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Logical aspects of equations and equation solving

Upper secondary school students' practices with equations

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Abstract

This thesis concerns danish upper secondary school students' practices with equations. Data for this thesis have been collected through questionnaires to which 161 students spread over eight classes from two different upper secondary schools responded.

The purpose of the thesis is to study how the students define the concept of equations, which difficulties they meet when solving equations and how they perceive the equality sign in equations. Moreover, the students' conceptions of equations are compared to the scholarly concept of equations, and some textbooks for upper secondary school are analyzed in terms
of their presentation of equations and compared to the students conceptions.

Of particular interest to this study is the four features: equality sign, variable, something to solve and whether the equation is always true, always false or conditionally true (ie true for some but not all possible values of the unknown).

The collected data show that students value the presence of a variable most important, the presence of an equality sign second most important, the fact that there is something to solve/rearrange third most import and the expression being conditionally true least important in order for something to be an equation.

Moreover, we have looked at students' mastery of operations applied to equations in order to solve them that cause an expansion or a reduction of the domain of the equation to be solved.

Data show that students are not aware of these types of mistakes at all. Moreover, they are likely to make calculation errors or get stuck before they even reach the point where the mistake could possibly be made.

Stephens et al. (2013) describe three different ways in which one can perceive the equality sign. This study is especially interested in the relational-operational understanding and the relational-structural understand. A person holding a relational-operational understanding is aware that the equality sign expresses a relation between the left hand side and the right hand side and confirms the relation by calculating. A person holding a relational-structural understanding views the equality sign as a symbol expressing equivalence between two expressions rather than between two calculations.

The data show that one fifth show a relational-structural understanding, and about three fifth show a relational-operational understanding at some point. But it is also found that the shown understanding differs from task to task, and very few students show a consistent understanding.

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FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Master thesis in Mathematics

Tanja Rosenberg Nielsen

Logical aspects of equations and equation solving

Upper secondary school students' practices with equations

Thesis for the Master degree in Mathematics
Department of Science Education, University of Copenhagen

Advisor: Carl Winsløw

Submitted: August 8th 2016

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1 Introduction

The matematical-didactical content of this thesis is equations. Equations are one of few mathematical objects which do not have an unambiguous definition. But they are at the same time one of the most used object throughout all of the mathematical fields and throughout all levels of teaching of mathematics: from primary school to university. This makes equations a particularly interesting object to deal with didactically.

There has been conducted a number of studies on primary and lower secondary school students' understanding of and practices with equations, equation solving, variables and the equality sign (Ngu & Phan, 2016; Stephens, Knuth, Blanton, Isler, Gardiner & Marum, 2013; Filloy & Rojano, 1989). These studies have focused on the students' conceptions and development of new conceptions while learning the concept of equations. In the studies, the word "conception" is used to describe personal knowledge while the word "concept" is used to describe scholarly knowledge. These definitions of the words will be adopted in this thesis. Further more, it have been studied how mathematics teachers and mathematics students understand the concept of equations (Tossavainen, Attorps & Väisänen, 2011; Attorps, 2006).

But there seems to be a hole in the research regrading to upper secondary school students. This is a group op people which are expected to be familiar with the concept of equations, opposite to primary and upper secondary school students, but at the same time not a group of people which are as well educated in mathematics as mathematics teachers and students. Hence, upper secondary school students practices with equations are particularly interesting to study for two reasons. One, because it has not been done before, and two, because the group are expected to be able to deal with equations but as reality shows, it is often the case that they are incapable of doing so.

This thesis will be based on the prior studies regarding students' practices with equations, but will focus on upper secondary school students' conceptions of and practices with equations. The thesis is divided into two natural parts. The first part focuses on students conceptions of equations. It will be investigated to what degree presence of equality sign, presence of a variable, truth value and that there is "something to solve" - meaning the equation is not on a form in which the solution set is immediately evident, like $x = 4$ - matter in students' definitions of the concept of equations. The second part of the thesis will investigate students ability to recognize and explain why some equations are equivalent and some are not. Hence, the thesis does not primarily - as prior studies - concern students' ability to use certain techniques to solve equations, but rather concern students' theoretical knowledge of equations and the process of solving them. Moreover, in a third - and shorter part - students' understandings the equality sign will be studied.

The thesis will build upon a light use of the Antropological Theory of Didactics.

1.1 Research questions

The thesis will aim to answer the following four questions:

- How do upper secondary school students' definitions of the concept of equations conform to the scholarly concept(s)?

The thesis will investigate how mathematics textbooks for upper secondary school define the concept of equations. Moreover students' own conceptions will be investigated, analyzed and compared to the scholarly concept.

- What difficulties related to biimplication do students experience when solving equations?

In the thesis, difficulties related to biimplication, which students' might face when solving equations, will be uncovered.

- In what ways do students perceive the equality symbol in equations?

Since the equality symbol is an essential feature in an equation, this will be investigated as well.

- Can upper secondary school students' understanding of and practices with equations be improved by introducing a more theoretical approach of teaching equations?

1.2 Teaching of equations

To give a picture of what is taught about equations in danish upper secondary schools, curricula (læreplanerne), the additional instructions (vejledningerne) and old exam tasks are studied briefly in the below sections.

1.2.1 Curricula and additional instructions

In the curricula (Undervisningsministeriet, 2013a) for mathematics on C, B and A-level in STX (upper secondary school), there are only one explicit mention of equations. Equation solving with analytic and graphical methods and use of it-tools belong namley to the core substance (kernestoffet).

The additional instructions (Undervisningsministeriet, 2010) say that through the study of rich material of examples students should gain a fundamental understanding of the balance principle in equations and build the insight that solving is done by repeated use of "inverse operations".

These very vague formulations do not say much about what to be expected of students' abilities to define the concept of equations. But the students should learn, it seems, that an equation expresses some sort of balance between two or more expressions, and that terms can be moved form one side of the balance to the other as long as signs/operations are inverted. What we can see is that treatment of equations are based on examples and, hence, the students' perception of equations may be based on intuition rather than on a clear definition. Moreover, the solving process is described as repeated use of inverse operations, but there is no mention of the domain of an equation, which is crucial when applying inverse operations. It is in addition not stated clearly at all how hard to solve the equations presented are supposed to be. Another thing to notice is that equations are described as a type of task rather than as an independent mathematical object.

The curricula (Undervisningsministeriet, 2013b) for matematics on C, B and A-level in HHX (higher commercial examination programme) state the exact same thing about equations as the curricula for STX do, but there is a difference in the additional instructions (Undervisningsministeriet, 2015b). They say that fundamental rules for solving equations including determination of domain and the set of solutions and correct use of mathematical notation should be taught. Here the domain is explicitly mentioned, but there is on the other hand no mentioning of theory of equations as in the STX instructions.

One thing is the guidelines that the Ministry of Education sets out another thing is what is going on in the classrooms. Teaching materials will give a more accurate picture of what is actually taught about equations. Therefore certain selected teaching materials will be analyzed in chapter 2.3.

1.2.2 Exam tasks

Besides ministerial documents and textbooks, the exam tasks for the educations can also be a good indicator of what is going on in the mathematics classes. The exam tasks without aids from 2010-2015 for STX have been reviewed. Equations appear in several tasks but not often as independent subjects of study. Only second degree equations, differential equations and in one case a linear equation system occurs as tasks only about equations and independent from other areas of mathematics such as functions or geometry. The second degree equations can be solved by using the quadratic formula - that is, in a rather mechanical way. In the tasks about differential equations, one typically just have to check if something is a solution to a given equation (Undervisningsministeriet, 2015a). So one do not need a deep knowledge and understanding of the theory of equations to answer any of the exam tasks.

1.3 Collection of data

As empirical foundation for this thesis a number of upper secondary school students have been asked to complete a set of tasks (see appendix 8.1). Moreover a few university students have been asked to complete a similar set of tasks (see appendix 8.2) to have something to compare the first set of data to.

1.3.1 Finding participants

The data have been collected in eight different upper secondary school classes at all three levels (C, B and A) at two schools in Copenhagen: N. Zahles Gymnasieskole (STX) and Niels Brock (HHX). N. Zahles Gymnasieskole is a private upper secondary school and Niels Brock is a upper secondary school offering completion courses in which a graduated student can upgrade the level of a completed course in e.g mathematics. In total 161 students have completed a set of tasks. The participating students were distributed as follows:

N. Zahles Gymnasieskole

- B-level, 26 first year students, taught by Carsten
- B-level, 26 first year students, taught by Peter
- B-level, 21 second year students, taught by Anders
- A-level, 15 first year students, taught by Marianne

Niels Brock

- 0→B-level, 16 students, taught by Jacob
- 0→B-level, 16 students, taught by Lars
- C→A-level, 22 students, taught by Henrik
- B→A-level, 19 students, taught by Jørn

The schools have been chosen because the researcher had connections at the two institutions. All teachers of the above eight classes have agreed to let the students participate in the data collection. Since N. Zahles Gymnasieskole is a private institution and Niels Brock only offers completion courses, neither of the schools represent the typical danish upper secondary school well. In order to make the results of this thesis apply for all upper secondary school students, the schools and the classes should have been chosen at random. But, since there is not that large a difference between private and public upper secondary schools in Denmark, since

Niels Brock gather students from different upper secondary schools and since the sample of students is sufficiently large, it is, with reservations of course, possible to say something in general about upper secondary school students' practices with equation from the collected data.

The data from university students have been collected at the University of Copenhagen. Nine students following the course ”Matematik i undervisningsmæssig sammenhæng” have answered a set of tasks. The university students all answered the same set of tasks since they were so few that it would not make sense to give them different sets of tasks.

1.3.2 Execution

There were four different sets of tasks (see appendix 8.1). Some of the questions in each set of tasks where the same and some where shuffled throughout the four sets. In each class students received two of the four sets, and no students sitting next to each other had the same set. During all data collection the researcher was present - and in some cases the teacher of the class was present too.

It was made very clear to the students prior to each collection that they were not assessed, but that it was their way of answering which was of interest. Therefore they were encouraged to write every thought down even though they were unable to give an answer.

Too motivate the students to do as good as possible and because of the teachers requests, the teachers of the classes got the opportunity to look at the students answers after they handed in the set of tasks. Even though knowing that their teacher were going to read their answers made students more eager to perform well, it could also have prevented them from including half solutions or solutions they thought were incorrect.

The sets of tasks were prior to the data collection tested on four participants to determine how much time the students should be given to solve the set. It took the participants between 30 and 40 minutes to complete the set, hence, the students received a full lesson, which on the two schools were 45 minutes, to solve the set of tasks after a brief introduction.

Due to practical considerations the university students only hand 30 minutes to solve the questionnaire.

2 What is an equation?

In this section, various textbooks will be investigated in order to find out how they define equations. There does not exist a commonly accepted, formal and unambiguous definition of equations, and the aim of this thesis is not to make one but merely to investigate different perceptions.

The equality sign, the notion of a variable and truth-value are invariably linked to equations which is why these objects will also be put under investigation in this section. Moreover, it seems that many primary and upper secondary school students tightly connect the task of solving (rearranging terms) and calculating with the concept of equations (Kieran, 1981), hence, this will be investigated as well.

It is difficult to talk about equations without talking about what it means to solve them. Chapter 3 will go deeper into theory of solving equations and analyzing difficulties connected to doing so, but what it actually means to solve an equation will be dealt with in this chapter.

2.1 The equality sign

The equality sign is closely related to equations. The presence of an equality sign is a necessary condition for something to be an equation, it is not a sufficient condition, though.

2.1.1 Scolarly concepts

The equality sign is widely used throughout all of mathematics. The equality sign has at least three different (but still closely related) and correct interpretations.

It can be regarded as a sign of definition like when we define a function, e.g $f(x) = 4x + 3$, or want to write something in a shorter way, e.g $d = b^2 - 4ac$. Here the equality sign signals that whenever we are writing $f(x)$ or d it should be regarded a short way of writing $4x + 3$ and $b^2 - 4ac$, respectively.

We can also use the equality sign as a sign for sameness or identity. In this case the equality sign expresses that whatever quantity is on the left hand side is always the same size as the quantity on the right hand side. We can divide identities into two categories: arithmetical identities like $3 + 4 = 7$ and algebraic identities like $\cos^2(x) + \sin^2(x) = 1$.

Moreover, the equality sign can be perceived as an equivalence relation - that is a relation which is reflexive, symmetric and transitive. This is how we want to think of the equality sign when working with equations. But it is not - it turns out - the way most students perceive it.

2.1.2 Grade school students' conceptions of the equality sign

Preschool children get a comparative notion of equality when they count the number of elements in a set and compare it to a number of elements in another set. Moreover, they are able to count sets together which gives them an operational - a "do something" - notion of equality. This notion of equality sticks around for a long time, and it is very difficult for school children to grasp the equality sign as an equivalence relation. Primary and lower secondary school students assign some sort of directionality to expressions with an equality sign: What is on the left hand side is some calculation to be carried out, and what is on the right hand side is the answer (Kieran, 1981). There is some development in the conceptual understanding of the equality sign between grade four and five but most students keep interpreting the equality sign as an operational symbol which give them great difficulties when solving non-standard equations like $7+4 = x+3$ (Stephens, 2013).

This duality of the meaning of the equality sign might be one of the reasons why many upper secondary school students have a hard time solving equations. If they are stuck in their operational understanding of the equality sign, it becomes very hard for them to understand what sort of object an equation is and to perform the legal manipulations needed in order to solve equations.

In section 4 we will take a closer look at the students' different perceptions of the equality sign.

2.2 Definition of the concept of equations

The definition of the concept of equations is not agreed upon throughout the literature. In fact, equations as mathematical objects are often not even formally defined in the mathematical literature. Since equations are needed almost right from the beginning when learning mathematics, they are very often not introduced formally. And later on a formal definition is not needed in order to be able to work with equations. It is easy to give an example of an equation but more difficult to explain which features make some mathematical expression an equation.

In this thesis the definition of equations given by Christiansen, Lichtenberg, & Pedersen (1964) will be introduced. The definition will be given and discussed in the following section.

2.2.1 Definition of equation

The book "Almene begreber fra logik, mængdelære og algebra" (Christiansen et al., 1964), which translates to "General concepts from logic, set theory and algebra", is a book which was adressed to teacher seminar students in Denmark. The book was written in order to embrace the drastical changes made in the mathematics curriculum for primary and lower secondary schools in the 60ties. The changes were an attempt to increase the students' mathematical understanding. The changes were highly criticized and therefore also brief and is today referred to as "new math". New math had a set theoretic and way more formal approach to school mathematics than ever seen before and after.

The reason for bringing this historical piece into play is because it actually contains a very formal definition of equations.

First Christiansen et al. (1964) define a proposition (udsagn).

Definition 2.1 (proposition). A proposition is a sequence of words which has the property of either being true or false.

Here we might want to ad that a proposition also can be a sequence of symbols and a combination of symbols and words. Moreover some mathematicians call an object with this property a closed proposition. Now Christiansen et al. (1964) define what an open proposition is.

Definition 2.2 (open proposition). Let $P(x)$ be an expression (udtryk) which contains a variable x, and let E be a set such that $P(x)$ is a proposition for every $x \in E$. Then $P(x)$ is an open proposition in the variable x with domain (grundmængde) E.

Since Christiansen et al. (1964) are very thorough it is surprising that they do not anywhere define what is meant by expression. By the above definition, an expression has to be some sort of collection of words and/or symbols such that when x is exchanged for an arbitrary element of E it becomes a proposition. Hence, the definition of an open proposition is as broadly formulated as possible. Christiansen et al. (1964) continues:

Definition 2.3 (solution set). The solution set of an open proposition $P(x)$ with domain E is the set $\{x \in E \mid P(x) \text{ is true}\}$.

Christiansen et al. (1964) stress that the domain of an open proposition is of great importance when determining the solution set of the open proposition. But in case the domain is clear from the context we can omit writing it. We will return to this.

Later on the definition of an open proposition is widened to also contain expressions which do not contain a variable. That is, if we consider a (closed) proposition P against some set E then it can be regarded as an open proposition. If P is true then we say that the solution set is all of E, and if P is false the solution set is empty. Finally Christiansen et al. (1964) are ready to give a definition of equations:

Definition 2.4 (equation). Let E and F be two sets and let $g(x)$ and $h(x)$ be two expressions containing the variable x (and here we might ad, according to the widening above, "zero or more times" which is also clarified later). Moreover,

$$\forall x \in E : g(x) \in F \wedge h(x) \in F.$$

We form the expression

$$g(x) = h(x), \tag{2.2.1}$$

which for each $x \in E$ clearly is a proposition and, hence, an open proposition with domain E. We call the expressions of the form (2.2.1) equations with domain E.

More intuitively and less accurate, we can say that an equation is an open proposition containing an equality sign. But this "definition" allows propositions which we do not want to perceive as equations, like

$$(x^2 = 4) \wedge (x > 0).$$

This is the reason of Christiansen et al. (1964) to making such a formal definition of equations as definition 2.4. It is not mentioned in the book, but definition 2.4 also assures that the two expressions $g(x)$ and $h(x)$ are comparable because they are assumed to belong to the same set F.

The approach to describe equations taken by Christiansen et al. (1964) (and the wave of new mathematics in general) rise the question: What is the purpose of making such a formal introduction to the concept of equations? Everybody (ie everybody who ever went to school) has an intuitive notion about equations, is that not enough?

The reason why we make definitions in mathematics is to secure unambiguousness and to be able to check whether some mathematical object is - in this case - an equation or not. But that is rarely what we want to do when we work with equations. The issue is not whether the object at hand is an equation or not. The issue almost always is to find the solution set to the given equation. Therefore, for all practical purposes we do not need a formal definition of the concept of equations to find out whether something is an equation or not since this almost never is the issue of interest. Hence, a formal definition has no applications in connection to work with equations. But thereby not said, that a more formal definition can not benefit students' general theoretical understanding of mathematics and specific understanding of the solving process of equations. A formal introduction like the one given by Christiansen et al. (1964) could possibly complement the students' technical understanding of and approach to the solving process of equations. A more theoretical understanding would make the students able to validate and explain their process when solving equations.

From definition 2.4 and 2.3 it follow that ”to solve an equation” means to find the solution set to the open proposition which the equation forms. In other words to find the values of $x \in E$, where E is the domain of the equation, for which the equation is a true proposition.

Here the notion of the domain of an equation plays a very important role as mentioned before. But the concept of a domain of an equation is very often omitted, too implicit or at least strongly neglected in many new textbooks of mathematics (see section 2.3). This unawareness might be one of the reasons why students sometimes make mistakes when solving equations. Especially mistakes of the type where they from manipulating the equation obtain an equation which is only equivalent to the original equation in some proper subset of the domain of the original equation. And hence, mistakenly find solutions which they regard as such, but which when inserted into the original equation yield expressions which are not a proposition but simply nonsense or propositions which are false. This speaks for a more formal definition of equations - or at least more awareness of the domain of an equation.

Moreover, Christiansen et al. (1964) contain a very thorough but also heavily detailed description of how to solve equations by calculating forward (fremadregning) and then afterwards either step for step or in the end calculating backwards (bagudregning) to secure biimplication between every new equation obtained by manipulating the original equation. In terms of the Antropological Theory of Didactics, we see that Christiansen et al. (1964) introduce techniques (how do we solve equations?) and theory (why can we solve equations like that?) connected to the task of solving equations parallel. In many new upper secondary school textbooks of mathematics - as we will see in section 2.3 - the main focus is on tasks and techniques and there is not mentioned much about theory in connection to solving equations.

An objection towards the notion of ”finding” the solutions of an equation that Christiansen et al. (1964) address is the fact that we do not have a standardized way of presenting the solution set to an open proposition - and hence to an equation. If we let $P(x)$ be an open proposition and E the domain of $P(x)$ then we can always write the solution set, L, like

$$L = \{x \in E \mid P(x) \text{ is true}\}.$$

But this is very rarely the most explicit or most useable way to represent the set L because if the equation only have a few solutions the most explicit way to present the set L would simply be to write a list of all of the solutions. In fact, very often we would not even consider the above representation as having "found" the set of solutions. So Christiansen et al. (1964) conclude that what it means to have "found" the set of solutions is very context dependent. Christiansen et al. (1964) describe it in weak terms as "the most satisfying representation of L".

In primary, lower secondary and upper secondary school "find the solutions" almost always means to explicitly write a list of all values of $x \in E$ which make the equation true. Often this list is very short - it only contains a few elements. Therefore students might have trouble dealing with equations which have either zero or infinitely many solutions because they have learned from experience that an equation has a few solutions.

2.2.2 Different kinds of equations

In definition 2.4 it is not specified what kind of expressions $g(x)$ and $h(x)$ should be. We must assume that it is implied that we deal with mathematical expressions of some sort. If we allow all mathematical expressions, we allow for example sums, limits and integrals to occur in equations. This is not wrong but a canonical example of an equation would properly no contain either sums, limits or integrals.

Likewise, we might think of the variable x as a number and in most cases in upper secondary school it is, unless we are dealing with differential equations. But Christiansen et al. (1964) actually allow the variable to be for example in the set of functions or some other set which is not necessarily a set of numbers.

The definition given by Christiansen et al. (1964) is as we have just seen very broad but it does not contain equations in multiple variables which in some sense is actually needed in upper secondary school. For example if we consider the linear function $y = 10x + 1$ as an equation in the variables y and x. This can be overcome by saying that we always consider one of the unknowns as a variable and the rest as constants. And this is also how Christiansen et al. (1964) indirectly throughout examples address this lack of broadness in the definition. Later on Christiansen et al. (1964) introduce propositions and then equations in multiple variables the same way they introduced propositions and equations in one variable.

In this thesis we consider equations to be as described in definition 2.4, but we only allow the set E to contain real numbers. Moreover, we let the set F consist of elements build from constants, numbers, arithmetic operations (addition, subtraction, multiplication and division) and functions which has been introduced in upper secondary school (for example exponential functions, power functions, logarithmic functions and trigonometric functions).

2.2.3 Features associated with equations

According to the definition given by Christiansen et al. (1964) there are four features invariably linked to the concept of equations:

1. The equality sign
2. The concept of variables
3. Truth value
4. The process of finding the set for which the equation is true

According to the definition given by Christiansen et al. (1964) the presence of an equality sign and the fact that an equation is a proposition and, hence, has a truth value are necessary conditions which has to be fulfilled for something to be regarded as an equation. Oddly these two features are those which are discussed least in the literature about equations for upper secondary school (see section 2.3).

An equation does not necessarily have to contain a variable, but all equations of interest in upper secondary school do. Hence, it is to be expected that students will mention the presence of a variable as a defining feature for equations since they rarely or never have met equations without variables. Moreover, the presence of a variable is tightly connected to the process of finding the solution set to a given equation. There are only very few types of tasks directly concerned with equations in upper secondary school, and finding the solution(s) to an equation is the most common. Hence, students may very well associate the concept of equations closely with finding the solutions.

While the process of solving an equation is just as tightly connected to truth value as it is to the concept of variables some textbooks still manage to formulate how equations are solved without mentioning of truth value. Hence, the textbooks introduce the technology of equation solving without relating it to the theory of equation solving (see section 2.3).

2.2.4 Balance and inverse methods

While as mentioned there is no commonly agreed upon definition of equations throughout the litterature, there do exist two strong analogies commonly used about equations and the process of solving them. Since - as we will see in the textbook analyses in section 2.3 - the introduction of equations at upper secondary school level mainly are done through examples these analogies become an important part of the explanation of what an equation is. The two analogies or methods for solving equations are the balance method which are primarily used in Europe, and the inverse method which are primarily used in Asia (Ngu & Phan, 2016).

The balance method explains equations as a balance or seesaw: when you add, subtract, multiply or divide (weight) you have to do it on both sides of the equality sign to retain the balance. This method though falls short when it comes to explaining more difficult operations which can be used on equations - like taking the root or raising to some power and even just explaining negative numbers. The inverse method emphasizes that operations can be inverted onto the other side of the equality sign.

2.3 Textbook analyses

In the below sections presentations of equations in various textbooks used to teach mathematics in danish upper secondary schools are studied. The textbooks chosen are common teaching materials in danish upper secondary schools. Moreover, they have been used in the specific upper secondary school classes in which data for the analysis of students' conceptions of equations are gathered for this thesis. The focus in the analyses are on the status of the equality symbol, on the mentioning of variables in the definition/description of equations and on the treatment of truth value in relation to equations. Moreover, there are focus on how equations are introduced (formal/informal, using the balance analogy, through examples ect.), on the use of implication arrows and their meaning and on the treatment or non-treatment of equivalence. Also, it is studied how the presentation of arithmetic (rules for calculating), equations and inequalities interact in each textbook.

2.3.1 ”MAT A1 stx”

The book ”MAT A1 stx” (Carstensen, Frandsen & Studsgaard, 2009) is a commonly used teaching material for first year A-level mathematics students in danish upper secondary schools.

In the first chapter about basic algebra we see an exclusive use of the equality sign as an operational symbol. The expressions to be calculated are always on the left hand side of the equality sign and the results are on the right hand side. The arithmetical operations addition, subtraction, multiplication and division are discussed but there is no mention of the status of the equality sign.

The second chapter are about equations. The definition of equations is given very broad as an expression which contains an equality sign. But the definition is quickly followed up by examples like $5 + 7 = 12$, $a + b = b + a$, $2x - 3y = 7$ and $3x - 4 = 11$. Moreover, Carstensen et al. (2009) stress that an equation often contains one or more unknown quantities. The examples (except from the second) preserve the idea that we have a calculation to be carried out on the left hand side and a result on the right hand side.

Very quickly Carstensen et al. (2009) move on to define what it means to solve an equation almost as if it should be regraded a defining feature, that an equation is something which can "be solved". According to Carstensen et al. (2009) to solve an equation means to find the values which "fit" into the equation ei which make it true. Carstensen et al. (2009) go on to define what it means for two equations to be equivalent (ensbetydende). They associate the biimplication symbol ($\Leftrightarrow$) with doing manipulations on an equation to obtain an equivalent equation ie an equation with the same solutions as the original equation. The sign could very well have been introduced as a combination of two implications ($\Leftarrow$ and $\Rightarrow$) and in terms of truth value of the two propositions, but this is not mentioned at all.

It is briefly mentioned that we are not allowed to divide or multiply both sides of an equation by zero if we want to obtain an equation equivalent to the original one but it is not explained why.

Also it is not mentioned to which set the unknown quantity belongs. It is an implicit assumption that the unknown belongs to the real numbers. There is one example in which the domain of the unknown is addressed. In the example, in order to solve

$$\frac{4}{13-3x} = \frac{20}{x+1}$$

there is multiplied by $13-3x$ and $x+1$ on both sides. After having found that the unknown x is equal to 4, Carstensen et al. (2009) mention that we must assume that $x \neq 1$ and $x \neq \frac{13}{3}$. But not assuming this is totally unproblematic in the equation in question since the solution found is not one of the two values. Hence, students might not see the significance of noticing this and quickly forget to keep track of the domain of the variable when solving other equations. Moreover, if the students have to solve problems from the real world, there might be some implicit assumptions about the unknown (like e.g a length in the real world can not be negative) which they could possibly overlook because they are not used to working with domains of unknowns.

In the end of the chapter there is a section called "Eksperimenter" which translates to "Experiments". In experiment 3 (see next page) two different absurd "proofs" are presented and it is up to the reader to find the mistakes in the "proofs". In both cases the mistakes are that in the end both sides of the equation is divided by zero. In the first "proof" zero is hidden as the expression $a-b+k$. Since k was defined to be a number such that $b = a+k$ we must have that $a-b+k = 0$ and, hence, we can not divide by it. In the second "proof" zero is hidden as $x+y+z$. The numbers x, y and z was assumed to meet the equation $x+y = -z$ and, hence, $x+y+z = 0$ so we can not divide by it.

EXPERIMENT 3
A couple of strange proofs. Once in a while pranksters present a "proof" which is obviously absurd. Vi shall look at a few here and you should try to find the mistakes.

Proof 1. We shall "prove" that two different numbers a and b are the same! We assume that $a < b$ and show that then we must have that $a = b$.

If $a < b$ we can add a positive number to a to get b (namely the difference $b - a$), ie there exists a positive number k such that $b = a + k$. Now we can rewrite the equation like this:

$$\begin{array}{rl} b = a + k \Leftrightarrow & \text{Multiply by } a - b \text{ on both sides} \\ b(a - b) = (a + k)(a - b) \Leftrightarrow & \text{Multiply out the brackets} \\ ab - b^2 = a^2 - ab + ak - bk \Leftrightarrow & \text{Add } bk \text{ to both sides} \\ ab - b^2 + bk = a^2 - ab + ak \Leftrightarrow & \text{Put common factor outside the brackets} \\ b(a - b + k) = a(a - b + k) \Leftrightarrow & \text{Divide by } a - b + k \text{ on both sides} \\ a = b & \end{array}$$

Where is the mistake in this "proof"?

Proof 2. We let x, y and z be three arbitrary numbers satisfying the equation

$$x + y = -z$$

We multiply by 4 and 5 on both sides - in the last case we switch the right and the left hand side. We obtain the equations

$$4x + 4y = 4z$$

$$5z = 5x + 5y$$

We add the equations

$$4x + 4y - 5z = 5x + 5y - 4z$$

Then $9z$ is added to both sides

$$4x + 4y + 4z = 5x + 5y + 5z \Leftrightarrow 4(x + y + z) = 5(x + y + z)$$

and this last equation is equivalent to $4 = 5$. Where is the mistake in this argument?
(Carstensen et al., 2007, pp. 79-80, authors translation).

These are great examples of traps students might step into if they not carefully keep track of the domain of the unknowns when solving equations. But the section ”Eksperimenter” is designed such that it can be skipped when teaching so very likely a lot of students in upper secondary school will never meet good examples like these. Also, the content of the section seems to be regraded as more difficult or at least as some kind of appendix for further investigation by the authors according to the preface of the book. Moreover, Carstensen et al. (2009) do not give any solution so if the readers can not find the mistakes by themselves they need someone to help them.

2.3.2 Gyldendals Gymnasiematematik

Gyldendal provides textbooks for first year teaching of mathematics at C-, B- or A-level (Clausen, Schomacker & Tolnø, 2006; Clausen, Schomacker & Tolnø, 2010a). Along with each of the textbooks follows a workbook (Clausen, Schomacker & Tolnø, 2005; Clausen, Schomacker, & Tolnø, 2010b). In both workbooks there are sections about equations and they are almost identical.

First off the section gives an example of an equation which includes an equality sign, is conditionally true, has a variable and something to solve on both sides of the equality sign. It is explained that for each choice for the variable the equation becomes a proposition which is either true or false. One example of choice of variable is given for which the proposition is false and one is given for which it is true. Now Clausen et al. (2005, 2010b) move on to define what a solution to an equation is. This is defined in terms of truth value, hence, a solution is a choice of the variable which leads to a proposition which is true. To solve an equation therefore means to find all solutions to the equation. It is underlined that one needs theoretical considerations, methods and techniques to do so - one might be able to guess one solution but one can not always find all solutions just by guessing.

This description of equations is leaning heavily on the concept of truth value. It makes it explicit and clear - without mentioning the term which might not be familiar to upper secondary school students - that an equation is an open proposition. But the description fails to mention explicitly that an equation always holds an equality sign. This is only apparent due to the introducing example. Moreover it seems to be a defining feature for an equation that it holds a variable but at the same time there is no mentioning at all of the domain of the equation; it seems as if the variable is a placeholder for everything - or at least for every number, which is not true in general.

Some general techniques (adding, subtracting, multiplying and dividing by the same number except zero when multiplying or dividing as well as the zero-rule) for solving equations are presented and illustrated by a drawing of a weight to which three are added to each side and in that way maintaining the equilibrium.

Then Clausen et al. (2005, 2010a) shortly introduce the biimplication arrow. The arrow can be used whenever one uses the rules/techniques described. There is no mentioning of the use of the single implication arrow.

The rest of the section is a collection of examples of equations and explanations of how to solve them. Between the examples it is mentioned that it is a good idea to insert the found solution(s) into the original equation to check whether the equation becomes true or not to catch possible mistakes. It is also mentioned that equations can be solved using CAS. There are no examples in which the domain of the equation matters in the solution process and, hence, no considerations about the domain are made at all. There are some exercises attached to the section. They can all be solved just by copying the procedure in the examples and no concerns about domain are relevant here either.

2.3.3 Vejen til matematik AB1

The book "Vejen til matematik AB" (Nielsen & Fogh, 2010) is a relatively new teaching material meant for first year teaching of students who take mathematics on either B or A level in upper secondary school.

The first mentioning of equations is in the end of the second chapter about basic calculation techniques. There is a scheme explaining some mathematical symbols and concepts. It is described how a solution set can be written either on a list or as $\{x \mid p(x)\}$, where $p(x)$ is a proposition. Moreover it is explained that the symbol "$\Rightarrow$" means "implies" or "if... then" while the symbol "$\Leftrightarrow$" means "equivalent" or "if and only if".

The third chapter concerns solving of equations and is also titled so. There is no definition or explanation at all of what an equation is, but two standard examples of equations are given. It is assumed that the reader is familiar with the concept of equations while at the same time the description of how to solve them is very thorough. The concept is introduced mainly through examples which all contain a variable, an equality sign, are conditionally true and in which is it possible to isolate the unknown quantity which is named x.

The chapter is divided into several subsections - first degree and second degree equations are separated in two different chapters, but inequalities are treated in the same chapter as first degree equations. The only reasonable explanation for this is that the inequalities presented are solved in almost the same way as the equations in the chapter. The solving procedure seems to be the controlling factor for the sectioning of the chapter in general. That indicates that the solving procedure and, hence, the presence of a variable are important to the authors' perceptions of equations.

While there is no definition of the concept of equations the concept of inequalities are actually briefly discussed. It is stated that an inequality is either true or false and that solving an inequality means finding the values of the unknown for which the inequality is true. This definition could very well apply for equations as well but is not introduced to do so.

It is mentioned as a theorem that one is allowed to add, subtract, multiply and divide by the same number on both sides of an equality sign but not multiply and divide if this number is zero. There is no further explanation of why one are allowed to do this. There is also no mentioning of the domain of an equation and the implication arrows and solution set introduced in the previous chapter are not revisited.

There are given three examples in which one actually should be careful not to divide or multiply by zero when dividing or multiplying by some expression containing a variable. But this issue is not addressed in the examples; the calculations are made such as if the equations in the examples all are equivalent. And the issue does not raise itself since the solutions all belong to the domain of the original equation.

In exercises 104 and 105 (Nielsen & Fogh, 2010) there is suddenly a mentioning of the domain (grundmængde) of the inequalities to be solved in the exercises. This word is not introduced anywhere in the book and hence must be strange to the reader.

Nielsen & Fogh (2010) proofs why one are allowed to cross-multiply in an equation in which both sides are a fraction. They end the proof by saying that since all calculations can be reversed the theorem is proven. This is of course true and a necessary consideration to finish the proof, but the reader have no prerequisites (provided by the book) to be able to understand what it means to reverse the calculations, why this is important to be able to do and why one can not always just reverse calculations.

Next to the proof there is a yellow box titled ”Udfordrende matematik”, which means ”Challenging mathematics”. It says the following:

Equation magic
Below it is proven by use of the rules of calculation for solving of equations that if $x = 2$ then $x = 0$. Which rules have been used? What went wrong?

$$x = 2 \tag{2.3.1}$$
$$2x + 2 = x + 4 \tag{2.3.2}$$
$$2x - 4 = x - 2 \tag{2.3.3}$$
$$\frac{2x - 4}{x - 2} = 1 \tag{2.3.4}$$
$$x \cdot \frac{2x - 4}{x - 2} = x \tag{2.3.5}$$
$$x \cdot 2 = x \tag{2.3.6}$$
$$x = 0 \tag{2.3.7}$$

(Nielsen & Fogh, 2010, p. 76, authors' translation and numeration of equations).

No further explanation or result to this task is provided in the book. The issue which this task addresses is the issue of implication. An issue which is not discussed anywhere else in the book except for - as already mentioned - very briefly in the end of the second chapter. To go from line (2.3.3) to line (2.3.4) the first equation is divided by $x - 2$ but to do so one must assume that $x \neq 2$.

To go from line (2.3.4) to line (2.3.5) the first equation is multiplied by x and hence we must assume that $x \neq 0$. But $x = 2$ and $x = 0$ are exactly the solutions to (2.3.1) and (2.3.7), respectively. So essentially what the calculations tell are that (2.3.1) and (2.3.7) are equivalent on $\mathbb{R}\backslash\{0, 2\}$ which is not surprising at all, but if the domain of each equation is not taken into consideration one might be convinced that the equations are equivalent on all of $\mathbb{R}$ which would be VERY surprising.

The above exercise shows nicely two of the problems (multiplying or dividing by zero) concerning implication one can meet when working with equations. But the book has not provided the reader with any language to talk about this issue since the issue has not been mentioned in the chapter. Hence, the task is very difficult compared to the level of the text and the level of the rest of the exercises attached to the chapter.

Moreover, the calculations made in the exercise is more difficult that the calculations presented in the examples. For example it is not very clear that $x + 2$ is added to go from (2.3.1) to (2.3.2). Also, there is no further explanation of the exercise so a reader which can not solve the exercise alone needs help which is not provided by the book to solve it.

2.3.4 Kernestof Mat C hhx

In the textbook Kernestof Mat C hhx (Gregersen & Skov, 2013) the treatment of equations and inequalities are in the same chapter. The chapter begins with seven introduction exercises about everyday problems containing little or no symbolic language.

Then Gregersen & Skov (2013) move on to define what an equality sign is and what an equation is. An equality sign is an assertion about the quantities on each side of it having the same size, and an equation are two quantities written on each side of an equality sign according to Gregersen & Skov (2013). A solution to an equation is a number that makes the equation true when inserted in stead of the unknown quantity.

Further Gregersen & Skov (2013) explain that an equation can be rearranged by using arithmetic operations as long as the same thing is done to both sides of the equation, and that the goal when solving equations is making rearrangements until the unknown is isolated on one side of the equality sign. Equivalence of equations are, hence, defined using the balance metaphor but not very explicit although a few examples and exercises have an explicit use of the balance model. It is not entirely true that everything can be done to an equation as long as it is done to both sides. Taking the square root can for example result in loosing a solution, or raising both sides to an even power can result in finding a solution which is not really a solution to the original equation.

After this explanation follows a bunch of examples and exercises in which it is only necessary to add, subtract or divide by nonzero numbers to isolate the unknown. Then follows a short examination of the degree of equations which is linked to the maximal number of solutions. There are also given two examples of equations with no solutions. Moreover there is one example and two exercises about graphical solutions of equations.

The zero-rule is introduced along with an example and an exercise. Then the chapter turns to a treatment of translating a word problem into an equation and a treatment of inequalities.

Attached to the chapter are some tasks very similar to many of the exercises in the chapter. One task though sticks out since it addresses the meaning of the equality sign. It says:

Task 228
A teenager looks in the mirror every morning and asks: "*Is that really me?*".

1. The answer is no! Explain why the equation *person=reflection* is not true.

2. The proposition $2 = 2$ seems true but the number 2 on the left hand side is printed in different inc and at another time than the number 2 on the right

hand side. Explain how the proposition $2 = 2$ still can be mathematically true.

3. Explain what the words *equal to* means in a mathematical context.

(Gregersen & Skov, 2013, page 31, authors' translation).

The exercise makes the students aware of the meaning of the symbol "=". But the exercise do not present the equality sign as an equivalence relation. Of course the reflexive property of the equality sign stands out but the symmetric and transitive properties are not included. The exercise just teaches the students to know the difference between something being identical/the exact same and something being a symbol covering for the same quantity. Though correct, it is kind of odd that the word equation is used in the first subtask while the word proposition is used in the second subtask as if $2 = 2$ is not really regarded to be an equation.

There are no text, examples or exercises in the textbook by Gregersen & Skov (2013) concerned with the domain of equations, a more precise definition of equivalence or use of implication arrows.

2.3.5 MAT C hhx

The textbook MAT C hhx (Bregendal, Nitschky Schmidt, & Vestergaard, 2005) contains an introductory chapter to arithmetic and sets and a chapter about equations and inequalities. This chapter contains a treatment of first degree equations, systems of equations with two equations and first degree inequalities. The introduction to the chapter says it will be dealing with finding solutions to first degree equations in one variable. Then it is defined what it means to solve an equation. It means to find that or those numbers which inserted instead of the unknown makes the equality sign true which is further explained as meaning the left hand side being equal to the right hand side. Hence, form the beginning the chapter is focused on an equation as something that can be solved and contains a variable and an equality sign. Moreover, the word "true" is mentioned but used in a very imprecise manner. It is not the equality sign that has to be true, but rather the equation.

Next the rules for solving an equation are presented. One allowed to add, subtract, divide by and multiply by the same number on both sides of the equality sign unless that number is 0 then one can not divide or multiply.

The next pages consist of examples and exercises. In all of the examples the operational lines show a use of the inverse method while the comments in the margin provide an explanation using the balance method. The examples are nonstandard linear equations meaning they contain both the variable and numbers on both sides of the equality sign. In the first example, after finding the solution it is inserted into the equation to check (gøre prøve) whether it is correct or not. The second example is an equation with no solutions and the symbol $\emptyset$ is introduced to describe the solution set. Example four has the variable x in the denominator so alongside with the equation is written that $x \neq 0$. Since the solution is $x = 4$ this does not really make any difference compared to the three first examples. Example five is a bit more interesting since we have $x - 2$ in the denominator and, hence, the condition from the beginning that $x \neq 2$ but the solution found is actually $x = 2$. It is concluded that the equation do not have any solutions, but there is not given any explanation why we reach the answer $x = 2$ by rewriting the equation. Example six has $x + 2$ in the denominator and is reduced to $0 = 0$ meaning that every number is a solution. Hence, the example concludes that the set of solutions is all of $\mathbb{R}$. This is a mistake since the set of solutions should be $\mathbb{R}\backslash\{2\}$. In between the examples are exercises which rise the same issues as the examples. To summarize the examples deal with expansions of the domain of the original equations due to multiplying by a variable, but do not explain why it can happen that the set of solutions found can be larger than the actual set of solutions.

The rest of the chapter deals with systems of equations and inequalities. The description of what it means to solve an inequality is a little more precise than the one for equations. It says that to solve an inequality means to find the numbers which inserted instead of the unknown makes the inequality true.

At one point inequalities with a variable in the denominator are introduced. The approach is first to determine the domain and then examine the signs of nominator and denominator. But the word "domain" is not defined anywhere in the book in relation to equations an inequalities. The word is only used this one time in the chapter.

In the chapter about polynomials there are a treatment of second degree equations. The formula for solving second degree equations are introduced, proofed and used to calculate some examples. The main concern of the chapter is to make the reader able to draw graphs for second degree polynomials by hand. The zero rule is introduced in order to solve second degree polynomials with the constant term equal to zero. Moreover, taking the root on both sides is used in an example in which the first degree term is zero. It is moreover mentioned that a second degree equation can have either 0, 1 or 2 solutions corresponding graphically to the number of intersection points with the x-axis.

2.3.6 Comparison of the textbooks

In this section the five textbooks analyzed above are compared to each other. We look at several parameters: (a) placement of the section about equations especially in relation to sections about arithmetic and inequalities; (b) presentation of the equality sign; (c) importance of the four features: equality sign, variables, solving process and truth value in the presentation of equations; (d) use of balance vs. inverse method; (e) examples and exercises; (f) explicit focus on equivalence; (g) use of implication arrows; (h) difference between HHX and STX.

In all five textbooks, the section about equations are distinct from the section(s) about arithmetic and general rules of algebra although there is a close relationship between the two since one needs algebra to rewrite an equation in order to solve it. On the other hand in three out of five books inequalities are treated in the same chapter as equations. The two concepts are linked together mainly by the solving process.

McNeil et al. (2006) found that American textbooks often present the equality sign in standard contexts (operations equals answer, e. g. $3 + 4 = 7$) and rarely in non-standard contexts (operations on both sides, e.g. $3 + 4 = 5 + 2$ or other contexts, e.g. $7 = 7$). Moreover, McNeil et al. (2006) suggest that students' interpretations of the equality sign depend on the context, and that the operations on both sides context is the one most likely to elicit a relational understanding of the equality sign which is essential in algebra. In the study of McNeil et al. (2006) a random sample of the textbooks for sixth-, seventh- and eighth-grade were analyzed. This thesis only looks at the use of the equality in the chapters concerning equations.

Carstensen et al. (2009) give in the beginning of the chapter about equations three standard (operations equals answer) examples of equations and one non-standard (operations on both sides) example. Clausen et al. (2005, 2010a) introduce equations using a non-standard example. In the rest of the section about first degree equations there are the same number of standard and non-standard equations. When Clausen et al. (2005, 2010a) move on to discuss systems of equations there are only standard equations but during the solving process some non-standard equations arise. Nielsen & Fogh (2010) start out by giving two standard examples of equations, but the rest of chapter contains the same amount of standard and non-standard examples of equations. Gregersen & Skov (2013) have no introductory examples only introductory exercises which consist of both standard and non-standard equations, algebraic expressions (which are not equations) and inequalities. In the following examples and exercises there are a slight overweight of standard equations but non-standard equations certainly occur. Bregendal et al.

(2005) have no examples in the introduction, but almost all of the following examples in the part about first degree equations are non-standard. Hence, it seems that there are a fair representation of the equality sign in non-standard connections in all of the textbooks although a larger part of the equations in the two HHX books are non-standard in comparison to the three STX books.

Since the equality sign is a defining feature of something being an equation all the textbooks do include the sign, but not all of them articulate it. Carstensen et al. (2009) explicitly mention that an equation has to hold an equality sign. Clausen et al. (2005, 2010a) do not mention the equality sign explicitly, but it becomes apparent through examples that it is something every equation hold. Nielsen & Fogh (2010) and Bregendal et al. (2005) have no definition of equations and, hence, the presence of the equality sign is only seen through examples. Gregersen & Skov (2013) actually give a definition of what an equality sign is, and mentions explicitly that it has to be present for something to be an equation.

There is a tight connection between the presence of a variable in an equation and the solving process since there is not much to solve if there is no variables in the equation. In that case the equation is just a closed proposition which is either always true or always false. In the definition of equations given by Carstensen et al. (2009) it is not a necessary condition for something to be an equation that it holds a variable, but Carstensen et al. (2009) stress that equations often hold one or more unknown quantities. In three out of four of the examples given in the introduction text there is one or more variables present in the equations. Clausen et al. (2005, 2010a) give an example of an equation holding a variable and further explain that for each choice of the variable the equation becomes a proposition. This could easily lead to the conclusion that an equation must hold a variable even though this is not stated explicitly. In the rest of the chapter there is no examples of equations without a variable. Nielsen & Fogh (2010) have a strong focus on the solving process which is seen in the title of the chapter "Solving of equations" and, hence, the presence of a variable is almost a necessity for something to be an equation. It is not mentioned that something can be an equation without a variable. Gregersen & Skov (2013) explain that in an equation the left hand side and the right hand side of the equality sign are quantities which can be put together by numbers and letters. So basically an equation does not have to have a variable according to Gregersen & Skov (2013), but in all of the following examples the equations do hold a variable. Bregendal et al. (2005) only deal with equations with one variable (probably in contrast to several variables). In example six the equation to be solved reduces to $0 = 0$ which is treated the same way as equations with a variable.

To summarize Carstensen et al. (2009) state most explicitly that an equation do not need to have a variable to be an equation. Then follows Gregersen & Skov (2013) since they do not explicitly mention the presence of a variable in the definition, but include a variable in all of the examples. Lastly Clausen et al. (2005, 2010a), Nielsen & Fogh (2010) and Bregendal et al. (2005) all exclusively operate with equations containing a variable.

Truth value is like variables tightly connected to the question of solving equations. Some books include a little theory of truth value and some books omit it. If the concept of truth value is discussed the solving of equations are put into a more theoretical frame than if the concept is omitted and solving just amount to "rearranging the equation" or "finding x". It might not have an influence on students' technical practices when solving equations, but it is important in order for the students to make critical reflections about their result and for them to interpret the result.

Carstensen et al. (2009) choose a slightly informal way of talking about solutions to an equation. A solution according to Carstensen et al. (2009) is a value which "fit" into the equation. Clausen et al. (2010) say that for each choice of the variable the equation becomes a proposition which is either true or false. A solution is then a value which leads to a true proposition. Clausen et al. (2010) point out that finding one solution is not the same as solving an equation since equations can have multiple solutions. Nielsen & Fogh (2010) do not define equations or what it means to solve them. Through examples it becomes apparent that solving an equation means finding a value for the unknown such that the value of the left hand side coincide with the value on the right hand side. The number of solutions is not discussed. In Gregersen & Skov (2013) a solution is defined as a number which makes the equation true. The solution set is only defined for inequalities probably since all equations in the chapter are of first degree. Bregendal et al. (2005) state that to solve an equation means to find that or those numbers which inserted instead of the variable make the equality sign true, that is such that the value of the left hand side equals the value of the right hand side of the equation.

Clausen et al. (2010) have the most exact and elaborate use of truth value, since they use the word proposition as well as the terms true and false. Moreover they distinguish between a solution and the set of solutions. Both Gregersen & Skov (2013) and Bregendal et al. (2005) use the term "true" but do not explicitly view an equation as a proposition. None of them state the difference between one solution and the solution set explicitly. Carstensen et al. (2009) do not use the term "true" but define a solution in the same way as Gregersen & Skov (2013) and Bregendal et al. (2005). Nielsen & Fogh (2010) do not connect truth value to the solution of equations at all.

Carstensen et al. (2009) mention that one can not divide or multiply by zero when rewriting an equation. Clausen et al. (2005, 2010a) formulate both in words and symbols that one can add, subtract, multiply or divide by the same number on both sides of the equality sign except zero when dividing or multiplying which supports the balance analogy also shown as a balance with weights on it representing each side of an equation. Moreover, the zero rule are given. In the examples Clausen et al. (2005, 2010a) use the inverse method and do not describe in words what happens from step to step in the solving process. Nielsen & Fogh (2010) formulate only in words that one can add, subtract, multiply or divide by the same number on both sides of the equality sign except zero when dividing or multiplying. Moreover, Nielsen & Fogh (2010) have a theorem about cross-multiplication. In the first example the balance method is used in the calculations as well as in the text describing what happens in each step. In the following examples the inverse method are used in the calculations, but the text attached to each operational line still supports the balance method. In some lines the calculations are not shown and hence we can not decide whether the inverse method or the balance method have been used. It seems that if the calculations are regarded easy (like adding a number) they are just conduced from one line to another, and if they are more difficult (like multiplying into a parenthesis) they are written and then calculated. Gregersen & Skov (2013) formulate the rules in words like Nielsen & Fogh (2010) as well as the zero rule. In the examples the balance method are used both in the operational lines and in the text attached to each operational line. The operations done to each side of the equation are written in red color which just underlines the balance principle. Throughout the chapter and in the tasks following the chapter both a double elevator and a seesaw are drawn to support the analogy. Bregendal et al. (2005) formulate the rules in words and symbols like Clausen et al. (2005, 2010a). Bregendal et al. (2005) use the inverse method in the operational lines of the examples, but the text attached to the lines give an explanation using the balance method.

All textbooks state the four general rules for solving equations using the balance analogy, moreover, some textbooks include the zero rule and one textbook includes a rule about cross-multiplication. No textbooks mention anything about other operations like squaring, taking the root or using some other function. Except for Gregersen & Skov (2013) which exclusively use the balance method throughout theory and examples, the textbooks mix the two methods but without making explicit that there in fact are two different methods or ways of thinking of equations during the solving process.

In all of the textbooks the main purpose of introducing equations is to show how they can be solved. In Gregersen & Skov (2013) there are some theory about turning a word problem into an equation, solve it and turn the mathematical solution back into an answer to the word problem. This does not seem to be that much of a concern in the four other textbooks. Even though Bregendal et al. (2005) have several tasks of exactly this type there is no mentioning of how to deal with them in the chapter. This slight shift in focus is one of the main differences between the books written for HHX and the books written for STX. HHX demands more applicability of the subjects taught than STX do.

Of special interest to this thesis is the treatment (or lack of same) of equivalence and biimplication in the textbooks. Carstensen et al. (2009) introduce the biimplication symbol which can be used between two equivalent equations. The domain of an equation is just discussed in one example in which it do not play a role in the solution process. Moreover, Carstensen et al. (2009) include the experiment described in section 2.3.1. It is a nice exercise but the book do not give the reader the proper prerequisites to solve it. Clausen et al. (2005, 2010a) say that the biimplication arrow can be used when general rules introduced to rewrite an equation are applied. Equivalence is not discussed further and the word is not mentioned at all. Nielsen & Fogh (2010) shortly introduce the implication arrow and the biimplication arrow in the chapter prior to the chapter about equations and two good examples are given of the use of each arrow, but the arrows are not used at all throughout the chapter about equations and equivalence are not discussed. Gregersen & Skov (2013) do not discuss equivalence and do not use the implication or biimplication arrows. Bregendal et al. (2005) do not use the implication or biimplication arrows and do not explicitly discuss equivalence, but in some of the examples the domain of the equation is not all of $\mathbb{R}$ and when using the general rules for solving equations one gets a ”solution” which is not in the domain for the equation.

There is no proper treatment of equivalence in any of the five textbooks examined. Carstensen et al. (2009) is the only textbook actually using the word equivalence, moreover, it is the only one of the textbooks which tries to show what can go wrong when one is not careful when dividing or multiplying by a variable, but since this is done through an exercise it is not knowledge which are available to every reader. Bregendal et al. (2005) contain some examples in which it is important to be aware of the domain of the original equation, but no theory are attached to the examples. Both Clausen et al. (2005, 2010a) and Nielsen & Fogh (2010) introduce the biimplication arrow but do not use it consistently throughout the chapters. Gregersen & Skov (2013) have neither any explicit nor implicit treatment of equivalence.

What seems to be more or less consistent throughout all of the textbooks is that equations and the different techniques for solving them are presented indirectly by examples. The word ”equation” is also used in different contexts than analyzed above e.g when differential equations are treated. So as presented in the textbooks ”equation” becomes a common name for a collection of types of tasks which each correspond to a distinct mathematical praxeology. But in the textbooks it is only a few features - like the presence of equality sign and a variable - that tie the praxeologies together. On a theoretical level some logical rules and a set theoretical description connect this large crowed of praxeologies, but this is treated highly informal and implicit if treated at all in the textbooks analyzed. Maybe because it is regarded unimportant or maybe because it is regarded too difficult.

2.4 Method

To explore how the students define the equation concept they are asked to

1. give three examples of equations.
2. write down short and precise what the word ”equation” means.
3. decide for each expression on a list whether it is an equation or not and indicate how certain they are of their answers.

The examples from question 1 will be used to find out what the students find important about equations even if they are not able to express it in question 2. This could be the case since it is often more difficult to describe a concept in words than to give examples of its members. Yet question 2 is included since three examples can not be expected to show the broadness in the students' definitions. The second question is also important because it reveals the students' own ideas and conceptions while the examples could just reflect the teachers' or textbooks' concepts which the students' might copy without understanding.

The third question will be used to test how important the students' regard the presence of equality sign and variables, whether there is something to solve (ie the variable is not already isolated and/or there are calculations to be performed before obtaining the solution(s)) or not and the truth value of the equation.

There are 13 different, meaningful combinations (from now on to be called types of expressions) of the four parameters mentioned above and for each type a representative example of an expression has been constructed. The examples of the different types of expressions can be seen in tables 1, 2 and 3 where $E \subseteq \mathbb{R}$ refers to the largest possible domains for the given equations. As an example of the type ”identity and something to solve” we also use $\cos^2(x) + \sin^2(x) = 1$ since this identity is well known and properly introduced as a formula rather than as an equation during teaching.

Table 1: Features: Expressions including both equality sign and a variable

	True $\forall x \in E$ (identity)	False $\forall x \in E$	True for some but not all $x \in E$ (conditional)
Something to solve	$2x = x + x$	$x + 1 = x$	$3x + 8 = 7$
Not anything to solve	$x = x$	-	$x = 4$

Table 2: Features: Expressions including equality sign but no variable

	True $\forall x \in E$ (identity)	False $\forall x \in E$	True for some but not all $x \in E$ (conditional)
Something to solve	$7 + 3 = 10$	$2 \cdot 4 = 12$	-
Not anything to solve	$8 = 8$	$0 = 1$	-

Since expressions without an equality sign do not have any truth value, we can make a combined table for expressions containing only variables and expressions which do not contain either equality nor a variable.

Table 3: Features: Expressions without equality sign

	Variable	No variable
Something to solve	$x + 3x - 2 \cdot 4$	$9 + 4 \cdot 3$
Not anything to solve	x	4

Moreover, it will be checked whether it matters what the name of the variable is and how students respond to definitions given by equations, inequalities and differential equations. Assuming that a canonical equation contains an equality sign and a variable, is true for some but not all possible values of the variable (a conditionally true equation) and that there is something to be solved, the equations made to asses students' characterization of expressions which are definitions given by equations, inequalities, differential equations or in which the variable is not named x are design to be as in table 4.

Table 4: Features: Other expressions

Definitions	$f(x) = 31 \cdot 1,23^x$, $y = 12x + 17$
Differential equation	$\frac{dy}{dx} = 8 - 2x$
Inequality	$45x \leq 23 + x$
Variable not named x	$6 + 9a = 20$

2.4.1 Coding of data

Since the students' answers to the first two questions are open, the given answers have been coded in order to make them analyzable.

In the examples given by the students in question 1 some features that an equation might posses have been counted. There has been looked for the presence of equality sign and inequality sign. Moreover, there has been looked for variables, but also for letters which most likely are to be perceived as constants like e.g the letters a, b and c in the equation $ax^2 + bx + c = 0$, which is an example from one of the answers to the questionnaire. Regarding letters it has also been detected whether there where multiple variables in an equation and whether the variable, if present, was called x. It was also checked if the given equations where differential equations. Moreover it was noted if the equations where identities or conditionally true and whether there was something to solve or not. It was noted if an example appeared to be a definition e.g of a function like $y = 7x + 3$ and if it contained any numbers. Expressions with powers like $a^2 + b^2 = c^2$ but no other numbers were not counted as containing numbers.

In the descriptions that the students have provided in question 2 some of the same as the above features and some other features have been detected. Again we have looked for the mentioning of variable or unknown and for the use of the word equality. We have also checked whether students have written something about ”finding x” or ”rearranging” or ”calculating” and coded it as ”something to solve”. Moreover answers containing phrases like ”there is the same quantity on both sides” or ”what is on the left hand side is the same as what is on the right hand side” have been coded as ”identity”. Answers including the words ”function” and ”definition” or ”formula” have been coded ”definition/function”. Some answers included the word ”connection” in phrases like ”it is a connection between numbers” and have been coded ”connection”. Lastly answers including the words ”true”, ”false”, ”truth value” or ”proposition” have been coded ”truth value”.

2.5 Results

2.5.1 Examples

First students were asked to give three examples of equations. If a student has used a feature like e.g an equality sign in all three given examples of equations we, assume that the student regard the feature as defining in order for something to be an equation ie as a necessary condition. If a student has used a feature at least once in the given examples but not in all three, we assume that the student allows the feature but does not find it defining in order for something to be an equation. If a feature do not occur at all, we can not assume that the student do not allow it, but it will show that it is something the student do not typically associate with equations or find particularily important.

Nine students did not answer the question, hence, the number of answers is 152. Out of all the examples holding a variable ($n = 442$) $93,4\%$ used the letter x. Table 5 shows the results of the question.

Table 5: Examples: Occurrence of features, $n = 152$

Feature	Occur 3 times	Occur 1-2 times	Do not occur %
Equality sign	86	12	2
Inequality sign	0	2	98
Variable	93	7	0
Name of variable is x	78	21	1
Multiple variables	10	26	64
Differential equation	0	1	99
Conditionally true	83	15	2
Identity	0	3	97
Something to solve	88	9	3
Definition/function	$10,5$	8	$81,5$
Numbers	79	$12,5$	$8,5$
Letters	4	11	85

The overall percentage of occurrence of the four main features (equality sign, variable, conditionally true, something to solve) from tabel 5 in all of the examples are shown in table 6.

Table 6: Examples: Overall occurrence of main features, $n = 456$

Feature	Occurence %
Equality sign	93
Variable	97
Conditionally true	92
Something to solve	93

With 93% the presence of a variable is what most students find defining in order for something to be an equation (see table 5). In fact, according to table 5 there is not a single student who do not include a variable in at least one of the three examples. Overall 97% of the examples given contain a variable (see table 6). This corresponds very well to the picture the five textbooks draw since almost all examples of equations hold a variable and the definitions do not make it very clear that it is not a necessity for an expression to be an equation.

Equality sign, the equation being conditionally true and the fact that there is something to solve occur three times in over 80% of the answers. This shows that the three features are regarded defining by many students as well. Since the equality sign is the only scholarly defining feature of the four, it is noteworthy that it do not occur three times most often (see table 5) and not even occur most in general (see tabel 6). This shows that like the textbooks students associate the word "equations" primarily with "finding x" or "finding the unknown" - even more than they associate it with the equality sign.

It is, moreover, interesting to see that $10,5\%$ of the students also associate the word "equation" primarily with definitions or functions (see table 5). This underlines the elasticity of the term "equation" - it is used in a lot of different branches of mathematics.

Since the textbooks typically present equations with numbers and one variable it is actually a bit surprising to see that $8,5\%$ gives three examples without numbers at all (see table 5).

To give a more accurate picture of truth value it has been counted how many of the expressions which hold an equality sign or an inequality sign are conditionally true, are an identity or are false since expressions without an equality sign or inequality sign do not have any truth value. The results can be seen in table 7. Of all the examples ($n = 456$) $93,2\%$ of them had a truth value and hence $6,8\%$ did not have a truth value.

Table 7: Examples: Truth value, $n = 429$

Truth value	Occurrence %
Conditionally true	97,9
Identity	0,9
False	1,2

Only $2,1\%$ of the examples or what corresponds to 9 examples are not conditionally true. This also depicts the textbooks very well since most examples given of equations are conditionally true. It is very likely that the students have given their examples without thinking of truth value since this do not seem to be an issue gaining much focus in upper secondary school. If they have just copied the form of the examples of equations they have been exposed to, they are very likely to give examples of conditionally true equations.

2.5.2 Explanations of the word equation

In question 2 students were asked to give an explanation of the word "equation". Some features has been chosen and counted for each explanation. There are 145 answers since 16 out of 161 students did not answer the question. The results are presented in table 8.

Table 8: Explanations: Occurrence of features, $n = 145$

Feature	Occurence %
Variable/unknown	58
Equality	56
Something to solve	50
Identity	20
Definition/function	7
Connection	1
Truth value	1

As in the examples more students mention the presence of a variable than presence of an equality sign. It is surprising that 20% mention something that could be interpreted as identity. But taking a closer look out of these 20% only 14% corresponding to just 4 students do not also mention equality sign. So the feature identity is probably just a description of the meaning of the equality sign: "the right hand side is the same as the left hand side" rather than a description of an equation as an actual identity. Half the students make it part of their explanation that something has to be solved in an equation. Almost nobody speaks of truth value and in particular no one mentions that an equation is a proposition.

To see if there is a connection between including equality and variable in the explanation a χ^2-test has been conducted (see table 9). The χ^2-test gave a p-value

Table 9: Explanations: Cross-tabulation of the features "equality" and "variable", $n = 145$, $p = 0.002500$

	Variable, yes	Variable, no
Equality, yes	38	43
Equality, no	46	18

of 0.002500 which means that there is dependence between including equality and variable in the explanation on a significance level of 1%. So we can reject the hypothesis that there is no connection between including the features "equality" and "variable" in the explanation. According to table 9 the connection seems to be that either students include equality or variable but typically not both or neither. This could reflect that they only had three lines to write the explanation in and hence could not include everything, or it could reflect two fundamentally different ways of thinking of equations.

Likewise, to see if there is a connection between including variable and something to solve a χ^2-test has been conducted (see table 10). The χ^2-test gave a

Table 10: Explanations: Cross-tabulation of the features "something to solve" and "variable", $n = 145$, $p = 0.000000$

	Variable, yes	Variable, no
Something to solve, yes	60	12
Something to solve, no	24	49

p-value of 0.000000 and, hence, we can reject the hypothesis that there is no connection between including "something to solve" and "variable" in the explanation. This would also be the expected outcome since it is very difficult to talk about solving something without talking about what to find.

2.5.3 Comparison of students' examples and explanations

In this section we will examine if there are compliance between students' examples and explanations of equations.

First a χ^2-test has been conducted to investigate if students which use equality sign in the examples also mention equality in the explanation (see table 11). Participants which either did not give examples or did not give an explanation have been excluded leaving 138 answers out of 161 possible answers. We found that

Table 11: Examples vs. explanations: Cross-tabulation of the feature "equality/equality sign", $n = 138$, $p = 0.117029$

	Equality in 3 examples	Equality in 0-2 examples
Explanation, yes	71	6
Explanation, no	51	10

$p = 0.117029$ and, hence, the hypothesis that inclusion of equality in examples and explanations are dependent can not be supported. This is kind of surprising since it shows that students are not consistent in their answers. One explanation for the independence is that it is more difficult to explain what an equation is than to give examples. We se from table 11 that 51 students include equality in the examples but not in the explanation, whereas only 6 students include equality in the explanation but not in all of the examples they give. Another explanation is that students simply do not find the equality sign to be a significant and defining feature for an equation. The equality sign is widely used in all branches of mathematics so maybe they do not feel the need to mention it as something important when giving their explanation because it is always there. In the examples, the students might just write up some expressions which they have seen during teaching of equations, but when they are asked to explain what an equation is they are not able to do so because they were never told what made all the examples equations.

Secondly a χ^2-test has been conducted to investigate if students who use a variable in the examples also mention a variable in the explanation (see table 12). We found that $p = 0.009658$ so the hypothesis that there is independence can

Table 12: Examples vs. explanations: Cross-tabulation of the feature "variable" in students' explanations and examples of equations, $n = 138$, $p = 0,009658$

	Variable in 3 examples	Variable in 0-2 examples
Explanation, yes	77	1
Explanation, no	53	7

be rejected at a significance level of 5%. This shows that students consistently include a variable in their answers regardless of how they are asked to describe an equation. But it should be taken into consideration that very few students only included a variable in $0-2$ of the examples they gave so the test might not be reliable. Nevertheless it is clear from looking at table 12 that most students include variable in the examples as well as in the explanation, but that there are some students which only include a variable in the examples.

2.5.4 Valuation of expressions

In question 3 students were asked to evaluate whether an expression was an equation or not and indicate their certainty of each of their answers measured from very unsure (1) to very sure (5). To each expression $158-160$ students gave an answer. The results appear in table 13.

Just by look at table 13 we see that all expressions either without an equality sign, a variable or both are only regarded to be equations by 43% of the students or less. Moreover, the only difference between $3x+8=7$ and $45x \leq 23+x$ is the connective symbol either being equality or inequality. But the first equation has 99% yes-answers and the latter only has 43% yes-answers suggesting that the equality symbol is important to the students.

As expected, the equation $3x+8=7$ which is a prototypical example of an equation is also regarded to be so by close to all students. It is prototypical because it contains an equality sign, a variable, is conditionally true and has something to solve. Looking only at the expressions which are the possible combinations of the four features, hence, the expressions from tables 1, 2 and 3, we see that $x+1=x$ and $2x=x+x$ come in second and third after $3x+8=7$ regarding yes-answers. These two equations only differ from the prototypical one in truth value since they are always false and always true, respectively. This could indicate that truth value is the one of the four feature which is regarded least important by the students in order for something to be an equation. The equation $x = 4$ takes a fourth place. The only difference between this equation and the prototypical one is the fact that it has nothing to solve indicating that having something to solve is the second least important feature for the students to make up an equation. The equation $x = x$ also has nothing to solve and is also an identity. The next expression on the list is $x + 3x - 2 \cdot 4$ and lacks only an equality sign (and hence also a truth value) compared to the prototypical equation suggesting that equality sign is the next most important feature for the students in order to call some expression an equation. This expression is followed by $7 + 3 = 10$ which compared to the prototypical equation is missing a variable making this feature the most important. But there is very little difference in the percentage of yes-answers to the two expressions $x + 3x - 2 \cdot 4$ and $7 + 3 = 10$ so nothing can yet be determined with absolute certainty.

Table 13: Valuations: Students' valuations of different expressions being an equation or not, $n = 158 - 160$

Expression	Yes (%)	Average certainty	No (%)	Average certainty
$3x + 8 = 7$	99	$4,7$	1	$3,0$
$6 + 9a = 20$	92	$4,3$	8	$2,9$
$\frac{dy}{dx} = 8 - 2x$	91	$3,8$	9	$3,1$
$\cos^2(x) + \sin^2(x) = 1$	86	$3,8$	14	$3,4$
$y = 12x + 7$	83	$4,2$	17	$3,9$
$x + 1 = x$	77	$4,2$	23	$3,3$
$2x = x + x$	75	$4,0$	25	$3,5$
$x = 4$	68	$4,0$	32	$3,7$
$f(x) = 31 \cdot 1,23^x$	64	$4,0$	36	$3,8$
$x = x$	50	$3,6$	50	$3,6$
$45x \leq 23 + x$	43	$3,6$	57	$3,7$
$x + 3x - 2 \cdot 4$	21	$4,5$	79	$4,3$
$7 + 3 = 10$	19	$3,6$	81	$4,3$
$8 = 8$	14	$3,1$	86	$4,3$
$2 \cdot 4 = 12$	12	$3,6$	88	$4,3$
x	11	$4,0$	89	$4,4$
$0 = 1$	8	$3,0$	92	$3,0$
$9 + 4 \cdot 3$	3	$4,5$	97	$4,4$
4	1	$3,0$	99	$4,6$

To find out more precisely which features are important to the students some cross-tabulations have been made of answers to some of the equations, and a χ^2-test has been performed for each cross-tabulation. The results will now be presented.

Something to solve
To find out whether it is important to the students if there is something to solve or not we compare the students' valuations of the expression $3x + 8 = 7$ and the expression $x = 4$ which both hold an equality sign and a variable and are conditionally true (see table 14). Hence, the only difference between the two expressions is that in the first there is something to solve and in the second there is not. A

Table 14: Something to solve: Answers to the expressions $3x + 8 = 7$ and $x = 4$, $n = 320$, $p = 0.000000$

	$x = 4$	$2x + 8 = 7$
yes	109	158
no	51	2

χ^2-test has been conducted and it showed that $p = 0.000000$ which means that the hypothesis that the valuation of $x = 4$ and $2x + 8 = 7$ would be the same can be rejected at a significance level of 1%. Table 14 shows that significantly more students regard $2x + 8 = 7$ to be an equation than $x = 4$. And since the only difference between the two expressions is that $2x + 8 = 7$ has something to solve and $x = 4$ do not this must be what makes the students answer like they do.

A χ^2-test has also been performed on the answers to $2x = x + x$ and $x = x$ (see tabel 15) which both have equality sign, a variable and are identities, on the answers to $7 + 3 = 10$ and $8 = 8$ (see table 16) which both have equality sign and are identities, on the answers to $2 \cdot 4 = 12$ and $0 = 1$ (see table 17) which both have equality sign and are false, on the answers to $x + 3x - 2 \cdot 4$ and x (see table 18) which both have variable but no equality sign, and on the answers to $9 + 4 \cdot 3$ and 4 (see table 19) which do not have either equality sign nor variable. Hence, the only difference between the two expressions in each of the five pairs is that one of them has something to solve and the other one do not.

Table 15: Something to solve: Answers to the expressions $2x = x + x$ and $x = x$, $n = 319$, $p = 0.000003$

	$x = x$	$2x = x + x$
yes	80	120
no	80	39

Table 15 shows with a p-value of 0.000003 that the hypothesis that students' valuate the expressions $x = x$ and $2x = x + x$ in the same way can be rejected at a significance level of 1%. Significantly more students regard $2x = x + x$ to be an equation than $x = x$. Even though both expressions are identities which is not as neutral as expressions being conditionally true, we still have a significant difference in the answers showing that it matters whether there is something to solve or not.

Table 16: Something to solve: Answers to the expressions $7 + 3 = 10$ and $8 = 8$, $n = 320$, $p = 0.225413$

	$8 = 8$	$7 + 3 = 10$
yes	22	30
no	138	130

Table 16 shows that the hypothesis that the valuations of $8 = 8$ and $7 + 3 = 10$ are the same can not be rejected since $p = 0.225413$. However this could have something to do with the expressions not holding a variable. Hence very few students answer yes, so the difference in the answers between the two equations do not become significant.

Table 17: Something to solve: Answers to the expressions $2 \cdot 4 = 12$ and $0 = 1$, $n = 318$, $p = 0.198247$

	$0 = 1$	$2 \cdot 4 = 12$
yes	12	19
no	146	141

Table 17 shows that the hypothesis that the valuations of $0 = 1$ and $2 \cdot 4 = 12$ are the same can not be rejected because $p = 0.198247$. But these expressions are both false and miss a variable, hence, very few students answered yes which is why the difference in the answers is not significant.

Table 18: Something to solve: Answers to the expressions $x + 3x - 2 \cdot 4$ and x, $n = 319$, $p = 0.014689$

	x	$x + 3x - 2 \cdot 4$
yes	17	33
no	142	127

Table 18 says that the hypothesis that the expressions x and $x + 3x - 2 \cdot 4$ are assessed in the same way can be rejected at a significance level of 5% because $p = 0.014689$. So even though the expressions are not equations because they both miss an equality sign, they are still assessed significantly different because $x + 3x - 2 \cdot 4$ has something to solve and x do not.

Table 19 tells that the hypothesis that 4 and $9 + 4 \cdot 3$ are assessed in the same way can not be rejected since $p = 0.174035$. But because the expressions lack both a variable and an equality sign almost nobody answered yes so the difference in

Table 19: Something to solve: Answers to the expressions $9 + 4 \cdot 3$ and 4, $n = 319$, $p = 0.174035$

	4	$9 + 4 \cdot 3$
yes	1	4
no	159	155

the answers is not significant.

To summarize; in the three comparisons in which the expressions contained a variable (tables 14, 15 and 18) there was a significant difference in the assessment of the expressions, meaning that expressions containing something to solve were more likely to be rated as equations than expressions not containing anything to solve.

In the three comparisons without a variable (tables 16, 17 and 19) it were not possible to reject the hypothesis that the expressions were assessed in the same way. In the comparison of assessments of 4 and $9 + 4 \cdot 3$ (table 19) the hypothesis that the two expressions were assessed in the same way can not be rejected. This is most likely the case because both were regarded not to be equations by almost all students because of the lack of the features variable and equality sign. Hence, whether there was something to solve or not did not change the students' answers. From this can be concluded that the presence of something to solve or calculate alone do not make up an equation in the students' minds - more is needed. In the comparisons of evaluations of $8 = 8$ and $7 + 3 = 10$ (table 16) and of $0 = 1$ and $2 \cdot 4 = 12$ (table 17) lack of a variable and in the second case also lack of the truth value being true led most students to not assess the expressions as equations.

Hence, it seems (which we will explore later) that the presence of a variable is a more deciding factor for the assessment than whether there is something to solve or not. But still when a variable is present whether there is something to solve or not matters to the students.

Truth value

To find out whether truth value matters in students' valuations of the expressions comparisons and χ^2-tests are made for some of the expressions. We have compared $3x + 8 = 7$, $2x = x + x$ and $x + 1 = x$ (see table 20) which all holds equality sign, a variable and have something to solve, but are conditionally true, always true and always false, respectively. Moreover $x = x$ and $x = 4$ (see table 21), $2 \cdot 4 = 12$ and $7 + 3 = 10$ (see table 22) and $8 = 8$ and $0 = 1$ (see table 23) have been compared too, since the only difference between the two expressions in each pair is the truth value. No expressions without equality sign have been compared since they do not have any truth value.

Table 20: Truth value: Answers to the expressions $3x + 8 = 7$, $2x = x + x$ and $x + 1 = x$, $n = 478$, $p = 0.000010$

	$3x+8=7$	$2x=x+x$	$x+1=x$
yes	158	120	123
no	2	39	36

Table 20 tells that the hypothesis that the valuations of the three equations $3x+8=7$, $2x=x+x$ and $x+1=x$ are the same can be rejected at a significance level of 1% because $p = 0.000010$. We see that the conditionally true expression $3x + 8 = 7$ are more likely to be regarded an equation by the students than the true and false expressions which are assessed to be equations almost equally often.

Table 21: Truth value: Answers to the expressions $x = 4$ and $x = x$, $n = 320$, $p = 0.000978$

	$x=4$	$x=x$
yes	109	80
no	51	80

Table 21 shows that the hypothesis that the expressions $x = 4$ and $x = x$ are assessed in the same way can be rejected at a significance level of 1% because $p = 0.000978$. So even though the expressions have nothing to solve it still makes a significant difference in the students' valuations that $x = 4$ is conditionally true and $x = x$ is an identity. The conditionally true expression are more often assed to be an equation than the identity.

Table 22: Truth value: Answers to the expressions $7 + 3 = 10$ and $2 \cdot 4 = 12$, $n = 320$, $p = 0.075596$

	$7+3=10$	$2\cdot 4=12$
yes	30	19
no	130	144

Table 23: Truth value: Answers to the expressions $8 = 8$ and $0 = 1$, $n = 318$, $p = 0.075738$

	$8=8$	$0=1$
yes	22	12
no	138	146

Table 22 shows that we can not reject the hypothesis that $7 + 3 = 10$ and $2 \cdot 4 = 12$ are assessed in the same way since $p = 0.075596$. Moreover Table 23 likewise tells that we can not reject the hypothesis that $8 = 8$ and $0 = 1$ are assessed in the same way because $p = 0.075738$. Put together we can conclude that the truth value do not matter in equations without a variable. But even though the difference in the answers is not significant there still is a little difference in favor of the true expressions being equations more often than the false equations. This indicates that truth value still might matter a little bit even though the expressions do not hold a variable, and that true expressions might be evaluated as equations a bit more often than false expressions.

To summarize; In expressions holding a variable the truth value definitely matters to the students' valuations. Conditionally true expressions are more likely to be assessed as equations than expressions which are always true or always false (tables 20 and 21). On the other hand in expressions without a variable truth value do not seem to matter even though true expressions are assessed to be equations a little bit more often than false equations (tables 22 and 23). But this difference might just be due to chance.

Comparison of something to solve and truth value

We would like to compare the feature "something to solve" to the feature "truth value". This is done by comparing the expression $x = 4$ to $2x = x + x$ (see table 24) and $x + 1 = x$ (see table 25), respectively. The expressions all have an equality sign and a variable but while $x = 4$ is conditionally true but has nothing to solve $2x = x + x$ and $x + 1 = x$ both have something to solve but are always true/false.

Table 24: Truth value vs. something to solve: Answers to the expressions $2x = x+x$ and $x = 4$, $n = 319$, $p = 0.144887$

	$2x = x + x$	$x = 4$
yes	120	109
no	39	51

Table 25: Truth value vs. something to solve: Answers to the expressions $x+1 = x$ and $x = 4$, $n = 319$, $p = 0.064102$

	$x + 1 = x$	$x = 4$
yes	123	109
no	36	51

In non of the tables 24 and 25 there is a significant difference between the answers. So it is not possible to finally conclude whether the expression being conditionally true or the expression having something to solve is more important. Although there seems to be a few more students answering yes to $2x = x + x$ and $x + 1 = x$ being an equation than to $x = 4$. This might indicate that that there is something to solve is a bit more important than the expression being conditionally true, but the difference is small and could just be due to chance.

Variable

To determine the importance of the presence of a variable in the students' evaluations of the expressions some comparisons and χ^2-tests have been made. Since an expression without a variable can not be conditionally true, and since this seems to be the most neutral setting for an expression with a variable according to table 13, we have compared $7 + 3 = 10$ both to $3x + 8 = 7$ (see table 26) and to $2x = x + x$ (see table 27). The first expression is conditionally true and ,hence, most neutral while the second expression is an identity like $7 + 3 = 10$.

Moreover we have compared $8 = 8$ to both $x = 4$ (see table 29) and $x = x$ (see table 30). In addition, $x + 1 = x$ and $2 \cdot 4 = 12$ (see table 28), $x + 3x - 2 \cdot 4$ and $9 + 4 \cdot 3$ (see table 31) and x and 4 (see table 32) have been compared, exhausting all possible pairs in which the presence of a variable is the only difference.

Table 26: Variable: Answers to the expressions $3x+8 = 7$ and $7+3 = 10$, $n = 320$, $p = 0.000000$

	$3x + 8 = 7$	$7 + 3 = 10$
yes	158	30
no	2	130

Table 27: Variable: Answers to the expressions $2x = x+x$ and $7+3 = 10$, $n = 319$, $p = 0.000000$

	$2x = x + x$	$7 + 3 = 10$
yes	120	30
no	39	130

Table 28: Variable: Answers to the expressions $x + 1 = x$ and $2 \cdot 4 = 12$, $n = 319$, $p = 0.000000$

	$x + 1 = x$	$2 \cdot 4 = 12$
yes	123	19
no	36	141

Table 29: Variable: Answers to the expressions $x = 4$ and $8 = 8$, $n = 320$, $p = 0.000000$

	$x = 4$	$8 = 8$
yes	109	22
no	51	138

Table 30: Variable: Answers to the expressions $x = x$ and $8 = 8$, $n = 320$, $p = 0.000000$

	$x = x$	$8 = 8$
yes	80	22
no	80	138

Table 31: Variable: Answers to the expressions $x + 3x - 2 \cdot 4$ and $9 + 4 \cdot 3$, $n = 319$, $p = 0.000000$

	$x + 3x - 2 \cdot 4$	$9 + 4 \cdot 3$
yes	33	4
no	127	155

Table 32: Variable: Answers to the expressions x and 4, $n = 319$, $p = 0.000098$

	x	4
yes	17	1
no	142	159

In tables 26-32 the p-values are incredibly low showing that the hypothesis that the presence of a variable does not matter can be rejected. It do not make a difference that the expressions are false (table 28), that there is nothing to solve (tables 29 and 30), that the equality sign is missing (table 31) or that there is nothing to solve and the equality sign is missing (table 32). This shows that the presence of a variable is very important in the students' assessments of the expressions.

An expression without a variable are generally assessed not to be an equation (tables 26-32). When the expression holds an equality sign (tables 26-30) it is generally assessed to be an equation if it also holds a variable. If the expression do not hold an equality sign (tables 31 and 32), even though most students' assess it not to be an equation significantly more students' assess it to be an equation if it at least holds a variable. It is surprising but also alarming that 33 and 17 students regard $x + 3x - 2 \cdot 4$ and x to be equations, respectively, and only 30, 19 and 22 students regard $7+3 = 10$, $2 \cdot 4 = 12$ and $8 = 8$ to be equations, respectively.

Equality sign
When comparing answers to equations with or without something to solve every other parameter (truth value, equality and variable) were being held equal. That is not possible when determining the significance of the equality sign since expressions without equality do not have a truth value and expressions with an equality always have a truth value. For expressions containing a variable and equality sign conditionally true expressions are regarded to be more neutral than identities and false expressions hence $3x + 8 = 7$ is compared to $x + 3x - 2 \cdot 4$ (see table 33) and $x = 4$ is compared to x (see table 34). For expressions containing an equality sign but no variable identities are considered to be more neutral than false expressions therefore $7 + 3 = 10$ is compared to $9 + 4 \cdot 3$ (see table 35) and $8 = 8$ is compared to 4 (see table 36). In all cases there is a significant difference between the answers at a significance level of 1%.

Table 33: Equality sign: answers to the expressions $3x + 8 = 7$ and $x + 3x - 2 \cdot 4$, $n = 320$, $p = 0.000000$

	$3x + 8 = 7$	$x + 3x - 2 \cdot 4$
yes	158	33
no	2	127

Table 34: Equality sign: answers to the expressions $x = 4$ and x, $n = 319$, $p = 0.000000$

	$x = 4$	x
yes	109	17
no	51	142

Whenever the expressions assessed hold a variable (tables 33 and 34) they are generally assessed to be equations by the students if they also have an equality sign. If they do not have an equality sign, they are generally assessed not to be equations even though they have a variable. So significantly more students' regard the expressions to be equations if they hold an equality sign as opposed to if they do not.

Table 35: Equality sign: answers to the expressions $7+3 = 10$ and $9+4\cdot3$, $n = 319$, $p = 0.000003$

	$7 + 3 = 10$	$9 + 4 \cdot 3$
yes	30	4
no	130	155

Table 36: Equality sign: answers to the expressions $8 = 8$ and 4, $n = 320$, $p = 0.000005$

	$8 = 8$	4
yes	22	1
no	138	159

If the expressions do not hold a variable (tables 35 and 36) they are generally assessed not to be equations. But still significantly more students assess the expressions to be equations if they hold an equality sign compared to if they do not.

Hence, the equality sign is very important in the students' evaluations of the expressions. But the equality sign alone is not enough for students in general to perceive something as an equation. The expression has to also hold a variable.

Other features

Lastly we see if there is any difference in students' perception of an identity with a variable and something to solve which is known as a formula, $\cos^2(x)+\sin^2(x) = 1$, and one which is not known as a formula, $2x = x + x$ (see table 37). Moreover we check if there is a difference between using x and a as the name of the variable (see table 38).

Table 37: Other: Answers to the expressions $2x = x + x$ and $\cos^2(x) + \sin^2(x) = 1$, $n = 318$, $p = 0.015469$

	$2x = x + x$	$\cos^2(x) + \sin^2(x) = 1$
yes	120	137
no	39	22

Table 37 says that the hypothesis that the the expressions $2x = x + x$ and $\cos^2(x) + \sin^2(x) = 1$ are assessed in the same way by the students can be rejected at a significance level of 5% because $p = 0.015469$. Students are more likely to assess the formula $\cos^2(x) + \sin^2(x) = 1$ to be an equation than the identity $2x = x + x$.

Table 38: Other: Answers to the expressions $3x + 8 = 7$ and $6 + 9a = 20$, $n = 319$, $p = 0.003479$

	$3x + 8 = 7$	$6 + 9a = 20$
yes	158	146
no	2	13

Table 38 tells that the hypothesis that the expressions $3x+8 = 7$ and $6+9a = 20$ which are alike but have different names for the variable are assessed in the same way can be rejected at a significance level of 1% since $p = 0.003479$. Hence, students prefer x as the name of the variable and a significant amount of the students find it so important that they do not assess an expression with a variable named a to be an equation.

2.5.5 Summary

The first two questions showed that the presence of a variable is actually what is most important for the students in order for something to be an equation. In the third question there seems to be a tie between variable and equality sign. Even though the presence of a variable is not a defining feature for something to actually be an equation it still makes good sense. All the textbooks analyzed in section 2.3 have a very heavy focus on solving equations and finding the unknown. Hence, students are led to think of equations as a process of finding the unknown/variable rather than as a mathematical object. The textbooks and therefore also the students almost never handle equations without variables and, hence, do not perceive expressions without a variable as equations.

The answers to the questions showed that second most important feature is the presence of an equality sign. This is actually the only feature of the four chosen features that are necessary for something to be an equation. But the equality sign is used in so many different contexts of mathematics that students might not realize that it is defining for equations and not just something which is always there and do not separate one kind of expression from another.

It is difficult to decide whether that the equations are conditionally true or that there is something to solve are more important. When students give an explanation of equations they do not mention truth value, but when they give examples almost all examples are conditionally true. On the other hand half of the students mention something about solving when they explain what an equation is, but this is also very closely related to variables. In the examples almost all equations given by the students have something to solve or calculate. It is kind of surprising since one has to mention truth value when defining equations formally whereas there being something to solve is not important. In the valuation of the equations it seems that truth value as well as the fact that there is something to solve matters equally as long as a variable and an equality sign are present. But when there is not an equality sign and/or a variable present in the expressions neither truth value nor that there is something to solve matters significantly in the students' valuations. In the textbooks there are exclusively given examples of equations which have something to solve and truth value are barely mentioned. On the other hand almost all examples of equations in the textbooks are conditionally true even though this is not mentioned explicitly. This could be the reason why students do not include truth value in their explanations, but still almost exclusively give examples of conditionally true equations. Put together something to solve being present in an expression seem to matter slightly more than the expression being conditionally true.

So if one were to range the four features starting with the one students find most important the order would be:

1. Variable present
2. Equality sign present
3. Something to solve present
4. Truth value being conditionally true

2.6 Discussion

In this short section the results from section 2.5 are discussed a little bit further.

2.6.1 Receptive and productive learning

One thing that can not pass unremarked is the fact that even though the students' answers to the three questions show more or less the same understanding of equations the answers are still very different.

Except nine all students were capable of providing three examples of equations in question 1. Seven percent of these examples did not have an equality sign and, hence, were not equations but the remaining 93% were actually equations. So from this one could conclude that students' are pretty confident with the concept of equations. But the explanations in question 2 tell another story. Only 56% include something about an equality sign. Only a single student mentions something about truth value. Most of the descriptions are inadequate and show a poor understanding of equations.

As already mentioned this could be because it is more difficult to describe a collection of items than to give examples of its members. But why is it so? Well, a parallel to language acquisition can be drawn. When small children are learning language they first develop a receptive vocabulary. That is, they learn to understand a certain amount of the language they are trying to learn before they start to use it; before they develop a productive vocabulary (Sprogpakken, 2011). It is the same situation when learning mathematics. One has to see examples and understand a certain amount of theory before one are able to express oneself about the subject at hand. Hence, without a fully developed language or in this case understanding of equations it is easier to give examples than to explain the rules that has to be met for something to be an equation. Students have some ideas (a receptive vocabulary) about what an equation is, but they have not yet developed a full understanding (a productive vocabulary in addition to the receptive vocabulary) of equations. It is like children who can recognize a car when they see it or even draw a car, but they can not describe exactly what features something has to hold in order to be a car and not e.g a bus. So because the productive vocabulary typically comes after the receptive vocabulary, students have a harder time answering question 2 than question 1.

2.6.2 Students' conceptions vs. the scholarly concept

There is certainly a difference between upper secondary school students' perception of equations, textbooks for upper secondary school's interpretation of equations and the scholarly concept of equations.

As we have seen in the above sections students' value the presence of a variable and equality sign highly. Moreover, the fact that there is something to solve seems important as well.The textbooks have a strong focus on the solving process almost so strong that they ignore defining the concept of equations properly. The scholarly concept focuses on the equality sign being present, but also on the fact that an equation is an open proposition and, hence, an expression with a truth value for each choice of an element in the domain of the equation.

It is not unusual that certain details of subjects taught are not discussed in upper secondary school. For example it is in its place not to mention that second degree equations always have two solutions - sometimes one or both are just a complex number - since this is difficult for the students to grasp, and unnecessary in order to understand and use the quadratic formula and relate the solutions to the graphs of parabolas. Of course there has to be a line somewhere between what is taught and what is not taught at a certain level of teaching. But the line that seems to have been drawn regarding the description of what an equation is appears arbitrary. It probably is so because equations are taught long before the students reach upper secondary school level, and because the concept of equations in itself is kind of hazy. There is no good reason though why students are not properly told what an equation is. On the other hand there are plenty reasons to teach theoretical aspects of equations. Equations are objects present not only as an individual subject of teaching, but present in every mathematical subject taught in upper secondary school and also present in other subjects like chemistry and physics. Moreover, it is not that difficult to properly explain what an equation is. It do not even have to be as rigidly explained as in Christiansen et al. (1964) in order to make sense and contribute to a better understanding. Looking at the exam tasks (see section 1.2.2) it is not surprising though that the theoretical aspects of equations are not devoted more time in upper secondary school since no tasks test this explicitly.

3 Solving equations

What we mean by ”solving an equation” is to explicitly find all the values of the variable(s) which make the proposition which the equation forms true.

There is (unfortunately) no general procedure to find the solutions when faced with an equation. But there are some strategies which work better than others and it is not arbitrary which operations mathematicians use when solving equations. It is just very hard to describe the process in general. In all the textbooks for upper secondary school level teaching analyzed in section 2.3 the task of solving equations is introduced mainly through examples. From these examples students are expected to be able to solve structurally alike equations but it is rarely explicit in the textbooks what is meant by this. Equations permeates almost all parts of the curriculum, and students might not discover the structural connection between equations from different parts of the curriculum on their own.

Some types of equations though are gathered and a general technique for solving them are explicitly presented to the students. This is for example the case with second degree polynomial equations. They are typically presented in the form

$$ax^2 + bx + c = 0$$

where $a, b, c \in \mathbb{R}$, and the quadratic formula are presented as a way of solving equations of this form. A second degree polynomial equation can always be rewritten into the above form, but if it is not presented in that form the technique might not be of great use for the weak students. Moreover, there are plenty of other ways to solve special types of second degree polynomial equations (factoring, completing the square, producing a reduced quadratic equation, Vieta's formula etc.) which are neglected in the books. Though the quadratic formula can be used on all second degree polynomial equations, it is not always the easies way to go about solving such equations.

When almost exclusively using such formulas the task of solving equations becomes a mechanical procedure disconnected from the concept of equations. Therefore equations which can not be solved directly by using a formula could be difficult for the students to handle.

This is the reason why this thesis will try to evaluate to what extend students are able to choose and perform legal operations on equations and detect mistakes regarding equivalence (biimplication) and domain in written solutions to equations. The main focus is not whether students are able to perform legal operations correctly or not, but whether they are able to choose the right operations and assess someone else's choices.

3.1 Equivalence of equations

Definition 3.1 (Equivalent equations). Two equations are said to be equivalent on some set E, if they have the same set of solutions on E.

Example 3.2. For $x \in \mathbb{R}$, consider the equations

$$2x^2 - 6x + 4 = 0 \tag{3.1.1}$$

$$(x-2)(x-1) = 0 \tag{3.1.2}$$

The two equations do not look the same at all, but we can easily check that $x = 1$ and $x = 2$ are solutions to both equations. And since both equations are second degree polynomial equations, we know by the fundamental theorem of algebra that they have exactly two complex solutions. Therefore the two equations have the same set of solutions on $\mathbb{R}$ namely the set $\{1, 2\}$, and therefore by definition 3.1 they are said to be equivalent. We can write:

$$2x^2 - 6x + 4 = 0 \Leftrightarrow (x-2)(x-1) = 0 \tag{3.1.3}$$

Thus an implication between two equations can be perceived as a set inclusion. Suppose $P(x)$ and $Q(x)$ are two equations with domains F and G respectively. Suppose, moreover, that we have a third set E such that $E \subseteq F \cap G$. Then $P(x) \Leftrightarrow Q(x)$ on E if and only if $P(x)$ and $Q(x)$ have the same set of solutions on E according to definition 3.1, hence, if and only if

$$\{x \in E \mid P(x) \text{ true}\} = \{x \in E \mid Q(x) \text{ true}\}.$$

From this follows that $P(x) \Rightarrow Q(x)$ on E if and only if

$$\{x \in E \mid P(x) \text{ true}\} \subseteq \{x \in E \mid Q(x) \text{ true}\}.$$

When solving an equation what we essentially do is that we make a sequence of equations equivalent to the original equation such that for each step in the sequence the solution set to the equation is more and more explicit. But what we see from the above is that if we only calculate forward and not give thought to whether the operations we use can be reversed, we can end up finding a set which only *contains* the solutions. Moreover, we also see that the equations only can be equivalent on the intersection of their domains. So if we apply an operation which decreases the domain of the original equation we might loose some solutions. On the other hand if we have not determined the domain of the equation before operating on it and apply an operation which increases the domain we might obtain solutions which are not contained in the domain of the original equation.

3.1.1 Legal operations

When solving equations we are aloud to subtract or ad any number an multiply or divide by a non-zero number on both sides of the equality sign. Using these operations we will always obtain an equation equivalent to the original equation. These operations are legal because equality is an equivalence relation (reflexive, symmetric and transitive) and antisymmetric.

Definition 3.3 (**Equivalence relation**)**.** A binary relation R on a set X is said to be an equivalence relation if and only if

1. aRa (reflexivity),
2. aRb if and only if bRa (symmetry),
3. If aRb and bRc then aRc (transitivity)

for all $a, b, c \in X$.

It clearly follows from definition 3.3 that equality is an equivalence relation on the real numbers.

Definition 3.4 (**Antisymmetry**)**.** A binary relation R on a set X is said to be antisymmetric if for all $a, b \in X$ it holds that if

$$aRb \text{ and } bRa \text{ then } a = b.$$

Less formally a relation is antisymmetric if no distinct pair of elements in X are related to one another. Equality is clearly an antisymmetric relation on the real numbers. This means that every equivalence class only consists of one element and, hence, that any two elements on $\mathbb{R}$ which are equal to one another can be interchanged in mathematical expressions. Note that this is not true in general for equivalence relations.

Theorem 3.5. *Let $a, b, c \in \mathbb{R}$. Then $a = b$ if and only if $a + c = b + c$.*

Proof. Let $a, b, c \in \mathbb{R}$. Assume first that $a = b$. By reflexivity of equality we have that $c = c$. Moreover, for some $k \in \mathbb{R}$ we have that

$$a + c = k.$$

By the assumption and antisymmetry the elements a and b are interchangeable so we obtain

$$b + c = k.$$

By symmetry we get

$$k = b + c.$$

And lastly by transitivity we can conclude that

$$a + c = b + c.$$

Assume now that $a + c = b + c$. By reflexivity we get $c = c$. For some $k \in \mathbb{R}$ we have that

$$a + c - c = k,$$

which is equivalent to $a = k$. By the assumption and antisymmetry the elements $a + c$ and $b + c$ are interchangeable, hence

$$b + c - c = k,$$

which is equivalent to $b = k$. By symmetry $k = b$ and by transitivity we therefore conclude that

$$a = b.$$

□

Since subtracting a number c is the same as adding $-c$ we also get by theorem 3.5 that we can subtract the same number from both sides of the equality sign. Likewise, we can show that we can multiply or divide by the same non-zero number on both sides of the equality sign and still obtain an equation equivalent to the original one.

Sometimes when solving equations we also need to square both sides of the equation or take the root on both sides - that is we need to apply some function to both sides of the equation - to solve it. If the functions we apply are continuous, one-to-one and defined everywhere (like $f_1(x) = x + c$, $f_2(x) = x - c$, $f_3(x) = c \cdot x$ and $f_4(x) = \frac{x}{c}$, for some $c \in \mathbb{R}$, where $c \neq 0$ for the last two functions), we will obtain an equation equivalent to the original equation.

But if we for instance apply a function which is not one-to-one we could gain solutions which in fact do not solve the original equation. If we apply a function which is not everywhere defined we could loose solutions to the original equation. This do not mean that we can not apply the functions. They will still be useful and legal as long as one keeps track of the domain of the equations. This problem will be addressed in section 3.2 below.

3.2 Difficulties related to biimplication between equations

This section is inspired by Sultan & Artzt (2011a; 2011b).

Definition 3.6 (**Original and modified equations**). In the following we will refer to the initial equation as the original equation, and to equations obtained by performing a series of legal operations on the original equation as modified equations.

Definition 3.7 (**Extraneous solutions**). Solutions to a modified equation which are not solutions to the original equation are called extraneous solutions.

We have to be aware that some of the operations we (normally) use when solving equations e.g squaring both sides or taking the root on both sides possibly can increase or decrease the set of solutions.

If we not very carefully keep track of the domain to which the variable quantity belongs, we might loose solutions to the original equation or gain extraneous solutions in the process of solving an equation.

The sections 3.2.1-3.2.5 will present examples of situations in which one has to be extra careful making sure that the operations applied do in fact yield an equivalent equation on the same domain as the original equation. These examples are also used in the questionnaires (see appendix 8.1) given to the students both as equations which the students have to solve by themselves, and as equations to which there are wrong solutions in which students have to find the mistake.

3.2.1 Applying a function which is not one-to-one

Example 3.8. Consider the equation

$$\sqrt{x} = 2x - 1 \tag{3.2.1}$$

In order to solve this equation we square both sides of it and obtain

$$x = (2x - 1)^2 \tag{3.2.2}$$

We multiply and rearrange the order of the terms in order to get

$$0 = 4x^2 - 5x + 1 \tag{3.2.3}$$

By use of the quadratic formula we obtain the solutions $x = 1$ and $x = \frac{1}{4}$. By inserting the solutions into the original equation, we see that only $x = 1$ is a solution and, hence, $x = \frac{1}{4}$ is extraneous.

In the original equation we have to have that $x > 0$ and that $2x - 1 > 0$ since $\sqrt{x}$ means the positive number which multiplied by itself gives x. The last inequality yields by rearranging that $x > \frac{1}{2}$. By squaring both sides of the equation we have increased the domain of the equation to all of $\mathbb{R}$ and therefore we could possibly get solutions to equation (3.2.2) which are not in the original domain and, hence, not solutions to the original equation.

Said in other words, we cannot reverse the implication without assuming that $x > \frac{1}{2}$. Therefore we have that the set of solutions to the original equation is a subset of the set of solutions to the modified equation, but not the other way around.

Example 3.9. Consider the equation

$$\sqrt{2x-4} = \sqrt{3x} \tag{3.2.4}$$

We square both sides of the equation and get

$$2x - 4 = 3x \tag{3.2.5}$$

which has the solution $x = -4$. But this is not a solution to the original equation. When we squared both sides, we increased the domain of the equation from being $[2; \infty]$ to all of $\mathbb{R}$. So the original equation has no solutions.

The examples 3.8 and 3.9 show what could happen when one is not careful handling the domain of the modified equations when applying a function to the original equation which is not one-to-one. The function f given by $f(x) = x^2$ is not one-to-one. Therefore when applied to an equation which can not be defined on all of $\mathbb{R}$ one might risk to increase the domain and, hence, obtain a modified equation with a larger domain than the original one. Moreover, one might also obtain extraneous solutions since the domain of the original equation is only contained in the domain of the modified equation so the solution set to the original equation will only be contained in the solution set to the modified equation.

3.2.2 Multiplying by zero

Example 3.10. Consider the equation

$$\frac{x^2-1}{x-1} = 0 \tag{3.2.6}$$

We multiply by $x - 1$ on both sides of the equality sign and get

$$x^2 - 1 = 0 \tag{3.2.7}$$

We rearrange the terms in order to get

$$x^2 = 1 \tag{3.2.8}$$

Hence, $x = \pm 1$. We see that only the negative solution is a solution to the original equation, hence, $x = 1$ is an extraneous solution. If we insert $x = 1$ in the original equation we divide by zero which is not allowed. By multiplying by $x-1$ we expand the domain of the equation from being $\mathbb{R}\backslash\{1\}$ to all of $\mathbb{R}$. If we choose to multiply by $x - 1$ we have to assume that $x \neq 1$ because in that case we are multiplying by zero.

Example 3.11. Consider the equation

$$\frac{3x+1}{x} = \frac{2x+1}{x} \tag{3.2.9}$$

We multiply by x on both sides to obtain

$$3x + 1 = 2x + 1 \tag{3.2.10}$$

Hence, $x = 0$. This is not a solution to the original equation. When we multiplied by x we expanded the domain of the equation from $\mathbb{R}\backslash\{0\}$ to all of $\mathbb{R}$.

The examples 3.10 and 3.11 show what happens when one mistakenly multiplies an equation by zero. It is of course commonly known that it is not allows to multiply by zero. If one do so anyway, the modified equation will be true for all elements in the domain of the original equation since one will obtain the equation $0 = 0$ which is always true. In other words, the solution set to the original equation will be a subset of the solution set to the modified equation. But since the solution set to the modified equation will be all of the domain we do not come closer to actually finding the solution set.

What happened in the examples 3.10 and 3.11 is that the zero with which we multiplied were in disguise as $x - 1$ and x, respectively. These expressions are equal to zero for certain values of x. Therefore the modified equations will be true for these values of x because they will be $0 = 0$. So the solution set to the original equation will only be a subset of the solution set to the modified equation. Moreover we know that the only possible extraneous solutions will be those for which the expression $x - 1$ and x respectively equals zero. Hence, the method of multiplying by some expression containing the variable is valid as long as it is checked for which values of the variable the expression is equal to zero.

3.2.3 Applying a function which is not everywhere defined

When applying a function which is not everywhere defined to an equation in which the unknown is everywhere defined, we can easily loose solutions if we are not very careful.

Example 3.12. Consider the equation

$$(x-3)^2 = (2x-3)^2 \tag{3.2.11}$$

We can solve this equation by taking the square root on both sides. We get

$$x - 3 = 2x - 3 \tag{3.2.12}$$
$$x = 0 \tag{3.2.13}$$

But we see that $x = 2$ is also a solution to the original equation. In the calculations we have assumed that the root function $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(x) = \sqrt{x}$ is the left inverse of the function $g : \mathbb{R} \to \mathbb{R}^+$ given by $g(x) = x^2$ on all of $\mathbb{R}$, but this is only the case on $\mathbb{R}^+$. Because since g is not one to one it does not have an inverse function on all of $\mathbb{R}$. So we can only be sure that equation (3.2.12) follows from equation (3.2.11) if $x - 3 > 0$ and $2x - 3 > 0$ or if $x - 3 < 0$ and $2x - 3 < 0$ since

$$\begin{aligned}(x-3)^2 = (2x-3)^2 \Leftrightarrow \\ (-1)^2(x-3)^2 = (-1)^2(2x-3)^2 \Leftrightarrow \\ (-(x-3))^2 = (-(2x-3))^2.\end{aligned}$$

That is as long as the numbers $x - 3$ and $2x - 3$ have the same numerical value. So to obtain equivalent equations, we must take the numerical value of both sides of equation (3.2.12). Then we have

$$(x-3)^2 = (2x-3)^2 \Leftrightarrow |x-3| = |2x-3|$$

from which we get both solutions.

Example 3.13. Consider the equation

$$x^2 = 9 \tag{3.2.14}$$

We take the square root on both sides of the equation to get $x = 3$. As in example 3.12, we may forget the other (and in this case negative) solution $x = -3$. Again to be precise we must write

$$x^2 = 9 \Leftrightarrow |x| = 3 \Leftrightarrow x = -3 \vee x = 3 \tag{3.2.15}$$

What we need to be careful about when applying a function which is not everywhere defined is that we could possibly decrease the set of solutions. This could happen if we use some function (in examples 3.12 and 3.13 the square root function) as an inverse function on a domain on which it is actually not an inverse.

3.2.4 Dividing by zero

It is common knowledge that we can not divide by zero when we solve equations, but we might do it without being aware of it. This can happen when we divide by an expression containing a variable and not just a number. In that case we can loose solutions. We loose solutions because we decrease the domain of the original equation if we divide by some expression containing the variable that are zero for one or more choices of the variable. The domain decreases since the modified equation will not be defined for those values. The method is useable when solving equations, but it has to be separately checked if the values are solutions afterwards. Two examples 3.14 and 3.15 are given below.

Example 3.14. Consider the equation

$$x^2 = 10x \tag{3.2.16}$$

To solve this equation we might be tempted to divide by x on both sides of the equality sign. We know that dividing by a nonzero number does not change the set of solutions. If we divide by x we get

$$x = 10 \tag{3.2.17}$$

We quickly see that the two equations are not equivalent. In the process we have lost the solution $x = 0$ because we forgot to keep track of the domain of the equation. In equation (3.2.16) the domain is all of $\mathbb{R}$ so if we choose to divide by x, we have to assume that $x \neq 0$. Hence, the domain of equation (3.2.17) is $\mathbb{R}\backslash\{0\}$. So what we get is that the solution sets of the two equations agree on $\mathbb{R}\backslash\{0\}$, but not on all of $\mathbb{R}$ therefore we have to check the set where they do not agree separately - in this case we have to check whether $x = 0$ is a solution to the original equation or not.

Example 3.15. Consider the equation

$$x(x-5) = x(x-5)(x-1) \tag{3.2.18}$$

We divide by $x - 5$ on both sides

$$x = x(x-1) \tag{3.2.19}$$

and then divide by x on both sides

$$1 = x - 1 \tag{3.2.20}$$

So $x = 2$. But we quickly realize that we have lost the solutions $x = 5$ and $x = 0$. When we divided by $x - 5$ we had to assume that $x \neq 5$, and when we divided by x we had to assume that $x \neq 0$ and, hence, we have decreased the domain of the equation. We therefore have to check the possibilities $x = 5$ and $x = 0$ separately.

3.2.5 Infinitely many solutions

Some equations have infinitely many solutions, but when we use legal operations or even sometimes CAS to solve the equations we only get one solution. The following examples 3.16 and 3.17 will not be used in the questionnaires (see appendix 8.1) explicitly.

Example 3.16. Consider the equations

$$\sin(x) = 1 \text{ and } \cos(x) = 1 \tag{3.2.21}$$

We can apply the inverse of sine and cosine and get

$$x = \sin^{-1}(1) = \frac{\pi}{2} \text{ and } x = \cos^{-1}(1) = 0 \tag{3.2.22}$$

But we might forget that because the trigonometric functions are periodic there are in fact infinitely many solutions to these equations namely $x = \frac{\pi}{2} + 2\pi n$ for all $n \in \mathbb{N}_0$ for the first equation, and $x = 2\pi n$ for all $n \in \mathbb{N}_0$ for the second equation.

Example 3.17. Consider the equation

$$3x - 6 = 3(x - 2) \tag{3.2.23}$$

$$3x - 6 = 3x - 6 \tag{3.2.24}$$

$$0 = 0 \tag{3.2.25}$$

Every $x \in \mathbb{R}$ is a solution to this equation but some students might interpret this as 0 being the (only) solution.

3.3 Method

In the questionnaires (see appendix 8.1) given to the students they are asked to solve some of the equations in section 3.2 above and to find and explain the mistakes made in some solutions to others. There are five types of difficulties related to biimplication of equations as described in section 3.2. The two tasks of the questionnaire expose students to the first four types: applying a function which is not everywhere defined, applying a function which is not one-to-one, dividing by zero and multiplying by zero.

The reason for formulating essentially the same task in to different ways is to make sure that poor technical skills of equation solving will not prevent the students from having a chance to make logical considerations about solution sets and equivalence in all of the given tasks.

A concern is that the equations are too difficult for the students. In upper secondary school students are explicitly introduced to first and second degree polynomial equations which they are required to be able to solve by hand, moreover, they are familiar with the root function, but they do not spend much time solving equations like the ones in section 3.2 by hand. There are no situations in upper secondary school in which the students are required to solve equations as the above without using CAS, hence, they do not spend much time on learing it during teaching.

To meet this concern the task of finding a wrongdoing when solving an equation is made even more accessible. Besides solving equations and finding mistakes in solutions, students have to decide for five pairs of equations whether the second is a legal rewriting of the first (ie whether they are equivalent or not). For each type of difficulty described in section 3.2 there are examples of one wrong rewriting and one right rewriting which are shuffled throughout the questionnaires (see appendix 8.1). The examples are as follows:

Applying a function which is not one-to-one:

$$\sqrt{x+1} = 3 \text{ and } x+1 = 9 \text{ (true)}.$$

$$\sqrt{x-4} = -3 \text{ and } x-4 = 9 \text{ (false)}.$$

Multiplying by a variable that might be zero:

$$\frac{x+1}{x} = \frac{2x}{x} \text{ and } x+1 = 2x \text{ (true)}.$$

$$\frac{3x+1}{x} = \frac{x+1}{x} \text{ and } 3x+1 = x+1 \text{ (false)}.$$

Applying a function which is not everywhere defined:

$$(4x)^2 = x^2 \text{ and } 4x = x \text{ (true)}.$$

$$x^2 = 16 \text{ and } x = 4 \text{ (false)}.$$

Dividing by a quantity that might be zero:

$$4x = 4x^2 \text{ and } x = x^2 \text{ (true)}.$$

$$-x^2 = 2x \text{ and } -x = 2 \text{ (false)}.$$

Infinitely many solutions:

$$x - 2 = x - 2 \text{ and } 0 = 0 \text{ (true).}$$

$$4x + 1 = 4x + 1 \text{ and } x = 0 \text{ (false).}$$

In each case students have to provide an explanation for their choices. The domain of the equations are omitted since students are not assumed to be familiar with this concept. The implicit assumption is that the domain in question is the largest possible domain which can be assign to the first equation in each case, but students can comment on this alongside with their answer.

3.3.1 Coding of data

In the following a mistake related to biimplication is defined as a mistake of the types described in section 3.2. That is a mistake in which no calculation errors or alike are made, but a mistake in which some operation is applied which either increases or decreases the domain of the original equation without this being pointed out and where it has consequences for the final solution.

3.4 Results to tasks of solving and finding mistakes

Just by glancing at table 39 it is evident that the equations have been too difficult for the students even if the issue about equivalence and biimplication had not been there. This documents that the students have very poor technical skills when it comes to solving non-standard equations. Still some students were able to solve the equations correctly.

Looking at the average scores in table 39, it seems that students' have the hardest time when there is a possibility of making mistakes connected to mulitiplying by zero, applying a function which is not one-to-one and dividing by zero. But since both the examples of equations in which one might be tempted to multiply by zero hold fractions, this could just as well explain the difficulties. Moreover, the two examples of equations in which one might use a function which is not one-to-one both have square roots in them. This might be the reason why students find these equations more complicated. On the other hand these are both situations in which one possibly could gain extraneous solutions whereas one in the two other situations (applying a function which is not everywhere defined and dividing by zero) could loose solutions. So it could be that students find it even harder to cope with finding a solution which is not really a solution than loosing one. But that do not explain why students also experience great difficulties with situations where one could be let do divide by zero.

Table 39: Solving equations: ”correct” refers to right answers, ”biimplication” refers to mistakes of the type described in section 3.3.1 and ”wrong” refers to any other kind of mistake. If there has been made both types of mistakes in an answer it is categorized as ”biimplication”. The numbers are given in percentage of all answers.

	Correct	Biimplication	Wrong (%)
Not one-to-one			
$\sqrt{x} = 2x - 1$ $(n = 42)$	4,8	16,7	78,6
$\sqrt{2x-4} = \sqrt{3x}$ $(n = 37)$	0	24,3	75,7
Average (n=79)	2,5	20,3	77,2
Multiplying by zero			
$\frac{3x+1}{x} = \frac{2x+1}{x}$ $(n = 37)$	0	35,1	64,9
$\frac{x^2-1}{x-1} = 0$ $(n = 44)$	0	11,9	88,1
Average $(n = 81)$	0	22,8	77,2
Not everywhere defined			
$(x-3)^2 = (2x-3)^2$ $(n = 44)$	9,1	13,6	77,3
$x^2 = 9$ $(n = 38)$	5,3	89,5	5,3
Average $(n = 82)$	7,3	48,8	43,9
Dividing by zero			
$x^2 = 10x$ $(n = 42)$	4,5	31,8	63,6
$x(x-5) = x(x-5)(x-1)$ $(n = 38)$	0	10,5	89,5
Average $(n = 80)$	2,4	22,0	75,6
Average (all equations)	3,1	28,6	68,3

In table 39 we also see how large a percentage of the solutions have a mistake related to biimplication. It differs a lot from equation to equation how many of the solutions hold a mistake related to biimplication. The equation with the highest percentage of biimplication mistakes is $x^2 = 9$. This makes sense since it essentially is the only mistake possible to make. On average $28,6\%$ of the solutions have a mistake related to biimplication. This is not much compared to the $68,3\%$ which are the solutions containing some other mistake. Everything needed - besides knowledge of domains and biimplication mistakes - to solve the equations should be something that the students have already learned. Hence, poor technical skills prevent the students from even reaching a point in the solving process where there is a possibility of making a mistake related to biimplication.

Before going into larger detail with the results in table 39, we will now turn our focus to the tasks in which students had to find mistakes in given solutions to equations. An overview of the results are given in table 40. In general a few more students $(5,9\%)$ were able to solve the tasks about finding mistakes in the equa-

Table 40: Finding mistakes: "explanation" refers to right answers, "wrong step" refers to answers in which the wrong step has been detected but no explanation have been given, and "wrong" refers to answers which are wrong or missing.

	Explanation	Wrong step	Wrong (%)
Not one-to-one			
$\sqrt{x} = 2x - 1$ ($n = 44$)	$0,0$	$0,0$	$100,0$
$\sqrt{2x-4} = \sqrt{3x}$ ($n = 38$)	$0,0$	$13,2$	$86,8$
Average (n=82)	$0,0$	$6,1$	$93,9$
Multiplying by zero			
$\frac{3x+1}{x} = \frac{2x+1}{x}$ ($n = 38$)	$0,0$	$5,3$	$97,4$
$\frac{x^2-1}{x-1} = 0$ ($n = 44$)	$4,5$	$2,3$	$93,2$
Average ($n = 82$)	$2,4$	$3,7$	$93,9$
Not everywhere defined			
$(x-3)^2 = (2x-3)^2$ ($n = 42$)	$7,1$	$11,9$	$81,8$
$x^2 = 9$ ($n = 37$)	$32,4$	$10,8$	$56,8$
Average ($n = 79$)	$19,0$	$11,4$	$69,6$
Dividing by zero			
$x^2 = 10x$ ($n = 42$)	$4,8$	$0,0$	$95,2$
$x(x-5) = x(x-5)(x-1)$ ($n = 37$)	$0,0$	$2,7$	$97,3$
Average ($n = 79$)	$2,5$	$1,3$	$96,2$
Average (all equations)	$5,9$	$5,6$	$88,5$

tions than the tasks about solving equations since only $3,1\%$ in general solved them right (table 39). The overall picture though is not very different from the one given by answers to the task of solving the equations (table 39).

In the following we will look more closely at some of the answers given by the students to the tasks of finding mistakes in solutions and solving equations. We will mainly focus on the answers which contain mistakes related to biimplication.

3.4.1 Not one-to-one

The typical mistake related to biimplication when students solve the equation

$$\sqrt{x} = 2x - 1$$

is to raise to the second power on both sides. None of the seven students who made this mistake did everything else right. Four of them reach an answer but non of the four tried to insert their solution into the original equation to test whether they reached the right solution or not. Three students did not reach an answer and, hence, they did not have the opportunity to validate their answer by inserting it into the original equation. The two students who gave the correct answer to the equation did not write any calculations down. It is most likely that they guessed the solution $x = 1$ and, hence, made no considerations about biimplication or the number of solutions.

No students detected the mistake in the wrong solution to the equation.

No students have solved the equation

$$\sqrt{2x-4} = \sqrt{3x}$$

right. Nine students have made a mistake related to biimplication. Eight of the students squared both sides of the equation and got the solution $x = -4$ which is extraneous since -4 is not in the domain of the original equation. One student just stated the answer $x = -4$.

Five students have detected the mistake in the solution to the above equation. All five wrote something like "he [Peter] can not just remove the square root". It is not an adequate explanation, but it is true. Maybe more true than the students themselves realize. The students might very well just have pointed to what looked most difficult in the solution, and since everything else was right (which they should be able to verify) they used the exclusion method to conclude that one simply can not cancel the square roots.

3.4.2 Multiplying by zero

No one gave the right answer to the equation

$$\frac{x^2-1}{x-1} = 0.$$

Out of the five who made mistakes related to biimplication two just gave the answer $x = 1$ which is wrong because the equation is not defined for $x = 1$. One inserted $x = 1$ and showed that the nominator and the denominator both gave zero. And two multiplied by $x - 1$ to begin with and then at the end forgot that $\sqrt{x^2} = \sqrt{1}$ also has the solution $x = -1$. So they just gave the wrong solution $x = 1$ without controlling whether it was right or wrong. Hence, the only mistakes these two students made were related to biimplication.

Three students identified the wrong step in the solution to the equation above. One figured there was something wrong when the fraction on the left hand side was calculated and gave an alternative but wrong way of doing so. Another noticed that the denominator just disappeared. Both of these students did not realize that there was multiplied by $x - 1$ on both sides of the equation. The last student argued that the obtained value $x = 1$ can not be a solution since one can not divide by 0 which is correct. But the student was not able to explain why this solution appeared. According to the student all calculations where done correctly.

No students gave the right answer to the equation

$$\frac{3x+1}{x} = \frac{2x+1}{x}.$$

Thirteen students made a mistake related to biimplication. One student just stated the answer $x = 0$ which is not true since the equation is not defined for $x = 0$. The twelve remaining students multiplied by x on both sides of the equation. Three of these students ended up not giving an answer, two gave a completely wrong answer and seven answered that $x = 0$. One of these students wrote next to the answer that it could not be right that $x = 0$ since one can not divide by 0, but that she had no time to redo the task. Hence, she actually considered whether her answer was valid or not.

Two students found the mistake in the solution to the above equation. One wrote "Can the equation be solved at all?" which actually is a precise concern since the equation has no solutions. The other one wrote "She [Mette] can not just remove x". This comment witnesses that the student identified the step in which things went wrong. There is no further argumentation, and it do not seem as if the student knows that both sides can be multiplied by x in order to remove it from the denominator.

3.4.3 Not everywhere defined

When solving

$$(x-3)^2 = (2x-3)^2$$

students seem to be doing two different things. Either they square both sides of the equation or they multiply out. Out of the six students who did something wrong related to biimplication four of them tried to square both sides of the equation and two of them succeeded. One of those reached the answer $x = 0$ which is one of the two solutions while the other one did something else wrong and delivered no answer. The remaining two students tried to multiply out the expression. One succeeded but later divided by x and, hence, only obtained the solution $x = 2$. The other one multiplied wrong but did at some point also divide by x. The four students who solved the equation correctly all multiplied out and got both solutions by using the quadratic formula. Hence, they had no need to make considerations about biimplication during the solving process.

Four students found the wrong step in the solution to the above equation and three students found the wrong step and gave some kind of explanation. The four students who just found the wrong step wrote that the parenthesis should be multiplied out which is not wrong, but they totally reject that one can take the square root on both sides of the equality sign. Two of the students who gave an explanation said that the equation is a second degree equation implying that such an equation always have two solutions which is not right in the field of real numbers at least. Although they have a point even though they do not argue why Peter did not find both solutions using the method he used. The last student wrote ”because he put it in a positive square root when it can be positive or negative” (authors translation). This is not very precisely formulated, but highly interesting since it is exactly what is at stake.

Even though the equation

$$x^2 = 9$$

are indisputably easier to solve than the rest of the equations only two students managed to find both the positive and the negative solution. 34 students made a mistake related to biimplication. Students either took the square root on both sides and got the result $x = 3$, they just wrote up the result $x = 3$ or they inserted $x = 3$ into the equation in order to show that it is a solution.

Sixteen students found the mistake in the solution to the equation. Four students just said with no further explanation that Peter forgot the solution $x = -3$. Nine students said that since minus times minus gives plus $x = -3$ must also be a solution. Three students explained that when taking the square root of a number there is always two solutions - a positive and a negative.

3.4.4 Dividing by zero

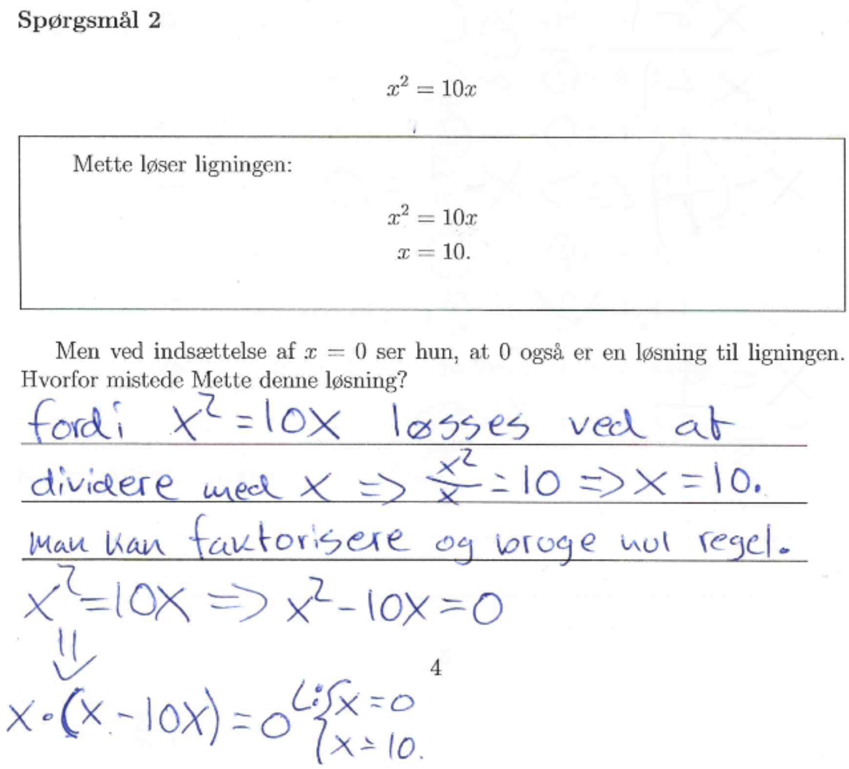
When solving the equation

$$x^2 = 10x$$

fourteen students made mistakes related to biimplication. Three students inserted $x = 10$ and saw that it fitted, and one just gave the answer $x = 10$. The remaining teen students all divided both sides of the equation by x which caused them to loose the solution $x = 0$. Eight of these students reached the answer $x = 10$ while two gave no answer.

Two students identified the mistake. One student just said that it is a second degree equation and, hence, implicitly says that it should have two solutions, but gives no explanation of why Mette only found one solution. The other student shows an alternative way of solving the equation in which one obtains both solutions (see figure 1). There is a mistake in the last line in which he puts x outside a parenthesis but keeps $10x$ inside the parenthesis instead of just 10, but it is probably just a writing mistake since he gets the right results. He is aware that Mette divides both sides of the equation by x, but do not explain why this cause her to loose a solution.

Figure 1: One solution to the equation $x^2 = 10x$. Translation: because $x^2 = 10x$ is solved by dividing by x (...) but one can factorize and use the zero rule (...).


Spørgsmål 2

$$x^2 = 10x$$

Mette løser ligningen:

$$x^2 = 10x$$
$$x = 10.$$

Men ved indsættelse af $x = 0$ ser hun, at 0 også er en løsning til ligningen. Hvorfor mistede Mette denne løsning?

fordi $X^2 = 10X$ løsses ved at dividere med $X \Rightarrow \frac{x^2}{x} = 10 \Rightarrow X = 10.$

Man kan faktorisere og bruge nul regel.

$X^2 = 10X \Rightarrow x^2 - 10X = 0$

$\Downarrow$

$X \cdot (X - 10X) = 0$ L: $\begin{cases} X = 0 \\ X = 10. \end{cases}$

4

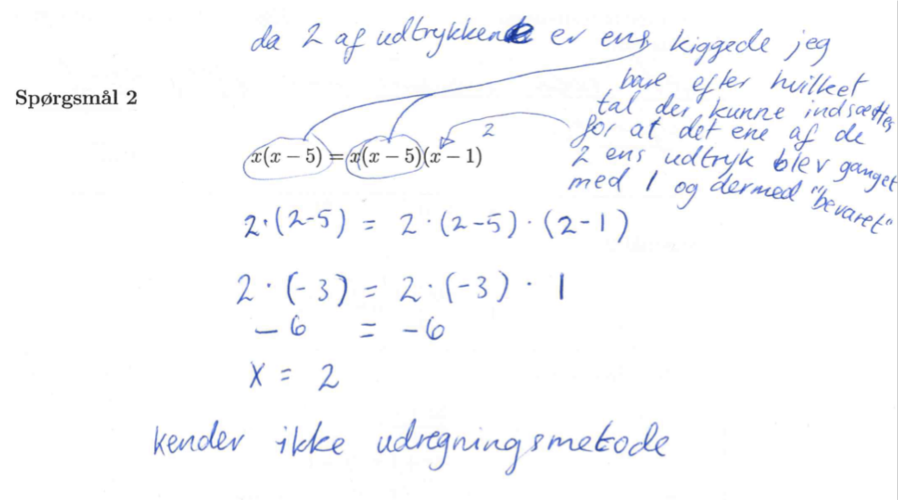
No student have given the right answer to the equation

$$x(x-5) = x(x-5)(x-1).$$

Four students have made a mistake related to biimplication. Two of these students have started out by dividing by $x(x-5)$ on both sides of the equality but have not given an answer. One student have divided by $x-5$ on both sides of the equality sign and have given a wrong answer. The last student have given a qualified and informed guess to a solution, namely $x = 2$. But have overlooked that the equation could have more than one solution (see figure 2).

Several of the analyzed textbooks (see section 2.3) introduce and give examples of use of the zero rule. It is surprising that no students at all tried to use this rule. The typical approach in the wrong solutions are to multiply out the parenthesis

Figure 2: One solution to the equation $x(x-5) = x(x-5)(x-1)$. Translation, top: since 2 of the expressions are the same I just looked for a number which could be inserted such that one of the 2 similar expressions was multiplied by 1 and hence "preserved". Translation, bottom: do not know the calculation method.



but since the students do not know how to solve a third degree equation they do not succeed.

One student where able to (sort of) identify the wrong step in the solution to the above equation. She just wrote "the zero rule" (authors translation) in her comment. What she must have seen is that if one uses the zero rule one obtains the solutions which Mette is missing. But the student were not able to explain why Mette lost the solutions.

3.5 Results to tasks about equivalence

In this section the results to the tasks concerning equivalence are presented and discussed. In table 41 the first equivalence presented for each type of mistake is true and, hence, the correct answer is yes while the second equivalence presented is false and, hence, the correct answer is no. In order to safe space the equivalences are not written out in 42 but just referred to as "true" or "false". The tasks were also presented in chapter 3.3.

In table 41 we see that for the first four types of difficulties it seems that whenever the equivalence in the task is true more students give a right answer than if the equivalence is false. For the fifth difficulty it is the other way around; more students give a correct answer to the false equivalence than to the true equivalence. Looking at the average scores the first tendency shows itself again; namely that if the equivalence is true students' are more likely to answer correctly

Table 41: Equivalence: The numbers are given in percentage of the total amount of answers to each task.

	Wrong answer	Right answer (%)
Not one-to-one		
$\sqrt{x+1} = 3 \Leftrightarrow x+1 = 9$ ($n = 83$)	37,3	62,7
$\sqrt{x-4} = -3 \Leftrightarrow x-4 = 9$ ($n = 74$)	58,1	41,9
Multiplying by zero		
$\frac{x+1}{x} = \frac{2x}{x} \Leftrightarrow x+1 = 2x$ ($n = 73$)	44,3	50,7
$\frac{3x+1}{x} = \frac{x+1}{x} \Leftrightarrow 3x+1 = x+1$ ($n = 81$)	69,1	30,9
Not everywhere defined		
$(4x)^2 = x^2 \Leftrightarrow 4x = x$ ($n = 79$)	45,6	54,4
$x^2 = 16 \Leftrightarrow x = 4$ ($n = 75$)	86,7	13,3
Dividing by zero		
$4x = 4x^2 \Leftrightarrow x = x^2$ ($n = 73$)	41,1	58,9
$-x^2 = 2x \Leftrightarrow -x = 2$ ($n = 82$)	50,0	50,0
Infinitely many solutions		
$x-2 = x-2 \Leftrightarrow 0 = 0$ ($n = 74$)	71,6	28,4
$4x+1 = 4x+1 \Leftrightarrow x = 0$ ($n = 82$)	59,8	40,2
Average of true ($n = 382$)	48,7	51,3
Average of false ($n = 394$)	64,5	35,5
Total average ($n = 776$)	56,7	43,3

than if the equivalence is false. The overall performance of the students in the true equivalences are no better than chance, and in the false equivalences students generally perform much worse than chance. This indicates that the students are not aware of the problems with biimplication in the false equivalences.

Table 42 displays the results in more detail. It shows that on average under half of the students who answered correctly provided a good explanation alongside with their answer. If we consider correct answers with wrong explanations and without explanations to be wrong, only 23,6% (7,2%+16,4%) answered correctly. On average of the true equivalences 38,7% (9,9%+28,8%) answered correctly, and on average of the false equivalences only 8,9% (4,6%+4,3%) answered correctly. For the first four types of mistakes, students who give a wrong answer have a tendency to provide a wrong explanation if the equivalence is false, but not provide any explanation if the equivalence is true. The wrong explanations typically concern biimplication mistakes. Students e.g write: ”one can take the square root on both sides” or ”one can multiply by x”, which is not true on the entire domain of the original equation.

Table 42: Equivalence: ”nothing” refers to a wrong answer with no explanation, ”explanation” refers to a wrong answer with a wrong explanation, ”bad” refers to a right answer without or with a wrong explanation, ”medium” refers to a right answer with a partly satisfying explanation and ”good” refers to a right answer with a satisfying explanation. The numbers are given in percentage of the total amount of answers to each task. The equivalences appear in table 41.

	Wrong answer		Right answer (%)		
	Nothing	Explanation	Bad	Medium	Good
Not one-to-one					
true ($n = 83$)	37,3	0,0	12,0	10,8	39,8
false ($n = 74$)	28,4	29,7	23,0	16,2	7,2
Multiplying by zero					
true ($n = 73$)	47,9	1,4	15,1	11,0	24,7
false ($n = 81$)	27,2	42,0	29,6	1,2	0,0
Not everywhere defined					
true ($n = 79$)	43,0	2,5	8,9	8,9	36,7
false ($n = 75$)	18,7	68,0	12,0	0,0	1,3
Dividing by zero					
true ($n = 73$)	41,1	0,0	13,7	16,4	28,8
false ($n = 82$)	30,5	19,5	47,6	2,4	0,0
Infinitely many solutions					
true ($n = 74$)	27,0	44,6	13,5	2,7	12,2
false ($n = 82$)	29,3	30,5	19,5	3,7	7,1
Average of true ($n = 382$)	39,3	9,4	12,6	9,9	28,8
Average of false ($n = 394$)	26,9	37,6	26,6	4,6	4,3
Total average ($n = 776$)	33,0	23,7	19,7	7,2	16,4

To explore the connection between the answers to the true and false equivalences for each type of mistake a χ^2-test have been conducted for each pair.

Table 43 tells that we can reject the hypothesis that answers to the equivalences

$$\sqrt{x+1} = 3 \Leftrightarrow x+1 = 9 \text{ and } \sqrt{x-4} = -3 \Leftrightarrow x-4 = 9$$

are independent at a significance level of 1%. The only difference between the equivalences are that while squaring both sides of the first equation in the first equivalence to obtain the second equation is unproblematic since the solution is in the intersection of the domains, doing the same thing to the second equivalence gives problems regarding the domain. The domain increases in such a way that an extraneous solution appears. That the answers to the equivalences are dependent shows that students do not realize the problems of equivalence. They treat the two equivalences in the same way.

Table 43: Equivalence: Applying a function which is not one-to-one, $n = 157$, $p = 0.009294$

	$\sqrt{x+1} = 3 \Leftrightarrow x+1 = 9$ (true)	$\sqrt{x-4} = -3 \Leftrightarrow x-4 = 9$ (false)
right answer	52	31
wrong answer	31	43

Table 44: Equivalence: Multiplying by a variable that might be zero, $n = 154$, $p = 0.012268$

	$\frac{x+1}{x} = \frac{2x}{x} \Leftrightarrow x+1 = 2x$ (true)	$\frac{3x+1}{x} = \frac{x+1}{x} \Leftrightarrow 3x+1 = x+1$ (false)
right answer	37	25
wrong answer	36	56

Table 44 shows that the hypothesis that the students' answers to the equivalences

$$\frac{x+1}{x} = \frac{2x}{x} \Leftrightarrow x+1 = 2x \text{ and } \frac{3x+1}{x} = \frac{x+1}{x} \Leftrightarrow 3x+1 = x+1$$

are independent can be rejected at a significance level of 5%. Hence, students' do not realize the problem of domains in the second equivalence where one actually divides by zero since $x = 0$ is a solution to the modified equation, but is not in the domain of the original equation.

Table 45: Equivalence: Applying a function which is not everywhere defined, $n = 154$, $p = 0.000000$

	$(4x)^2 = x^2 \Leftrightarrow 4x = 4$ (true)	$x^2 = 16 \Leftrightarrow x = 4$ (false)
right answer	43	10
wrong answer	36	65

Table 45 compares the answers to the equivalences

$$(4x)^2 = x^2 \Leftrightarrow 4x = 4 \text{ and } x^2 = 16 \Leftrightarrow x = 4.$$

The hypothesis that the answers are independent can be rejected at a significance level of 1%. A large amount of students give a wrong answer to the second equivalence. The second equivalence is very simple since the modified equation is also the answer. It is just not the only answer, but most students do not see that. In the false equvialences in the two first cases (tables 43 and 44) a solution which was in fact not a solution was obtained. In this case a solution is lost, but there is nothing wrong with the solution found. This might fool the students to think that the rewriting is correct.

Table 46: Equivalence: Dividing by a quantity that might be zero, $n = 155$, $p = 0.266743$

	$4x = 4x^2 \Leftrightarrow x = x^2$ (true)	$-x^2 = 2x \Leftrightarrow -x = 2$ (false)
right answer	43	41
wrong answer	30	41

In table 46 the hypothesis that the answers to

$$4x = 4x^2 \Leftrightarrow x = x^2 \text{ and } -x^2 = 2x \Leftrightarrow -x = 2$$

are independent can not be rejected. In table 42 we see that almost none of those who gave a correct answer to the false equivalence gave an acceptable explanation. From studying the answers it becomes apparent that students do not get that both sides have been divided by x. Those answering correctly either just states their answer or gives a wrong explanation like "the square can not just be moved like that". Therefore it is difficult to conclude anything about students' awareness of biimplication problems since poor technical skills seem to play a larger role in the answering process.

Table 47: Equivalence: Infinitely many solutions, $n = 156$, $p = 0.119823$

	$x - 2 = x - 2 \Leftrightarrow 0 = 0$ (true)	$4x + 1 = 4x + 1 \Leftrightarrow x = 0$ (false)
right answer	21	33
wrong answer	53	49

In table 47 the answers to the equivalences

$$x - 2 = x - 2 \Leftrightarrow 0 = 0 \text{ and } 4x + 1 = 4x + 1 \Leftrightarrow x = 0$$

are compared. The hypothesis that the answers are independent can not be rejected. A lot of the students who gave a wrong answer to the true equivalence actually proposed the solution $x = 0$, and some argued that it should be so since the same thing was on both the right hand and the left hand side of the equality sign. Some of the students who gave a wrong answer to the false equivalence said that $x = 0$ is a solution which is true, it is just not the only solution, but they failed to see that. Even though there are no significant difference in how the tasks are solved, it is evident from the large proportion of wrong answers to the two tasks that students are not good at handling equations with infinitely many solutions.

3.6 Summary and discussion

The results to the tasks of solving equations clearly show that students are not equipped to work with equations like the given. Even though all operations required to solve the equations are some they know about, most students are not able to apply them. It was expected that students would have some trouble solving the equations since they are non-standard, but not that so many of them would be unable to work with the equations at all. Maybe some of the equations are above upper secondary school level, but the students should be able to handle equations like $x^2 = 9$, $(x-3)^2 = (2x-3)^2$ and $x^2 = 10$. Regarding biimplication mistakes the conclusion clearly must be that the students are not aware of these kinds of problems when solving equations. The tasks of finding mistakes in solutions show the same thing.

Students perform better in the tasks about deciding whether a rewriting of an equation is correct or not. But because the questions are yes/no-questions and only $43,3\%$ answered correctly on average the results are not impressive. Actually a better result could have been obtained by just guessing randomly. But the question also shows that students have a tendency to give a correct answer to the true equivalences and a wrong answer to the false equivalences. This shows that they are not aware of mistakes related to biimplication.

4 Understandings of the equality sign

This chapter discusses students' knowledge and perceptions of the equality sign and is tightly connected to the two preceding chapters.

Stephens et al. (2013) have conducted an investigation which addressed 3rd, 4th and 5th graders understanding of the equality symbol. Since the equality sign has an important role in equations, this is of interest to this thesis. Stephens et al. (2013) found that many students have an operational view of the equality sign. They see it as a "do something" signal rather than as a relational symbol expressing equivalence. Some students hold a relational-operational view of the equality sign. They know that equality expresses a relation between two sides of an equation, and they confirm the relation by calculating. Later this type of students have more success in solving first degree equations than students who have an operational view. The last type of students showed what Stephens et al. (2013) call relational-structural understanding of the equality sign. They see equality as a symbol expressing equivalence between two expressions rather than two calculations. Not many students showed this understanding.

The data gathered by Stephens et al. (2013) show that the higher the grade the better the understanding of the equality sign. If we assume that this development continues it must be expected that upper secondary school students primarily show relational-structural understanding of the equality sign. But given their poor solving skills (see section 3.4), this might not be the case after all.

Consider the equation

$$5555 + 8888 = x + 7777,$$

which is used in the questionnaires (see appendix 8.1) among others. According to Stephens et al (2013) it would be so that students which hold an operational view of the equality sign would tend to carry out the calculation on the left hand side, $5555 + 8888 = 14443$, and take this as the answer, hence, conclude that $x = 14443$. This is so because they have a feeling of directionality from left to right. So they regard what is on the left hand side as the task to be solved, and what comes directly after the equality sign on the right hand side as the result.

Students which hold a relational-operational view of the equality sign would tend to carry out the calculation on the left hand side and then rearrange the equation to isolate x, probably by subtraction 7777 from their result of the calculation on the left hand side.

A student holding a relational view of the equality sign would immediately "see" that the solution must be 6666, and might be able to give an explanation such as: "Because 7777 is 1111 less than 8888, we must have that x is 1111 more than 5555, hence, $x = 6666$".

4.1 Method

In this thesis the students' understanding of the equality sign is tested by letting the students solve some equations which are significantly easier solved by recognizing the structure than by computing. The eight equations below are different variations of equations which are easily solved if the relationship between the numbers are discovered, and the equality sign is understood in a relational-structural way. The equations can be solved by computing, but it will be much more difficult. In each questionnaire (see appendix 8.1) four equations appear - one for each arithmetical operation.

$$5555 + 8888 = x + 7777$$

$$x + 65419 = 3131 + 65417$$

$$134679 - x = 134678 - 1675$$

$$123321 - 5665 = x - 5666$$

$$623 \cdot x = 1246 \cdot 17$$

$$53 \cdot 132 = 106 \cdot x$$

$$\frac{1488}{x} = \frac{744}{2}$$

$$\frac{6}{129} = \frac{x}{43}$$

The students were asked to explain how they reached their conclusion - either by writing down calculations or an explanation in words.

A disadvantage of evaluating the students understanding of the equality sign by letting them solve the above equations, is that the students need a well developed intuition about numbers and ability to see the relation between the numbers involved in each equation to show a relational-structural understanding of the equality sign. So maybe the tasks test the students' intuition about numbers just as much as they test their understanding of the equality sign.

An advantage on the other hand - one that Stephens et al. (2013) pointed out but did not make use of themselves - is that since the equations are hard to solve by straight forward calculations students who in some situations have a relational-structural understanding of the equality sign would be likely to apply that understanding in the situations of solving the above equations since it would make the task easier for them. In the research of Stephens et al. (2013) one could not be sure that students who solved the equations by straight forward calculations were not able to solve the equations in any other way. And since the two ways of solving the equations (structural considerations or forward calculations) were about equally difficult in the investigation of Stephens et al. (2013) it was not possible to assume that students would favor one method over the other.

In this study on the other hand, it can deservedly be assumed that it is easier to solve the equations when applying a relational-structural understanding of the equality sign. Hence, if students are able to do this it is likely that they will.

One thing working against this argument though is that students are trained to solve equations using certain mechanical techniques. In the textbooks analyzed (see section 2.3) we see no evidence that students work with equations in any other way than applying a small set of rules saying that one can do the same thing to each side of the equation sign. The students are not trained (by the textbooks at least) to see patterns, be creative and to work with the equality sign as a relational symbol when solving equations.

4.2 Results

The tables 48 and 49 show the results of the students' solutions to the equations presented in chapter 4.1. It is not particularly interesting whether students gave the correct answer or not to the equations. What is interesting is how they approached the tasks.

Table 48 shows the percentage of the students who explicitly showed a relational-structural or an relational-operational understanding of the equality sign. On average $22,2\%$ $(20,5\%+1,7\%)$ of the students showed a relational-structural understanding while $34,4\%$ $(18,4\%+16,0\%)$ showed a relational-operational understanding. We see from the table that more of the students who showed a relational-structural understanding actually solved the equations (reached a correct answer) than students who showed an relational-operational understanding. This for one tells that it is important to be able to perceive the equality sign in a relational-structural way since this apparently improves the chance of being able to solve an equation. The large hindrance for students who showed a relational-operational view was that they where not able to carry out the calculations. Some of the students actually wrote that alongside with their answers.

Table 48: Answers showing a relational-structural or relational-operational understanding of the equality sign. ”answer” refers to solutions with a right answer and argumentation and ”no answer” refers to solutions which include correct argumentation but no final answer. The numbers are in percentage of all answers.

	Relational-structural		Relational-operational	
	answer	no answer	answer	no answer (%)
Addition				
$5555 + 8888 = x + 7777$ $(n = 86)$	14,0	0,0	46,5	8,0
$x + 65419 = 3131 + 65417$ $(n = 75)$	33,3	6,7	30,7	7,0
Average $(n = 161)$	23,0	3,1	39,1	9,9
Subtraction				
$134679 - x = 134678 - 1675$ $(n = 86)$	19,8	1,2	27,9	7,0
$12321 - 5665 = x - 5666$ $(n = 75)$	37,3	5,3	18,7	8,0
Average $(n = 161)$	28,0	3,1	23,6	7,5
Multiplication				
$623 \cdot x = 1246 \cdot 17$ $(n = 86)$	19,8	0,0	4,7	31,4
$53 \cdot 132 = 106 \cdot x$ $(n = 75)$	6,7	0,0	5,3	36,0
Average $(n = 161)$	13,7	0,0	5,0	33,5
Division				
$\frac{1488}{x} = \frac{744}{2}$ $(n = 86)$	19,8	0,0	8,1	4,7
$\frac{6}{129} = \frac{x}{43}$ $(n = 74)$	14,9	1,4	2,7	22,0
Average $(n = 160)$	17,5	0,6	5,6	13,1
Total average $(n = 643)$	20,5	1,7	18,4	16,0

There is a large difference in the percentage of students’ who show a relational-structural understanding of the two equations including addition. There is no obvious explanation for this difference. Maybe students where better at recognizing the relationship between the numbers 65419 and 65417 than between the numbers 8888 and 7777. Or maybe the calculation $5555 + 8888$ where considered easier than the calculation $3131 + 65417$ and, hence, some students who could have applied a relational understanding just carried out the calculation instead.

The difference in the percentage of students’ who showed a relational-structural understanding of the two equations including subtraction might be a bit easier to explain. It gave trouble in the relational-structural reasoning that there were a minus in front of x so students tended to carry out the calculations instead.

There is a large difference in the percentage of students able to answer the two multiplication equations using a relational-structural approach. It must have been easier for students to see the relationship between 623 and 1246 than the relationship between 53 and 106. Moreover, in the first equation students had to double 17 which might be a bit easier than dividing 132 by two. Looking at the proportions of students giving a solution using the relational-operational method it is evident that students know what to do, but are unable to carry out the calculations which prevents them from giving an answer. In this case the solving procedure would be significantly easier for the students if they knew how to view the equality sign in a relational-structural way.

Regarding the equations including division one would assume that the first equation is more difficult to solve than the second at least if it is solved using a relational-operational method. In that case one has to perform two or three steps in order to isolate x and carry out at least two computations in order to solve the first equation. The second equation can be solved by performing one step to isolate x and carry out two calculations. The assumption is verified since more students are able to describe a computational solving procedure (though without being able to perform it) corresponding to a relational-operational understanding for the second equation than for the first. On the other hand there is only little difference in the proportion of students able to solve the two equations using a relational-structural method. When applying a relational-structural view of the equality sign, it do not make a large difference whether x appears in the nominator or denominator.

On average more students' reached an answer to the equations including addition than the equations including subtraction. And more students reach an answer to the equations with addition and subtraction than to the equations including multiplication and division. This was expected since division and multiplication are regarded more difficult than addition and subtraction and are typically introduced later. Hence students are more familiar with addition and subtraction than they are with multiplication and division.

On average more students were able to reach an answer to the equations containing division than the equations containing multiplication which at first is a bit surprising since division in general are more difficult than multiplication. The proportion of students able to give an answer using the relational-structural method to the equations with division might be larger than the proportion of students able to give an answer using the relational-structural method to the equations with multiplication because students are used to working with fractions and prolonging and shortening them and less familiar with factorizing numbers. Overall more students were able to argue correctly for the solution method to the equations with multiplication than for the equations with division.

Table 49 shows the percentage of students who only gave a correct answer but no argumentation and students who gave a solution which is wrong.

Table 49: Answers not explcitly showing neither a relational-operational nor a relational-structural understanding of the equality sign. ”only answer” refers to solutions with a correct answer but no argumentation and ”wrong/nothing” refers to solutions which have wrong or no answers and argumentation. The numbers are in percentage of all answers.

	Only answer	Wrong/nothing (%)
Addition		
$5555 + 8888 = x + 7777$ $(n = 86)$	$9,3$	$18,6$
$x + 65419 = 3131 + 65417$ $(n = 75)$	$4,0$	$17,3$
Average $(n = 161)$	$6,8$	$18,0$
Subtraction		
$134679 - x = 134678 - 1675$ $(n = 86)$	$8,1$	$36,0$
$12321 - 5665 = x - 5666$ $(n = 75)$	$4,0$	$26,7$
Average $(n = 161)$	$6,2$	$31,7$
Multiplication		
$623 \cdot x = 1246 \cdot 17$ $(n = 86)$	$5,8$	$38,8$
$53 \cdot 132 = 106 \cdot x$ $(n = 75)$	$2,7$	$49,3$
Average $(n = 161)$	$4,3$	$43,5$
Division		
$\frac{1488}{x} = \frac{744}{2}$ $(n = 86)$	$9,3$	$58,1$
$\frac{6}{129} = \frac{x}{43}$ $(n = 74)$	$1,4$	$56,8$
Average $(n = 160)$	$5,6$	$57,5$
Total average $(n = 643)$	$5,8$	$37,6$

It is not possible to determine how students perceive the equation sign when they have not included any explanation or calculations. One possibility is that they have a relational-structural understanding and, hence, just could ”see” the answer immediately and have not included any explanation even though they were asked to do so. Another possibility is that they made calculations on a separate piece of paper and just transferred the answer to the questionnaire even though they were asked to write everything in the questionnaire.

It is evident that most students could not solve the equations containing division. In general $57,5\%$ did not solve these equations. $43,5\%$ of the students could not solve the equations containing multiplication. Only 31, 7% could not solve the equations with subtraction which is under the overall average of 37, 6% and 18, 0% could not solve the equations with addition. This ranking of difficulty is not surprising. It is a bit discouraging though that 37, 6% of the students' answers in general were wrong.

Maybe in some cases it would have been possible to detect what kind of understanding the students' showed in their answers even though they were wrong, but this has not been dealt with. It is not the impression from working with the data material though that a significant amount of the students approached the equations with a operational ("do something") understanding of the equality sign. Primarily the students' made calculation errors, wrong operations in order to solve the equations or did not try at all.

Table 50 shows how many of the students show a relational-structural and a relational-operational understanding of the equality sign in at least one of the four questions and in all the questions. Moreover, it shows how many of the students either just give an answer or give a solution which have wrong or no answer and argumentation why it is difficult to determine which understanding of the equality sign they have.

Table 50: Students' overall understanding of the equality sign. "Rest" refers to the answers from table 49 in which the understanding of the equality sign can not easily be determined, $n = 161$

	Relational-structural	Relational-operational	Rest (%)
At least one time	41	60	81
In all four questions	4	5	20

We see that 41% of the students at least once show a relational-structural understanding even though only 22, 2% of the solutions in general showed a relational-structural understanding of the equality sign. Moreover, 60% of the students showed a relational-operational understanding in at least one of the questions, while only 34, 3% of the solutions showed a relational-operational understanding in general. Furthermore, 81% of the students delivered at least one answer in which it could not easily be determined which understanding was shown whereas 43, 4% (5, 8%+37, 6%) of the solutions in general were of this type.

Very few students exclusively showed either a relational-structural or a relational-operational understanding showing that the understanding can be different not just from person to person but also from situation to situation. Moreover, 20% of the students delivered four solutions which either just included an answer or had wrong or no answer and explanation.

5 Comparison and discussion

This section will contain a comparison of the data collected from upper secondary school students presented in this thesis and the data collected from university students. Moreover, it will shortly be discussed what the gains and losses are by teaching equations in the way it is done in danish upper secondary schools at the present time.

5.1 University students

To have something to compare the collected data to a questionnaire (see appendix 8.2) containing some of the same tasks that were given to the upper secondary school students has been given to nine university students following the course ”Matematik i undervisningsmæssig sammenhæng” (Mathematics in educational context). The nine students is a diverse group since seven of them major in different subjects than mathematics and have math as a minor and two major in mathematics. The group is too small to say something in general about university students' practices with equations and to make a valid comparison to upper secondary school students' practices with equations. But still the data can give a nice perceptive to the results regarding upper secondary school students.

5.1.1 Examples, explanations and valuations

The inclusion of features in the examples given by the university students (see table 51) is not very different from the examples given by upper secondary school students. Everyone includes equality sign and variables (which typically are named x) and give examples which are conditionally true. Almost everyone gives equations in which there is something to solve. Since the sample of students is very small it is not possible to say which of the four main features the universtiy students find most important.

Table 52 shows that most of the university students include equality in their explanation of what an equation is. Moreover, three students mention something about truth value, namely that an equation is a proposition. Only a few students mentions that equations should hold a variable and can be solved.

Even though the sample of university students is small it seems that they in general give more theoretical and less practical descriptions of the concept of equations. The university students seem to consider an equation to be an object which is a proposition including an equality sign whereas upper secondary school students perceive equations more as a task, as something that can be rearranged until the unknown is found.

Table 51: University students: occurrence of features in examples of equations, $n = 8$

Feature	Occur 3 times	Occur 1-2 times	Do not occur
Equality sign	8	0	0
Variable	8	0	0
Name of variable is x	7	1	0
Multiple variables	1	4	3
Differential equation	0	0	8
Conditionally true	8	0	0
Something to solve	7	1	0
Definition/function	0	1	7
Numbers	7	1	0
Letters	0	1	7

Table 52: University students: occurrence of features in explanations of the word "equation", $n = 8$

Feature	Occurence (number of explanations)
Equality	5
Truth value	3
Variable/unknown	2
Definition/function	2
Something to solve	1
Identity	1
Connection	0

Table 53 shows the university students' evaluations of whether the expressions are equations or not. The prototypical equations $3x + 8 = 7$ and $6 + 9a = 20$ were regarded to be equations by all students who answered the question. Not surprising no students evaluated the expressions without a relational symbol (4, $9 + 4 \cdot 3$, x and $x + 3x - 2 \cdot 4$) to be equations. Perhaps more surprising, all expressions without a variable ($8 = 8$, $2 \cdot 4 = 12$, $0 = 1$ and $7 + 3 = 10$) were not regarded to be equations either. Everyone regarded the expression $x = 4$ to be an equation even though there is nothing to solve in it. This is very different from the upper secondary school students' evaluations.

Two students did not give answers to $x + 1 = x$, $2x = x + x$ and $x = x$. Only half of the total amount of students agreed that the three expressions were in fact equations. This is surprising since they both contain a variable and an equality sign. The only thing not prototypical about these equations keeping in mind that everyone accepted $x = 4$ as an equation is that they are not conditionally true but always false and true, respectively. Moreover, the formula (which is also an identity) $\cos^2(x) + \sin^2(x) = 1$ is considered an equation by everyone. So the students do not seem to be consistent about their choices. This could be explained by the fact that the latter expression is a well known formula presented as an equation during teaching situations.

The distribution of yes- and no-answers for the expressions $y = 12x + 7$, $f(x) = 31 \cdot 1,23^x$, $\frac{dy}{dx} = 8 - 2x$ and $45x \leq 23 + x$ are about fifty-fifty.

So overall students seem to find the presence of equality sign and variable important. Moreover the truth value being conditionally true seems to matter more than the expression having something to solve.

Table 53: University students: valuations of different expressions being an equation. The numbers in "yes" and "no" do not always sum to 8 since not everyone answered, $n = 8$

Expression	Yes	Average certainty	No	Average certainty
$3x + 8 = 7$	8	5,0	0	-
$x = 4$	8	4,1	0	-
$6 + 9a = 20$	6	4,5	0	-
$\cos^2(x) + \sin^2(x) = 1$	6	4,5	0	-
$x = x$	5	5,0	2	4,0
$x + 1 = x$	4	5,0	2	4,5
$2x = x + x$	4	3,8	2	4,0
$y = 12x + 7$	3	4,7	2	2,5
$45x \leq 23 + x$	3	4,3	2	1,5
$\frac{dy}{dx} = 8 - 2x$	3	3,3	3	3,6
$f(x) = 31 \cdot 1,23^x$	1	5,0	4	3,3
$7 + 3 = 10$	1	1,0	7	4,1
$0 = 1$	0	-	5	4,0
$8 = 8$	0	-	6	3,7
$9 + 4 \cdot 3$	0	-	6	4,5
$2 \cdot 4 = 12$	0	-	7	4,1
$x + 3x - 2 \cdot 4$	0	-	8	4,6
x	0	-	8	4,6
4	0	-	8	4,6

5.1.2 Equation solving, finding mistakes and equivalence

The university students are generally much better at solving and finding mistakes in equations than the upper secondary school students (see tables 54 and 55) which was to be expected since they have more training. More than 3/4 of the solutions to the equations $(x-3)^2 = (2x-3)^2$ and $x^2 = 10x$ are correct. The university students have some trouble giving a good explanation of why the solution to $\sqrt{x} = 2x - 1$ is wrong but 2/3 succeed in explaining why the solution to $\frac{x^2-1}{x-1} = 0$ is wrong.

Table 54: University students: solutions to the equations. ”correct” refers to right answers, ”biimplication” refers to answers in which there is a mistake related to biimplication, and ”wrong” refers to any other kind of mistake. If there has been made both types of mistakes in an answer it is categorized as ”biimplication”.

	Correct	Biimplication	Wrong
Not everywhere defined			
$(x-3)^2 = (2x-3)^2$ $(n = 9)$	7	1	1
Dividing by zero			
$x^2 = 10x$ $(n = 9)$	7	2	0
Total $(n = 18)$	14	3	1

Table 55: University students: solutions to the tasks of finding mistakes in solutions to equations. ”explanation” refers to right answers, ”wrong step” refers to answers in which the wrong step has been detected but no explanation has been given, and ”wrong” refers to answers which are wrong.

	Explanation	Wrong step	Wrong
Not one-to-one			
$\sqrt{x} = 2x - 1$ $(n = 8)$	1	3	4
Multiplying by zero			
$\frac{x^2-1}{x-1} = 0$ $(n = 9)$	6	3	0
Total $(n = 17)$	7	6	5

A lot of the students did not give answers to the tasks about finding out whether an equivalence was right or wrong. An explanation could be that the university students had a little less time (30 minutes) than the upper secondary school students to solve the questionnaire. Moreover, they probably used more time on the tasks of solving equations and finding mistakes in solutions because they were actually able to solve the tasks whereas many of the upper secondary school students just skipped the tasks. Under any circumstances this makes it difficult to interpret the data. The results are presented in table 56.

About half of the tasks were not answered. Of the tasks which were answered about 3/4 were answered correctly and 1/4 wrongly. Almost nobody give a wrong answer to the true equivalences whereas a lot of wrong answers are given to the wrong equivalences. This is a tendency also seen in the answers of the upper secondary school students.

When looking closely at each solution it is clear that the university students provide much better and more detailed explanations for their answers.

Table 56: University students: answers to the task about whether two equations are equivalent or not, n=9.

	Right answer	Wrong answer	No answer
Not one-to-one			
$\sqrt{x+1} = 3 \Leftrightarrow x+1 = 9$	6	1	2
$\sqrt{x-4} = -3 \Leftrightarrow x-4 = 9$	1	1	7
Multiplying by zero			
$\frac{x+1}{x} = \frac{2x}{x} \Leftrightarrow x+1 = 2x$	3	0	6
$\frac{3x+1}{x} = \frac{x+1}{x} \Leftrightarrow 3x+1 = x+1$	4	2	3
Not everywhere defined			
$(4x)^2 = x^2 \Leftrightarrow 4x = x$	6	1	2
$x^2 = 16 \Leftrightarrow x = 4$	2	3	4
Dividing by zero			
$4x = 4x^2 \Leftrightarrow x = x^2$	2	0	7
$-x^2 = 2x \Leftrightarrow -x = 2$	5	3	1
Infinitely many solutions			
$x-2 = x-2 \Leftrightarrow 0 = 0$	2	0	7
$4x+1 = 4x+1 \Leftrightarrow x = 0$	3	1	5
Total of true ($n = 45$)	19	2	24
Total of false ($n = 45$)	15	10	20
Total of all ($n = 90$)	34	12	44

5.2 Gains and losses

As we saw in the textbook analyses in section 2.3, equations are primarily taught through examples. The textbooks spend only little time on explanations of equations as mathematical objects. Moreover, these explanations are superficial and imprecise. Equations are described more as a type of tasks than as an object. Little or no theory about equations and domains of equations are presented. Some techniques for solving equations are presented and exemplified in the textbooks, but the treatments do not go beyond this mechanical description of equations and the solving procedure. The way of teaching presented in the textbooks probably equips the students to solve equations similar to the ones presented in the books.

That is, first degree equations of the form $ax + b = cx + d$ and second degree equations of the form $ax^2 + bx + c = 0$, where a, b, c, d are real constants and the unknown $x \in \mathbb{R}$. By reading the textbooks students should become familiar with the mechanical rearrangements of terms in order to isolate the unknown. If this is the aim - then everything is fine as it is.

But from this thesis it is pretty obvious what the students do not learn. They do not learn to think for them selves. Meaning, if an equation is not on one of the forms above students have no clue how to solve it. And they fail to perform just simple operations like multiplying out a parenthesis with two terms, multiplying both sides of an equation by x or dividing by a factor in common on both sides of the equality sign. Moreover they are not capable of checking whether a reached answer really is an answer to the original problem they were given. This is because they do not really know what an equation is and what it means to find the solution to an equation. They do not think of an equation as a proposition with a truth value, but rather as some kind of calculation to be carried out.

Hence, students have actually no knowledge connected to equations that are applicable in more than just a selected few cases. They will have great difficulties transferring their knowledge to other school subjects or situations. Moreover, they do not really learn to reason mathematically, they just learn to copy a selected handful of procedures.

Since equations occur in all parts of mathematics and also in other school subjects and situations, the gained knowledge is unsatisfactory compared to what the students possibly could learn from working with equations. Barahmand & Shahvarani (2014) have studied ninth and tenth grade Iranian students' interpretations of equations, solutions to equations and the relationship between students' equation understanding and solving. They have found that there is a significant relationship between understanding and solving equations. It is difficult to know if understanding of equations implies that one is good at solving equation, if it is the other way around or if the implication goes both ways. But the research by Barahmand & Shahvarani (2014) and the poor achievements of the students in this study suggest, that a slight change of perspective from solving procedures to theory when teaching equations could possibly help to expand the students' outcome from the teaching of equations from just limited practical abilities to skills applicable in all sorts of different situations. The theory of equations is not too difficult to be introduced to upper secondary school students. And the extra time it might take to give a deeper presentation of equations would undoubtedly be well spend since it must slow down a lot of the other teaching in mathematics that the students have so much trouble solving equations.

6 Conclusions

We will now return to the research questions which were presented in section 1.1 and answer them one by one.

- How do upper secondary school students' definitions of the concept of equations conform to the scholarly concept(s)?

There is definitely a difference between the students' conceptions of equations and the scholarly definition. Of the four features equality sign, variable, truth value and something to solve students value the presence of equality sign and variable most important. Moreover, students are more convinced that something is an equation if it has something to solve. The students explanations focus on an equation being a type of task in which the goal is to find the unknown by solving the equation. Absolutely no one mentions that an equation is a preposition.

The textbooks for upper secondary school have a strong focus on the solving process, but they also have vague definitions of equations - some of them even including the fact that an equation is a proposition. Hence, it is not odd that students' perceive equations as a type of task rather than as an object since this is what is the main focus in the books mediating the knowledge of equations to the students.

The scholarly concept focuses on the equality sign being present - like the students often also do - but also on the fact that an equation is an open proposition. An open proposition that becomes a closed proposition for each choice of an element in the domain. Some choices will lead to a true proposition and are, hence, considered to be solutions while others will lead to a false proposition and are, hence, considered not to be solutions. The set of elements leading to true propositions is the solution set.

What the students' and the scholarly definitions have in common is the equality sign. But while students see equations as some sort of task or calculation, equations really are open propositions on some domaine. Moreover, students speak about solving equations in terms of "finding x" or "isolating the unknown" while the scholarly concept of solutions is easier defined using a set theoretical description.

- What difficulties related to biimplication do students experience when solving equations?

Four difficulties related to biimplication have been described in section 3.2: applying a function which is not one-to-one, multiplying by zero, applying a function which is not everywhere defined and dividing by zero. Either the applied operation makes the domain of the equation smaller and, hence, one could loose solutions or it makes the domain larger, hence, one could obtain extraneous solutions. According to the results of the data collection, students experience all four types of difficulties. But it is evident that they more often are unable to apply operations correctly, miscalculate when reducing one side of the equation or are unable to decide which operation to apply in a given situation. Moreover, we have found that they are generally not aware of problems due to extending of decreasing the domain of the original equation.

- In what ways do students perceive the equality symbol in equations?

We saw that about one fifth of the students showed a relational-structural understanding of the equality sign at least once and about three fifth showed a relational-operational understanding at least once. We also saw that only about one out of twenty had a consistent either relational-structural or relational-operational understanding of the equality sign. Hence, it depends very much on the type of task which understanding is shown. Moreover, the solutions showing a relational-structural understanding more often had a final answer than the solutions showing a relational-operational understanding. No particular amount of students seemed to show a purely operational understanding of the equality sign. But about two fifth of the solutions were so wrong that any understanding of the equality sign could not clearly be determined.

- Can upper secondary school students' understanding of and practices with equations be improved by introducing a more theoretical approach of teaching equations?

What this study has shown is that students do not perform well outside their (narrow) comfort zone when it comes to solving equations. Moreover, they are not capable of clearly explaining what an equation is even though it is something they have worked with since primary or at least since lower secondary school. The way the students are introduced to equations by the textbooks are mechanical and based on examples. Students do not obtain prerequisites to solve equations much different from the narrow selection shown through examples. A more theoretical introduction would definitely make students better suited for explaining what an equation is. And since there is a connection between students understanding of equations and their ability to solve them (Barahmand & Shahvarani, 2014), they would probably also be better suited for solving them. At least a theoretical overview would make the students capable of understanding what is going when an equation is solved. But they still have to master some techniques for solving equations and be able to calculate and reduce algebraic expression which they at the present time are not according to this study.

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8 Appendices

8.1 Questionnaires for upper secondary school students

The four questionnaires which where solved by upper secondary school students appear in this section. The questionnaires are in danish since the participants were danish speakers.

LIGNINGER (1A)

Følgende er et opgavesæt, der handler om ligninger. Når du har besvaret opgaverne på første side, må du ikke bladre tilbage.

Hvis du forsøger at løse en opgave, men giver op, skal du sætte et kryds ud for opgaven.

Generel information

Skole:

____

Klassetrin og niveau:

____

Navn:

____

Hvad er en ligning

Spørgsmål 1

Giv tre eksempler på ligninger:

____

____

____

Spørgsmål 2

Skriv kort og præcist hvad ordet ”ligning” betyder:

____

____

____

Spørgsmål 3

Hvilke af nedenstående udtryk opfatter du som ligninger? Sæt ring om ja eller nej. Ranger desuden dine svar fra 1 (meget usikker) til 5 (helt sikker).

1.	x	*ja*	*nej*	1	2	3	4	5
2.	4	*ja*	*nej*	1	2	3	4	5
3.	$x + 3x - 2 \cdot 4$	*ja*	*nej*	1	2	3	4	5
4.	$3x + 8 = 7$	*ja*	*nej*	1	2	3	4	5
5.	$x + 1 = x$	*ja*	*nej*	1	2	3	4	5
6.	$x = x$	*ja*	*nej*	1	2	3	4	5
7.	$x = 4$	*ja*	*nej*	1	2	3	4	5
8.	$7 + 3 = 10$	*ja*	*nej*	1	2	3	4	5
9.	$2 \cdot 4 = 12$	*ja*	*nej*	1	2	3	4	5
10.	$\frac{dy}{dx} = 8 - 2x$	*ja*	*nej*	1	2	3	4	5
11.	$45x \leq 23 + x$	*ja*	*nej*	1	2	3	4	5
12.	$2x = x + x$	*ja*	*nej*	1	2	3	4	5
13.	$8 = 8$	*ja*	*nej*	1	2	3	4	5
14.	$\cos^2(x) + \sin^2(x) = 1$	*ja*	*nej*	1	2	3	4	5
15.	$f(x) = 31 \cdot 1,23^x$	*ja*	*nej*	1	2	3	4	5
16.	$y = 12x + 7$	*ja*	*nej*	1	2	3	4	5
17.	$6 + 9a = 20$	*ja*	*nej*	1	2	3	4	5
18.	$9 + 4 \cdot 3$	*ja*	*nej*	1	2	3	4	5
19.	$0 = 1$	*ja*	*nej*	1	2	3	4	5

Kommenter evt. dine svar:

Løsning af ligninger 1

Løs de følgende ligninger. Skriv alle dine mellemregninger og tanker ned.

Spørgsmål 1

$$\sqrt{x} = 2x - 1$$

Spørgsmål 2

$$\frac{x^2 - 1}{x - 1} = 0$$

Fejlfinding

Forklar hvad der er gået galt i løsningen af nedenstående ligninger.

Spørgsmål 1

$$(x-3)^2 = (2x-3)^2$$

Peter løser ligningen:

$$\begin{aligned}(x-3)^2 &= (2x-3)^2\\ x-3 &= 2x-3\\ -3+3 &= 2x-x\\ x &= 0.\end{aligned}$$

Men ved indsættelse af $x = 2$ ser han, at 2 også er en løsning til ligningen. Hvorfor mistede Peter denne løsning?

Spørgsmål 2

$$x^2 = 10x$$

Mette løser ligningen:

$$\begin{aligned}x^2 &= 10x\\ x &= 10.\end{aligned}$$

Men ved indsættelse af $x = 0$ ser hun, at 0 også er en løsning til ligningen. Hvorfor mistede Mette denne løsning?

Ensbetydende ligninger

Afgør for hvert par af ligninger nedenfor, om man må omskrive ligning A til ligning B (når man skal løse A)? Forklar hvorfor eller hvorfor ikke.

Spørgsmål 1

$$A : -x^2 = 2x \qquad B : -x = 2.$$

Spørgsmål 2

$$A : \sqrt{x+1} = 3 \qquad B : x + 1 = 9.$$

Spørgsmål 3

$$A : (4x)^2 = x^2 \qquad B : 4x = x.$$

Spørgsmål 4

$$A : \tfrac{3x+1}{x} = \tfrac{x+1}{x} \qquad B : 3x + 1 = x + 1.$$

Spørgsmål 5

$$A : 4x + 1 = 4x + 1 \qquad B : x = 0.$$

Løsning af ligninger 2

Find/gæt en løsning til hver af nedenstående ligninger. Forklar med ord hvordan du fandt løsningen.

Spørgsmål 1

$$5555 + 8888 = x + 7777$$

Spørgsmål 2

$$134679 - x = 134678 - 1675$$

Spørgsmål 3

$$623 \cdot x = 1246 \cdot 17$$

Spørgsmål 4

$$\frac{1488}{x} = \frac{744}{2}$$

LIGNINGER (1B)

Følgende er et opgavesæt, der handler om ligninger. Når du har besvaret opgaverne på første side, må du ikke bladre tilbage.

Hvis du forsøger at løse en opgave, men giver op, skal du sætte et kryds ud for opgaven.

Generel information

Skole:

Klassetrin og niveau:

Navn:

Hvad er en ligning

Spørgsmål 1

Giv tre eksempler på ligninger:

Spørgsmål 2

Skriv kort og præcist hvad ordet "ligning" betyder:

Spørgsmål 3

Hvilke af nedenstående udtryk opfatter du som ligninger? Sæt ring om ja eller nej. Ranger desuden dine svar fra 1 (meget usikker) til 5 (helt sikker).

1.	x	*ja*	*nej*	1	2	3	4	5
2.	4	*ja*	*nej*	1	2	3	4	5
3.	$x + 3x - 2 \cdot 4$	*ja*	*nej*	1	2	3	4	5
4.	$3x + 8 = 7$	*ja*	*nej*	1	2	3	4	5
5.	$x + 1 = x$	*ja*	*nej*	1	2	3	4	5
6.	$x = x$	*ja*	*nej*	1	2	3	4	5
7.	$x = 4$	*ja*	*nej*	1	2	3	4	5
8.	$7 + 3 = 10$	*ja*	*nej*	1	2	3	4	5
9.	$2 \cdot 4 = 12$	*ja*	*nej*	1	2	3	4	5
10.	$\frac{dy}{dx} = 8 - 2x$	*ja*	*nej*	1	2	3	4	5
11.	$45x \leq 23 + x$	*ja*	*nej*	1	2	3	4	5
12.	$2x = x + x$	*ja*	*nej*	1	2	3	4	5
13.	$8 = 8$	*ja*	*nej*	1	2	3	4	5
14.	$\cos^2(x) + \sin^2(x) = 1$	*ja*	*nej*	1	2	3	4	5
15.	$f(x) = 31 \cdot 1,23^x$	*ja*	*nej*	1	2	3	4	5
16.	$y = 12x + 7$	*ja*	*nej*	1	2	3	4	5
17.	$6 + 9a = 20$	*ja*	*nej*	1	2	3	4	5
18.	$9 + 4 \cdot 3$	*ja*	*nej*	1	2	3	4	5
19.	$0 = 1$	*ja*	*nej*	1	2	3	4	5

Kommenter evt. dine svar:

Løsning af ligninger 1

Løs de følgende ligninger. Skriv alle dine mellemregninger og tanker ned.

Spørgsmål 1

$$(x-3)^2 = (2x-3)^2$$

Spørgsmål 2

$$x^2 = 10x$$

Fejlfinding

Forklar hvad der er gået galt i løsningen af nedenstående ligninger.

Spørgsmål 1

$$\sqrt{x} = 2x - 1$$

Peter løser ligningen:

$$\begin{aligned}\sqrt{x} &= 2x - 1\\ x &= (2x-1)^2\\ x &= 4x^2 + 1 - 4x\\ 0 &= 4x^2 - 5x + 1\end{aligned}$$

$$d = b^2 - 4ac = (-5)^2 - 4 \cdot 4 \cdot 1 = 25 - 16 = 9$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{d}}{2a} = \frac{5 \pm 3}{8}\\ x &= 1 \quad \text{eller} \quad x = \tfrac{1}{4}.\end{aligned}$$

Men ved indsættelse ser han, at kun $x = 1$ er en løsning til ligningen. Hvorfor har Peter fundet en løsning for meget?

Spørgsmål 2

$$\frac{x^2-1}{x-1}=0$$

Mette løser ligningen:

$$\begin{aligned}\frac{x^2-1}{x-1}&=0\\ x^2-1&=0\\ x^2&=1\\ x&=\pm 1.\end{aligned}$$

Men ved indsættelse ser hun, at kun $x=-1$ er en løsning til ligningen. Hvorfor har Mette fundet en løsning for meget?

Ensbetydende ligninger

Afgør for hvert par af ligninger nedenfor, om man må omskrive ligning A til ligning B (når man skal løse A)? Forklar hvorfor eller hvorfor ikke.

Spørgsmål 1

$$A : -x^2 = 2x \qquad B : -x = 2.$$

Spørgsmål 2

$$A : \sqrt{x+1} = 3 \qquad B : x + 1 = 9.$$

Spørgsmål 3

$$A : (4x)^2 = x^2 \qquad B : 4x = x.$$

Spørgsmål 4

$$A : \tfrac{3x+1}{x} = \tfrac{x+1}{x} \qquad B : 3x + 1 = x + 1.$$

Spørgsmål 5

$$A : 4x + 1 = 4x + 1 \qquad B : x = 0.$$

Løsning af ligninger 2

Find/gæt en løsning til hver af nedenstående ligninger. Forklar med ord hvordan du fandt løsningen.

Spørgsmål 1

$$5555 + 8888 = x + 7777$$

Spørgsmål 2

$$134679 - x = 134678 - 1675$$

Spørgsmål 3

$$623 \cdot x = 1246 \cdot 17$$

Spørgsmål 4

$$\frac{1488}{x} = \frac{744}{2}$$

LIGNINGER (2A)

Følgende er et opgavesæt, der handler om ligninger. Når du har besvaret opgaverne på første side, må du ikke bladre tilbage.

Hvis du forsøger at løse en opgave, men giver op, skal du sætte et kryds ud for opgaven.

Generel information

Skole:

Klassetrin og niveau:

Navn:

Hvad er en ligning

Spørgsmål 1

Giv tre eksempler på ligninger:

Spørgsmål 2

Skriv kort og præcist hvad ordet ”ligning” betyder:

Spørgsmål 3

Hvilke af nedenstående udtryk opfatter du som ligninger? Sæt ring om ja eller nej. Ranger desuden dine svar fra 1 (meget usikker) til 5 (helt sikker).

1.	x	*ja*	*nej*	1	2	3	4	5
2.	4	*ja*	*nej*	1	2	3	4	5
3.	$x + 3x - 2 \cdot 4$	*ja*	*nej*	1	2	3	4	5
4.	$3x + 8 = 7$	*ja*	*nej*	1	2	3	4	5
5.	$x + 1 = x$	*ja*	*nej*	1	2	3	4	5
6.	$x = x$	*ja*	*nej*	1	2	3	4	5
7.	$x = 4$	*ja*	*nej*	1	2	3	4	5
8.	$7 + 3 = 10$	*ja*	*nej*	1	2	3	4	5
9.	$2 \cdot 4 = 12$	*ja*	*nej*	1	2	3	4	5
10.	$\frac{dy}{dx} = 8 - 2x$	*ja*	*nej*	1	2	3	4	5
11.	$45x \leq 23 + x$	*ja*	*nej*	1	2	3	4	5
12.	$2x = x + x$	*ja*	*nej*	1	2	3	4	5
13.	$8 = 8$	*ja*	*nej*	1	2	3	4	5
14.	$\cos^2(x) + \sin^2(x) = 1$	*ja*	*nej*	1	2	3	4	5
15.	$f(x) = 31 \cdot 1{,}23^x$	*ja*	*nej*	1	2	3	4	5
16.	$y = 12x + 7$	*ja*	*nej*	1	2	3	4	5
17.	$6 + 9a = 20$	*ja*	*nej*	1	2	3	4	5
18.	$9 + 4 \cdot 3$	*ja*	*nej*	1	2	3	4	5
19.	$0 = 1$	*ja*	*nej*	1	2	3	4	5

Kommenter evt. dine svar:

Løsning af ligninger 1

Løs de følgende ligninger. Skriv alle dine mellemregninger og tanker ned.

Spørgsmål 1

$$\sqrt{2x-4} = \sqrt{3x}$$

Spørgsmål 2

$$\frac{3x+1}{x} = \frac{2x+1}{x}$$

Fejlfinding

Forklar hvad der er gået galt i løsningen af nedenstående ligninger.

Spørgsmål 1

$$x^2 = 9$$

Peter løser ligningen:

$$\begin{aligned} x^2 &= 9 \\ x &= 3. \end{aligned}$$

Men ved indsættelse af $x = -3$ ser han, at -3 også er en løsning til ligningen. Hvor mistede Peter denne løsning?

Spørgsmål 2

$$x(x-5) = x(x-5)(x-1)$$

Mette løser ligningen:

$$\begin{aligned} x(x-5) &= x(x-5)(x-1) \\ x &= x(x-1) \\ 1 &= x-1 \\ x &= 2. \end{aligned}$$

Men ved indsættelse af $x = 5$ og $x = 0$ ser hun, at også 5 og 0 er løsninger til ligningen. Hvorfor mistede Mette disse løsninger?

Ensbetydende ligninger

Afgør for hvert par af ligninger nedenfor, om man må omskrive ligning A til ligning B (når man skal løse A)? Forklar hvorfor eller hvorfor ikke.

Spørgsmål 1

$$A : 4x = 4x^2 \qquad\qquad B : x = x^2.$$

Spørgsmål 2

$$A : \sqrt{x - 4} = -3 \qquad\qquad B : x - 4 = 9.$$

Spørgsmål 3

$$A : x^2 = 16 \qquad\qquad B : x = 4.$$

Spørgsmål 4

$$A : \tfrac{x+1}{x} = \tfrac{2x}{x} \qquad\qquad B : x + 1 = 2x.$$

Spørgsmål 5

$$A : x - 2 = x - 2 \qquad\qquad B : 0 = 0.$$

Løsning af ligninger 2

Find/gæt en løsning til hver af nedenstående ligninger. Forklar med ord hvordan du fandt løsningen.

Spørgsmål 1

$$x + 65419 = 3131 + 65417$$

Spørgsmål 2

$$123321 - 5665 = x - 5666$$

Spørgsmål 3

$$53 \cdot 132 = 106 \cdot x$$

Spørgsmål 4

$$\frac{6}{129} = \frac{x}{43}$$

LIGNINGER (2B)

Følgende er et opgavesæt, der handler om ligninger. Når du har besvaret opgaverne på første side, må du ikke bladre tilbage.

Hvis du forsøger at løse en opgave, men giver op, skal du sætte et kryds ud for opgaven.

Generel information

Skole:

Klassetrin og niveau:

Navn:

Hvad er en ligning

Spørgsmål 1

Giv tre eksempler på ligninger:

Spørgsmål 2

Skriv kort og præcist hvad ordet ”ligning” betyder:

Spørgsmål 3

Hvilke af nedenstående udtryk opfatter du som ligninger? Sæt ring om ja eller nej. Ranger desuden dine svar fra 1 (meget usikker) til 5 (helt sikker).

1.	x	*ja*	*nej*	1	2	3	4	5
2.	4	*ja*	*nej*	1	2	3	4	5
3.	$x + 3x - 2 \cdot 4$	*ja*	*nej*	1	2	3	4	5
4.	$3x + 8 = 7$	*ja*	*nej*	1	2	3	4	5
5.	$x + 1 = x$	*ja*	*nej*	1	2	3	4	5
6.	$x = x$	*ja*	*nej*	1	2	3	4	5
7.	$x = 4$	*ja*	*nej*	1	2	3	4	5
8.	$7 + 3 = 10$	*ja*	*nej*	1	2	3	4	5
9.	$2 \cdot 4 = 12$	*ja*	*nej*	1	2	3	4	5
10.	$\frac{dy}{dx} = 8 - 2x$	*ja*	*nej*	1	2	3	4	5
11.	$45x \leq 23 + x$	*ja*	*nej*	1	2	3	4	5
12.	$2x = x + x$	*ja*	*nej*	1	2	3	4	5
13.	$8 = 8$	*ja*	*nej*	1	2	3	4	5
14.	$\cos^2(x) + \sin^2(x) = 1$	*ja*	*nej*	1	2	3	4	5
15.	$f(x) = 31 \cdot 1,23^x$	*ja*	*nej*	1	2	3	4	5
16.	$y = 12x + 7$	*ja*	*nej*	1	2	3	4	5
17.	$6 + 9a = 20$	*ja*	*nej*	1	2	3	4	5
18.	$9 + 4 \cdot 3$	*ja*	*nej*	1	2	3	4	5
19.	$0 = 1$	*ja*	*nej*	1	2	3	4	5

Kommenter evt. dine svar:

Løsning af ligninger 1

Løs de følgende ligninger. Skriv alle dine mellemregninger og tanker ned.

Spørgsmål 1

$$x^2 = 9$$

Spørgsmål 2

$$x(x-5) = x(x-5)(x-1)$$

Fejlfinding

Forklar hvad der er gået galt i løsningen af nedenstående ligninger.

Spørgsmål 1

$$\sqrt{2x-4} = \sqrt{3x}$$

Peter løser ligningen:

$$\begin{aligned}\sqrt{2x-4} &= \sqrt{3x}\\ 2x-4 &= 3x\\ x &= -4\end{aligned}$$

Men ved indsættelse ser han, at $x = -4$ faktisk ikke er en løsning til ligningen. Hvad gik galt?

Spørgsmål 2

$$\frac{3x+1}{x} = \frac{2x+1}{x}$$

Mette løser ligningen:

$$\begin{aligned}\frac{3x+1}{x} &= \frac{2x+1}{x}\\ 3x+1 &= 2x+1\\ x &= 0\end{aligned}$$

Men ved indsættelse ser hun, at $x = 0$ faktisk ikke er en løsning til ligningen. Hvad gik galt?

Ensbetydende ligninger

Afgør for hvert par af ligninger nedenfor, om man må omskrive ligning A til ligning B (når man skal løse A)? Forklar hvorfor eller hvorfor ikke.

Spørgsmål 1

$$A : 4x = 4x^2 \qquad B : x = x^2.$$

Spørgsmål 2

$$A : \sqrt{x-4} = -3 \qquad B : x - 4 = 9.$$

Spørgsmål 3

$$A : x^2 = 16 \qquad B : x = 4.$$

Spørgsmål 4

$$A : \tfrac{x+1}{x} = \tfrac{2x}{x} \qquad B : x + 1 = 2x.$$

Spørgsmål 5

$$A : x - 2 = x - 2 \qquad B : 0 = 0.$$

Løsning af ligninger 2

Find/gæt en løsning til hver af nedenstående ligninger. Forklar med ord hvordan du fandt løsningen.

Spørgsmål 1

$$x + 65419 = 3131 + 65417$$

Spørgsmål 2

$$123321 - 5665 = x - 5666$$

Spørgsmål 3

$$53 \cdot 132 = 106 \cdot x$$

Spørgsmål 4

$$\frac{6}{129} = \frac{x}{43}$$

8.2 Questionnaire for university students

The questionnaire which where solved by the university students appear in this section. The questionnaire is in danish since the participants were danish speakers.

LIGNINGER (1B+)

Følgende er et opgavesæt, der handler om ligninger. Når du har besvaret opgaverne på første side, må du ikke bladre tilbage.

Generel information

Har du matematik som hovedfag eller sidefag?

____

Startår:

____

Navn:

____

Hvad er en ligning

Spørgsmål 1

Giv tre eksempler på ligninger:

____

____

____

Spørgsmål 2

Skriv kort og præcist hvad ordet ”ligning” betyder:

____

____

____

Spørgsmål 3

Hvilke af nedenstående udtryk opfatter du som ligninger? Sæt ring om ja eller nej. Ranger desuden dine svar fra 1 (meget usikker) til 5 (helt sikker).

1.	x	*ja*	*nej*	1	2	3	4	5
2.	4	*ja*	*nej*	1	2	3	4	5
3.	$x + 3x - 2 \cdot 4$	*ja*	*nej*	1	2	3	4	5
4.	$3x + 8 = 7$	*ja*	*nej*	1	2	3	4	5
5.	$x + 1 = x$	*ja*	*nej*	1	2	3	4	5
6.	$x = x$	*ja*	*nej*	1	2	3	4	5
7.	$x = 4$	*ja*	*nej*	1	2	3	4	5
8.	$7 + 3 = 10$	*ja*	*nej*	1	2	3	4	5
9.	$2 \cdot 4 = 12$	*ja*	*nej*	1	2	3	4	5
10.	$\frac{dy}{dx} = 8 - 2x$	*ja*	*nej*	1	2	3	4	5
11.	$45x \leq 23 + x$	*ja*	*nej*	1	2	3	4	5
12.	$2x = x + x$	*ja*	*nej*	1	2	3	4	5
13.	$8 = 8$	*ja*	*nej*	1	2	3	4	5
14.	$\cos^2(x) + \sin^2(x) = 1$	*ja*	*nej*	1	2	3	4	5
15.	$f(x) = 31 \cdot 1,23^x$	*ja*	*nej*	1	2	3	4	5
16.	$y = 12x + 7$	*ja*	*nej*	1	2	3	4	5
17.	$6 + 9a = 20$	*ja*	*nej*	1	2	3	4	5
18.	$9 + 4 \cdot 3$	*ja*	*nej*	1	2	3	4	5
19.	$0 = 1$	*ja*	*nej*	1	2	3	4	5

Kommenter evt. dine svar:

Løsning af ligninger 1

Løs de følgende ligninger. Skriv alle dine mellemregninger og tanker ned.

Spørgsmål 1

$$(x-3)^2 = (2x-3)^2$$

Spørgsmål 2

$$x^2 = 10x$$

Fejlfinding

Forklar hvad der er gået galt i løsningen af nedenstående ligninger.

Spørgsmål 1

$$\sqrt{x} = 2x - 1$$

Peter løser ligningen:

$$\begin{aligned} \sqrt{x} &= 2x - 1 \\ x &= (2x-1)^2 \\ x &= 4x^2 + 1 - 4x \\ 0 &= 4x^2 - 5x + 1 \end{aligned}$$

$$d = b^2 - 4ac = (-5)^2 - 4 \cdot 4 \cdot 1 = 25 - 16 = 9$$

$$x = \frac{-b \pm \sqrt{d}}{2a} = \frac{5 \pm 3}{8}$$
$$x = 1 \quad \text{eller} \quad x = \tfrac{1}{4}.$$

Men ved indsættelse ser han, at kun $x = 1$ er en løsning til ligningen. Hvorfor har Peter fundet en løsning for meget?

Spørgsmål 2

$$\frac{x^2-1}{x-1}=0$$

Mette løser ligningen:

$$\begin{aligned}\frac{x^2-1}{x-1}&=0\\ x^2-1&=0\\ x^2&=1\\ x&=\pm 1.\end{aligned}$$

Men ved indsættelse ser hun, at kun $x=-1$ er en løsning til ligningen. Hvorfor har Mette fundet en løsning for meget?

Ensbetydende ligninger

Afgør for hvert par af ligninger nedenfor, om man må omskrive ligning A til ligning B (når man skal løse A)? Forklar hvorfor eller hvorfor ikke.

Spørgsmål 1

$$A : -x^2 = 2x \qquad B : -x = 2.$$

Spørgsmål 2

$$A : \sqrt{x+1} = 3 \qquad B : x + 1 = 9.$$

Spørgsmål 3

$$A : (4x)^2 = x^2 \qquad B : 4x = x.$$

Spørgsmål 4

$$A : \tfrac{3x+1}{x} = \tfrac{x+1}{x} \qquad B : 3x + 1 = x + 1.$$

Spørgsmål 5

$$A : 4x + 1 = 4x + 1 \qquad B : x = 0.$$

Spørgsmål 6

$$A : 4x = 4x^2 \qquad\qquad B : x = x^2.$$

Spørgsmål 7

$$A : \sqrt{x-4} = -3 \qquad\qquad B : x - 4 = 9.$$

Spørgsmål 8

$$A : x^2 = 16 \qquad\qquad B : x = 4.$$

Spørgsmål 9

$$A : \frac{x+1}{x} = \frac{2x}{x} \qquad\qquad B : x + 1 = 2x.$$

Spørgsmål 10

$$A : x - 2 = x - 2 \qquad\qquad B : 0 = 0.$$