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Modeling and Measuring Teachers' praxeologies for teaching Mathematics

A comparative study of three contemporary assessment instruments

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Abstract

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UNIVERSITY OF COPENHAGEN
FACULTY OR DEPARTMENT

Master Thesis

Sara Lehné

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A comparative study of three contemporary assessment instruments

Supervisor: Carl Winsløw

Submitted on: July 31st 2017

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Teachers mathematical and didactical knowledge has over the last century become of greater interest to research and institutions surrounding the educational system. Despite this interest in teachers' knowledge a clear definition of what makes a good teacher are still to be given. This interest has resulted in the development of instruments for measuring teachers' knowledge. In this thesis, I will present three of such instruments which are used today to measure praxeology's for teaching mathematics along with the framework surrounding them. Afterward items from the instruments will be analysed using the Anthropological Theory of Didactics. This analysis will result in a comparison of the mathematical and didactical content in the three instruments along with an evaluation of whether the instruments corresponds to their theoretical frameworks.

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Introduction

In the summer of 2016 I engaged in a translation of a, assessment instrument for mathematics educators developed by professor Patrick Thompson. This translation was followed by a crash course for evaluating this instrument. This experience gave rise to wonderings of what was measured, how this matches with the evaluation model used by Thompson and for what purpose this type of instrument can be used. Initial questions were focused on this specific instrument but as other instruments were introduced it became interesting to study similarities and differences. Even more interesting is it to see, if the structure of the different instruments reveal anything of what the developers find relevant for teachers to know. Seen in a broader perspective the meeting with these different types of instrument rise a question, well-known within educational research – What should a teacher know?

Education of teachers are developing and different models appears around the world. We see retraining programs in Denmark educating people from other professions, in Brittan and America we see teaching certifications achieved by internships and we see countries in East Asia combining university studies in education with internships (Winsløw, 2017). As Winsløw ask in his article – how do we know which of these models present the best? Before answering this question, we need to decide on a measurement. To make this decision we will need to answer the above question.

The structure of the thesis:

In the following text, I will first present a historical introduction into the development of the role of the teacher throughout history. How has the role and responsibility of the teacher changed? What are the expectations today and how do we educate teachers? Afterwards a short introduction to the Anthropological Theory of Didactics (ATD) will be presented. This theory will constitute the theoretical framework for a reference model. This model will be used to analyse three instrument which are being used today, to measure praxeology's for teaching mathematics.

The first instrument presented is developed by Deborah Ball, an American educational researcher, it was developed to serve as an instrument which can be used as a standard measure in the United States. The second instrument, Teacher Education and Development Study in Mathematics (TEDS-M), is developed internationally based partly on the theory developed by Ball. This instrument is, like Balls', used as a standard measure but on an international scale.

The third instrument is developed by the American professor Patrick Thompson. This instrument is not intended as a measure for comparing teachers' knowledge. It is developed as an assessment tool to be used in teachers' professional development.
The framework for these three instruments will be elaborated further and will be followed by an analysis of three items from each.

In a historical perspective – From Plato to constructivism

The role of the teacher has developed throughout time. From the ancient Greeks to todays' constructivist approach to teaching. At the time of Plato and Socrates there existed little written work and even fewer people able to read it and a great part of passing on knowledge would naturally be to recite it. Besides this we see the role of Socrates as a guide to help his students *recall* certain knowledge, as the idea of the Platonists were that we all possess knowledge, "learning" is simply a matter of recalling. Thus, here one might need a guide – which then becomes the role of the teacher. Another thing which we can see in the descriptions of the Socrates dialogs is that teaching is described as 1:1 situations. Thus, one teacher may have just one student. There might be an audience, which to some extend will recall some knowledge ass well, though this is not the goal of the learning situations as described e.g. in the work Menon(Sedley & Long, 2010).
The role as a reciter did not change much the following couple of thousand years. Not until education became a right to laymen. In this new world of mass education teaching became a profession. And teaching and learning became something which were to be done in larger groups. Now one person was responsible of teaching many at the same time.
As oppose to before the teacher now had a shared responsibility of the student's success. This change in responsibility called for new tools in education. This is happening in the late 16'th century. Following the French revolution, thus following this the French required that all (boys) were to learn how to read and write. Note that this is at the level of lower secondary and primary school. The privileged of higher education were still for the few.
A hundred years later we see another change with Klein. At this point more people were entering secondary schools and this called for yet another change in the role of the teacher. Higher education was no longer reserved for the rich but were becoming an option for the common man, were he gifted and lucky enough to find one to finance his education. This called for a more applied use of the mathematics taught in schools. The need of linking the different topics was needed – especially when educating e.g. engineers. This change in the role of the teacher posed

new problems. Klein describe the core of these problems as a double discontinuity. This term he describes in the following way:

"The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fell in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching.

There is now a movement to abolish this double discontinuity, never having been helpful either to the school or to the university." (Klein, 2016)

This is a difficulty still to be solved. But what is required to solve this? In his book, Klein offers suggestions as to how different topics can be presented and tries to link the different mathematical objects in a greater context. Thus, connecting the different areas of mathematics but still at a level understandable to student of other sciences than mathematics. To solve the problem with the double discontinuity this presentation need to always be renewed and fitted to the given situation. This is calling for a type of teacher which acts autonomously. A teacher who can construct and contextualize mathematics in different contexts.

Now, mass schooling is becoming common in the western world and along with this the role of the teacher is changing. As opposite as to before teachers now need to teach many students at ones. Now teaching becomes an actual occupation for secondary level as well. Along this new profession comes the education of such professionals.

Teaching primary school, you are educated in a professionally oriented manner which focus on the didactical and pedagogical aspects of the teaching profession. Teachers often teach more than one subject and the education is composed of both theory and practice/internships.

Are you supposed to teach at I higher level e.g. higher secondary school you are required to have an education which focus lies within the profession of the subject(s) you will be teaching. The education will contain less didactics and no pedagogics and teachers rarely teach more than two subjects at most. (Winsløw, 2017). According to (Winsløw, 2017) the teacher education for lower education and higher education has very different focuses. And this has not changed much from the time of Klein till todays teacher education system. At least in western Europe and North America.

But what has changed is that at the time of Klein only few students would continue studying through high school. Whereas today a high percentage of student continue some type of further education. And the secondary schools of today must embrace a much wider range of students going on to many different professions(Carl Winsløw, 2014). So, how does this change for the role of today's secondary teachers. Before, the problems described by Klein concerned a small and specialized group of students and might therefore have been as visible.
Today these problems become evident for a great part of students and institutions surrounding secondary school expect the greater part of a generation to complete education above the level of lower secondary. Then, what is needed for the teacher to know and master to handle this problem? One of the first attempts to highlight and answer this question was given by Begle (1972). He finds:

"The most significant information, of course, comes from the regressions of the effectiveness scores on the two teacher scores. These indicate that teacher understanding of modern algebra (groups, rings, and fields) has no significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra. Teacher understanding of the algebra of the real number system has no significant correlation with student achievement in algebraic computation. However, teacher understanding of the algebra of real number system does have a significant positive correlation with student achievement in the understanding of ninth grade algebra."(Begle, 1972)

I.e. Begle show that there is no significant correlation between the teachers' achievement in algebra and that of their students unless it was algebra related to the algebra taught at the level which they were teaching.
Later Ball publishes an article which support this theory. She concludes like Begle, that the teachers' ability to master techniques used at the level of which they teach, seems to have a positive effect on their students achievement (Heather, Brian, & Deborah Loewenberg, 2005). This implication leads to the question which was first presented – What should a teacher know?

Theoretical framework – Introduction to ATD

In this section, the Anthropological Theory of Didactics (abbreviated ATD) will be presented. This theory will provide the framework for the theoretical reference model which will be used in the analysis of the three different instruments, which all have their own theoretical framework. ATD consists of concepts such as praxeology's, mathematical organizations and didactical organizations. All of which will be elaborated in the following section.
ATD is introduced by Yves Chevallard around the change of the millennium and has its roots in the ideas of *Theory of Didactical Situations* (TDS) developed by Guy Brousseau in the 1970's (Marianna Bosch, 2006). And with Brousseau's ideas of TDS it has taken its inspirations for learning situations in ATD. They are described in the following way by Chevallard (translation by Barbé et al.):

"The first moment of study is that of the first encounter with the organisation O at stake. Such an encounter can take place in several ways, although one kind of encounter or 're-encounter', that is inevitable unless one remains on the surface of O, consists of meeting O through at least one of the types of tasks T_i that constitutes it. [. . .] The second moment concerns the exploration of the type of tasks T_i and elaboration of a technique τ_i relative to this type of tasks. [...] The third moment of the study consists of the constitution of the technological–theoretical environment [. . .] relative to τ_i. In a general way, this moment is closely interrelated to each of the other moments. [. . .] The fourth moment concerns the technical work, which has at the same time to improve the technique making it more powerful and reliable (a process which generally involves a refinement of the previously elaborated technique), and develop the mastery of its use. [. . .] The fifth moment involves the institutionalisation, the aim of which is to identify what the elaborate mathematical organisation 'exactly' is. [. . .] The sixth moment entails the evaluation, which is linked to the institutionalisation moment [. . .]. In practice, there is always a moment when a balance has to be struck, since this moment of reflection when one examines the value of what is done, is by no means an invention of the school, but is in fact on a par with the 'breathing space' intrinsic to every human activity."(Barbé, Bosch, Espinoza, & Gascón, 2005)

As this thesis will not be concerned with learning situations but on task and techniques for teachers, which again, won't lead directly to a learning situation I will go on to other aspects of ATD.

Didactic transposition

ATD was developed as an aid in studying the field of didactical transposition and the development of mathematics in different independent institutions(Winsløw, 2010). The term didactical transposition describes the transition of knowledge from one institution to that of the school and further into the classroom and then into the mind of the student, represented in the figure below.

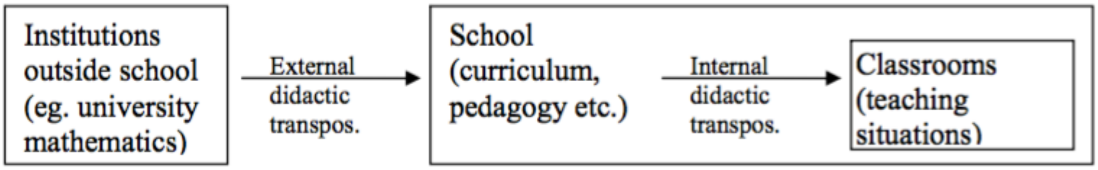


Figure 1: Didactical transposition (Winsløw, 2010).

In figure 1 two terms are presented: External didactic transposition: considerations about the school as an institution, done outside the school e.g. at the university or work done by politicians. This is what Chevallard named the "noo-sphere". And internal didactic transposition, the thinking done inside the school e.g. by teachers' considerations about the curriculum and how to convey the curricular to the students and the student absorbing the knowledge available in the lessons. This is not to be confused with the double-discontinuity which we shall return to later.

Praxeologies and organizations

To describe how mathematics are structured in the different institutions Chevallard includes another term – praxeology. The origin of the word is as follows:

"The word praxeology has been around for (at least) two centuries in the sense recorded by most dictionaries, in which it is held to refer to the "study of human action and conduct", to the "study of practical or efficient activity", or to the "science of efficient action".(Yves Chevallard)

As read above the word praxeology, which is central to the theory of ATD, determ the study of human action. Though this is not exactly the definition in ATD terms. So, what is it a praxeology in ATD? Well, ATD is based on the assumption that the learning and teaching of mathematics can be viewed and framed in the same way as any other human activity.(Marianna Bosch, 2006) She quotes Chevallard:

"A praxeology is, in some way, the basic unit into which one can analyse human action at large. [...] What exactly is a praxeology? We can rely on etymology to guide us here — one can analyse any human doing into two main, interrelated components: praxis, i.e. the practical part, on the one hand, and logos, on the other hand. "Logos" is a Greek word which, from pre-

Socratic times, has been used steadily to refer to human thinking and reasoning" (Marianna Bosch, 2006)

I.e. A praxeology is a set of two blocks – a practical block consisting of type of tasks and techniques to solve them. And a theoretical block of a technology which explanations and justifications of the techniques along with a theory. More exact, a praxeology consists of a four-tuple (T, τ, θ, Θ) – tasks (T), techniques (τ), technologies (θ) and theory (Θ).

To clarify let's look at an example of a mathematical praxeology:

A task could be: t: "*What is the sum of* $\frac{a}{b}$ *and* $\frac{c}{d}$ *e.g.* $\frac{4}{9} + \frac{8}{12}$. "

This task belongs to a type of mathematical tasks: T: "*calculating with fractions"*

And for this type of tasks there typically belong a certain, sometimes more than just one, technique τ, which can be used to solve this class of tasks. In this example, a technique would be: *"finding a common denominator, extend the fractions and then add them."* This pair (T(calc. with. fractions), τ(finding/using common denominator)) constitutes the practical part, the "know how"

Used to justify and/or explain the use this technique we have the technologies, in this case a technology could be: *"We can extend the fraction by a natural number* $n \neq 0$ *and still have the same fraction e.g.* $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$ *and thus we can add the fractions after finding their common denominator. "*

To explain the technology we have the theory, in this example a theory could be the explanation and introduction to the rational numbers: $\mathbb{Q} = \{\frac{x}{y} \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{N}\}$

This pair (θ, Θ) constitutes the theoretical/logos part.

Hence these two pairs together constitute a mathematical praxeology for adding fractions.

Praxeology's often relate to other praxeology's. This can be done on different levels and organisations are named based on the level of which the praxeologies are related.

Punctual – "*A punctual mathematical organisation or praxeology consists of these four elements: a type of task, a technique, a technology and a theory, where each element corresponds to the previous one."(Viviane Durnad-Guerrier, 2010).* Hence, a punctual organization is another term for praxeology

A local organization are praxeology's which find their common features in their technology. Thus, if we have a set of praxeology's unified by their justification and explanations of their techniques we have what is called a *Local Organization*

At last, if we have a set of praxeology's unified by a common theory we have a *Regional Organization(Winsløw, 2010).*

These are the three types of organizations in ATD and when the theme of the elements is concerned with mathematics they are denoted as *Mathematical Organizations*.

Thus, the mathematical organizations are concerned with what Shulman denotes Subject Matter Knowledge.(Shulman, 1986) Now for teaching mathematics, mathematical organizations cannot stand alone. Hence, they are accompanied by *Didactical Organizations (DO).* The didactical organizations consist of the same components at the mathematical organizations. I.e. they have Task, techniques, technologies and theory as well. And hence, they consist of didactical praxeologies. Using the same example as above let's exemplify the DO:

T: Pose the question "*What is the sum of* $\frac{a}{b}$ *and* $\frac{c}{d}$ *e.g.* $\frac{4}{9}+\frac{8}{12}$*.*" In such a way that it will be available to the students in question.

τ: Consider the MO of the student and decide e.g. whether it should be presented as a general statement at first or whether the students should be eased into the generalization.

θ: Teachers must consider the MO available to students and adjust their teaching accordingly such that the mathematics are presented such that it is available to students.

Θ: Learning theory based on Vygotsky's theory on zone of proximal development

Furthermore, there exists two types of didactic praxeologies, the didactical praxeologies belong to the students called the students didactic praxeology. And the praxeology which covers what Shulman determ *pedagogical content knowledge*. This is the didactic praxeology of the teacher who want to help the students achieve a proper MO (Barbé et al., 2005). This DO is to be conceived as teacher's part of the learning situation. The task for the teacher is e.g. to prepare and consider which (student)tasks should be presented and how. Now, which techniques does the teacher use to help the students "attach" the task and work with it the way intended ~ the teachers' techniques. This is determined by the teacher's didactical technologies which is again based on a didactical theory. These praxeology's constitutes the Didactical Organization(DO). And thus, a collection of praxeology's we determ a didactical organization for teaching mathematics.

Both the students' and the teacher's DO are contained within the institution in which the students are taught. Another relation of interest is the relation between the DO of the teachers' education program and this DO's relation to the institution where the teacher will eventually teach represented in the figure below.

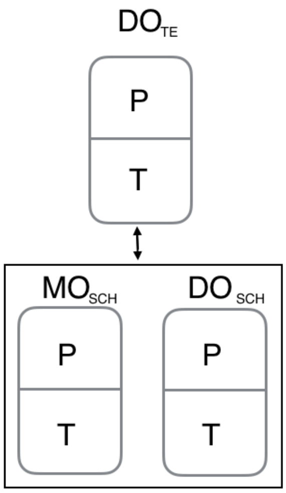


Figure 2; DO of the different institutions including the relation between the teachers' DO and the school.

In this thesis, I will focus on the teachers' didactic organizations. Unless anything else is written the reader might all ways assume that this is the praxeology or organization which is referred to. Now, the MO and the DO now might appear to be two things which can be separated from one another. This is not the case. At second glance, it is quite clear that it does not make sense to talk of a didactical organization without a mathematical organization – thus ones need to have a deep insight into the MO to create a proper DO. In the word of Chevallard: the two are co-determined. The different organizations are bound to the institution in which they are created, e.g. an institution forms its own praxeologies which is adapted to the circumstances under which it operates. Within such an institution there exist relations between student, the praxeology taught in the institution, which in ATD is noted $R_{Institution}(individual, praxeology)$.

With this notation, we can model Klein's problem concerning the transition of a pupil becoming university student and then a teacher. $R_S(p, O) \rightarrow R_U(s, \Omega) \rightarrow R_S(t, O)$ Where O and Ω represent the praxeologies taught at the school and university respectively. Klein's double-discontinuity addresses transition represented by the last arrow. I.e. The relationship between the different mathematical organizations contained within the different institutions and how the student who has now become the teacher can relate these.

ATD as an assessment tool

ATD served as a strong theoretical tool for researchers working with many different aspects of mathematics education. And has been used as a frame for reference models before (e.g. (Michèle Artigue, 2010)). Though this theory serve as a strong tool for creating reference models it goes beyond that. The anthropological theory of didactics present a framework for the structure of knowledge implemented in institutions and how this is transferred to individuals. I.e. ATD is based on TSD, which offers a framework for the situation of learning (the internal didactic transposition), and from this theory ATD describe and theorize the elements which constitute knowledge within institutions ant the transition between these. From this theory, a vocabulary for describing mathematical and didactical knowledge arise and establish a strong tool for studying the knowledge within these institutions. ATD is used within educational science but as the theory is generic in the sense that none of the concepts are specific for mathematics it is not distant to imagine the theory used in a different context. In the following analysis ATD will be used to describe the three frameworks and selected items from these, described in the following sections. This description will be used to frame the instruments and the items in the theory of ATD.

Research Questions

In the summer of 2016 I was hired to translate Thompson's instrument from English to Danish. During the translation process and the following course on how to evaluate the different classes of items I began wondering about several things; Do these items really test what they set out to? How does such an instrument find its shape? And how do these items differ from familiar mathematical tests? After further consideration and talks with my supervisor the idea of analysing and, to the extent possible, comparing different instruments came along. At first it was the intention to try out Thompson's instrument among Danish high school teachers. But after considering how this would contribute to the thesis I decided not to do so. This was due to the lack of empirical data which would be possible to collect and analyse as well at the nature of my interest. Hence, I wanted to make a theoretical analysis of the items and therefore did not believe, that the data, which would have been possible to collect and analyse, would not contribute enough compared to the time and resources it would demand to do a data collection. These processes lead to these final research questions:
RQ1: What didactical and mathematical tasks, in terms of ATD, are contained in the items?
RQ2: How do the three frameworks theorize knowledge for teaching mathematics? And how do the items relate to the theoretical framework in terms of measuring teacher knowledge.
RQ3 – Looking at the didactical and mathematical items, how do the three instruments differ and agree?

Mathematical Knowledge for Teaching (MKT) – Deborah L. Ball

In the context of the increased mathematical demands of the Common Core State Standards and data showing that many elementary school teachers lack strong mathematical knowledge for teaching, there is an urgent need to grow teachers' MKT. With this goal in mind, it is crucial to have research and assessment tools that are able to measure and track aspects of teachers' MKT at scale. (Selling, 2016)

These are the first words of the abstract of an article by Ball clearly stating the motivation for the development of this instrument. Thus, this instrument is thought as an assessment tool for teachers around the US. The basis for the theoretical framework is found in the work of Shulman (Selling, 2016) Here the idea of Subject matter knowledge and pedagogical content knowledge is introduced for the first time. And from this idea Ball has developed the theory described below as a framework for the concept *"Mathematical Knowledge for Teaching"* (MKT)

Theoretical framework

Before I can represent the full MKT model, a few concepts will have to be introduced. These are Balls' concepts of subject matter knowledge and pedagogical content knowledge along with the subcategories of the two. After these have been accounted for Balls' model will be introduced and I will give a short account of how the assessment tool by Ball was developed. Lastly a couple of items will be presented as to exemplify items which are meant to assess the different areas of the MKT model.

Subject matter knowledge:

As the name indicate subject matter knowledge is concerned only with the disciplinary knowledge, in this case, the mathematics. Subject matter knowledge is divided into three sub-categories. Common content knowledge (CCK) is the mathematics which is commonly known, by teachers and non-teachers alike. E.g. that 0 is neither even or uneven, that subtracting a negative number is the same as adding the numerical value of that number or a subtraction algorithm. The Horizon Content Knowledge (HCK) is the knowledge needed to connect different topics within mathematics, e.g. *"an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First- grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later."(Ball, Thames, & Phelps, 2008)* HCK is closely related to another of the categories which we shall see in the following paragraph.

As the last component within the subject matter knowledge we have Specialized Content Knowledge (SCK). This is, in this context, a special kind of subject knowledge which is needed to teach, and usually not in any other context. This type of knowledge is categorized as knowledge which is specific for teaching, this could for example: *"Teachers must know rationales for procedures, meanings for terms, and explanations for concepts. Teachers need effective ways of representing the meaning of the subtraction algorithm — not just to confirm the answer but to show what the steps of the procedure mean and why they make sense. "(Ball et al., 2008)* As Ball describes it in her own terms. Usual examples of teachers SCK could be: "Presenting mathematical ideas", "Responding to students' "why" questions", "Finding an example to make a specific mathematical point" or "Recognizing what is involved in using a particular representation" (Ball et al., 2008).

Pedagogical content knowledge:

In the words of Shulman, from where Balls theory arises:

The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. . . . Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons.(Ball et al., 2008)

Aside from subject matter knowledge, teachers also need Pedagogical Knowledge for Teaching combines the mathematical knowledge with other areas of teachers' knowledge such as their knowledge about students' content domain and their way of thinking or which approaches to different topics is most appropriate for different groups of students. *"For example, in teaching integers, teachers need to appreciate that notions of "debt," "assets" and "net worth" are unfamiliar to elementary age learners and that therefore financial contexts are not likely to be useful as a representation of integer arithmetic."(Selling, 2016)*

Hence the teacher need the ability to foresee what content knowledge students possess, this is termed *"Knowledge of Content and Students" (KCS): "is knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing."(Ball et al., 2008)* Hence, the teacher must be able to place himself in the mind-set of the student in order to understand the students difficulties. I.e. the teacher must know what might be challenging based on the mathematics known to the student.

The second part of PCK is the Knowledge of Content and Teaching (KCT). *"[…], knowledge of content and teaching (KCT), combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction "(Ball et al., 2008)* Hence, this knowledge cover e.g. the ability needed by the teacher to choose which kind of examples should be included in the instructions or the choice of which tools should be used in the instructions given to students.

At last we have Knowledge of Content and Curriculum (KCC): This is another category taken from Shulman, He described the curricular such: "*The curricular and its associates materials are the materia medica of pedagogy, the pharmacopeia from which the teacher draws those tools of teaching that present or exemplify particular content and remediate or evaluate the adequacy of student accomplishment."(Shulman, 1986)* Hence, the curricular is the basis from where the teacher can retrieve or develop tools for teaching. At the same time, it is a detailed description of the mathematics which should be told at a given grade. This includes a thorough description of to which extend the different element are expected to be taught. Furthermore, Shulman divides this curricular knowledge into two subcategories:

"[…] lateral curriculum knowledge and vertical curriculum knowledge. Lateral knowledge relates knowledge of the curriculum being taught to the curriculum that students are learning in other classes (in other subject areas). Vertical knowledge includes "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Ball et al., 2008)

Which leads one thought back to the HCK – and not unjustified. Hence, Ball has chosen to place KCC under pedagogical content knowledge but is aware that KCC might stretch over several of the categories.

The MKT model

These two overall categories, SMK and PCK, constitute what Ball determ as Mathematical Knowledge for Teaching (MKT) In the figure below Ball presents the model visually. It is not explicitly mentioned by Ball whether the size of each area represents the important of each part respectively, the desired distribution of focus or nothing at all.

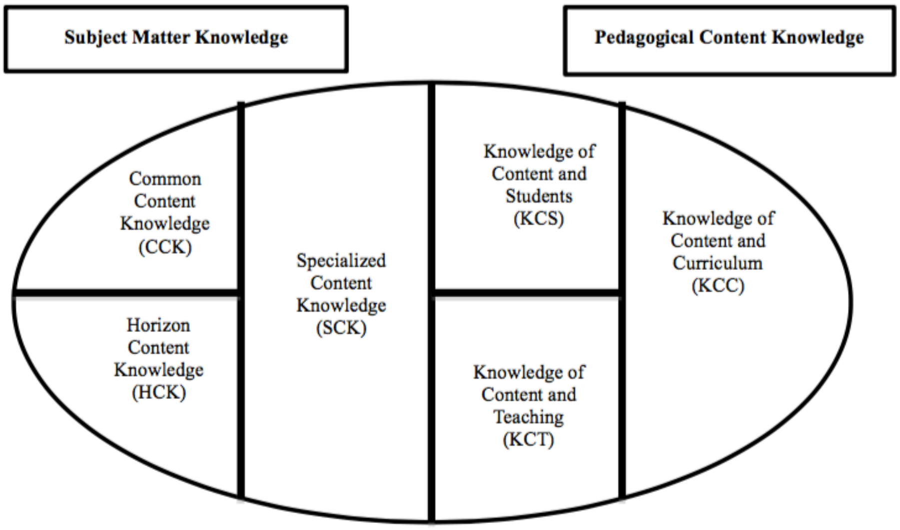


Figure 3; MKT model (Ball et al., 2008)

To exemplify the theory above let us study the following item:

13. Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students focused on particular difficulties that they are having with adding columns of numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

I)	II)	III)
1	1	1
38	45	32
49	37	14
+ 65	+ 29	+ 19
142	101	64

Which have the same kind of error? (Mark ONE answer.)

a) I and II

b) I and III

c) II and III

d) I, II, and III

Figure 4; Example of item presented by Ball (Hill, 2004)

In this case, the common content knowledge would the ability to use a general algorithm so solve the task of addition. Hence, the respondent would need this common content knowledge to even get started on this item. Horizontal content knowledge could be to know how a similar representation or how to introduce a corresponding algorithm for subtraction from this. This type of knowledge does not seem directly connected to this item but could be needed in a learning situation where the teacher/respondent would need to introduce other techniques to students struggling. SCK here is the ability to explain every step of the algorithm. Such as when to write one, two etc. above the next column, or know how to explain why beginning at the far-right

column is a good idea. E.g. knowing and explaining the positional system and how this is related to the algorithm.
For the Pedagogical Content Knowledge, we have the three sub categories: Knowledge of content and student (KCS), here the respondent must place him/her self in the thoughts of the students. How did the students approach the task? And when/what went wrong. This the teacher must be able to answer, and these considerations are the teacher's knowledge of student and content. From this item. the need to help students correct their misunderstanding could arise and the teacher's knowledge of which tools to represent to the students is the Knowledge of Content and Teaching (KCT). Knowledge of Content and Curriculum can in this item be represented by the teacher's knowledge of what he can expect from students at this grade and to which extend the students should be able to justify the algorithm they are using.

Methodology

Ball and her team developed the item based on theory developed for the project *Learning Mathematica for Teaching* along with analysis of curriculum, examples of students' work and personal experience. In 2001 researchers could present 138 items. These items were concerned with content knowledge or knowledge of students and content and were all within the topics "number concepts", "Operations" or "patterns, functions & algebra". The category of "patterns, functions & algebra – Knowledge of student and content" was dropped due to lack of literature on students learning patterns within these topics. Items within this category was developed later as research became available.(Hill, 2004) Now an analysis of the items were made and revealed that some items covered more than one of the remaining categories and some were more specific within topics which lead to design of items investigating teachers knowledge within topics specific to the curriculum they were teaching(Hill, 2004). It is now studied which of the MKT categories each item can be associated with. Interviews were made to test whether the responses matched what the respondent had understood by the item and if this matched what the item was intended to study. Whether these interviews followed a trail of the items is unclear.
Later the theoretical framework for item writing has developed and a greater focus is placed on *mathematical work for teaching* (for table over this framework see appendix). This mean that items are developed around the mathematics as opposite to the pedagogics as it turned out that item writers had problems with these items(Selling, 2016). For more on this framework and the effect it has on item development see Selling 2016.
The trial of new items developed on this framework is not described by Ball and can therefore not be described further, but can be expected to be similar to that of the first items.

Thoughts and open questions

When reading articles by Ball concerning MKT and the instrument developed is it noticeable that the concept of knowledge is used widely in the theory presented. However, it seems unclear what this means to Ball. Does knowledge cover all sorts of skills and reflections within a field or does it need to be more specific. E.g. knowledge of content and student – above it is described as the knowledge of students' meeting with the discipline and what might be confusing. Still, what does this mean? Can a teacher's personal perception of the difficulties of a topic be categorized as PKN or does the thoughts and actions connected to teaching this topic need to be founded on a theoretical framework? A more detailed description of what is meant by knowledge would have made it clearer what exactly each item is measuring.

Looking at Ball description of PCK seems to be closer related to what is known as stoffdidaktik in Europe than to the European definition of pedagogics. I.e. this term seem to be covering the students' meeting with mathematics and the considerations which comes along – This is what can be said to be didactics. Pedagogics in European settings denotes everything surrounding this situation.

Teacher Education and Development System in Mathematics (TEDS-M) – EIA[1]

TEDS-M is the result of a collaboration between several countries and is developed as a mean to compare different educations systems. In this instrument, we see a greater interest in the educational background and professional experience of the respondent than in both Ball's and Thompson's. The reason for this interest is, that for this instrument to compare entire educational systems across culture and economy one needs much more information to make a proper statistical analysis of potential correlated parameters.

The TEDS-M program deviates from the other instruments presented because is it developed with the purpose of being an assessment instrument used in this one research project rather than an instrument which should be useable in more than one context. This means that the research questions and the elements which is studied is very specific and the instrument and its items are directed clearly towards answering these.

Theoretical framework

The TEDS-M program is constructed of three main components: (Tatto, 2008)

- COMPONENT I: Studies of teacher education policy, schooling, and social contexts at the national level.
- COMPONENT II: Studies of primary and lower secondary mathematics teacher education routes, institutions, programs, standards, and expectations for teacher learning.
- COMPONENT III: Studies of the mathematics and related teaching knowledge of future primary and lower secondary school mathematics teachers.

Clearly, these components cover more than what the items in what in TEDS-M terms are called a booklet, and what in this thesis is named an instrument can contain.

The first component is concerned with the institutions surrounding the school, i.e. the institutions which affect the external didactic transposition.

The second component is a mapping of the structure of teacher education in the participating countries along with a comparison of what the curricular in the different teacher educations contain. These two first components are studied using, among other things, questionnaires given to the respondent of the booklets. These questionnaires contain questions concerning e.g. the

[1] International Association for the Evaluation of Educational Achievement

respondent educational background and their view on the nature of mathematics. (Blömeke, 2013)

The third component of the TEDS-M program is the component which contain the instrument which will be analysed. This component is concerned with mathematical knowledge along with knowledge for teaching mathematics: This term is described the following way:

"Knowledge for teaching mathematics consists of two constructs: mathematics content knowledge and mathematics pedagogical content knowledge (see the introduction of the framework for the theoretical origins of these constructs)."(Tatto, 2008)

The theoretical origin of the constructs are the ideas of Shulman, which is the ideas we see in the theory of ball as well. Tatto writes:

"The roots of the concept of knowledge for teaching can be traced to ideas expressed by Lee Shulman in his 1985 presidential address to the American Educational Research Association. Later, Shulman (1987) identified three categories of teachers' knowledge: subject-matter knowledge, pedagogical content knowledge, and curricular knowledge. According to Shulman, subject-matter or content knowledge is the set of fundamental assumptions, definitions, concepts, and procedures that constitute the ideas to be learned. Pedagogical content knowledge (PCK) includes useful forms of representation of those ideas, powerful analogies, examples, and explanations of a subject, insights into what makes the learning of specific topics easy or difficult, and the conceptions that students of different ages and backgrounds bring with them to the learning of the topic. Curricular knowledge involves understanding how the topics are arranged over time across schooling experiences."(Tatto, 2008)

Which is the same idea Ball's theoretical framework and instrument is built on. Hence, the idea of what teacher knowledge is, or should be, is like Ball's. But from here on the TEDS-M framework is based on the work of the TIMSS project. TIMSS is, like TEDS-M, an international comparative study which is also cross national.

In the case of the mathematical content knowledge TEDS-M based on the theory used in the TIMSS 2007 and TIMSS Advanced 2008 programs. The mathematical topics which were used by TIMSS were used as inspiration for the content of the TEDS-M items and cover the mathematical areas described in table 1:

Subdomain	Content Areas
Number and Operations	Whole numbers (*ps*) Fractions and decimals (*ps*) Number sentences (*ps*) Patterns and relationships (*ps*) Integers (*ps*) Ratios, proportions, and percentages (*ps*) Irrational numbers (*ps*) Number theory (*ps*)
Geometry and Measurement	Geometric shapes (*ps*) Geometric measurement (*ps*) Location and movement (*ps*)
Algebra and Functions	Patterns (*ps*) Algebraic expressions (*ps*) Equations/formulas and functions (*ps*) Advanced topics, e.g., limits, continuity, matrices (*s*)
Data and Chance	Data organization and representation (*ps*) Data reading and interpretation (*ps*) Chance (*ps*)

Note: *p* = primary level; *s* = lower-secondary level.

Table 1, TEDS-M ***Mathematics content knowledge framework: content knowledge subdomains and content areas(Kiril Bankov et al., 2013)***

After studying the syllabus for different participating countries, the weight of the different domains was adjusted accordingly(Kiril Bankov et al., 2013).

Hence, the TIMSS and TEDS-M programs are similar when considering the content of the items and the level of difficulty but differs when considering the distribution of the different topics. number of items within each topic.

TEDS-M focus on the content which the teachers, or teacher students are required to teach along with curricular content for the following two grades. Higher levels of mathematics are included as well but to a lesser extent. The TEDS-M program adopted the cognitive domains used in the TIMSS program for describing the knowledge held by the teachers. I.e. the teachers' mathematical skills, they are described as so: *Understanding a mathematics topic consists of having the ability to operate successfully in three cognitive subdomains. The first domain, knowing, covers the facts, procedures, and concepts students need to know, while the second, applying, focuses on the ability of students to make use of this knowledge to select or create models and solve problems. The third domain, reasoning, goes beyond the solution of routine problems to encompass the ability to use analytical skills, generalize, and apply mathematics to unfamiliar or complex contexts.(Kiril Bankov et al., 2013)*

A detailed description of the cognitive framework, the domains and subdomains, for mathematics used by TEDS-M are described in the tables below:

Knowing	
Recall	Recall definitions; terminology; number properties; geometric properties; notation.
Recognize	Recognize mathematical objects, shapes, numbers and expressions; recognize mathematical entities that are mathematically equivalent.
Compute	Carry out algorithmic procedures for addition, multiplication, division, subtraction with whole numbers, fractions, decimals, and integers; approximate numbers to estimate computations; carry out routine algebraic procedures.
Retrieve	Retrieve information from graphs, tables, or other sources; read simple scales.
Measure	Use measuring instruments; use units of measurement appropriately; estimate measures.
Classify/Order	Classify/group objects, shapes, numbers, and expressions according to common properties; make correct decisions about class membership; order numbers and objects by attributes.
Applying	
Select	Select an efficient/appropriate operation, method, or strategy for solving problems where there is a known algorithm or method of solution.
Represent	Display mathematical information and data in diagrams, tables, charts, or graphs; generate equivalent representations for a given mathematical entity or relationship.
Model	Generate an appropriate model, such as an equation or diagram, for solving a routine problem.
Implement	Follow and execute a set of mathematical instructions; draw figures and shapes according to given specifications.
Solve Routine Problems	Solve routine or familiar types of problems (e.g., use geometric properties to solve problems); compare and match different representations of data; use data from charts, tables, graphs, and maps to solve routine problems.
Reasoning	
Analyze	Determine and describe or use relationships between variables or objects in mathematical situations; use proportional reasoning; decompose geometric figures to simplify solving a problem; draw the net of a given unfamiliar solid; visualize transformations of three-dimensional figures; compare and match different representations of the same data; make valid inferences from given information.
Generalize	Extend the domain to which the result of mathematical thinking and problem-solving is applicable by restating results in more general and more widely applicable terms.
Synthesize/Integrate	Combine (various) mathematical procedures to establish results, and combine results to produce a further result; make connections between different elements of knowledge and related representations, and make linkages between related mathematical ideas.
Justify	Provide a justification for the truth or falsity of a statement by reference to mathematical results or properties.
Solve Non-routine Problems	Solve problems set in mathematical or real-life contexts where future teachers are unlikely to have encountered closely similar items, and apply mathematical procedures in unfamiliar or complex contexts; use geometric properties to solve non-routine problems.

Table 2, TEDS-M; Mathematics Framework: Cognitive Domains (Tatto, 2008)

These are the different mathematical skills which the items are structured to measure. I.e. table 2 give a set of skills which the development team can structure the item around as well as a vocabulary for describing both the content of items as well as the outcome of the project.

Now, it might seem that TEDS-M and TIMSS are the same but with different wording of the framework. Anyhow, this is not the case. What separate the two are the fact that TIMSS and TEDS-M differ in the theoretical framework for what TEDS-M denotes pedagogical content knowledge. In the case of pedagogical content knowledge, the TEDS-M framework is founded, among others, in the ideas of Ball. From this stand-point the framework developed and along another categories the category: *"knowledge of planning for mathematics teaching and learning"*. Along the development of Ball's categories some were dropped and the TEDS-M framework ended up with the categorization of pedagogical content knowledge showed in the figure below:(Tatto, 2008)

Mathematical Curricular Knowledge[a]	Establishing appropriate learning goals Knowing different assessment formats Selecting possible pathways and seeing connections within the curriculum Identifying the key ideas in learning programs Knowledge of mathematics curriculum
Knowledge of Planning for Mathematics Teaching and Learning (pre-active)	Planning or selecting appropriate activities Choosing assessment formats Predicting[b] typical students' responses, including misconceptions Planning appropriate methods for representing mathematical ideas Linking the didactical methods and the instructional designs Identifying different approaches for solving mathematical problems Planning mathematical lessons
Enacting Mathematics for Teaching and Learning (interactive)	Analyzing or evaluating students' mathematical solutions or arguments Analyzing the content of students' questions Diagnosing typical students' responses, including misconceptions Explaining or representing mathematical concepts or procedures Generating fruitful questions Responding to unexpected mathematical issues Providing appropriate feedback

Table 3, Mathematics Pedagogical Content Knowledge (MPCK) Framework

And finally, TEDS-M introduce the concept of *General Knowledge for Teaching*. This term covers the following categories: *"(a) students (i.e., knowledge of students), including the influence of socioeconomic status on teaching and learning; (b) classroom environment; (c) instructional design; and (d) diagnostics and assessment."(Tatto, 2008)*

And are included since this was in the curricular of many education programs in countries participating in the pilot studies. The study of this concept in the final TEDS-M booklets are in the form of questionnaires. I.e. these areas are not included in the items which will be studied in the following analysis.

Let's take an example of a mathematical item from the TEDS-M instrument:

Prove the following statement:

If the graphs of linear functions

$$f(x) = ax + b \text{ and } g(x) = cx + d$$

intersect at a point P on the x-axis, the graph of their sum function

$$(f + g)(x)$$

must also go through P.

Figure 5; item from the TEDS-M program (Blömeke, 2013)

An appropriate answer for this item could be as follows:

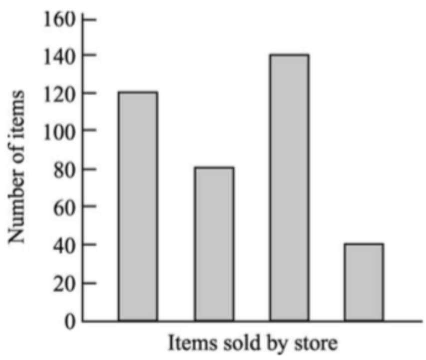
If $f(x)$ and $g(x)$ both intersects the x-axis at be we have: $f(P) = g(P) = 0$ Since both functions are linear we have: $f(x) + g(x) = (f + g)(x)$ and hence $f(P) + g(P) = (f + g)(P) = 0 + 0 = 0$ and we can conclude that $(f + g)(x)$ goes through the point P.

Now consider table 2. The answer above reflects the domain of knowing as the property for linear functions lies within the subdomain of recalling. As the answer conclude that the composite function goes through P with reference to the property stated this part of the proof can be placed in the subdomain of justifying. Hence, to give a proper answer to this item the respondent will need both the domains of "knowing" and "reasoning".

For an example of a didactical item consider the following:

The following problem was given to children in <primary> school.

The graph shows the number of pens, pencils, rulers and erasers sold by a store in one week.



The names of the items are missing from the graph. Pens were the item most often sold. Fewer erasers than any other item were sold. More pencils than rulers were sold.

(a) How many pencils were sold?

Check one box.

A.	40	□1
B.	80	□2
C.	120	□3
D.	140	□4

(b) Some <primary> students would experience difficulty with a problem of this type. What is the main difficulty you would expect? Explain clearly with reference to the problem.

Figure 6; item from the TEDS-M program (Blömeke, 2013)

In this item, the respondent must identify potential student difficulties or misconception. Considering table 3 these reflections corresponds to the category of *"*knowledge of planning for mathematics teaching and learning*"*. In the scoring rubric (see appendix) for this item reveal that the explanation which the respondent is asked to give should contain the reason for the students' difficulties. Including this the items also covers the category of "enacting mathematics for teaching and learning" as this explanation would correspond to the subcategory of diagnosing typical students' responses including misconceptions.

Methodology

The TEDS-M program faces several problems like those of the other programs when it comes to development of items. Now that teachers' knowledge has been classified how is it measured in terms of these classifications? Tattoo write:

"The conceptual challenges of measuring the outcomes of teacher preparation in terms of teacher knowledge and belief are considerable. TEDS-M builds on our development study (i.e., MT21), which produced an earlier and shortened version of a questionnaire for future lower secondary teachers to measure knowledge of (i) mathematics, (ii) mathematics pedagogy, and (iii) general knowledge for teaching. A number of belief scales and preparedness scales based on the literature were also included. These instruments were trialed on a small-scale basis in six countries with promising results and served to inform the instrument development in TEDS-M."(Tatto, 2008)

Thus, it is on these preliminary questionnaires and belies scales which the items are developed. From this first trial the first set of booklets were designed. To clarify: Booklet are the entire handout given to respondents. Booklets consist of: questionnaires investigating teachers' mathematical beliefs and general knowledge for teaching, questionnaires mapping the respondents' education i.e. what topics within mathematics has the respondent studied and items. The booklets were now adjusted such that all the subcategories of the cognitive domains were represented in the levels they were appropriate at.

The TEDS-M program has lent items from the "Knowing mathematics of Teaching Algebra (KAT) project at Michigan University, the "learning Mathematics for Teaching (LMT) project at Michigan University, researchers in Australia and participating countries(Tatto, 2008). The item is then examined by an expert panel, what this panel consist of is not clear. But the examination has considered aspect such as validity and classification within the framework. The items are then taken to a field trial adjusted and brought before the panel once again. This process is repeated up till five times – depending on the number of test trials the items were submitted for. Scoring rubrics were developed for all items and answers were collected to serve as examples.(Tatto, 2008) How these rubrics were developed is not elaborated in the article by Tattoo. The test study was made for lower secondary level and items were later adjusted such that it fitted the level of primary school in the booklets for these.

After the items were completed they were distributed into different block which would then be combined in the different booklets focusing on the distribution of items covering different cognitive domains such that each booklet would contain approximately the same number of items or points within each topic. (for an example of a block design see appendix)

Thoughts and open questions

TEDS-M seems to be categorizing the techniques into three domains, the knowing, applying and reasoning. Which look like a categorization which describe different levels of mathematical skills – ranged in the order which they are mentioned. It is also worth noting the similarities to the categories of Bloom's taxonomy in the naming of the categories as well as their ranking. Beside this, parallels can be drawn to the theory of ATD and the categorization here. As it could look like the domains "knowing" and "applying" could correspond to the practical and "reasoning" to the theoretical block of the MO. Looking at the pedagogical domains presented in table 3 a similar parallel can be drawn to the practical block of the DO. However, there is no elements of explanation or justification in table two and there can therefore be drawn no parallels to the theoretical block of the DO. For the development of the items TEDS-M state that: *The process of item development for TEDS-M has been thorough, extensive, and rigorous.(Tatto, 2008)* The initial development of items i.e. first drafts is not described and it becomes unclear if they are developed ad hoc and later adjusted. Especially since they have requested for items from other programs it becomes unclear whether the items were initially useable within the framework of TEDS-M.

Mathematical Meaning for Teaching Secondary Mathematics (MMTsm) – Patrick W. Thompson

In contrast to the other two instruments presented in this thesis, this instrument is not developed as an instrument for comparison of institutions. It was developed as an instrument to be used in assessing teachers participating in professional development programs.

Another difference is the focus of the instrument. MKT and TEDS-M attempt to measure the teacher's mathematical skills as well as their didactical approach. Whereas MMTsm works with the concept of meaning. Thompson's definition of meaning will be elaborated below. The reason for focusing on meaning rather than knowledge and skills is obvious for Thompson: "In one sense, the issue of meaning is irrelevant to mathematics education—if we accept the current state of mathematics education. It is rare for a mathematics teacher, at least one in the United States, to be concerned with meaning, either intended or conveyed. If we believe the results of TIMSS classroom studies (e.g., Hiebert, Stigler, Jacobs, Givvin, Garnier, Smith, Hillingsworth, Manaster, Wearne, & Gallimore, 2005; Schmidt, Houang, & Cogan, 2002; Schmidt, Wang, & McKnight, 2005; Stigler & Hiebert, 1999), the main goal of most U.S. mathematics teachers is that students learn to perform prescribed procedures. Issues of meaning are largely irrelevant. But if we intend that students develop mathematical understandings that will serve them as creative and spontaneous thinkers outside of school, then issues of meaning are paramount.(P. W. Thompson, 2013)"

This change in focus means that, when used in comparative context, this instrument can be expected to give a different perspective than more classical instruments would. Even though Thompson's instrument is not intended to be an instrument for comparison, comparative studies have been made. These have been made by Thompson and his colleges and the limits of the instrument have been considered. This study was made to compare educators from the United States and South Korea(P. W. Thompson, 2015). Alongside this the instrument has been translated to Norwegian and Danish but no publications have been made along with these translations.

As a last relevant point, this is the only one of the instruments which does not attempt to measure teachers' pedagogical or didactical knowledge. Thompson is developing another instrument which will be used to assess the quality of instruction in secondary mathematics classrooms. As I have not seen this instrument there is no saying what it contains.

But it might concern itself with teachers' didactical praxeologies. An element which the MMTsm lack in contrast to the MKT and TEDS-M instruments.

Theoretical framework

The framework for Thompson's instrument differs considerably from the framework of Ball and TEDS-M. The first thing which I shall clarify is Thompson's definition of the word *meaning*. Since, for Thompson this is the interesting thing to study when working with developing and studying teacher's professional development.

Thompson believes that understanding meaning rather than skills are the key to develop a better educational system within mathematics. But, why do we need to look at teachers mathematical meaning when assessing them? Could we not simply look at their mathematical performance, if they can perform mathematically well can't we assume that the meaning they attribute a mathematical object is the correct one? According to Thompson; *Understanding what people mean gives more insight into their thinking than does understanding what they believe to be true.* And; *[The shift from studying MKT to MMT] is essentially from a philosophically mainstream view of knowledge as justified, true belief and about things external to the knower to a Piagetian perspective in which meaning and knowledge are largely synonymous, and both are grounded in the knower's schemes. This shift allows us to move, for example, from asking what teachers know about equations to what teachers mean by an equation.(Patrick W. Thompson, 2016)* Hence, for a meaningful insight what teachers possibly convey to their students we need, according to Thompson, to be looking at the mathematical meaning they have. But what is to be understood by the word *meaning*? Thompson's definition of meaning roots in the idea of Paget's assimilation to schemes, he writes:

> *Standard meanings of "assimilate" all entail some sense of something being absorbed by something else. As Piaget famously said, "A rabbit that eats a cabbage doesn't become cabbage; it is the cabbage that becomes rabbit—that's assimilation.(P. W. Thompson, 2013)*

And he goes on explaining how this is interpreted, this assimilation is to be thought of as a cognitive action. Hence it is knowledge which is being transferred from one person to another. The person receiving it, being the rabbit, and absorbing the knowledge, such that the knowledge is incorporated into the person's own scheme. Now, it may sound as if meaning is something for the individual to develop and hence, meaning cannot be categorized as right or wrong. This is not the case, *"Is meaning on a printed page? Written on a whiteboard? Does it appear on a computer screen? Is meaning conveyed to students by directing their attention to "real world" referents? Each of these stances puts meaning in the world, so that there are "correct" meanings to be had and any meanings that depart from them are incorrect." (P. W. Thompson, 2013)* Thus, the meanings which do not correspond, in the case of mathematics, to the meaning of the rest of the mathematical society, or the meaning of the institution which the individual acts within, is incorrect. In his article from 2016 Thompson gives the example of the meaning of the word “over” in a mathematical context. In the MMTsm instrument an item for investigating this is given:

A college science textbook contains this statement about a function *f* that gives a bacterial culture’s mass at moments in time.

The change in the culture’s mass over the time period Δx is 4 grams.

Part A. What does the word “over” mean in this statement?

Part B. Express the textbook’s statement symbolically.

Figure 7; item from the MMTsm-instrument, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

According to Thompson there are two classical interpretations of the word in this context.

1. One believe that the word “over” refers to the placement of the symbols when calculating the average growth during a time period (growth/time).
2. The event of two things happening simultaneously.

As for description of this item I will let Thompson speak for himself:

“The purpose of Part A in Figure 1 [Figure 8] was to have teachers commit to a meaning of “over” in a context where, when interpreted normatively, it means “during”. The purpose of Part B was to give teachers an opportunity to show how they interpreted the context in which “over” occurred while expressing it symbolically. Since the statement is about a change in mass, the symbolic representation of it should reflect a change in mass that happened as time passed from one moment in time to another. Since the function f gives the culture’s mass at moments in time, and since the change in time is represented by “Δx”, one representation of the change in mass would be $f(x0 + \Delta x) - f(x) = 4$ or $f(x0 + \Delta x) - f(x0) = 4$, where x0 refers to a specific moment in time.”(Patrick W. Thompson, 2016)

Looking at the scoring rubric (figure 9) for this item we see that Thompson have taken these two meanings to be the most interesting, though it is not clear whether these are the only meanings attached to the word. However, what we can derive from the rubric these two meanings are in the highest three levels and a fourth level is set for all answers which does not reflect these two possibilities. Hence, these two are the only ones regarded as acceptable to Thompson.

Part B, as Thompson writes, are there to see whether teachers, which have the desired meaning of the word "over" will likewise be able to express the statement symbolically. I.e. does the meaning they state, fit the mathematics they present in part B?

Level A3 Response:	The response conveys that "over" means "during," or otherwise refers to the passage of time while the culture's mass is changing.
Level A2 Response:	The response conveys the meaning of "over" as the equivalent of "elapsed time" or "amount of time".
Level A1 Response:	The response conveys that "over" means division, i.e. "*a* over *b*" means "divide *a* by *b*".
Level A0 Response:	***Any*** of the following: – The response does not fit a higher level. – The scorer cannot interpret the response. – The response consists of scratch work with no clear indication of a final answer. – The response does not address the prompt; that is, the response is off-topic (see Purpose and Rationale). – The page contains no work, but does contain at least one mark to suggest that the teacher saw this item.
IDK Response:	The response consists only of "I don't know", or something equivalent that suggests that the teacher is unsure of how to respond. If the teacher stated uncertainty and gave an additional response, score the response ignoring the uncertainty.
X Response:	The page is completely blank.
Level B3 Response:	***Any*** of the following: – The teacher represented the difference of 4 grams in the culture's mass at beginning and end of a time period. If the response contains a variable other than *m* or *y* to stand for mass, it *must* be defined. – The teacher presented a graph whose symbolic equivalent would fit the first bullet.
Level B2 Response:	***Any*** of the following: – The teacher represented a change in the culture's mass, but does not refer to the passage of time. – The teacher wrote a quotient that is equivalent to representing a change in mass (e.g., $\Delta m/\Delta x=4/\Delta x$ or $m/\Delta t=4/\Delta t$).
Level B1 Response:	The response does not fit Level B2 and contains a quotient or an algebraically equivalent statement (e.g., $m/\Delta x=4,\ m=4\Delta x$).
Level B0 Response:	***Any*** of the following: – The response does not fit a higher level. – The scorer cannot interpret the response. – The response consists of scratch work with no clear indication of a final answer. – The response does not address the prompt; that is, the response is off-topic (see Purpose and Rationale). – The page contains no work, but does contain at least one mark to suggest that the teacher saw this item.
IDK Response:	The response consists only of "I don't know", or something equivalent that suggests that the teacher is unsure of how to respond. If the teacher stated uncertainty and gave an additional response, score the response ignoring the uncertainty.
X Response:	The page is completely blank.

Figure 8; Scoring rubric for the item shown in figure 8; © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

According to Thompson, the teachers' meanings are what is conveyed to their students. The transitions of these meanings i.e. the learning situations are explained by Thompson as a two-sided situation where the teacher and student both must consider the intention and meaning which the other part has. The transfer of mathematical meaning from one person to another there are things which should be considered. Thompson's idea of conveying meaning he expresses in the figure below:

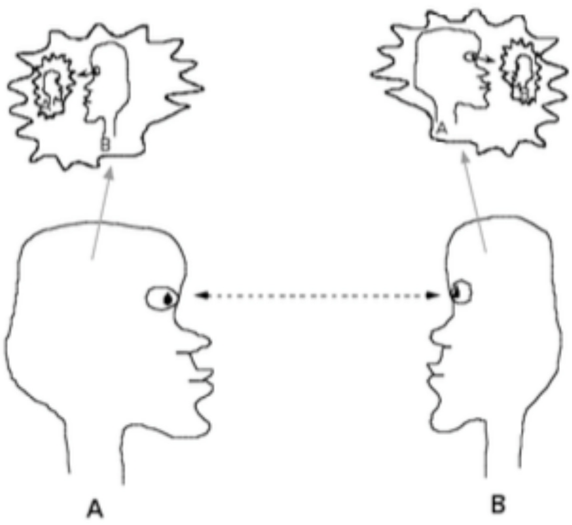


Figure 9; Thompon's model of conveying meaning

Persons A and B attempting to have a meaningful conversation. Person A intends to convey something to Person B. The intention is constituted by a thought that A holds that he wishes B to hold as well. The figure shows A not just considering how to express Persons A and B attempting to have a meaningful conversation. Person A intends to convey something to Person B. The intention is constituted by a thought that A holds that he wishes B to hold as well. The figure shows A not just considering how to express his thought, but considering how B might interpret A's utterances and actions. It is worthwhile noting that A's action towards B is not really towards B. A's action towards B is towards A's image of B. In a sophisticated conversation, A's action towards B is not just towards B, but it's towards B with some understanding of how B might hear A. Likewise, B is doing the same thing. He assimilates A's utterances, imbuing them with meanings that he would have were he to say the same thing. But B then colors those understandings with what he knows about A's meanings and according to the extent to which A said something differently than B would have said it to mean what B thinks A means. B then formulates a response to A with the intent of conveying to A what B now has in mind, but B colors his intention with his model of how he thinks A might hear him, where the model is updated by anything he has just learned from attempting to understand A's utterance. And so on.(P. W. Thompson, 2013)

Thus, for a person to convey one's own meaning to another, you must consider the status of the receiving part. The person receiving the new information must try to place themselves in your mind-set and try to understand your perception of them, and how this might influence how you explain concepts. This process Thompson determ as *reciprocal assimilation*.

Thus, for Thompson there is a great difference in the definition of meaning, knowing and understanding. He presents the following scheme to summarize the assimilation process:

Construct	*Definition*
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning. What Harel previously called *Way of Understanding*
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

Figure 10; phases of the assimilation prosses (Hatfield, 2014)

Methodology

So, how does an instrument for measuring mathematical meaning develop? And how does Thompson and his team develop scoring rubrics for the items?

"To assess teachers' mathematical meanings for teaching requires that the assessment designers have a theory of the meanings they intend to assess."(Patrick W. Thompson, 2016) But how does this theory come about? Thompson explained at his seminar in Kristiansand, Norway in the fall of 2016 that these categories of meanings was developed the following way. Developers' own ideas of which meanings might occur was illustrated or expressed in other ways, these ideas could come from their own meaning of a given phenomenon or experience from prior research. These illustrations were then taken out to groups of mathematics educators and interviews of their meaning on the given topic were taken after a presentation of the illustrations. These understandings would then either fit into one of the already constructed categories presented by Thompson's team or might be completely new to the researchers. If a new meaning was presented and accepted this was taken along to the next trial.

An example of mathematical meaning could be the meaning of continuous variation. What does it mean for a variable to vary continuously? Thompson has the following proposal:

Meanings of Continuous Variation	
Level	*Description*
Smooth Continuous Variation	The individual thinks of variation of a variable's value as the variable's magnitude increasing in bits while anticipating simultaneously that within each bit the variable's value varies smoothly.
Chunky Continuous Variation	The individual thinks of variation of a variable's magnitude as increasing by intervals of a fixed size. The individual imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 to (and so on). Values between 0 and 1, between 1 and 2, between 2 and 3 "come along" by virtue of each being part of a chunk, but the quantity does not have them as a value in the same way it has 0, 1, 2, etc. as values. Chunky continuous variation is *not* just thinking that increases happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on.
Discrete Variation	The individual thinks of a variable as taking specific values. The individual sees the variable's value changing from *a* to *b* by taking values $a_1, a_2, \ldots, a_n$, but does not envision the variable taking any value between a_i and a_{i+1}.
No Variation (NV)	The individual envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.
Variable as Letter (VL)	A variable is a letter. It has nothing to do with variation.

Figure 11, Meanings of continuous variation (Patrick W. Thompson, 2016)

Where, what is by Thompson, perceived as the most productive meaning is at the top row and the least productive at the bottom.

Now that a set of possible meanings for a given topic has been defined, development of items for assessing the topic can begin. Item writers attempt to design items such that respondent answer in a way which will reflect their meaning instead of what they believe the items asking for – i.e. the item writers attempt to imagine how respondent might read the item and consider this when choosing the wording(Patrick W. Thompson, 2014). Thompson describe the overall design process: A draft item is proposed and then tested among a small test group which is then interviewed to see whether the response fits what the test persons describe as their perception/meaning of a given topic in the interview. The item is also revised by a panel of mathematicians and math. educators. The item is now edited if needed and the interview are repeated if necessary. Now a trial is run and responses are analysed. Items are discarded if they don't meet the benchmark by this trial. This process of trial, interviews and sorting are repeated and finally scoring rubrics are developed.(Patrick W. Thompson, 2014) (for full description see appendix)

Thoughts and open questions

After studying articles by Thompson I am still left somewhat puzzled. To clarify how I have understood Thompsons ideas let's try to explain them in terms of ATD. Thompson has a very physiological approach to learning theory which is explicit as he refers and quote Piaget in many of his articles. Hence, in terms of ATD Thompson, it seems, would say that everyone would have his/her own praxeologies, contrary to the theory of ATD itself where praxeologies belong to institutions. Thus, the students are to assimilate their praxeologies to that of their teachers, which by no guarantee is like that of the institution which they act within.

Methodology

When framing the instruments in terms of ATD and describing their content in terms of task technique etc. a description of what is found in each of the selected item is presented. Hence, if I don't see a mathematical task in the item this will not be described – it will be described if the mathematics needed to solve the item is above the level of preschool. But for some items it is assumed that any respondent will have no problems with the mathematics included.

Research questions

Research question one reads: *What didactical and mathematical tasks, in terms of ATD, are contained in the items?* To answer this question an analysis of selected items from the different instruments will be executed. This will be done in terms of ATD which is elaborated in the section "analysis of an item". Afterwards a summary of the different task types and techniques are presented along with reflections of the results.

After describing the items using the ATD analysis they will be presented in their respective framework to clarify how the items refer to the framework in which they have originated. This part of the analysis is made to answer research question two: *How do the three frameworks theorize knowledge for teaching mathematics? And how do the items relate to the theoretical framework in terms of measuring teacher knowledge.*

In the summary of the analysis, the structure of the items will be compared using the ATD analysis. This part of the summary will result in the answer to research question three: *Looking at the didactical and mathematical items, how do the three instruments differ and agree?*

Analysis of an item

First an item is categorized with respect to whether it is a mathematical, didactical or a mix of the two. This is done by attempting to create a mathematical question posing the exact same content. If this can be done without loss of any nuances an item will be categorized as mathematical. An item cannot be categorized as didactical in the same matter as the didactics will not stand alone. Hence didactical tasks (and DOs) are co-determined with the MO and will always have a mathematical content. A purely mathematical task could be the following from TEDS-M:

A class has 10 students. If at one time, 2 students are to be chosen, and another time 8 students are to be chosen from the class, which of the following statements is true?

Check <u>one</u> box.

A.	There are more ways to choose 2 students than 8 students from the class.	$\square_1$
B.	There are more ways to choose 8 students than 2 students from the class.	$\square_2$
C.	The number of ways to choose 2 students equals the number of ways to choose 8 students.	$\square_3$
D.	It is not possible to determine which selection has more possi-bilities.	$\square_4$

Figure 12, item from TEDS-M instrument (Blömeke, 2013)

The fact that the question is framed in terms of class rooms and students does not make this thin item more didactical than any other typical mathematics task. This item could might as well have been phrased: *A bag contains ten different marbles. Does there exist more way to choose two or eight different marbles from the collection?* And hence, the item can be categorized as mathematical. Items for which we cannot do this is items like the following:

A mathematics teacher wants to show some students how to prove the quadratic formula.

Determine whether each of the following types of knowledge is needed in order to understand a proof of this result.

Check <u>one</u> box in each <u>row</u>.

		Needed	**Not needed**
A.	How to solve linear equations.	$\square_1$	$\square_2$
B.	How to solve equations of the form $x^2 = k$, where $k > 0$.	$\square_1$	$\square_2$
C.	How to complete the square of a trinomial.	$\square_1$	$\square_2$
D.	How to add and subtract complex numbers.	$\square_1$	$\square_2$

Figure 13, item from TEDS-M instrument (Blömeke, 2013)

To give a proper response to this item the teacher needs to be able to prove the statement themselves, which is a mathematical task. But the item does not require a proof. It requires a reflection as to which mathematical techniques student's needs to complete the proof. Hence, we cannot rewrite this item into a purely mathematical task without loss of content. Even though we are not told anything about the students in this item either, the reflections of which techniques are needed to solve certain mathematical task is not a part of the MO itself.

Which mean that the item lies within the practical block of the didactical organization. In terms of Ball we would be in the sector of *Knowledge of Content and Student*.
An example of an item which lies between the purely mathematical and predominantly didactical items and which is found more challenging to categorize could be the following:

Mrs. O'Neill gave this problem to her students.

> Inger Miller ran the 200-meter dash in 1999 with a time of 21.77 seconds. Alice Cast ran the 200-meter dash in 1922 with a time of 27.80 seconds. Suppose that they ran their races against each other. Approximately how many meters would have been between them when Inger Miller crossed the finish line?

Tanya explained her reasoning this way:

> Assume they ran at constant speeds. 21.77 is about 78% of 27.80. So Alice would have been about 44 meters behind Inger.

Explain to the class what 21.77 being 78% of 27.80 has to do with solving this problem.

Figure 14, item from MMTsm instrument, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

To answer this question the respondent must convey information to students. Even though we are not told much about the student's the respondent is given an elaborative student answer. This does reveal some information about the level of the student's, at least if we let the rest of the students be at approximately the same level as Tanya. And thus, it is possible that the respondent might consider how to convey the mathematics appropriately. And hence, it is not possible to ask this question only in terms of mathematics. On the other hand, the respondent must assess part of his own MO to follow the thoughts of Tanya and the item has the potential of assessing the respondent MO as well. Some items are separated into two part which can be of different character. In case of this, an analysis will be made of the two parts separately.

Now we need to look at whether the items are within the theoretical or practical block in terms of ATD. The mathematical items, or the mathematical part of the items, is categorized simply by looking at which mathematical skills are needed to give a proper answer,
The item will be described in terms of ATD e.g. the task, technique, technology and theory associated with the item will be stated, and in cases where not clear explained. In some items, especially the open-ended items, a proposition for an answer is given to clarify how task and

techniques associated with an item has been obtained. To exemplify this description of an item in terms of ATD we'll look at an example which is a pure mathematical item:

The equation $3x^5 - 2x^2 + 4 = 0$ has $x = -0.942$ as a solution.

What is a solution to $3(x+1)^5 - 2(x+1)^2 + 4 = 0$?

Figure 15, item from the MMTsm instrument, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

T_i: Find a solution to the equation given

τ_i: Use the method of substituting variables – Thus if we substitute x by y = x+1 the equation will still have the same solution and hence: $y = -0.942$ giving $x + 1 = 0.942 \iff x = -1.942$

Θ_i: Justification for why we can substitute the variables – substituting variables does not change the expression as long as the transformation is a bijection.

θ_i: Theory on linear transformations homeomorphisms.

This description of the content is now used to place the item in either the practical or theoretical block. This placement is determined by the elements which the respondent need to answer the item. If the respondent need the practical elements the item is placed in the practical part and likewise in the theoretical part if these are the element needed by the respondent. In the example above (see fig 16) the respondent will need the only the technique, not the technology and hence the item would be placed in the practical part of the MO.

Analysis of teachers' didactical theory are not as easily done as of their mathematical theory. Analysing teachers' theoretical DO might not be completely impossible, though many believe that one needs to be along the teacher as they prepare or evaluate on their own teaching practice to assess this. The categorization of an item within the DO will in the following analysis be made like that of the MO. The item will be described in terms of ATD and the simplest possible tool which is needed for the respondent to answer the item will be the category of the item.

An item which lie within the practical block of the DO could be that in figure 14. As we note here the respondent is asked to address which techniques and technologies must be accessible to his student for them so solve a specific task. He is not asked to justify his answers nor is he asked to explain why his considerations are necessary for him to convey the mathematics to his students. A theoretical didactical question might be: *In what order should the different arithmetical operations be introduced? Explain your considerations.*

Here the respondent must explain how he resulted with the techniques which he states as an answer to the first question and hence, we are addressing the respondent's technology.

When an item has been categorized it will be placed, according to the categorization, in the model shown below:

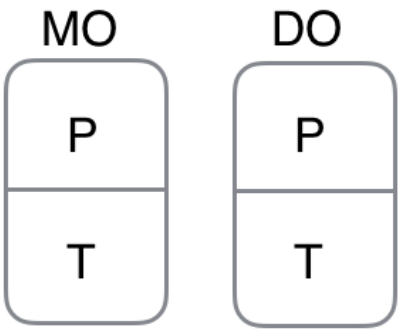


Figure 16; The two boxes represent the mathematical and didactical organization and the P and T in each represent the theoretical and practical block respectively. Each item selected will be placed in a category in terms of ATD

After the ATD analysis of an item the item will be analysed in the framework in which it is originally presented. This will be done using the results from the ATD analysis. With these results in mind the item will be placed and described in the original framework. In the cases where the items have been categorized by the author's themselves this categorization will not be determining where the items will be placed in my analysis. The author's categorization will be compared to my placement as a reference as to whether the item follows the original framework or not. For analysis of the MMTsm items placing it in the framework of Thompson is challenging as each mathematical concept has its own set of meanings defined additionally these definitions of different meanings is not available. So instead of trying to extract these meanings from the items the analysis of Thompsons items will clarify how Thompson is attempting to extract the meaning i.e. what parts of the ATD components are focused on in the items as well as the scoring rubrics.

For the MKT-items the analysis in Balls framework will place the item within the MKT-model presented earlier using the results from the ATD analysis and then compared to Ball's own categorization. The same will be done with the TEDS-M-items using the cognitive domains for the mathematical items and the framework presented in table 3 for the pedagogical/didactical items.

Selection of items for analysis

Each item taken for analysis is chosen based on the mathematical or didactical content of the item. The content is chosen based on what topics is represented in the instruments as well as the overlap of topics in all three. Three sets of items have been chosen containing three items each giving a total of nine items. The triplets are made up of items which has at least one common feature.

After a preliminary analysis, the items where "re-chosen" hence a second evaluation of the choice was made to ensure that the triplets still fell into similar categories either in terms of didactical or mathematical content. The preliminary analysis consists of the first categorization of the content which is described above, e.g. the categorization of whether an item is mathematical didactical or both. Each of the three instruments consist of different ratios of multiple choice items. In the instrument by Ball all items are multiple choice whereas the TEDS-M and Thompson instrument where both a mix of multiple choice and open ended questions. Therefor the items selected from the two instruments lastly mentioned is a mix of these type of items as well. Since only three items are taken from each instrument for analysis the exact ratio of multiple choice items cannot be represented fully. Below the three triplets are presented. The items contained in the triplet is described shortly in the beginning of each subsection followed by a description of the commonalities which make the selected items the triplet. The full version of the items can be found the section *Framing the instruments*.

1. *Triplet*

Item 1: An open-ended question where the respondent is asked to analyse and identify possible misconceptions in a student's work on solving equations with one unknown variable.

Item 2: A multiple choice item presenting different options for a student's misconception which is presented with an example of the students work.

Item 3: This item consists of two parts. The first part is a multiple-choice item where the respondent must evaluate a student's response to a given task. The second part is an open-ended question where the respondent is asked to outline how the student has arrived at this conclusion.

These three items have been chosen to constitute the first of three triplets. Each of these items is, at first glance, tasks of evaluating student performance. Hence these items are chosen based on their didactical content. As we shall see in the following analysis the item from Thompsons instrument is evaluated as a mathematical item. Despite this we shall keep it here among the didactical items. This is done since the item, after the preliminary analysis, still shows potential of a didactical content. The formulation of the items is also similar in the items from Ball and TEDS-M. Hence these two items are more alike than the MMTsm-item. Hence these two items are both asking for the respondent's thoughts on what the imaginary student might have thought of. Whereas the MMTsm-item are merely asking for the mathematical error which might occur. These three items all engage in different mathematical topics. They are concerned with positional system, representation forms and equations/algebra respectively. Two items from the TEDS-M instrument were considered for this triplet (for discarded item see appendix). The one which was discarded was also concerned with student misconception. In this item, the respondent is asked to offer a possible student misconception along with a drawing which should help the student correct this misconception. Since this structure is not seen as often as the structure of item 3 it was discarded.

2. Triplet

Item 4: The item present a situation where the slope of a line is calculated to be 3.04 and the respondent is asked to convey the meaning of 3.04 to a group of students.
Item 5: A multiple-choice item where a group of student present different explanations as to why $a - (b + c) = a - b - c$. The respondent must now choose the explanation they find best.
Item 6: A multiple-choice item: Three different proofs are presented and the respondent must evaluate, for each proof, whether it is valid or not.
The coherence between these three items are the technique used to investigate the mathematical knowledge of the respondent. The MKT and MMTsm are here the two items closely related. Initially another item from MMTsm where chosen to be the item used. But after the preliminary analysis, this item was replaced by the one above as it turned out not to have the format first assumed (see appendix for discarded item). The MKT and MMTsm items both evaluate the items as mathematical items. For Thompson, this is the case for all items, but Ball has items which is by her categorized either as pedagogical or mathematical. This item she categorizes as a "middle school content knowledge item". For the TEDS-M item, this is categorized as a "Mathematikkdidaktik"-item. But since all the TEDS-M items which is categorized only as mathematical items are always presented as so I have chosen to include this item here. This decision was made since this item demand a good mathematical understanding of proofs and different techniques and hence, this item has a high mathematical content despite the TEDS-M own categorization. Hence, this item can be categorized as a mathematical item, this argument will be elaborated in the analysis in the following chapter.

Hence these three items are presented as didactical items where respondents are asked to evaluate students answers but the purpose of the items are to assess the respondents' mathematical knowledge. Aside from this the MKT and TEDS-M item are both multiple choice, the first MMTsm-item also has an element of multiple choice but here the respondent is expected to explain their choice. The final MMTsm-item is an open-ended item and hence, the response form is not a common factor for the selected items.

3. Triplet

Item 7: A multiple-choice item. The respondent is presented with a situation and must determine the correct ratio given different options.
Item 8: A multiple-choice item. The respondent is presented with four different situations. All situations are concerned with two different ways of measuring and in each case the respondent must determine whether the ratio stay the same when measured in different measurements.
Item 9: This item consists of two parts, both open ended. In the first part the respondent must solve two tasks concerned with ratio. The two tasks pose the same problem in different wording. In part two the respondent must explain why one task is more complicated than the other.

In the third triplet, the mathematical content was the primary focus for choosing the items. The three instruments all treat a various selection of mathematical topics. The three items above all focus on the topic ratio/proportionality. This topic was chosen since many items in Ball and Thompsons instruments are concerned with these topics. In the items, available from the TEDS-M program there is no topics which is obviously represented stronger than other and the choice of topic therefore landed on ratio. In the TEDS-M item ratio might not be obviously included, but since this is included in the evaluation of solutions for part an I have chosen to accept this item. Besides from this I wished to have one pure multiple choice item from Thompson so that these would be represented as well. Since this was the only item from Thompson which both represented the multiple-choice format as well as a well-represented topic the choice of ratio was made, despite a compromise on the TEDS-M item.

Framing the instruments

In the following section, an analysis will be made of the selected items introduced above. The analysis will follow the description given in the methodology section along with any new considerations the analysis of a given item might has given rise to.

Item 1, MMTsm

Mr. Abri posed this problem to his students.

Solve for x in $xn - 2x = xn + x$.

Baruti submitted this work:

$$xn - 2x = xn + x$$
$$x(n-2) = x(n+1)$$
$$\frac{x(n-2)}{x} = \frac{x(n+1)}{x}$$
$$n - 2 = n + 1$$
$$-2 = 1 \text{ Contradiction!! No solution.}$$

Explain Baruti's mathematical error, if there is one.

Figure 17, Item 1, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

In this item, the respondent is not told anything about Baruti and additionally are not asked to explain his mistake to Baruti himself but as to anyone. Thus, the didactical element of considering the students abilities are not present any longer. Hence, this item is categorized as a mathematical item. The desired answer for this item yields that the respondent notice that division by x is not possible without noting that x in that case must be different from 0, and that x= 0 is a solution to the equation. Hence, the practical block of this item would be:

T_1: Solving first degree equations.

τ_1: Calculating with letters and isolating a variable.

And the theoretical block:

θ_1: Explanations of how/why operations are done simultaneously on the left and right side of the equation. Including explanations as to why not all operations can be executed on both sides of the equality.

Θ_1: Explanation of why division by variables are limited by certain restrictions along with a justification of why division by 0 impossible along with the algebra of the basic algebraic operations. E.g. what it the theory of the distributive and associative law etc.
Since the respondent is asked to explain the problem of dividing by a variable we are beyond the level of techniques and we can place the model in the theoretical block of the MO. Had the respondent only been asked to state his mistake and not explain it the item would have been at the practical level. Thus, in the model the item is placed likewise:

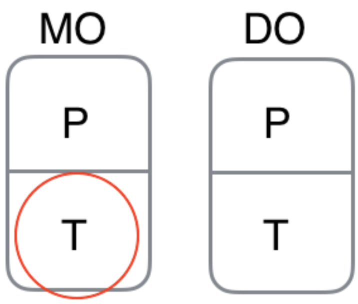


Figure 18: MMTsm item 1, 1. Triplet placed in reference model

Framing this item in the original framework we'll first try to do so through the ATD analysis above. In this case, we are looking at the task of solving equations and the techniques of algebraic manipulation. Hence, Thompson seems to be looking for the respondent meaning on the algebraic manipulation of dividing by variables. Which meanings are associated with different answers and which meanings are perceived as the most productive we will need the scoring rubric to determine:

Purpose:
The goal of this item is to see if teachers can identify and explain a non-equivalence-preserving transformation (in this case, dividing by x without considering if $x = 0$).

Rationale:
Baruti's work is correct except for one conceptual slip. When he divided both sides by x he should have stated the condition $x \neq 0$. So there indeed is no solution when $x \neq 0$. But $x = 0$ is a solution. We added "mathematical" so that when teachers say "He should have solved it differently" we have greater assurance that they missed the mathematical error. Additionally, we included the prompt to explain the error's significance as we found that teachers' explanations gave us better insight into their meanings than just identifying Baruti's mistake.

Figure 19: Scoring rubric for item 1, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

So, the purpose of item is as the ATD analysis indicate. In the rationale above we note that it is stated that the explanation is included to access the teachers meaning or technology as it is noted in the ATD analysis. So, which responses can be expected for this item? Let's take another look at the scoring rubric:

Level 3 Response:	***Any*** of following: – The response is consistent with explaining that Baruti's work eliminates a solution to the equation. – The response is consistent with explaining that Baruti's work does not assume that $x \neq 0$.
Level 2 Response:	***Any*** of the following: – The teacher wrote that Baruti's error was to divide by the variable *x*. – The teacher wrote that Baruti's error was to divide by zero.

Figure 20, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

Here is an example of two types of answers which are closely related but give rise to different meanings. The level three responses indicate that the respondent has understood what consequences the division of zero has in this example – the solution is eliminated. This coincide with what is categorized as reasoning in the cognitive domains by TEDS-M which is again like the theoretical block of the MO.

The second level responses simply state Baruti's error – they do not explain them. This could that the respondent has not fully understood the error Baruti made. I.e. the respondent knows that division by 0 or a variable is problematic but cannot explain why. This response would only suffice to answer the question: *What is Baruti's error?* And then the item would, in terms of ATD be placed in the practical block of the MO.

Item 2, MMTsm

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. Convey to Mrs. Samber's students what 3.04 means. **Part B.** Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04. A student explained the meaning of 3.04 by saying, "It means that every time *x* changes by 1, *y* changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if *x* changes by something other than 1?" What would be a good answer to Mrs. Samber's question?

Figure 21, Item 2, © Patrick Thompson, University of Arizona

This item is, in the instrument, presented on two pages. On the first page, Part A where respondent is supposed to answer the first question: *"Convey to Mrs. Samber's students what 3.04 means"* and part B which replicate the part A and give a hypothetical student answer. Which the respondent is now asked to respond to.

An example of a proper answer from the respondent to item 2 could be:

Part A: 3.04 means that when x increase by one y increases by 3.04

Part B: It still means the same – so whenever x changes by an amount, let's say n y changes by $n \cdot 3.04$

The mathematics contained in this item can be descripted in terms at ATD as such:

T_2: Calculate the slope of a line

τ_2: Divide the two changes in values - This mathematical technique which could have been present in this item has already been presented and solved in the text for the item.

θ_2: Explain the result of the given calculation – what does the result represent?

Θ_2: Explanation of why dividing these given numbers result in the change in y when x changes by one and not e.g. two. Relation to the calculation of other, non-linear slopes.

For part A of this item the respondent is asked to convey a piece of information to students. This might look like a teacher task, and hence, a didactical task. But since the respondent is not given any information about the students, other than that they are at an introduction class, the respondent cannot make profound didactical considerations as to how the MO should be conveyed. Hence, the teacher is asked to give a direct instruction, i.e. their own interpretation of the concept slope. Thus, the item could have been posed the following way:

To calculate the slope of a line, 8.2 is divided by 2.7 getting 3.04. What does 3.04 mean?

Hence this part of the item can be categorized as mathematical.

Part B: Here the respondent is placed at the role of the student. Since we have not been given any new information regarding the students the respondent still does not have the possibility of considering different student answers and evaluating these based on the level of the students and he is again left with the option only to give his/her best possible answer. The information that it is a student who answers the first question does not change the character of the question. Without any loss of content the second part of this item could have been reduced to Ms. Samber question and the peeled version of the item would look as so:

"Part A: The slope of a given line is 3.04, what does 3.04 mean?

Next page -

Part B: What does it mean if x changes by anything other than one?"

And like part a part b is categorized as mathematical, thus the entire item is a mathematical one. The respondent is not asked to do any calculations but they are asked to explain the outcome of the calculations. Though they are not asked to explain why the algorithm works or in which cases. The interpretation of results lie within the technology cf. the ATD content description. And hence, the item can be placed in the theoretical block of the MO in the reference model:

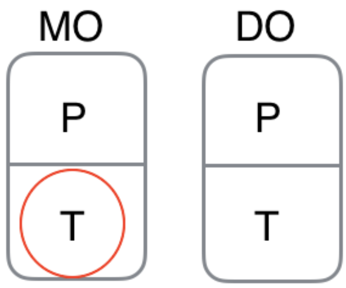


Figure 22: MMTsm item 2, 2. Triplet placed in reference model

In terms of meaning, the ATD analysis give us the impression that item 2 is studying the respondent's technology. What meanings Thompson extract from these technologies i.e. the respondent's explanation as to what the calculation given is representing. Is not clear from the ATD analysis. To clarify this, we need again to look at the scoring rubric for the item:

*"**Purpose:** The purpose of Part A is to characterize the meaning of slope teachers would teach their students. The purpose of Part B is to see if teachers explained a meaning of slope that explains the relationship between any sized change in x and the associated change in y.*

Rationale: *Part B anticipates that many teachers will have answered Part A in terms of Δx being 1, thus putting them in the place of the student described in Part B. Being able to envision that x can change by any amount (especially tiny amounts) allows the teacher to think about input values varying continuously instead of in jumps of a fixed amount, and thus to convey that same meaning to students." (Project ASPIRE: Patric W Thompson)*

So, according to Thompson, part A is used to analyze the technology of the teacher, which is then assumed to be the technology they convey to their students. These technologies are described and ranged in different levels in the scoring rubric but in terms of ATD there are none of the levels which exceed the level of technology. Hence, we can conclude that the ATD analysis coincide with the intended content of the item.

For part B, Thompson state that the respondent is invited to think of x as a variable variating continuously. In the ATD analysis this change of the variable is not clear. I.e. In the ATD analysis the changing role of the respondent did not change the character of the question. And for part B it must be concluded that the ATD analysis does not agree with the intended purpose of the item.

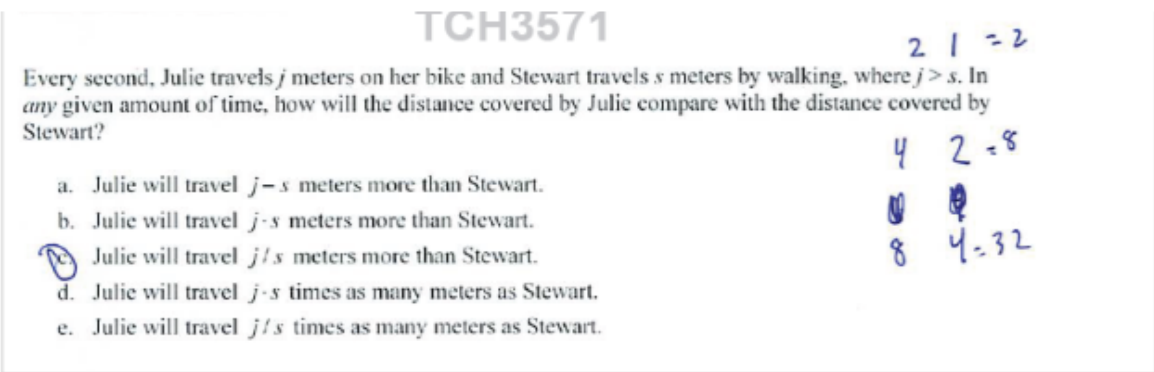
Item 3, MMTsm

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.
f. I don't know.

Figure 23, Item 3, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

As this is a multiple-choice item potential responses are obvious. But, it would not be surprising to see respondents doing calculations before answering as an aid to finding the correct answer. This is seen e.g. in the example of an answer in the scoring rubric for item 3:


TCH3571

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.

Figure 24, extract from the scoring rubric for item 3, © 2016 Arizona Board of Regents, Project Aspire, P. W. Thompson, PI. Used with permission.

Even though these calculations are not considered when scoring the item with Thompsons rubrics they will be considered as a potential technique in the ATD analysis of the item since they are explicitly seen as a used technique in the example above. I terms of ATD the following element can be associated with item 3:

T_3: State which expression describe the relationship between the two variables. e.g. how are the proportionality or ratio between the two?

τ_3: Calculate different scenarios to get an impression of the relationship. And from these calculations the respondent can generalize into a general expression.

θ_3: Knowing what different types of ratios or what kind of proportionality describes different given situations.

Θ_3: Theory of proportionality and ratio and these subjects use in mathematical modelling.

The wording of the item emphasizes the difference between "as many times" and "more" which indicate that the researchers expect this focus of the topic to be confusing to respondents. Whether this is due to empirical data is unknown. As mentioned earlier Thompson is not including possible techniques directly in the evaluations rubric. But looking at the comments for the different answers in the rubric it is evident that the technology which the respondent is the main-focus of item. For example, for the highest scoring option the following comment is attached: *"[the respondent selected e] This response suggests an awareness of the proportional relationships described in the rationale. Not only did the teacher select the quotient j/s, but the wording also suggested thinking of j/s as multiplicative comparison of two quantities as opposed to additive comparison." (Patrick W. Thompson, MMTsm)* This comment suggests that Thompson and his team, in constructing these rubrics, are associating certain techniques to a specific technology which is formulate by the research team as opposite to the respondent.
So, the rubric is constructed in such a way that it, without access to the scoring rubric (marked with red in fig 26) would place itself in the practical block of the MO in our model. But considering the comments available in the scoring rubric (marked with blue in fig 26) it places itself in the theoretical block of the MO:

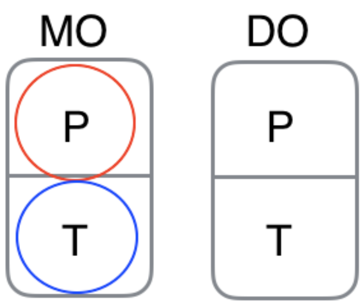


Figure 25, MMTsm item 3, 3. Triplet placed in reference model

As the comment from the scoring rubric indicate Thompson associate a specific meaning with the different answers. In case the respondent answered (e) they think of the ratio as a multiplicative proportional one, if the respondent chose (c) they have the proportionality right but think additively instead of multiplicatively and so forth for each option(Project ASPIRE: Patric W Thompson). How this form a full description of the respondent meaning on proportionality and ratios are unclear. This might be clarified if descriptions of the different meanings associated with proportionality were available.

Item 4, MKT

12. You are working individually with Bonny, and you ask her to count out 23 checkers, which she does successfully. You then ask her to show you how many checkers are represented by the 3 in 23, and she counts out 3 checkers. Then you ask her to show you how many checkers are represented by the 2 in 23, and she counts out 2 checkers. What problem is Bonny having here? (Mark ONE answer.)

a) Bonny doesn't know how large 23 is.

b) Bonny thinks that 2 and 20 are the same.

c) Bonny doesn't understand the meaning of the places in the numeral 23.

d) All of the above.

Figure 26, MKT item 4, 1. Triplet (Hill, 2004)

This item is another example of a multiple-choice item. Which as with item 3 makes the possible responses finite. Anyhow, the item above present a teaching situation with a student and the content of the item in terms of ATD are:

T_4: Identify Bonny's misconception.

τ_4: Analyse Bonny's actions when asked which checkers represent what in the number 23.

θ_4: Justification for the results of the analysis based on the observations.

Θ_4: Theory on diagnostic tasks.

Since it can be expected that all respondent for this instrument fully understand the positional system this item contains no mathematical task. Hence, it is impossible to rewrite this item to become a mathematical task. Thus, the task technique etc. described belongs to the DO. For this item, the diagnostic question a respondent might use to solve the task (i.e. part of the technique) has already been presented. Hence, the respondent does not need to come up with the diagnostic tools him/her self but must be able to make an analysis based on Bonny's answers. The justification for the result are not expected to be presented by the respondent and thus, the item is at the practical level of the DO:

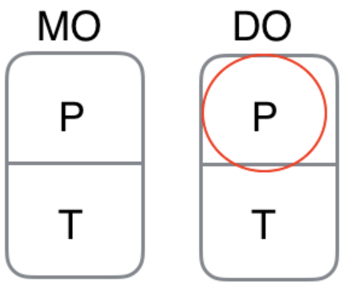


Figure 27, MKT item 4, 1. Triplet placed in reference model

In the framework of Ball this item will place itself in the pedagogical content knowledge for the same reasons that it is place in the DO of the ATD model. The mathematics involved in this item is within the CCK and at such a basic level that this will not be included any further in the analysis of the item. Regarding the placement of the item within the PCK: As the item is concerned only with identifying Bonny's misconception and not with helping her correct it this item can be placed in the subcategory KCS as it is partly defined as a category covering teachers' ability to anticipate what student are likely to think and what they will find confusing.(Ball et al., 2008) Had the item included the task of helping Bonny correct this misconception it could have been categorized as a KCT-item as it would have required the respondent to consider which mathematical tools should be given to Bonny for her to develop better understanding of the positional system.

Item 5, MKT

32. Students in Mr. Carson's class were learning to verify the equivalence of expressions. He asked his class to explain why the expressions a – (b + c) and a – b – c are equivalent. Some of the answers given by students are listed below.

Which of the following statements comes closest to explaining why a – (b + c) and a – b – c are equivalent? (Mark ONE answer.)

a) They're the same because we know that a – (b + c) doesn't equal a – b + c, so it must equal a – b – c.

b) They're equivalent because if you substitute in numbers, like a=10, b=2, and c=5, then you get 3 for both expressions.

c) They're equal because of the associative property. We know that a – (b + c) equals (a – b) – c which equals a – b – c.

d) They're equivalent because what you do to one side you must always do to the other.

e) They're the same because of the distributive property. Multiplying (b + c) by –1 produces –b – c.

Figure 28, MKT item 5, 2. Triplet (Hill, 2004)

As for categorizing this item into mathematical or didactical we return to the same question: Can this task be asked without the setting of the classroom. In the written text of the item the respondent is placed in the role of the teacher and is asked to evaluate the answer presented by the students. But as the respondent is told nothing about the students and are not asked to give feedback or even give an explanation to his/her answer the setting of the classroom become irrelevant. And thus, the item can be reduced to the question and the options of choice. Hence, the item can be categorized as purely mathematical and therefor belong to the MO.

To determine which part of the MO let's look at the content of the item:

$T_{5.}$: Verify that the statements are equivalent.

$\tau_{5.1}$: A direct proof. (This could be what was meant, however unclear stated, in d)

$\tau_{5.2}$: Using the argument presented in e) perhaps with an elaboration of why

$$-(b+c) = -1(b+c)$$

θ_5: Here the basic algebraic properties will be used to justify the use of both the associative and distributive law used in the two first techniques same technology can be used when justifying the validity of mathematical statements.

Θ_5: Theory on abstract algebra (binary operations)

Above two of the techniques from the item is presented as techniques for solving the task. One might say, that the other options in the item might as well be techniques for solving the task. But as they were either incorrect or vague in their argument they were not included as accepted techniques. What is need to answer this item, is for the respondent to take the explanation of the techniques given and evaluate choose which one is the best. Hence the respondent must state which technology he finds most satisfying. Thus, the item belongs in the theoretical part of the MO:

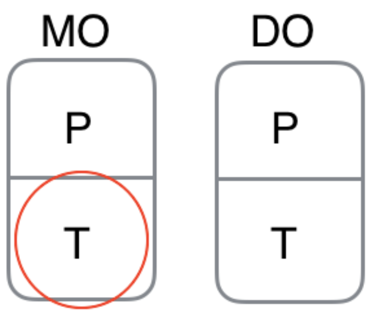


Figure 29, MKT item 5, 2. Triplet placed in reference model

Ball places this item in the category "Middle school content knowledge"(Hill, 2004) which corresponds to the placement of the item in MO. Ball does not state explicitly which part of the content knowledge she places the item. Therefore, I have placed it according to what the analysis using ATD has revealed. Since the item is looking at the technology it has the potential of being a SCK. But since the techniques which the respondent is to evaluate are techniques which are explained in middle school as well this item is categorized as CCK.

Item 6, MKT

30. Mr. Garrison's students were comparing different rectangles and decided to find the ratio of height to width. They wondered, though, if it would matter whether they measured the rectangles using inches or measured the rectangles using centimeters.

As the class discussed the issue, Mr. Garrison decided to give them other examples to consider. For each situation below, decide whether it is an example for which different ways of measuring produce the same ratio or a different ratio. (Circle PRODUCES SAME RATIO, PRODUCES DIFFERENT RATIO, or I'M NOT SURE for each.)

	Produces same ratio	Produces different ratio	I'm not sure
a) The ratio of two people's heights, measured in (1) feet, or (2) meters.	1	2	3
b) The noontime temperatures yesterday and today, measured in (1) Fahrenheit, or (2) Centigrade.	1	2	3
c) The speeds of two airplanes, measured in (1) feet per second, or (2) miles per hour.	1	2	3
d) The growths of two bank accounts, measured in (1) annual percentage increase, or (2) end-of-year balance minus beginning-of-year balance.	1	2	3

Figure 30, MKT item 6, 3. Triplet (Hill, 2004)

In this item, we are again presented with a hypothetical teaching situation. Mr. Garrison is presenting his students with a task. But opposite to item 5, the respondent is not expected to consider student responses. The respondent is placed in the role of the student. And hence, this item could have been presented replacing the explanatory text with: *"Decide whether each of examples below produces the same ratio"* Thus, the item is categorized as mathematical. Now let's look at the mathematical content of the item:

T_6: Determine whether the two measurement produces the same ratio.

$\tau_{6.1}$: Make numerical examples.

$\tau_{6.2}$: Analyse the calculations of the conversion between the measurements.

$\theta_{6.1}$: Justification for this technique is simply that the two calculations would produce the same ratio in case this is true for this example.

$\theta_{6.2}$: Justification for the analysis made of the conversion formulas.

Θ_6: Theory on linear transformations – mathematical analysis.

Had the item been structured such that the respondent should explain when two ways of measuring produces the same ratio it could have been categorized as theoretical as the respondent should have revealed thoughts about his/her technology. But as this is not the case the item is categorized as a practical item and it is placed in the model:

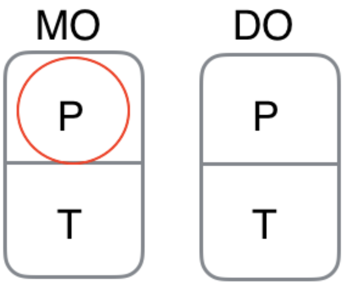


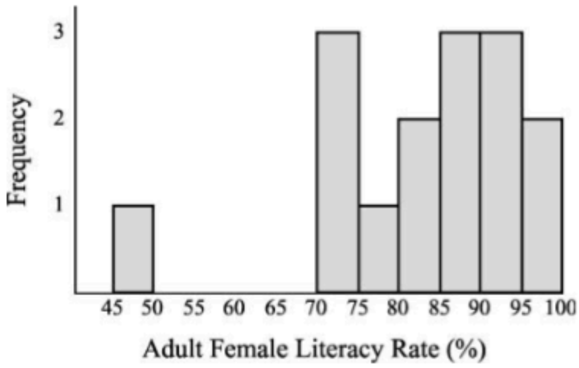
Figure 31, MKT item 6, 3. Triplet placed in reference model

Placing the instrument in Balls model using the results from the ATD analysis yields that we place it in the SMK main category. As to which sub-category it belongs we need to look at the item once again. This item could be placed in both the category CCK and STK. Whether the item lies within the category of the CCK or SCK depends on which of the two techniques presented above is of interest to the researcher. If the item had been structured in the way described in the ATD analysis which laid focus on the technology the item could be placed in the category of SCK. This could have been done since the respondent would need to justify and explain their thoughts on which of the different conversions/transformations would result in equal ratios.

But, like we had to place the item in the practical block of the MO the item will be placed in the category of CCK since the respondent need nothing more that the skills to produce numerical ratios and perform linear transformations.

Item 7, TEDS-M

The following graph gives information about the adult female literacy rates in Central and South American countries.



Suppose you ask your students to tell you how many countries are represented in the graph. One student says, "There are 7 countries represented."

	Check one box.	
	Right	**Wrong**
a) Is the student right or wrong?	$\square_1$	$\square\square_2$

b) In your opinion, what was the student thinking in order to arrive at that conclusion?

Figure 32, TEDS-M, item 7, 1. Triplet (Blömeke, 2013)

This item consists of two parts which are of different character, one multiple choice and one open ended. The two part will be analysed separately.

Part a): The respondent is asked to evaluate the student's response, but as the question is a right or wrong question and we have no information about the student it could have been phrased: Is the graph representing statistics over seven countries? And we are left with a mathematical question. The content of the item is as follows:

$T_{7.a}$: Determine the number of countries represented in the graph.

$\tau_{7.a}$: Counting the number of countries represented in each column and add the numbers to get a total.

$\theta_{7.a}$: The respondent must use knowledge about representation forms in simple statistical plots. Reading the axes and interpreting the graph accordingly

$\Theta_{7.a}$: Theory on applied statistics – naive plots and representation forms.

To answer this item the respondent will only need the practical aspects of the MO as the respondent is not asked to explain how he arrived at his conclusion. And thus, the item can be placed in the reference model:

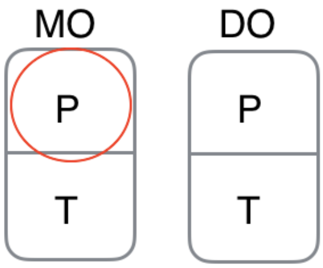


Figure 33, TEDS-M item 7 - part a, 1. Triplet placed in reference model

Part b): Here the respondent is asked to reflect on the thoughts of the student. Hence there is no possibility of asking the question as a mathematical one and the item belong to the DO. Which part of the DO the item belong to depends on the didactical content:

$T_{7.b}$: Assess the students answer and clarify which technique the student has used to solve the problem.

$\tau_{7.b}$: Addressing the student's MO and finding which techniques are available to the student.

$\theta_{7.b}$: Explanation as to why the student might have thought of the number of columns as the number of countries represented based on the MO available to the student.

$\Theta_{7.b}$: E.g. Vygotsky's theory on zone of proximal development

For the respondent to answer this item meaningfully he will not need to justify the origin of the student's thoughts or how to help the student realize his/her misconception. And the respondent never goes beyond the descriptive level of the student's challenge. Hence, the respondent only need the DO at the practical level and this is where the second part of the item is placed:

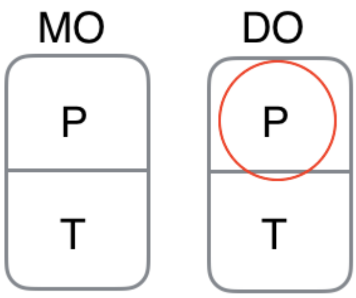


Figure 34, TEDS-M item 7 - part b, 1. Triplet placed in reference model

TEDS-M own categorization of this item is didactical which corresponds to the categorization made in the ATD analysis for part b. This categorization coincides with the ATD categorization as the TEDS-M evaluation does not consider part a (see appendix for full item including scoring instructions). To frame the item in terms of TEDS-M framework recall the table describing the cognitive domains (see table 2) In these terms the item falls under the cognitive domain of "knowing" and the subcategory "retrieve" in terms of the mathematics needed to evaluate the student's response. And as noted earlier this coincide with the analysis of placing the item within the practical block of the MO. To frame the didactical part of the item, recall table 3. The task of evaluating students' answers corresponds to the pedagogical category of Enacting Mathematics for Teaching and Learning which again, as noted earlier, draw parallels to the practical block of the DO. When looking at the scoring instructions for this item correct responses are all responses for which:

"Response indicates that the student thought that each bar represented one country.
Example: The student counted the number of bars, and concluded that the answer (7) represented the number of countries."(Blömeke, 2013)

All other responses are considered incorrect. This is another indicator that the item does not reach the level of theoretical didactics since the respondents are not expected justify why he thinks that the student arrived at this misconception. Note that there is no category within the pedagogical framework which covers the theoretical didactics and therefore there is no theoretical frame for evaluating this.

Item 8, TEDS-M

Some <lower secondary school> students were asked to prove the following statement:

When you multiply 3 consecutive natural numbers, the product is a multiple of 6.

Below are three responses.

[Kate's] answer

A multiple of 6 must have factors of 3 and 2.
If you have three consecutive numbers, one will be a multiple of 3.

Also, at least one number will be even and all even numbers are multiples of 2.

If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

[Leon's] answer

$1 \times 2 \times 3 = 6$

$2 \times 3 \times 4 = 24 = 6 \times 4$

$4 \times 5 \times 6 = 120 = 6 \times 20$

$6 \times 7 \times 8 = 336 = 6 \times 56$

[Maria's] answer

n is any whole number

$n \times (n+1) \times (n+2) = (n^2+n) \times (n+2)$

$= n^3 + n^2 + 2n^2 + 2n$

Canceling the *n*'s gives $1+1+2+2=6$

Determine whether each proof is valid.

Check one box in each row.

		Valid	Not valid
A.	[Kate's] proof	$\square_1$	$\square_2$
B.	[Leon's] proof	$\square_1$	$\square_2$
C.	[Maria's] proof	$\square_1$	$\square_2$

Figure 35, TEDS-M, item 8, 2. Triplet (Blömeke, 2013)

This item is staged in the settings of a classroom and students who hand in proofs. As seen before we are told little about these students. Aside from having little information about the students the respondent is not required to use this information in his/her answer to the item. Hence the information becomes redundant. I.e. the item could have been presented the following way without loss of content: Below are three proofs of the statement *When you multiply 3 consecutive natural numbers, the product is a multiple of 6.* Determine for each proof whether it is valid or not. Hence, the item is categorized as mathematical. The content of the item can be described in the following way:

T_8: Determine validity for each of the three proofs.

τ_8: The respondent ca executes the procedure of different mathematical proofs. E.g. direct proofs or proof by induction or contraposition.

$\theta_{8.1}$: Justification of when the different proofs are appropriate.

$\theta_{8.1}$: Explanations for each step in the different procedures. E.g. Proof of induction (Lützen, 2013):

> Let $p(x)$ be a predicate where x runs over $\mathbb{N}$. If $p(x)$ have the following two properties:
>
> 1. $p(1)$ is true
> 2. for every $m \in \mathbb{N}$ $p(m+1)$ can be concluded from $p(m)$
>
> Then $p(n)$ is valid for all $n \in \mathbb{N}$. The first step is called "the basis" and the second "the inductive step". I.e. in the second step one is assuming $p(m)$ and then concluding $p(m+1)$ Hence, $p(m)$ is called the induction assumption.

The only valid proof in the item is the proof provided by Kate and is a direct proof. Direct proofs are proofs which follow a series of valid logic statements to arrive at the conclusion. E.g. if we want to proof a property of a type of numbers e.g. the prime numbers a series of operations or manipulations can be made with a x representing all prime numbers as long, as we only use properties already known to be valid for all prime numbers.

Θ_8: To complete a proof and justify all the steps included the theory will be concerned mainly with the mathematics of which the proof is concerned. In item 8 the theory would be theory on natural numbers along theory on factorization of numbers.

To answer this item, it is not enough for respondent to know procedures for classic proofs, this would have been sufficient if the respondent were to proof the statement himself. But as the respondent is supposed to evaluate the work of others the respondent need to validate each step of the proof. The respondent will have to draw on his/her knowledge on both justification of the different steps but likewise on the knowledge concerning natural numbers and factorization. Hence, the item is placed in the theoretical block of the MO:

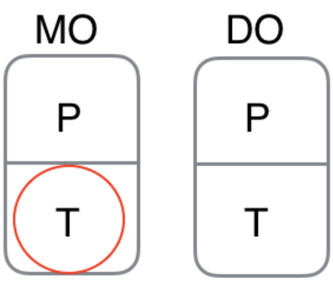


Figure 36, TEDS-M item 8, 2. Triplet placed in reference model

Placing this item in the TEDS-M framework using the ATD analysis direct us to first find the mathematical cognitive domain of which the item belongs to. Recall table 1. Here we notice the subcategory of “Justification: Provide a justification for the truth or falsity of a statement by reference to mathematical results or properties.” Will cover the verification of the proofs since the mathematical results which the respondent should refer to can be the arguments of the proofs themselves. In the cases where the proofs are invalid the arguments might as well be a falsification of the statement but as noted above, the proofs could have been from anyone so the validation of such a proof is still a task of mathematical validation. Hence, the item in placed in the domain of reasoning. And as noted earlier this domain has several commonalities as the theoretical block of the MO.

Furthermore, TEDS-M own categorization of this item is mathematical-didactical. The scoring tables which is linked to this item only has statistics showing percentage of correct and incorrect answers along with tables informing about the wrong answers. Hence here there is no further information as how this item is measuring respondents’ didactical knowledge. Considering table 3 the description which comes closest to this item is: “*Analysing or evaluating students’ mathematical solutions or arguments”* (Blömeke, 2013) Again, the setting of students proof in this item is not relevant to the task as the respondent is not asked to give feedback nor state what are wrong with the invalid proofs and what might have caused this.

Item 9, TEDS-M

The following problems appear in a mathematics textbook for <lower secondary school>.

1. [Peter], [David], and [James] play a game with marbles. They have 198 marbles altogether. [Peter] has 6 times as many marbles as [David], and [James] has 2 times as many marbles as [David]. How many marbles does each boy have?

2. Three children [Wendy], [Joyce] and [Gabriela] have 198 zeds altogether. [Wendy] has 6 times as much money as [Joyce], and 3 times as much as [Gabriela]. How many zeds does each child have?

   (a) Solve each problem.

   (b) Typically Problem 2 is more difficult than Problem 1 for <lower secondary> students. Give one reason that might account for the difference in difficulty level.

Figure 37, TEDS-M, item 9, 3. Triplet (Blömeke, 2013)

As this item consist of two parts so will the analysis. The first part of the item is stated as a mathematical question and we do not have to do any further work before associating it with the MO:

$T_{9.a}$: Tasks are stated in the item. Note that the two questions are alike.

$\tau_{9.a.1}$: Create an equation representing the number of marbles of one of the children. E.g. in task two: let $w = Wendy'snumber\ of\ marbles$ then: $w + \frac{1}{6} \cdot w + \frac{1}{3} w = 198$ and solve for w then multiply to get the different numbers for the different girls.

$\theta_{9.a}$: Justification for the setup of the equation. In the case of the girls the fractions are equivalent to saying that "Wendy has six times as many marbles as Joyce" and similar for Gabriella.

$\Theta_{9.a}$: Theory on equations and fractions.

In this item, the respondent will need nothing more than the technique so answer the question and the item can be placed in the practical block:

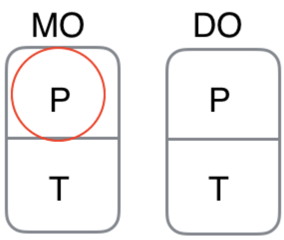


Figure 38, TEDS-M item 9.a, 3. Triplet placed in reference model

For the same reason that the item is placed in the MO we shall look at the cognitive domains of the TEDS-M theory to refer to this framework. Recall table 2 and note the categories:

Select: Select an efficient/appropriate operation, method, or strategy for solving problems where there is a known algorithm or method of solution.

Model: Generate an appropriate model, such as an equation or diagram, for solving a routine problem. (Tatto, 2008)

Which could both cover the technique from the ATD analysis along with the methods considered in the scoring rubric:

- 1) Using *one variable*, setting up *one equation* and solving. *Example* (Problem 1): Let *m* = the number of marbles that David has. Then Peter has 6*m* and James has 2*m*. Therefore, 6*m* + 2*m* + *m* = 198, and *m* = 22.
- 2) Using *more than one variable*, establishing a *system of equations*, performing substitutions, and solving. *Example* (Problem 1): Let *p* = the number of marbles that

Peter has, d = the number of marbles that David has, and j = the number of marbles that James has $p = 6d$ and $j = 2d$, $p + d + j = 198$.

- 3) Trial and error or guess and check
- 4) Ratio or other arithmetic methods
- 5) Representation/diagram

The second part of the item is concerned with the reason why students find some tasks more challenging than others. Hence, this question cannot be stripped down to mathematics as the respondent must go beyond the MO to explain the level of difficulty and especially the cause. Note that as in earlier items we are told little about the students in question. All there is stated is that it is material meant for lower secondary students. Even though this has caused items to be categorized as mathematically before it must be considered that even if remove the students from the description and ask the respondent to answer which question he finds the most challenging and why the respondent will need to investigate his own MO and conclude what makes one harder than the other – i.e. the teacher must investigate what was earlier mentioned as the students DO (the student is in this case the teacher). This item is therefore categorized as didactical. The didactical content of the item is as follows:

$T_{9.b}$: Identify potential student difficulty in the two tasks and rank them from easiest to hardest.

$\tau_{9.b}$: Answer the question and reflect upon the students' mathematical skills meanwhile possibly by the help of students' conceptual maps.(Winsløw, 2006)

$\theta_{9.b}$: Using the students' conceptual maps may help the respondent identify where student's lack a connection between two elements needed to solve the task.

$\Theta_{9.b}$: Theory on metacognition.(Winsløw, 2006)

For answering this item, the respondent must not only identify students' potential difficulties but must also consider which difficulties students will find more difficult. Determining the difference in difficulty and determining which of the items are the most challenging for students can be done using only the technique. An analysis of why students have more problems with one problem than another will again be possible for the respondent to answer by a technique as the technologies of the DO cover the justification for the techniques it is not needed to solve this item. And hence, this part of the item is placed in the practical part of the DO:

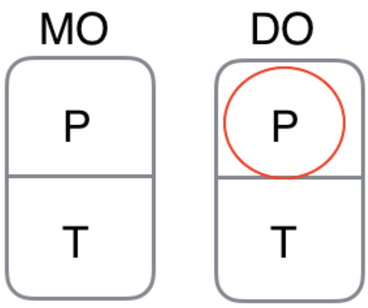


Figure 39, TEDS-M item 9.b, 3. Triplet placed in reference model

This placement of the item in the ATD model urge us to look at table 3 for placement in the TEDS-M framework. In this table, the statement which corresponds best to the task and technique presented is phrased: *Predicting typical students' responses, including misconceptions.(Tatto, 2008)* which is a subcategory of *Knowledge of Planning for Mathematics Teaching and Learning.* Which to some extend can be linked to the task in terms of how the respondent would help students approach the problem or considering how this should be presented to students. In the scoring rubric for this item the following are how correct responses for this item is defined:

"Reason clearly expresses a difference in the mathematical or cognitive complexity of the two problems."(Blömeke, 2013)

What is meant by cognitive complexity is not clear but might refer to the different cognitive domains described in table 2. I.e. respondents need to state which task require which techniques and then rank the complexity of the techniques needed.

If this is this the case, the scoring rubric fits with how the ATD analysis has categorized the item.

Summary of the analysis

In the table below an overview of the categorization of the items from the ATD analysis along with a column showing whether the ATD analysis coincides with the placement of the item in the original framework is shown:

Item no	$MO_{Practical}$	$MO_{Theoretical}$	$DO_{Practical}$	$DO_{Theoretical}$	Original categorization coincides with ATD analysis
1		x			Yes
2		x			Yes & No
3	x	(x)			No
4			x		Yes
5		x			Yes
6	x				Yes
7	x		x		Yes
8		x			Yes
9	x		x		Yes

We note that most of the items are considering the mathematical knowledge of the respondents. When it comes to Thompsons instrument this is not surprising as the instrument is concerned with the mathematical meaning of the respondent only. As for the other instrument this cannot be interpreted the distribution of items, as it was stressed earlier in this thesis: the real ratio of didactical and mathematical items cannot be represented with such a small sample of items.
An overview of the content of the items can be found in the appendix. In this overview we note, looking at the types of tasks: The task of identifying students' misconceptions and task of verifying mathematical statements are tasks which are repeated in several items across instruments. Though the techniques and technologies which are identified in the ATD analysis has similarities it is not possible to know whether this reflects the same similarity in the instruments as it turns out that the instruments do not have detailed evaluation rubrics. Aside from these two types of tasks the items does not coincide across instruments considering topic or focus of the items. This might be due to the small sample of items selected. What is remarkable is that none of the items are exploring the theoretical part of the DO. For the TEDS-M framework this is not surprising as it, as mentioned earlier, do not have a theoretical frame for describing the theoretical part of the DO. As for the other instruments, this area might be

considered impossible to measure but nothing concerning this is stated anywhere and nothing can be concluded.

Most items fit the ATD analysis when placed in their original framework. This is not surprising as the ATD analysis was used to place the items in the original frameworks such that this placement would not be done ad hoc. But, how the items were framed and analysed or scored by developers were also considered which cause the inconsistencies in the table. The two items which did not coincide in the two frameworks both originated from the MMTsm instrument – this reason will be discussed in the next section.

For the MKT-items scoring rubrics are not available and the framing of the items in their original framework becomes less convincing as this is based on the ATD analysis only and there is no telling which of the subcategories Ball herself would have placed the items within. Hence, all the MKT-items is concluded to coincide with the framework. In the case of MKT, it shall therefore be understood as possibility of placing items within the MKT framework based on the ATD analysis.

Many of the items we see are staging teaching situations. This is also seen in items where it is of no relevance. There are items where this staging could be misunderstood but especially for the MKT and TEDS-M items this is rarely the case. The reason for this staging is still not clear after this analysis.

Discussion

Thoughts on the instruments and their frameworks

In the beginning of this thesis the origin of the question concerning what knowledge is needed to teach was presented. So, what does the three instruments offer as an answer to this question: Thompson's theoretical framework results in a standpoint which disregards the classic measurements as he believes that is it the mathematical meaning a teacher has which is the core of what the teacher conveys to his/her students. All that the teacher has of theory, technology and techniques are determined by the meaning of a given object held by the teacher. What a teacher should know, according to Thompson, I can only conclude on the mathematics as the MMTsm instrument is not concerned with the didactics or pedagogics. For Thompsons opinion on these topics we should look at the IQAsm-instrument[2]. The meanings Thompson has defined for different mathematical topics are not all published, what we do know is that the meanings are constructed the same way as the items. Afterwards they are ranked according to which are more constructive for students. So, how does Thompson measure mathematical meaning? Analysis of the items and the associated scoring rubrics indicate that Thompson associate certain techniques and technologies with different meanings. How this association is made is not clear from his framework. Likewise, would it have been nice to have access to the meanings which are described by Thompson. The ATD analysis of the items and rubrics does not clarify this association. Which indicate that either the reference model is not strong enough or this association might not be theoretically justified in the framework presented by Thompson. Another indication that the ATD instrument might would have enjoyed being developed further, is the trouble of translating the items into term of ATD as the scoring rubrics where included in the analysis.

The MKT framework by Ball has a division of the different domains of teachers' knowledge. These domains are based on those of Shulman some of which has stayed close to the original definitions and others which has developed of moved from the domain of subject matter knowledge to pedagogical content knowledge. The categories covering pedagogical knowledge resemble stoffdidaktik closer than pedagogics in a European context. What is noticeable looking at these categories are the blurred lines between different categories. This is noted by Ball in her

2 Instrument for Quality of Instruction Assessment for secondary mathematics

own description of these categories, though this does not make it easier on the reader to separate the topics when trying to clarify which categories are to be used when or to which extend.
Another thing which should be mentioned when discussing the MKT framework is the broad definitions linked to the different categories. Having such broad definitions of what teachers' knowledge for teaching is, and no definition of "knowledge", makes the measuring of this difficult to frame for the reader without the help of examples from Ball's own instrument. How Ball intent to measure this knowledge is not clear when reading about MKT and this does not become completely clear as one study the items. Though the theory for scoring the items are not available and hence this cannot help to make such an impression as it did with the other instruments. It is not unclear when studying the items how they correspond to the framework but as the framework is unclear as to what e.g. knowledge of content and student covers in terms of what techniques and technologies are expected of teachers so becomes the items.
The TEDS-M program offers a two-part theoretical framework. One for the didactics which is closely related to that of the MKT. They differ here as the TEDS-M categorization of didactical knowledge are categorized according to what looks like what could be a timeframe for in what order the knowledge is needed. The theoretical framework for mathematical knowledge is like that of Bloom and is detailed to a much greater extent than that of the didactics. This leaves me wondering: Why have different frameworks for the different areas? How does didactical knowledge part itself from that of natural science? In terms of ATD didactical and mathematical knowledge can be described using the same terms and descriptions. I believe that this separation in types of knowledge is a problem for the TEDS-M framework. Either it should be stated why knowledge concerning mathematics and didactics differ in nature i.e. why couldn't the same structure e.g. the levels of Bloom be used to describe both types of knowledge. Or, it should be clarified how the assessment of the different kinds of items differ. In the framework for TEDS-M there is nowhere a description of how the items are assessed. Such a description would be appreciated as the evaluation of the didactical items are, at least for some, so broad that one could question whether the analysis made from the responses manage to describe all the categories within pedagogical content knowledge.
Common for TEDS-M and MKT instruments is that they are not concerned with the teachers' didactical theory. Maybe they, like Thompson does with the mathematics, expect to read the theory out of the practice when it comes to didactics? Though there is no indication of this in the TEDS-M scoring rubric and as there is no scoring instructions for MKT available this cannot be assumed to be the case. Or maybe they simply do not care? It is unknown whether this is also the case for Thompsons pedagogical instrument, though it would weaken his framework if it were as

the strategy for measuring teachers' mathematical and pedagogical/didactical meaning would be inconsistent. I certainly do not hope that the lack of measurement on teachers' didactical theory is not due to lack of interest as this part of the DO must be considered as important as that of the MO. Hence, this part of the DO is what leads the teachers to act and teach as they do.

As none of the frameworks offer a clear image of what they expect a good teacher to know it becomes difficult to create instruments which measure on these exact parameters. Though the instruments still seem to have a ranking of teachers answers whether these are made on a part of the frameworks not clearly stated or made ad hoc is not clear.

However, we are still left with the question of how we should educate teachers unanswered. And until the frameworks can give an answer to this question it seems premature to create measurement instruments for this kind of knowledge or meaning.

Considerations concerning the ATD reference model and ATD as a common frame.

ATD works in this thesis as a strong tool for describing and framing the items as well as the other frameworks. It seems that ATD is more consistent and precise as opposite to the other instruments. This might be due to the need which has breaded the different frameworks. Thompsons framework arises as an attempt to contrast the existing views on what is conveyed to students. I.e. Thompson tries to let the psychological approach and idea of learning as an individual action shine through in assessment for this knowledge in teachers. MKT is developed based on the need to measure teachers' knowledge. Hence, the teaching situation and profession becomes a focus of this framework making the potential of generalization hard to see. TEDS-M challenges with the double theoretical framework is described in the above paragraph. The ATD has its origin in the situation of learning situations as well. However, ATD manages to combine this origin with a theory of actions in a way which makes it possible for the theoretical framework to become generic for all human activity. That is, ATD takes the forces of the MKT in terms of having its origin in a specific problem and the force of MMTsm in being founded on a theory which is not limited to teaching situations. With this combination, it bypasses the problem faced by the TEDS-M framework.

The reference model used in this thesis was simple in its visual form but worked well for creating an overview as well as categorizing the items. The rest of the analysis was used using the vocabulary of ATD and not the visual model, which worked for describing the content of the items.

The categories of MKT and TEDS-M are easily associated with the different elements of ATD whereas the framework for MMTsm are too specific within the different topics to have the same ability of categorization.

How to study the measuring of teachers' praxeologies for teaching further

For further study of these instruments a throughout analysis of the scoring methods could be natural next step as the justification for the scoring is lacking in the frameworks. Had the full instrument of MKT been available it could be interesting to see if the same structure and balance between topics and difficulty is present in the MKT and MMTsm are present as it is in the TEDS-M instrument. For further study of these instruments using ATD as a frame a further development of the reference model would be necessary. It would be a necessity to develop a model for assessing the scoring rubrics as well as further development of the existing model such that it could frame the concept of meaning as it is framed by Thompson as the current model has difficulties translating the content of the items along the focus of teachers personal praxeologies.

Conclusion

The types of mathematical and didactical tasks seen in the different items span over many different topics and on different levels. The different task types are framed and presented differently in the different instrument. After the analysis two task types appear to be recurrent across the instruments. Task types are those of "identifying students' misconception" for the didactical tasks and for the mathematical tasks the task of "validating mathematical statements" are recurring. For the didactical tasks, it is the techniques which is assessed by the instruments there is no cases of items assessing the theoretical block of the DO. For the mathematical items both the practical and theoretical block are assessed but within the theoretical block only the technologies are assessed.

MKT frames teachers' knowledge in six main categories. Three for within Subject Matter Knowledge, categorized according to the knowledge expected to be held by different groups of people, and three for Pedagogical Content Knowledge which are categorized according to what the content are to be related to. The MKT-items are easily placed within one of the main categories. The further categorization is more challenging as the categories are highly co-determined. As it is not explicit how Ball score the items it is not certain if the scoring responses corresponds to the analysis presented.

The TEDS-M program presents two different frameworks for the pedagogical/didactical and mathematical items. The pedagogical aspect is rooted in the theory of Shulman and Ball's development of this. Though it differs from Ball theory as it categorizes the different block of knowledge according to the situation in which it is used. The theory of mathematical knowledge presented by TEDS-M frame mathematical knowledge in different levels; knowledge, applying and reasoning which resembles that of Bloom's taxonomy. The items analysed from the TEDS-M instrument are easily identified with one of the categories. But looking at the scoring instructions for the items it becomes unclear to which extend the responses are used beyond the level of determining whether the respondent can answer correctly or not and not what type of problems respondents have.

The MMTsm framework roots in the theory of Piaget and is built around the idea that individuals possess their own praxeologies i.e. praxeologies belong to the individual instead of the institution. Thompson are interested in the teachers' mathematical meaning and hence it is the mathematical meaning which Thompson is attempting to measure. For each specific mathematical concept measured in the MMTsm instrument a set of meanings have been defined and ranked. For this reason, the items cannot be categorized the same way as for the other instruments. After analysis of these items including the scoring rubric it appears that Thompson

analyse the respondent techniques and technologies and associate different type of responses with different meanings.
Considering the three instrument they agree somewhat on the topics which they assess. They do, as mentioned above, agree on a few types of tasks. Whether there are more recurring types of tasks would require an analysis of the full instruments. Aside from the time perspective the only instrument for which I have had full access to all items are the MMTsm instrument, which makes such an analysis impossible. Aside from this the instruments agree in their formulations of many items as they all stage items in school settings as well as the areas of the ATD model which they assess. Though it should be noticed that the MMTsm has its primary focus on the theoretical block of the MO whereas the two other instruments have a more balanced distribution of items within both the theoretical and practical block.
The instruments differ to a higher degree when looking at the scoring of the items. This is not surprising considering the different purposes of the instruments.
The MKT and TEDS-M are the only instruments considering pedagogical or didactical knowledge. As they share a theoretical basis for this area they do not differs greatly in content but for the items analysed the items format differ. I.e. Ball's didactical items are multiple choice and the didactical items included in the analysis, from the TEDS-M instrument are both open ended questions.
Another structural difference between the items can be seen when considering the MMTsm-items as they are less direct than that of the other instruments. Even though all instruments stage teaching situations the TEDS-M and MKT instruments does not require an analysis of the scoring rubrics to see the purpose of the items. Whereas for some of the MMTsm items the ATD analysis did not fit how the items were framed by Thompson which can be caused by the MMTsm framework.

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Appendix:

Mathematical work of teaching framework organized by (1) mathematical objects, (2) actions with and on those objects in teaching, and (3) specific examples. (Selling, 2016)

	MWT: Actions with and on objects	Examples
Explanations (includes justifications & reasoning)	**Comparing explanations** to determine which is more/most valid, generalizable, or complete explanation	Given two explanations, choose which is more complete Given multiple student explanations, determine which is most valid Given several explanations, choose the best explanation. Given conflicting explanations, determine which is valid and why. Select an explanation that best captures an underlying idea.
	Critiquing explanations to improve them with respect to completeness, validity, or generalizability.	Given an incomplete but valid explanation, determine what, if anything, is missing or needs to be added to be more complete.
	Critiquing explanations with respect to validity, generalizability, or explanatory power.	Given an explanation, determine if it is mathematically valid. Given several explanations, determine which ones are valid. Given a text, determine what may be misleading about an explanation.
	Writing mathematically valid explanations for a process, conjecture, relationship, etc.	Write a mathematically valid explanation for a process or concept. Write a mathematically valid explanation for a conjecture. Given student strategies, determine properties that could be used to justify the strategy's validity.
Mathematical structure	**Determining, analyzing, or posing** problems with the same (or different) mathematical **structure**	Given a set of problems, determine which have the same structure. Given a set of problems, choose the description of the structure type. Given a set of problems, determine which does NOT have the same structure. Write a problem that has the same structure as given problems. Given a description of a structure, determine which problems fit that structure.
	Analyzing structure in student work by determining which strategies or ideas are most closely connected with respect to mathematical structure	Given a set of student strategies, determine which have similar mathematical structure. Given a set of student strategies most of which use the same core idea but slightly differently, determine which one does not fit. Given a set of strategies and a structure, determine which strategies fit the structure.
	Matching word problems and structure	Given a structure, choose a word problem with that structure. Given a word problem, choose another problem with the same structure.
Representations	**Connecting or matching representations**	Match a representation to a given interpretation of an operation. Determine how different representations are connected. Given two claims about a representation, determine which is correct and why.
	Analyzing representations by identifying correct or misleading representations in a text, talk, or written work.	Given a written representation (e.g., number line, table, diagram), determine what may be misleading. Given a set of representations, choose which does or does not show a particular idea (table?)
	Selecting, creating, or evaluating representations for a mathematical purpose	Create a representation for a given number or operation. Select a representation that highlights a particular mathematical idea.
	Talking a representation (i.e., using words to talk through the meaning of a representation and connecting it to the key ideas)	Given a suggested way to talk about a representation in a text, evaluate whether the talk clearly connects the representation and the ideas. Given a colleague's request for feedback, determine how their talking about a representation could be improved to highlight mathematical meaning

Scoring rubric for item shown in figure 7:

Code	Response	Item: MFC502B
	Correct Response	
20	Responses that refer to reading and comprehension difficulties related to the **complexity of the language** used in the question with **reasons** and/or **references to specific examples.** Examples: • *The language used is quite challenging. Example, "fewer than any other" and "more pencils than rulers".* • *Students would be challenged by the difficulty/complexity of the wording in the question such as 'most often' 'fewer'. There is a considerable load on their 'higher order' skills as they are required to organise, interpret and relate back to the graph.* • *The items described in the text are listed in a different order to the bars on the graph creating logistic or sequencing challenges.*	
	Partially Correct Response	
10	Less detailed responses that recognize that the language is likely to be a difficulty for children but **without reasons or examples.** *Examples:* • *They would have trouble with the language used in the question.* • *Reading and comprehending the text would be difficult for many children.* • *There is a considerable amount of information to read, organize, sequence and relate to the graph.*	
11	A statement describing difficulties attributable to the graph rather than the text. *Examples:* • *They would have trouble reading the graph.* • *The names are missing from the graph and they wouldn't have experienced this before.*	
12	A statement attributing difficulties to the level of **problem-solving** or **analysis** required without explaining how/why. *Examples:* • *They would have trouble analyzing the information in the problem.* • *The problem requires problem-solving strategies and they would have trouble with that.*	
	Incorrect Response	
79	Incorrect (including crossed out, erased, stray marks, illegible, or off task)	
	Non-response	
99	Blank	

Scoring rubric for item number MFC502 B, (Blömeke, 2013)

Booklet design for the TEDS-M program:

	Novice	Intermediate	Advanced	Total No. of Questions
Algebra	3	5	4	12
Geometry	3	7	2	12
Number	1	6	2	9
Data*	2	1	1	4
Mathematics Pedagogy	7	5	0	12

Note: Although we have included items on data, "data" in itself is not a reporting sub-domain.

Example of booklet design based on difficulty

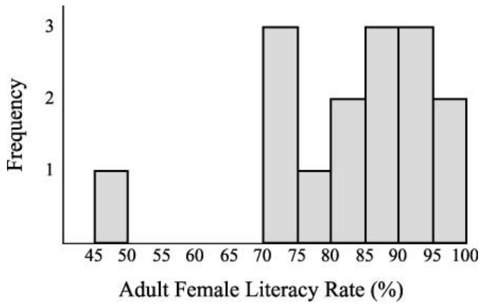
Full item design description by Thompson

"(1) Create a draft item, interview teachers (in-service and pre-service) using the draft item. A panel of four mathematicians and six mathematics educators also reviewed draft items at multiple stages of item development. In interviews, we look for whether teachers interpret the item as being about what we intended. We also look for whether the item elicits the genre of responses we hoped (e.g., we do not want teachers to think that we simply want them to produce an answer as if to a routine question); (2) Revise the item; interview again if the revision is significant; (3) Administer the collection of items to a large sample of teachers. Analyse teachers' responses in terms of the thinking they reveal; (4) Retire unusable items; (5) Interview teachers regarding responses that are ambiguous with regard to meaning and it is important to settle the ambiguity; (6) Revise remaining items according to what we learned from teachers' responses, being always alert to opportunities to make multiple-choice options that teachers are likely to find appealing according to the meaning they hold;3 (7) Administer the set of revised items to a large sample of teachers; (8) Devise scoring rubrics and training materials for scoring open-ended items; revise items only when absolutely necessary."(Patrick W. Thompson, 2014)

TEDS-M item 7 including scoring rubric (Blömeke, 2013):

Teil 2: Beispielitems zur Erhebung mathematischen Wissens 399

The following graph gives information about the adult female literacy rates in Central and South American countries.



Suppose you ask your students to tell you how many countries are represented in the graph. One student says, “There are 7 countries represented.”

	Check <u>one</u> box.	
	Right	**Wrong**
a) Is the student right or wrong?	☐ 1	☐☐ 2

b) In your opinion, what was the student thinking in order to arrive at that conclusion?

Code	**Response**	**Item: MFC806B**
	Correct Response	
10	Response indicates that the student thought that each bar represented one country. *Example:* *The student counted the number of bars, and concluded that the answer (7) represented the number of countries.*	
	Incorrect Response	
79	Incorrect response (including crossed out, erased, stray marks, illegible, or off task).	
	No response	
99	Blank	

Item number MFC806A and MFC806B, (Blömeke, 2013)

Item discarded for 1. Triplet (Blömeke, 2013):

[Jeremy] notices that when he enters 0.2 × 6 into a calculator his answer is smaller than 6, and when he enters 6 ÷ 0.2 he gets a number greater than 6. He is puzzled by this, and asks his teacher for a new calculator!

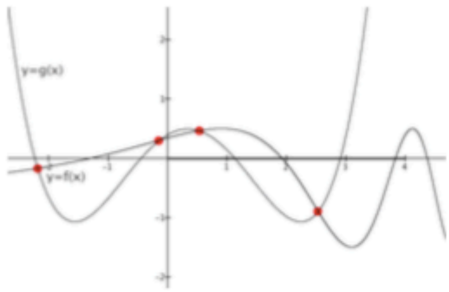
(a) What is [Jeremy's] most likely misconception?

(b) Draw a visual representation that the teacher could use to model 0.2 × 6 to help [Jeremy] understand **WHY** the answer is what it is?

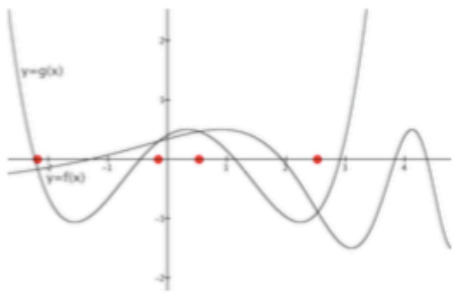
MMTsm- Item discarded for 2. Triplet:

Three students drew graphs to represent the solutions to an equation of the form $f(x) = g(x)$. Which of these three ways of thinking would you prefer your students to have about solutions to $f(x) = g(x)$? Why?

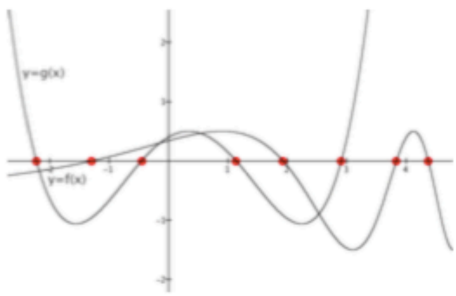
Student 1:
Since the solutions to $f(x) = g(x)$ are ordered pairs we should highlight the points of intersection on the graphs of f and g.



Student 2:
Since the solutions to $f(x) = g(x)$ are values of x, we need to highlight the x values of the points of intersections on the graphs of f and g.



Student 3:
A solution is where a graph crosses the x-axis, so we need to highlight those places where either graph crosses the x-axis.



Your answer and explanation:

Project ASPIRE: Defining and Assessing Mathematical Knowledge (Meanings) for Teaching Secondary Mathematics Pat Thompson (PI), Marilyn Carlson (Co-PI), Mark Wilson (Co-PI) This version is cut up in the columns for readability, for unedited version see: (Project ASPIRE: Pat Thompson)

1. Overview

Project ASPIRE will use *mathematical meanings for teaching* as an interpretative framework for understanding why a secondary mathematics teacher

- Makes particular didactic choices,
- Formulates particular learning trajectories, and
- Interacts mathematically with students in particular ways.

The project recognizes the need for assessments of mathematical meanings for teaching and instructional quality at the secondary level as well as a need to align teacher knowledge assessments and teacher practice assessments. Thus two related instruments are under development. The first will assess teachers' *mathematical meanings for teaching secondary mathematics* (MMTsm) by drawing upon research on teachers' and students' understandings of key mathematical ideas. The second assessment will extend the *Instructional Quality Assessment* to focus on *secondary mathematics* (IQAsm) by drawing upon co-PIs recent research on transforming secondary teachers' classroom mathematical practices. A focus on the extent to which a teacher attempts to convey meaning to students will play a key role in the relationship between the two instruments, linking secondary teachers' mathematical knowledge with their instruction.

Project ASPIRE will enhance the research infrastructure by providing what could become common measures for assessing mathematical meanings for teaching secondary mathematics, for assessing the quality of secondary mathematics instruction, and for assessing the impact of teacher education and professional development programs.

Project ASPIRE will support efforts to improve secondary mathematics teacher preparation and teacher professional development by

- its instruments and frameworks being used as didactic objects in professional development
- broadening the national conversation about goals of mathematics instruction and mathematics teacher education.

2. Why *MMT* and not *MKT* ?

Investigations of MKT rarely say what they mean by "knowledge" and rarely explain how knowledge connects with action.

To connect knowledge with action requires a cognitive theory in which "knowledge" is actionable. We see the notion of *meaning*, defined properly, as providing a nexus from knowing to acting.

Project ASPIRE takes ideas of meaning in the vein of Piaget and Dewey as its theoretical foundation. In this perspective, meaning and understanding are two sides of a coin. A person's *understanding* of a word, object, sentence, utterance, mathematical inscription, or situation is the result of assimilating it to a scheme of actions, operations, and implications. A person's *meaning* for a word, object, sentence, utterance, mathematical inscription, or situation is the scheme to which it is assimilated. Our use of *meaning* and Harel's use of *way of understanding* are essentially the same. Harel's phrase *way of thinking* is, in our terms, a person's habitual employment in reasoning of a particular set of meanings. We take it as axiomatic that a teacher's instructional actions are both enabled and constrained by the mathematical meanings by which he or she operates.

Definition of *Mathematical Meanings for Teaching*

A teacher's meanings for particular mathematical words, phrases, symbolic expressions, topic names, or concept names that are expressed in the teacher's instructional actions are that teacher's *mathematical meanings for teaching*. These are not necessarily identical with the teacher's mathematical meanings. A teacher might have many meanings for "fraction", and if so selects, perhaps unawarely, one or more as what he or she wishes to convey to students.

A mathematical meaning for teaching needn't entail any ideas. A teacher's memory of a procedure (such as "cross-multiply"), together with the teacher's image of problems for which that procedure can be used, could be the meaning he or she wishes to convey to students.

3. Examples of *Mathematical Meanings for Teaching*

Teachers' Meanings for Angle Measure and Trig Function, and their expression in Teaching

Teacher's Meanings

- Angle numbers are indices for one direction relative to another. "Straight right" is 0. "Straight up" (or "perpendicular to") is 90. All other angle numbers refer proportionally to an amount of turn, where 90 is 1/4 turn. π (pi) is straight left, or half a turn. 2π and 360 are a full turn. Any number with "π" in its representation is "a radian". Any whole or decimal number between 0 and 360 is degrees.
- Sine, cosine, and tangent are ratios of sides in a right triangle. In the teacher's thinking, the meanings of sine, cosine, and tangent are unrelated to a meaning of angle measure.
- In the expression "sine of __", teacher expects "__" to be filled in with *the name of an angle that is ensconced within a triangle.* If a number is in the blank, then it is an index of a named angle within a triangle. But it is an image of a triangle that is most pronounced in the teacher's meaning of "sine of __".

Observed Expressions of these Meanings in Teaching

- The teacher teaches "degrees" and "radians" as entirely unconnected topics.
- The teacher, when teaching trigonometric functions, imposes a triangle on every situation so that "*x*" in "sin(*x*)" is the name of an angle. It is not a number that gives a measure of some attribute of an angle.
- "*x*" in "sin(*x*)" is static. It does not vary except that it could name a different angle. But it makes no sense to think of *x* having a numeric value that varies continuously, and if *x* does have a numeric value, it is unrelated imagistically to a meaning of sin(*x*).

Teachers' Meanings for Graph, Linear Function, and Slope

Teacher's Meanings

- Variables vary continuously in chunks. When *x* varies continuously through the non-negative reals, its value goes, for example, from 0 to 1, from 1 to 2, from 2 to 3 to (and so on). Numbers between 0 and 1, between 1 and 2, between 2 and 3 "come along" by each being part of a chunk, but *x* does not have them as a value in the same way it has 0, 1, 2, etc. as values.
- A linear graph is a pseudo-geometric object. If it has points, they are a fixed distance apart and they are connected by line segments or it is a line without points that passes through two points. The line segments and the line have graph-points only if you put them there.
- Slope of a line is determined by "so many over and so many up".

Observed Expressions of these Meanings in Teaching

- A teacher has these meanings and is trying out a new approach that uses rate of change to teach the point-slope formula for linear functions and to generalize the point-slope formula to the two-point formula for linear functions. The method works like this:
  Suppose a linear function passes through the point (3, 5) with rate of change 2. Its rate of change being 2 means that whenever x changes by some amount, y changes by 2 times that amount. So, to find the y intercept, pretend that we change x from 3 to 0 (a change of -3). y will change from 5 by -6, so when x is 0, y is -1. So the linear function is y = 2x - 1.
- The method generalizes this one-point approach to two points by determining the function's rate of change from the two points on its graph and then using the point-slope formula with one of the two points.
- The teacher states this problem: *The graph of a linear function passes through the points (3, 1) and (7, 4). Represent this linear function in standard form.*
- He draws the diagram in Figure 1 while saying, "These two points being on the graph means that it goes over 4 and up 3. So if we go back 4 from 3 … (long pause). We'll finish this tomorrow. Here's your homework. Do just the problems for one point."

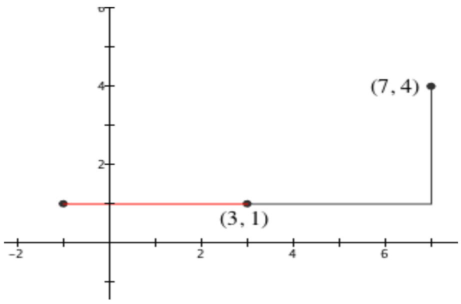


Figure 1. Teacher draws this figure on whiteboard while discussing two-point method.

4. Developing the Instruments

Developing the MMTsm

Drawing upon the work of the Berkeley Education and Assessment Research (BEAR) group, Project ASPIRE will make use of the four building blocks of the BEAR Assessment System.

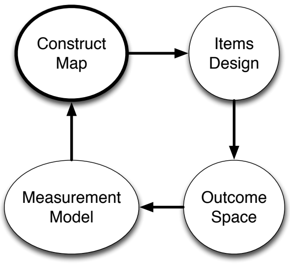


Figure 2. The four building blocks of the BEAR Assessment System. (Wilson, 2005)

Figure 2 shows the four building blocks that are being used to develop the Mathematical Meanings for Teaching secondary mathematics instrument. The first building block is the construct map. The construct map is the specific, unidimensional definition of an element of a person's cognition that a researcher wants to measure. The structure of a construct map is an ordered set of qualitatively different levels of performance with the content stemming from what research has told us about individuals' meanings for the given topic. For example, Figure 3 shows a construct map developed by Project ASPIRE for mathematical meanings of division based upon research (Ball, 1990; Simon, 1993; Thompson & Saldanha, 2003) as well as a pilot study (Byerley, Hatfield, & Thompson, 2012).

Division Construct Map V4.0

Level	What the Individual Means		
RM	**Relative Magnitude** The individual understands the operation of division as a multiplicative comparison of two quantities' magnitudes and the result of division as a measure of the magnitudes' relative size. She sees that a measure induces a partition and a partition induces a measure.		
P2	**Partitive** The individual anticipates the operation of division as "equal sharing" and anticipates the result of dividing as giving the number of items per group when all items are shared equally with all groups.	Q2	**Quotative** The individual anticipates the operation of division as making a partition and anticipates the result of dividing as producing a measure.
P1	The individual is able to communicate that division is a sharing process but needs to mentally/physically enact the sharing process in order to give a meaning for the quotient. Individual can give meanings for dividend and divisor.	Q1	The individual is able to communicate that division is a measuring process but needs to mentally/physically enact the measuring process in order to give meaning to the quotient.
O	**Other** The individual gives a response that does not fit with other categories. Examples: response is purely driven by algorithms and rote procedures; an expression of the belief that "Division Makes Smaller."		
NR	**No Response**		

Figure 3. Draft version of a construct map for meanings of division as used in teaching high school topics of slope, rate of change, rational function, etc.

5. Challenges

Design of MMTsm items

- Assessment items are typically about performance, not about meanings. Meanings are difficult to discern. A person's meanings are most often enacted in the process of *understanding* a statement or a problem. Thus, current emphasis is on designing tasks and asking about interpretations of them, or designing tasks whose success demands meanings.
  Example: What is cos(35°), in degrees?
- Mathematical meanings for teaching are most easily notice when someone is teaching. Classroom observations, and discussions of what teachers "had in mind" at various points, is inspirational both for potential assessment items and for potential distractors.
- We will use iPads to administer MMTsm in order to have teachers interpret video excerpts of students' or teachers' activities and answer questions about students' or teachers' meanings in open response formats.

Overview of the analysis

Item	Task type - T	Technique - τ	Technology -θ	Theory – Θ
1	Solving first degree equations	Use of regular algorithm.	Explanation of wrong use of algorithm.	Operating with variables
2	Calculation of a (linear) slope.	Use of regular algorithm.	Analysis and explanation of result.	Function analysis.
3	Calculation of ratio.	Deciding on a proper algorithm for the give situation.	Explaining why this algorithm fits the scenario.	Proportionality and ratio.
4	Identify student's misconception	Analysis of student behaviour.	Justification for conclusion based on observations.	Theory on diagnostic tasks.
5.1	Verification of mathematical equivalence	Direct proof	Justification by explaining algebraic operations.	Theory on abstract/modern algebra
5.2	Evaluate mathematical statements.	Determine mathematical validity	Justification by explaining algebraic operations.	Theory on abstract/modern algebra
6.1	Determine equivalence.	Use numerical examples	Explanation to why this is a valid technique	Theory on linear transformations
6.2	Determine equivalence.	Analyse linear transformations.	Explanation for steps used in the analysis	Theory on linear transformations
7.a	Extract information from a bar chart.	Count height and number of	Explanation of how bar charts	Representation forms and naïve

		bars.	are constructed.	plots.
7.b	Identify student's misconceptions.	Analyse student's answer in connection to the MO	Justifying what elements of the student's MO might have caused the misconception.	Vygotsky's zone of proximal development.
8	Verification of mathematical statements.	Analysis and assessment of proofs.	Explanation on each step of the given proofs.	Theory on the properties of natural numbers.
9.a	Classic problem solving task	Setting up one equation and solve for chosen variable.	Explaining how base variable was chosen and the structure of the equation.	Theory on equations.
9.b	Identify potential student misconceptions.	Assessing students' MO and find potential gaps.	Using students' conceptual maps to justify the gaps identified.	Theory on metacognition.