INSTITUT FOR NATURFAGENES DIDAKTIK
KØBENHAVNS UNIVERSITET

The Trigonometric Functions - The transition from geometric tools to functions

Lotte Nørtoft
Kandidatspeciale

Februar 2016

IND's studenterserie nr. 47

INSTITUT FOR NATURFAGENES DIDAKTIK, www.ind.ku.dk
Alle publikationer fra IND er tilgængelige via hjemmesiden.

IND's studenterserie

1. Ellen Berg Jensen: 15-åriges viden om klimaforskelle (2007)
2. Martin Sonnenborg: The Didactic Potential of CAS (2007)
3. Karina Søgaard og Sarah Kyhn Buskbjerg: Galoisteori i Gymnasiet (2007)
4. Ana Hesselbart: Mathematical reasoning and semiosis (2007)
5. Julian Tosev: Forskningslignende situationer (2007)
6. Niels Nørskov Laursen: En Covarians-tilgang til Variabelssammenhænge i gymnasiet (2007)
7. Katja Vinding Petersen: Lyd og Liv (2007)
8. Jesper Bruun: Krop og computer i fysikundervisning (2008)
9. Jakob Svendsen: Matematiklærerens forberedelse (2009)
10. Britta Hansen: Didaktik på tværs af matematik og historie (2009)
11. Nadja Ussingkær: En didaktisk undersøgelse af brudte lineære funktioner i rammerne af tysk fritidsmatematik (2009)
12. Thomas Thrane: Design og test af RSC-forløb om vektorfunktioner og bevægelse
13. Flemming Munch Hansen: Samspil omkring differentialregningens elementer i gymnasiets matematik og fysik (2009)
14. Hasan Ademovski og Hatice Ademovski: Proportionalitet på mellemtrinnet - Design af didaktiske situationer baseret på stofdidaktisk analyse (2009)
15. Philipp Lorenzen: Hvem er de nye studenter? Baggrund, interesse & uddannelsesstrategi (2010)
16. Signe Ougaard: Logiske strukturer i matematisk analyse på gymnasieniveau. Et forløb om kvantorer og ε-δ-definition af grænseværdi (2010)
17. Jesper Winther Sørensen: Abstrakt algebra i gymnasiet - design, udførelse og analyse af undervisning i gruppeteori (2010)
18. Sofie Stoustrup: En analyse af differentialligninger på A-niveau i STX ud fra den antropologiske didaktiske teori (2010)
19. Jan Henrik Egballe Heinze: Eksponentialfunktioner i STX (2010)
20. Mette Beier Jensen: Virtuelgalathea3.dk i biologiundervisningen i gymnasiet (2010)
21. Servet Dönmez: Tosprogede elever og matematik i gymnasiet (2011)
22. Therese Røndum Frederiksen: Designing and implementing an engaging learning experience about the electric sense of sharks for the visitors at Danmarks Akvarium (2011)
23. Sarah Elisabeth Klein: Using touch-tables and inquiry methods to attract and engage visitors (2011)
24. Line Kabell Nissen: Matematik for Sjov – Studie- og forskningsforløb i en eksperimentel kontekst (2011)
25. Jonathan Barrett: Trigonometriske Funktioner i en Gymnasial Kontekst – en epistemologisk referencemodel (2011)
26. Rune Skalborg Hansen: Et studie i læringsopfattelse og -udbytte – Om fysik C kursister på Københavns VUC (2011)
27. Astrid Camilus: Valideringssituationer i undervisningsforløb om differentialligninger og radioaktivitet (2012)
28. Niven Adel Atie: Didaktiske situationer for fuldstændiggørelse af kvadratet i andengradsligningen (2013)
29. Morten C. B. Persson: Kvantekemi i gymnasiet - Tilrettelæggelse, udførelse og evaluering af et undervisningsforløb (2013)
30. Sofie Birch Jensen: Køn, evaluering og The Force Concept Inventory (2013)
31. Simone Gravlund Nielsen: Når børn forsker i matematik (2013)
32. Henrik Egholm Wessel: Smartphones as Scientific Instruments in Inquiry Based Science Education (2013)
33. Nicole Koefoed: Et didaktisk design om definition, eksistens og eksakt værdi af bestemt integral (2013)
34. Trine Louise Brøndt Nielsen: From Master's programme to labour market – A study on physics graduates' experience of the transition to the labour market (2013**)**
35. Rie Hjørnegaard Malm: Becoming a Geologist – Identity negotiations among first year geology students (2013)
36. Mariam Babrakzai Zadran: Gymnasiealgebra I et historisk perspektiv – Matematiske organisationer I gymnasiealgebra (2014)
37. Marie Lohmann-Jensen: Flipped Classroom – andet end blot en strukturel ændring af undervisningen? (2014)
38. Jeppe Willads Petersen: Talent – Why do we do it? (2014)
39. Jeanette Kjølbæk: One-dimensional regression in high school (2015)
40. Anders Wolfsberg: A praxeological investigation of divergence – Exploring challenges of teaching and learning math-in-physics (2015)
41. Asger Brix Jensen: Number tricks as a didactical tool for teaching elementary algebra (2015)
42. Katrine Frovin Gravesen: Forskningslignende situationer på et førsteårskursus I matematisk analyse (2015)
43. Lene Eriksen: Studie og forskningsforløb om modellering med variabelsammenhænge (2015)
44. Caroline Sofie Poulsen: Basic Algebra in the transition from lower secondary school to high school
45. Rasmus Olsen Svensson: Komparativ undersøgelse af deduktiv og induktiv matematikundervisning(2016)
46. Leonora Simony: Teaching authentic cutting-edge science to high school students(2016)
47. **Lotte Nørtoft: The Trigonometric Functions - The transition from geometric tools to functions (2016)**

Abstract

In the third year of high school Danish students are introduced to the trigonometric functions. Before, they have been working with sine and cosine as tools to investigate angles and other dimensions in (essentially) triangles, and this transition from geometric tool to function is often challenging. The aim of this thesis is to elucidate why the students have problems with the above transition. This is first done through a subject-matter didactic analysis where the complexity and diversity of the trigonometric functions are elaborated. Here it is suggested that, to hinder the problems concerning the transition, the teaching of trigonometric function could involve a simplified introduction to the natural parametrization of the unit circle. Thus the students would realize the ono-to-one correspondence between the arc length on the unit circle and the x-axis.

IND's studenterserie *består af kandidatspecialer og bachelorprojekter skrevet ved eller i tilknytning til Institut for Naturfagenes Didaktik. Disse drejer sig ofte om uddannelsesfaglige problemstillinger, der har interesse også uden for universitetets mure. De publiceres derfor i elektronisk form, naturligvis under forudsætning af samtykke fra forfatterne. Det er tale om studenterarbejder, og ikke endelige forskningspublikationer.*

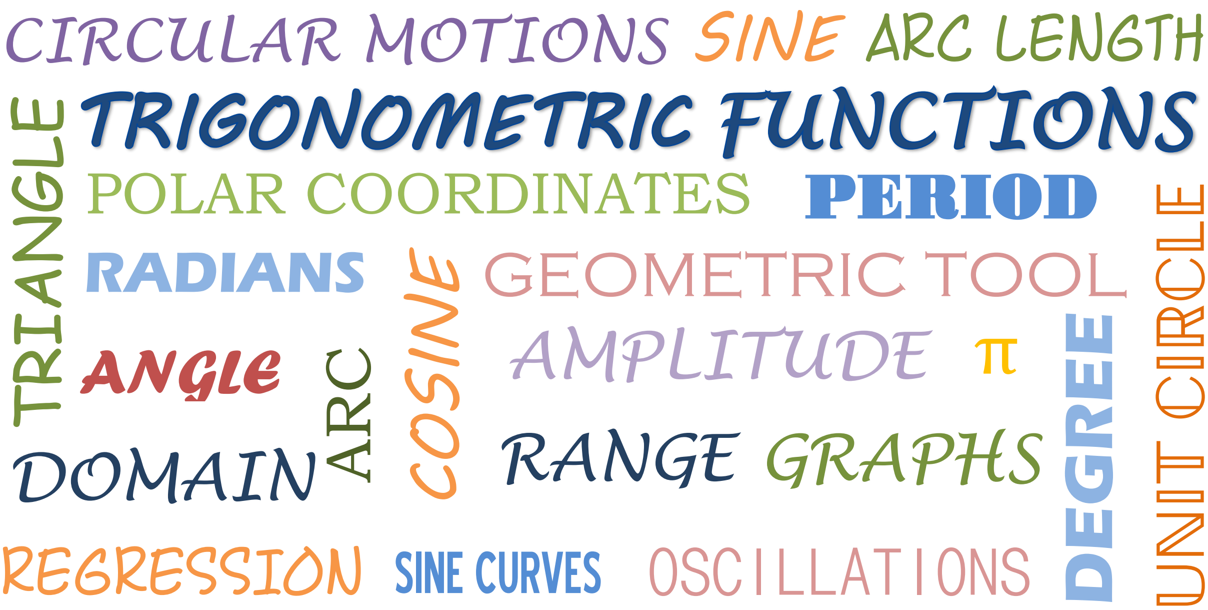
Se hele serien på: www.ind.ku.dk/publikationer/studenterserien/

UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

THE TRIGONOMETRIC FUNCTIONS

THE TRANSITION FROM GEOMETRIC TOOLS TO FUNCTIONS

Lotte Nørtoft



Thesis for the Master degree in Mathematics

Department of Science Education, University of Copenhagen

Supervisor: Carl Winsløw

Submitted: 1 February 2016

ABSTRACT

In the third year of high school Danish students are introduced to the trigonometric functions. Before, they have been working with sine and cosine as tools to investigate angles and other dimensions in (essentially) triangles, and this transition from geometric tool to function is often challenging. The aim of this thesis is to elucidate why the students have problems with the above transition. This is first done through a subject-matter didactic analysis where the complexity and diversity of the trigonometric functions are elaborated. Here it is suggested that, to hinder the problems concerning the transition, the teaching of trigonometric function could involve a simplified introduction to the natural parametrization of the unit circle. Thus the students would realize the ono-to-one correspondence between the arc length on the unit circle and the x-axis.

The subject-matter didactic analysis works, together with the theory of didactical situations, as a tool for analyzing concrete teaching episodes, observed at a Danish high school. Since the simplest and most common way to introduce sine and cosine as functions is as graphs, the analysis is focused on the students' interactions with the graphic milieu.

Both the subject-matter didactic analysis and the observations showed that primarily two elements inhibit the students' acknowledgement of sine and cosine as functions. Firstly a lack in the perception of functions, secondly that focus is not on the actual transition, neither in the official material, nor in the concrete teaching. Instead focus is on the possibilities the trigonometric functions gives us, illustrated though the sine curves.

ACKNOWLEDGEMENTS

The completion of this thesis would not have been possible without the support from a lot of people to whom I owe a big thank you. I would like to give a special thanks to some of these people.

Fist I would like to thank Birgitte Erskov Halland and her class at Gefion High School for letting me observe the teaching of trigonometric functions. Without their help, there would not have been any data to analyze.

Great thanks to Carl Winsløw for inspiring and profound supervision. Thank you for always being disposal with answers, reading material and comments to improve my thesis.

Thanks to my family and friends for listening when it was needed. Especially a warm thanks to my parents for their support over the years, not at least in my education life.

A special thanks to Casper for his great support. Thank you for reading and commenting on the thesis and thank you for always being helpful, understanding and forbearing.

Finally big thanks to my fellow students at the thesis-office for creating a fantastic atmosphere to work in. Thank you for great advice, discussions and cosiness.

CONTENTS

1. INTRODUCTION

If we try to go back to the years of high school, and remember what we learned about sine and cosine we will remember that they were used to find sides and angles in triangles. We remember sine and cosine as functions taking an angle as input. But did we not learn that sine and cosine also works as functions in the domain of real numbers? Yes we did, but most of us have probably forgotten, either because we did not understand the notion back then, or because we could not see the link between the useful geometric tool and the more abstract function. In high schools trigonometric functions are introduced through three different contexts:

- Triangle trigonometry, where sine and cosine are defined as ratios between sides in a right triangle.
- Unit circle trigonometry, where sine and cosine are defined as coordinates to a point on the unit circle.
- Trigonometric functions, where sine and cosine are defined as functions in the domain of the real numbers.

The focus in high school is mainly on the first two contexts, but in the third year the students are introduced to sine and cosine as functions in the domain of real numbers. Teachers present the three contexts separately and this often results in students getting a fragmented perception of the trigonometric functions (Weber, 2005; Demir, 2012; Orhun, n.d.). The students get an angle-measure-dominant conception of the functions, they cannot really understand the trigonometric graphs, nor do they build important connections between the unit circle and the graphs, such as the transition from angle to real numbers through the radian concept (Demir, 2012, p. 1).

In this thesis we will try to examine why high school students have problems with the transition from working with sine and cosine as geometric tool, in (essentially) triangles, to working with them as functions. Is it because they do not understand the notion of function? Is it because they do not understand why they need sine and cosine as functions? Is it due to the way trigonometric functions are taught, or could there be another reason? If we find the answer, the teaching of trigonometric functions may be adjusted such that the transition becomes more manageable and the fragmented perception diminishes.

1.1 The structure of the thesis

The thesis consists of 11 chapters. After this introduction chapter, the second one contains a subject-matter didactic analysis of the trigonometric functions in order to realize how they can be introduced in a Danish high school. The chapter includes an analysis of the external didactic transposition of trigonometric functions together with an analysis of the transition from geometric tool to function, built on the presentation of the trigonometric functions. Thus the subject-matter didactic analysis works both as an independent presentation of how diverse and complex trigonometric functions are, but also as a tool for a later analysis of concrete teaching episodes.

Chapter three presents the theory of didactical situations, which foundation is that learning is a social activity; hence it is a perfect tool for analyzing teaching, especially validation situations, because they are built on interactions, either between students or between students and teacher.

Chapter four is a presentation of the research questions, built on the theory of didactic situations and the subject-matter didactic analysis, which specify what we will analyze in the concrete course in trigonometric functions. In chapter five the methodology is described.

Chapter 6-9 is a presentation of the course of study, followed by the analyzes of three distinct situations. Chapter 10 is a discussion of these situations and chapter 11 is a conclusion collecting all of the above.

2. ANALYSIS OF THE TRIGONOMETRIC FUNCTIONS

In this chapter we will analyze how one can introduce trigonometric functions in Danish high schools. This requires an insight in what these functions contains. As we will see the trigonometric functions can be defined in several ways. We will present some of these definitions, as a way of showing the diversity and importance of the trigonometric functions. We will see that not all definitions are suitable for high school; therefore the analysis will also include considerations concerning which definitions are needed for the high school students to acknowledge the trigonometric functions as functions, and which will be too difficult or mess with their ideas of functions. This requires that we know how other functions are introduced to the students and not at least an elaboration of what the notion of function is.

2.1 The notion of function

In this paragraph we will present what is needed for us to define a function. We will start with a brief historic overview leading to the modern definition of functions. At the end we will suggest what is needed to introduce a new function in high schools.

The first time the word "function" was used was in 1748 by Euler. He defined a function of a variable quantity as an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities (Katz, 2009, p. 618). So according to Euler the word "function" means an "analytic expression". Euler distinguishes between curves and function in the way that any function can be translated into geometry to determine a curve in the plane, but a curve cannot always be determined by a single function. This distinction is also used today; just recall the curve representing the circle, which is a curve but cannot be determined by a single function. Euler categorizes curves in to classes; continuous curves, which can be expressed in terms of a single function, and discontinuous functions, which require different functions of x for its expression. According to Euler discontinuous curves cannot be expressed by a constant law, but are formed from several continuous parts (Ibid., p. 618). Here Euler implicitly associates a function with a law. What he means by this we do not know, but the connection between law and function will occur again later. Euler also divides the set of functions into two classes, namely algebraic and transcendental. The former are formed by using the usually algebraic operations on the variables and constants. The latter covers the cases of trigonometric, exponential and logarithmic functions. An important tool in Euler's discussion of functions is that of the power series. So when he says, "composed in any way whatsoever" he includes the notion of infinite series, infinite products and infinite continued fractions. He was convinced that any function could be expressed by a power series, but gave no proof. Instead he showed how to expand any algebraic function as well as various transcendental functions into such a series (Ibid., p.618-620). Later we will see how the trigonometric functions are expressed by power series, but now let us focus on the notion of function. Throughout the 18th century Euler and some of his colleagues, including Jean d'Alembert and Daniel Bernoulli, had a debate concerning the solution to the partial differential equation called "the wave equation". They could all agree that the solution must be a function, and they also agreed that a function was an analytic expression. But the problem concerned what kinds of initial functions there could be permitted. Bernoulli suggested that the solution could be expressed as an infinite sum of trigonometric functions. The initial position function is then represented by the infinite sum (Ibid., p. 610):

$$y(0,x) = \alpha \sin\left(\frac{\pi x}{l}\right) + \beta \sin\left(\frac{2\pi x}{l}\right) + \gamma \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

Which Bernoulli believed represented an arbitrary initial position function with appropriate choices of $\alpha, \beta, \gamma, \ldots$, even though he was not able to deduce the coefficients for this trigonometric series. In 1807 Joseph Fourier proved Bernoulli right and moreover he found an expression for the coefficients. Fourier proved that an arbitrary function can be expressed as a sum of an odd and an even function. He also proved that a sine series can describe an arbitrary odd function on the interval $[-\pi, \pi]$ and a cosine series can describe an even function on the same interval. The result is that an (to Fourier) arbitrary function can be expressed in the interval $[-\pi, \pi]$ by (Godiksen, Jørgensen, Toldbod, & Hanberg, 2014, p. 44):

$$f(x) = \frac{1}{2}b_o + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

Where $n \in \mathbb{N}$ and the coefficients a_n and b_n are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

This is clearly not a simple analytic expression and Fourier also defined the notion of a function in a different way than his predecessors: "In general the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there is an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or null… (Katz, 2009, p. 782). So there is no equality between the function and its analytic expression. Notice here that Fourier, in contrast to Euler, requires that the amount of function values $f(x)$ is equal to the number of values of x. This could be seen as an expression for the requirement of uniqueness when we think of functions. Gustav Dirichlet defines in 1829 a function in almost the same way as Fourier, but is a bit more concrete:

> *"One thinks of a and b as two fixed quantities and x as a variable quantity, which gradually will take all values between a and b. If now a unique finite y corresponding to each x, and moreover in such a way that when x ranges continuously over the interval from a to b, $y = f(x)$ also varies continuously, then y is called a continuous function of x for this interval. It is not at all necessary here that y be given in terms of x by one and the same law throughout the entire interval, and it is not necessary that it be regarded as a dependence expressed using mathematical operations "(Godiksen et.al., 2003, p. 61, own translation).*

Here the requirement for uniqueness is made clear, and in contrast to both Euler and Fourier the function is defined on a certain interval. What happens outside this interval is to Dirichlet irrelevant, whereas Euler believed that the law in a given interval would continue outside the interval. Dirichlet also defines a continuous function in a different way than Euler. Recall that for Euler a continuous function was a function determined by one single analytic expression, so one law. For Dirichlet a continuous function is dependent on the continuous relationship between x and y, and the continuous function could easily be determined by more than one law.

The concept of continuity is still a bit vague, but has later been defined more formally. For example in terms of limits where the function $f(x)$ is continuous if $\lim_{x \to c} f(x) = f(c)$ or by Weierstrass' "epsilon-delta" definition from 1872 saying that: A function f is continuous in a point a, belonging to the domain, if the following holds: For any number $\varepsilon > 0$, however small, there exists some number $\delta > 0$, such that when x is in the domain and $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$ (Lindstrøm, 2006, p. 212).

To sum up, we now have that a function on a given interval is a relationship between all the x values and the uniquely corresponding y values. But still the interpretation of the term function varies. To Euler a function is represented as an analytic expression. It means that the function changes the variable quantity depending on how the analytic expression is composed. A function is to Euler a dynamic process where the input is changed by the function to get the output. Also Dirichlet has a dynamic interpretation of functions, when he says that if x varies continuously also y will varies continuously. In contrast, Fourier interprets the notion of function as static: the function is a succession of ordinates, and the amount of ordinates depends on the amount of abscissa. Here the function is a complete object and not changing at all. This interpretation can be associated with the function as a table with abscissa in one column and ordinates in another or with the function as a graph. This static conception of function is also seen in the set theoretic approach to the notion of function.

In 1888 Dedekind defines a set S as a collection of different things which can be considered from a common point of view (Katz, 2009, p. 794). And a function ϕ on a set S he defines as a law according to which to every determinate element s of S there belongs a determinate thing called the transform of s and denoted by $\phi(s)$ (Ibid., p. 783). Again a function is associated with a law. The Dedekind definition of a function is very similar to the one presented in modern books about set theory, just with the term law expressed more formal. Kiming (2001) describes the law as a relation. If A and B are sets, then any subset of $A \times B$ is called a relation between A and B (Kiming, 2001, p. 66). This means that a relation is simply a set of ordered pairs $(a, b) \in A \times B, a \in A, b \in B$. This leads to the definition of a function (Ibid., p.104):

Let A and B be non-empty sets. A function f from set A to set B is a relation between A and B satisfying the following conditions:

1. *For each* $a \in A$ *there exists* $b \in B$ *such that* $(a, b) \in f$
2. *If* (a, b) *and* (a, c) *are in f, then* $b = c$

If $a \in A$*, then the unique element* $b \in B$ *for which* $(a, b) \in f$ *is denoted by* $f(a)$*.*

If f is a function, then an element in A corresponds to exactly one element in B. In other words, a function is a relation between two sets, with the rule that each element in the former set corresponds uniquely to an element in the latter set. The former set is called the domain of the function f and the latter the codomain. The set containing the elements of B satisfying $b = f(a)$ is called the range of f (Ibid., p. 105).

Let us consider two relations, first the relation between a side in a square and the area of the square. The side could be all positive real numbers, and the same could the area, so we have a relation from $\mathbb{R}^+$ to $\mathbb{R}^+$. Since the area of a square is the side squared, we can express the relation as $y = x^2$, where x is the side and y the area. For each x we get exactly one y, so this relation is a function. If we consider the relation between height and weight in a high school class, then we again have a relation from $\mathbb{R}^+$ to $\mathbb{R}^+$, but one height does not have to relate to only one weight. You could have two students who are 170 cm high, but one weights 65 kg and the other 75 kg. So this relation is not a function, one input cannot give you a specific output.

This set theoretic definition of a function interprets the function as a static object. The function does not transform a to b, but relate them. When we, in the example above, express the relation as $y = x^2$, we present the function as an analytic expression and hence a dynamic process. The same function can be presented in different ways, and behind each representation is either a static or a dynamic approach to the notion of function. Sierpinska (1992) distinguish the two conceptions by saying that the former is when you think of the *relationship between variables* and the latter is when you think of *mappings between variables*. The first one occurs when we, as Euler and Dedekind did it, think of the function as a law describing for example a scientific phenomenon. Sierpinska (1992) describes it as:

> *"When, in mathematics, we think of curves represented in systems of coordinates we also think of relationships between the coordinates of points that belong to the curve. […] Sometimes the relationship is given by an equation which describes the conditions under which a point belongs to the curve. If the curve is already there,*

the equation "unveils" the pre-existing relationship between the coordinates. Here, the image we have of functions is "static" in the sense that these "laws" are not defined by us, we do not make them; rather they are discovered by us." (Sierpinska, 1992, p. 29)

With a static approach to the notion of function we conceive the function as an object on its own. We can discover it and describe it, but we do not have any influence on it. In contrast the dynamic approach creates a picture of the function as a process on given objects:

> *"One thing is mapped into or onto another thing, it is transformed into a representation that, at the time, serves best our purposes. For example, we project three dimensional objects onto two dimensional objects to obtain representations on sheets of paper. […] We do it: We process the objects or sets of objects to obtain other objects. This gives a more dynamic image of a function. This dynamic image is also present when we plot the graph of a function: we process the independent variables to obtain the related variables. "* (Sierpinska, 1992, pp. 29-30)

When introducing the notion of function to high school students an idea could be to do as we did above, give them an example and a counterexample. This would give the student an idea of the specific term "function", and hereafter different examples of functions can be introduced. This could be done by an analytic expression and a graph; hereby giving the students both a dynamic and a static approach to the function. In order to convince the students that this analytic expression is in fact a function the graph and the so called vertical line test can be used. No matter where you insert the vertical line in the coordinate system it must only hit the graph once. The vertical line is represented by an x-value, and if the line hits the graph more than once, we have two y-values for one x-value, and thus not a function. This vertical line test ensures that the students keep thinking of a function as a relation between some x-values and some y-values, with the requirement that each x is only related to one y.

As a part of the notion of function, the students will also be introduced to the family of functions. A family of functions is a set of functions defined by the same formula. As an example we have the parabola, which in general is represented by the formula $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ are arbitrary constants. This formula represents a family of functions. For each specific value of the constants we have one function. As another example we have the class of trigonometric functions. Later on we will see that in Danish high schools the students are introduced to trigonometric functions as the functions: $\sin(x)$, $\cos(x)$ and $\tan(x)$ and then to a general sine curve represented by the formula $f(x) = a\sin(bx + c) + d$. This general sine curve is in fact a family of functions, which we will call the class of trigonometric functions:

$$\mathcal{F}_{trig} = \{f_{a,b,c,d} | a, b, c, d \in \mathbb{R}\} \text{ where } f_{a,b,c,d}(x) = a\sin(bx + c) + d$$

It contains all the sinusoidal functions, which are functions that can be contained from the parent function $f(x) = \sin(x)$. If we look at it graphically the function $f(x) = a\sin(bx + c) + d$ is obtained by changing the amplitude of $f(x) = \sin(x)$ to a, move the graph by the size of d in the positive direction of the y-axis, by the size of c in the negative direction of the x-axis and finally change the period to $\frac{2\pi}{b}$. Also the function $f(x) = \cos(x)$ is a part of this family, because $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$, so here the amplitude is 1, the period is 2π and the curve is displaced by $\frac{\pi}{2}$ in the negative direction of the x-axis.

If the notion of function is difficult for the students, so will the notion of a family of functions be. Not only do the students have to identify a given object as a function, they also have to argue whether the object has the characteristics required for being in the given family. It can also be a challenge to realize that different functions belong to the same family, just consider $\sin(x)$ and $\cos(x)$, which both are sine curves.

The class of trigonometric functions is special, because the parent function $f(x) = \sin(x)$ does not have an analytic expression, so in order to verify trigonometric functions as functions the students must at first rely on the graphic milieu and the vertical line test. In the following we will examine different definitions of the trigonometric functions in order to find a suitable way to present these functions to high school students in order for the students to acknowledge them as functions.

2.2 The trigonometric functions

There are six basic trigonometric functions, sine, cosine, tangent, cotangent, secant and cosecant. The last four can be expressed in terms of sine and cosine and, as we will see later on, every point on the unit circle can be expressed in terms of sine and cosine. Therefore this thesis will only concern these two functions, so from now on, whenever we speak of the trigonometric functions, we only refer to the functions sine and cosine (Unless we talk about the class of trigonometric functions, which involves all the sinusoidal functions). The problem is that trigonometric functions cannot be expressed analytically, so the link between the expression $f(x) = \sin(x)$ and the graph can be difficult to see. If one understands this link, it is easy to see that sine is a function, just by using the vertical line test. But without the graph, how do we convince ourselves that the expressions $\sin(x)$ and $\cos(x)$ are in fact functions? In this paragraph we will try to answer that question by reviewing different definitions of the trigonometric functions. For each definition we will point out why it represents a function, together with their advantages and disadvantages. Since sine and cosine often are defined in the same way we will mainly look at sine and then mention cosine whenever it seems necessary.

Before consulting the different approaches to sine and cosine as functions we will consider how they are defined in geometry and in analytic geometry. In geometry sine and cosine take an angle as input and in analysis they take a real number. To realize that this transition is legal we need a specification of the word "angle".

2.2.1 The notion of angle

The general conception of an angle is; as the space between two lines. But how is this space defined, how does it look and how can we measure it? How does an angle which measurement is 10 look like? These questions will be examined here.

The word angle comes from the Latin word "angulus" meaning "corner", and in plane geometry an angle is a figure formed by two rays, called the sides of the angle, sharing a common endpoint, called the vertex of the angle, hereby forming a corner (Wikipeida_angle, 2015). But if we should be able to use the angle as an input in our sine function, the angle must be some kind of quantity. We need to be able to measure our angle. Some would say that angle measurement is the amount of rotation you need to move one side around the vertex to get to the other side. Hence the size of an angle would be measured in degrees, since a degree is $\frac{1}{360}$ of a full rotation (Wikipedia_degree, 2015). Others would say that the size of an angle is the space between the sides, but then the question arise whether the two angles below is the same or not.

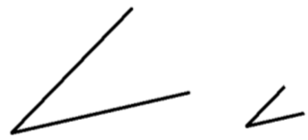


Figure 2.1

The answer is that they are the same, but we need a more formal definition of angle measurement to see this. To measure the size of an angle the vertex of the angle is placed at the center of a circle. The size of the angle is then the ratio between the arc intercepting the two sides and the radius of the circle, independent of the size of the circle. This relationship between the arc length subtending the angle and the radius is called the angle's radian number; hence an angle can be measured either in degrees or in radians.

Already now we see the complexity of the notion of angle. We have an angle as a geometric figure, but we can also think of it as a rotation around the vertex. So being an angle is a quality and this quality can be viewed as either static or dynamic. Moreover the angle can be measured, giving it a measure and thinking of it as a quantity. And finally this angle measurement can be determined as a relation between the arc length subtending the angle and one of the angle's sides. The transition from the unformal notion of angle as a geometric figure, to the more formal in terms of an angle measurement requires absorption of the angle as both a quality, quantity and a relation and this could lead to confusions for many students.

When attending high school the students would probably associate an angle with a space between two sides (often in a triangle) which is measured in degrees by a protractor. Here the angle is a static quality and the angle measurement is just a quantity they can read off. During the first year the students learn that they can calculate the angles in a right triangle by using the ratio of the sides. In the third year of high school the unit radians is introduced. An obstacle for the students here is both the transition from degrees to radians, but also this new notion of arc length. To complete the confusion the notion of angle is extended to also include negative angles and angles beyond 360 degrees. This is done in order to be able to consider angles of arbitrary size. If we consider an angle in a coordinate system, with the vertex in (0,0), one side lies at the x-axis and the other has an arbitrary position, together with a unit circle, centered at (0,0), we see that the radian number of an angle is equal to the arc length subtending the angle on the unit circle.

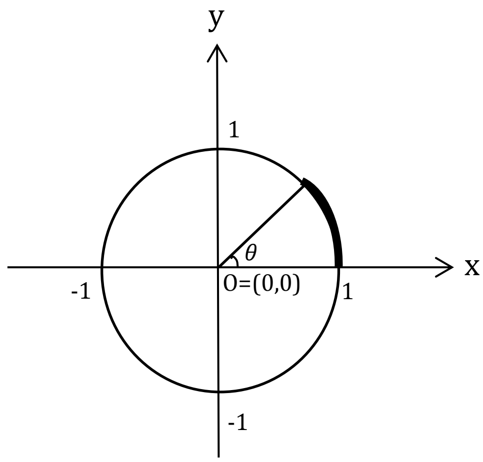


Figure 2.2: The arc length subtending the angle on the unit circle

This gives us the opportunity to talk about an angle as a real number without some mystic unit as degree or radians. Hence we can translate between angles and other measurements such as high, temperature, volume and so on. We are able to answer the question of how the angle with measurement 10 looks like. It is one full rotation plus approximately 0,6 rotation.

However the question is still whether this arc length always exists, and how we measure it. When describing an angle in terms of its radian number, you actually just move the problem of angle measurement to a problem of arc length measurement. A formal notation of an angle could be:

An angle is measurable if the arc length on the unit circle subtended by the angle exists, and if so the arc length is the angle's measurement.

The existence of this arc length is examined in section 2.2.4. But first let us consider the introduction to sine and cosine in the geometric sector.

2.2.2 Sine and cosine in the triangle

In the geometry sector sine and cosine are defined as ratios between sides in a right triangle. The right triangle consists of one right and two acute angles. The side opposite the right angle is called the hypotenuse and the other two are called cathethi. If v is one of the acute angles sine and cosine of this angle are defined as:

$$\sin(v) = \frac{opposite\ cathetus}{hypotenuse}, \qquad \cos(v) = \frac{adjacent\ cathetus}{hypotenuse}$$

From this definition it is possible to find unknown sides and angles in any arbitrary triangle, because drawing the height in an arbitrary triangle divides the triangle into two right triangles:

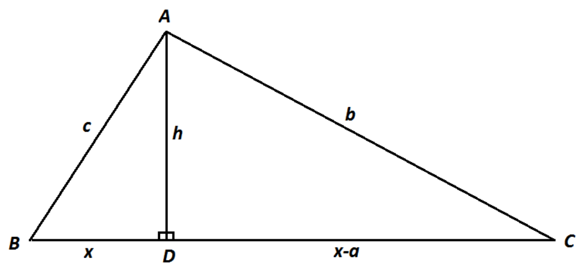


Figure 2.3

Figure 2.3 shows us that:

$$\sin(B) = \frac{h}{c} \Leftrightarrow h = \sin(B) * c$$

Likewise the height can be expressed in terms of $\sin(A)$ or $\sin(C)$. The area of triangle ABC is then:

$$\frac{1}{2}a(c\sin(B)) = \frac{1}{2}b(c\sin(A)) = \frac{1}{2}a(b\sin(C))$$

Multiplying by 2 and dividing by abc gives us:

$$\frac{\sin(B)}{b} = \frac{\sin(A)}{a} = \frac{\sin(C)}{c}$$

This is called the sine relations. By these relations we can find all sides and angles in a triangle given that we know three elements already, not all sides or angles. If we know three sides we can use the so called cosine relations explained here:

Considering triangle ABC in Figure 2.3 we see that the height creates two right triangles which by Pythagoras theorem give us two equations:

$$\Delta ABC: h^2 + (x - a)^2 = b^2$$

$$\Delta ABD: h^2 + x^2 = c^2$$

Subtracting these two equations from each other gives us:

$$h^2 + (x - a)^2 - (h^2 + x^2) = b^2 - c^2 \Rightarrow$$

$$h^2 + x^2 + a^2 - 2xa - h^2 - x^2 = b^2 - c^2 \Rightarrow$$

$$2xa = a^2 + c^2 - b^2$$

Recall that:

$$\cos(B) = \frac{x}{c} \Leftrightarrow x = c\cos(B)$$

Inserting this in the above equation gives us:¨

$$2ac\cos(B) = a^2 + c^2 - b^2 \Rightarrow$$

$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$

This is one of the cosine relations. The other two can be found in the same manner and are:

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

Hence in the geometry sector sine and cosine works as tools to find the unknown sides and angles in triangles. Since all figures with straight edges can be divided into triangles, sine and cosine play an important role in the geometry sector.

2.2.3 The unit circle

In analytic geometry the trigonometric functions are defined as the coordinates to a given point on the unit circle. Considering a point P on the unit circle, it can be described as $P = (x, y)$, but it can also be described by a vector $\overrightarrow{OP}$, which has length $|\overrightarrow{OP}|$ and direction angle θ.

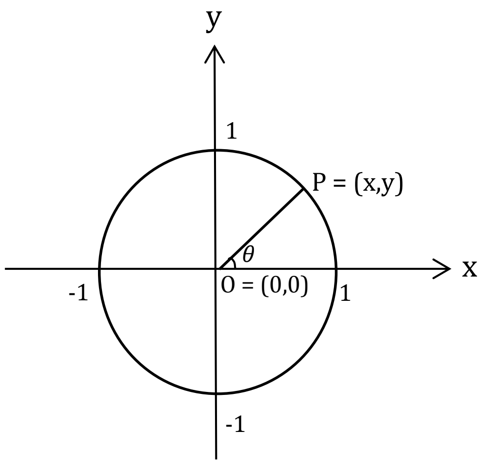


Figure 2.4

This angle can be measured in either degrees or radians. Recall that the angle measurement in radians is equal to the length of the arc it covers on the unit circle. Since the circumference of the unit circle is 2π, the entire circle equals an angle of 2π radians or 360 degrees. So in order to calculate between radians and degrees we have the ratio:

$$\frac{d}{360°} = \frac{r}{2\pi}$$

The point P is arbitrary chosen, so we can choose other points on the circle, either by moving clockwise or counterclockwise. If we move counterclockwise we call it the positive direction and the clockwise the negative direction. In this way we can have negative angles, namely if our point P lies in the third or fourth quadrant. The angle concept is not only extended to including both positive and negative measurement, but also angles bigger than 360 degrees. If we move 1½ times around in the unit circle we get and angle of 540°or 3π**.** In this way any real number can be considered as an angle. Notice that the point P where the direction angle is 3π is $(-1,0)$. This point could also be described by the direction angle π or $-\ \pi$, but this relation we will come back to later.

As mentioned sine and cosine can be defined as coordinates to the point P on the unit circle. To see this recall that in a right triangle sine and cosine is defined as:

$$\sin(v) = \frac{opposite\ side}{hypotenuse}, \qquad \cos(v) = \frac{adjacent\ side}{hypotenuse}$$

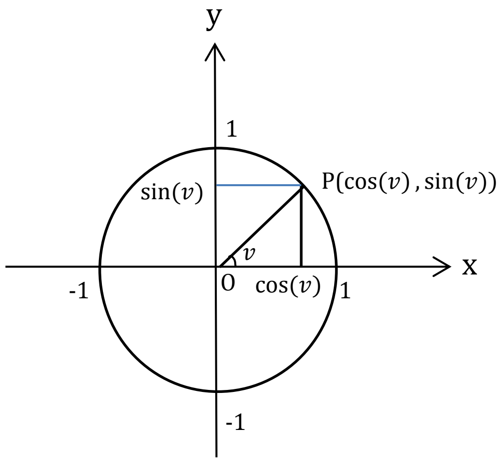


Figure 2.5

If we consider the right triangle, in Figure 2.5, formed by the vector $\overrightarrow{OP}$, the x-axis and the vertical line from P to the x-axis, then we see that the hypotenuse $\overrightarrow{OP}$ equals 1, and hereby we get that cosine represents the adjacent side, which is the length of the x-axis from O to the point where the line from P cuts the x-axis, i.e. the first coordinate to P. In the same way sine represents the opposite side, which is equal to the length from O to the point where the horizontal line from P cuts the y-axis, i.e. the second coordinate to P. In other words, cosine and sine is defined as the first and second coordinates to a point on the unit circle, given the direction angle of this point:

$$P = (\cos(v), \sin(v))$$

This definition gives us some properties of trigonometric functions we could not see only be using a triangle. First we have that:

$$\cos^2(v) + \sin^2(v) = 1$$

This follows from Pythagoras theorem. Next we can prove that:

$$\sin(-v) = -\sin(v)$$
$$\cos(-v) = \cos(v)$$

Consider the angle v in Figure 2.5. The negative angle $-v$ is the angle reflected in the x-axis. Let the direction point to v be $(\cos(v), \sin(v))$, then the direction point to $-v$ is: $(\cos(-v), \sin -(v)) = (\cos(v), -\sin(v))$ and the above properties are proved. These properties tell us that sine is an odd function and cosine is even.

Lastly we can prove the addition formulas just by considering the triangle in Figure 2.5 and using the trigonometric properties from section 2.2.2. Here we will just present the addition formula for some angles v, w without proof:

$$\sin(w-v) = \cos(v)\sin(w) - \sin(v)\cos(w)$$
$$\sin(w+v) = \cos(v)\sin(w) + \sin(v)\cos(w)$$
$$\cos(w-v) = \cos(v)\cos(w) + \sin(v)\sin(w)$$
$$\cos(w+v) = \cos(v)\cos(w) - \sin(v)\sin(w)$$

2.2.4 The natural parametrization of the unit circle

To define sine and cosine as functions from $\mathbb{R}$ to $\mathbb{R}$ we must in some way move from an angle measurement as input to a real number. With sine and cosine as the coordinates to the points on the unit circle, it is possible to find a finite number of points on the unit circle, using the trigonometric laws and identities. But not all directions angles can easily be deduced to a point in the unit circle. So how do we ensure that no matter what real number we use, as input, we get a point on the unit circle as output? And how do we even know that all real numbers can represent an angle? To ensure this we must consider the notion of angle. In section 2.2.1 we saw that an angle can be expressed as the arc length on the unit circle subtending the central angle. To ensure that every real number can represent an angle, we must ensure that a curve that traces the entire unit circle exists. Let the set $S^1 \subset \mathbb{R}^2$ be the unit circle:

$$S^1 = \{(x, y) | x^2 + y^2 = 1\}$$

We want to construct a map $\gamma: \mathbb{R} \to S^1$, which traces the entire unit circle. That is, when $t \in \mathbb{R}$ traverses an interval $I \subseteq \mathbb{R}$, will the corresponding point $\gamma(t)$ traverses an arc length on the unit circle.

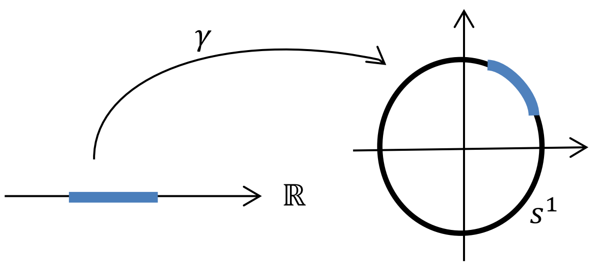


Figure 2.6: The map γ

Eilers et al. (2014) calls this map an "angle mapping". This refers to the fact that the traveled arc length equals the central angle it subtends. By definition this angle has the size of I, so the map γ has the characteristic that a line segment in $\mathbb{R}$ is mapped into a same sized arc length in S^1. This characteristic makes γ a natural parametrization, because a parametrization $\gamma: I \to \mathbb{R}^2$ is called natural if for all $t_1, t_2 \in I$ where $t_1 < t_2$ the arc length from $r(t_1)$ to $r(t_2)$ equals $|t_1, t_2| = t_2 - t_1$ (Eilers et al., 2014, p. 219). As we will see this natural parametrization has the arc length as the parameter, so we need to make sure that this arc length exists.

Considering an arbitrary continuous curve given by the parametrization $r: [a, b] \to \mathbb{R}^2$, we want to define the curve length. By subdividing the interval $[a, b]$ with the use of the points $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$, and connecting the points $r(t_o), r(t_1), r(t_2)$ and so on by straight lines we get the length:

$$l(D) = \sum_{j=1}^{k} \left| r(t_j) - r(t_{j-1}) \right|$$

where D is the subdivision of $[a, b]$. It is well known that the shortest way between two points is the straight line, so $l(D)$ will always be a lower bound for the length of the curve r, but if D is very fine we have a good approximation to the curve length. Therefore we define the arc length as (Ibid., p. 2015):

$$l = \sup\{l(D) | D\ is\ a\ finite\ subdivision\ of\ [a, b]\}$$

This supremum only exists if $l(D)$ has an upper bound, and luckily $l(D)$ has, because the length of the curve r will always be greater than $l(D)$. Thus for a continuous curve $r: [a, b] \to \mathbb{R}^2$ the curve length does exist. If the curve is C^1, i.e. one time continuous differentiable, as the unit circle is, the curve length is exactly given as:

$$l = \int_a^b |r'(t)| dt$$

To verify this let us make a simplified proof. The formal proof is very similar, but requires that you go deeper in to the mathematical analysis. This will not be done here, because the simplified proof gives us the idea we want, and may also be suitable for a clever high school class. Let us look at the proof. Since r is a map from $\mathbb{R}$ to $\mathbb{R}^2$, it is on the form $r(t) = (x(t), y(t))$, hence:

$$l = \int_a^b |r'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Considering the Figure 2.7 we want an approximated expression for the curve length Δl between $a = r(t)$ and $b = r(t + \Delta t)$.

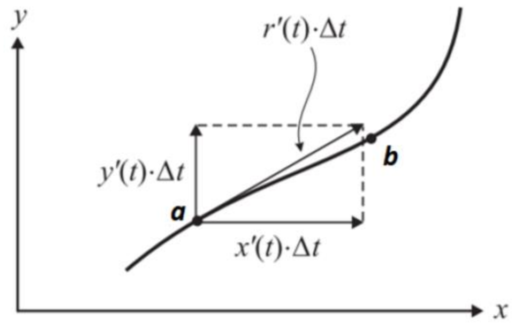


Figure 2.7: Sketch of the proof (reproduced from Vestergaard, 2007, p. 8)

If Δt is small the straight line from a to b is a good approximation of Δl. The length from a to b equals $|r(t + \Delta t) - r(t)| = |\Delta r|$, therefore:

$$\Delta l \approx |\Delta r| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Since the derivative $x'(t)$ is the limit of the difference quotient $\frac{\Delta x}{\Delta t}$ when Δt approaches zero, we say that $x'(t) \approx \frac{\Delta x}{\Delta t}$, hence $\Delta x \approx x'(t)\Delta t$, which gives us:

$$\Delta l \approx |\Delta r| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x'(t)\Delta t)^2 + (y'(t)\Delta t)^2} = \sqrt{(x'(t))^2 + (y'(t))^2} * \Delta t$$

$$\Downarrow$$

$$\frac{\Delta l}{\Delta t} \approx l'(t) \approx \sqrt{(x'(t))^2 + (y'(t))^2}$$

By integration we get:

$$l(D) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b |r'(t)| dt = l$$

Thus by making some approximations we accept the statement. The curve length is then just a quantity describing the length between two fixed points on the curve. What if one of the points is not fixed? What if we want to follow the trace of the curve as we traverse the interval $[a, b]$? This is here we want the arc length. Consider the C^1- curve $r: I \to \mathbb{R}^2$. For every bounded, closed subinterval $[t_1, t_2] \subset I$ we denote the corresponding curve length as $l(t_1, t_2)$. The arc length $s = s(t)$ calculated with direction from a fixed curve point $r(t_0)$ to an arbitrary curve point $r(t)$ is then (Eilers et al., 2014, p. 219):

$$s = \begin{cases} l(t_0, t) & for\ t > t_0, t \in I \\ 0 & for\ t = t_0 \\ -l(t, t_0) & for\ t < t_0, t \in I \end{cases}$$

And by the above:

$$s(t) = \int_{t_0}^{t} |r'(\tau)| d\tau$$

The fundamental theorem of calculus says that the integral is an antiderivative to the integrand:

$$s'(t) = |r'(t)|$$

As mentioned before the parametrization $r: I \to \mathbb{R}^2$ is called natural if for all $t_1, t_2 \in I$ where $t_1 < t_2$ the arc length $l(t_1, t_2)$ from $r(t_1)$ to $r(t_2)$ equals $|t_1, t_2| = t_2 - t_1$ (Ibid., p. 219). Using that $s = s(t)$ is the arc length from a fixed curve point $r(t_0)$ to an arbitrary curve point $r(t)$ we get:

$$l(t_1, t_2) = \int_{t_1}^{t_2} |r'(t)| dt = \int_{t_1}^{t_2} s'(t) dt = s(t_2) - s(t_1)$$

So for a parametrization to be natural we have the following conditions, where each is sufficient and all equivalent:

1. $l(t_1, t_2) = t_2 - t_1$ for $t_1 < t_2, t_1, t_2 \in I$
2. $s(t_2) - s(t_1) = t_2 - t_1$ for $t_1, t_2 \in I$
3. $s'(t) = 1$ for $t \in I$
4. $|r'(t)| = 1$ for $t \in I$

The equivalence of the first and third condition is seen by proving:

$$\forall t_1, t_2, \int_{t_1}^{t_2} s'(t) dt = t_2 - t_1 \iff s'(t) = 1$$

Let $s(t)$ be the antiderivative to $s'(t)$. Then:

$$s(t_2) - s(t_1) = t_2 - t_1 \ \forall t_1, t_2 \Rightarrow s(t) = t \Rightarrow s'(t) = 1$$

The other way around is clear. □

We want to construct a natural parametrization of the unit circle, but the question is whether this parametrization exists. According to Eilers et al. (2014) it does if the unit circle is a smooth C^1-curve, because "Any smooth C^1- curve has a natural parametrization" (Ibid., p. 221). Let us prove this statement and afterwards examine whether the unit circle is smooth.

Assume that the curve with the parametrization $r: I \to \mathbb{R}^2$ is a smooth C^n- curve, where $n \geq 1$. Then $r'(t)$ is a C^{n-1}- curve and the same is $|r'(t)|$, because if r is smooth $r'(t) \neq 0$ for all $t \in I$. Since the arc length $s(t)$ is the antiderivative to $|r'(t)|$, $s(t)$ must be a C^n- function. Furthermore $s'(t) > 0$ as we have $r'(t) \neq 0$. The function $t \to s(t)$ is then strictly increasing and since it is continuous it maps the interval I into an interval $J \subset \mathbb{R}$. The inverse function $s^{-1}: J \to I$ is also a C^n- function with the derivative:

$$(s^{-1})'(u) = \frac{1}{s'(s^{-1}(u))} > 0$$

Using the inverse function as a reparametrization we define $\tilde{r}(u) = r\left(s^{-1}(u)\right)$. What we have made here is a reparametrization $s^{-1}: J \to I$, such that the arc length becomes the parameter. Both r and $\tilde{r}$ trace the same curve and since the reparametrization is increasing they both have the same direction. By the chain rule we get:

$$\tilde{r}'(u) = r'\left(s^{-1}(u)\right) * (s^{-1})'(u) = \frac{r'\left(s^{-1}(u)\right)}{s'(s^{-1}(u))} = \frac{r'\left(s^{-1}(u)\right)}{|r'(s^{-1}(u))|} \qquad for\ u \in J$$

It follows from here that $|\tilde{r}'(u)| = 1$, which was one of the conditions for being a natural parametrization. Hence any smooth C^n- curve has a natural parametrization. □

We now return to the unit circle. The parametrization $r_1(x) = \left(x, \sqrt{1-x^2}\right), x \in [0,1]$ represents the curve of the upper half of the unit circle. It is a function, because each input gives one and only one output, it is continuous and since ${r_1}'(x) \neq 0$ for all $x \in [0,1]$ it is smooth. In the same way we can parametrize other pieces of the unit circle, and since they are all smooth the unit circle itself is smooth. So the unit circle has a natural parametrization, we can parametrize it with the arc length as the parameter. The question is then how this natural parametrization looks like. It would be difficult to construct it analytically since this would involve finding an expression for the inverse of the arc length function. Let us instead construct it in the Cartesian coordinate system. But before constructing it let us define the number π.

Normally we think of π as the ratio between the diameter and circumference of a circle, but this definition is not rigorous, because it would demand that we prove that this ratio is the same in all circles. Let us instead define π as the arc length of the upper unit circle. That is the arc length from -1 to $+1$ of the curve parametrized by $(x) = \left(x, \sqrt{1-x^2}\right), x \in [0,1]$:

$$\pi = \int_{-1}^{1} |r'(x)| dx = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1-x^2}}\, dx = \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$

This integral is improper since we do not know whether it exists or not at ± 1. In order to prove this let us consider the integral $\int \sqrt{1-x^2}\, dx$. By partial integration we get:

$$\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\, dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\, dx - \int \sqrt{1-x^2}\, dx$$

$$\Downarrow$$

$$2\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\Downarrow$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = 2\int \sqrt{1-x^2}\, dx - x\sqrt{1-x^2}$$

Since the integral $\int_{-1}^{1} \sqrt{1-x^2}\, dx$ clearly exists, the right hand side is meaningful. Setting the limits to $-1+\varepsilon$ and $1-\varepsilon$ and next let ε approach 0 we get an integral:

$$\lim_{\varepsilon \to 0} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\, dx = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

Which clearly converge and the result is what we will call π.

Let us now begin the construction of the natural parametrization of the unit circle. Consider $\mathbb{R}^2$, i.e. the Cartesian coordinate system. For $x > 0$ let us start in the point (0,1) and move in the positive direction on S^1 until we have traveled the distance x. This point we call $\gamma(x)$. Since the circumference of the unit circle is 2π, by the definition of π, we will pass the starting point if $x > 2\pi$, but that does not matter. Notice that $\gamma(x + 2\pi) = \gamma(x)$, because constructing $\gamma(x + 2\pi)$ when $x \geq 0$ we just travel an extra full turn around the unit circle and ends in the point $\gamma(x)$.

For $x < 0$ we again start in the point (0,1) but now we move in the negative direction until we have traveled the distance $|x|$. This point we call $\gamma(x)$. Also here $\gamma(x + 2\pi) = \gamma(x)$, which is easily seen if $x \leq -2\pi$, because it is the same construction as before, just in negative direction. If $x \in (-2\pi, 0)$ the problem is that $\gamma(x)$ is constructed by moving in the negative direction, and $\gamma(x + 2\pi)$ by moving in the positive direction. But the sum of the distance traveled in negative direction and in positive direction equals 2π, so we end up in the same point.

Finally for $x = 0$ we consider the point $\gamma(0) = (1,0)$, where we does not move at all. Recall that each point on the unit circle is determined by cosine and sine to the direction angle, and that the arc length traveled equals the direction angle measured in radians. Hence cosine and sine can be defined as the first and second coordinate function to the map $\gamma: \mathbb{R} \to \mathrm{S}^1$, which is the natural parametrization of the unit circle:

$$\gamma(x) = (\cos(x), \sin(x)), \qquad x \in \mathbb{R}$$

Hence sine and cosine are functions from $\mathbb{R}$ to $\mathbb{R}$. Since $\gamma(x) \in S^1$ for all $x \in \mathbb{R}$ we have that:

$$\cos^2(x) + \sin^2(x) = 1, x \in \mathbb{R}$$

As well as both $\cos(x)$ and $\sin(x)$ lies in the in the interval $\lfloor -1,1 \rfloor$. Since $\gamma(x + 2\pi) = \gamma(x)$ for all $x \in \mathbb{R}$ we also get the relations:

$$\cos(x + 2\pi) = \cos(x), \qquad \sin(x + 2\pi) = \sin(x), x \in \mathbb{R}$$

This tells us that the trigonometric functions are periodic.

The parametrization also tells us, that our definition of direction is correct. Considering the arc lengths $\frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$, each one longer than the previous. The three arc lengths represent the points:

$$\gamma\left(\frac{\pi}{4}\right) = \left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\gamma\left(\frac{\pi}{3}\right)=\left(\cos\left(\frac{\pi}{3}\right),\sin\left(\frac{\pi}{3}\right)\right)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

$$\gamma\left(\frac{\pi}{2}\right)=\left(\cos\left(\frac{\pi}{2}\right),\sin\left(\frac{\pi}{2}\right)\right)=(0,1)$$

Plotting these points in $\mathbb{R}^2$ shows us that the positive direction on the unit circle is counterclockwise.

Finally the existence of the natural parametrization of the unit circle gives us a formal way to define the notion of angle. We saw in section 2.2.1 that an angle is measurable if the arc length on the unit circle subtended by the angle exists, and if so the arc length is the angles measurement. By the natural parametrization we have shown that the arc length does exist, and in fact we can define the notion of angle as the inverse of the natural parametrization, that is the function $\gamma^{-1}: S^1 \rightarrow \mathbb{R}$ taking a point P on the unit circle and mapping it to the arc length from $(0,1)$ to P.

This paragraph has shown us that the definition of sine and cosine as trigonometric functions from $\mathbb{R}$ to $\mathbb{R}$ is based on the natural parametrization of the unit circle, with the arc length as the parameter. This is clearly a dynamic approach to the notion of function. We map the real line to the arc of the unit circle. It also clarifies why sine and cosine can be accepted as functions from the domain of real numbers, but the theory underlying this natural parametrization is far above the mathematical level expected in high school. A possibility could be to simplify the theory in order for the students to realize the connection between the unit circle and the graphs of the trigonometric functions. We will return to this discussion, but let us first introduce other definitions of the trigonometric functions.

2.2.5 Differentiation of the sine function

Now we have the trigonometric functions and since the functions clearly are continuous, they are also differentiable. To find the derivative of sine we must first find the limit to the difference quotient. The difference quotient is:

$$\frac{f(x+h)-f(x)}{h}=\frac{\sin(x+h)-\sin(x)}{h}$$

The following identity:

$$\sin(x) - \sin(y) = 2 * \cos\left(\frac{x+y}{2}\right) * \sin\left(\frac{x-y}{2}\right)$$

Gives us that:

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{2 * \cos\left(\frac{2x+h}{2}\right) * \sin\left(\frac{h}{2}\right)}{h} = \frac{2 * \cos\left(x + \frac{h}{2}\right) * \sin\left(\frac{h}{2}\right)}{h}$$

$$= \frac{\cos\left(x + \frac{h}{2}\right) * \sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos\left(x + \frac{h}{2}\right) * \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

Now let us consider the limit. The limit of a product is equal to the product of the limits, so let us look at the two limits. Since cosine is continues $\cos\left(x + \frac{h}{2}\right) \to \cos(x)$ when $h \to 0$, and since $\frac{h}{2}$ is just a constant we just need to consider the limit:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

Since the limit of the denominator is zero, we cannot find the limit by using the normal theorems for limits. Instead we need to consider the unit circle:

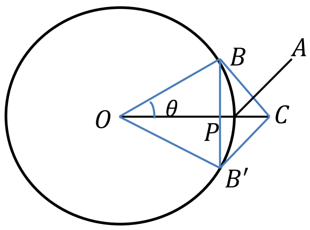


Figure 2.8: Sketch of the proof (Reproduced from Spector, 2015)

Let O be the center of the unit circle and let θ be the central angle BOA measured in radians. Then the arc length AB is equal to θ. Draw angle AOB' equal to angle BOA, making arc length AB' equal to arc length AB and draw the straight line from B to B' cutting OA in the point P. Finally draw the straight lines BC and $B'C$ as tangent to the circle. Then:

$$BB' < arc\ length\ BAB' < BC + B'C$$

It is easily seen by triangle computations that $BB' < BC + B'C$. But to see that the arc length lies in between is more difficult. The first inequality can be accepted by recalling the Euclidian definition of a straight line as the shortest way between two points. Then

then arc length BAB' will at most be equal to BB', never less than. To accept the last equality we must trust our eyes by looking at the figure. But a figure is not a valid proof, thus the inequality is still up for some discussion, but let us continue using it.

By the unit circle definition of sine and tangent we get $PB = PB' = \sin(\theta)$, which leads to $BB' = 2\sin(\theta)$, and $BC = CB' = \tan(\theta)$, since $OB = OB' = 1$. Therefore we get the inequality:

$$2\sin(\theta) < 2\theta < 2\tan(\theta)$$

Since $\tan(\theta) = \frac{sin\theta)}{\cos(\theta)}$, dividing by $2\sin(\theta)$ gives us:

$$1 < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)}$$

Taking the reciprocal and changing the sign will change the sense of the inequality two times and give us:

$$-1 < -\frac{\sin(\theta)}{\theta} < -\cos(\theta)$$

If we add 1 to each term we get:

$$0 < 1 - \frac{\sin(\theta)}{\theta} < 1 - \cos(\theta)$$

If θ approaches 0, then $\cos(\theta)$ approaches 1 and $1 - \cos(\theta)$ approaches 0. So $1 - \frac{\sin(\theta)}{\theta}$ is squeezed in between to quantities approaching zero, thus also $1 - \frac{\sin(\theta)}{\theta}$ approaches zero, giving us that:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

The derivative of $f(x) = \sin(x)$ then becomes:

$$f'(x) = \lim_{h \to 0}(\frac{f(x+h) - f(x)}{h}) = \lim_{h \to 0}\left(\cos\left(x + \frac{h}{2}\right) * \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right) = \cos(x)$$

□

Likewise we can show that the derivative to $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$. Even though differentiation is a branch of the analytic mathematic, we have to use the geometric world to fully understand and explain the behavior of trigonometric functions. This is one of the reasons why trigonometric functions are so complex and difficult to fully understand.

As a counterpart to the differentiation let us mention the integration. A function $F(x)$ is said to be the antiderivative to $f(x)$ if $F'(x) = f(x)$. Since $\sin(x)' = \cos(x)$, the antiderivative to $\cos(x)$ must be $\sin(x)$ and likewise since $\cos(x)' = -\sin(x)$, the antiderivative to $\sin(x)$ is $-\cos(x)$. Remember that when differentiating functions containing constants the constants just vanish, so in fact also $-\cos(x) + c$ is an antiderivative to $\sin(x)$. We have:

$$f(x) = \sin(x) \Rightarrow F(x) = -\cos(x) + c$$
$$f(x) = \cos(x) \Rightarrow F(x) = \sin(x) + c$$

where c is a constant. The derivative and antiderivative show us that there is an interesting connection between the two trigonometric functions. Even though this is interesting it is not the most useful definition of the trigonometric functions.

2.2.6 The inverse of sine

With the parametrization of the unit circle we used the arc length to define sine and cosine. In this section we will use the integral. As seen in section 2.2.5 the integral of sine is no simpler than sine itself. However the integral of the inverse of sine, also called arcsine, turns out not to involve trigonometric functions, so we will define arcsine and then define sine and cosine in terms of arcsine. The arcsine function is the function $\arcsin: [-1,1] \to \mathbb{R}$ defined by:

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

for all $x \in [-1,1]$. The integral is improper when $x = 1$ and $x = -1$ (Bloch, 2011, p. 373). Hence the proof of the convergence in ± 1 is similar to the proof of the fact that $\pi = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx$ converges at ± 1, shown in section 2.2.4. In order to construct the sine function from the arcsine some properties must apply to the arcsine. We assume that arcsine is the inverse of sine when $x \in [-1,1]$ and vice versa, indirectly assuming that arcsine has an inverse, ergo must arcsine be bijective. In order to be bijective let us first show that arcsine is continuous:

From the definition of arcsine and the fundamental theorem of Calculus, saying that the integral is an antiderivative to the integrand, we know that arcsine is differentiable on $(-1,1)$ with $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ for all $x \in (-1,1)$. Hence arcsine is continuous on $(-1,1)$. The integral $\arcsin(1)$ is improper meaning that:

$$\arcsin(1) = \int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \lim_{y \to 1^-} \int_0^y \frac{1}{\sqrt{1-t^2}} dt = \lim_{y \to 1^-} \arcsin(y)$$

So for $x = 1$ the limit exists and $\lim_{y \to 1^-} \arcsin(y) = \arcsin(1)$, so in $x = 1$ arcsine is continuous from left. The same argument holds for $x = -1$, making the entire arcsine function continuous.

Next let us consider the range of arcsine, in order to show that arcsine is surjective. The domain is $[-1,1]$, so let us first consider:

$$\arcsin(1) = \int_0^1 \frac{1}{\sqrt{1-t^2}} dt$$

Recall that the definition of π is the improper integral:

$$\pi = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = 2 * \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

where the last equation follows from the fact that the unit circle is symmetric around the y-axis. Hence $\arcsin(1) = \frac{\pi}{2}$. To find $\arcsin(-1)$, we first show that $\arcsin(-x) = -\arcsin(x)$ for all $x \in [-1,1]$: Let $x \in [-1,1]$. By using the substitution $u = -t$, which implies that $du = -dt$ we see that:

$$\arcsin(-x) = \int_0^{-x} \frac{1}{\sqrt{1-t^2}} dt = -\int_0^{x} \frac{1}{\sqrt{1-(-t)^2}} (-1) dt =$$

$$-\int_0^{x} \frac{1}{\sqrt{1-u^2}} du = -\arcsin(x)$$

The above gives us that $\arcsin(-1) = -\arcsin(1) = -\frac{\pi}{2}$. So the range of arcsine is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, hence the function $\arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is surjective.

Since $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} > 0$ for all $x \in (-1,1)$ arcsine is strictly increasing, hence injective, which is seen in the following proof:

Let $x_1, x_2 \in [-1,1]$ and $x_1 \neq x_2$. Then $x_1 < x_2$ or $x_1 > x_2$ and since arcsine is strictly increasing it implies that $\arcsin(x_1) < \arcsin(x_2)$ or $\arcsin(x_1) > \arcsin(x_2)$, hence $\arcsin(x_1) \neq \arcsin(x_2)$ □

To sum up we have shown that the function $\arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is bijective and hence have an inverse function $\arcsin^{-1}: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$. The sine function is the function $\sin: \mathbb{R} \to \mathbb{R}$ defined as the periodic extension of the function $f: \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \arcsin^{-1}(x) & if\ x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \arcsin^{-1}(\pi - x) & if\ x = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{cases}$$

The function f is well-defined, because $\arcsin^{-1}\left(\frac{\pi}{2}\right) = \arcsin^{-1}\left(\pi - \frac{\pi}{2}\right)$. By definition, a periodic extension of a function $g: [a,b] \to \mathbb{R}$ is the function $h: \mathbb{R} \to \mathbb{R}$ with period $b - a$, such that $h|_{[a,b]} = g$, under the assumption that $g(a) = g(b)$ (Ibid., p. 372). In the definition of sine we have:

$$f\left(\frac{3\pi}{2}\right) = \arcsin^{-1}\left(\pi - \frac{3\pi}{2}\right) = \arcsin^{-1}\left(-\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right)$$

Hence sine is a well-defined periodic extension of the function f. From this definition we can state some of the well-known facts about the sine function. First of all we see that sine has the period $\frac{3\pi}{2} - \left(-\frac{\pi}{2}\right) = 2\pi$. Hence we have that $\sin(x + 2\pi) = \sin(x), \forall x \in \mathbb{R}$. Then no matter what $x \in \mathbb{R}$ we choose we can end up in the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ just by adding or subtracting 2π. Hence $\sin(x) = f(x), \forall x \in \mathbb{R}$. If we want to calculate some of the values of sine let us first state that sine is strictly increasing because arcsine is. Consider a strictly increasing function $g: \mathbb{R} \to \mathbb{R}$ and its inverse $g^{-1}: \mathbb{R} \to \mathbb{R}$. The definition of strictly increasing implies that $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$ for some $x_1, x_2 \in \mathbb{R}$. Since g^{-1} is the inverse we have by definition $g^{-1}(g(x_1)) = x_1$ and $g^{-1}(g(x_2)) = x_2$, hence $g(x_1) < g(x_2)$ implies that g^{-1} is strictly increasing since $g^{-1}(g(x_1)) = x_1 < g^{-1}(g(x_2)) = x_2$. Hereby $\arcsin(x) < \arcsin(y) \Rightarrow \sin(x) < \sin(y)$for all $x, y \in [-1,1]$.

Recall that $\arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hereby $\arcsin^{-1}(x) = \sin|_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$. Now we are able to find some values of sine:

$$\sin\left(-\frac{\pi}{2}\right) = -1, \sin\left(\frac{\pi}{2}\right) = 1, \sin(0) = 0$$

Where the last result follows from the fact that $\arcsin(0) = 0$ and hence also $\arcsin^{-1}(0) = 0$. In fact we can say that:

$$0 < \sin(x) < 1, \forall x \in \left(0, \frac{\pi}{2}\right)$$
$$-1 < \sin(x) < 1, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$-1 \leq \sin(x) \leq 1, \forall x \in \mathbb{R}$$

This leads to the definition of cosine. By use of the unit circle and its natural parametrization we defined sine and cosine independently, here we will define cosine in terms of sine. The cosine function is the function $\cos: \mathbb{R} \to \mathbb{R}$ defined by

$$\cos x = \begin{cases} \sqrt{1 - \sin^2 x} & if\ x \in \left[-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right]\ for\ some\ n \in \mathbb{Z} \\ -\sqrt{1 - \sin^2 x} & if\ x \in \left[\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right] for\ some\ n \in \mathbb{Z} \end{cases}$$

This definition is well-defined since sine is never greater than 1 and neither is sine squared. From this definition we can deduce the famous trigonometric identity: $\cos^2(x) + \sin^2(x) = 1$, and we can also prove the derivative of the two trigonometric functions. We will not go in to details here, but just conclude that both sine and cosine can be defined in terms of the function arcsine.

The name arcsine may seem strange, but an explanation is found in section 2.2.4, concerning the natural parametrization of the unit circle. One parametrization of the unit circle is $r(t) = \left(\sqrt{1 - t^2}, t\right), t \geq 0$ where the arc length from $(1,0)$ to the unique point $(x, y)$where $x \geq 0$ is:

$$\int_0^y |r'(t)| dt = \int_0^y \sqrt{\frac{t^2}{1 - t^2} + 1}\, dt = \int_0^y \frac{1}{\sqrt{1 - t^2}} dt = \arcsin(y), y \geq 0$$

Thus for $x, y \geq 0$ arcsine determines the arc length on the unit circle, hence arcsine gives a kind of angle between circle points. Finally sine takes $\arcsin(y)$ to the second coordinate of the direction angle's point on the unit circle, ensuring that the unit circle definition of sine still holds.

This section has shown us a rigorous definition of sine and cosine as functions. The approach to the notion of function is here dynamic, first of all because both sine and cosine are defined by analytic expressions, but more intuitively we can, by sine taking the $\arcsin(y)$ to the second coordinate, see the function as a process. The section has also shown us a definition with technical complications such as working with improper integrals, defining arcsine in terms of an integral instead of just the inverse of sine and finally the fact that we need an extension of the arcsine function in order to define the entire sine function in terms of arcsine. All of these are good reasons for not introducing this definition in a Danish high school.

2.2.7 Differential equations

The trigonometric functions can also be defined as the solution to some differential equations. In fact sine is the solution to the initial value problem:

$$y'' = -y, y(0) = 0, y'(0) = 1$$

and cosine is the solution to:

$$y'' = -y, y(0) = 1, y'(0) = 0$$

To verify this let us first consider the general homogenous second order differential equation:

$$ay'' + by' + cy = 0$$

where a, b and c are constants. A possible solution to this equation is a function where a constant multiplied by the second derivative plus a constant multiplied by the first derivative plus a constant multiplied with the function itself equals zero. The exponential function $y = e^{rx}$ has the property that $y' = re^{rx}$ and $y' = r^2e^{rx}$, hence the exponential function is a solution if:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \Rightarrow (ar^2 + br + c)e^{rx} = 0$$

e^{rx} is never zero, so $y = e^{rx}$ is a solution if r is a root of the equation:

$$ar^2 + br + c = 0$$

This equation is called the characteristic equation and in the case with the differential equation $y'' = -y$ we get:

$$r^2 = -1 \Leftrightarrow r^2 + 1 = 0$$

This equation has the irrational roots $\pm i$ so $y_1 = e^{ix}$ and $y_2 = e^{-ix}$ are solutions to the differential equation. Hence:

$$y(x) = Ce^{ix} + De^{-ix}$$

is a solution for all choices of the constants C and D. Using Euler's formula, which connects the complex exponential to the trigonometric functions we can rewrite the solution as:

$$y(x) = C(\cos(x) + i\sin(x)) + D(\cos(x) - i\sin(x))$$

$$= (C + D)\cos(x) + i(C - D)\sin(x) = E\cos(x) + F\sin(x)$$

where $E = (C + D)$ and $F = i(C - D)$ are still just constants. Let us consider the initial values $y(0) = 0$and $y'(0) = 1$. This gives us:

$$y(0) = E\cos(0) + F\sin(0) = E \Rightarrow E = 0$$

Differentiating our solution gives us:

$$y'(x) = -E\sin(x) + F\cos(x) \Rightarrow$$

$$y'(0) = -E\sin(0) + F\cos(0) = F \Rightarrow F = 1$$

So the solution to the initial value problem, $y'' = -y, y(0) = 0, y'(0) = 1$, is:

$$y(x) = \sin(x)$$

Likewise we can consider the initial values $y(0) = 1, y'(0) = 0$ and get:

$$y'(0) = -E\sin(0) + F\cos(0) = 0 + F = 0 \Rightarrow F = 0$$

$$y(0) = E\cos(0) = 1 \Rightarrow E = 1$$

So the solution to the initial value problem, $y'' = -y, y(0) = 1, y'(0) = 0$, is:

$$y(x) = \cos(x)$$

The uniqueness theorem for second order differential equations (Lindstrøm, 2006, p. 538) tells us that both sine and cosine are the only solutions to the given initial value problems. Defining sine and cosine as these solutions makes them static objects, and since we in High school would normally have a dynamic approach to functions, this static approach would not help the students grasp sine and cosine as functions. Second order differential equations are not a part of the curriculum in Danish high schools, so it would demand a considerable amount of new theory to present the trigonometric functions as solutions to second order differential equations. Nonetheless, we have included the definitions here because they could be presented as an interesting result when the students are introduced to mathematical modeling. As Camilus (2012) observes in her thesis the traditional teaching of differential equations focuses on finding solutions, just as we have done here. However, if we instead focus on the differential equation as a modeling tool and the solutions as concrete descriptions of scientific processes, then it would be easier for the students to validate the solutions (Camilus, 2012, p. 42). As supplementary material one could introduce the second order differential equation, for example in connection with a physics course on harmonic oscillations, where the students would see the differential equations as a useful model and moreover they would see that trigonometric functions can be used to describe the wave pattern we often see in the real world. Hence they get a good argument for knowing these functions.

2.2.8 Power series

Defining sine in terms of a power series would create a type of power series called a Taylor series. A Taylor series is a function expansion containing an infinite sum of terms calculated from the values of the function's derivatives in a given point. Since both sine and cosine are infinitely differentiable we can create their Taylor polynomials. A function's Taylor polynomial is the highest nth-degree polynomial, which has the same function value and the same first n derivatives as the function it represents in a point a:

$$T_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Let us consider $f(x) = \sin(x)$ around the point 0, if we differentiate this function we get:

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f^{(3)}(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

The fourth time we differentiate we end up where we started, so this process will repeat itself if we continue. If we evaluate these derivatives in the point 0 we get: $\sin(0) = 0, \cos(0) = 1, -\sin(0) = 0, -\cos(0) = -1, \sin(0) = 0$ and so on. Hence the first terms in the Taylor polynomial would be:

$$T_n \sin(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 - \frac{1}{7!}x^7 \ldots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \ldots$$

and with n terms we get the Taylor polynomial:

$$T_n \sin(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}$$

In the same manner we can find the Taylor polynomial for $f(x) = \cos(x)$:

$$T_n \cos(x) = \sum_{k=0}^{n} (-1)^{k} \frac{x^{2k}}{(2k)!}$$

If we let n go to infinity we get the Taylor series and as we will see both sine and cosine equal their Taylor series. If we consider the Taylor polynomials, then it is only locally around 0 that $T_n \sin(x) = \sin(x)$. The amount by which $T_n \sin(x)$ differs from $\sin(x)$ is called the remainder and is defined as:

$$R_n \sin(x) = \frac{1}{n!} \int_0^x sin^{(n+1)}(t)\,(x-t)^n dt$$

Hence:

$$\sin(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!} + R_n \sin(x)$$

In order for sine to equal its Taylor series the remainder must approach zero when n approaches infinity. Remember that sine and its $n+1$ derivative are continuous on the interval $[0, x]$. Let M be the number such that $\left|\sin^{(n+1)}(t)\right| \leq M, \forall t \in [0, x]$, and assume $x > 0$. Then we have:

$$|R_n \sin(x)| = \frac{1}{n!}\left|\int_0^x sin^{(n+1)}(t)\,(x-t)^n dt\right| \leq \frac{M}{n!}\int_0^x (x-t)^n\,dt = \frac{M}{n!}\left(\frac{x^{n+1}}{n+1}\right)$$
$$= \frac{M}{(n+1)!}x^{n+1}$$

Since all the derivatives of sine are either $\pm\sin(x)$ or $\pm\cos(x)$ then $M = 1$ is an upper bound and we get:

$$|R_n \sin(x)\,| \leq \frac{1}{(n+1)!}|x|^{n+1}$$

The left side clearly approaches zero when n approaches infinity and so does the remainder, so we get:

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

In the same manner we can show that cosine equals its Taylor series. In order for the functions to go from $\mathbb{R}$ to $\mathbb{R}$ we need to ensure that the series converges in the entire $\mathbb{R}$, i.e. have a radius of convergence that equals infinity. Using the ratio test (Lindstrøm, 2006, p. 638) we get:

$$\lim_{k\to\infty}\frac{(-1)^{k+1}\frac{x^{2(k+1)+1}}{(2(k+1)+1)!}}{(-1)^k\frac{x^{2k+1}}{(2k+1)!}} = \lim_{k\to\infty}\frac{(-1)^{k+1}x^{2k+3}(2k+1)!}{(2k+3)!\,(-1)^k x^{2k+1}}$$

$$= \lim_{k\to\infty}\frac{(2k+1)!}{(2k+3)!}|x|^2 = \lim_{k\to\infty}\frac{|x|^2}{(2k+1)!\,(2k+3)!} = 0$$

Since the limit is zero for all values of x the Taylor series converge for all x and we say that the radius of convergence equals infinity. The same argument can be used for cosine, so the trigonometric functions can be defined as power series. Now let us see what use we can make of them. First we see that we can easily find some of the known properties for both sine and cosine:

$$\sin(0) = \sum_{k=0}^{\infty}(-1)^k \frac{0^{2k+1}}{(2k+1)!} = 0 + 0 + \cdots = 0$$

$$\sin(-x) = \sum_{k=0}^{\infty}(-1)^k \frac{(-x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{-(x^{2k+1})}{(2k+1)!} = -\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = -\sin(x)$$

$$\cos(0) = \sum_{k=0}^{\infty}(-1)^k \frac{0^{2k}}{(2k)!} = 1 + 0 + 0 + \cdots = 1$$

$$\cos(-x) = \sum_{k=0}^{\infty}(-1)^k \frac{(-x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} \cos(x)$$

Since the radius of convergence is infinite the Taylor series are differentiable, and the derivative is found by differentiating term by term. From this we find that:

$$\sin'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(2k+1)x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos(x)$$

$$\cos'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}(2k)x^{2k-1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} x^{2k+1}$$
$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = -\sin(x)$$

This proof of trigonometric functions' derivatives is based purely on analysis, so in contrast to the earlier proof of the derivatives we do not have to use the unit circle here. Hence this proof seems more rigorous and simultaneously it is brief, so one may think that this would be favored. However, in order to make this proof rigorous we need to prove that a power series is differentiated term by term, and that a power series is differentiable if it has an infinite radius of convergence. A drawback is that these proofs involve numerous statements about the convergence of different series, hence in order to reach the desired knowledge concerning trigonometric function we would need a great deal of power series theory. This has a theoretical level way beyond what is expected for high school students. In addition the connection from this power series definition to the unit circle is difficult to realize, which could provoke the students' fractional idea of the trigonometric functions. The connection between the power series and the unit circle requires the complex numbers, which we will now consider.

2.2.9 Complex numbers

The connection between the power series and the unit circle is realized when sine and cosine is defined in the field of complex numbers. The Cartesian representation of the complex number is:

$$z = a + ib$$

where a and b are both real numbers and i is the imaginary unit satisfying $i^2 = -1$. a is called the real part and denoted $Re(z)$ and b the imaginary part, denoted $Im(z)$ of the complex number z. The mathematician Casper Wessel defined the complex number to be a vector in the plane, so $z = a + ib$ just represent the vector, i.e. the point (a, b) in the plane. In this way the complex number is an extension of the real line to the complex plane. Notice that $z = a = a + i0$ is the point $(a, 0)$, which lies on the real line, so all real numbers are included in the complex (Lindstrøm, 2006, p. 114). The complex numbers can also be explained by polar coordinates (r, θ). Here r represent the length of the vector, and θ the angle between the vector and the x-axis. Recall that the geometric interpretation of cosine and sine is respectively the x- and y- coordinates to the points on the unit circle. The connection between Cartesian and polar coordinates is seen in the Figure 2.8.

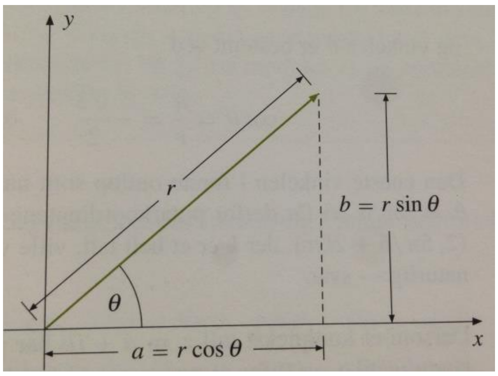


Figure 2.8: (Retrieved from Lindstrøm, 2006, p.115)

So if the point (a, b) represents the complex number $z = a + ib$ we get the polar form:

$$z = r * \cos(\theta) + ir * \sin(\theta) = r * (\cos(\theta) + i\sin(\theta))$$

Let us now see how sine and cosine are defined in the field of complex numbers. Rudin (1987) begins his book "Real and Complex Analysis" by stating that the most important function in mathematics is defined, for every complex number z, by the formula:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The ratio test shows that this series converges for every complex numbers, and this leads to the fact that the important addition formula $e^a e^b = e^{(a+b)}$ is valid for all complex numbers a, b. Next he defines sine and cosine as the real respectively the imaginary part of e^{ix}:

$$\cos(x) = Re(e^{ix}), \qquad \sin(x) = Im(e^{ix}) \qquad x \in \mathbb{R}$$

This is equivalent to Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x)$$

Comparing Euler's formula with the polar form of the complex number we get that e^{ix} represents all the numbers lying on the unit circle in the complex plane. Hence $(\cos(x), \sin(x))$ are points on the unit circle, which coincide with the geometric interpretation of sine and cosine. Also the addition formulae for the trigonometric functions can be deduced from Euler's formula:

$$\cos(x+y) + isin(x+y) = e^{i(x+y)} = e^{ix} * e^{iy} =$$

$$\cos(x) + isin(x) * (\cos(y) + isin(y)) =$$

$$\cos(x)\cos(y) + i^2 \sin(x)\sin(y) + isin(x)\cos(y) + icos(x)\sin(y) =$$

$$\cos(x)\cos(y) - \sin(x)\sin(y) + i(\sin(x)\cos(y) + \cos(x)\sin(y))$$

Splitting up in real and imaginary parts we get the addition formulas:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

This section shows us that through the complex numbers we have a relation between exponential growth and circular movement, and this relation makes it possible to deduce rules about well-known phenomena. The wider our mathematical knowledge is the more we see the connection between things and get the ability to compute knowledge without having to remember everything.

One thing is missing in the above, namely an argument for switching from θ as the argument to x and y. The argument is the answer to why an angle can be represented by an arbitrary real number. This answer was found in section 2.2.4 where we showed that the angle is defined as the arc length on the unit circle.

2.2.10 Fourier series

In section 2.1 we saw that the development of the notion of function happened through the study of trigonometric series. The mathematician Joseph Fourier investigated, in connection to his investigation of heat diffusion, the representation of various functions by trigonometric series. As mentioned earlier Fourier proved that an (to him) arbitrary function can be expressed as a sum of an odd and an even function. He also proved that a sine series can describe an arbitrary odd function on the interval $[-\pi, \pi]$ and a cosine series can describe an even function on the same interval. The result is that an arbitrary function can be expressed in the interval $[-\pi, \pi]$ by (Godiksen et.al., 2003, p. 44):

$$f(x) = \frac{1}{2}b_o + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

Where $n \in \mathbb{N}$ and the coefficients a_n and b_n are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Here we must be aware of Fourier's notion of a function. He asserts that he has proved that arbitrary functions can be expressed as a series of trigonometric function, but in fact he only shows it for piecewise-continuous functions. Moreover his proof only concerns the interior of a given interval. The values at the endpoint could easily be calculated, and the periodicity of trigonometric functions enabled one to extend the original function to the entire real line (Katz, 2009, p. 782). Hence the functions Fourier considers are continuous periodic functions. Another lack in Fourier's investigation is that he does not show that his series convert. This is first proven by Dirichlet in 1829.

The theory of Fourier series is way beyond the mathematical level in high school, and in fact it does not give us a clear definition of trigonometric functions, but it is briefly presented here because it shows that trigonometric functions are important functions, both now and in the historic development of mathematics.

2.3 The trigonometric functions in Danish high schools

In the previous sections we have seen several different approaches to the trigonometric functions, not all suitable for high school students. The transition from the scientific knowledge to knowledge we can expect the students to gain is called the external didactical transposition. In this section we will examine the result of this transposition, namely how trigonometric functions are introduced at Danish high schools and what the students are expected to learn. We will only consider A-level students. Jonathan Barret (2012) creates in his thesis an epistemological reference model concerning the trigonometric functions in high school (A- and B-level). His model builds upon two formularies and several written exam tasks, in other words the official specifications regarding the teaching of trigonometric functions. This section is built on Barret's model, but we also consult the written exams from 2011 until now, in case we see a change in tasks. Moreover we will consult the textbook used in the class we will observe. The textbook is built upon the Syllabus[1] issued by the Danish ministry of education, and often the teacher will use the book instead of the Syllabus when the teaching is planned (Winsløw, 2006, p. 83). In Denmark it is the teacher who decides which book she will use in her teaching, therefore the book chosen gives us, first of all a picture of the official specifications, but also a picture of how the teacher thinks the teaching should be planned, based on the expectation that she follows the book in her teaching.

Even though the teaching in high schools can be organised without consulting the Syllabus we will here briefly examine the extent to which the trigonometric functions appear herein. The central material in mathematics on A-level has ten categories, where the trigonometric functions appear in four of those (Matematik A - stx, juni 2013):

- Ratio calculations in right triangles and trigonometric calculations in arbitrary triangles

[1] The Syllabus is as set of official documents, which describes the purpose and guidelines for a given teaching subject, here Mathematics. Syllabus replaces the Danish word "Læreplan".

- The notion $f(x)$, characteristics of the elementary functions, including sine and cosine, characteristics of these functions graphic path and the use of regression
- The derivative of the elementary functions
- The antiderivative of the elementary functions

The central material alone does not cover all the academic targets. The students need to be taught in supplementary material. The Syllabus does not concretize this material but mentions general topics which should be taught in order to reach the academic targets (Ibid.). As an example where trigonometric functions could be included in the supplementary material we have a course in differential equations models or a course in the history of mathematics. In the former one could introduce the second order differential equation and in the latter one could work with Euler's formula and hereby include trigonometric functions, exponential growth and complex numbers. But implementation of trigonometric functions in the supplementary material is the teacher's choice so let us not focus more on this. The fact that trigonometric functions appear in four out of ten categories shows that it is an important subject in high school. Barret (2012) creates an overview of the trigonometric functions in high school by consulting a formulary[2] issued by the Danish ministry of education. The formulary gives an overview over the formulas and symbols related to the central material presented in the Syllabus (Barret, 2012, p. 14). After examining the formulary and the exam tasks Barret analyzes the teaching of trigonometric functions in terms of three sectors; a geometric, an algebraic and an analytic. The geometry sector involves the calculation of line segments and angles, the algebra sector the trigonometric identities and the calculation of exact values of the trigonometric functions by use of formulas, and the analysis sector involves graphs, harmonic functions, differential calculus and integral calculus (Barret, 2012, p. 30). Here we will only consider the last sector because until this part it is not required to acknowledge sine and cosine as functions.

The formulary contains the graphs of sine and cosine, as well as the derivative and antiderivative to both functions (Ibid., p. 20). The harmonic function is not presented in either the formulary or the exam tasks Barret consulted, but he includes it in his analytic

[2] A formulary is a collection of formulas, presenting the students with an idea of what formulas they should be able to use in the written exams. Formulary is replacing the Danish word "formelsamling".

sector because of their application in physics in connection with modelling of periodic phenomena (Ibid., p. 41). With the term "harmonic function" Barret refers to the class of functions represented by the analytic expression: $y = \text{a}\sin(bx + c)$, and if we consider these functions we find them represented in at least one exam task each year for the last five years. The exam tasks are a picture of what the students are expected to learn, hence a guideline for the teachers, so there must have been a change in the contents in the teaching of the trigonometric functions over the last years, and the analytic expression is represented in various textbooks used in Danish high schools today (Nielsen & Fogh, 2006; Carstensen, Frandsen & Studsgaard, 2008).

If we consider the textbook (Nielsen and Fogh, 2006) used in the teaching, we will observe that the chapter called "the functions sine and cosine" starts by introducing radians as a new angle measure, followed by an extension of the notion of angles to also include angles larger than 360° and negative angles, which arise if you move clockwise on the unit circle. In this way it is possible to extent the notion of angle to include all real numbers. Next is the graphs of the trigonometric functions presented and the definition of a periodic function is established. The next subchapter is called "sine curves in general" and starts by introducing the general formula for a sine curve:

$$f(x) = a\sin(bx + c) + d$$

Then the four constants are explained. The constant d displaces the curve in the direction of the y-axis. The constant a is called the amplitude and indicates the maximal oscillation from the line $y = d$. The constant b indicates the number of oscillations on the line segment 2π on the x-axis and the constant c indicates the displacement in the direction of the x-axis (Ibid., p. 43). In the end of the chapter is an example where a given dataset is plotted by using a CAS-tool. The plot looks like a sine curve, so the next step is to do sine regression on the data. The result is a function with concrete values for a, b, c and d. The derivatives and antiderivatives are not introduced in this chapter, but in separate chapters concerning respectively differential and integral calculus. In both chapters the derivative/antiderivative is just presented in a box together with other functions' derivative/antiderivative. Nowhere is a proof or an explanation for the derivative/antiderivative to the trigonometric functions presented. The reason is probably that the Syllabus only demands that the students know the derivative and antiderivative, not the proofs behind (Matematik A - stx, juni 2013). So even though the proof of the derivative (see section 2.2.5) has a suitable level for a high school class, it is not prioritized.

Another interesting thing in the textbook is the quick step from the specific functions sine and cosine to the class of functions called sine curves. Nowhere is it explained why the graphs of sine and cosine looks like they do. Instead the textbook focuses on how we can change the form and placement of a sine curve in the coordinate system by changing the constants. The reason could be that the Syllabus requires that the students learn the use of sine regression (see p. 40). Regression is about finding the best model to describe a relationship between a dependent and an independent variable. In the situation of sine regression that is to find the function $f \in \mathcal{F}_{trig}$that describes the given dataset best. In order to validate such a result the students must be able to explain what the different constants in the class of trigonometric functions indicate.

The Syllabus also requires that the students know the characteristic graphical path of sine and cosine (see p. 40). The best way to acknowledge these graphs is to realize the relation between them and the unit circle definition of sine and cosine. As we will see in the next section this realization is not easy.

2.4 The transition from geometric tool to function

In Danish high schools trigonometry is introduced in three different contexts; first as a geometric tool for solving triangle problems, here sine and cosine are defined as ratios between sides in a right triangle, secondly as the coordinates of the intersection between a ray and the unit circle and finally as functions with real numbers as input. This transition from a geometrical tool to an analytic function may create some obstacles in the student's interpretation of sine and cosine. These possible obstacles will be analyzed here.

The first obstacle could arise with the notion of function. If the students shall be able to acknowledge sine and cosine as functions, it is significant that they have an idea of the notion of function. Demir (2012) made in connection with his thesis "Students' Concept Development and Understanding of Sine and Cosine Functions" a test of Dutch students' interpretation of functions. Here he saw that the students were not able to give a formal definition of a function. Out of twenty-three students, eighteen referred to a formula to explain what a function is and seven also mentioned a graph. Only five referred to an input-output mechanism, although other students might have meant a similar mechanism when talking about formulas where you can fill in x to get y (Demir, 2012, p. 57). This shows that most of the students have a dynamic interpretation of the notion of function. The test also showed that the students tended to mark graphs they did not know as non-functions. We would expect to see the same result in a Danish high school because, like at Dutch high schools, the students are not presented to a formal definition, but only to different examples of functions.

When introducing sine and cosine as functions the big problem is that the functions do not have an analytic expression. There is no explicit formula which can give the students a y when they plug in a specific x. Hence the students must rely on the graphs in their validation of sine and cosine as functions. But a graph is just a representation of a function, not a formal definition. So in order to accept sine and cosine as functions the students must realize the relation between the definition of sine and cosine as coordinates on the unit circle to graphs of sine and cosine. This is another big challenge. It would require an insight in the change from an angle measurement to the real numbers as input. But still one will not be able to make an exact correspondence between the points on the unit circle and the graphs of the trigonometric functions, because not all real numbers will give appropriate outputs, and moreover who can tell whether the functions are defined in the entire $\mathbb{R}$? This would require some of the mathematics mentioned in section 2.2.4, mathematics too difficult for high school. The question is how to minimize the gap between the mathematical insights needed to realize the relation between the unit circle definition of sine and cosine and the graphs, and the mathematical insight high school students are able to comprehend? We will try to answer this as a closure to this chapter.

Since the big problem seems to lie in the transition from unit circle coordinates to the graphs of the trigonometric functions, it could be convenient if we could present the functions outside the geometric field and without dealing with the notion of angle. This is possible if we define sine and cosine in terms of either the inverse of sine, differential equations or power series. The problem with both the inverse of sine and the power series is that the mathematical knowledge needed is too difficult for high school students. Moreover the definition in terms of power series has no clear link to the unit circle definition, which could provoke student's fractional idea of the trigonometric functions. The differential equations is good as supplementary material, but seeing sine and cosine as solutions to differential equations only gives a reason for why we need sine and cosine as functions and not a better perception of these functions. So none of the three approaches mentioned would be suitable for high school. The complex numbers could be introduced, but again we would have to deal with the notion of angle, and the link between a part of a complex number and a function from $\mathbb{R}$ to $\mathbb{R}$ will also be too difficult in high school. Hence the only way to improve the students' conception of sine and cosine as trigonometric function is through a reduction of the gap between the unit circle coordinates and the graphs of the trigonometric functions. This could happen through an introduction of the natural parametrization of the unit circle. Of course all the theory mentioned in section 2.2.4 will be too much to handle. However it could be explained that in order to make the transition we need to be able to trace the entire unit circle, and this is possible if we can construct a natural parametrization of the unit circle. That is, when $t \in \mathbb{R}$ traverses an interval $I \subseteq \mathbb{R}$, the corresponding point $\gamma(t)$ will traverses an arc length on the unit circle. The simplified proof of the curve length sketched in Figure 2.7 could, together with the definition of arc length, be introduced to a (clever) high school class. The existence proof could be left out and instead one could move directly to the construction of the natural parametrization using the unit circle. Most A-level high school students should be able to follow this construction. Hence this presentation will reduce the gap, but it is not possible to remove it completely. The rest of this thesis concerns how this problem is handled in a concrete course of trigonometric functions at a Danish high school.

3. THEORY

3.1 The didactical transposition

To elucidate the problematic we will use Guy Brousseau's theory of didactical situations (from now on shortened to TDS). This theory is chosen because its foundation is that learning is a social activity, hence it is perfect as a tool for analyzing interactions between students internally and with the teacher. Before presenting the theory we will establish what a didactical situation is and how it arises. The first step is the didactical transposition from mathematics as a science to mathematics as a teaching subject. When a mathematician does research she sets up hypotheses, tries to prove them, discovers mistakes, and finds less difficult proofs and so on. But before communicating new knowledge to the public, she suppresses all these digressions and finds the most suitable way to present her work, often in an axiomatic way. She must conceal all personal reasons which lead her in these directions and contextualize all her remarks. She must find the most general theory within which the results remain valid. Thus she decontextualizes and depersonalizes her knowledge as much as possible (Brousseau, 1997, p. 22).

In the classroom the student's work is very similar to the mathematician's. The epistemological hypothesis in subject matter didactics, says that human knowledge can be formulated as answers to problems (Winsløw, 2006, p. 38). So in order for the student to gain new knowledge she must be presented with a problem to which this new knowledge is the answer. In the process towards this knowledge the student formulates hypotheses, makes proofs, exchange ideas with others and so on, all activities we also see in the mathematician's work.

To make such a learning situation possible the teacher must ensure that the student has the right tools, in terms of old knowledge, and that the problem given ensures the construction of the new knowledge. In fact the teacher has to do the exact opposite of the mathematician. She must recontextualize and repersonalize the knowledge, such that it can become the student's knowledge (Brousseau, 1997, p. 23). This is a central part of the didactical transposition. Next she must, together with the student, redecontextualize and redepersonalize the knowledge in order for it to be official knowledge.

In fact we will call the above the internal didactical transposition, insinuating that we also have an external. The internal didactical transposition is the transposition from the official teaching-based knowledge, such as curriculum and textbooks, to the didactical situation created by the teacher. The external didactical transposition is the transposition from the scientific knowledge to the official teaching-based knowledge (Winsløw, 2006, p. 19). We saw this transition in chapter 2, by first presenting the scientific approaches to the trigonometric functions and next the teaching-based approach.

3.2 The didactical (and adidactical) situation

As mentioned above, it is the teacher's job to create a learning situation where the student has the best possibilities for constructing personal knowledge. Such a situation occurs if the teacher creates a milieu where the student has the ability to start using a strategy based on her existing knowledge, but the obstacles in the milieu soon makes her realize that she has to change her strategy to adapt to the milieu, thereby creating new personal knowledge. It is important that the knowledge is drawn from the student's own interaction with the milieu, hence the teacher cannot give the answer, and it is best if the teacher is sometimes absent from the milieu. A situation where the teacher does not intervene is called an adidactical situation. If the teacher does not intervene, it is important that the milieu created has the ability to give the student feedback on whether she is right or wrong. The extent to which a milieu has this ability is called the adidactical potential. The word "adidactical" does not indicate that the situation has no didactical intention. In fact there is a lot of work connected to devolving an adidactical situation that provides the student with the most independent and most fruitful interaction possible. The teacher must decide which information she needs to communicate to the student, what questions she should ask, and what method is the best, and so on. Thus the teacher is involved in a game with the system of interaction between the student and the milieu provided by the teacher. This game, or broader situation, is the didactical situation (Brousseau, 1997, p. 31).

Winsløw (2006) describes the learning situation as a combination of two games: The student's interaction with the milieu created by the teacher in order to personalize the official knowledge and the teacher's work initiating the student's game and afterwards making the results official knowledge. Winsløw calls this the didactical double game (Winsløw, 2006, p. 137).

3.3 The five phases of TDS

The didactical game in a learning situation can, according to TDS, be divided into five phases. Later on we will analyze specific learning situations drawing on TDS, hence it is essential to look at these five phases. As an example to illustrate the phases we will use the puzzle problem described by Brousseau (1997). The students get a puzzle where the pieces are as shown in the figure:

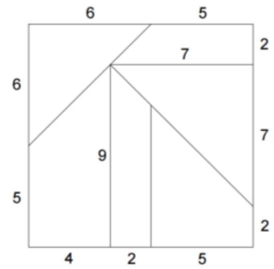


Figure 3.1: The puzzle presented to the students (retrieved from Måsøval, 2011, p. 29)

The task is to enlarge the puzzle such that the sides measuring 5 cm on the given puzzle will measure 8 cm in the new puzzle. The target is to extend the students' knowledge about proportionality. Let us consider the phases of the learning situation.

3.3.1 Devolution

The teacher introduces the task and explains the rules of the game. In our example she could decide that the students shall work in groups of four and that every student must enlarge at least one piece of the puzzle. She must also make it clear that the task is not solved until the new pieces fit together. In this introduction the teacher devolves the didactical milieu to the students. Hence the students also have a responsibility for understanding the given task. They can ask clarifying questions which the teacher can answer to make sure everyone knows what is going to happen. The teacher makes the students accept the responsibility for the coming (adidactical) learning situation and accepts the consequences of this transfer of responsibility (Brousseau, 1997, p. 230). The devolution is a didactical situation.

3.3.2 The situation of action

In this phase the students interact with the milieu, they play the game. An expected observation would be that the students add 3 to all sides of their puzzle, but when trying to assemble the puzzle they will see the pieces are not compatible. Hence the students must change their strategy in order to adapt to the milieu and win the game. The teacher does not intervene in this phase; hence we have an adidactical situation. But if the obstacles from the milieu seem too big, the teacher can devolve a modified milieu (Winsløw, 2006, p. 138). In our example she can display a table on the blackboard showing the side lengths in original and in enlarged puzzle:

Original piece	2	4	5	6	7	9
Enlarged piece			8			

Figure 3.2: Side lengths in original and enlarged puzzle (Retrieved from Måsøval, 2011, p. 30)

This could make the action look clearer and lead the students to argue something like: "If 5 becomes 8, then 1 must become 8/5. Hence each side length must be multiplied by 8/5". This is a formulation of a strategy to win the game, which lead us to the next phase.

3.3.3 The situation of formulation

In this phase the students formulate and compare their observations and different strategies to win the game. The knowledge developed in the situation of action is personal knowledge, but here the knowledge needs to be shared, hence the students must try to formulate the strategies developed. The teacher may re-enter the game here in order to make sure that the formulations are available to all. Hence this phase is often didactical.

3.3.4 The situation of validation

Now we have one or more explicit strategies to win the game and the aim is to validate these strategies. This is done through argumentation and sometimes experimentation. Brousseau (1997) claims this phase is about establishing theorems in a broad sense. To state a strategy as a theorem you must confirm that what you say is true. The students must be able to use the mathematics as a reason for accepting or rejecting a theory (Brousseau, 1997, p. 15). They need to be able to make a kind of proof. This ability to prove is not innate, but developed out of the desire to convince others about the truth of a statement or property. As Brousseau (1997) asserts*:*

> *In mathematics, the "why" cannot be learned only by reference to the authority of the adult. Truth cannot be conformity to the rule, to social convention like the "beautiful" and the "good". It requires an adherence, a personal conviction, an internalization which by definition cannot be received from others without losing its very value. We think that knowledge starts being constructed in a genesis of which Piaget has pointed out the essential features, but which also involves specific relationships with the milieu, particularly after the start of schooling. We therefore consider that for the child, making mathematics is primarily a social activity, not just an individual one (Brousseau, 1997, p. 15).*

Thus the construction of mathematical knowledge emerges through the conviction of the truth and this conviction is ultimately socially established. The teacher can decide whether she wants the students to validate each others' strategies or if she will do the validation herself.

3.3.5 Institutionalization

In the previous phases the knowledge to be taught has functioned as the solution to a problem given to the student under conditions which allow her to find the solution herself. In this phase the personal contextualised knowledge constructed in the previous phases is made public by a decontextualisation. The teacher is the main actor here. She presents to the students the official formulations, the definitions and theorems considered important for the contextualised knowledge to gain the status of cultural knowledge (Måsøval, 2011, p. 53). As a part of the institutionalization the teacher may present the students for the same problem, but in a new context to make sure that the knowledge is decontextualized. In the example with the puzzle the teacher must state that similarity is a multiplicative structure. To enlarge a figure you must multiply each side by a fixed factor. The decontextualisation can be made by presenting the students with another puzzle.

It is important to establish that these five phases do not always come in the order above. And sometimes the same phase appears more than one time within a learning situation. For example in the situation with the puzzle the situations of action and formulations can be repeated to enable and strengthen the validation. Just imagine that the teacher decides to validate one of the formulations by letting the students create new pieces.

3.4 The milieu

Above, the milieu has been mentioned without really defining it. We will do that now. As mentioned above the teacher devolves a learning situation to the student consisting of the student and a milieu which the student can act on. This milieu consists of the problem to be solved and the tools available to do so. The milieu is a subset of the student's environment consisting only of those features that are relevant for learning a given piece of knowledge (Måsøval, 2011, p. 55). This means that the students do not construct the knowledge of algebra in interaction with the same milieu as the milieu in which they construct the knowledge of probability theory (Brousseau, 1997, p. 23). Moreover, the milieu is not stable but changes throughout a learning situation. Briefly explained the situation of action is the milieu for the situation of formulation and so on. This is illustrated in the figure below in terms of the different roles of the student (S) and the teacher (T).

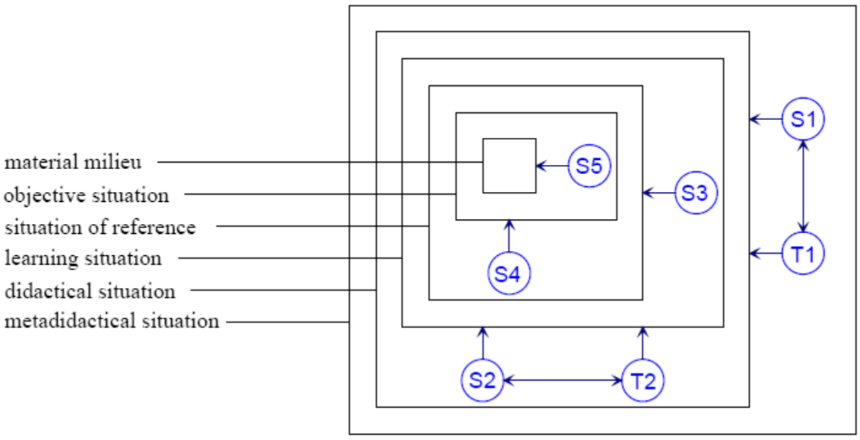


Figure 3.3: The different roles of the student and the teacher in the didactical milieu (Retrieved from Måsøval, 2011, p. 56)

Let us look at it from the inside and out. First we have the objective situation consisting of the mathematical problem given to the student. The milieu for the objective situation is the "reality" in the problem and the student (S5) interacting with this milieu is a hypothetical person, the person who is in the problem, for example: "A group must enlarge a puzzle..." The objective situation is the milieu for the situation of reference, where a student (S4) tries to solve the problem. The next level is the learning situation where the student (S3) tries to reflect on the situation with S4 trying to solve the problem. Here the situation of reference is the milieu. S3 and S4 can be the same student. Next we have the didactical situation where a student (S2) and a teacher (T2) act on the learning situation as a milieu and finally we have a metadidactical situation where a student (S3) and a teacher (T3) have the didactical situation as the milieu to act on. It is in this situation that the teacher constructs her lesson and it is also here that the rules between the teacher and student are negotiated, i.e. the didactical contract is created (see section 3.5). If we compare this figure with the five phases of TDS we may say that the objective situation is the milieu devolved by the teacher, the situation of reference can be associated with the situation of action, where the student acts on the problem and the learning situation can be associated with situation of formulation. The didactical situation is the situation where the teacher may appear for the first time, and this can be associated with the situation of validation. It is here one reflects on what has been learned.

The teachers fundamental task is to design a milieu in such a way that the student is able to interact with it, but the milieu also has to challenge the student so the student has to think in new ways to adapt to the milieu, because knowledge is gained by adapting to a given milieu (Brousseau, 1997, p. 30)

As a part of the objective milieu we will in this thesis focus on the graphic milieu. That is the part of the milieu involving graphs. It differs from the rest of the milieu by giving the students a visual idea of the task they are handling. In this thesis the graphic milieu is important because trigonometric functions cannot be represented by a simple analytic expression, hence the graphs are needed.

3.5 The didactical contract

In order to make sure that the game in the learning situation is won by the students, i.e. the students learn something, it is important that both the teacher and the students participate in the game and follow the rules of the game. These rules form the didactical contract between student and teacher, and this contract is often implicit. To describe the didactical contract we will use the structure used by Hersant and Perrin-Glorian (2005). They distinguish between four dimensions of the didactical contract: The domain of the knowledge, the didactic status of the knowledge, the characteristics of the didactic situation and the distribution of responsibility (Hersant & Perrin-Glorian, 2005, p. 118). These dimensions are not independent, especially the three last ones depends on each other, as we will see later.

The domain is a dimension in the contract because certain techniques will be favoured and others will be improbable within a certain mathematical field. The teacher can change the domain by referring to a mathematical field the students had not thought of, or she can translate the problem to a new mathematical context, hereby enhancing the learning (Ibid., p. 118). The second dimension, the status of the knowledge, can be distinguished as entirely new knowledge, entirely old knowledge or knowledge in development. This dimension is related to the distribution of responsibility, because the teacher can leave more responsibility with the students in the case of old knowledge, than in the case of new knowledge. But if the status of knowledge is new the teacher still has the possibility to delegate the responsibility to the students if the situation has a milieu endowed with an adidactical potential. Hence the distribution of knowledge is depending on both the status of knowledge and the characteristics of the didactical situation.

Besides the four domains, Hersant and Perrin-Glorian (2005) distinguish three levels of the didactical contract: The macro-, the meso- and the mico-contract. The macro-contract is mainly concerned with the teaching objective or the target knowledge, the meso-contract is concerned with the realization of an activity, e.g. solving an exercise and the micro-contract is concerned with a concrete episode, e.g. a specific question in an exercise.

On each level some of the domains stay stable, but only at the level of micro-contract they all stay stable and by contrast only few or none of them stay stable at the level of macro-contract. So in order to analyze the macro-contracts we must start by analyzing the micro- and the meso-contracts. The micro-contracts are defined mainly by the distribution of responsibility between teacher and student and the meso-contract is deduced from two dimensions: The existence of a milieu with adidactic potential and the status of knowledge at stake (Ibid., p.120). The structure of the didactic contract is illustrated in Fig. 3.4.

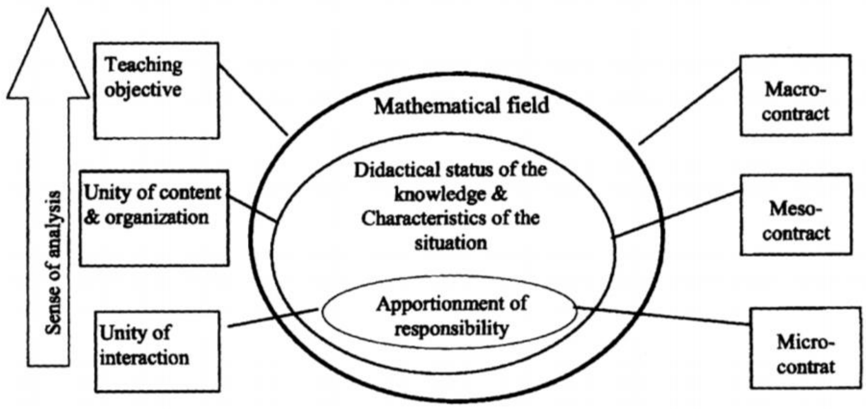


Figure 3.4: Structure of the didactic contract (Retrieved from Hersant and Perrin-Glorian, 2005, p. 120)

In this thesis we will mainly analyze small episodes with associated micro-contracts in order to reach the larger meso-contract in each situation. Hence the distribution of responsibility is very important. According to the knowledge at stake the teacher can delegate the responsibility in didactic situations of formulation, validation and the institutionalization in two different ways, creating two different micro-contracts:

<u>Micro-contract of collective production:</u>

Here the knowledge at stake is available for most students. The teacher asks questions and most students raise their hands to answer and the teacher allows one student, then another one and so on. The teacher may also ask a specific student to check her knowledge; others wait and may intervene in case of an error. This is a micro-contract of individual production, inside the micro-contract of collective production (Ibid., p. 136).

<u>Micro-contract of agreement:</u>

Here the knowledge at stake is only available to a few students. The teacher relies on these few students and let them speak and the rest of the students are then supposed to agree more or less tacitly.

3.6 Effects of the didactical contract

The didactical contract is the implicit rules by which we can win the didactical game, but sometimes the teacher's eager attempts to fulfil her part of the contract results in situations where the game is "won" without the students gaining the target knowledge. Here we will look at two of these effects of the didactical contract.

3.6.1 The Topaze effect

This effect appears when the teacher, in an attempt to avoid mistakes from the students, gradually reduces the difficulty of the task, such that she in the end more or less just gives the students the answer (Winsløw, 2006, p. 148). This effect can be seen for instance when the teacher uses very leading questions.

3.6.2 The Jourdain effect

This effect appears when the teacher agrees to recognize the indication of a scientific insight in the student's behaviour or answer even though the student is just following the instructions from the teacher (Ibid., p.148). This effect can be seen in the work with CAS-tool were the students just follow a guide and click on the right button. In the end it is concluded that the student has now learned to make regression, for example.

3.7 Obstacles in the teaching

The best learning situation appears when the student works in a milieu that provides her with some obstacles she must overcome to reach the target knowledge. Hence it is an important part of the a priori analysis to realize what obstacles the student may run into. In the following we will consider some of the obstacles. TDS distinguishes between three different types of obstacles:

1. Obstacles of epistemological origin
2. Obstacles of ontogenic origin
3. Obstacles of didactic origin

An epistemological obstacle is when old knowledge struggles with new knowledge (Warfield, 2006). As an example we can look at a child's knowledge concerning multiplication of numbers. The child knows that multiplying two numbers gives a bigger number, but this knowledge becomes false when the notion of numbers is expanded to include fractions and negative numbers. As Camilus (2012) explains the epistemological obstacles are related to the nature of the knowledge, hence they are built into a piece of knowledge independently of the student, teacher and situation (Camilus, 2012, p. 13). Often one uses a historic approach to identify epistemological obstacles and in this thesis the aim is that the subject-matter didactic analysis could help identifying some of the epistemological obstacles.

The ontogenic obstacles are related to the student's cognitive level. It is not possible to teach a child to add and subtract before she has learned to count. In this thesis we will not focus on the ontogenic obstacles.

The didactical obstacles are those who are produced by the teaching system. They can originate from the curriculum, the textbook, the teacher, the methods etc. As an example we can use the number π. In the early years of school students learn that $\pi = 3{,}14$. The teacher tells them that, and it might even figure in the textbook. When they reach higher levels they have to think of π as the ratio between diameter and circumference in a circle. The "old knowledge" could lead to a didactical obstacle to the "new knowledge".

4. RESEARCH QUESTIONS

The purpose of this study is to examine how students in Danish high schools make the transition; from working with sine and cosine as tools to investigate (essentially) triangles to working with them as functions. Hence a fundamental question is what students acknowledge as a function, and as we saw in the subject-matter didactic analysis this question does not have a clear answer. Thus the first research question is:

RQ1: What didactical strategies does the teacher deploy to help the students make the above transition?

The chapter concerning the subject-matter didactics shows that the teaching of trigonometric functions always will involve graphic milieus in some way. Therefore the next two questions are asked:

RQ2: How are graphic milieus presented to the students?

RQ3: What role do the graphic milieus play in the students' transition? In particular what adidactical potentials are found in the milieus of the episodes we observe?

The didactic literature on trigonometric functions all agrees that students have difficulties in the above transition (Weber, 2005; Demir, 2012; Orhun, n.d.). Hence the research questions could lead up to a discussion whether the role of the graphic milieu could be changed in order to make the transition from geometrical tool to function easier.

5. METHODOLOGY

In order to answer the research questions, I have followed a third year A-level class in their course on trigonometric functions at Gefion High school in Copenhagen. The course lasts four lessons and the content of the lessons is described in chapter 6. In this chapter the method for data collection and data analysis are described.

5.1 Method for data collecting

Prior to the course I received the teachers' plan for the course together with a worksheet which was handed out to the students. This material together with the observations made during the four lessons is the data used for answering the research questions. Before each lesson I placed a Dictaphone on the catheter, one in the middle of the classroom, one in the back and finally one on myself. In this way I could compile all conversations, but most important I was sure that the conversations I overheard were recorded. An exception was the last lesson. Here the students made group work throughout the whole lesson; hence I had Dictaphones placed at three groups and one on myself. The sound recordings have been useful to capture the dialogues between students, but also between students and teacher. This has eased the process of identifying micro-contracts in the different episodes.

During whole class teaching I sat in the back and made notes. From here I could capture the choices made by the teacher, including what was presented on the smart board, which students was asked, the type of work mode and how much time was devoted to the different tasks. When the students worked in groups I walked around between them intercepting their conversations. When I found something interesting I took pictures of their computer screens or even sometime videos. This visual data has been useful to understand what the students were looking at through their discussion and it has been useful to identify the visual feedback from the milieu. Throughout the course I tried to make my presence as invisible as possible, in order to make the data as realistic as possible. Only few times did the students address me in person and often it was just to ask about my project or my education.

5.2 Method for data analysis

I have selected three different situations to be analyzed. First a situation where the students work in groups with sine regression, next a situation consisting of both group work and institutionalization of the constants in the formula for the general sine curve, and finally a situation where a group of students tries to institutionalize sine curves in general. The first situation is chosen to show how obstacles of the milieu sometimes can be too big. The second situation is chosen to show the interaction between students and teacher and the last situation is chosen to identify how much the students have learned through this course.

The situations are divided into small concrete episodes determined by a specific question such as “why does the curve look differently than we expected?”. The dialogues in these episodes help us identify the micro-contracts. The aim of this study is to examine the students’ perception of sine and cosine as functions. This is done by analyzing their interactions, both internally and with the teacher. That is why TDS is chosen as the theoretical tool. The term didactical contract reveals the responsibility between student and teacher, hence the status of the knowledge at stake. The status of the milieu also reveals the status of the knowledge at stake and moreover the student’s interaction with this milieu reveals something about the knowledge available for the student.

In the transcriptions and translations of the dialogues I came across spoken language and phrases not possible to translate from Danish to English. Hence I have given myself the liberty to reformulate some sentences making sure not to lose the content. I have tried to make the transcriptions as realistic as possible and since they represent the spoken language some of them may be confusing to read. I believe this will not influence the understanding of the content.

6. THE COURSE OF STUDY

To get insight into the teaching of trigonometric functions in a Danish high school, I have followed a third year A-level class in their course on trigonometric functions. The class was composed by students from two different classes. The two first years, the students have had mathematics on level B, but now they have chosen to upgrade to A-level. The teacher has taught half of the class the first two years, whereas the other half is new to her. The first half of the class is used to work with the CAS-tool Nspire, because this is the tool the teacher prefers, whereas the other half is used to work with the CAS-tool Maple. Even though the teacher is not familiar with Maple, she has chosen to let the students themselves decide which CAS-tool they will use. I have followed the class in 4 lessons and present an overview in Table 6.1.

Lesson	Teaching targets[3]	Structure of Lesson
1 (90 min)	Applications of the trigonometric functions Use the notion of radians Try to work in practice with the trigonometric functions	First a short introduction to periodic functions. Next group work on radians using a worksheet[4] handed out by the teacher. Then an introduction on the smart board to sine and cosine to a number instead of an angle. In the last half an hour the students work experimentally by exploring how a point on a circle moves, when the circle rolls forward.
2 (90 min + 55 min)	Draw the graphs of the trigonometric function using the CAS-tool Argue for periodicity Do sine regression using the CAS-tool Create a model by using a trigonometric function	First a recap of last lessons activity of examining a point's movement on a circle. Next the drawing of sine, cosine and tangent, first in groups and next in plenum on the smart board followed by a discussion of periodicity. Then an introduction to makes sine regression in Maple and Npsire,

[3] The teaching targets are the targets set by the teacher for each lesson. In Appendix 2 they are formulated by the teacher, but she changed her plan during the course, so the targets in the Appendix and the ones here are not identical.

[4] The worksheet is presented in Appendix 1.

		followed by group work were the students make sine regression on the data collected last time. The last one and a half hour the students work with the function class $f(x) = a\sin(bx + c) + d$ to create a model of the Ferris wheel in Tivoli.
3 (90 min)	Differentiate and integrate expressions containing the trigonometric functions.	First the students are asked to refresh the knowledge concerning the four constants in the expression for a general sine curve. Next the students are asked to find the derivative of sine and cosine using their CAS-tool and share the result in plenum. The same procedure is used for finding the antiderivative. The rest of lesson is used to solve exercises looking like the ones in the written exam.
4 (90 min)	Communicate your knowledge of the trigonometric functions to your classmates.	This lesson was a self-study lesson, and the students were asked to create documents for their own homepage[5]. The teacher divides the students into groups, gives each group their own topic concerning the trigonometric functions and leaves the class. The rest of the lesson the students work on their documents in groups.

Table 6.1: An overview of the course of study

[5] The homepage can be viewed at: http://www.infogeist.dk/html/gefiongymnasium/infogeist-1oma/matematikbogen2/topic_1.html

7. ANALYSIS OF SITUATION 1

7.1 Context

This situation takes place in lesson 2. Lesson 2 was actually two teaching periods divided by a short break, and it consisted of a variety of assignments and a lot of changes in the work mode. Therefore the lesson is divided into different parts depending on the work mode. The situation we here will consider is marked with yellow in Table 7.1. The situation marked with orange will be analyzed in next chapter.

Duration (min:sec)	**Activity**	**Work mode**
1	Repetition of the term "radians"	Whole class teaching
7	A recap of last lessons activity of examining a point's movement on a circle, when the circle is moving forward. In particular a determination of the relationship between diameter, amplitude and period.	Whole class teaching
11	Drawing sine, cosine and tangent in the CAS-tool	Individual work
5:30	Drawing sine, cosine and tangent on the smartboard	Whole class teaching
5	Group discussion about the relations $\sin(x) = \sin(x + p\ 2\pi)$ and $\cos(x) = \cos(x + p\ 2\pi)$	Group work
5:30	A recap of the discussion together with a discussion of the term "periodicity"	Whole class teaching
2.30	Introduction to make sine regression in Maple and Nspire	Whole class teaching
18.30	Making sine regression on the data collected last time	Group work
7:30	Presentation of exercise 8 in the work sheet: The Ferris wheel in Tivoli. Including a presentation of the function class $f(x) = a\sin(bx + c) + d$	Whole class teaching
7:30	Attempt to determine the values a, b, c, and d in the concrete exercise with the Ferris wheel.	Group work
1:30	Determination of the values a and d	Whole class teaching
2	Introduction on how to determinate the four constants' impact on a sine curve, by drawing different sine curves where one constant is changed at a time.	Whole class teaching

14:30	The determination of the four constants' impact on a sine curve, by use of the CAS-tool	Group work
5	Break	
6	Continued group work with the determination of the constants' impact.	Group work
11	Collective determination of the four constants' impact on a sine curve	Whole class teaching
Approx. 5	Determination of the value *b* in the concrete exercise with the Ferris wheel. Hereafter the students are free to go.	Group work

Table 7.1: An overview of lesson 2

In the previous lesson the students did an activity where they rolled a wheel along a table and measured the corresponding x- and y- values, and next plotted them in either Nspire or Maple. The target for the students was to get insight into what a circular movement looks like graphically. In the present situation the assignment is *to do sine regression on the data collected last time by the use of a CAS-tool*. The teacher has introduced how to do it both in Nspire and Maple and the students are now supposed to do it using the data they collected last time. Prior to the observed situation the students have drawn the graph of the trigonometric functions both individually in their CAS-tool and in plenum, with the teacher as "the secretary" writing on the smartboard. The result is shown in Fig. 7.1.

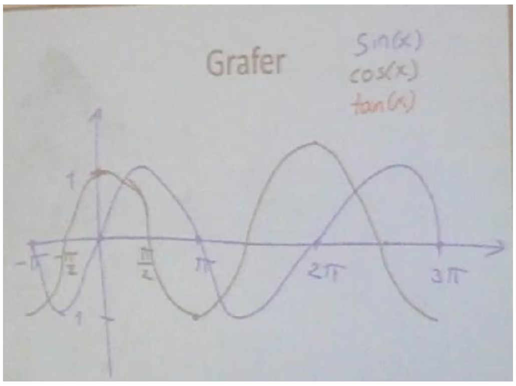


Figure 7.1: A picture of the smartboard (tan(x) has not yet been drawn)

The teacher introduces the task by saying: *"Now let us return to the wheel we rolled. We shall try to figure out whether it is a sine function. Is it possible to get something which looks like these curves? [Points at the smartboard] "*. This tells the students that the result of the sine regression must be a function represented as a graph, which looks like one of those in Fig. 7.1.

7.2 A priori analysis of situation 1

Before examining what went on in the situation we will make an a priori analysis, determining the target knowledge of the situation, what knowledge the students possessed and the content of the milieu they were working in. Let us first consider the task "to do sine regression on the data collected last time by use of the CAS-tool". A problem arises, because the task from last time was misunderstood by the students resulting in data not modeling a sine curve but a cycloid. The teacher was not present in the last part of the previous lesson and therefore not able to correct them. In the worksheet the task was formulated as: *"We will examine how a point on a circle moves as the circle moves forward. [...]For each circulation you must have 7-10 values of x and y. Measure 2 to 3 circulations."* (Appendix 1, p. 2). The problem is that the students interpret the x- value as the x-coordinate to the point on the circle, whereas the x- value is in fact the traveled distance of the circle. The difference is illustrated in the Fig. 7.4.

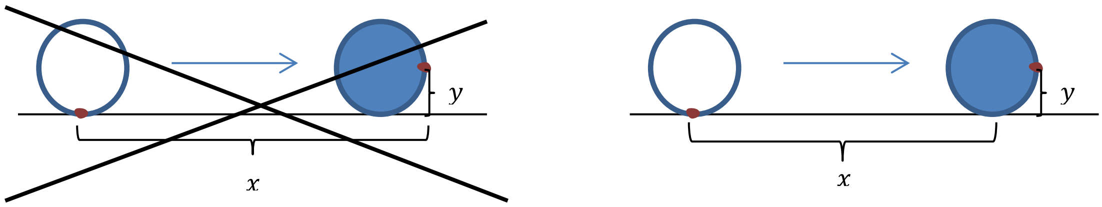


Figure 7.2: Two different interpretations of the task in exercise 4 in the work sheet.

The two different interpretations lead to two different graphs. The former gives a cycloid whereas the latter gives the desired sine curve.

7.2.1 Target knowledge

The teacher's intention with the task is to make the students comfortable with doing sine regression in a CAS-tool. The target knowledge is to learn the instrumented techniques attached to sine regression and to be able to validate the result. In this situation we will observe the students' work in Nspire. Here they choose the application "lists and spreadsheets" and plug in their data in two lists. Hereafter they choose the tool "statistics" and next the button "sine regression". The result could look like:

	A dist	B height	C	D	E
=				=SinReg(dist,heigh	
1	0	0	Titel	Sinusregression	
2	2	0.8	RegEqn	a*sin(b*x+c)+d	
3	4	2.5	a	2.20129	
4	6	4.1	b	0.447185	
5	8	4.3	c	-1.65094	
6	10	3	d	2.22157	
7	12	1	Resid	{-0.02734305598...	
8	14	0			

Figure 7.3: An example of sine regression in Nspire

This looks very confusing, and since the students have not been introduced to the expression for a general sine curve they will not be able to compare the values of the constant in the expression with the graph on the smartboard. Moreover the teacher implicitly said that the result of the regression would be a graph looking like those on the smartboard, hence the students are not done yet. To get a graphic representation they choose the application "Diagrams and statistics", which plots the data from the two lists in a coordinate system. From here the students can press a button called "examine the data" and next "show sine regression". The result is a sine curve approximating the measured points and an analytic expression for the curve:

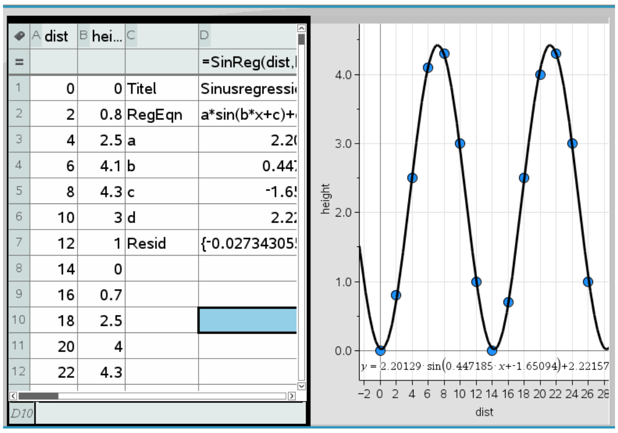


Figure 7.4: An example of sine regression in Nspire

The students are not supposed to learn the theory behind sine regression, but they know that regression is a method for finding the best function (within a given class of functions) to model a given data set. Graphically this could be rephrased to find the best curve (of a given type) through the given points. Hence they can validate their result visually just by looking at the graph and the measured points; if these coincide the regression must be good. The comparison with the graph on the smartboard could lead to some obstacles, since the students may expect their result to be exactly as the graph of $\sin(x)$. The realization that the result of sine regression is not, necessarily, the function $\sin(x)$, but an element f in the class of functions $\mathcal{F}_{trig}$ (section 2.1), is also a part of the target knowledge.

7.2.2 Students' knowledge

The students have been working with CAS-tools for over two years and they have previously learned how to do linear regression in Nspire. The sine regression only requires the use of the "sine" button instead of the "linear" button, so the knowledge concerning the instrumented techniques is essentially old knowledge.

The knowledge concerning sine regression must be considered entirely new knowledge. The students have just been introduced to sine as a function and so far all they have seen is the graphic representation. The teacher has no intention of explaining the theory behind sine regression, and the students know that linear regression gives the linear curve that best models the data set given; hence they may conclude that sine regression gives the curve of $\sin(x)$. Also the graphs on the smartboard give the students the idea that the result of the sine regression is the graph of $\sin(x)$. This is not the case since the result of the sine regression is a general sine curve (see p. 11), which has not yet been introduced to the students. Hence also the graphic representation of sine regression is considered new knowledge.

The fact that the students must validate a result as being a sine curve, without knowing what a sine curve is, must be an obstacle to them. It is didactic because it is the teacher's choice to introduce sine regression before sine curve. The intention can be to create this obstacle on purpose in order for the students to extend their knowledge concerning sine curves.

7.2.3 The Milieu

The objective milieu consists of the problem to be solved, in this case "Use Nspire to make sine regression on the data collected last time" and the tools available for solving this problem. The primary tool is Nspire. By using the teacher's guide for sine regression the students will be giving the analytic expression $f(x) = a\sin(bx + c) + d$, which they can plot to get a sine curve. Besides Nspire, the milieu is also supported by the graph of sine on the smartboard. The students can easily compare the graph in Nspire with the one on the smartboard. Hence the graphic part of the milieu has a good feedback potential.

Another element of the milieu is the data collected last time. Since the data does not model a sine curve it may be difficult to get a graph, which approximates the measured data, and this could be an obstacle when the students have to validate the result of the sine regression. The obstacle is didactic because it originates from the formulation of the task from the previous lesson.

Doing sine regression on the data will create a sine curve, even though the data should be modelled by a cycloid. If a sine curve does not oscillate around the x- axis, the students may reject it as a correct result because they would expect it to look like the graph of sine presented on the smartboard. The idea that al sine curves look like the sine function may be an obstacle. Moreover the evident black box status of the sine regression may also be an obstacle. If the feedback from the milieu does not show what the students expect, the students would normally develop their old knowledge to adapt to the milieu, but here the students do not have any knowledge concerning the sine regression, so all they can do is change their input in Nspire. Even though the graphic part of the milieu has a good feedback potential, the adidactical potential in the milieu is weakened by the student's lack of knowledge about what the graphic milieu represents.

7.3 A posteriori analysis of situation 1

The a posteriori analysis will build on dialogues in the class room. The situation we observe involves a group, working in Nspire, consisting of a girl and three boys, in the dialogues named boy 1, 2, 3 and a girl from another group named Kimmi, plus the teacher. Uninteresting parts are left out of the dialogues and marked by //. The entire dialogue is presented in appendix 3. The students in the group have done the regression, and drawn the graph together with the points from last time:

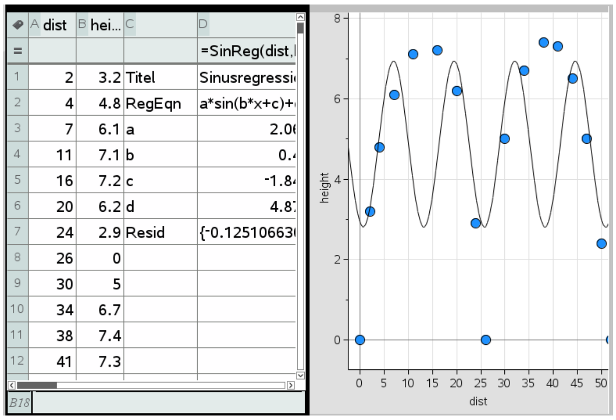


Figure 7.5: A picture of the students' screen

The situation we will observe is supposed to be adidactical. The teacher has devolved a milieu where the students should be able to do sine regression, and get an appropriate result. But as we will see the students find themselves unable to adapt their knowledge to the milieu, and the teacher has to intervene making the situation didactical. Our analysis will concern, first the adidactical part of the situation, and next the didactical part. The girl, Kimmi, from the other group comes to look at the boys' result:

1. Boy 1 We can't make it work.
2. Kimmi Oh. You can't make it work. But there is one missing.
3. Boy 1 We have our points here, but the graph doesn't want to go to zero.
4. Kimmi It is the same problem with ours. Our regression… It just says that in Nspire it will not do it apparently. It must be because it doesn't approximate.
5. Boy 2 But it is not even close to the points if you see.
6. Kimmi Oh. Can I see?
7. Boy 2 It is almost the opposite of the points.
8. Boy 1 Doesn't it look like it is the half of the points?
9. Boy2 Yes, that's what it does.
10. Kimmi What? Is it the half of the points? [Go gets her own computer]
11. Boy 2 [Looks at girl's computer, which shows a graph not hitting the x-axis, but with the same period as the points plotted.] Okay, yours also looks weird.
12. Kimmi So ours up here. Here it works fine enough...

The first quote tells us that the boys believe that their graph is a bad model of the data points. Looking at their graph (Fig. 7.5) we must agree. In quote 2 we do not know what Kimmi means when she says that there is one missing, but maybe she refers to the fact that the graph shows the point (0,0) as a measured point, whereas this point is not visible in the list of data. This is because the boys did not realize that the movement started in the point (0,0) before typing in all the points, so they just added the point (0,0) in the end. In quote 3 we see that boy 1 expects the graph to approach the x-axis. This is either because he knows that regression makes the best fitted curve through the measured points, and since the point plot shows three points on the x- axis he would expect the curve to approach these, or it is because the graph of $\sin(x)$ oscillates around the x-axis, and his expectations to sine regression is a result looking like the graph of $\sin(x)$. Kimmi informs the boys that they are not the only ones with that problem. Her explanation is that it is because the curve does not approach the points on the x- axis, hence she knows that regression is about finding the curve that models the data points best. Boy 2 is also aware of this, because he questions the fact that the curve is not even close to the points. In quote 7-9 boy 1 and 2 agrees that the curve is half of the points (Fig. 7.5). They do not use the term "period" in their formulations. Even though trigonometric functions was introduced as periodic functions, and the students have had time to work with the connection between the unit circle and the period of $f(x) = \sin(x)$, the knowledge concerning periodicity is still new knowledge. In quote 11 the boys see Kimmi's graph, which may look like the one in Fig. 7.6.

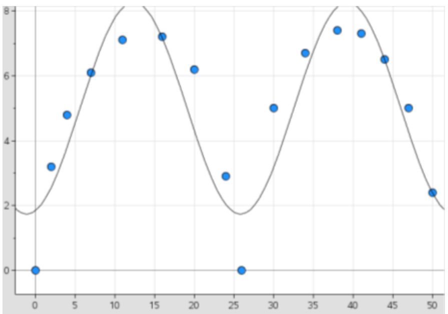


Figure 7.6: A suggestion to how Kimmi's graph may look like

Even though the period here is correct, boy 2 still thinks that the curve is wrong. It shows that the curve's closeness to the x- axis is an important element in the students' validation of the curve. Kimmi, on the other hand, accepts her graph to be correct even though it does not approach the x-axis. Either she knows that a general sine curve does not have to oscillate around the x-axis, or more likely she just accepts that the points on the x-axis are too far away for the curve to approach these, just like for linear regression, where there could be "outliers".

The entire dialogue shows students trying to explain what they see, but no one tries to change anything. This is probably because the students' knowledge concerning the mathematical theory behind sine regression is insufficient in order for them to adapt to their situation. They do not know what to change.

A few minutes later the teacher comes by and the struggle with the period continues. The students try to do regression one more time, but this time with the point (0,0) as the first data point in the list, in contrast to earlier where they had just typed it as the last point. This makes the period of the graph change, so it becomes the same as the period of the data points (Fig. 7.7). But the group is still not satisfied:

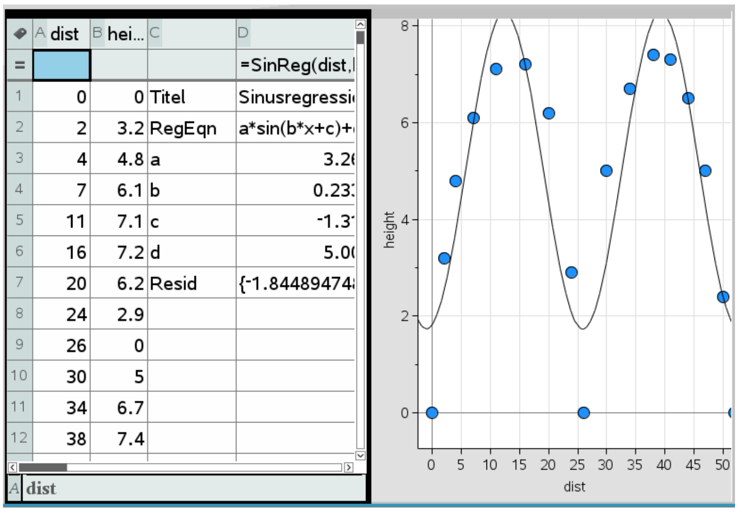


Figure 7.7: A picture of the students' screen

1. Boy 2	It still doesn't hit zero.
2. Kimmi	It still doesn't hit. Look we get the same [shows her computer to the teacher]
3. Teacher	But what is the period here? It matches. It has two arcs.
4. Boy 1	Yes, that is correct. It matches.
5. Teacher	Yes. The other thing is just because you have chosen exactly one above, that's why it doesn't come down. But this one, it matches [points at the graph]. //
6. Teacher	It isn't because you used a ruler there didn't started in zero, but a little bit over?
7. Boy 1	No, we used my ruler and it added 0,6, so we subtracted 0,6 from all points.
8. Kimmi	Maybe the paper has been moved a bit.
9. Boy 2	Yes, there are a lot of sources of errors.

Quote 1 shows that hitting the x-axis is still important for the students. In quote 3 the teacher wants the students to focus on the period and in quote 5 she says that the reason why it does not hit the x-axis is that the students must have added 1 to all the points besides the ones where the wheel hit the table. This seems a bit confusing, but she probably means that it is possible to choose whether the graph should hit the x-axis or not, just by adding or subtracting a given number to all your points, thus implicit also saying that sine curves do not have to hit the x-axis. At the end she validates the students graph by saying that the new curve matches the measured points. In quote 6 the teacher tries to find reasons why the students' graph does not hit the x-axis. She tries to help the students in their attempt to get the graph down to the x-axis, but one could question whether this would help the students reach the target knowledge; to do sine regression and validate the result. Maybe the teacher believes that if the students see the graph oscillating around the x-axis, they would accept it as a correct result and hereafter they will be able to see how the graph can be moved in the direction of the y-axis without changing its character as a sine curve.

It is worth noticing that the teacher does not point out that the data collected does not model a sine curve. This may be because she has not realized that the task from the previous lesson could be misunderstood. Most likely she realizes that the data collected does not model a sine curve when she looks at the students' screen. She may think that this is due to measuring mistakes made by the students, but to redo the task from last lesson will take too much time and she believes that the target knowledge of the present situation can be reached even though the data does not model a sine curve.

The teacher has in this situation devolved an adidactical milieu leaving much of the responsibility for learning to the students. This is accepted by the students because the primary work is technical work in Nspire, which should be old knowledge. But since the mathematical knowledge in the situation is entirely new the students do not have the tools to change their actions. The meso-contract of the situation is the technical action in the CAS-tool and a part of the meso-contract is that if obstacles occur the teacher must intervene, because the knowledge to be taught is still just recently being introduced.

8. ANALYSIS OF SITUATION 2

8.1 Context

This situation also takes place in lesson 2. It is marked with orange in Table. 7.1. Since the work mode changes in the situation we observe we will consider the situation as two phases (not phases in the sense of TDS). The first phase is just before the break and we observe a group working with sliders in Nspire, this phase is called phase 2A. Next we observe the teachers institutionalization of the four constants, this is called phase 2B. The assignment in the situation is to *determine how the four constants impact on a sine curve*. The teacher shows the importance of this by giving the students an exercise where they need to know the constants' impact. The exercise given is:

8. The Ferris wheel in Tivoli has a diameter of 15 m. One round takes 15 sec. The tour starts 5 m above the earth.
   a) You have to figure out how the height of the basket you are sitting in changes in time.
   b) How high above the earth are you after 20 sec.?
   c) At what time are you 17 m above the earth for the first time?

   (Appendix 1, own translation)

The transition from the group work on sine regression to the presentation of exercise 8 is a monologue by the teacher:

> *"One can see that every time something oscillates like this [refers to the data, which the students have just made sine regression on] then one can describe it as a function like $a\,sin(bx + c) + d$ [writes the formula and a sine curve on the smartboard] In everyday language one calls it a sine curve, something waving like this [points at the curve on the smartboard]. This is regardless of how the period is, how big the difference is and how it is displaced in both the direction of x and y. Then one will get an expression like this. That is why, when someone has something looking like this and one has points [draws points on the sine curve] from a collection of data, then one will be able to do sine regression and get the expression. But sometimes you have a verbal description and from this you can produce the expression. That is what we are going to do now."*

When presenting exercise 8 the teacher draws the information on the smartboard together with the information drawn during her monologue (Fig. 8.1):

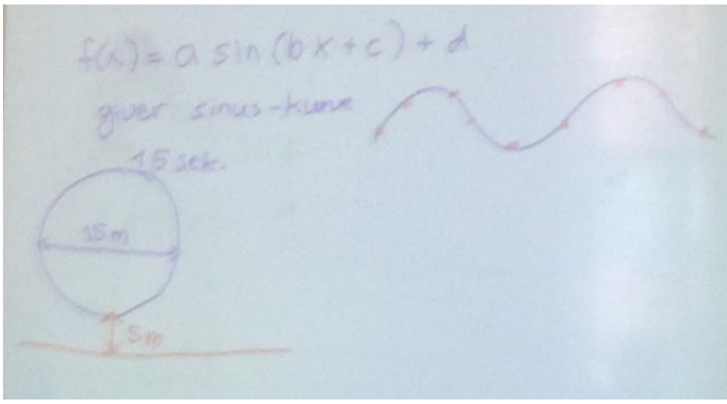


Figure 8.1: The smartboard during exercise 8

In order to solve subtask a) you must figure out a way to describe the height above the earth as a function of time. After the teacher's monologue the students know that the function must be of the form $f(x) = a\sin(bx + c) + d$, but the question is what the four constants are.

At first the teacher lets the students try to find the constants in groups. They find the constants a and d to be respectively 7,5 and 12,5, but when they are going to find b and c the teacher changes the concrete assignment; "Describe the height above the earth as a function of time" to the more general assignment; " Use the CAS-tool to describe the four constants' impact on a sine curve". To see what a constant does to the curve, the teacher asks the students to draw different sine curves where they change one constant at a time. Recall that some of the students have Nspire as CAS-tool and others have Maple. In Nspire you can create so-called sliders for each constant, hereby changing the constants continuously. In Maple this would require some programming, or else the students have to manually change the constants in the expression. In the situation we observe, the sliders in Nspire are used to explain the constants' impact, both in phase 2A and 2B.

8.2 A priori analysis of situation 2

Even though the situation consists of two phases I have decided to make one a priori analysis of the entire situation, primarily because the target knowledge is the same in the two phases. In the cases where we have some deviations we will make the deviation clear by referring to “phase 2A” or “phase 2B”.

8.2.1 Target knowledge

The target knowledge in this situation is to be able to use the CAS-tool to explain how the four constants a, b, c and d impact on the class of trigonometric functions $\mathcal{F}_{trig} = \{ f(x) = a\sin(bx + c) + d | a, b, c, d \in \mathbb{R}\}$ called sine curves. This is relevant because one of the teacher’s targets for this lesson is that the students are capable of creating a model using a trigonometric function (Appendix 2). In order to do so you must know the meaning of the different elements in the model. The task is presented as a subtask to solve exercise 8. Thus the teacher clarifies, through a concrete example, the importance of knowing these constants’ behavior in order to create a model using a trigonometric function.

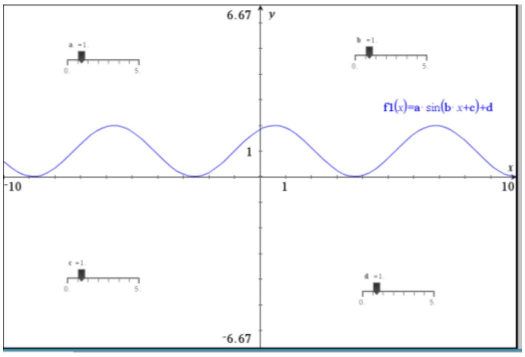


Figure 8.2a

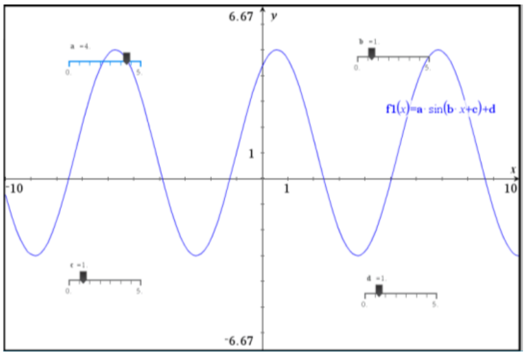


Figure 8.2b

The benefit of the sliders in Nspire is that the students immediately get an explicit picture of the constant’s behavior. When changing the sliders for a from a small value to a greater one, they will immediately see that the amplitude of the sine curve becomes greater (Fig. 8.2a+b). The target knowledge is to see that the amplitude is two times the value of a. In the same way the student will easily see that d displaces the curve on the y-axis. When using the sliders for b the students will see that the period of the function is changed. Hence an intuitive description of b could be that “b is the length of the period”, but here one has to notice that when b is made greater the period becomes smaller. The period and b is inversely proportional. If we call T the period we have the relationship:

$$T = \frac{k}{b}$$

where k is the proportionality factor. To find k one can let b equals 1, and then read off the period. The period will be 2π, giving the relation:

$$T = \frac{2\pi}{b}$$

This makes good sense since we know the period of $f(x) = \sin(x)$ is 2π.

When changing the slider for c, one will see the curve moving in the direction of the x-axis. The question is whether it moves in the positive or the negative direction. The answer is negative direction, which is observed by looking at one of the curve's intersections with the x-axis. If $c = 0$ (and $d = 0$) then the intersection is in 0. If we change c to 2, the intersection is in -2, for $c = 4$, the intersection is in -4 and so on. Hence c displaces the curve, in such a way, that if c is made greater the curve is displaced in negative direction. These mathematical insights achieved by using the CAS-tool are also a part of the target knowledge.

8.2.2 Students' knowledge

In phase 2A we consider a small group of students working in Nspire and in phase 2B we observe the entire class, including both students working in Nspire and students working in Maple, hence the students' instrumented techniques are not the same in the two phases. In phase 2A the small group of students is asked to create sliders in Nspire, the teacher does not explain how to do this, and the students do not seem to have any problem with this task, so here the knowledge concerning the CAS-tool is old knowledge. In phase 2B the case is a bit different. The teacher presents the sliders in Nspire on the smartboard, but we now have students who have never worked in Nspire, so the use of sliders is probably a new technique to them. But these students working in Maple do not have to create the sliders themselves, and the use of the sliders is intuitively clear, so we do not see this as an obstacle to reach the target knowledge.

Prior to this situation the students have drawn the graph of $\sin(x)$ and they have made sine regression in their CAS-tool where the result was a function of the form $f(x) = a\sin(bx + c) + d$ and a corresponding graph. Here the students tried to validate their results with statements such as "It is not a sine function, because the graph does not hit the x-axis". This is a reasonable conclusion because they have only seen sine curves oscillating around the x-axis. So far their knowledge concerning sine curves does not involve curves oscillating around the line $y = d$, but still they have an idea of the graph as periodic with a wave-like shape. So the graph of sine curves must be considered as knowledge in development.

The students have just been introduced to sine as a function, and now they also have to acknowledge sine as an element in a family of functions. This could be an epistemological obstacle to some of the students, because their perception of functions does not include family of functions. Neither does the textbook or Syllabus indicate that the students should relate to this.

In phase 2A the students have only worked with concrete examples of sine curves and are (partly) able to decide whether a given graph is a sine curve or not. But they have not worked with the general characteristics of a sine curve, so the meaning of the constants a, b, c and d must be considered as new knowledge in phase 2A. In phase 2B, the students have had time to work with these constants in Nspire, so here the knowledge is in development.

8.2.3 The milieu

The objective milieu consists of the problem to be solved, here "explain how the four constants a, b, c and d impact on a general sine curve $f(x) = a\sin(bx + c) + d$" and the tools available. In phase 2A the students first of all have the CAS-tool Nspire at their disposal. They draw the graph of $f(x)$ and create four sliders. By using the slider for the constant a, they may for instance see that the amplitude is changing (Fig. 8.2a+b). In this way the milieu can help the students formulate hypotheses about the constants' behavior and moreover the milieu will provide the students with feedback concerning their hypotheses. If the hypothesis about b is that "b is the length of the period", then one will expect the period to be greater when b is made greater. Testing this in Nspire will show the opposite; the period will become smaller when b is made greater. Hence Nspire can be used in both the formulation and validation of hypotheses concerning the constants' behavior, thus the milieu supported by Nspire has a high adidactical potential when working with this particular task. The milieu also consists of information written on the smartboard (see Fig. 8.1).

In phase 2B the milieu is changed, because here we have students who are used to work in Maple and students who are used to work in Nspire, so the milieu of each student depends on the CAS-tool he/she uses. Common to all is the presentation on the smart board, where the teacher uses Nspire to create four sliders and the graph of $f(x) = a\sin(bx + c) + d$. The students working in Nspire will not experience a big change in the milieu, but the students working in Maple will get a whole new visual response from the milieu. In Maple they had to create a new graph each time they changed a constant, hereby letting the milieu provide them with lots of different graphs. Now they can see one graph changing form as the different constants are changed, and this could be an advantage. The graphs created in Maple are all static representations of functions, but when we use the sliders in Nspire we change this static representation to a dynamic one. We have a family of functions, where each function is a static object, but the graph is dynamic. It is changing depending on which function it represents. This dynamic approach to the graphs of the sine curves may give the students a feeling of controlling the process, a "hands on" feeling and hereby ease the learning. As mentioned before the notion of a family of functions can be difficult to interpret, and here we could actually talk about a family of a family of function. If we consider one parameter at a time, and let the other be fixed we get one family of functions, for example:

$$f_a(x) = a\sin(x)$$

Since we have 4 parameters we can consider four of these families, and on top of that we have to consider them as one family, namely:

$$\mathcal{F}_{trig} = \{\, f_{a,b,c,d} | a, b, c, d \in \mathbb{R}\} \text{ where } f(x) = a\sin(bx + c) + d$$

It is no wonder that this could be difficult, and by splitting up, and looking at one constant at a time the students only have to focus on one thing at a time. But it is important, that the pieces are collected, and the students get an idea of what the family of trigonometric functions represents. The graphic milieu with the sliders will hopefully give them the idea that the family is all the functions looking sinusoidal. We can displace, stretch and compress any sinusoidal function in both the direction of x and y, and it is still a sinusoidal function belonging to the family of trigonometric functions.

But to sum up, the milieu in both phases is supported by an algebraic part; the family of functions $\mathcal{F}_{trig}$ and a graphic part; the graph window with sliders in Nspire.

8.3 A posteriori analysis of situation 2

We analyze phase 2A and 2B separately. Each phase is split into episodes defined by the micro-contract at stake and represented by the dialogue in the given episode. In the dialogues, uninteresting phrases are left out. This is marked by //. The whole dialogue is presented in appendix 4 and 5.

8.3.1 Phase 2A

We observe a group consisting of two girls and a boy. Kimmi, one of the girls, sits in the middle and is the only one typing on her computer. Just before the observed situation Kimmi has orally formulated the behaviour of the four constants to the teacher:

> *"a changes the length of top and bottom in the curve, b is the period, or the length of the period. The wave length, makes it greater or smaller. c moves the curve on the x-axis and d moves the curve on the y-axis."*

The group continues to change the sliders in order to validate their formulations. Since c displaces the curve on the x-axis Kimmi expects that the curve goes through $(0{,}0)$ when c is zero. But Nspire presents her with a graph going through $(0{,}1)$, so she asks the teacher for help:

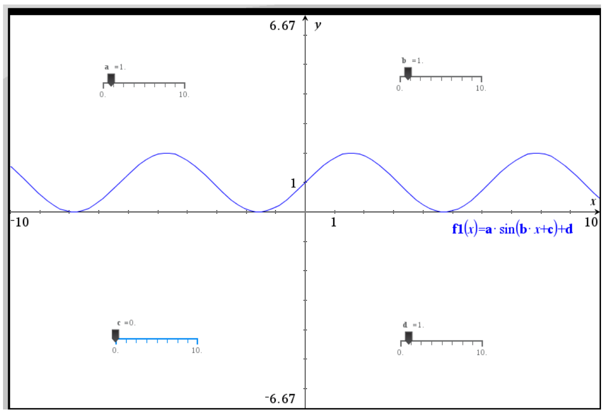


Figure 8.3: Kimmi's screen

1. Kimmi	Bente...Why is it, that when I start by letting the c-value be zero, then my graph doesn't starts in zero?
2. Teacher	It has something to do with the sine part.

3. Kimmi	So, this is our graph [points at her screen]. But if one lets c be zero, because it displaces it.
4. Teacher	Yes, then the whole sine part becomes zero.
5. Kimmi	No, because it is just added by zero. So it just.. This one just becomes "sine to bx"
6. Teacher	Yes, and x is zero here, right? At the y-axis x is zero.
7. Kimmi	Yes, then the whole sine part becomes zero.
8. Teacher	Yes, then the whole sine part becomes zero. But then there is still d left, so in zero you start in d.
9. Kimmi	Okay, so if we let this one be zero [refers to the slider for d] // Now we start in zero.

In this episode we have a micro-contract of individual production, since the development of knowledge involves only the teacher and one student. The episode starts with Kimmi asking the teacher a question, showing the teacher that the milieu devolved has some obstacles. The teacher has created an adidactical situation were the students should be able to validate their formulations using the CAS-tool. But the fact that Kimmi needs to ask the teacher for help, hereby making the situation didactical, indicates that either the feedback potential in the milieu or the students' knowledge is not as good as expected.

The teacher says that it has something to do with the sine part, referring to the formula for a sine curve, or in fact just the part "$a\sin(bx)$". Kimmi continues to consult the graph window, explaining that if one lets c be zero, then the graph should not have moved, because c displaces the graph, and if c is zero, it is the same as doing nothing. The teacher agrees by saying yes, but refers again to the algebraic part of the milieu by saying that the sine part becomes zero. Now Kimmi also consults the formula, but disagrees with the teacher and needs to be reminded that x is not x, but zero, since she is interested in a point on the y-axis. Kimmi is so focused on the constant c that she forgets to include the meaning of the other constants. The graph on her screen is representing the function $y = \sin(x) + 1$, but to Kimmi the graph represents the function $y = a\,sin(bx + 0) + d$, where the constants a, b and d are not in focus. She is not aware that a formula including a constant actually represents a family of functions, so when considering an expression as $y = a\,sin(bx + 0) + d$ she must first fix the constants a, b and d, and be aware of the fixed values impact on the curve, before being able to validate the behaviour of the constant c. Kimmi does not grasp this at first, but with the teacher's help she gets the idea. Then she tries to let d equal zero in her graph window, and then she gets a graph going through (0,0). Kimmi uses the graphic milieu to validate her hypothesis that in order to have a sine graph going through (0,0) you must let both c and d be zero.

It may seem a bit misleading that the teacher says that the reason why the graph does not goes through (0,0) has something to do with the sine part, when it is in fact the constant d, which influences on the graphs displacement. But if the teacher had just informed that d was the reason why, we would have had a clear Topaze effect were the teacher's desire to let Kimmi win the didactical game would make her just give away the answer. By letting Kimmi use the formula of a sine curve to realize that when both x and c is zero we get $y = d$, Kimmi creates the personal knowledge needed to accept that the graph does not pass (0,0), including the fact that the graph is affected by all the involved constants at the same time. This creation of personal knowledge is important in order to reach the target knowledge, because if the knowledge is not personalized to the student, it does not become a part of the student's knowledge, but just some institutionalised knowledge the student may not relate to any situation. The conversation between the teacher and the group continues:

1.	Kimmi	But how can we then calculate the real b- and c- values?
2.	Teacher	Yes, you may consider that. You can consider that if you have the period here [points at the graph on the screen]. What does that have to do with b?
3.	Kimmi	It lasts b, a period. Or half a period.
4.	Teacher	It has something to do with the period.
5.	Boy	It is how long it takes for each wave.
6.	Kimmi	The length of the period
7.	Boy	Yes the length of the period.
8.	Teacher	Mmm..
9.	Kimmi	It was 15 seconds it took.

The situation has now changed because the question no longer concerns the behavior of the graph, but how to find the values of b and c in exercise 8. The boy in the group interferes, but still we have a micro-contract of individual production, because the knowledge at stake is not available to most of the students in the class, mainly because they are not part of the conversation. The responsibility for solving the problem is with

Kimmi and the boy. In quote 4 the teacher makes a partial (implicitly negative) validation, but she does not state whether Kimmi is right or wrong, but encourages her to continue looking at the period. In quote 8 the teacher just mumbles an “Mmm”, indicating that she agrees, but does not want to validate the students’ formulations. We have a micro-contract saying that the teacher must not clearly validate the student’s formulation, because the responsibility of creating the knowledge in development is with the students. Quote 9 refers to exercise 8 and the fact that it took 15 seconds to complete a period, but instead of focusing on the exercise the teacher wants them to look at the graph on the screen:

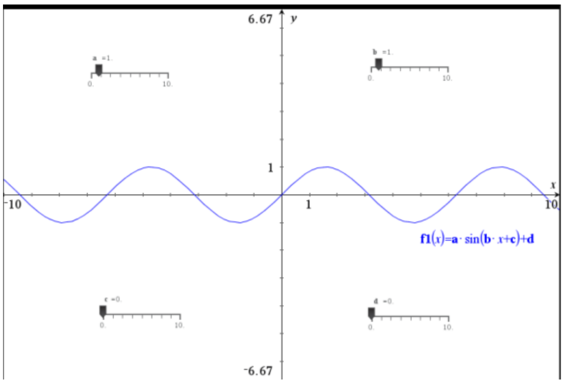


Figure 8.4: Kimmi’s screen when both *c* and *d* are zero.

1. Teacher	Then try to look here. What is the period here?... In relation to what *b* is.
2. Kimmi	*b* is 1 here and the period is… [Tries to read the period directly from the screen] 1, 2, 3, 4, 5.. A little bit above 5, maybe. I cannot quite see it.
3. Teacher	No, but then you can find two maxima.
4. Kimmi	[Works in NSpire for a while] Okay, 6,28
5. Teacher	Mmm, and what is that?
6. Kimmi	That is the length
7. Teacher	What is 6,28? It is a very nice number.
8. Girl	It is two times π
9. Teacher	Yes. So now you can consider why it is exactly 2π.
10. Girl	That is because it is the circumference
11. Boy	It is the circumference.

In this episode the teacher wants the students to see the relation between the period and the constant *b*. The teacher guides the students by asking questions, hereby taking more of the responsibility, probably because she knows this is entirely new knowledge.

In quote 1 the teacher wants the students to use the graphic milieu, maybe because she in the previous episode realized that the students have not realized that b is not the length of the period, but reversal proportional to this length. Girl 1 knows that b is 1, because that is what the slider is set to, and she knows that the period is the length from wave top to wave top, so she starts using the graphic milieu to count the distance between the two wave tops. In mathematics, reading off a graph is less valid, than finding the value algebraically. One reason is, as here, that a reading will not always be precise. Thus if possible one should find the maxima algebraically, but in this case this is not yet possible, because the students have not learned how to differentiate sine and cosine, so the next best thing is to find the maxima using the CAS-tool (Fig. 8.5a). In quote 4-8 we see that it requires some efforts from the teacher before the students realize, or more likely guess, that the period is 2π. In fact we are approaching a Topaze effect, where the teacher almost gives the correct answer. In quote 9 we are approaching a Jourdain effect. The teacher realizes that the students do not reach the target knowledge as quickly as she has expected, so she starts guiding them by giving instructions on what to do. If this continues the result could be that the students do not construct personal knowledge, but just becomes actors in the teacher's presentation of the knowledge. These effects of the didactical contract show us that the teacher is very eager to let the students grasp the relation between b and the period. In quote 10 the girl claims that the reason why the period is 2π is because it is the circumference. We do not know to what it should be the circumference, but the girl probably sees a link between this period and the circumference of the unit circle, which the students have spent a lot of time on the lesson before, when they were converting between degrees and radians. The teacher does not confirm this statement, but instead she wants the students to consider another value of b:

1. Teacher	So when b is 1 you get 2π. What if b is 2, what will you get then?
2. Boy	Then, we will get... Then we will get…
3. Kimmi	[Changes b to 2 in Nspire] Then we get the same.
4. Boy	Yes
5. Girl	Do we get the same?
6. Kimmi	Yes, we get the same. It is the only change we have made, the one with b.

7. Teacher [Looks at the screen and sees that Nspire now have found maximum of the first and third wave top (Fig. 8.5b)] But, try to look here. This isn't the distance between the neighbor holes.

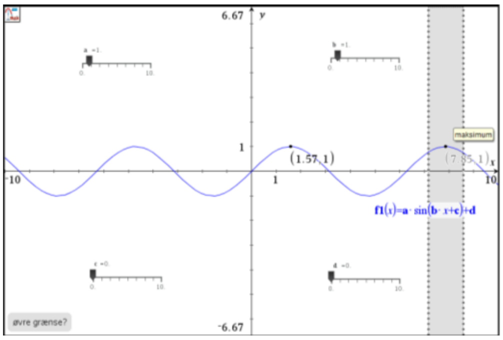


Figure 8.5a: The maxima when $b = 1$

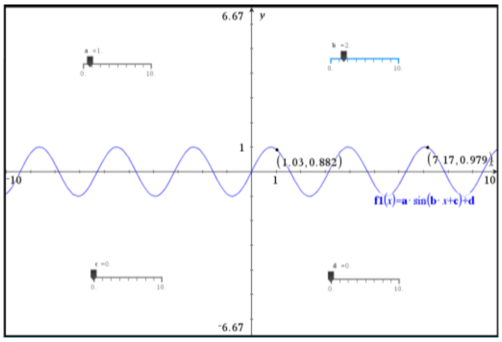


Figure 8.5b: The (local) maxima when $b = 2$

8. Kimmi Oh yes, it has measured the third hole.

9. Teacher But you are allowed to take these to values, but then you just have to remember that there are two periods.

10. Kimmi Yes. So we have to divide by 2, then we get π.

//

11. Kimmi Okay, now we have it on 4.

12. Teacher You could try to guess

13. Kimmi [Hesitantly] It must be the half? One quarter?

14. Teacher So when you make b

15. Kimmi Greater

16. Teacher Then..

17. Kimmi π becomes smaller.

18. Teacher Then the period becomes smaller.

The first part of this episode shows us that trusting the CAS-tool too much can be misleading. In the previous episode, Kimmi found the maxima by highlighting small parts of the graph, where she could see that the maxima would be, and hereafter letting Nspire find the maximum. If you change the period, then the graph changes and so does the placement of the maxima. But the parts you have highlighted in Nspire do not change, so the maxima giving to you by Nspire, is just the maxima in the highlighted parts, not necessarily the maximum of your graph. In our situation the change in period still results in wave tops in both highlighted parts, so the students do not see that it is not the first and second wave top, but the first and third.

The second part of the episode contains a formulation of the correspondence between the constant *b* and the period. Again we have a micro-contract of individual production.

In quote 12 the teacher wants Kimmi to guess what the period would be, giving her the responsibility for the knowledge to be taught. Kimmi's answer in quote 13 shows the teacher that Kimmi knows that the period will be smaller than before, hence the teacher helps her, in quote 14-17, formulate the relation, hereby taking much of the responsibility for the formulation of the relation. In quote 18 the teacher chooses to correct Kimmi's answer by repeating it with the word "period" instead of π. She could have given a negative validation and said: " No, it is not π, which becomes smaller, but the period", but instead she chooses the more positive solution, maybe because she knows that Kimmi meant period and not π or maybe because she thinks that the context is more important than the correct word. A negative validation may have given Kimmi the idea that she was all wrong, whereas the partial positive validation makes her feel that she almost had the correct answer.

Most part of phase 2A concerns the teacher's attempt to lead the students towards certain points in order for them to gain their own personal knowledge about, the correspondence between the constant b and the period. The meso-contract of the situation is the formulation of new knowledge. A part of the meso-contract is the fact that the students must produce the knowledge themselves using the questions from the teacher as guidelines. Towards the end of the situation the teacher has a tendency to break the contract, both when her questions turn into instructions and when they become to leading.

8.3.2 Phase 2B

This phase takes place in the end of lesson 2, right after the students' group work. It is whole class teaching and the teacher leads the conversation. The teacher has projected her computer screen on the smartboard, so now everyone can see a sine curve and four sliders:

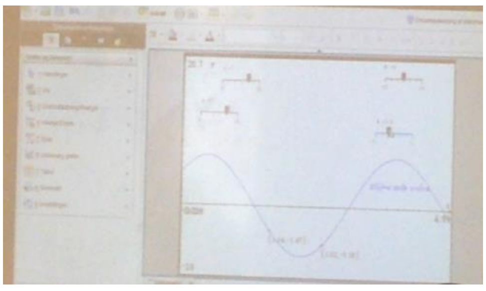


Figure 8.6: The smartboard during the institutionalization

1. Teacher	Okay, I think we will make a recap // We did more or less agree what happens with a, right? [Pause] Camilla?
2. Camilla	Eeh, the highest and lowest y-value.
3. Teacher	Yes. We can do this here [changes the slider for a]. Now a becomes a greater number, and we can see that the oscillations becomes greater. So what you can say is that a determines the oscillation in relation to the equilibrium position. If we try to find maximum and minimum for this one [Uses Nspire], then we see that minimum is 2,5 and maximum is 12,5, so there is a difference on 10. This difference is exactly $2a$, because it oscillates a both ways, right? When a is 5 it can go 5 up and 5 down, and that corresponds to 2,5 and 12,5.
4. Girl	What was it you said? It determines the oscillations in relation to what?
5. Teacher	The equilibrium position. So, I can move it a little bit up, then it goes from 0,5 to 14,5, that is a difference on 14 and now a is set to 7, right? You could of course move… [She changes d to zero in Nspire] // So this was a, and d we already have discussed. Now d is zero, so it oscillates equally on both sides.

The first thing to notice here is that the teacher has long monologues. This is due to the fact that she now wants to institutionalize the four constants' impact on a sine curve. She wants to make sure that all students know the meaning of the constants, and are able to formulate the meaning using the official terms. The pause in quote 1 is a pause where the teacher, after asking her question, waits for the students to raise their hands. In this way she gets an idea of how many students the knowledge is available to. The majority of the class raised their hands creating a micro-contract of collective production. In quote 3 the teacher validates Camilla's answer by using the sliders on the smartboard, hereby taking responsibility for the knowledge at stake. She could have asked another student to validate, but by doing it herself she can incorporate the official formulation: "a determines the oscillation in relation to the equilibrium position." hereby institutionalizing the constant a. The teacher does not focus on the fact that Camilla's answer is incorrect; in fact she gives the answer a positive validation by saying "yes", but a is not the highest and lowest y-value, and neither is it the distance between these two values, but only the half of it. This is either because the teacher accepts Camilla's answer since she can connect a to the highest and lowest y-value, or because the teacher finds it more important to institutionalize the official formulation to the entire class. Either way Camilla's role in the building of knowledge becomes unimportant.

The term "equilibrium position" is probably used because the teacher also teaches in physics, and we see in quote 4 that this term is confusing, and maybe new, to some students. The teacher does not explain the term, but changes the value of d to zero, hereby moving the equilibrium position to the x-axis, maybe to reduce the confusion. Her focus is on clearly showing how the graph changes when a changes and hereafter she moves to the constant b.

1. Teacher	Then we have b and c. Some of you examined what happens with b, right? What happens when you changes b? Eric?
2. Eric	The length of the period changes
3. Teacher	Changes, yes. So when you make b greater…
4. Eric	Then the period becomes smaller
5. Teacher	Yes. We can see that by changing this one. Now I make b greater, then I get a smaller period, and if I make b small, then I get a very long period. So in a way you can say that b and the period.. [Is interrupted by a student, but the interruption is left out here].

Here we have a micro contract of individual production inside the micro-contract of collective production. Eric answers to the teacher's question, and she gives a positive validation by saying yes, but still she wants Eric to be more explicit in his formulation, so she starts a sentence and let Eric finish it. She did the same thing earlier when Kimmi had to explain the relation between b and the period, which indicates that a part of the didactical contract is that the teacher implicitly can ask questions through unfinished sentences. The teacher lets Eric elaborate his answer and hereby have the responsibility for the formulation. This indicates that the teacher knows that the impact of b is more difficult to formulate than the impact of a, so she wants to ensure that Eric, representing the entire class, has constructed a personal knowledge of the relation between b and the period. In quote 5 the teacher validates Eric's formulation by use of the sliders and after a short interruption she continues the institutionalization.

1. Teacher	But we agree that when you make b greater the period becomes smaller. And the period is often called T, at least in

physics. Then one can say that b and the period must be inversely proportional. That is, if you make b twice as big, then the period becomes the half and vice versa. // If they are inversely proportional, then their product must be some constant. What constant is this? Here we can just see what happens if for example b is 1 [She uses the sliders to illustrate]. How can I read the period most accurately? Martin?

2. Martin	Isn't it from wave top to wave top?
3. Teacher	Yes. [She uses Nspire to find the two maxima]. How can we then calculate the period? I have found the two points where there is a maximum. Here x is equal to 0,854 and here x is equal to 7,14. The period is how far there is between these two points, and what is that? Nanna?
4. Nanna	It is 7 minus 0,854.
5. Teacher	Yes, $7{,}1 - 0{,}854$. And if we calculate it what do we get? I think. Didn't some of you do it? Kimmi?
6. Kimmi	It is 2π
7. Teacher	It is in fact 2π. So the product of b and the period is 2π, or one can say that b is 2π divided by the period. So if you know the period you can figure out b, and also the other way around, if you know b you can figure out the period.

In quote 1 the teacher has a long monologue of pure institutionalization. She talks about the term "inversely proportional" and she does not involve the students, so we may assume that she expects this to be old knowledge. She ends her monologue by asking a question concerning the reading of the period. This indicates that here she wants the students to take responsibility. The knowledge concerning the period of a function is knowledge in development and by letting the students participate in the production of knowledge the teacher first of all sees to whom the knowledge is available. In quote 5 the teacher asks Kimmi what the period is, because she knows that Kimmi has the right answer. By letting Kimmi give the answer the teacher lets the knowledge constructed in the group work become available for the entire class, hereby avoiding a repetition of the scenario where the students do not connect 6,28 with 2π. In quote 7 the teacher again institutionalizes, but this time it is new knowledge. This should be easy to follow if all students have explored the relationship as the group in phase 2A did, and this is probably what the teacher had in mind when she designed the teaching.

1. Teacher	So that was the b. Then we only lack the c. This we have up here. What happens when I change the c? Nille?
2. Nille	Then the curve displaces itself on the x-axis.
3. Teacher	Yes, so it just changes. It doesn't changes shape or size, it is just displaced. So it has something to do with where the curve starts. If I let c be zero, where does it start then?... Now I look a bit confused, I thought it would start in zero. [Changes her sliders in Nspire] Maja?
4. Maja	Then it starts in zero
5. Teacher	Yes, then it starts in zero. When c is zero it starts in zero.
6. Kimmi	But only if the d-value also is zero. //
7. Teacher	Oh, yes. d also have an imfluence. But now I also have d to be zero here. So the equilibrium point will be in zero. But the c-value you can use to make it have a certain initial value. If we consider the case with the balloon swing, then the value of c depends on where you are, when it starts to turn around. But I just think we will write that c is the displacement on the x-axis.

Here we again have a micro-contract of collective production, where the teacher asks one of the students, raising their hand, and next reformulate the answer so it becomes more clear and formal. In quote 3 the teacher says that the c-value has something to do with where the curve starts. This word "start" is then used several times by the teacher and the students, but a curve like the sine curve approaches infinity in both the negative and positive direction, so to talk about a starting point is absurd. We may assume that the teacher refers to an appropriate point from where we can describe our curve. In quote 6 we see that Kimmi wants to share the knowledge she developed in phase 2A, and the teacher agrees with her. In quote 7 the teacher tries to explain how you can use the c-value to give the sine curve a certain starting point. The terms equilibrium point and initial value are again terms that are frequent in physics, but maybe less familiar to the students. But she decides to just institutionalize the c-value as the displacement on the x-axis.

The entire dialogue in phase 2A shows a classroom discourse of triadic dialogue, defined by Lemke (1990) as a three part Question-Answer-Evaluation pattern (Lemke, 1990, p. 23). The teacher asks a question, waits for the students to raise their hands and then calls on a student. The student answers and the teacher evaluate the answer, either by repeating what the student just said or simply by saying “yes”. The evaluation is here often followed by an explanation and an official formulation. In other words, the didactical milieu the teacher has created consists of small formulations followed by a more spelled out validation, which leads to the institutionalization. The benefit of this triadic dialogue is that the teacher can lead the conversation towards the target knowledge and simultaneously observe which students the knowledge is available to, and to which it is not. A drawback is that the students do not influence the dialogue. The teacher decides which topics are discussed as we saw in the first episode with the question concerning the equilibrium position.

The meso-contract in phase 2B is the institutionalization of new knowledge. The triadic dialogue is a part of the contract and the students know that the responsibility for validating the formulations is with the teacher.

9. ANALYSIS OF SITUATION 3

9.1 Context

This situation takes place in lesson 4, the self-study-lesson where the students had to create documents for their own homepage. The class has a homepage where they write about the subjects taught in class. The goal is to create an E-book, which can be used when they have to study for the exam. Before leaving the class the teacher divides the students into eight groups and gives each group their own topic concerning trigonometric functions:

1. Graphs of sine, cosine and tangents
2. Radians
3. Periodicity
4. Sine curves in general
5. Sine regression in Maple
6. Sine regression in Nspire
7. The derivative of sine, cosine and tangens
8. The antiderivative of sine and cosine

The group observed got the topic “Sine curves in general”, hence the assignment in the situation can be interpreted as *select and formulate the knowledge concerning the trigonometric functions you find relevant for explaining “sine curves in general” to your class mates*. Lesson 4 was the only one where the students had homework to prepare. They were asked to read the chapter “The functions sine and cosine” (Nielsen & Fogh, 2006, pp. 40-45). So besides the personal knowledge learned in the previous lessons, they also have the institutionalized knowledge from the textbook at their disposal.

9.2 A priori analysis of situation 3

9.2.1 Target knowledge

The students are supposed to review the knowledge taught through three previous lessons and select the parts they find relevant for explaining sine curves in general. The target knowledge is to redepersonalize (see section 3.1) personal knowledge and make it official, i.e. produce an institutionalization of sine curves in general.

If we consider the textbook (Nielsen & Fogh, 2006) we find a section called "Sine curves in general" and we expect this to be the institutionalized knowledge the students should learn about sine curves in general. The section starts by presenting the general formula for a sine curve:

$$f(x) = a\sin(bx + c) + d$$

The constants are explained as:

> *"d displaces the graph in the direction of the y- axis. Instead of oscillating symmetrically around the x- axis, the graph will oscillate around $y = d$. The constant a is called the amplitude and is the maximal oscillation from the line $y = d$. The constant b is called the cyclic frequency and indicates the number of oscillations corresponding to the length 2π on the x-axis. The length that one oscillation fills on the x-axis is called the oscillations period T. The relation between b and T is:*
>
> $$b = \frac{2\pi}{T}$$
>
> *The constant c is the displacement in the direction of the x-axis. "(Nielsen & Fogh, 2006, p. 42, own translation).*

The students should be able to formulate the behavior of these constants. Likewise the students should be able to explain that the general sine curve is not just one function, but a class of functions. So when considering one specific sine curve you consider an element f in the class of trigonometric functions $\mathcal{F}_{trig}$. As mentioned in section 2.1 it can be difficult to realize that *one* general analytic expression represents a great amount of functions, and hence a great amount of graphs. Likewise it can be difficult to explain to others.

Moreover a presentation of sine curves in general will also include an explanation of what a sine curve is, including the graph's wave-looking pattern and the term "periodicity". If a graph is periodic it means that it repeats itself after a given time. In the case of a general sine curve it repeats itself after one oscillation, hence the period of a general sine curve is T. The ability to formulate the above is a part of the target knowledge.

9.2.2 The students' knowledge

Sine as a curve in a coordinate system was introduced to the students for the first time in lesson 2, and the same goes for the expression $f(x) = a\sin(b * x + c) + d$. As we saw in situation 2 the students had time to work with the constants' influence on the curve and especially the influence of the constants a and d will be considered knowledge in development and even old knowledge. The students have not had time to personally work with the constants b and c, so the influence of those is considered new knowledge.

The term periodicity has been mentioned to the students in the very first introduction to trigonometric functions, and in lesson 2 they had time to work with the relation between the period of the sine function and the circumference of the unit circle, hence the knowledge concerning periodicity is considered knowledge in development.

In general we will say that the knowledge concerning general sine curves is knowledge in development. As we described in section 2.4 the transition, from sine as the y-coordinate to a point on the unit circle to sine as a function, is difficult to realize. Through the last three lessons the students have had a lot of information concerning this sine function and in particular the more general sine curves. To process all this new information and decide what is relevant for a general sine curve, and what is not, may be a challenge for the students.

9.2.3 The milieu

The milieu is always attached to a task with criteria as to when the task is solved. In this situation the task to be solved is "create a document about sine curves in general", and this does not have these criteria. There is no milieu around this task, only a media in terms of the textbook. Since the textbook has a section dedicated to this specific topic the book becomes an authority the students can rely on.

In order to solve the task the students must create smaller tasks, which have the wanted criteria and hereby making small milieus. An example could be the task "describe the constants in the formulae for a general sine curve". A part of this milieu is the CAS-tool Nspire. By means of the sliders in Nspire the students can easily change the constants in the expression for a general sine curve and immediately get a visual response in terms of a changed curve. Moreover this milieu is also supported by the documents the students have produced earlier in Nspire. The students can consult these documents and see how they have solved a given problem before and use this as a guideline for the present problem.

Since the teacher is not present in this situation the objective milieu plays a crucial role, because in an adidactical learning situation only the milieu can react on the students' actions. The milieus' ability to provide this feedback is called the milieus' adidactical potential. With the task "describe the constants in the formulae for a general sine curve" Nspire has a very high feedback potential. If the students have a hypothesis about, for example, a's behavior, the sliders in Nspire immediately tells them whether the hypothesis is correct or not. If the students are able to create these smaller tasks, they can create milieus with high adidactical potential. The teacher believes that the students are able to do so, because the knowledge concerning general sine curves has already been personalized through previous lessons.

9.3 A posteriori analysis of situation 3

The observed situation involves a group consisting of a boy and two girls, named girl 1 and girl 2 in the dialogue. As before uninteresting elements are left out of the dialogue and marked by //. The entire dialogue is presented in appendix 6. The situation takes place half an hour into the lesson. It lasts 23 minutes and consists of three different episodes. First an episode concerning which graph illustrates a general sine curve best, next an episode where they compare a textbook example of a sine curve with the graph of $\sin(x)$, and finally an episode where the boy struggles with the feedback from Nspire.

In the first episode the group has begun writing their document and has written that the sine function is periodic with the period 2π. After some discussions on whether they should add a graph with one period or multiple periods, they decide to add the former and then explain how this graph shows one turn around the unit circle, but in fact the graph could continue to infinity. They present the graph:

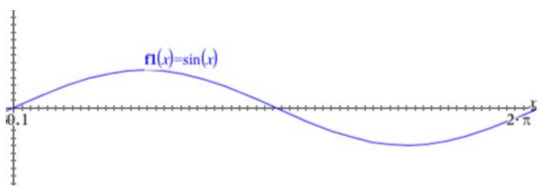


Figure 9.1: The graph in the students' document

But they are still not sure that this is the best way to present a general sine curve. Their doubt is due to the fact that the textbook presents two graphs; the function $\sin(x)$ and an example of a general sine curve:

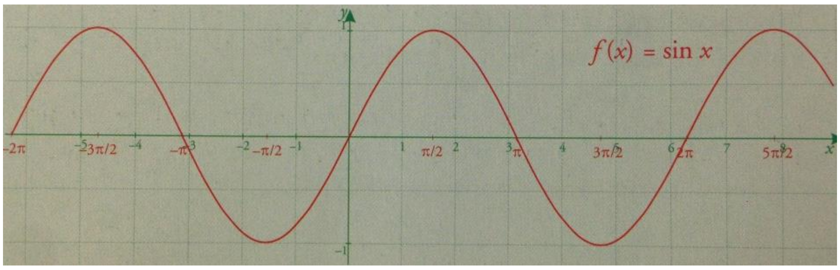


Figure 9.2: The graph of $f(x) = \sin(x)$ (Nielsen & Fogh, 2006, p. 42)

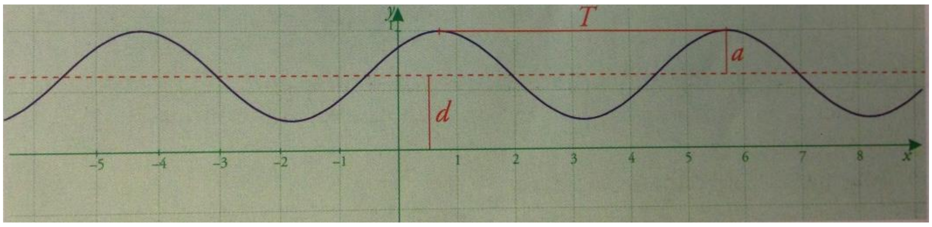


Figure 9.3: An example of a general sine curve (Nielsen & Fogh, 2006, p. 43)

The students have changed the media to a milieu with the task "choose the graph, which represent a general sine curve best". Let us consider their dialogue:

1. Boy	But shouldn't we have a picture of this function instead of this? [Points first at the graph for a general sine curve and next at the graph of $\sin(x)$]
2. Girl 1	No, why?
3. Boy	Yes, because this one [points at the general curve] looks a lot different. It doesn't intersect the x-axis.
4. Girl 2	Oh, but isn't that just because…
5. Girl 1	It MUST intersect the x-axis.
6. Girl 2	... They have inserted a d?
7. Girl 1	Yes. It is the d that does it. The thing where it intersects.
8. Girl 2	But beside that, isn't it just the same principle?
9. Girl 1	Yes, yes. But I think this one [points at the general curve] is more correct than that one [points at $f(x) = sin(x)$]. Because our Nspire, it wouldn't intersect the x-axis. I remember that from last time. [Small pause] But let us just explain what a, b and c is.
10. Girl 2	Yes, eeh.. Okay, a… [Looks in the textbook]
11. Boy	But I just don't think this one is a sine curve. It is a sine function. That is why it doesn't make any sense.

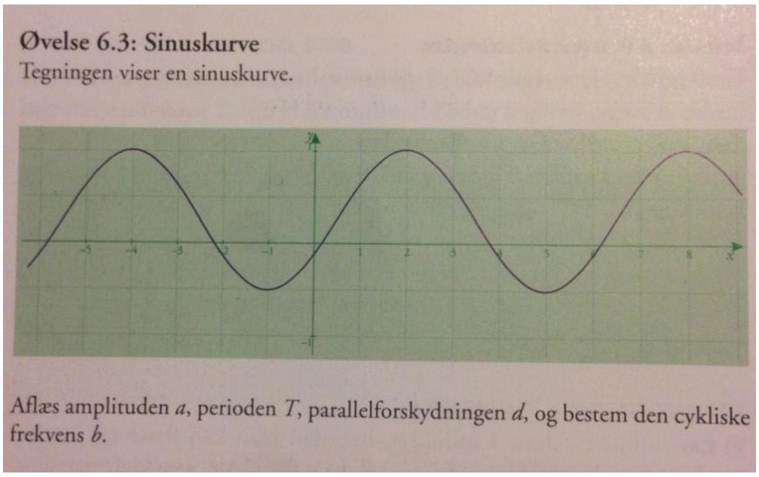
12. Girl 1	Of course it is. It is this one. Isn't it?
13. Boy	No, because this one is just $f(x) = sin(x)$. It is just the sine function. This one [points at the general curve] is sine CURVES. Here you work with the constants a, b, c and d.
14. Girl 1	Mmm..
15. Boy	Try to make one like this in Nspire

The boy thinks that the graph presented in Fig. 7.3 is a more correct presentation of a sine curve, than the one presented in Fig. 7.2. His argument in quote 3 tells us that he does not believe that two different graphs can both represent a sine curve. One of them has to be more correct then the other. The boy interprets a sine curve as one function and not as a class of functions. He leans towards the example of a general curve probably because this one is presented in the section called "sine curves in general" and the boy knows that the textbook represents the knowledge to be taught.

Girl 1 is not willing to reject the graph of $\sin(x)$. She believes that a sine curve must intersect the x-axis. This is a result of the previous lessons. In lesson 2 the class was presented to the sine function oscillating around the x-axis, and in the same lesson Girl 1 was in the group from situation 1, where they were confused because their sine regression would not hit the x-axis. Thus Girl 1 has got the impression that sine curves oscillate around the x-axis. But when Girl 2 suggests that the difference between the two graphs is the constant d, Girl 1 remembers that changing d can prevent the curve from intersecting the x- axis. The girl's knowledge concerning sine curves is still in development. In quote 9, Girl 1 argues that the general curve must be the correct one, because Nspire would not intersect the x-axis either, referring to the previous lesson where her group could not get the sine regression in Nspire to hit the x-axis. Girl 1 cannot rely on her own knowledge yet, so she has to use elements of the milieu to validate her hypothesis, here the fact that Nspire would not make the graph hit the x-axis, so most likely the graph should not hit the x-axis. Even though Girl 1 gets to this conclusion, she does not engage to replace the graph of $\sin(x)$ but just suggests that they explain what the constants in the expression for a general sine curve do. She believes that this is a part of the assignment, and either she finds this more important than the choice of graph or she chooses this part of the task because this is old knowledge to her or because it is explicitly described in the textbook.

In quote 11 the boy rejects the sine function to be a sine curve because of its status as a function. The boy's knowledge concerning the notion of function has the status of knowledge in development. He accepts the graph in Fig. 7.2 to be a function, but he does not realize that a sine curve also represents a function from the class of trigonometric functions. The boy makes a clear distinction between functions and curves, most of all to convince himself that it is okay that he cannot see the connection between the two graphs.

The girls continue describing the constants and the boy finds his computer and starts to construct a general sine curve in Nspire with the use of sliders. When the girls are done Girl 1 asks if there is more in the textbook. The boy flips the page and discovers an exercise about a sine curve:


Øvelse 6.3: Sinuskurve
Tegningen viser en sinuskurve.

Aflæs amplituden *a*, perioden *T*, parallelforskydningen *d*, og bestem den cykliske frekvens *b*.

Figure 9.4: Exercise in textbook (Nielsen & Fogh, 2006, p. 43)

1. Boy	Okay, there is a proper example
2. Girl 1	But that is the same as the one we have made.
3. Boy	No..
4. Girl 1	Yes it is.
5. Boy	I am quite sure it isn't. Because our function has the same oscillations on both. Our function goes from the period T, 2T [long pause]. But I think it is because it has a *d* we don't have.
6. Girl 1	Mmm..
	//
7. Boy	But then it is wrong all what we wrote in the beginning. Because here it doesn't have the period 2T all the time.
8. Girl 1	No
9. Girl 2	Oh..

10. Boy	Argh!
11. Girl 1	it is bad, huh?
12. Girl 2	What was it we had done wrong?
13. Boy	Look here. It doesn't have the same oscillations on both sides of the x-axis. So the period where we say it goes from 0 to 2T and from 2T to 4T, it isn't correct, because it only goes from 3,8 to 6,1. I think it is because they have a d. [Long pause]
14. Boy	I think that what we wrote is correct. It has to be. Because if we move from wave top to wave top there is 2π.
15. Girl 1	Yes there is.
16. Boy	But it just doesn't intersect the x-axis in π.

In quote 1 the boy probably calls this exercise a good example, because the text says "The drawing shows a sine curve". Here the group has a graph they know for sure is a sine curve. In quote 2 Girl 1 links the graph in the example with the one in Fig 7.1. It is a bit surprising that she immediately links a graph with one period to a graph with several periods. Here we must remember that the students have their CAS-tool right in front of them, so even though Girl 1 has inserted the graph with only one period in the document, she has seen it looking like the graph of $\sin(x)$ in the textbook. So in fact the girl compares the graphs in Fig 7.2 and Fig. 7.4, and the similarity here may be the fact that they both intersect the x-axis. The girl's knowledge of sine curves is still centered on this intersection.

In quote 5 the boy will not accept the two graphs to be the same because the oscillations do not look the same. The sine function oscillates around the x-axis and the sine curve in the example does not. He refers to the period of the sine function, probably because he will use this as an argument for the graphs' dissimilarity, but instead he concludes that the dissimilarity is due to the constant d. This is either because the constant d has been mentioned before in their discussion or because the knowledge concerning d is more consolidated in the boys knowledge of sine curves, than for example the knowledge of c. When trying to explain a phenomenon one will always try to find the answer in the personal old knowledge.

In quote 7 the boy confuses T with π. The function $\sin(x)$ has the period $T = 2\pi$, not $2T$. So what they have written in their document is not wrong, but they must not confuse the period of $f(x) = \sin(x)$ with the period of a general sine curve, and this is what the boy does, both here and in quote 13. When confusing T with π he gets the idea that since the sine function has the period $2T$ and oscillates around the x-axis, then all function must oscillate around the x-axis if they have the period $2T$. The boy's knowledge concerning periodicity is still new. Again he refers to the constant d as the reason why the sine curve in the example does not oscillate around the x-axis, and he is right.

After a long pause the boy finally accepts what they wrote in the document. In quote 14 he realizes that his interpretation of the period was wrong and states that the period in the example is 2π, showing us that he did confuse T with π before. But first of all the period is not 2π but 6, and secondly the boy still confuses the period of a general sine curve with the period of the sine function. In quote 16 he mentions why he got confused. If the period should be 2π, the boy would expect the curve to intersect the x-axis in π, forgetting that both the constant c and d can influence on this.

This episode is characterized by the boy's attempt to explain the difference between the graph of the sine function and the graph of a general sine curve, both to the girls but not at least to himself. The episode has a micro-contract of agreement, because the girls do not contribute at all.

Later on the boy again gets confused because his graph in Nspire does not match the one in the textbook. He does not say it, but maybe it is the meaning of period which is confusing him again.

1. Boy	Now it doesn't make any sense at all. I don't understand it
2. Girl 1	Should we add pictures of everything?
3. Boy	No, but it looks odd. [He is very focused by what he sees on his computer screen. He is changing the constants by use of the sliders].
4. Girl 2	Why?
5. Boy	It still doesn't fit. [Long pause] But we have explained what they do?
6. Girl 1	Yes, but you wanted a picture [both girl laughs].
7. Boy	So. Now it looks nicer. [He has changed the constants to a = 5, b = 2, c = 5, d=0, so the graph is as the one presented in Fig. 7.5c]

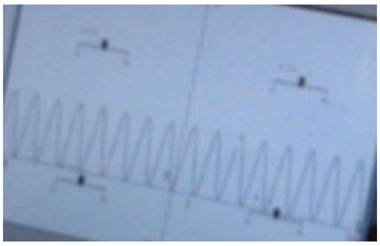


Figure 9.5a: The screen in quote 1

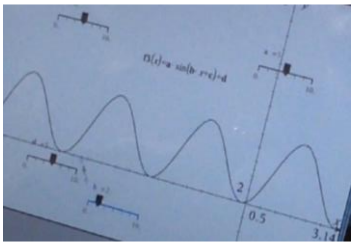


Figure 9.5b: The screen in quote 5

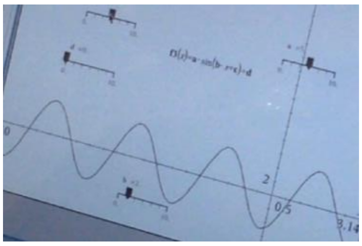


Figure 9.5c: The screen at the end

The boy wants the graph to look as much as possible as the sine function. If the function has a small period or is too far from the x-axis they boy does not accept it as a proper sine curve. To him the perfect sine curve is the sine function. He accepts his graph when it oscillates around the x-axis, so apparently $d = 0$ is still a resistant requirement to him.

The entire situation is characterized by the fact that the students easily get confused and easily change their minds. This is due to the fact that the knowledge at stake is still currently being learned. The term "sine curves" is a huge abstraction because it is a family of functions with four independent parameters. It requires a lot of work and time to interpret all four constants. Every time the students get confused they have to change their interpretation and hereby extend their knowledge. Moreover the realization of sine curves as more than one function is challenged by the fact that both the book and Nspire only presents one curve at a time.

The meso-contract of the situation is the institutionalization of knowledge concerning sine curves in general. A part of the meso-contract is that the knowledge at stake is available for the students, either as their own personal knowledge or in the milieu. Otherwise the teacher would not have left them alone. The students know that the book presents the official knowledge and use it both explicitly when they have to formulate the impact of the constants, and more implicitly when the boy for example thinks that a general sine curve is more correct to present than the sine function. Also Nspire has an authority role. What Nspire produces is considered correct by the students. If Nspire does not give the feedback the students expect, they are convinced that they have made something wrong or misunderstood something. This is seen both in the first episode where Girl 1 is convinced that a sine curve should not hit the x-axis, because it does not in Nspire, and in the last episode where they boy struggles with the feedback from Nspire.

10. DISCUSSION

In this chapter we will discuss the observed situations in order to answer the research questions. The chapter will also include a discussion of the CAS-tool and a discussion of the graphic milieu.

In situation 1 the students are challenged by the fact that Nspire produces a graph not hitting the x-axis. This is an obstacle to the students because they believe that sine curves must oscillate around the x-axis. Normally an obstacle would make the students change their actions in order to adapt to the milieu, but in this situation their old knowledge is insufficient. The question is why the teacher has chosen to give much of the responsibility for learning to the students, when clearly the knowledge at stake is new. The teacher's intention with the task was probably, that by using the well-known instrumented techniques the students could get a sine curve fitting the measured points. The fact that this sine curve most likely does not oscillate around the x-axis, will make the students extend their knowledge of sine curves. The problem was that the data collected did not make the sine curve fit the data points, hence the step towards accepting sine curves not oscillating around the x-axis is blocked by the unexpected fact that the curve does not approach the data points. The teacher tries to find reasons why the students' graph does not hit the x-axis, instead of just making it clear that sine curves do not have to oscillate around the x-axis. This might be because she believes that a graph which is neither fitting the data point nor oscillating around the x-axis is too big a challenge for the students. If the graph could just oscillate around the x-axis, the students may accept it as a sine curve, and from here the step towards acknowledging a sine curve oscillating around $y = d$ is small.

In situation 2 we saw that the teacher finds the knowledge concerning the constants in the general sine curve very important. She devotes one and a half hour to ensure that the students acknowledge these constants' impact on a sine curve. There could be several reasons for this; firstly the teacher knows that the acknowledgement requires a personal need to realize these constants' impact, a need which only appears if the students are faced with a problem where the answer lies within the constants. Secondly the teacher knows that the acknowledgement will require some time. As mentioned in section 8.2.3 the students have to interpret sine curves as a family of functions with four parameters, this requires the ability to both acknowledge the constants separately, but also acknowledge their compound impact on a sine curve. Moreover the students have to work with both an analytic and a graphical representation of a function and this switch between representations may also require extra time. Thirdly the teacher must find the knowledge concerning the general sine curve very important. This is probably due to the fact that sine curves have been represented in the written exams for the last five years (section 2.3).

Situation 3 gives us an insight in the students' personal knowledge after a course in trigonometric functions. An interesting part is that the assignment given is so open that the students have to create smaller tasks in order to create milieus wherein they can work. This assignment is only possible because the knowledge the students are supposed to (re)institutionalize is already made personal and partly institutionalized by the teacher. The question is then whether the students learn something during this (re)institutionalization. The assignment alone does not ensure that knowledge is developed, but in the situation we saw that the students did extend their knowledge concerning sine curves. Even though the constants in the expression for a general sine curve were institutionalized by the teacher, the students had a hard time comparing different sine curves and realizing that the difference was due to the constants. The boy realizes that there is a distinction between a function and a curve, but he does not realize that a curve is a representation of a function. He considers two graphs and concludes that one is a function and the other a curve. His knowledge concerning functions does not include different representations of functions, because if it did, he would have known that both graphs are curves representing different functions. His knowledge concerning a family of functions is also weak, and that is why he so desperately wants to distinguish the two graphs. He does not realize that they are both elements in $\mathcal{F}_{trig}$. The fact that the notion of function is abstract to the boy may inhibit his acknowledgement of sine and cosine as functions, and this could also affect his acknowledgement of the sine curves.

A reason why the teacher chooses an assignment like the one in situation 3 is probably that she believes that the argumentation and discussion, concerning the selection of which knowledge is suitable for the given subject, is helping the students developing their knowledge. Moreover the documents created can be used as an evaluation of the students' knowledge, but only few students participated in creating each document. More likely the assignment ensures the teacher that the students have proper notes for the exam.

In all three situations periodicity is a challenge for the students. This is probably linked to the transition from unit circle to graph. If the students are aware that the graph of sine emerges as we traverse the unit circle, then they will be able to connect the period of a given sine curve to the circumference of the circular motion it models. On the other hand, the more abstract this transition is to the students the harder it gets to understand the periodicity.

Linking the three situations we see that the teaching is focused on sine curves instead of the function sine. The focus is to acknowledge the possibilities connected to a sine curve more than to acknowledge the transition from unit circle coordinates to trigonometric functions. This is probably because the teacher uses the textbook as a guideline for her teaching and here the transition is not explained at all (Nielsen & Fogh, 2006). This didactical choice made both by the official instances and the teacher is most likely caused by the fact that the mathematical knowledge required for realizing this transition is way beyond what is expected in high school (see section 2.2.4). But maybe this realization could help the students acknowledging sine and cosine as functions. Most likely the students interpret a function as an input-output mechanism. If they can realize that sine tells them how far they are from the x-axis when they have travelled a given distance on the unit circle, then sine can be interpreted as a function. Likewise cosine can be interpreted as the function taking the distance on the unit circle as input and giving the distance to the y-axis as the output. If the students realize this, they also realize the transition from the unit circle to the graph. Thus the question is what is needed to make this realization. The natural parametrization mentioned in section 2.2.4 is the mathematical foundation for the existence of sine and cosine as functions. Most likely all of this does not have to be acknowledged in order to interpret sine and cosine as functions. But the fact that there is a one-to-one correspondence between the real line and unit circle may help the students in the process.

Another reason for skipping a detailed explanation of the transition could be that the teacher finds it more important that the students know that sine and cosine are elements in the family of functions called "sine curves". Besides mathematics the teacher also teaches physics and knows how useful sine curves are to model a great amount of periodic phenomena like oscillations, sound waves, the temperature throughout the years etc. In continuation of this the result of sine regression is a sine curve, as we saw in situation 1. This is another reason for focusing on sine curves instead of the sine function.

The transition from sine as a value, either in a triangle or on the unit circle, to sine as a function is a big abstraction. But the next step; to realize sine as a family of function is an even bigger abstraction for the students. The students are challenged by the fact that they have to realize that different functions can all be the solution to a specific task, because they are all a part of the same family of functions; sine curves. Moreover they have to relate to four different parameters when they validate a given sine curve. These challenges concerning sine curves may be the reason why the teacher chooses to spend most of her teaching on those.

Even though the teacher does not spend much time on the transition from unit circle coordinates to functions, she uses most part of the first lesson to introduce the angle measure "radians". Her motivation for this is that we want to give sine and cosine real numbers as inputs instead of degrees, and this requires radians. But why does a radian number equals a real number? Here the natural parametrization of the unit circle becomes convenient again. The radian number is a measure for the arc length subtending an angle in the unit circle, and this arc length has a one-to-one correspondence to the real line because the natural parametrization of the unit circle exists.

10.1 The status of the CAS-tool

In all the observed lessons the students had their computers right in front of them. The computer is the one tool the students always bring along and the teacher also uses it as an integrated part of the teaching. When the students have to realize the graphic path of sine and cosine in lesson 2, the teacher asks them to plot the functions in the CAS-tool and thereafter explain to her what they see. Then she explains how this graph relates to points on the unit circle. Before computers were a part of teaching, the realization of the graphic path would have proceeded strictly opposite. The students would have had to create the graphs by considering the x- and y- coordinates after traveling certain distances on the unit circle. The use of the CAS-tool in a task like this may inhibit the students' interpretation of sine and cosine as functions. Also when the students are introduced to the derivative and antiderivative of sine and cosine it is done through the CAS-tool. The teacher simply asks them to differentiate and integrate the functions using the CAS-tool and next they discuss the results in plenum. It seems like the presence of the CAS-tool makes it legal to use it anytime possible, in fact it is in favour over for example the textbook. This may also be the reason why the students give the CAS-tool an authority role as we saw in situation 3. They have observed the teacher using the CAS-tool both in her introduction to new knowledge, in her validation of knowledge in development and in the institutionalization. Thus the students assume that the CAS-tool is an authority they can rely on, and in most situations it is.

10.2 The graphic milieu

In all three situations the students interact with a graphic milieu, and in all of them the interaction concerns a general sine curve. The graphic milieu makes the interpretation of the constants in $f(x) = a\sin(bx + c) + d$ easier. Especially the constant c is difficult to explain without the sliders in Nspire. It may also be easier to relate to a family of curves instead of the family containing all the functions $f_{a,b,c,d}$. The sine curves is just all the curves we can get when stretching, squeezing and moving the sine function both horizontal and vertical. But to grasp that every time we change one of the constants in $f_{a,b,c,d}$ we get a new function, but still an element in

$$\mathcal{F}_{trig} = \{f_{a,b,c,d} | a, b, c, d \in \mathbb{R}\} \text{ where } f_{a,b,c,d}(x) = a\sin(bx + c) + d$$

is difficult. Just the notation will confuse the students. This is probably also the reason why both the textbook and the teacher talks about sine curves instead of the class of trigonometric functions. The graphic milieu is put on privileged form with the term "curves". Still it can be a problem that the media, such as the textbook and Nspire, only display one curve at a time. It strengthens the students' idea of *the* sine curve instead of *a* sine curve as an element in the family of sine curves. However, this is improved with the sliders in Nspire, where one function is easily changed into another. The students just have to remember that every time the curve changes just a bit it represents a new function.

The question is whether this graphic milieu can ease the transition from sine and cosine as unit circle coordinates to functions. As we described earlier the transition is explained by the teacher focusing on the graph and then relating it to the unit circle. Here the graph is just a media, not a milieu the students can act on. This prevents the students from making the knowledge personal. Instead the students should consider the unit circle and then by themselves create the graphs, either in Nspire or on a piece of paper. Thus the graphic milieu can ease the transition, but it requires that the students are familiar with the notion of function. If this is not the case it will be difficult to acknowledge sine and cosine as functions.

Isabelle Bloch (2003) has made a study proving that students working within a graphic milieu became able to think of functions as objects and much better to formulate and discuss mathematical knowledge. Bloch creates a tool called "paths" and with these paths the students can prove whether a given graph represents a function or not. The rule is that a graph represents a function if there is only one direct path from each point of the x-axis (Bloch, 2003). With a tool like this the students could easily acknowledge the graphs of sine and cosine as functions.

11. CONCLUSION

The aim of this thesis was to elucidate why high school students have problems with the transition from working with sine and cosine as geometric tools in (essentially) triangles to working with them as functions. In the subject-matter didactic analysis we saw that one of the challenges was to move from angles to real numbers as input. We also saw that sine and cosine can be defined through different branches of mathematics; in fact they can be defined in terms of both arcsine and power series, eliminating the "angle-to real number" problem. The problem here is that the mathematical theory behind this definition is too difficult for high school students. Hence the students have to face the transition from angle to real number. We suggested in the end of chapter 2 that this transition could be clarified with an introduction to the natural parametrization of the unit circle, and the fact that there is a one-to-one correspondence between the real line and the unit circle. The subject-matter didactic analysis also showed that the problems with the transition from geometric tool to functions may lie within the students' conception of the notion of function. The students are not introduced to a formal definition of functions, but only to examples of functions. They lack a clear explanation to why a certain expression or graph represents a function, an explanation, which can be used to accept sine and cosine as functions.

The observations at Gefion high school showed that the teacher has chosen to let the students acknowledge sine and cosine as functions through sine curves. She does not focus on the transition from unit circle to graph, but focus instead on how to make the students work with, and acknowledge the possibilities of sine curves. This is a result of the external didactical transposition. Neither the textbook nor the Syllabus mentions this transition as important to realize. Instead the transposition focuses on the sine curves, both through the requirement of knowing sine regression in the Syllabus and the presence of sine curves in exam tasks. The students do extend their knowledge concerning sine curves, but as we saw in situation 3 this does not help them acknowledging sine as a function. Instead they make a separation between curves and function and only focus on curves.

Even though the focus is not on the transition from unit circle to graph, the teacher does explain the connection. From the theory of didactical situations we know that in order to learn, the student has to produce the knowledge personally. This is not the case here. A reason why the students have problems with the transition is because they do not realize, through personal actions, that sine tells them how far they are from the x-axis when they have traveled a given distance on the unit circle, and cosine tells them how far they are from the y-axis. If the students could realize this they could produce the graphs of sine and cosine. A challenge here is to connect the arc length of the unit circle to the real line. This could be solved by introducing the natural parametrization of the unit circle, as mentioned in the subject-matter didactic analysis. When the students have acknowledged the graphs as representing sine and cosine, they can use the vertical line test to acknowledge the graphs as functions.

The fact that focus is on sine curves and not functions, shows us that the graphic milieus are preferred. They are first of all presented to the students through the CAS-tool. In all three situations the students produce sine curves in Nspire or Maple, either as a result of sine regression or by the use of sliders. Also the smartboard is a part of the graphic milieu, but in fact it only displays what the students have already produced themselves in their own CAS-tool. It is not until the last lesson, that the textbook becomes a part of the graphic milieu. It is in situation 3 where the students try to compare two graphs presented in the book.

The graphic milieus play an important role in acknowledging sine and cosine as functions; first of all because their status as functions are stated by the teacher through graphs, secondly because sine curves are used as the representation of the trigonometric functions. In all the observed situations the graphic milieu had a high adidactical potential, especially when the milieu involved the use of sliders in Nspire. Hence the graphic milieus helps the students extend their knowledge of sine curves, but since the students do not know when to accept a curve as a function, this knowledge does not improve their acknowledgement of sine and cosine as functions.

To sum up, both the subject-matter didactic analysis and the observations at Gefion high school has shown that there is at least two problems concerning the transition of sine and cosine from geometric tools to functions. Firstly a lack in the perception of functions, secondly the fact, that focus is not on the actual transition, but more on the possibilities the trigonometric functions gives us, illustrated though the sine curves.

Thus, if we want the students to acknowledge sine and cosine as functions, we may first of all introduce them to a formal definition of function, and not just examples. Next we must explain to them how traveling the unit circle can be interpreted as traveling the x-axis, and then let them try to produce the graphs of sine and cosine. If they understand a function as an input-output mechanism they will realize that if we give sine a real number as input we get the distance to the x-axis as output. Hence sine can be acknowledged as a function in the domain of the real numbers.

BIBLIOGRAPHY

Barret, J. (2012). *Trigonometriske funktioner i en gymnasiel kontekst - en epistemologisk referencemodel.* Copenhagen: Deparment of Science Education.

Bloch, E. D. (2011). *The real numbers and real Analysis.* New York: Springer.

Bloch, I. (2003). Teaching functions in a graphic milieu: What forms of knowledge enable students to conjecture and prove? *Educational Studies in Mathematics, 52*, 3-28.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics: Didactique des mathématiqes 1970-1990.* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.) Dordrecht: Kluwer Academic Publisher.

Camilus, A. K. (2012). *Validerignssituationer i undervisnignsforløb om differentialligninger og radioaktivitet.* Copenhagen: Department of Science Education.

Carstensen, J., Frandsen, J., & Studsgaard, J. (20008). *MAT A2.* Århus: Systime.

Demir, Ö. (2012). *Student's Concept Development and Understanding of Sine and Cosine Functions.* Amsterdam: Universiteit van Amsterdam.

Eilers, S., Hansen, E., & Madsen, T. G. (2014). *Indledende matematisk analyse* (5th ed.). Copenhagen: Department of Mathematical Science.

Godiksen, R. B., Jørgensen, C., Toldbod, B., & Hanberg, T. M. (2014, November 25). Fourier og Funktionsbegrebet - overgangen fra Eulers til Dirichlets funktionsbegreb. Roskilde.

Hersant, M., & Perrin-Glorian, M.-J. (2005). Characteriation of an ordinary teaching practice with the help of the theory of didactic situations. *Educational Studies of Mathematics, 59*, 113-151.

Katz, V. J. (2009). *A History of Mathematics - An Introduction* (3rd ed.). Pearson Education.

Kiming, I. (2007). *Mathematical Method.* Copenhagen: Pearson .

Lemke, J. (1990). *Talking science. Languages, Learning and Values.* Norwood: Ablex Publishing Corporation .

Lindstrøm, T. (2006). *KALKULUS* (3rd ed.). Oslo: Universitetsforlaget.

Matematik A – stx, juni 2013. (2015, November 25). Retrieved from stx-bekendtgørelsen: https://www.retsinformation.dk/Forms/R0710.aspx?id=152507#Bil35

Måsøval, H. S. (2011). *Factors Constraning Student's stablishment of Algebraic Generality in Shape Patterns: A Case Study of Didactical Situations in Mathematics at a University College.* Faculty of Engineering and Science. Kristiansand: University of Agder.

Nielsen, K. E., & Fogh, E. (2006). *Vejen til matematik A2.* Silkeborg: Forlaget HAX.

Orhun, N. (n.d., Januar 15). *Students' Mistakes and Misconceptions on teaching of Trigonometry.* Anadulo University Science Faculty Mathematics Deparment.

Rudin, W. (1987). *Real and Complex Analysis.* Singapore: McGraw-Hill Book Co.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky, & H. Guershon, *The concept of function: Aspects of epistemology and pedagogy* (pp. 25-58). Washington, DC: Mathematical Association of America.

Spector, L. (2015). *Derivatives of Trigonometric functions*. Retrieved January 29, 2016, from www.themathpage.com: http://www.themathpage.com/acalc/sine.htm#sine

Vestergaard, E. (2007). *www.matematiksider.dk/vejgeometri/vejgeometri.pdf.* Retrieved January 27, 2016, from matematiksider.dk: http://www.matematiksider.dk/vejgeometri/vejgeometri.pdf

Warfield, V. M. (2006). *Invitation to Didactique.* Seattle: University of Washington.

Weber, K. (2005). Students' Understanding og Trigonometric Functions. *Mathematics Education Research Journal*, 91-112.

Wikipedia_degree. (2015, December 3). Retrieved from Wikipedia.

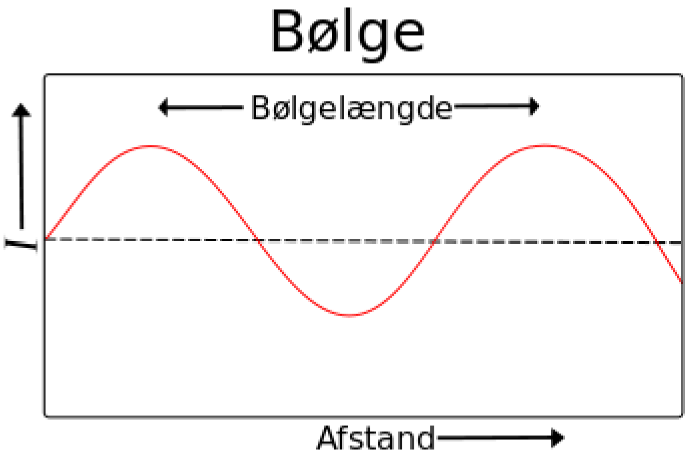
Wikipeida_angle. (2015, December 1). Retrieved from Wikipedia: https://en.wikipedia.org/wiki/Angle

Winsløw, C. (2006). *Didaktiske elementer - En indføring i matematikkens og naturfagenes didaktik.* Copenhagen: Forlaget biofolia/Forlaget Samfundslitteratur.

APPENDIX 1

Trigonometriske funktioner

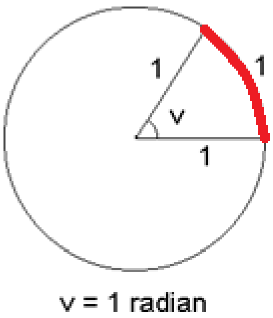
Vi støder ofte på fænomener, der varierer periodisk i løbet af tiden f.eks. tidevand, vekselstrøm, planetbevægelser og de fleste former for bølger.

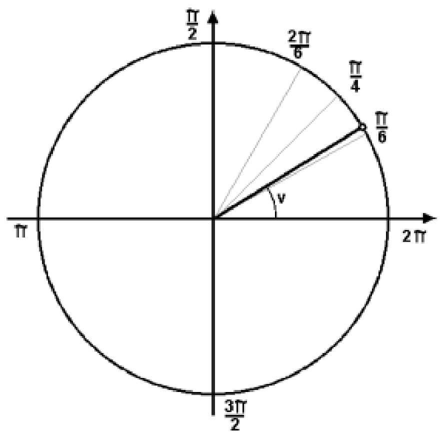


Kurver, der varierer pænt er ofte såkaldte sinusbølger. Vi vil i det følgende arbejde med sådanne kurver.

For at beskæftige sig med disse funktioner, har vi brug for et andet mål for en vinkel end grader. Vi indfører vinkelmålet radianer.

1. Sinus og cosinus blev defineret ud fra enhedscirklen altså en cirkel med centrum i (0,0) og radius 1. Hvis vi lægger en vinkel ind i koordinatsystemet, så dens toppunkt er i (0,0) og højre vinkelben er ud af z-aksen, vil vinkel afskære et buestykke på enhedscirklen. Vi indfører nu et nyt mål for en vinkel som er længden af dette buestykke. Vi betegner det radiantallet for vinklen. Se figuren.





Gradtal	30	45	60	90	18(			
Radiantal								

a) Bestem radiantal for følgende vinkler:

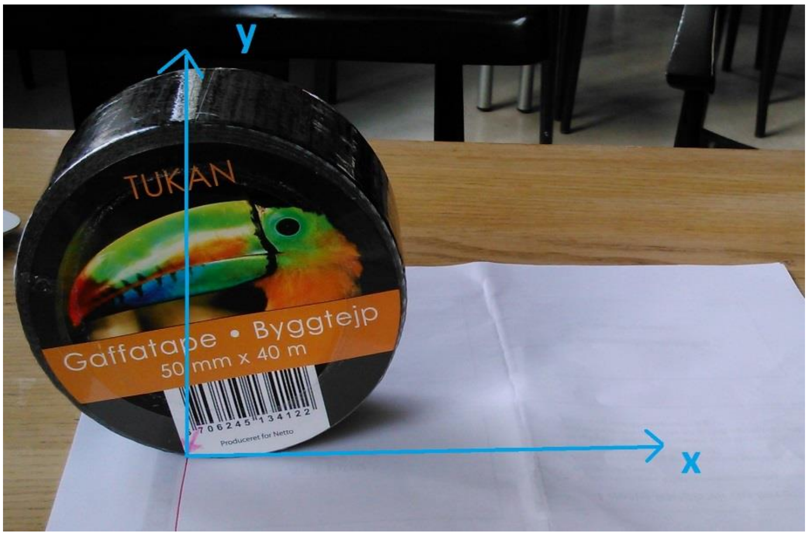
Når man regner i radiantal, kan "vinklen" godt være negative eller over 360°. Til ethvert reelt tal svarer således en vinkel.

2. Oversæt følgende tal, x til vinkler v (mellem 0° og 360°).

x	1	2	π	2π	7	10	20	-4
v								

3. Argumenter for, at man kan tale om sinus og cosinus til et tal.

4. Vi vil undersøge hvordan et punkt på en cirkel bevæger sig, når cirklen bevæger sig fremad. I skal trille en dåse eller lignende, der har markeret et punkt på periferien af cirklen.



For hvert omløb, skal I have 7-10 værdier af x og y. Mål 2 til 3 omløb.
Bagefter skal I lave et punktplot af (x,y).

a) Hvad er funktionens periode? – hvordan hænger det sammen med "hjulets" diameter?

b) Hvad er forskellen på laveste og højeste y-værdi? – og hvordan sammenlignet med "hjulets" diameter?

5. Tegn grafer for funktionerne $f(x) = \sin(x)$ og $g(x) = \cos(x)$. I Nspire skal vinklen være indstillet til at regne i radianer. I Maple skal man skrive sin og cos med små bogstaver.

6. Argumenter for, at $\sin(x + p \cdot 2\pi) = \sin(x)$ og $\cos(x + p \cdot 2\pi) = \cos(x)$.

7. Gå nu tilbage til opgave 4, hvor du arbejdede med hjulet. Prøv at lave en sinusregression på data fra eksperimentet.

8. Ballongyngen i Tivoli har en diameter på 15 m. Et omløb tager 15 sek. Turen starter 5 m over jordoverfladen.
   a) I skal finde ud af, hvordan højden af kurven man sidder i ændrer sig med tiden.

   b) Hvor højt er man efter 20 sek.?

   c) På hvilket tidspunkt er man første gang 17 m over jorden?

9. Plot funktionen $f(x) = \tan(x)$ i [-2π; 2π].

   a) Hvad sker der i punktet x = $\frac{\pi}{2}$?

   b) Er der andre tilsvarende punkter?

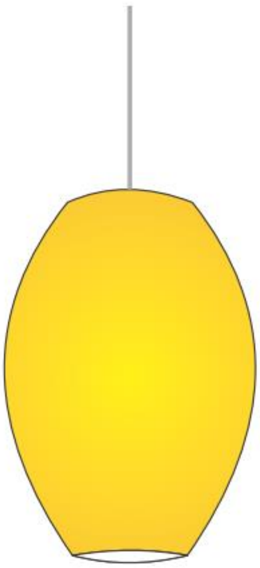
   c) Angiv definitionsmængden for funktionen $f(x) = \tan(x)$.

10. En funktion f er givet ved

$$f(x) = 6{,}5\sin(0{,}0849x) + 6\,.$$

Grafen for f afgrænser sammen med koordinatakserne og linjen med ligningen $x = 38$ et område M, der har et areal.

a) Skitsér grafen for f, og bestem arealet af M.



11. En loftslampes ydre har samme form, som overfladen af det omdrejningslegeme, der fremkommer, når M drejes 360° omkring førsteaksen, idet enheden på koordinatsystemets akser er cm.

Det oplyses, at overfladearealet af dette omdrejningslegeme kan beregnes ved integralet

$$O = 2\pi \int_0^{38} f(x) \cdot \sqrt{1 + f'(x)^2}\, dx\,.$$

b) Bestem lampens overfladeareal.

12. Funktionen $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Benyt brøkreglen til at finde differentialkvotienten af $\tan(x)$.

13. Løs følgende opgaver:

    a) Bestem den afledede funktion $f'(x)$ til funktionen $f(x) = 1 + 3{,}22\sin(2{,}14t - \pi)$.

    b) Bestem den afledede funktion $f'(x)$ til funktionen $f(x) = 2\cos(3x - 4)$.

    c) Bestem den afledede funktion $f'(x)$ til funktionen $f(x) = 3x - 1 + \cos(x)$.

    d) Bestem integralet $\int_0^\pi \sin(x)\,dx$.

    e) Bestem integralet $\int_0^\pi \mathrm{x} \cdot \sin(2x^2 - \pi)\,dx$.

    f) Bestem rumfanget af omdrejningslegemet der fremkommer, når grafen for $f(x) = 2\sin(x) + 1$ i [0;π] drejes 360° om x-aksen.

APPENDIX 2

Trigonometriske funktioner - forløbsplan

Onsdag 16.9

Mål:

Eleverne skal have en fornemmelse af hvad trigonometriske funktioner bruges til.

De skal forstå og kunne anvende begrebet radianer og omsatte mellem grader og radianer.

De skal prøve at arbejde i praksis med trigonometriske funktion (opgave 3)

Kort intro om anvendelser – periodiske funktioner.

Elever arbejder med opgave 1 om hvad radianer er og omregninger begge veje. De introduceres til sinus og cosinus til et tal i stedet for en vinkel.

Opgave 1-4 (Arbejd selv fra 9.00).

Hvis der er tid skal de også tegne grafer for $f(x) = \sin(x)$ og $g(x) = \cos(x)$.

Torsdag 17.9

Mål:

Kunne tegne trigonometriske funktioners grafer vha. CAS-værktøj.

Kunne argumentere for periodicitet.

Kunne foretage sinusregression i CAS-værktøj.

Kunne opstille en model ved anvendelse af en trigonometrisk funktion.

Vi snakker om graferne for $f(x) = \sin(x)$ og $g(x) = \cos(x)$. Eleverne regner opgave 6 og vi snakker om funktionernes periodicitet.

Vi vender tilbage til opgave 4 og foretager sinusregression på CAS. Opsamling.

Vi regner opgave 8 om ballongyngen (model).

Evt. regnes opgave 10.

Fredag 18.9

Mål:

At kende tangensfunktionens graf og definitionsmængde.

At kunne regne opgaver med trigonometriske modeller.

Opsamling fra sidst.

Opgave 9 om tangensfunktionen regnes. Opsamling.

Vi regner opgave (10 og) 11.

Mandag 21.9

Mål:

At kunne differentiere og integrere udtryk med trigonometriske funktioner.

Opsamling fra sidst Lektie s.40-45 (opgave 10 og 11).

Differentialkvotient af sin, cos og tan findes vha. CAS.

Stamfunktion af sin, cos og tan findes vha. CAS.

Opgave 12 og 13 regnes.

APPENDIX 3

Transcription of the dialogue in situation 1

Boy 1: We can't make it work.

Girl 1: Oh. You can't make it work. But one is missing.

Boy 1: We have our points here, but the graph doesn't want to go to zero.

Girl1: That is the same problem with ours. Our regression.. It just says that in Nspire it will not do it apparently. It must be because it doesn't approximate.

Boy 2: But it is not even close to the points if you see.

Girl 1: Oh. Can I see?

Boy 2: It is almost the opposite of the points.

Boy 1: Doesn't it look like it is the half of the points?

Boy2: Yes, that's what it does.

Girl 1: What? Is it the half of the points? [Go gets her own computer]

Boy 2: [Looks at girl's computer, which shows a graph not hitting the x-axis, but with the same period as the points plotted.] Okay, yours also looks weird.

Girl 1: So ours up here. Here it works fine enough.. It doesn't hit a point.

Boy 2: It still doesn't hit zero.

Girl 1: It still doesn't hit. Look we get the same [shows her computer to the teacher]

Teacher: But what is the period here? It matches. It has two arcs.

Boy 1: Yes, that is correct. It matches.

Teacher: Yes. The other thing is just because you have chosen exactly one above, that's why it doesn't come down. But this one, it matches [points at the graph].

Girl 1: But how can it be that Nspire…

Teacher: Try to plot your points in this window [points at the graph again].

Teacher: [To girl] it is because the distance is too far from the others to that point.

Girl 1: Ohh.. It was just because I thought it was a bit weird.

Teacher: It isn't because you used a ruler there didn't started in zero, but a little bit over?

Boy 1: No, we used my ruler and it added 0,6, so we subtracted 0,6 from all points.

Girl 1: Maybe the paper has been moved a bit.

Boy 2: Yes, there is a lot sources of errors.

APPENDIX 4

Transcription of the dialogue in phase 2A

Girl 1	Bente...Why is it, that when I start by letting the c-value be zero, then my graph doesn't starts in zero?
Teacher	It has something to do with the sine part.
Girl 1	So, this is our graph [points at her screen]. But if one lets c be zero, because it displaces it.
Teacher	Yes, then the whole sine part becomes zero.
Girl 1	No, because it is just added by zero. So it just.. This one just becomes "sine to bx"
Teacher	Yes, and x is zero here, right? At the y-axis x is zero.
Girl 1	Yes, then the whole sine part becomes zero.
Teacher	Yes, then the whole sine part becomes zero. But then there is still d left, so in zero you start in d.
Girl 1	Okay, so if we let this one be zero [refers to the slider for d].. No, that we can't.. Yes! Now we start in zero.
Girl 2	But then it goes under the earth.
Girl 1	No, but that was because I wanted to see why it was that if one had c, which displaces this one, why was it that you didn't started in zero and that was simply because of d at the end, which is the decisive factor.
Teacher	Mmm
Girl 1	But how can we then calculate the real b- and c- values?
Teacher	Yes, you may consider that. You can consider that if you have the period here [points at the graph on the screen]. What does that have to do with b?
Girl 1	It lasts b, the period. Or half the period.
Teacher	It has something to do with the period.
Boy	That is how long it takes for each wave.
Girl 1	The length of the period
Boy	Yes the length of the period.
Girl 1	It was 15 seconds it took.
Teacher	Then try to look here. What is the period here?... In relation to what b is.
Girl 1	b is 1 here and the period is…
Boy	The period is also the diameter
Girl 1	[Tries to read the period directly from the screen] 1, 2, 3, 4, 5.. A little bit above 5, maybe. I cannot quite see it.
Teacher	No, but then you can find to maxima.
	[the girl works in NSpire]
Girl 1	Okay, 6,28
Teacher	Mmm, and what is that?
Girl 1	That is the length
Teacher	What is 6,26? It is a very nice number.
Girl 2	It is two times π
Teacher	Yes. So now you can consider why it is exactly 2π.
Girl 2	That is because it is the circumference
Boy	It is the circumference.

Teacher	So when b is 1 you get 2π. What if b is 2, what will you get then?
Boy	Then, we will get... Then we will get…
Girl 1	[Changes b to 2 in Nspire] Then we get the same.
Boy	Yes
Girl 2	Do we get the same?
Girl 1	Yes, we get the same. It is the only change we have made, the one with b.
Teacher	[Looks at the screen and sees that Nspire now have found maximum of the first and third wave top] But, try to look here. This isn't the distance between the neighbor holes.
Girl 1	Oh yes, it has measured the third hole.
Teacher	But you are allowed to take these to values, then you just have to remember that there are two periods.
Girl 1	Yes. So we have to divide by 2, then we get π.
Teacher	So this here was if it was x and that one [Tries to point on something above the screen] was if it was $2x$.
Girl 1	Yes, So now we need to have a period which is 15.

[The teacher laughs a little, and tries to find a value of b that can help the students in their formulation of b. She repeats the quote "What if it was.." without suggesting anything. The students is focused on Nspire]

Girl 1	Okay, now we have it on 4.
Teacher	You could try to guess
Girl 1	[Hesitantly] It must be the half? One quarter?
Teacher	So when you make b…
Girl 1	Greater
Teacher	Then..
Girl 1	π becomes smaller.
Teacher	Then the period becomes smaller.

[The rest of the talk disappears in noise, since it is time for the break]

APPENDIX 5

Transcription of the dialogue in phase 3B

Teacher	Okay, I think we will make a recap. [Projects her computer screen on the smartboard] I have made it with sliders because it is probably the easiest. It is quite inconvenient in Maple, so that I haven't done. But here you have some sliders, so you can change the values of a and b continuously and see what happens with them. And we did more or less agree what happens with a, right? Camilla?
Camilla	Eeh, the highest and lowest y-value.
Teacher	Yes. We can do this here [changes the slider for a]. Now a becomes a greater number, and we can see that the oscillations becomes greater. So what you can say is that a determines the oscillation in relation to the equilibrium position. If we try to find maximum and minimum for this one [Uses Nspire], then we see that minimum is 2,5 and maximum is 12,5, so there is a difference on 10. This difference is exactly $2a$, because it oscillates a both ways, right? When a is 5 it can go 5 up and 5 down, and that corresponds to 2,5 and 12,5.
Girl	What was it you said? It determines the oscillations in relation to what?
Teacher	The equilibrium position. So, I can move it a little bit up, then it goes from 0,5 to 14,5, that is a difference on 14 and now a is set to 7, right? You could of course move… So, if I want it to lies around the equilibrium position, then we agreed that d must be zero, right? I must move d down to zero, and then I just have to change my graph window [She does that]. Yes, now it fits. So this was a, and d we already have discussed. Now d is zero, so it oscillates equally on both sides. Then we have b and c. Some of you examined what happens with b, right? What happens when you changes b? Eric?
Eric	The length of the period changes
Teacher	Changes, yes. So when you make b greater…
Eric	Then the period becomes smaller
Teacher	Yes. We can see that by changing this one. Now I make b greater, then I get a smaller period, and if make b small, then I get a very long period. So in a way you can say that b and the period.. [Is interrupted by a student, but the interruption is left out here]
Teacher	But we agree that when you make b greater the period becomes smaller. And the period is often called T, at least in physics. Then one can say that b and the period must be inversely proportional. That is, if you make b twice as big, then the period becomes the half and vice versa. I am sure you will discover this. I think some of you already touched the idea. So the question is. If they are inversely proportional, then there product must be some constant. What constant is this? Here we can just see what happens if for example b is 1 [She uses the sliders to illustrate]. How can I read the period most accurately? Martin?
Martin	Isn't it form wave top to wave top?
Teacher	Yes. [She uses Nspire to find the two maxima]. How can we then calculate the period? I have found the two point where there is a maximum. Here x is

	equal to 0,854 and here x is equal to 7,14. The period is how far there is between these two points, and what is that? Nanna?
Nanna	It is 7 minus 0,854
Teacher	Yes, $7{,}1 - 0{,}854$. And if we calculate it what do we get? I think. Didn't some of you do it? Kimmi?
Kimmi	It is 2π
Teacher	It is in fact 2π. So the product of b and the period is 2π, or one can say that b is 2π divided by the period. So if you know the period you can figure out b, and also the other way around, if you know b you can figure out the period.
Teacher	So that was the b. Then we only lack the c. This we have up here. What happens when I change the c? Nille?
Nille	Then the curve displaces itself on the x-axis.
Teacher	Yes, so it just changes. It doesn't changes shape or size, it is just displaced. So it has something to do with where the curve starts. If I let c be zero, where does it start then?... Now I look a bit confused, I thought it would start in zero. [Changes her sliders in Nspire] Maja?
Maja	Then it starts in zero
Teacher	Yes, then it starts in zero. When c is zero it starts in zero.
Kimmi	But only if the d-value also is zero.
Teacher	No, that doesn't matter. If c is zero it says $\sin(bx)$ and if x is zero bx becomes zero. Then it says $\sin(0)$, which is 0, So if c is zero, it will go through the point (0,0). If I for example says that c is 2π…
Kimmi	There is a plus between bx and c.
Teacher	Yes it says b times x plus c, and if c is zero…
Kimmi	You told us that it was about the d. You told us just before, because we had c to be zero and then it went through -1, and that was because of our d.
Teacher	Oh, yes. d also have an influence. But now I also have d to be zero here. So the equilibrium point will be in zero. But the c-value you can use to make it have a certain initial value. If we consider the case with the balloon swing, then the value of c depends on where you are, when it starts to turn around. But I just think we will write that c is the displacement on the x-axis.

APPENDIX 6

The transcription of the dialogue in situation 3

Boy	But shouldn't we have a picture of this function instead of this? [Points first at the graph for a general sine curve and next at the graph of the function $f(x) = sin(x)$]
Girl	No, why?
Boy	Yes, because this one [points at the general curve] looks a lot different. It doesn't intersect the x-axis.
Girl 2	Oh, but isn't that just because..
Girl 1	It MUST intersect the x-axis.
Girl 2	.. they have inserted a d?
Girl 1	Yes. It is the d that does it. The thing where it intersect.
Girl 2	But beside that, isn't it just the same principle?
Girl 1	Yes, yes. But I think this one [points at the general curve] is more correct than that one [points at $f(x) = sin(x)$]. Because our Nspire, it wouldn't intersect the x-axis. I remember that from last time. [small pause] But let us just explain what a, b and c are.
Girl 2	Yes, eeh.. Okay, a… [Looks in the textbook]
Boy	But I just don't think this one is a sine curve. It is a sine function. That is why it doesn't make any sense.
Girl 1	Of course it is. It is this one. Isn't it?
Boy	No, because this one is just $f(x) = sin(x)$. It is just the sine function. This one [points at the general curve] is sine CURVES. Here you work with the constants a, b, c and d.
Girl 1	Mmm..
Boy	Try to make one like this in Nspire
Boy	Okay, there is a proper example
Girl 1	But that is the same as the one we have made.
Boy	No..
Girl 1	Yes it is.
Boy	I am quite sure it isn't. Because our function has the same oscillations on both. Our functions goes from the period T, $2T$ [long pause]. But I think it is because it has a d we don't have.
Girl 1	Mmm..
Boy	Then I don't understand how we do. Because then we need some points, and in the same time we should be able to find a,b,c and d and insert the regression.
Girl 1	Yes, the sine regression.
Boy	So I don't understand what we are supposed to say about sine curves in general. Should we insert a picture? Because did you have the points from last time? Then we could just use them couldn't we? If we want to add an example.
Girl 1	Yes, I have the ones we made in the group.
Boy	You can just make that then. Then insert that one.
Girl 1	Yes, we can do that

Boy	But this it is wrong all what we wrote in the beginning. Because here it doesn't have the period $2T$ all the time.
Girl 1	No
Girl 2	Oh..
Boy	Argh!
Girl 1	it is bad, huh.
Girl 2	What was it we had done wrong?
Boy	Look here. It doesn't have the same oscillations on both sides of the x-axis. So the period where we say it goes from 0 to $2T$ and form $2T$ to $4T$, it isn't correct, because it only goes from 3,8 to 6,1. I think it is because they have a d.
Girl 2	Yes, it is because there is missing a d. We need to show where d is here [points at the example in the textbook].
	Girl 1 types on her computer
Girl 1	Look. Her is a sine regression, right?
Boy	Mmm. [Long pause] I think that what we wrote is correct. It has to be. Because if we move from wave top to wave top there is 2π.
Girl 1	Yes there is.
Boy	But it doesn't intersect the x-axis in π.
Boy	Now it doesn't make any sense at all. I don't understand it
Girl 1	Should we add pictures of everything?
Boy	No, but it looks odd. [Is very focused by what he sees on his computer screen. He is changing the constants by use of the sliders].
Girl 2	Why?
Boy	It still doesn't fit. [Long pause] But we have explained what they do?
Girl 1	Yes, but you wanted a picture [both girl laughs].
Boy	So. Now it looks nicer. [He has changed the constants to $a = 5, b = 2, c = 5, d = 0$, so the graph is as the one presented in Fig. 7.5]
Girl 1	That one you can't really show in any way. Or maybe you can if you insert a before- and an after- picture.