avoided confronting *why* the student had interpreted f as a power function. In fact, considering all the participants' corrections given in task 1b it appears that all of the techniques, which were identified *a priori*, were present, except one, namely:

$\tau_{1.2-5}$*: State why f is not a power function (i.e. the difference between a power function and an exponential function).

Meaning, that also T1 and S7 who explicitly identified the student's problem in identifying f correctly in 1a, did not try to remedy the student's wrongful identification through the correction in 1b. S7 identified explicitly the student's misunderstanding related to the different notation of constants and variables in 1a. However, S7 did not address this in the correction in task 1b. S7 chose instead, to focus the correction on which type of function f *is not* and which rules *cannot* be used and thus, neglected to state what type of function f *is*, why it is so and upon this, which rules are relevant. Illustrated below is the answer of S7.

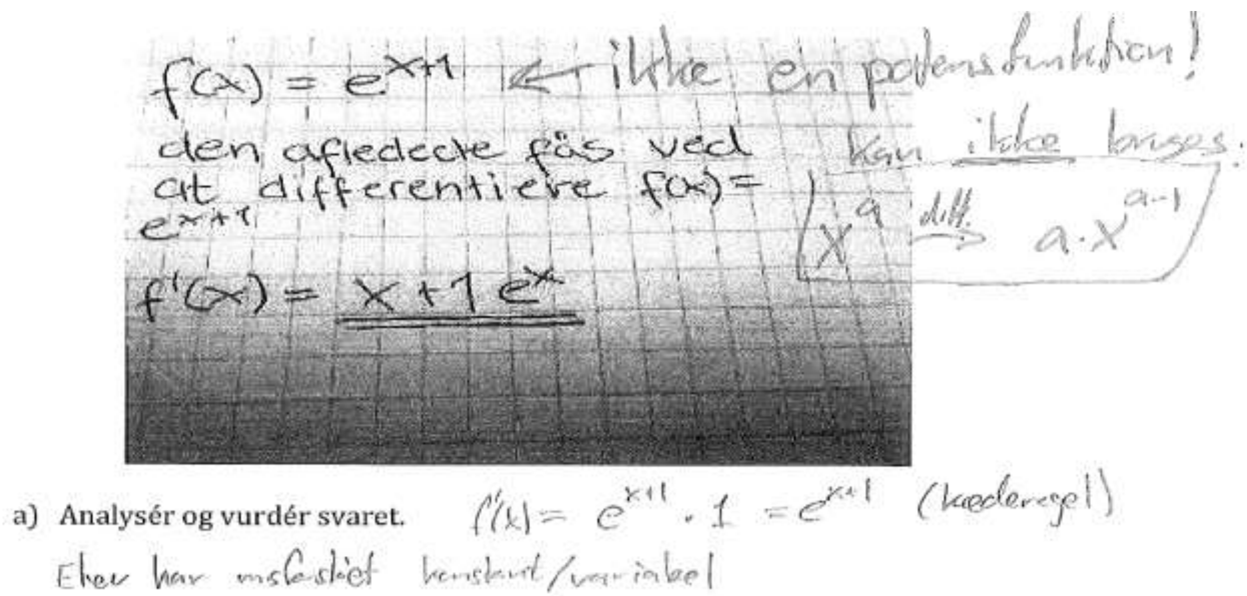


Figure 24: S7's answer to task 1a and 1b.

S5 solved 1a correctly, saying (accessible via audio recordings) "[the function] is differentiated using rule for power [functions]". However, in answering 1b, S5 wrote (see figure 25) "The rule $(x^a)' = ax^{a-1}$ does not work, since the expression is of the type a^x. Differentiate through use of the rule for composite function". S5 did not confront the possible reason for the failed distinguishment between exponential and power functions. Furthermore, S5 first identified f as an exponential function and then stated that it should be differentiated as a composite function, which must be considered as a confusing statement.

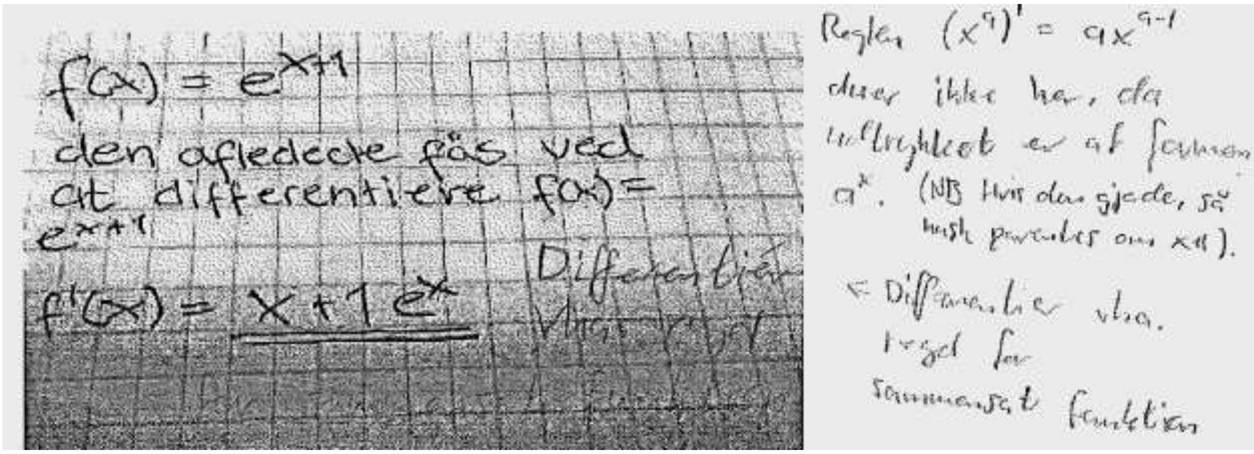


Figure 25: S5's correction in task 1b.

T1 also answered correctly in 1a and answered 1b using technique $\tau_{1.2-1}$*: state that the answer is wrong (in symbols: ÷) identified in the *a priori* analysis, with the addition of a technique (not *a priori* identified):

$\tau_{1.2-10}$*: Ask the student to contact you.

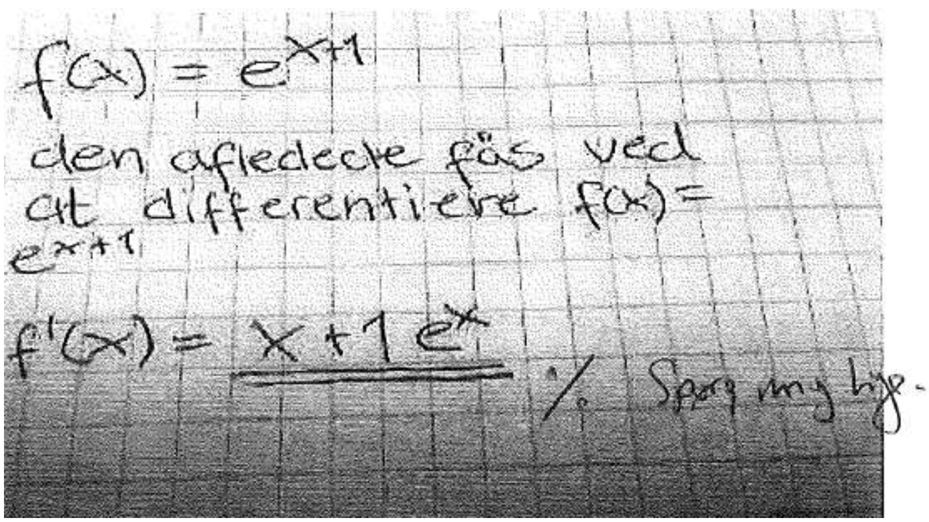


Figure 26: T1's answer to task 1b.

T1 explains (on audio recordings) how the choice base on the conviction that "It is very difficult to explain in writing what the problem is here because he mixes functions together". Which shows that T1 want to address the student's identification problem. However, this answer is given 0 points based on the two criteria of assessment, given in the a priori analysis (section 6.2.1). It is however, not denied that in some cases students' challenges are better addressed in dialogue and that this might be one of those cases.

Below is a table, showing the participants' answers to 1c and it shows how a variety of functions was suggested. An in depth analysis of the suggestions will not be presented here, as each of them should be considered in light of the related analysis and correction. However, it is noted, that although the majority of the suggestion seems meaningful, the points were generally low, due to lack of correspondence to the corrections in 1b and the analysis in 1a.

Participant	Tasks (All of the type: Differentiate the function …)
T1	$x^3 + 2x^2 + x - 7,\ e^x,\ e^x + x^2,\ \ln(2x+1),\ \sqrt{x^3 - 7},\ e^{x+1}$
T2	$\sqrt{2x+1},\ e^{x^2}$
T3	$e^x,\ x+1,\ e^{2x}, (x+3)^2$
T4	$\sqrt{x+2}$
S5	$x^{a+1},\ a^{3x}$
S7	$10^x,\ \sin(x^2),\ x^5$
S8	$\sin x^2$

Table 2: The participants' answers to exercise 1c (participant S6 and S9 did not respond).

For example, T3 provided a correction (figure) using technique $\tau_{1.2-3}$*: write the correct calculations and a version of $\tau_{1.2-9}$*: state the correct differentiation rules associated with $t_{1.1}$ (figure 27). To uncover whether the student understood the correction, T3 posed the task: "Differentiate the following functions: $e^x,\ x+1,\ e^{2x},\ (x+3)^2$" among which, one can be solved by a direct copy of the correction $(x+1)$ and only one requires the rule illustrated in the correction. Moreover, it was asserted by T3 in 1a that the primary problem was the distinguishment between functions of the type e^{kx} and e^{x+k}, why e^x, $x+1$ and $(x+3)^2$ appears unable to uncover whether the correction remedied this particular challenge.

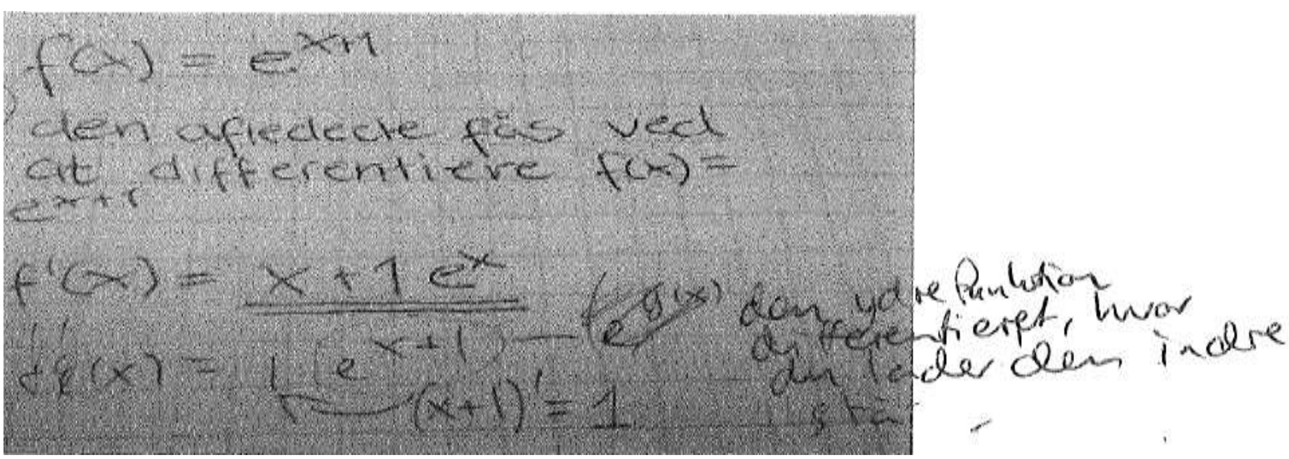


Figure 27: T3's answer to exercise 1b.

Another example is T4 who, upon the analysis (amounting to an assertion that the student had forgotten the chain rule) corrected the work by providing the chain rule. As it appears in figure, this in fact reproduced, wrongly (see p. 40):

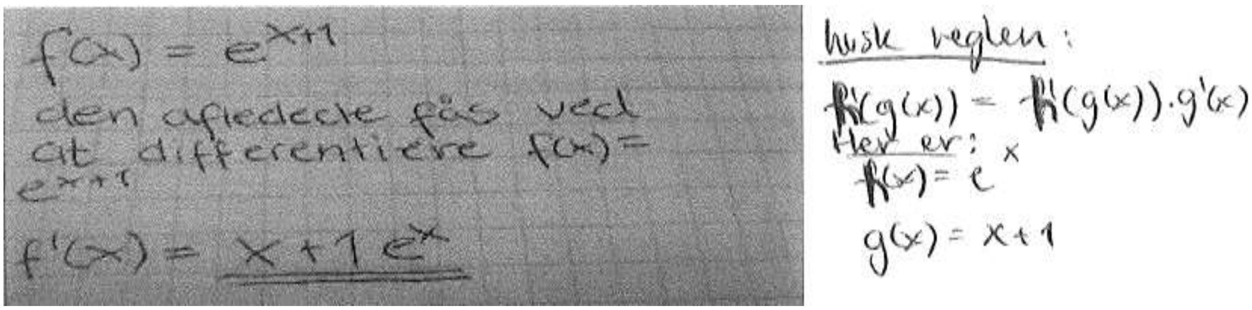


Figure 28: T4's answer to task 1b

The function provided in 1c, to be differentiated, was $\sqrt{x+2}$. T4 was given 1 point for this answer because it provides an opportunity for the student to work with the chain rule. However, since the correction does not provide any tool for the student to transfer the rule to another scenario (i.e. from an composite function with outer function e^x to a composite function with outer function $\sqrt{x}$) it makes little sense to assert that $\sqrt{x+2}$ should uncover anything but the fact that the student has 'remembered' the rule.

The participants' answers to 1c illustrates furthermore, that none of the participants were able to determine and pose just a single task to 'test' the student's primary problem. Those who did pose *one* problem (T4, S8: S8 suggested the function $\sin x^2$) are not testing whether the student can distinguish between an exponential and a power function. Furthermore, the generating of multiple task (for example T1, who constructed six functions to be differentiated) also suggests some difficulty for the participants in determining a single task, which encompass precisely the challenges they have identified. In all, the answers to HTT 1 does not necessarily signify that the participants do not hold knowledge of MO_3', but they signify that appropriate didactical techniques related to the specific student answer are not easily mobilized.

7.3 Performances on HTT 2

We saw in chapter 6 how HTT 2 can be solved in a variety of ways, however the most simple being based on the recognition of the equality $\sqrt{x^2} = |x|$. Overall, HTT 2, challenged the participants more than expected. Task 2a posed the participants three mathematical tasks. In the first task ($t_{2.1}$), the participants were to determine the derivative of $f(x) = \sqrt{x^2}$; in the *a priori* analysis two main techniques to perform this task was identified: $\tau_{2.1-1}$ and $\tau_{2.1-2}$, the latter depending on technological component $\theta_{2.1-2}$ ($\sqrt{x^2} = |x|$).

Only four participants (and thus, less than half: T3, S5, S7 and S8) used one of these correctly. Among the three students, S5 and S7, used a version of technique $\tau_{2.1-1}$, determining f' through a use of the chain rule. Both of these participants also explicitly stated $\theta_{2.1-2}$, however without using it to determine f'. Below is the work of S5 and S7.

Figure 29: S5 determines $f'$by using $\tau_{2.1-1}$

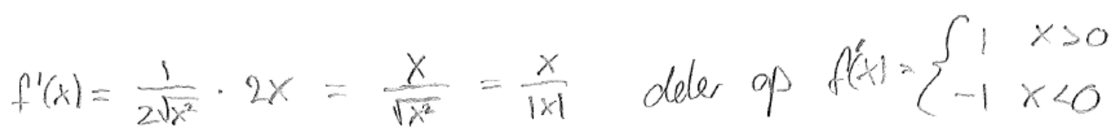


Figure 30: S7 determines f' by using $\tau_{2.1-1}$.

It is not explicit which techno-theoretical components S5 and S7 used for determining the domain of f', however the presence of $\theta_{2.1-1.2}$: a fraction cannot take 0 in its denominator, appear plausible. The other two participants, T3 and S8, used $\tau_{2.1-2}$; first they identified the equality $f(x) = x$ for $x \geq 0$ and $f(x) = -x$ for $x < 0$ ($\theta_{2.1-2}$) and following, attaining $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$. It does not appear on recordings or in writing, which technique S8 used to determine f' (contrary to instructions, S7 kept a personal paper for notes and calculations). T3 initially wrote: "$f(x) = -1$ when $x < 0$ and $f(x) = 1$ when $x \geq 0$" (it is however clear that T3 means f' and not f) and thus a version, not completely correct, of the technological component $\theta_{2.1-2}$ is present, as f' is defined for all real numbers. Upon this, T3 drew the correct graphs using a version of $\tau_{2.2/2.3-1}$: graphs are constructed by plotting various coordinates (x_i, y_i) and drawing a curve going through these coordinates; though no coordinates was actually plotted, possibly due to the simple nature of the function expression. First then, did T3 write: "$f'(x)$ is not defined when $x = 0$", which is explained by the technological component $\theta_{2.1}$: a function is not differentiable in a point if the graph has an 'edge' in that point.

The remaining five participants did not solve $t_{2.1}$ correctly. T1 defines f' for $x > 0$ and following concludes that $f'(x) = 1$. T1's technique is illustrated below (figure 31). It appears from audio recordings related to the exercise 2b, how T1 explained this technique, in particular the condition $x > 0$. T1 said:

> When I differentiate I initially get the same [the same as the student in 2b; referring to $\frac{x}{\sqrt{x^2}}$] It might be, that you are thinking, that I, as a teacher, will let the square root and the exponent take each other right away and then differentiate. I do not [do that] because it says that x belongs to the real numbers. So this inside [x^2] will always be larger or equal to zero and when I use that [referring to the quotient $\frac{x}{\sqrt{x^2}}$] I must not divide by zero, so this have to be included [referring to the condition $x > 0$].

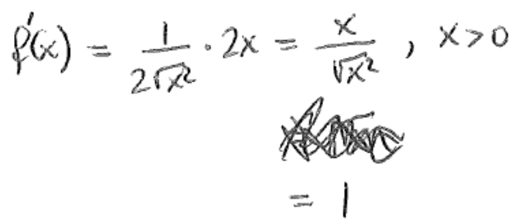


Figure 31: T1's answer to the first mathematical task in exercise 2a.

T1 thus justified the used technique by:

$\theta_{2.1-1.3}$⁻: The domain of a composite function is given by the intersection of the range of the inner function and the domain of the outer function.

T1 proceeded to draw the graph of f ($t_{2.2}$) and f' ($t_{2.3}$), which is performed correctly with the exception of the domain of f':

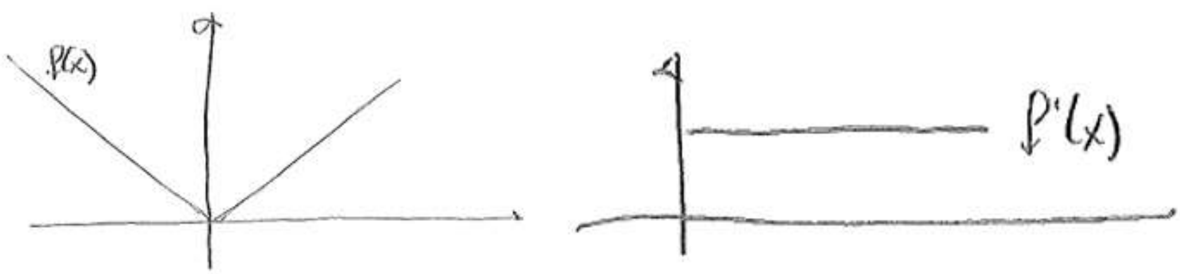


Figure 32: T1's graph of f and f' related to exercise 2a.

T1 did not explicitly recognize the relation $\sqrt{x^2} = |x|$ and thus, did not explicitly show to hold $\theta_{2.1-2}$. However, the graph of f is drawn without use of plotted points, corresponding to a version of technique $\tau_{2.2/2.3-1}$, similar to the one used by T3, but in the absence of the simpler function expression. It is therefore possible that T1 did hold $\theta_{2.1-2}$. However, this appears contradictory with the technique used for $t_{2.1}$; as well as the technique used for task $t_{2.3}$: draw the graph of f'. The graph of the derivative function corresponds to a direct translation of $f'(x) = 1, x > 0$ from an algebraic to a graphical setting. At no point, did T1 consider the different domains of f and f', in particular T1 did not explicitly consider the alike (symmetrical) nature of f for $x > 0$ and $x < 0$ and the contradiction in asserting that the derivative exists for the one side of f but not the other.

T2, T4 and S9 interpreted f to be equal to x, meaning they hold the technological component:

$\theta_{2.1-2.1}{}^{-}$: $\sqrt{x^2} = x$ for all $x \in \mathbb{R}$.

Consequently, the derivative was determined as $f'(x) = 1$. The graphs are in accordance to this result. The technique applied for $t_{2.1}$ is thus:

$\tau_{2.1-3}{}^{-}$: $f'(x) = (x)' = 1$ for all $x \in \mathbb{R}$.

An example of the corresponding graphs, is given below (figure 33).

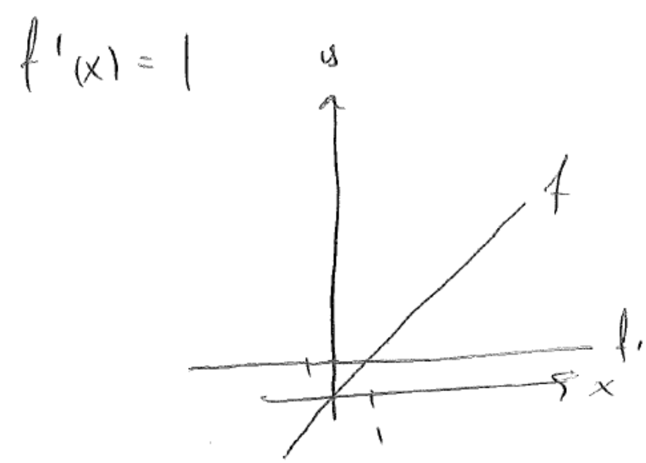


Figure 33: T2's graph of $f(x) = x$ and $f'(x) = 1$ in the same coordinate system.

The work of S9 however, contains additional incorrect elements. Firstly $f(x) = \sqrt{x^2}$ was defined for $x \neq 0$. This contradicts both the information given in task 2a, but moreover it contradicts S9's following work. S9 establishes $\theta_{2.1-2.1}{}^{-}$: $\sqrt{x^2} = x$ from the technique:

$\tau_{2.1-4}{}^{-}$: $f(x) = \sqrt{x^2} = x^{\frac{2}{2}} = x^1 = x$

This calculations only holds for $x \geq 0$ (and thus contradicts the restriction $x \neq 0$) and furthermore, attaining $f(x) = x$, also contradicts the restriction on the domain as x is defined on all of ℝ. The work of S9 is illustrated in figure 34 and as it appears, the derivative graph does not correspond to the achieved derivative function. No technology is detectible in the written work or on audio recordings, which could explain this.

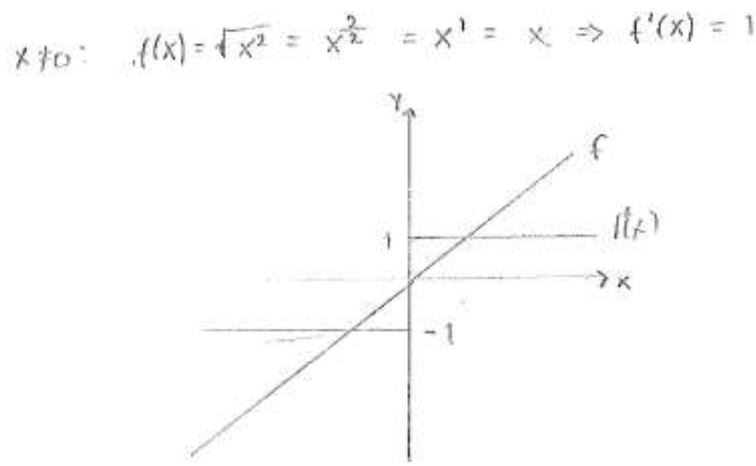


Figure 34: S9's answer to exercise 2a.

The performance of S6 in exercise 2a should also be mentioned, as this performance was given 1 point. The written work of S6 is illustrated in figure 35 below.

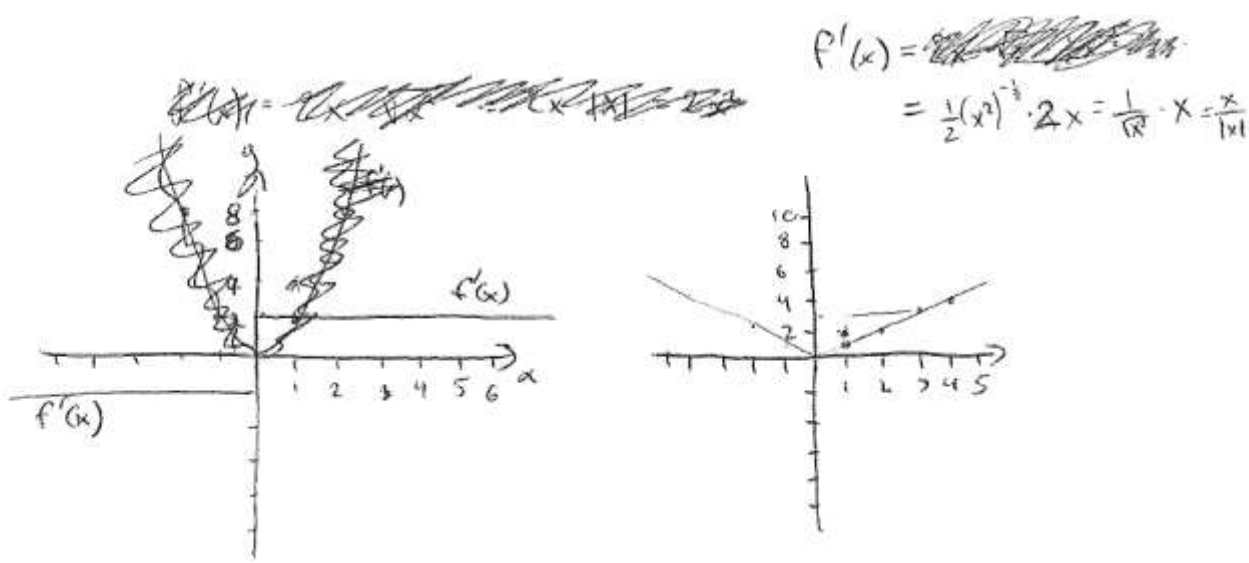


Figure 35: S6's answer to exercise 2a.

From the picture, it appears that S6 answered correctly on the task. However, as it partly appears by the work in the picture (and supported by audio recording), S6 did not initially mobilize appropriate mathematical techniques for solving $t_{2.1}$ (differentiate $\sqrt{x^2}$). Following is an extract from the audio recordings, where S6 is in dialogue with S5.

S6: "What was it you said? Differentiate the inner [function] and multiply with the outer [function]?"
S5: "Yes … oh no that was not what I said. You differentiate the outer [function] and let the inner [function] be, you differentiate the inner [function] and multiply it on."
S6: "Okay, one more time …"
S5: [repeats].
S6: "Oh yeah … All right … it just sounds like a lot of steps."

In the next, approximately 9.5 minutes, S6 determined $f'(x) = 2x^2$ (this is scratched out, but is still visible in the picture, just above the graph of f'). This result correspond to the initial (scratched out) graph of f'. The plotted points on this, as well as for the graph of f, shows that S6 used technique $\tau_{2.2/2.3-1}$: graphs are constructed by plotting various coordinates (x_i, y_i) and drawing a curve going through these coordinates. Then S6 asked:

S6: "Is this all wrong?"
S5: "Hmm, yes, you sort of have something more than you should … no wait … I can not quite recognize … You differentiate the outer [function] and let the inner [function] be … Hmm"
S6: "Oh … this gives a half, so this becomes x^2 of course, in minus a half … I forgot that."
S5: "And then the inner multiplied on …"
S6: "Yes like this …. I forgot what we talked about earlier … This is then the same as one over … Is this $\frac{1}{\sqrt{x^2}}$?"
S5: "Hmm … Yes"

The conversation continued a short while. S6 obtained the result $f'(x) = \frac{x}{|x|}$ and drew the correct graph. S6's result is explained by $\theta_{2.1-2}$ ($\sqrt{x^2} = |x|$), however it does not appear as if the meaning of $|x|$ is recognized since S6 insisted on using the chain rule in spite of the fact that S6 did not remember this rule. Also through audio recordings, it is evident that S5 was the one, who mentioned explicitly $\theta_{2.1-2}$ and it is thus possible that S6 did not know why this equality holds, but included it because S5 did. These are however, speculations. Finally, S6 did not specify for which $x \in \mathbb{R}$ f' is defined; neither related to the algebraic expression nor related to the graph of f'.

Five of the participants (3 teachers and 2 students) was thus unable to mobilize the relevant mathematical techniques to solve an exercise 2a. For these participants, the problem arise already in task $t_{2,1}$, a task belonging mainly to MO_3', deviating only from a typical task in the transposed MO, because the function is not differentiable for all $x \in D_f$. However, the problem does not necessarily lie in the algebraic manipulations associated with the chain rule but more so in the interpretation of the function expression; in particular regarding considerations of the domain and range of the function and in this respect also considerations regarding the prerequisites necessary when using the chain rule. Concerning the latter, none of the participants who used the chain rule, explained the domain by $\theta_{2.1-1.1}$: the prerequisites of the chain rule (V).

Task 2b in HTT 2 especially proved as a challenge for those who did not answer exercise 2a correctly. T1 did not answer 2a, saying, "I simply don't know what to say ... It is very hypothetical". T2 wrote, as a response to the student "Overkill. If it is possible to reduce then do it, also before you get to your result and seriously did CAS give you that?" The reduction that T2 mentions refers to $\theta_{2.1-2.1}{}^{-}$: $\sqrt{x^2} = x$ for all $x \in \mathbb{R}$. As is apparent from this respond, T2 did not mobilize any mathematical techniques for solving 2b and T2's answer to 2a completely hindered any analysis of the student's result. The answer of T4 also bases on the technological component $\theta_{2.1-2.1}{}^{-}$, while S9 wrote, "The student must have written f wrongly" (when plugging it into the CAS-tool).

Among the participants who mobilized (independently) appropriate techniques and solved all three tasks in 2a completely correct (T3, S5 and S7), none where given three points in 2b, because not all the techniques $\tau_{2.1-1}{}^{*}$ – $\tau_{2.1-1.1.1}{}^{*}$ and $\tau_{2.1-2}{}^{*}$ – $\tau_{2.1-2.1.1}{}^{*}$ (or some version of them) were activated. T3 left out the technique:

$\tau_{2.1-1.1}{}^{*}$: Ask/tell the student what it means that f' is not defined in $x = 0$.

While S7 wrote: "What about $x = 0$?" which is interpreted as referring to the domain of the function presented by the student, and with the additional comment: "can it be written any smarter? What does $\sqrt{x^2}$ give? $(= |x|)$", it is asserted that also S7 neglected using technique $\tau_{2.1-1.1}{}^{*}$. Meanwhile, S5 left out the techniques:

$\tau_{2.1-2.1/2.11}{}^{*}$: Ask the student whether $\sqrt{x^2}/\frac{x}{\sqrt{x^2}}$ can be expressed any different.

The hypotheses stated in the introduction of this chapter, in particular (1), is not supported by the participants' responses to HTT 2. It appeared that three teacher did not the mathematical knowledge related to the mathematical task in 2a and thus, they were unable to activate appropriate techniques for the didactical task in 2b. Among the students who held the necessary mathematical knowledge to solve 2a correctly, all used a version of $\tau_{2.1-1}$*: ask the student whether $f'(x) = \frac{x}{\sqrt{x^2}}$ is defined for all x, but no one used $\tau_{2.1-1}$* – $\tau_{2.1-1.1.1}$* addressing *why* f' is not defined for all x, techniques related to MO4. It can however, not be concluded that these students do not hold the mathematical knowledge necessary; only that they did not activate didactical techniques related to this knowledge.

7.4 Performances on HTT 3

The majority of the techniques identified in the *a priori* analysis are present in the participants' collected answers. However, only one participants (T1) was given 3 points in both 3a and 3b. Firstly, T1's performance in HTT 3 will be presented. Upon this some examples of answers will be given, which neglected key techniques identified in the *a priori* analysis.

The written answer of T1 provides little information about what techniques were used and the answer is quite sparse itself (figure36).

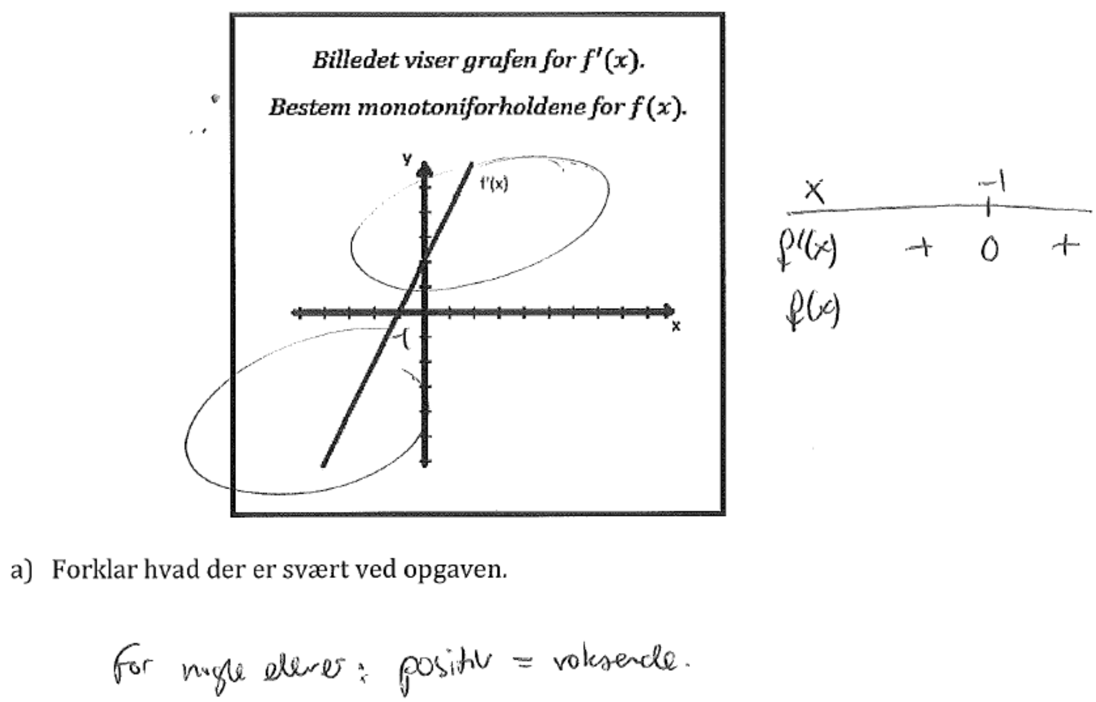


Figure 36: T1's answer to 3a – "For some students: positive = increasing".

However, from audio recordings it is clear that T1 used all of the techniques identified in section 6.2.3 or some version of them:

$\tau_{3.1-1}$*: Identify the relevant techniques associated with $T_{3.1}$.

$\tau_{3.1-2}$*: Identify the theoretical components related to the relevant techniques.

$\tau_{3.1-3}$*: Identify challenges related to the above identifications.

$\tau_{3.1-4}$*: Identify the difference between $T_{3.1}$ and other tasks relating monotonicity that students typically find easy/easier.

T1 said upon reading task 3a:

> What is difficult for the students here is that they don't have a very good understanding of what it means that a graph is ... or a function is positive and what it means that a function is increasing. They mix those terms together… [$\tau_{3.1-3}$*] So here we can see [drawing on the picture in HTT 3, see figure] that $f'(x)$ is positive here, negative here and zero here [$\tau_{3.1-1}$*] but for them $f'(x)$ is increasing, so some of them will think that $f'(x)$ is positive everywhere. Some of them understand well, that if $f'(x)$ is positive then it is the slope of the tangent and then the original function is increasing but that is not the same as to say that they necessarily can translate this graphical expression to what they know [$\tau_{3.1-3}$*]. So, it is the translation which is difficult. It is the translation of this [the graph] to how they can use the monotonicity theorem [$\tau_{3.1-2}$*/$\tau_{3.1-3}$*]. Maybe it is also difficult because, well at least I don't, i don't practice the visual impression, it is more the other way, they have a function, they differentiate and so on and then they get this [referring to the drawn monotonicity scheme in figure] and then they make a conclusion [$\tau_{3.1-4}$*].

Notice that the monotonicity theorem, which T1 mentions is the transposition of Corollary 1 and Corollary 2. T1's answer to 3b corresponds directly to the above analysis, as T1 suggested working with the graphical (visual) interpretation of the term positive function and negative function; specifically to assign to each term a graphical interpretation:

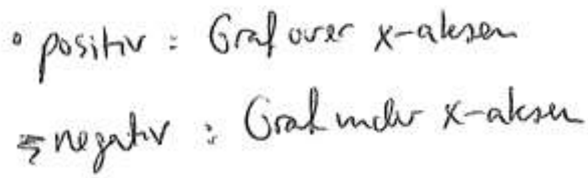


Figure 37: The first part of T1's answer to 3b – "positive: Graph above the x-axis / negative: Graph under the x-axis".

T1 moreover suggested posing tasks wherein students should use this graphical interpretation to determine where a function is positive and where it is negative. Thus, T1 used the following technique, not included in the a priori analysis:

$\tau_{3.2-7}$*: Provide a visual image for the terms *negative* and *positive*.

Aiming to avoid the confusion regarding the actual words, as T1 explained: "It is the word 'positive' that is difficult, it sounds too much like 'increasing'. It is common that students use the word positive to describe an increasing function".

The rest of the performances of the participants in HTT 3 suggests (quite like HTT 2) that the identification of a primary problem and the selection of a specific technique to work with this problem is a challenge. Many participants were quite vague in 3a. For example T2 wrote: "Typically, the students see graphs of functions $f(x)$, not so often for functions derivatives $f'(x)$", while S5 wrote that it is difficult "to translate info about f' to info about f" and S9 wrote: "The student have to remember that the graph shows the derivative of f and not f itself". While, in 3b the participants suggested a variety of tasks aiming at incorporating in general the graphical representations more in teaching, using techniques such as $\tau_{3.2-5}$*: ask the students to draw f and f' when working with the functions algebraic expressions, a technique not targeted any specific challenge, but aims to include graphical representations in general. While, the technique $\tau_{3.2-4}$*: ask the students to draw the graph of f given the graph of f', which requires the same interpretation of the graph of f' as task $T_{3.1}$, is in not employed by any of the participants. Contrary to this technique, T3 used the following:

$\tau_{3.2-7}$*: Ask the students to draw f', given the graph of f.

Which also appears as unrelated to the *specific* challenges associated with $T_{3.1}$, but possibly accomplishes in a more general sense, familiarity with the relation between the graph of f and the graph of f'.

S6 proposed the following student challenge as the only participant: "[I] would believe that there can be confusion between $f'(x)$ intersection with the x-axis and possibly the y-axis and $f(x)$'s". Such confusion can naturally be considered as included in the assertion that students assume resemblance between the graph of a function and its derivative. However, the analysis of S6 is incomplete as the confusion between the other characteristics of f and f' is not considered. Furthermore, the difficulties identified by S6 are asserted not to be primary since the students need to identify where $f' = 0$, $f' < 0$ and $f' > 0$; and while the meaning of the terms *positive* and *negative* has counterparts in the terms *increasing* and *decreasing*, respectively, the meaning of $f' = 0$ does not have the same alike counterpart.

Some participants concluded that it is difficult for students to 'go the other way'. For example, S6 wrote that one "has to go the 'opposite' way than usual, to say something about f" while T4 wrote "The difficulty is, that one has to go 'reverse' [omvendt] compared to usual". These statements reveal that T4 and S6 did not activate (properly) the technique

$\tau_{3.1-1}$*: identify the relevant techniques associated with $T_{3.1}$, since these do not involve going 'reverse' or 'opposite' (see section 6.2.3).

As a group, the teachers performed slightly better than the students did. In all, the students' answer to HTT 3 were short and poorly elaborated (except possibly S5's answer), while the teachers' answers were more comprehensive (excluding T2's answer to 3a). Hypotheses (1) is considered as weakly supported by the respondents' answer to HTT 3.

7.5 Performances on HTT 4

HTT 4 is the task with the lowest average percentage of collected points among all the participants (11.1 % and 14.8 % for task 4a and 4b, respectively); among the students the average was 0 % and 13,3 %, respectively, while the average among the teachers was 25 % and 16.7 %, respectively. Task 4a posed a didactical task, namely $t_{4.1}{}^{*} \sim$ MO4', MO4:

$t_{4.1}{}^{*}$: Your student presents the graph of a function, which she claims to be a derivative function. What do you say to your student?

T1 did not answer HTT 4 at all. Half of the remaining participants answered task 4a by posing a question to the student. T2, T4, S5 and S6 used the following technique:

$\tau_{4.1-3}$*: Ask the student what the original function is.

While T2 additionally used the following technique:

$\tau_{4.1-4}$*: Ask the student whether she thinks it makes sense that $f'(1) = 0$ and just after it is 1 and increasing.

Although posing such questions can qualify as an appropriate didactical technique, it is essential that the participants are capable in assessing whatever answer the student may provide; which appears (also by audio) not to be the case. That means in particular, that *none* of the mentioned participants explicitly activated techniques for the mathematical task:

$t_{4.1}$: Given the graph of a function, in particular f', determine if it is a derivative function.

Nor the task $T_{4.1.1}$: can a derivative function have a jump discontinuity? which is considered especially relevant for T2, when using $\tau_{4.1-4}$* stated above.

While T1, T2, T4, S5 and S6 did not activate any appropriate techniques for these tasks, S7 and S8 answered task $T_{4.1.1}$, however justified by a faulty technological component, namely $\theta_{4.1.1}{}^{-}$: a derivative function cannot be discontinuous. Based to this, they mobilized the faulty didactical technique:

$\tau_{4.1-5}{}^{*\,-}$: Tell the student that all derivative functions are continuous.

This is incorrect and it seemingly shows that S7 and S8 do not hold the unit of knowledge belonging to MO_4:

$\Theta_{4.1.1}$: Theorem 2.

However, since the derivative in HTT 4 has a jump discontinuity, it is possible, that it is this particular type of discontinuity, to which they are referring. Alternatively, as we saw in sections 5.3, discontinuous functions are presented in upper secondary school, only as functions with jumps, and it is thus possible that S7 and S8 used the term continuous as it is understood in MO_4' existing in upper secondary school. These are however, speculations.

T3 was the only participant given 3 points in 4a; indeed T3 was the only participant whom explicitly activated mathematical techniques, in order to determine how to respond to the student. T3 first said: "It cannot be defined here [referring to $x = 1$, see figure below] because in order for a function to be differentiated in a point it has to be differentiable in that point, and it is clearly not differentiable in this point". It is not explicit how T3 explains this conclusion. Upon this, T3 mobilized a mathematical technique for task $t_{4.1}$: given the graph of a function, in particular f', determine if it is a derivative function. This technique appears in figure 38. The determination of the domains of $y = -x + 1$ and $y = x$, might convey that T3 interpreted the function as continuing 'outside the picture' or that T3 simply defined the functions on domains to large; no conclusion can be made, in this regard. The technique activated by T3, though not completely correct, as the integration constants are absent, belongs to MO_5. Following the above calculations T3 said, "We have to determine whether this function is continuous by inserting *1*" and verified the equation: $-\frac{1}{2}1^2 + 1 = \frac{1}{2}1^2$, concluding that the antiderivative was continuous. Then T3 said, "But it is not differentiable in that point [$x = 1$] and that is because it has an edge in that point, so she has differentiated a function which is not differentiable and by principle she cannot". The techno-theoretical components explaining and justifying T3's techniques are very similar to those associated with $t_{5.1}$ identified in the *a priori* analysis of HTT 5:

$\theta_{5.1-2}$: The graph shows straight lines on (0,1] and (1,2): they can be considered on their natural domain $\mathbb{R}$.

$\Theta_{5.1-2}$: Function theory: The straight line: $y = ax + b$, $a = \frac{y_2 - y_1}{x_2 - x_2}$ and b.

As well as techno-theoretical discourses belonging to MO_5 and MO_6.

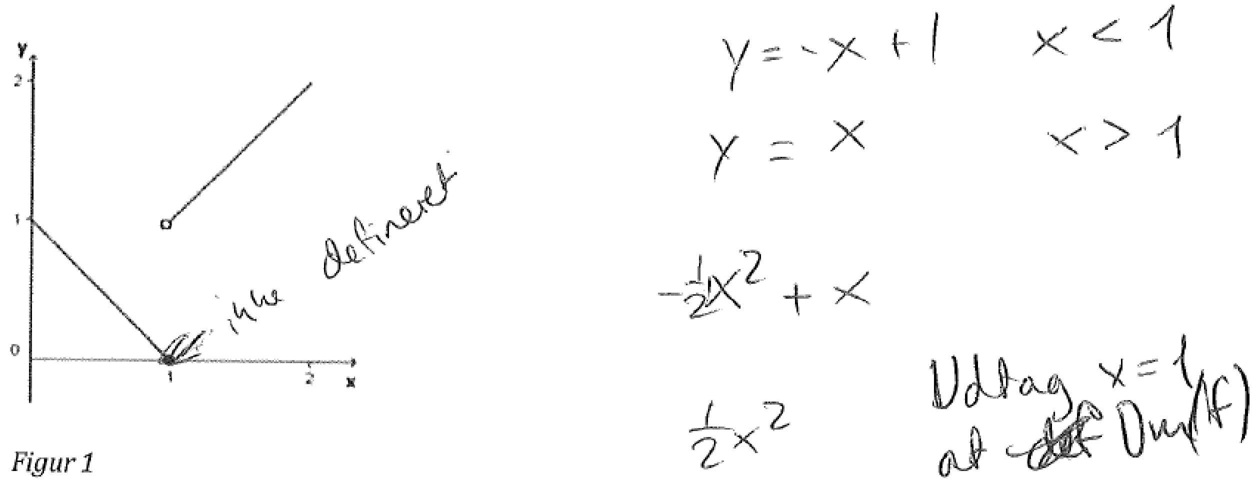


Figure 38: T3's answer to task 4a

The work of T3, though not completely correct, shows that this participant is capable of mobilizing mathematical techniques to gain insight into a students work and thereby place T3 in a better position to respond to the student's claim.

Exercise 4b posed a didactical task, namely $t_{4.2}{}^{*} \sim MO_4{}'$, MO_5:

$t_{4.2}{}^{*}$: Your student shows you two functions presented graphically (figure 2 and 3 in HTT 4) and claims that these are antiderivative functions for the function presented in figure 1 (in HTT 4). Provide exhaustive feedback to your student.

The majority of the participants was given 0 points for their performance in task 4b. T1,T4, S7, S9 did not answer the task. T2 did not activate appropriate mathematical techniques for the tasks:

$t_{4.2}$: Given the graph of f, determine if f is differentiable.

$t_{4.3}$: Are g and h, presented graphically in figure 2 and 3, antiderivatives functions of f'?

This is evident in the feedback provided by T2 in 4a. T2 wrote, related to figure 2 in HTT 4: "A continuous, differentiable function does not have [har vist ikke] a discontinuous function as derivative". Related to figure 3 in HTT 4, T2 wrote: "Be careful when calling that [the function in figure 3] an antiderivative, because there are certain rules for discontinuous functions' derivatives". Thus, T2 did not activate any of the necessary techniques related to MO_4 and furthermore, T2 conveyed not to hold the relevant techno-theoretical components belonging to MO_4'

The written work performed by S5 is quite sparse and inconclusive. It is explicit however; on audio recordings, that S5 activated relevant techniques for task $t_{4.2}$ (restated above) and subsequently for $\mathcal{T}_1$: what is the derivative of f in a point $a \in D_f$? S5 used the following technique to solve task $t_{4.2}$:

$\tau_{4.2}$: Reading the graph to determine that the limit of the difference quotient in $x = 1$ is different approaching from the left and approaching from the right.

Justified by the definition of the derivative; this appears from the following transcription where S5 asked S6: "If there is to be a derivative in one [$x = 1$] shouldn't the limit of the differential quotient be the same, if you take it from the right and from the left (…) if you let h approach from the right and from the left? So the two functions in the fork [the piecewise function] agree upon the slope in that point (…) it looks like the first parabola have slope equal to zero [in $x = 1$]". At this point in the conversation, S5 was interrupted by S6, who did not appear to accept the argument, due to the length between the two indicated points on the y-axis. The objection of S6 is not valid since a correct proportional distance between the indicated points on the y-axis, simply would vertically 'stretch' the 'second branch' and thus produce a slope even steeper. The graph in question is reprinted below (figure).

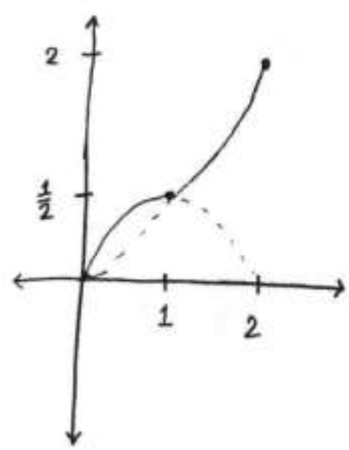


Figure 39: One of the graphs presented by a student in HTT 4 (reprinted here for completeness).

S5 continued later: "This one to the right [the second branch] has slope one in that point there [$x = 1$] if we should give it a slope and the other one [first branch] has slope zero [in $x = 1$]

(…) but is it differentiable then?" Trying to confirm this with S6, however without luck. Perhaps this 'disagreement' with S6 is the reason why these considerations of S5 did not appear in the written answer.

T3 included all elements identified in the *a priori* analysis of exercise 4b, except one aspect, which concerns the impossibility of a positive function to have a decreasing antiderivative. This point was not included in any of the respondent verbal considerations or written answers. A possible explanation for this could be the focus on the concept of the derivative. The participants might have fixated on the given functions, as related by the slope of the original functions and thus, neglected to think of the functions in figure 2 and 3 as related to the function in figure 1, by the area under its curve.

The most striking aspect of the participants' responses is the general lack of activated relevant mathematical techniques; only one participant activated techniques to determine the original function in 1a and thereby making it possible to determine what to say to the student. In all, only three participants, one teacher (out of four) and two students (out of five) activated techniques explained and justified within MO_4' and he hypothesis (2) is thus not supported the participants' answers to HTT 4.

7.6 Performances on HTT 5

As a group, the teachers were given few points in their performance on the mathematical task posed in 5a, namely $t_{5.1}$: given the graphical representation of f' draw the graph of f with the additional assumption that $f(0) = 0$. A task, which can be solved using techniques belonging to MO_3' or techniques belonging mainly to MO_5. Among the teachers, T2, T3 were given zero points, while T1 and T4 were given 1 point and 2 points, respectively. Among the students, S5, S6, S7 and S8 were given 3 points, while S9 was given 2 points. The participants' answers to 5b do not support hypothesis (2) of the introduction, as only 2 of the students pointed to the non-differentiability of f while this was only the case for one of the teachers. Meanwhile, the most comprehensive performance in the didactical task 5b was that of participant T3. The answers for 5a and 5b are highly connected, so the analysis will encompass the answers to both tasks for each relevant participant.

T2's answer in 5b focused almost completely on the arrows in the ends of the curve: "Without the arrows I would believe that the student thinks that the function is decreasing all the time [on the entire domain] but with the arrows it is tough [to figure out]". This statement imply that T2 wrongfully considered the graph presented by the student in 5b as decreasing. Moreover, T2 wrote, "The student has tried to show that something mysterious is going on in $x = 1$". This analysis highly relates to T2's answer in exercise 5a, illustrated below (figure 40). It is apparent that T2 did not mobilize any of the appropriate mathematical techniques identified in the *a priori* analysis for this task. The performance in 5b signifies that T2, besides not being able to activate the appropriate techniques associated with $t_{5.1}$, did not recognize when these techniques are used. This is supported by the following questions posed to the student by T2: "What do the arrows mean? Where did you get the points from? What does the 'edge' [hakket] in the graph of f' mean to f?"

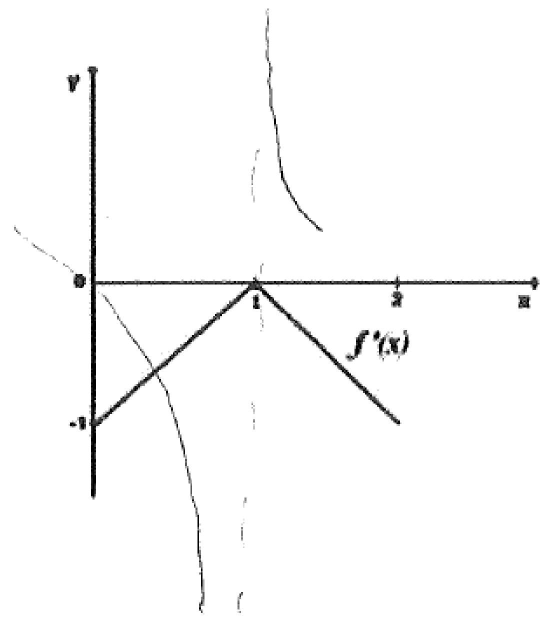


Figure 40: T2' answer to exercise 5a.

As in HHT 4, T2 did not reveal any activation of mathematical techniques for these tasks, except the latter question, to which the answer (based on the above graph) is believed by T2 to be, that the original function f is not differentiable and the technology explaining T2's answer to $t_{5.1}$ is thus:

$\theta_{5.1-3}{}^{-}$: If $f'(x_0) = 0$ the original function is not defined in x_0.

T4, similarly, did not activate any techniques that could provide insight into the student's answer in exercise 5b. T4 wrote, "The student think that when there is an 'edge' on the graph of the derivative it means that the graph of the function is not continuous", but did not elaborate, to establish whether this was correct or not, and did not consider the coordinates on the graph. Related to exercise 5a, T4, activated technique $\tau_{5.1-1}$, concerning the determination of the graph of f based on its monotonicity characteristics displayed by the graph of f'. The result is illustrated below (figure 41).

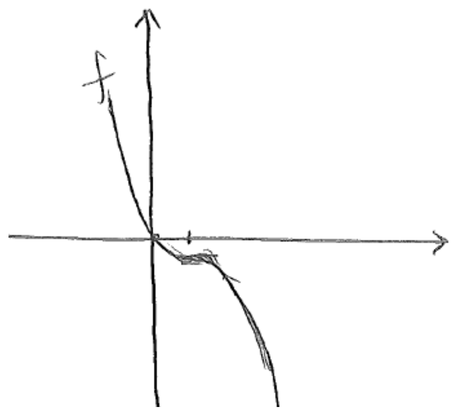


Figure 41: T4's answer to exercise 5a.

As it appears, T4 correctly drew a function which is decreasing and convex in [0,1), has an inflection point in $x = 1$ and is decreasing and concave on (1,2], however with the mistake of drawing f on a larger domain than [0,2] (why T4 was given only 2 points). A choice of this technique over technique $\tau_{5.1-2}$ (extracting the algebraic expression of f', integrating this and determining the function value in the inflection point), does not necessarily mean that T4 is not capable of activating the latter technique, however, due to the incorrect analysis of the student's graph in 5b it is asserted that this is actually the case.

T1 initially solved task 5a correctly. First, constructing the algebraic expression of f', integrating this and proceeding to draw the graph of f leaving k_1 and k_2 undetermined (both denoted k by T1) and thereby also $f(1)$. T1 said "It has to be decreasing on the whole interval and it has to have $f'(x) = 0$ here [$x = 1$] and it is two 2nd degree polynomial combined (...) this one has a positive *a* value [referring to the algebraic expression of the first 'branch'] and this one has a negative *a* value [referring to the algebraic expression of the second 'branch]".

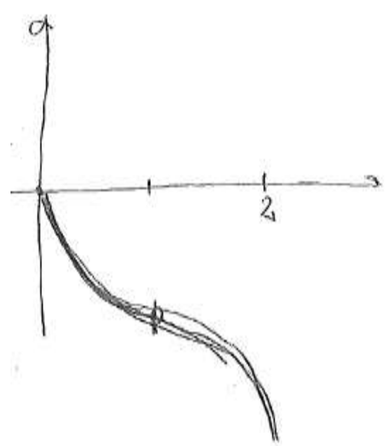


Figure 42: T1's initial answer to exercise 5a.

The technique activated by T1 is thus a combination of $\tau_{5.1-1}$ and $\tau_{5.1-2}$ (section 6.2.5). The result is illustrated above (figure 42). T1 did not explicitly activate the part of technique $\tau_{5.1-1}$ concerning the reading of f' to establish that f' is decreasingly negative on (0,1) and increasingly negative on (1,2) and thereby that the first and second ‘branch’ are convex and concave, respectively, but determined this based on the ‘a values’ in the two function expressions for f. However, the considerations that T1 make regarding exercise 5b leads to a change of heart and T1 drew the graph in figure 43 as representing f.

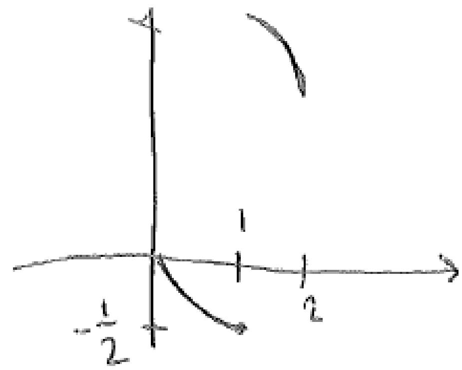


Figure 43: T1’s second (and final) answer to exercise 5a.

T1’s considerations related to 5b shows that T1 used the following didactical techniques:

$\tau_{5.1.1-4}$*: Read the student’s graph and the graph constructed in 5a to see that the two graphs have the same progress in slope.

$\tau_{5.1.1-3}$*: Comparing the student’s graph with the correct graph or the graph of f' to see that the student has drawn f on a domain too large.

$\tau_{5.1.1-1}$*: Reading the graph to establish that the student has drawn the graph of a discontinuous function and conclude that it is correct.

T1 said: “The student … pretty much have the same answer as I have, if you remove some of this … [scratching out the part of the graph which is outside the interval [0,2]] and then, the function is not connected but it does not necessarily needs to be connected. We cannot know if it is”*.* T1 thus shows not to hold the knowledge of MO_4’ necessary in this context. T1 concluded that “This is a very creative answer in which it is understood that k [k_2] is arbitrary and (…) this [the second branch: the graph f_2] can slide op and down however you want it to”. In answering 5b, T1 moreover activated techniques for the following task:

$t_{5.1.2.1}$*: How did the student produce the coordinates $(1, -\frac{1}{2})$ and $(1, \frac{1}{2})$?

T1 mobilized the following mathematical technique:

$\tau_{5.1.2.1-3}{}^{-}$: $f_1(1) = \frac{1}{2}1^2 - 1 + k_1 = -\frac{1}{2} \Rightarrow k_1 = 0$

$f_2(1) = -\frac{1}{2}1^2 + 1 + k_2 = \frac{1}{2} \Rightarrow k_2 = 0$

And upon this, the didactical technique:

$\tau_{5.1.2.1-}{}^{*-}$: Identify that the student have fixed these.

However technique $\tau_{5.1.2.1-2}{}^{*}$: identify the use of the condition $f(0) = 0$ in the determination of the integration constant in the algebraic expression of f defined on (1,2], is absent. T1 proceeded by correcting T1's own answer to exercise 5a. First by setting $k_1 = 0$ (using the assumption) and confirming that $f_1(1) = -\frac{1}{2}$ and secondly by drawing a graph with even greater distance between the graphs in the point of discontinuity in order to demonstrate (the wrong conclusion) that k_2 can vary (figure 43). T1 did not activate any techniques belonging to or justified by MO_4' when solving task 5b and did not address the two function values in $x = 1$. The faulty technology explaining the technique is:

$\theta_{5.1}{}^{-}$: A differentiable function does not need to be continuous.

T3's answer to 5a was given 0 points because T3 did not actually draw the graph of f'. However, T3 did mobilize a version of technique $\tau_{5.1-2}$. As it appears in figure below, the techno-theoretical discourse explaining the technique is correct, however not the technique itself.

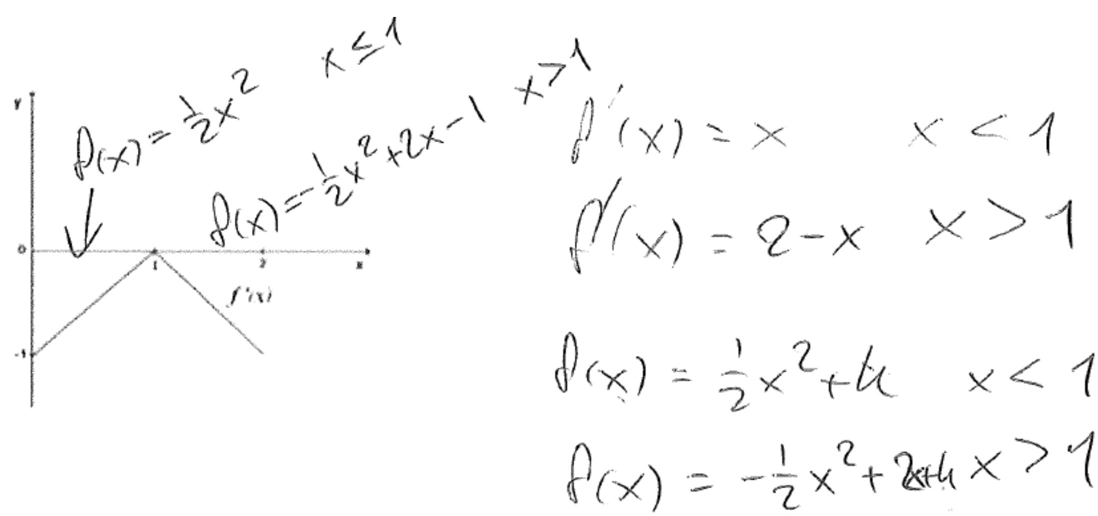


Figure 44: T3's answer to task 5b

The activated technique (or rather the present technology) in T3's work in 5a is highly related to T3's analysis in 5b. T3 activated all the relevant techniques identified in the *a priori* analysis, associated with task $t_{5.1.1}$*: analyse and assess the student's answer to $t_{5.1}$, and in particular $t_{5.1.2.1}$*: how did the student produce the coordinates $(1, -\frac{1}{2})$ and $(1, \frac{1}{2})$? T3 did not activate techniques addressing the domain of f, the arrows or the two function values in $x = 1$ and therefore T3 was given 2 points.

S5, S6, S7, S8 and S9 all constructed similar graphs in exercise 5a (the graphs of S6 and S7 are shown in figure). The majority of these participants used a version of $\tau_{5.1-1}$ (reading the graph of f'). S7 additionally used $\tau_{5.1-2}$, determining the exact algebraic expressions of f_1 and f_2, using the assumption, while S8 used a versions of $\tau_{5.1-2}$ and drew f based on plotted points determined by using the algebraic expressions. This appears from the points marked on the graph (graph to the right in figure), however again S8 used a personal paper for notes, so the exact techniques of S8 can not be determined.

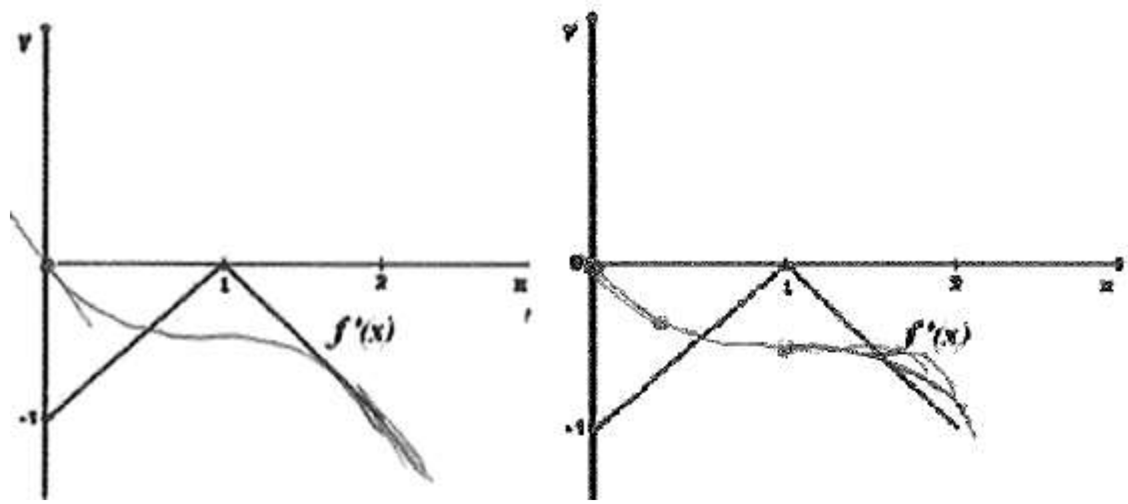


Figure 45: S6 (left) and S8's (right) answers to exercise 5a (similar to the answers of S5, S8 and S9)

Regarding exercise 5b, none of the student participants identified the student's technique including the use of the condition $f(0) = 0$ on both 'branches'. S7, S8 and S9 wrote that f has to be continuous. However, from audio recordings it appears that S9 did not activate techniques related to MO_4. Upon working with HTT 5 individually, S7 and S9 engaged in dialogue and S7 said, "The interesting thing is, why they have to be connected [referring to the graph in 5b] and that is probably because one cannot differentiate something that is not connected", upon which S9 said, "I have not considered at all, that it could be a problem". S7 wrote, "Forgets that f has to be continuous (forgets the integration constant)", which is recognised as a possible interpretation, why S7 was given 2 points contrary to S8 who only pointed to the problem of discontinuity. Contrary to this, S6 wrote: "Since it [f] (most likely) is a continuous function the pencil has to stay on the table [holde blyanten i bordet]", wherein the main thing to notice is the parenthesis, which reveals that S6 holds the same technology as T1, namely $\theta_{5.1}{}^{-}$: a differentiable function does not need to be continuous. The same appears in the answer provided by S5: "Don't know why the graph jumps, but except the fact that there are two function values in the same point, there is nothing wrong with that", and thus implicitly stating that a function with a point of discontinuity is differentiable. Compared to S5's consideration on the audio recordings related to task 4b, where S5 stated (roughly) the definition of differentiability and asserted that the two functions expressions in the piecewise function should agree upon the slope' in $x = 1$; it is possible that S5 did not have any objections in this case because S5 believed that the graphs *did* 'agree upon the slope' in $x = 1$. This is incorrect, since none of the 'branches' have a slope in that point and thus, in the context of HTT 5, S5 was not able to activate the necessary techniques related to MO_4'. A correct element in the answer of S5 is the point regarding the two functions values of one point. S5 is the only participant who activated this technique ($\tau_{5.1-8}$*).

8 Discussion

The subject matter analysis of chapter 5 and the *a priori* analysis of the HTTs, presented in chapter 6, showed that the tasks included in the five HTTs could be solved by activating techniques belonging to MO_3, MO_4 and MO_5 and techniques belonging to organisations of functions theory. The latter are however, considered inherent in MO_3, MO_4 and MO_5, as they comprise of interpreting function expressions and translating between functions' algebraic and graphical representation. Furthermore, the specific knowledge components which justifies the necessary techniques are present (in varying degrees) in the transposed organisations MO_3', MO_4' and MO_5'. The latter was not included in the analysis of chapter 6; however, the specific knowledge required merely encompassed how to integrate elementary functions and the significance of the integration constant, which undoubtedly is a part of high school curricula (see Clausen et al., 2011a, pp. 54-73). The only knowledge component, relevant in working with the HTTS, which was not present in MO_4', was that of Theorem 2. However, as we saw, this knowledge component was not *necessary*, as activating techniques related to MO_3' could lead to the same conclusion. Furthermore, it was established that the majority of the tasks included in the HTTs did not relate to, or belong to the 'typical' types of task in the transposed MO_3' and MO_4'. However, based on the educational background of the participants, presented in subsection 4.2.1, the *a priori* analysis of chapter 6 and the assessment performed by a subject matter adviser of Mathematics in Danish high schools, it was expected that the participants, students as well as teachers, would be able to activate relevant techniques for the mathematical tasks, which were explicitly or implicitly, included in the HTTs.

The *a posteriori* analysis presented in chapter 7, revealed that the participants in general were challenged by the tasks; indeed, to such a degree that one has to question whether other factors, aside from the participants' mathematical and didactical knowledge, contributed negatively to their performances. It is possible that such 'other factors' originated from the data collecting methods, i.e. that the methods had negative impact on the participants' performances. However, two data collecting methods were employed; why it is necessary to consider the possible shortcomings of each method. The teacher participants answered the HTTs individually, while the author was sitting across from them (the arguments as to *why* the researcher was present, was given in subsection 4.2.2). This consequently meant that the teacher participants did not have anyone to discuss the tasks with, which according to Durand et al. is an "important real-life channel for the development and exchange of didactical technology (and to some extent, theory)" (Durand-Guerrier et al., 2010, p. 7). Furthermore, the presence of the researcher (author) possibly created some 'pressure' to perform. Three of the teacher participants expressed the latter, during the test as well as afterwards, in particular, one teacher claimed that *professional pride* was the main source of this feeling of pressure, resulting in a performance not reflecting properly the capability and knowledge held by the participant (teacher participant, personal communications, June 6, 2016). On the other hand, the student participants answered the HTTs in groups of two and three. The same amount of time was offered to the students as there were to the teachers, however, it became clear that the student participants spent much time on discussing the task with each other and although this was the intended; for some of the participants much time was spent, in particular, on presenting arguments to (and convincing) their fellow group member(s). This left less time to consider all the tasks thoroughly and to formulate precise and comprehensive answers, which could explain the many rather short written answers provided by the students. Moreover, though the group constellation has its advantages, some disadvantages also appeared. For example, S6 generally let S5 speak first and in some cases, it was not entirely clear, if S6 held the knowledge which explained and justified the activated techniques or if S6 activated these only because S5 did. If the researcher had been present during their work with the HTTs, this could possibly have been decided. Furthermore, it is asserted, that to surely gain something from the group constellation, the members must be comfortable, i.e. feel free to speak their mind. For example, related to the majority of the tasks, S9 did not engage in dialogue with S7 and S8, contrary to instructions. Therefore, nothing on the audio recordings could help assess S9's rather uncomprehensive answers, while audio recordings were used extensively in assessing other participants' answers. This also resulted in an uneven data material available when analysing the respondents' performances as well as S7's personal note sheet did.

Concerning the hypothesis (1): the teachers will activate didactical techniques, which are more appropriate than will the university students, it appeared that the mathematical content of the tasks challenged the participants such that, the HTTs inhibited the participants in showing their capability in activating techniques of a more didactical character. It was established in chapter 2 that teacher knowledge is more than mathematical subject matter knowledge alone, however since many of the participants stumbled in the mathematical tasks, the HTTs did not provide an opportunity for those participants to activate techniques related to didactical knowledge exterior to pure mathematical knowledge. The hypothesis (1) was supported slightly by the empirical data, in particular by the participants' responses to HTT 3.

The hypothesis (2): the university students will to a greater extent activate appropriate mathematical techniques related to MO_4 and provide more relevant answers to tasks that belongs to this MO, than will the teachers, was not supported by the collected empirical data. Though T3 was the only teacher who activated correct techniques justified by MO_4, only two participants among the students showed to hold knowledge of MO_4 in task 5b, namely S7, S8. In addition, S5 showed to hold knowledge of MO_4; however only on audio recording and only related to task 4b. In task 5b, S5 did not activate techniques justified by MO_4, in spite of being able in doing so related to 4b.

The empirical data does not support the hypotheses (1) and (2). Moreover, no other patterns appeared in the data. This consequently means that the participants' various teaching experience did not appear. Among the teachers, the participant with the best performance was T3, followed by T1, T2 and T4 (in that order), while the students, with no teaching experience, in average performed better than the teachers. That the participants' various teaching experience is not reflected in the data can immediately signify one of three things. The first possibility is, that no general difference exists between the *teacher knowledge* held by students, who have finished their minor in mathematics, and by teachers or between two teachers with different amounts of teaching experience. Such an assertion is however, to far reaching in this context and additionally, it is not supported by research. For example, *knowledge of content and student* (KCS), proposed by Ball et al. as constituting an area of mathematical knowledge for teaching, is said to be derived " … from experience with students and knowledge of their thinking" (Ball et al., 2008, p. 9). While Bromme writes, "The integration of knowledge originating from various fields of knowledge, discussions with colleagues, and *experience* is an important feature of the professional knowledge of teachers …" (Bromme, 1994, p. 86, italics added). The first possibility can however also be considered in a limited version, more relevant to the current context, namely that no general difference exists between the teacher knowledge associated with the specific HTTs, held by students, who have finished their minor in mathematics, and by teachers or between two teachers with different amounts of teaching experience. Some further comments to this interpretation is presented in the next section. The second possibility is that the designed HTTs was not fit to capture the difference in teacher knowledge held by the participants. Considering the hypotheses (1) and (2), which were not supported by the empirical data, this possibility is conceivable; again possibly due to the general struggle with the mathematical tasks inherent in the HTTs. Lastly, it is possible that the participants' various teaching experiences are not reflected in the data, in particular the student' non-existing teaching experience and the teachers' numerous years of experience, because the different collecting methods have provided the participants with conditions so distinct, that the empirical data from the two groups are not comparable. For example, acknowledging the 'pressure' upon the teachers, created by the presence of the researcher, the data collected from the teachers might convey an inaccurate and poorer picture of their technology and theory related to the relevant organisations. However, the teacher participant who collected the most points, namely T3, did not explicitly articulate any feeling of pressure, why it is possible that the causality is opposite. Specifically, that the 'pressure' did not cause the participants' to activate fewer or irrelevant techniques but that the incapability of mobilizing appropriate techniques caused the 'pressure'.

Another factor contributing negatively to the overall performance of the participants, besides 'pressure' and time constraints for the teachers and the students, respectively, could possibly be the untypical character of the tasks. Meaning, the deviance of the HTTs from the tasks typically presented in high school within the theme. Related to the *a priori* analysis of the mathematical task on which HTT 3 centred, it was argued that students might find the mathematical task difficult simply because it is a non-typical task and therefor unrelated to a specific and practised algorithm. The same phenomena might apply in this respect. However, in light of the participants' educational background, it is quite surprising that a deviation from the typical tasks, to the extent contained in the HTTs, created such uncertainty in the respondents. Moreover, if this is a general phenomenon, the dominance of the typical algebraic tasks in high school is unlikely to be challenged; based on the assertion, that teachers are unlikely to include tasks in their teaching praxis, which they are not confident in solving themselves.

8.1 Limitations of the study

For the purpose of answering Research Question 2, it is clear that the empirical study presented in this thesis, was not aiming at producing generalizable results. For the purpose of answering this research question, the performed study was sufficient, i.e. from the collected data it was possible to determine that the participants' teaching experience was not reflected in their answers to the HTTs. However, an immediate question upon this result, must be *why*? Some possible answers for this question was considered in the last section, but to determine which of these possibilities constitutes the correct answer, requires more generalizability than the study offers. For example, to conclude that there appeared no general difference in the data between the two groups because no general difference exists between the teacher knowledge associated with the specific HTTs, held by students, who have finished their minor in mathematics and by teachers, the study is overwhelmingly limited. In particularly, in terms of the number of participants, but also in terms of the different collecting methods.

Regarding the number of participants, it might also be relevant to consider, whether a larger number of participants, would have made the pattern and differences between the two groups more visible. This is likely to be the case, why stating, "This study was sufficient" to answer Research Question 2 should be reconsidered. Since this research question requires comparison between the two groups, the result would have been more substantiated if the study had included not only a larger number of participants, but also a group of teachers, which were more homogenous in terms of teaching experience. Furthermore, such a comparison suffers greatly from the difference in data collecting methods.

8.1.1 A redesign

A redesign would not necessarily include an increase in the number of participants; not if the study should fit the size of a thesis. It would entail however, if possible, a more careful selection of participants, such that they formed two groups, more homogenous than was the case in the present study. Furthermore, it would entail using *one* data collecting method. Using the method employed in collecting data from the students, with all participants, would require some care in establishing groups, as pairing teachers, who do not know each other would possibly magnify the disadvantages seen with in students' groups, i.e. the limited engaging in dialogue. A redesign of the study would also entail a verification of the given points by an outside source, to avoid the current study's subjectivity related to the distribution of points.

9 Conclusion

The aim of this thesis was to investigate the concept of HTTs building on the principles of ATD, and the use of such tasks to access and assess teacher knowledge within the theme of differential calculus. In particular, the aim of the thesis was to design such HTTs in order to explore how teachers' and teacher students' responses to these, reflect their different amount of teaching experience and thereby explore the potential of the designed HTTs, as well.

Constituting the thesis' answer to Research Question 1, five HTTs along with an *a priori* analysis of each task was presented. The analysis of the HTTs employed a reference model constructed through a subject matter analysis of the relevant theme and an investigation of the *knowledge to be taught* in Danish high schools. The analysis of the designed HTTs showed how these required techniques primarily relating to MO_3' and MO_4', while techniques belonging to MO_6' was necessary for a full response in one task.

For the purpose of answering the thesis Research Question 2, the designed HTTs was presented to five teacher students at the University of Copenhagen and four upper secondary school teachers. It was found that the participants' different amount of teaching experience was not reflected in their answers to the HTTs. Each performance was given points and based on this distribution, no substantial difference between the performances of the two groups appeared. Furthermore, based on an in depth analysis of the participants' answers, it appeared that neither of the hypotheses concerning the characteristics of the groups' performances was supported. Thus, the teachers' did not, 'on average', activate didactical techniques more appropriate than did the teacher students and the teacher students did not activate more appropriate techniques related to MO_4 than did the teachers. Furthermore, no other general characteristic of the two groups appeared in the data.

The empirical study is however, limited in regards to concluding *why* the participants' teaching experience did not appear in their responses to the HTTs, as the study contain many variables which could have influenced this result; it can certainly not be concluded that there *is* no difference. It is asserted, that the main source of the study's result is the occurring absence of appropriate techniques for the mathematical tasks inherent in the HTTs, which then leaves the participants unable in activating appropriate didactical techniques. However, since it was highly expected that the participants could 'handle' the mathematical content, it is also probable that the absence of appropriate mathematical techniques is a consequence of the data collecting methods: not providing enough time for the teacher students and not creating optimal working conditions for the teachers.

10 Literature

Abbey, K. D. (2008). Students' understanding of deriving properties of a function's graph from the sign chart of the first derivative. Orono: Science in Teaching, University of Maine. A thesis.

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. Educational Studies in Mathematics, 33(3), 301-317.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of teacher education, 59(5), 389-407.

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools. Beyond the Apparent Banality of the Mathematics Classroom, 59(1/3), 235-268.

Berry, J. S., & Nyman, M. A. (2003). Promoting Students' Graphical Understanding of the Calculus. The Journal of Mathematical Behavior, 22(4), 479-495.

Bosch, M., Chevallard, Y. & Gascón, J. (2005). Science or magic? The use of models and theories in didactics of mathematics. In: Bosch, M. (Orgs.). Proceedings of the IVth Congress of the European Society for Research in Mathematics Education (CERME 4). Barcelona: Universitat Ramon Llull Editions, 2005. p. 1254–1263.

Bosch, M. & Gascón, J. (2006). Twenty-five years of didactic transposition. ICMI Bulletin, 58, 51-65.

Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In: Biehler, R., Sholz, R., Straaer & Winkelmann, B. (Eds.), Didactics of mathematics as a scientific discipline (p. 73-88). Dordrecht: Kluwer Academic Publisher

Clausen, F., Schomacker, G. & Tolnø, J. (2011a). Gyldendals Gymnasiematematik Grundbog B2. (Vol. 2). Gyldendal.

Clausen, F., Schomacker, G. & Tolnø, J. (2011b). Gyldendals Gymnasiematematik Arbejdsbog B2. (Vol. 2) Gyldendal.

Cooney, T. J. (1999). Conceptualizing teachers' ways of knowing. Educational studies in mathematics, 38(1-3), 163-187.

Derivative of Cosine Function. (n.d.). Retrieved August 3, 2016, from https://proofwiki.org/wiki/Derivative_of_Cosine_Function

Derivatives cannot have jump discontinuity. (n.d.). Retrieved August 5, 2016, from http://math.stackexchange.com/questions/563771/prove-that-if-a-function-f-has-a-jump-at-an-interior-point-of-the-interval-a

Durand Guerrier, V., Winsløw, C. & Yoshida, H. (2010). A model of mathematics teacher knowledge and a comparative study in Denmark, France and Japan. Annales de didactiques et de sciences cognitives, 15, 141-166.

Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. C. (2010). Contrasting Cases of Calculus Students' Understanding of Derivative Graphs. Mathematical thinking and learning, 12(2), 152-176.

Hähkiöniemi, M. (2006). The role of presentations in learning the derivate. Jyväskylä: Finland: University of Jyväskylä.

Hill, H. C., Rowan, B. & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American educational research journal, 42(2), 371-406.

Hill, H. C., Ball, D. L. & Schilling S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. Journal for research in mathematics education, 39(4), 372-400.

Holm, A, & Pelger, S. (2015). Mathematics Communication within the Frame of Supplemental Instruction–SOLO & ATD Analysis. In: Bosch, M., Chevallard, Y., Kidran, Y., Monaghan, J. & Palmé, H. (Eds.), CERME 9 (pp. 87-97). Prague: European Association for Research in Mathematics Education.

Hunter, J. K. (2014). An introduction to real analysis. Retrieved August 5, 2016, from https://www.math.ucdavis.edu/~hunter/intro_analysis_pdf/intro_analysis.pdf

Introduktion til de matematiske fag (MatIntroMat). (2016, March 9). Retrieved August 3, 2016, from http://kurser.ku.dk/course/NMAB10002U

Jessen, B. E., Holm, C., & Winsløw, C. (2015). Matematikudredningen: Udredning af den gymnasiale matematiks rolle og udviklingsbehov. København: Ministeriet for Børn, Undervisning og Ligestilling.

Klein, F. (1932). Elementary Mathematics – from an advanced standpoint. Translated by Hedrick, E., R. & Noble, G., A. New York: Macmillan.

Lindstrøm, T. (2006). Kalkulus (Vol. 3). Universitetsforlaget.

Matematik – stx. (2013, November 6). Retrieved July 26, 2016, from https://www.uvm.dk/Uddannelser/Gymnasiale-uddannelser/Fag-og-laereplaner/Fag-paa-stx/Matematik-stx

Matematik A – stx. (2013, June). Retrieved July 26, 2016, from https://www.retsinformation.dk/Forms/R0710.aspx?id=152507#Bil35

Matematik B – stx. (2013, June). Retrieved July 26, 2016, from https://www.retsinformation.dk/Forms/R0710.aspx?id=152507#Bil36

Matematik C – stx. (2013, June). Retrieved July 26, 2016, from https://www.retsinformation.dk/Forms/R0710.aspx?id=152507#Bil37

Mean Value Theorem Explanation. (n.d.). Retrieved July 26, 2016, from https://www.wyzant.com/resources/lessons/math/calculus/differentiation/mean_value_theorem

Miller, J. (n.d.). Retrieved July 26, 2016, from http://www.solitaryroad.com/c355.html

Ministeriet for børn, undervisning of ligestilling. (2016) Evaluering af de skriftlige prøver i matematik på STX og HF ved sommereksamen 2015.

Miyakawa, T. & Winsløw, C. (2013). Developing mathematics teacher knowledge: the paradidactic infrastructure of “open lesson” in Japan. Journal of Mathematics Teacher Education, 16(3), 185-209.

Nemirovsky, R., & Rubin, A. (1992). Students' tendency to assume resemblances between a function and its derivative. Cambridge, Massachusetts: TERC Communications, 2-92

Pino-Fan, L., Godino, J. D., Font, V., & Castro, W. F. (2012). Key epistemic features of mathematical knowledge for teaching the derivative. In: Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education, (vol. 3 pp. 297-304). Taipei, Taiwan: PME

Proofs of derivatives of ln(x) and e^x. (n.d.). Retrieved August 3, 2016, from https://www.khanacademy.org/math/differential-calculus/taking-derivatives/der-common-functions/v/proofs-of-derivatives-of-ln-x-and-e-x

Proof of the Squeeze Theorem. (n.d.). Retrieved August 3, 2016, from http://educ.jmu.edu/~ohmx/squeeze_proof.pdf

Pædagogikum. (n.d.). Retrieved July 7, 2016, from http://www.gl.org/loenogans/nygymnasielaerer/paedagogikum/Sider/Hjem.aspx

Retningslinjer for universitetsuddannelser rettet mod undervisning i de gymnasiale uddannelser. (2006, January 20). Retrieved August 3, 2016, from https://www.retsinformation.dk/Forms/R0710.aspx?id=29265

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. Journal of educational psychology, 93(2), 346.

Santos, A. G., & Thomas, M. O. J. (2003). Representational ability and understanding of derivative. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), Proceedings of the 27th Conference of PME (Vol. 2, pp. 325-332). Honolulu: University of Hawaii

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. In: Educational studies in mathematics, 22(1), 1-36.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational researcher, 15(2), 4-14.

Sidefag i matematik. (n.d.). Retrieved August 3, 2016, from http://www.science.ku.dk/uddannelser/andre-tilbud/sidefag/matematik/

Stx-bekendtgørelsen. (2013, June 28). Retrieved July 26, 2016, from https://www.retsinformation.dk/Forms/R0710.aspx?id=152507

Sådan bliver du gymnasielærer. (n.d.). Retrieved July 7, 2016, from http://www.gl.org/loenogans/Nygymnasielaerer/Sider/blivgymnasielaerer.aspx

Thim, J. (2003). Continuous nowhere differentiable functions. Luleå: University of Technology, Department of Mathematics. Master Thesis.

The Derivative. (n.d.). Retrieved August 7, from http://math.colgate.edu/math323/dlantz/extras/notesC5.pdf

Weierstrass function. (n.d.). Retrieved July 11, 2016, from https://en.wikipedia.org/wiki/Weierstrass_function

Winsløw, C. (2013). ATD and other approaches to a classical problem posed by F. Klein. Copenhagen: University of Copenhagen.

Winsløw, C. (2015). Mathematical analysis in high school. In: Bergsten, C. & Sriraman, B. (Eds.), Refractions of Mathematics Education (p. 197-213). Copenhagen: Information Age Publishing, incorporated.

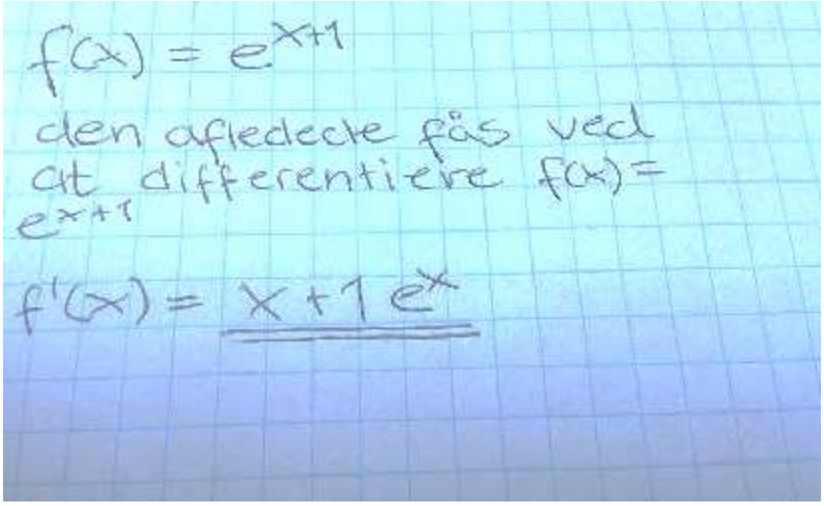
Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. CBMS Issues in Mathematics Education, 8, 103-127. Webpages

Appendix A

The content of this appendix was originally four national written-exam tests: two tests for B-level students (28th of May 2015 and 22nd of May 2015) and two tests for A-level students (22nd of May 2015 and 28th of May 2015); these are left out in this version, because they are not published yet.

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



a) Analysér og vurdér svaret.

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

Opgave 2

Givet $f(x) = \sqrt{x^2}, \ \ x \epsilon \mathbb{R}$

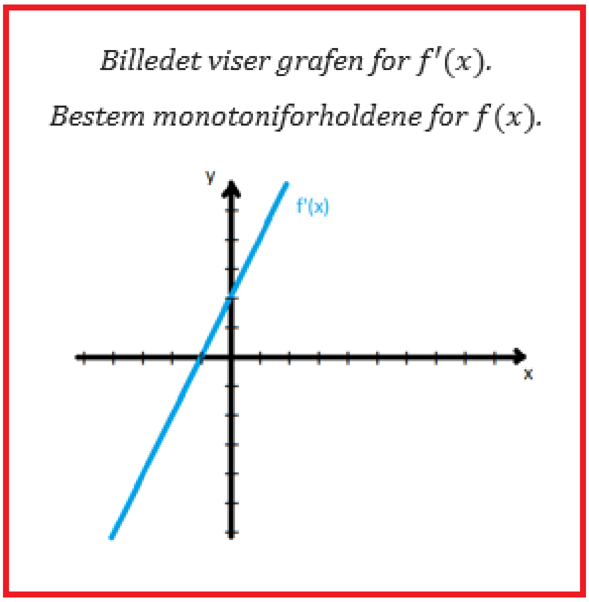
a) Bestem f' (uden brug af CAS) og tegn graferne for f samt f'.

b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

Appendix B

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

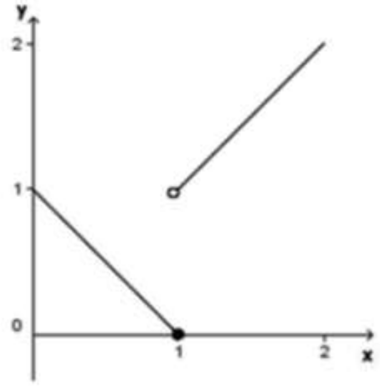


a) Forklar hvad der er svært ved opgaven.

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

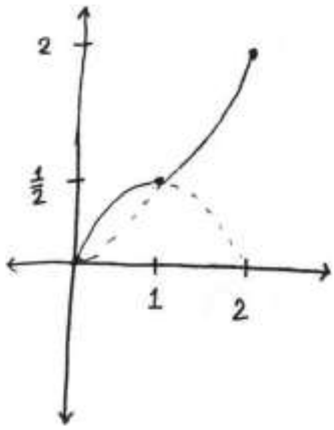
Opgave 4

a) Marie har tegnet følgende graf *(figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?

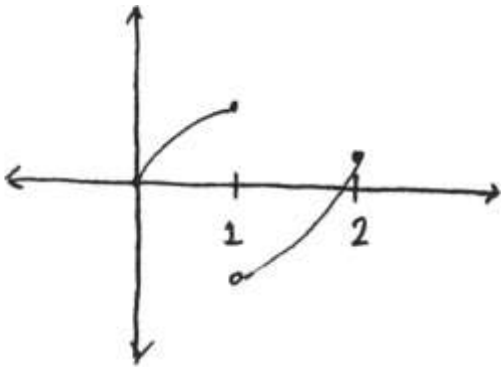


Figur 1

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



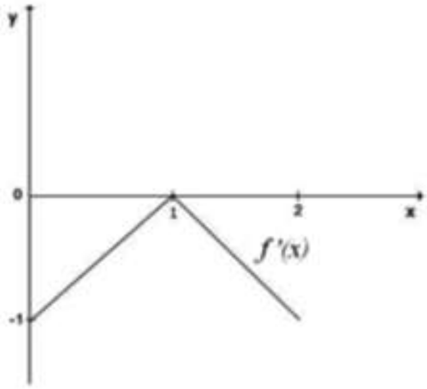
Figur 2



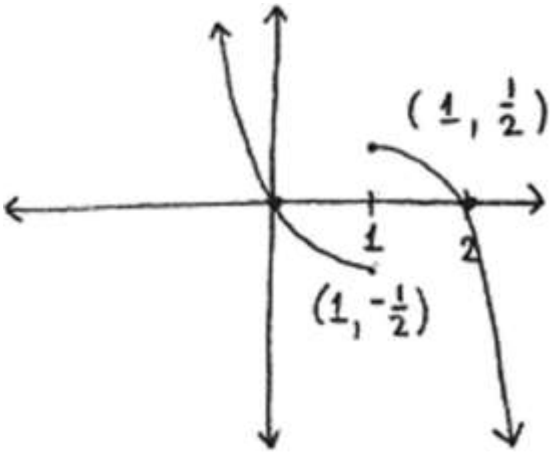
Figur 3

Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

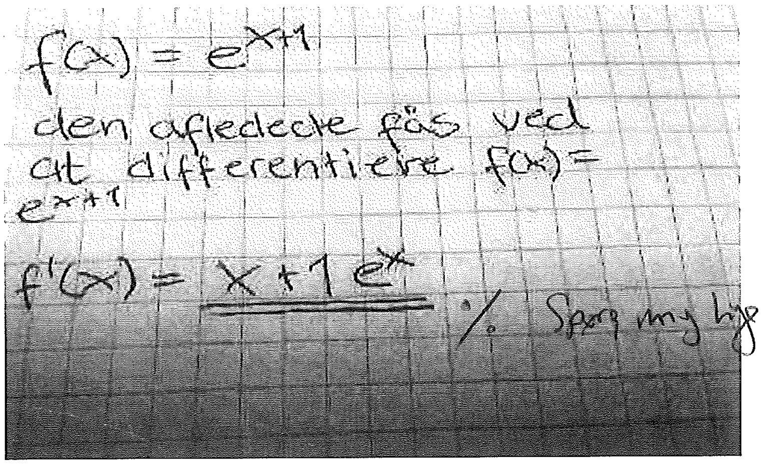


c) Hvad siger du til din elev?

Appendix C1

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



a) Analysér og vurdér svaret.

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

Opg. 1

a) $f(x) = x^3 - 2x^2 + x - 7$

b) $f(x) = e^x$

c) $f(x) = e^x + x^2$

Opg. 2

a) $\ln(2x+1)$

b) $\sqrt{x^3-7}$

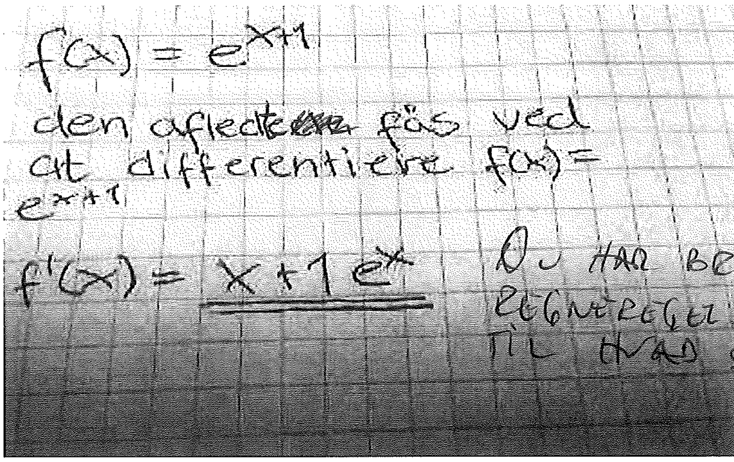
c) ~~e^{x+2}~~ e^{x+1}

I

[redacted]

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.


$f(x) = e^{x+1}$
den afledte fås ved at differentiere $f(x) = e^{x+1}$
$f'(x) = x+1\, e^{x}$

DU HAR BRUGT DEN FORKERTE REGNEREGEL. HUSK, AT e OPLØFTET TIL HVAD SOM HELST, GIVER e OPLØFTET TIL HVAD SOM HELST OG HVIS "HVAD SOM HELST" ER EN FUNKTION SKAL MAN BRUGE KÆDEREGLEN

a) Analysér og vurdér svaret.

HAN HAR BRUGT REGNEREGLEN FOR HVORDAN MAN DIFFER EN POTENS. HAN HAR ALTSÅ GANGET EKSPONENTEN NED (OG [illegible] GANGET PÅ) OG TRUKKET 1 FRA I EKSP..

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

✓

JEG VILLE NOK OGSÅ SKRIVE KÆDEREGLEN.

c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

DIFF. FUNKTIONERNE: ~~$f(x) = $~~

$f(x) = \sqrt{2x+1}$

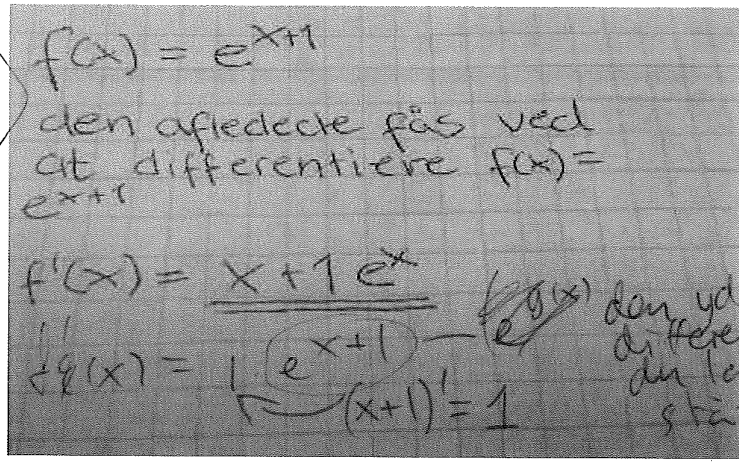
OG

$g(x) = e^{x^2}$

Opgave 1

Peter har i sin skriftlige aflevering differentieret funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.

Fandt: på regneregel $(e^{ux})' = ue^{ux}$



den ydre funktion differentieret, hvor du lader den indre stå

a) Analysér og vurdér svaret.

$f'(x) = 1 \cdot e^{x+1}$

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

c) Stil en ny opgave til eleven som kan afdække om din elev har forstået din rettelse:

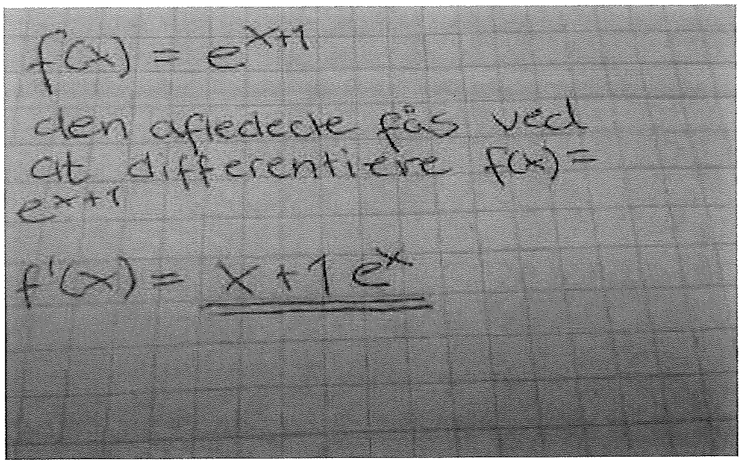
Differentier følgende funktioner

1) $f(x) = (x+3)^2$
2) $f(x) = e^{2x}$
3) $f(x) = x+1$
4) $f(x) = e^x$

omvendt rækkefølge.

Opgave 1

Peter har i sin skriftlige aflevering differentieret funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



husk reglen:

$f'(g(x)) = f'(g(x)) \cdot g'(x)$

Her er:

$f(x) = e^x$

$g(x) = x+1$

a) Analysér og vurdér svaret.

Han ved at han skal differentiere.
Og han ved, at han skal kigge på indre og ydre funktion.
Han ved også, at han skal sætte den indre funktion "nedforan".
Han glemmer reglen for diff. af sammensat fkt.

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

c) Stil en ny opgave til eleven som kan afdække om din elev har forstået din rettelse:

$h(x) = \sqrt{x+2}$

hvis svag elev: fortælle hvilken funktion der er den indre og hvilken der er den ydre.

$f(x) = \sqrt{x}$

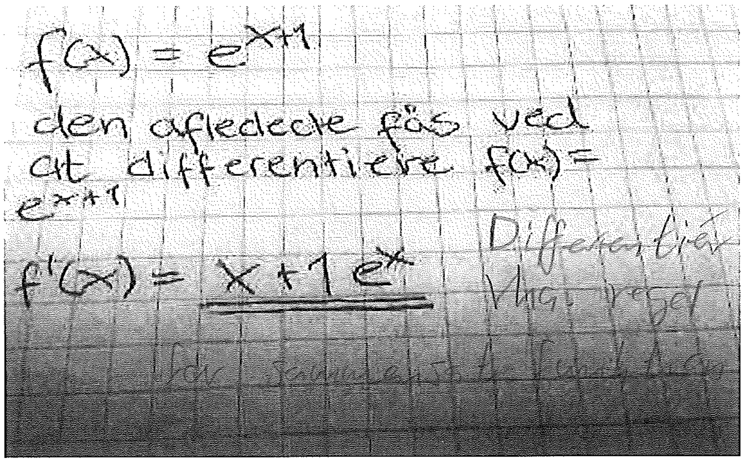
$g(x) = x+2$

[redacted]

13

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



Regler $(x^a)' = a x^{a-1}$ duer ikke her, da udtrykket er af formen a^x. (NB Hvis den gjaldt, så husk parentes om $x+1$).

⇐ Differentiér vha. regel for sammensat funktion

Alg.

a) Analysér og vurdér svaret.

Det er forkert. Er diff. vha. regneregel for potenser. - og endda forkert

Rigtigt svar: $e^{x+1} \cdot 1 = e^{x+1}$

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

Differentiér x^{a+1}

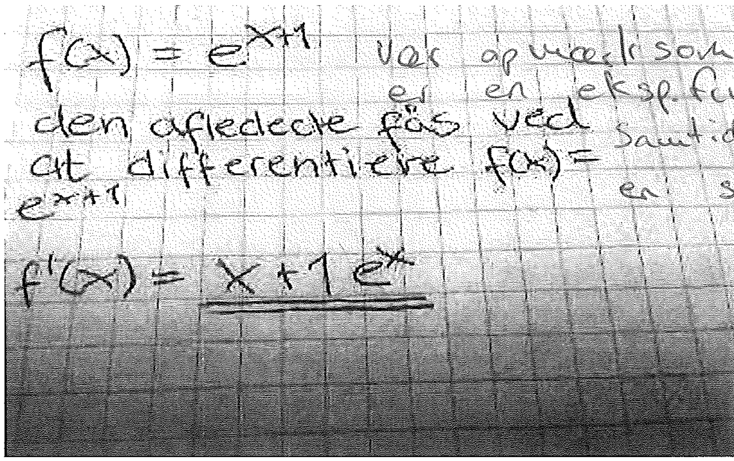
a^{3x}

V

[redacted] nr. 7.

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



Vær opmærksom på at f(x) er en eksp.funktion, når du differentierer. Samtidig er der tale om en sammensat funktion.

a) Analysér og vurdér svaret.

$(e^{x+1})' = (x+1)' \cdot e^{x+1} = 1 \cdot e^{x+1}$

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

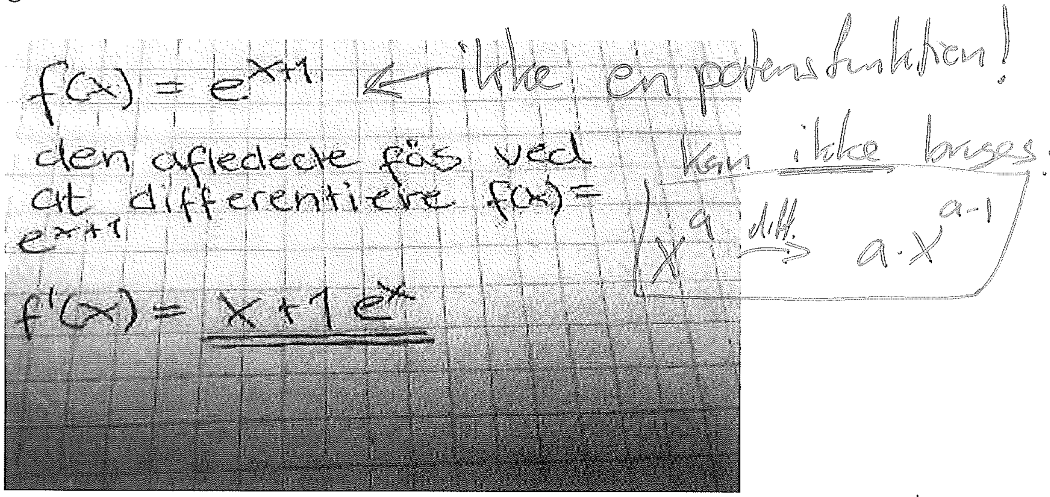
c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

[redacted]

$x^a \rightarrow a \cdot x^{a-1}$

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



a) Analysér og vurdér svaret. $f'(x) = e^{x+1} \cdot 1 = e^{x+1}$ (kædeRegel)

Elev har misforstået konstant/variabel

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

Tænkt på $g(x) = e^x$ og $h(x) = x+1$ som hhv. ydre og indre funktion, og find en regl til at differentiere sammensatte funktioner.

c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

$f(x) = 10^x$. Find $f'(x)$

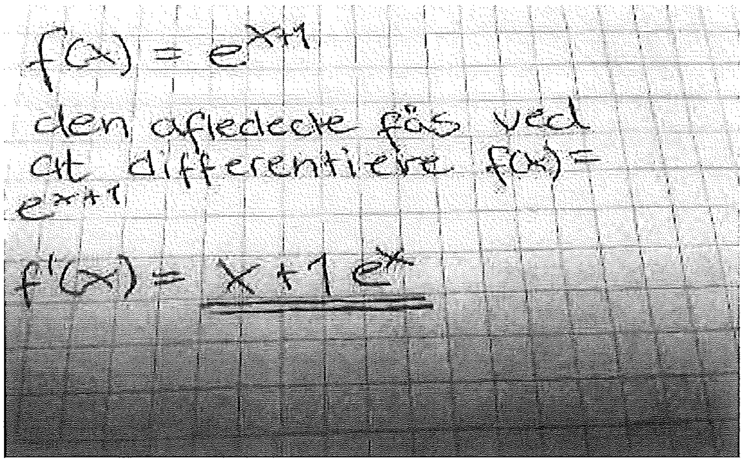
$f(x) = \sin(x^2)$ Find $f'(x)$

$f(x) = x^5$ Find $f'(x)$

Svend 13

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



a) Analysér og vurdér svaret.

ELEVEN HAR BRUGT REGEL FOR DIF. ~~AA~~ AF x^n.
MEN SÅ GLEMT PARENTES OM ~~x+1~~ x+1.

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

DU HAR TÆNKT PÅ x^{n+1}. MÅSKE ER DET NEMMERE
AT SE SOM YDRE OG INDRE FUNKTION, HVIS
DU SKRIVER $F(x) =$ ~~EF~~ $EXP(x+1)$

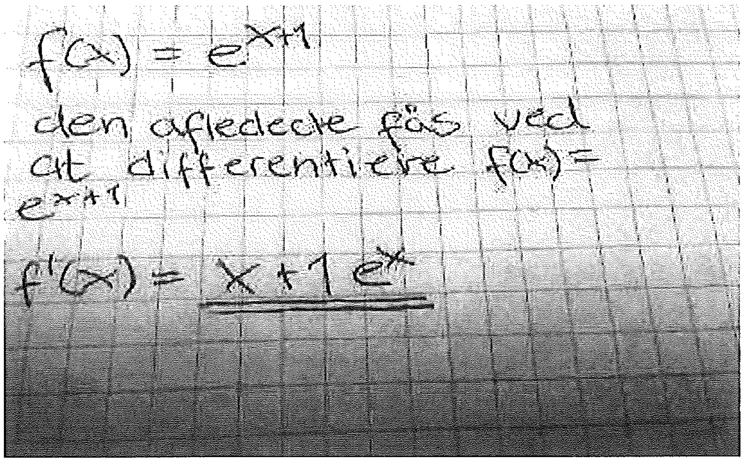
c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

DIF. $SIN(x^2)$

[redacted]

Opgave 1

Peter er i sin skriftlige aflevering blevet bedt om at differentiere funktionen $f(x) = e^{x+1}$. Hans arbejde fremgår af nedenstående billede.



a) Analysér og vurdér svaret. Eleven har ikke differentieret korrekt

b) Hvordan vil du rette besvarelsen? (Skriv gerne på billedet, som du ville have skrevet i en elevs aflevering) Evt. kommentar:

Eleven har anvendt differentiale reglerne vedrørende potenser og ikke e.

c) Stil en ny opgave til eleven, som kan afdække, om din elev har forstået din rettelse:

Appendix C2

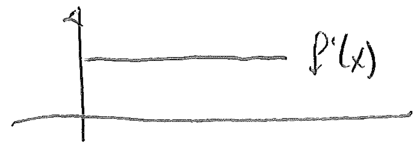
Opgave 2

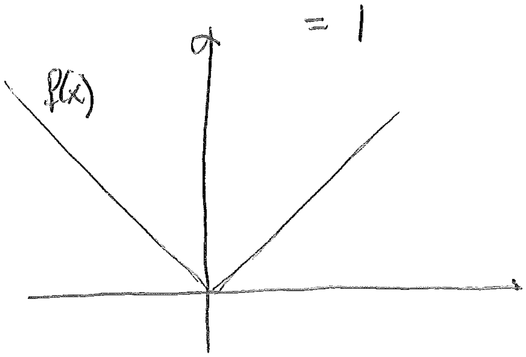
Givet $f(x) = \sqrt{x^2}, \;\; x \in \mathbb{R}$

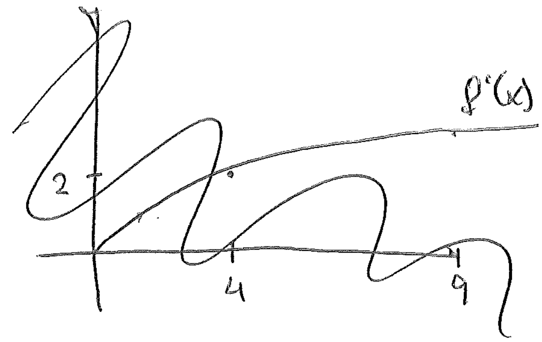
a) Bestem f' (uden brug af CAS) og tegn grafenerne for f samt f'.

$$f'(x) = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{\sqrt{x^2}} \;, \quad x > 0$$

$$= 1$$







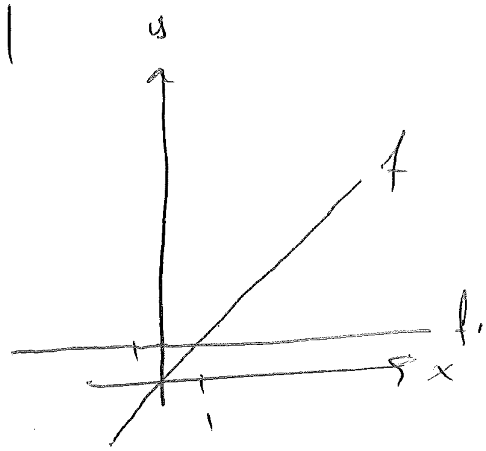
b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

Opgave 2

Givet $f(x) = \sqrt{x^2}, \quad x \in \mathbb{R}$

a) Bestem f' (uden brug af CAS) og tegn grafe~~n~~erne for f samt f'.

$f'(x) = 1$



b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

OVERKILL.

HVIS MAN KAN REDUCERE, SÅ GØR DET, OGSÅ INDEN DU NÅR TIL FACIT

OG

SERIØST? HAR CAS GIVET DIG DET DER?!

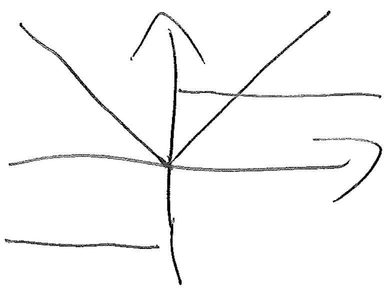
Opgave 2

Givet $f(x) = \sqrt{x^2}, \ x \in \mathbb{R}$

a) Bestem $f'(x)$ (uden brug af CAS) og tegn grafen for $f(x)$ samt $f'(x)$.

$f(x) = |x|$

$f(x) = -1$, når $x < 0$ og $f(x) = 1$ når $x \geq 0$



$f'(x)$ ikke defineret når $x = 0$.

b) En af dine elever har udregnet $f'(x)$ vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

lad være med at bruge.
Jeg ville referere til afstande og brugen af $\sqrt{x^2}$ til at beregne $|x|$.
Bede eleven om at tegne $f(x) = x$ og $f(x) = -x$.
og snakke om $f(x) = |x|$.

$\frac{x}{|x|} = +1$ når $x \geq 0$ og -1 når $x < 0$

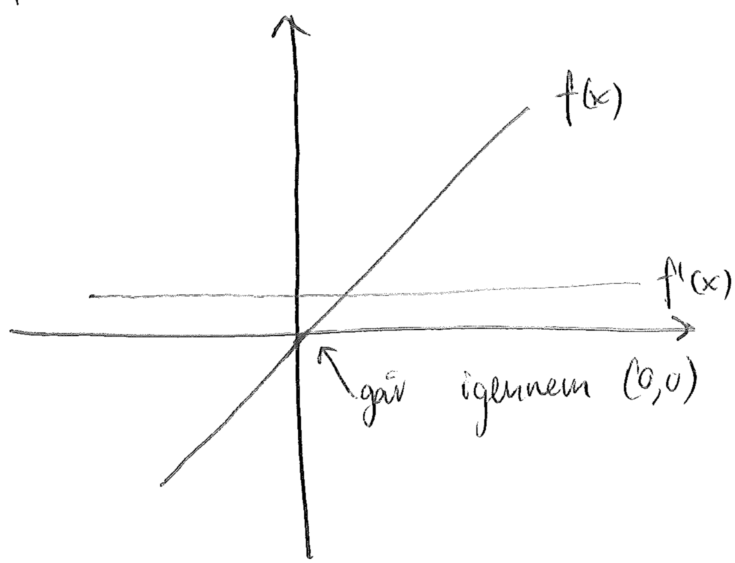
Opgave 2

Givet $f(x) = \sqrt{x^2},\ x \in \mathbb{R}$

a) Bestem $f'(x)$ (uden brug af CAS) og tegn grafen for $f(x)$ samt $f'(x)$.

$f(x) = \sqrt{x^2} = x$

$f'(x) = 1$



b) En af dine elever har udregnet $f'(x)$ vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven? En ad gangen ville jeg sige følgende:

1) Jeg ville sige, at den kunne reduceres yderligere.

2) Hvis eleven stadig ikke forstod hvad jeg mente, ville jeg tale om omvendte funktioner (+ overfor −), (· overfor ÷) osv.

3) Til sidst sige at der kan gøres noget ved nævneren, så brøken kan reduceres.

4) Spørge ind til hvad fx $\frac{4}{4}$ er eller $\frac{a}{a}$ osv.

Opgave 2

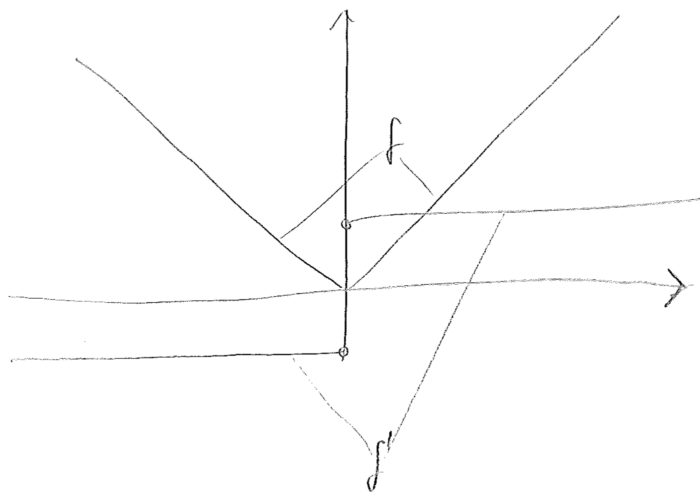
$f(x) = (x^2)^{\frac{1}{2}}$

Givet $f(x) = \sqrt{x^2}, \ \ x \in \mathbb{R}$

a) Bestem f' (uden brug af CAS) og tegn grafenerne for f samt f'.

$f'(x) = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} \quad , \ x \neq 0$

$\sqrt{x^2} = |x|$



b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

At det for så vidt er rigtigt, men hvad sker der hvis $x=0$?

(f er ikke diff. i $x=0$)

$x^{\frac{1}{2}}\,' = \frac{1}{2}x^{-\frac{1}{2}}$

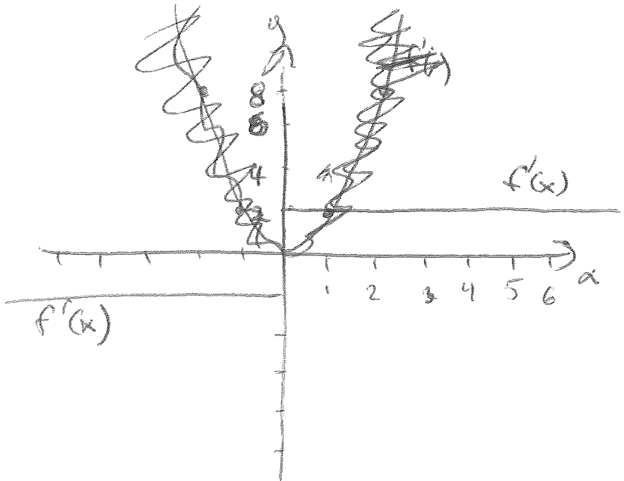
$2x\sqrt{x}$

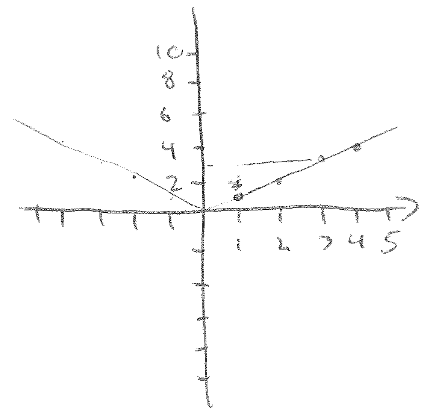
Opgave 2

Givet $f(x) = \sqrt{x^2}$, $x \in \mathbb{R}$

a) Bestem f' (uden brug af CAS) og tegn grafenerne for f samt f'.

$f'(x) = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x = \frac{1}{\sqrt{x^2}} \cdot x = \frac{x}{|x|}$





b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

Overvej $\sqrt{x^2}$. Evt. Regn eksempler ud. Hvad for $x=0$?

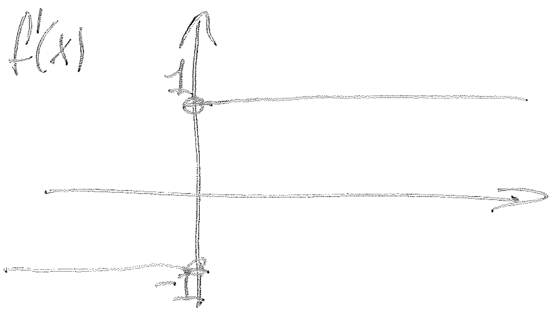
Opgave 2

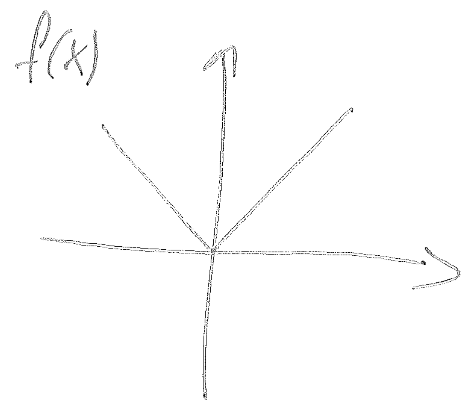
Givet $f(x) = \sqrt{x^2}$, $x \in \mathbb{R}$

a) Bestem f' (uden brug af CAS) og tegn grafenerne for f samt f'.

$$f'(x) = \frac{1}{2\sqrt{x^2}} \cdot 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

deler op $f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$





b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

Hvad med $x = 0$?

Kan det skrives smartere?

Hvad siger $\sqrt{x^2}$? $(= |x|)$

Opgave 2

Givet $f(x) = \sqrt{x^2},\ \ x \in \mathbb{R}$

a) Bestem f' (uden brug af CAS) og tegn grafenerne for f samt f'.

$f(x) = x, \quad x \geq 0$

$f(x) = -x, \quad x < 0$

$f'(x) = 1, \quad x > 0$

$f'(x) = -1, \quad x < 0$

EJ. DIFF. F. $x = 0$

b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

BEMÆRK AT X ~~IKKE~~ (X) KAN VÆRE 0.

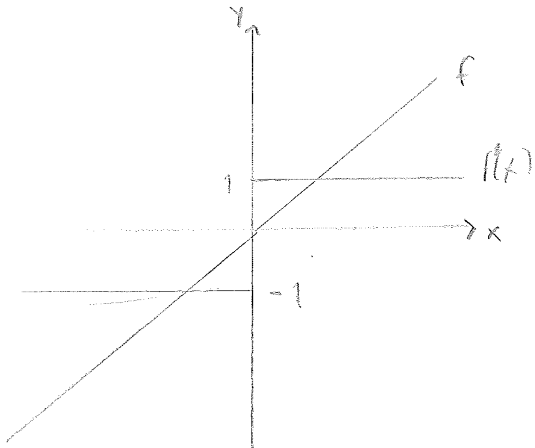
KUNNE DU HAVE OPDELT FOR ~~X,~~ $x > 0$, $x < 0$?

Opgave 2

Givet $f(x) = \sqrt{x^2}$, $x \in \mathbb{R}$

a) Bestem f' (uden brug af CAS) og tegn grafenerne for f samt f'.

$x \neq 0$: $f(x) = \sqrt{x^2} = x^{\frac{2}{2}} = x^1 = x \Rightarrow f'(x) = 1$



b) En af dine elever har udregnet f' vha. CAS og har fået $f'(x) = \frac{x}{\sqrt{x^2}}$. Hvad siger du til eleven?

Eleven må have opskrevet f på en forkert måde

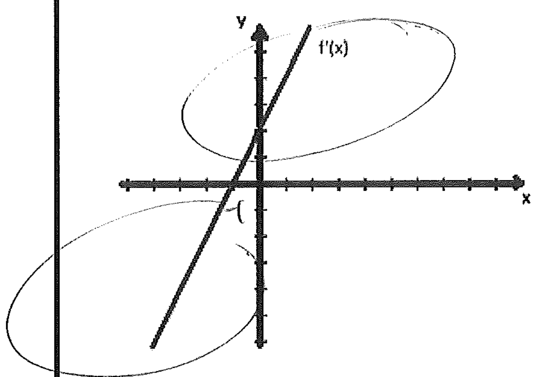
Appendix C3

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



x		-1	
f'(x)	+	0	+
f(x)			

a) Forklar hvad der er svært ved opgaven.

For nogle elever: positiv = voksende.

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

- positiv : Graf over x-aksen
- negativ : Graf under x-aksen

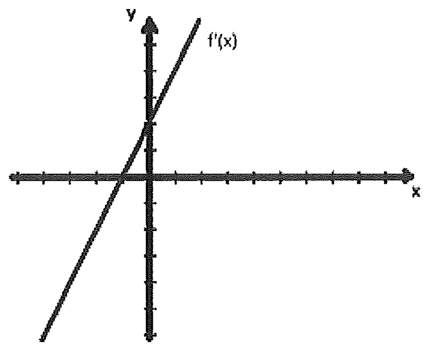
- Opgaver, der først spørger til graf over/under x-aksen, dernæst til om funktion er positiv/negativ.

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

ELEVERNE SER TYPISK GRAFER FOR FUNKTIONER f(x), IKKE SÅ OFTE FOR FUNKTIONERS AFLEDTE, f'(x).

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

- TEGNE FLERE FUNKTIONERS AFLEDTE, NÅR MAN HAR OM MONOTONIFORHOLD.
- SKABE EN GRAFISK KOBLING MELLEM DIFF.- OG INTEGRALREGNING.

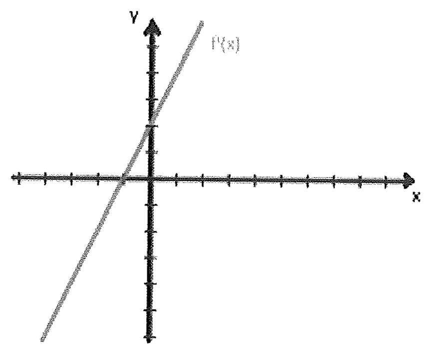
↳ NOGET I RETNING AF, AT DEN OPR. FUNKT. JO OGSÅ ER EN ANDEN FUNKTIONS DIFFERENTIEREDE.. ALTSÅ: HVIS f(x) ER TEGNET ER DET JO BARE F'(x)..

Opgave 3

Til den skriftlige eksamen svarede mere end 25% af eleverne forkert på en opgave som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

Eleverne er vant til at arbejde med grafen for f(x) og så tegne f'(x) eller bestemme fortegn for den. Ikke omvendt. De kan også tegne grafer ud fra skema over fortegn. Men grafen for f'(x) har de svært ved at forholde sig til - specielt når f'(x) vokser og f(x) aftager.

b) Hvordan man kan arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

1) Stille opgaver - det er et spørgsmål om tilvænning.

2) Relatér f'(x)-kurven til hastighed og tilbagelagt f(x) til en afstand der bliver tilbagelagt. Elever kan ofte godt relatere til sammenhæng mellem hastighed og afstand.

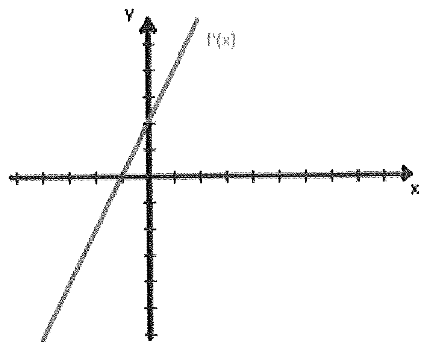
3) Øve eleverne i at tegne f'(x) ud fra f(x), så de ser, hvordan de 2 hænger sammen, med eksempler.

Opgave 3

Til den skriftlige eksamen svarede mere end 25% af eleverne forkert på en opgave som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

Det er svært for dem, fordi de ikke har grafen for f, men kun graf for f'.

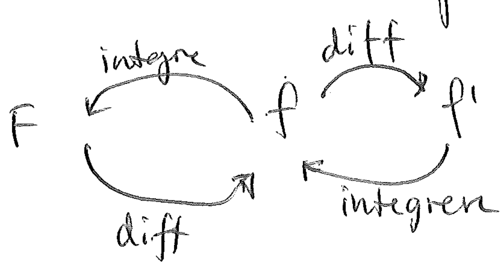
Det svære er at man her skal gå "omvendt" af hvad "man plejer".

Man går normalt fra $f(x)$ til $f'(x)$ men glemmer (som lærer) den anden vej, i undervisningen, - men den kommer til integralregning.

Det er svært at huske at hvor $f'(x) > 0$ er $f(x)$ voksende og hvor $f'(x) < 0$ er $f(x)$ aftagende. for $f(x) = 0$ har $f(x)$ vandret tangent.

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

- bede eleverne plotte graf for f og f' i samme koordinatsystem hver gang de har differentieret. + kommentere på sammenhængen.
- bede eleverne lave fortegnsvariationsundersøgelse.
- komme ind på det ifm. integralregning

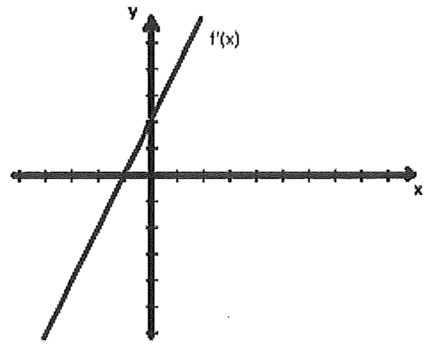


Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

Et bud kunne være at eleverne har svært ved at omsætte info om f' til info om f.

En anden ting er, at man plejer at få givet og evt. vist grafen for f, mens man her pludselig får grafen for f' givet.

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

1. Tolkning af f' som hældning af tangent
2. Hvad betyder det for f, at f' er pos., 0, hhv. neg.?
3. Hvad siger grafen for f' og f? Specielt: hvad sker der omkring dens skæring med x-aksen?

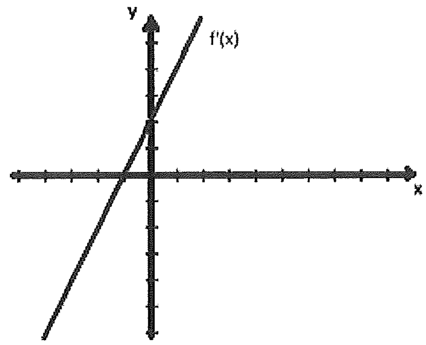
V

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

Vil tro at der kan være forvirring mellem f'(x)'s skæring med x-akse (og måske y-akse) og f(x)'s.

Man får givet f'(x) og skal gå modsat vej af vanligt for at sige noget om f(x).

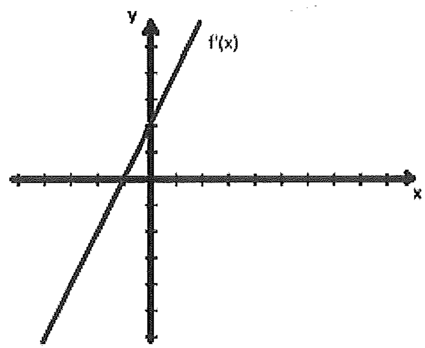
b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

svært at skelne mellem "f'(x)" og "hældningen af f'(x)."

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

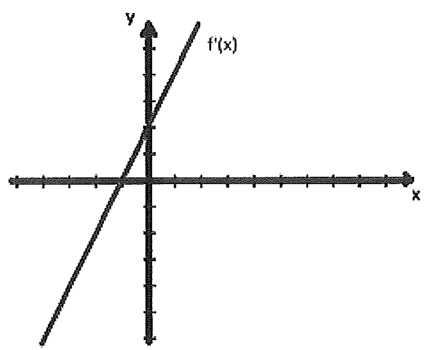
tegn f og f' i samme koordinat-system for "en masse" forskellige grafer

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

HÆLDNING AF F' BLANDES SAMMEN MED HVAD F' ER.

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

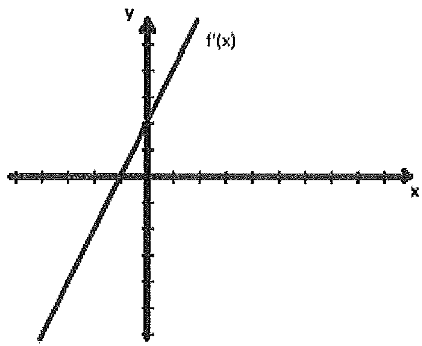
MASSER AF EKSEMPLER.

Opgave 3

Til den skriftlige eksamen i Mat A 2015 (delprøven uden hjælpemidler) svarede mere end 25% af eleverne forkert på en opgave, som ligner den følgende.

Billedet viser grafen for $f'(x)$.

Bestem monotoniforholdene for $f(x)$.



a) Forklar hvad der er svært ved opgaven.

Eleven skal huske at grafen viser den afledte af f og ikke f selv.

b) Hvordan kan man arbejde videre med de udfordringer som er identificeret i a)? Skriv evt. i punktform.

Som lærer kan man arbejde med hvordan vi udfra billedet af den afledte kan læse hvordan funktionen selv ser ud

$f'(x) = 0 \iff f(x)$ konstant

$f'(x) > 0 \iff f(x)$ voksende

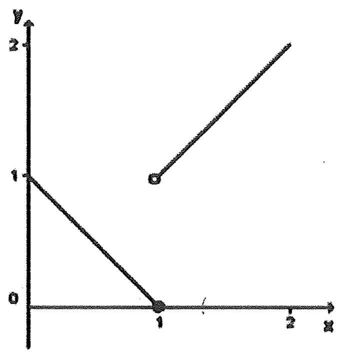
$f'(x) < 0 \iff f(x)$ aftagende

Kan arbejde med f og f' i samme koordinatsystem

Appendix C4

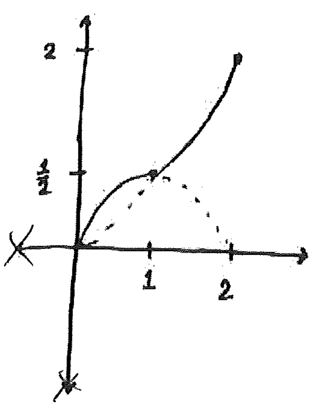
Opgave 4

a) Marie har tegnet følgende graf *(figur 1)*, som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?

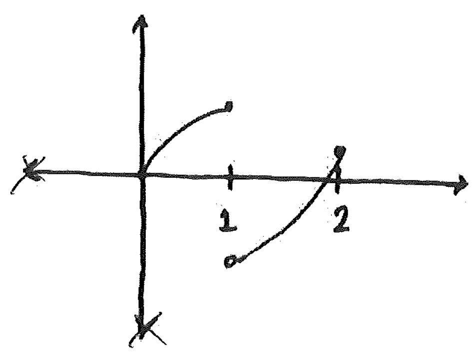


Figur 1

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



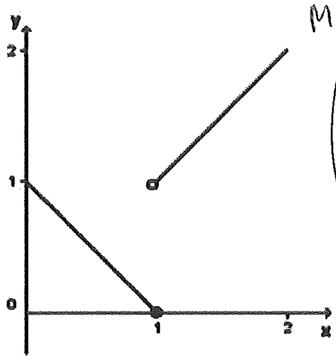
Figur 2



Figur 3

Opgave 4

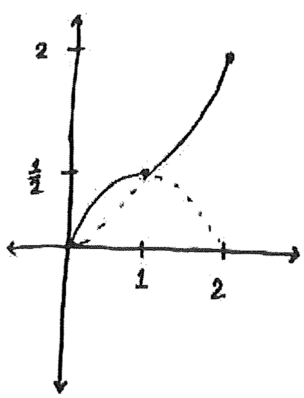
a) Marie har tegnet følgende graf (*figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



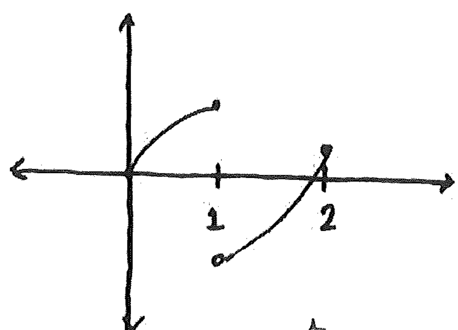
Figur 1

Mig: Hvad er det nu man kan se på grafen for en afledt funktion?
- Om hældningen på den opr. funkt. er voksende eller aftagende -
Mig: Hvad er hældningen for den opr. fkt?
Hvad har du offe'et for at få det fkt? Giver det mening for dig, at f'(1)=0 og lige bagefter er den 1 og voksende?

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



Figur 2



Figur 3

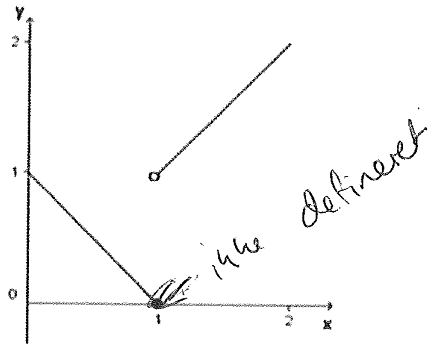
En kontinuert, differentiabel funktion har vist ikke en diskontinuert funkt. som afledt.

Pas på med at kalde den for en stamf., for der gælder bestemte regler for diskontinuerte funktioners afledte..

Måske.. spørg mig igen imorgen Marie..

Opgave 4

a) Marie har tegnet følgende graf (*figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



Figur 1

$y = -x + 1 \quad x < 1$

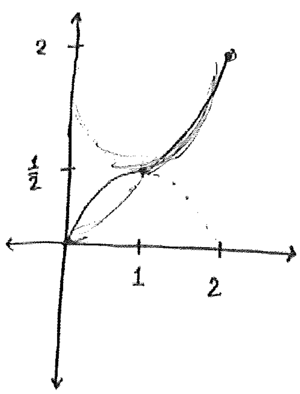
$y = x \quad x > 1$

$-\frac{1}{2}x^2 + x$

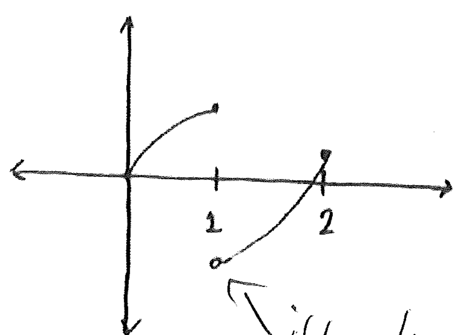
$\frac{1}{2}x^2$

Udtag $x=1$ af Dm(f)

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



Figur 2



Figur 3

ikke kontinuert og ikke differentiabel
Udtag $x=1$ af Dm(f) og så går det.

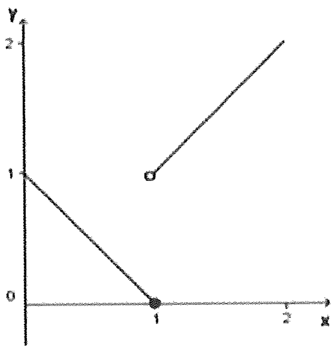
Mundtlig feedback

ikke differentiabel da den er kontinuert, men har ét knæk ved $x=1$. hvor hældningskoefficienten er forskellig ud fra hvilken retning du bevæger dig hen imod punktet (Hun kan godt differentiere funktionen for resten af definitionsmængden, bare ikke for $x=1$.)

III

Opgave 4

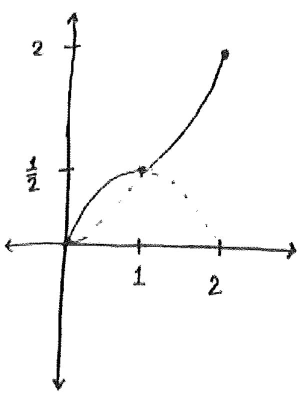
a) Marie har tegnet følgende graf (*figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



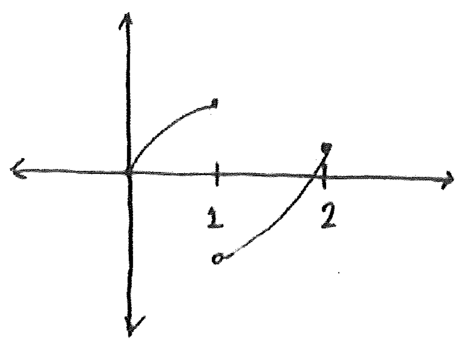
Hvilken funktions afledte er det? Argumenter for hvorfor den afledte ser ud som du siger den gør.

Figur 1

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



Figur 2

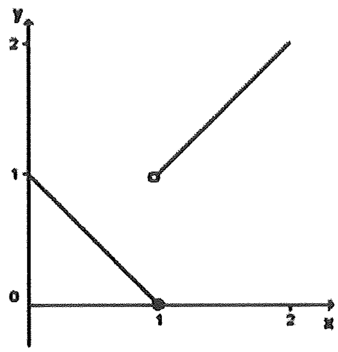


Figur 3

Opgave 4

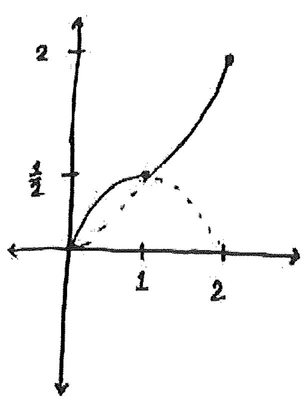
a) Marie har tegnet følgende graf *(figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?

Hvilken funktion er det en afledt for?

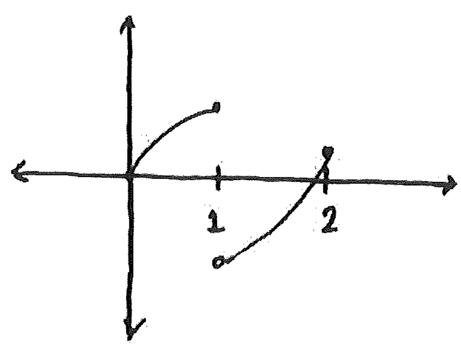


Figur 1

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



Figur 2



Figur 3

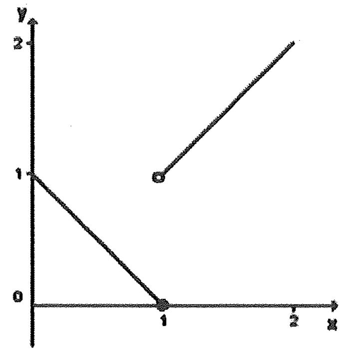
Ville lige læse op på differentiabilitet og så vende tilbage til Marie.

Det springende punkt er om funktionen overhovedet er differentiabel i 1.

V

Opgave 4

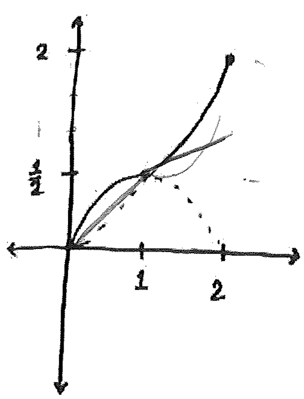
a) Marie har tegnet følgende graf *(figur 1)*, som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



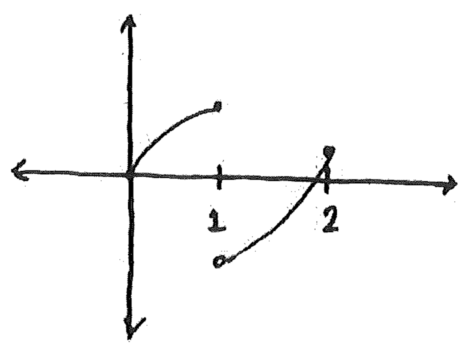
Bede hende om at beskrive/tegne den fkt. hun tænker på.

Figur 1

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



Figur 2

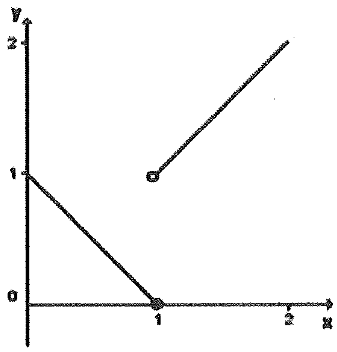


Figur 3

- Husk ensartet akser for overskuelighed
- Nævn +k ved integration
- Vende den med lærestab

Opgave 4

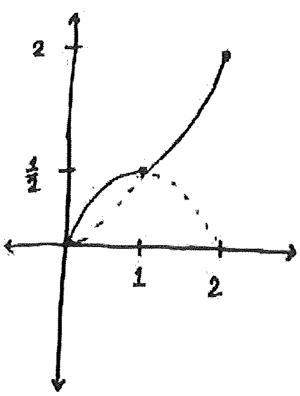
a) Marie har tegnet følgende graf (*figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



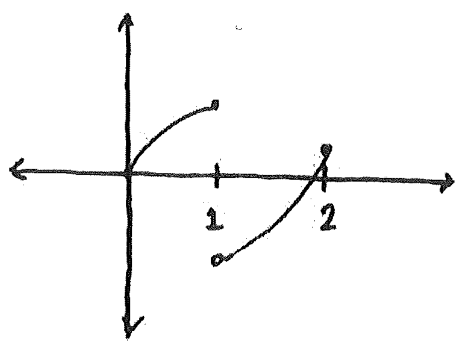
Figur 1

Alle grafer for afledte funktioner skal være kontinuerte, ellers var din oprindelige funktion slet ikke differentiabel (i $x=1$)

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



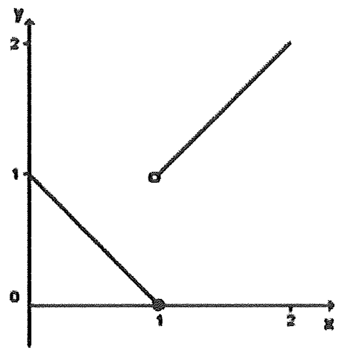
Figur 2



Figur 3

Opgave 4

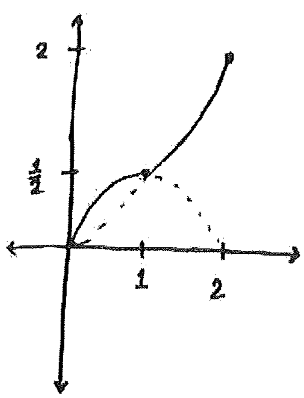
a) Marie har tegnet følgende graf *(figur 1)*, som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



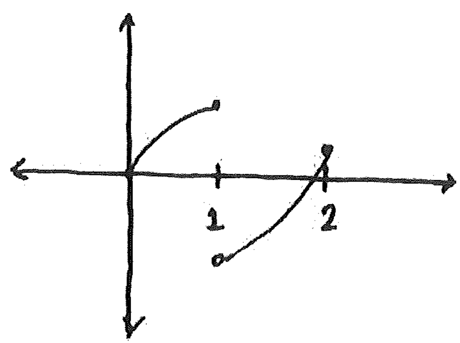
Figur 1

TÆNK PÅ AT ALLE AFL. FUNK. ER KONT.

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.



Figur 2



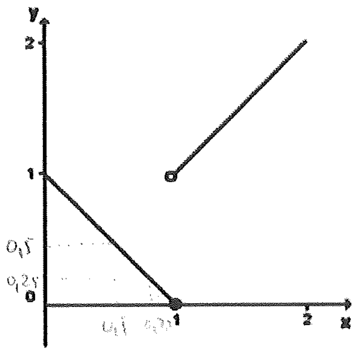
Figur 3

GODT, DOG (D).
DIF. I 1.

DO.

Opgave 4

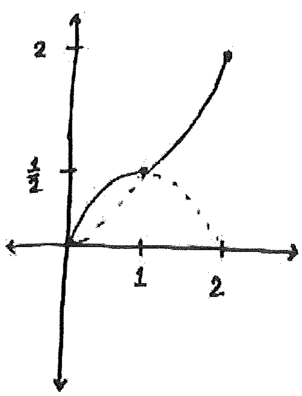
a) Marie har tegnet følgende graf (*figur 1*), som hun påstår er grafen for en afledt funktion. Hvad siger du til Marie?



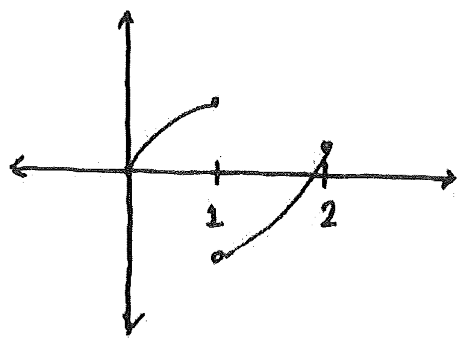
Denne funktion er stykvis kontinuert. Det kan godt være en afledt, men i pkt. x = 1 er der et problem ...

Figur 1

b) Marie kan også tegne grafen for flere stamfunktioner til den funktion, som hun tegnede på Figur 1. På billederne (*Figur 2* og *Figur 3*) ses nogle af disse stamfunktioner. Giv udtømmende feedback til Marie.

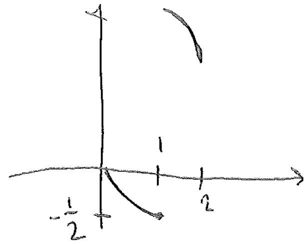


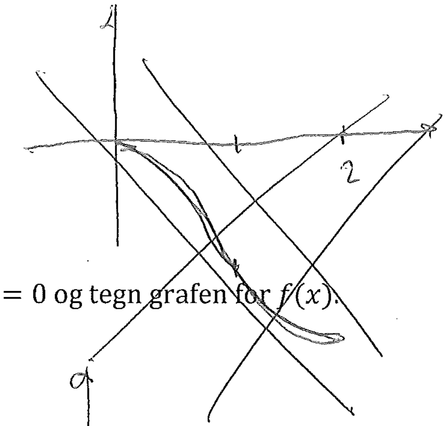
Figur 2



Figur 3

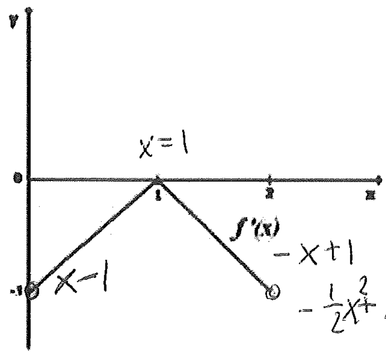
Appendix C5





Opgave 5

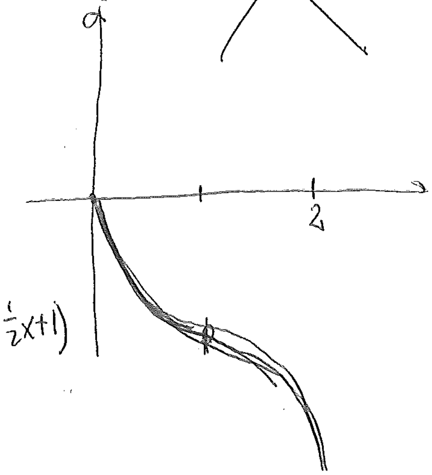
a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



$-\frac{1}{2}x^2 + x + k = x(-\frac{1}{2}x+1)$

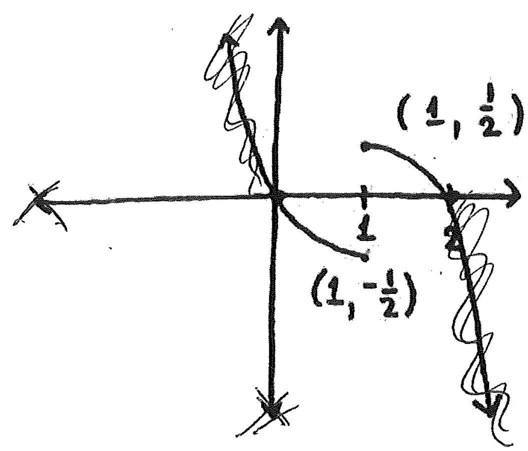
$\frac{1}{2}x^2 - x + k = x(\frac{1}{2}x-1)$

0



b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

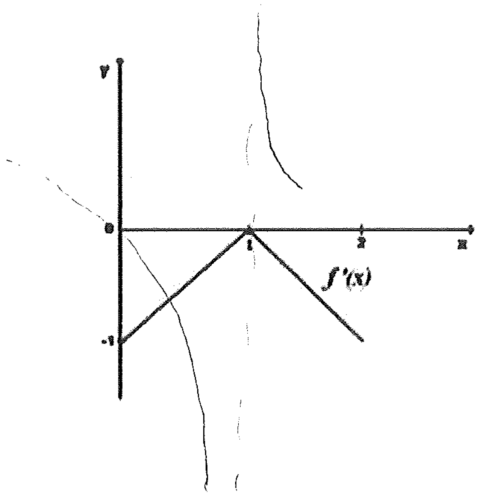
– eleven fastlægger k=0



c) Hvad siger du til din elev?

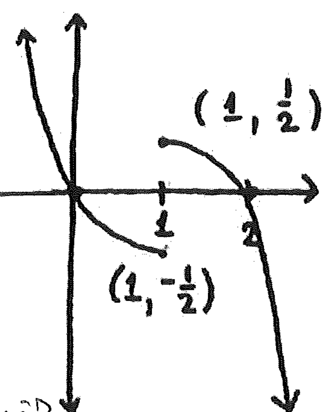
Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

DE DEN PILE ER FORVIRRENDE FOR, OM ELEVEN FORSTÅR SIN EGEN BESVARELSE. UDEN PILE VILLE JEG TRO, AT ELEVEN TÆNKER, AT FUNKTIONEN f(x) ER AFTAGENDE HELE TIDEN, MEN MED PILE ER DEN SVÆR.



HVIS ~~BOLLERNE ER FYLDTE, HVORFOR~~, SÅ?
HVORFOR ~~KØRER~~ GRAFEN TIL

ELEVEN HAR PRØVET AT VISE, AT DER SKER MYSTISKE TING I x = 1, UD AT ~~[illegible]~~ STOPPE ÉN DEL AF GRAFEN ET STED OG STARTE ET ANDET.

c) Hvad siger du til din elev?

HVAD BETYDER PILENE?
HVOR HAR DU PUNKTERNE FRA?
HVAD HAR HAKKET I GRAFEN FOR f' AF BETYDNING FOR f?

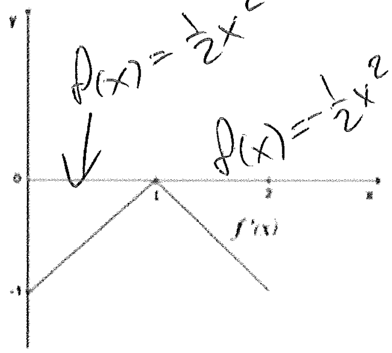
SKAL VI LIGE LÆSE OP PÅ DET SAMMEN..?

Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.

$f(x) = \frac{1}{2}x^2 \quad x \le 1$

$f(x) = -\frac{1}{2}x^2 + 2x - 1 \quad x > 1$



$f'(x) = x \quad x < 1$

$f'(x) = 2 - x \quad x > 1$

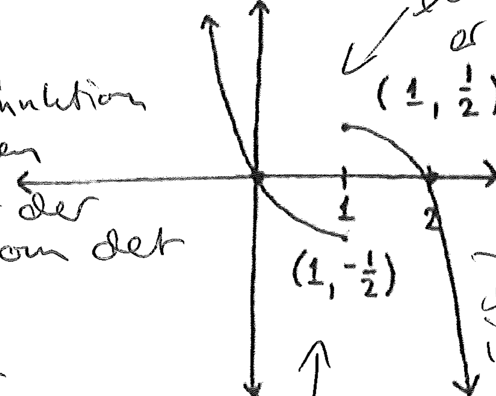
$f(x) = \frac{1}{2}x^2 + k \quad x < 1$

$f(x) = -\frac{1}{2}x^2 + ~~2x~~ + k \quad x > 1$

b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

1) fejltype 1.
Eleven har bestemt $k = 0$ i den første funktion og indsat $k = 0$ i den anden i den tro, at der SKAL være tale om det samme k.

2) Eleven har glemt at indsætte k.



Denne funktion er ikke differentiabel (og heller ikke kontinuert)

Du kan ikke differentiere denne funkt.

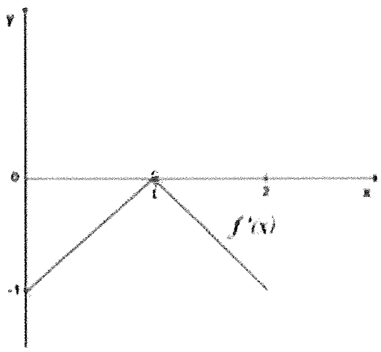
eleven har glemt at tænke på k − muligvis i begge funktioner.

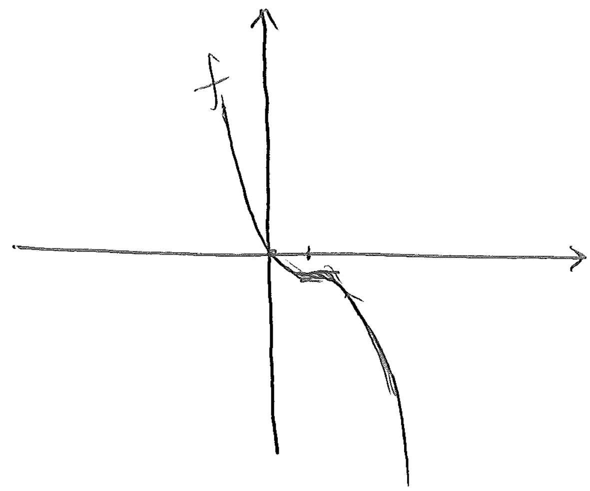
c) Hvad siger du til din elev?

Først afdækkes om elevens fejltype er af type 1 eller 2. Derefter snakker jeg med eleven om betydningen af k-erden for ALLE funktioner (fordi eleven glemmer det) eller når der er mere end ét k, at de så ikke behøver at være det samme.

Opgave 5

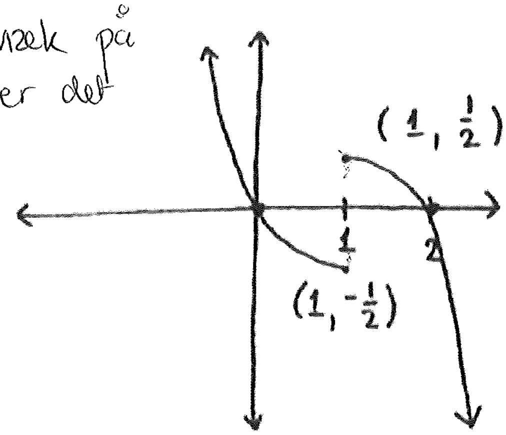
a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.





b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

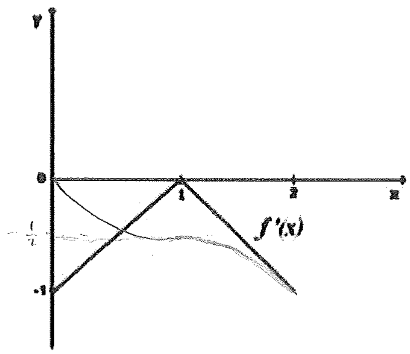
Eleven tror, at når der er knæk på grafen for den afledte medfører det at grafen for funktionen er ikke kontinuert.



c) Hvad siger du til din elev?

Opgave 5

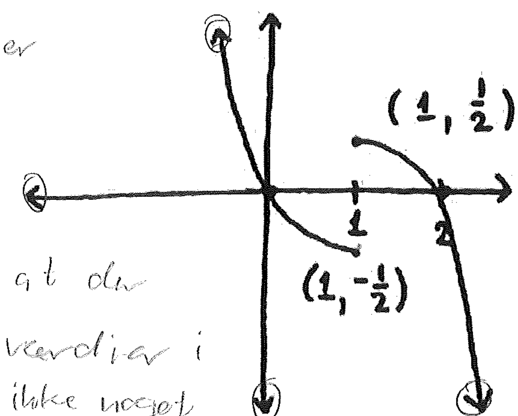
a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

Forståelse af tangenthældning er rigtig nok.

Ved ikke hvorfor grafen springer, men bortset fra, at der ikke kan være to funktionsværdier i samme punkt, er der heller ikke noget galt med det.



c) Hvad siger du til din elev?

Drop pilene (de markerede)

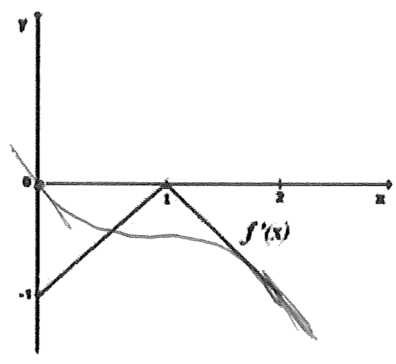
Vi kan ikke sige noget om f, der hvor vi ikke kan se graf for f'.

Husk at en fkt. kun kan have én fkt.-værdi i et punkt

(• vs. V.)

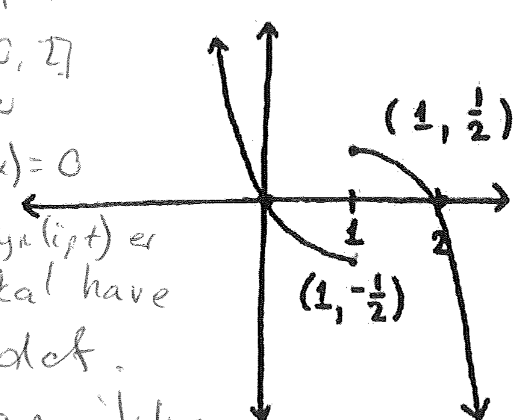
Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

Vedkommende har forstået forløbet på intervallet [0, 2] i hovedtræk men blive forvirret der hvor $f'(x)=0$ ~~[illegible]~~ Da det højst sandsynligt er en kont. funktion skal have holde blyanten i bordet. Yderligerere kan han ikke ~~[illegible]~~ sige noget for $x<0$ og $x>2$ da det ikke fremgår af $f'(x)$.

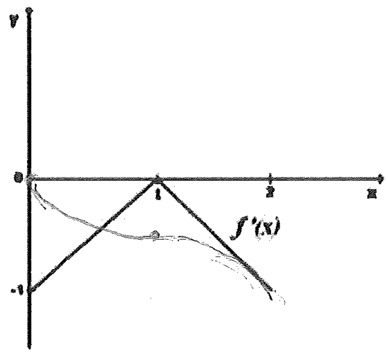


c) Hvad siger du til din elev?

Lidt af ovenstående

Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.

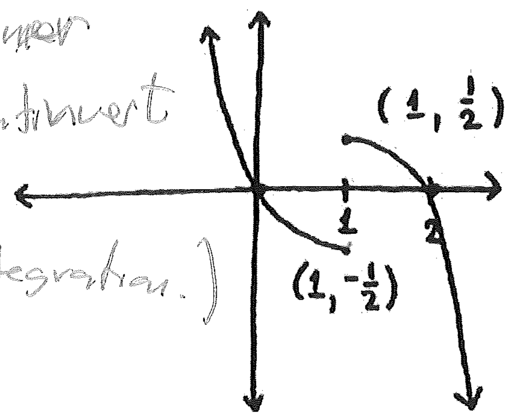


$$f'(x) = \begin{cases} x-1 & 0 < x < 1 \\ -x+1 & 1 \leq x < 2 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2}x^2 - x \\ -\frac{1}{2}x^2 + x \; (-1) \end{cases}$$ ← kontinuitet

b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

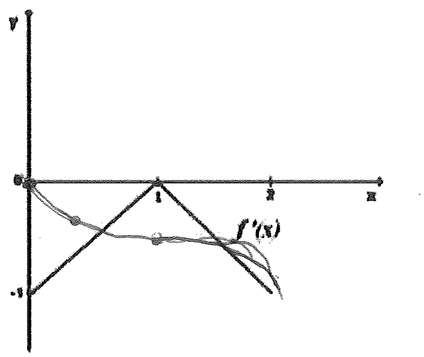
fint svar, men glemmer at f skal være kontinuert (glemmer konstant ved integration.)



c) Hvad siger du til din elev?

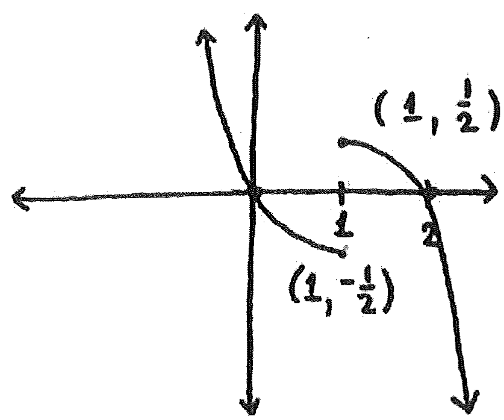
Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

RIGTIG BORTSET FRA AT ELEVEN EJ SET AT SKAL VÆRE DIF.

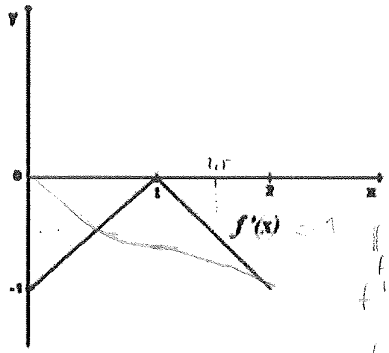


c) Hvad siger du til din elev?

TÆNK PÅ AT F SKAL VÆRE DIF.

Opgave 5

a) På billedet vises grafen for $f'(x)$. Antag at $f(0) = 0$ og tegn grafen for $f(x)$.



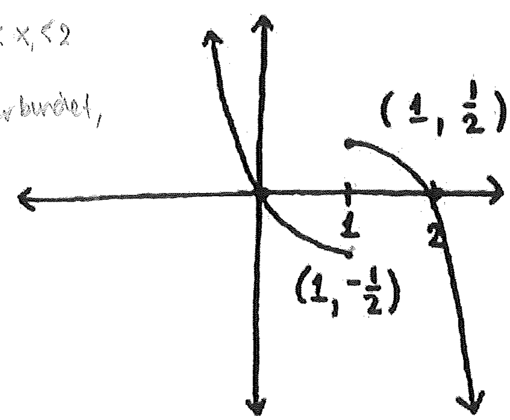
f'(0) = -1 = aftagende
f'(0,5) = -0,5 aftagende med en mindre stejl hældning
f'(1) = 0 = konstant
f'(2) = -1 = aftagende

b) På billedet vises en af dine elevers svar på ovenstående opgave. Analysér svaret.

Vi kan ikke sige noget om f(x) når 0 < x < 2

f er kontinuert, derfor er grafen forbundet,

så der kan ikke være et spring



c) Hvad siger du til din elev?