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Situations for modelling Fermi Problems with multivariante functions

Niels Andreas Hvitved
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Abstract

This thesis in didactics of mathematics examine whether Fermi problems can be used as an introduction to mathematical modelling and functions of one or more variables in Danish high schools at C-level. To answer this question, a teaching sequence dealing with Fermi problems is developed and tested in a Danish 1st year high school class.
Using the theory of didactical situations, both an á priori analysis and an á posteriori analysis of the teaching sequence is conducted. Through this analysis, the didactical opportunities and challenges concerning Fermi problems as a tool for introducing mathematical modelling is investigated.

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UNIVERSITY OF COPENHAGEN
DEPARTMENT OF SCIENCE EDUCATION

Situations for modelling Fermi Problems with multivariate functions

Master Thesis

Niels Andreas Hvitved

Supervisor: Carl Winsløw

Submitted on: 8. August 2017

1. Contents

2. Abstract

This thesis in didactics of mathematics examine whether Fermi problems can be used as an introduction to mathematical modelling and functions of one or more variables in Danish high schools at C-level. To answer this question, a teaching sequence dealing with Fermi problems is developed and tested in a Danish 1st year high school class.

Using the theory of didactical situations, both an á priori analysis and an á posteriori analysis of the teaching sequence is conducted. Through this analysis, the didactical opportunities and challenges concerning Fermi problems as a tool for introducing mathematical modelling is investigated.

Resumé

Dette speciale i matematikdidaktik har til formål at undersøge, hvorvidt Fermi problemer kan anvendes i forbindelse med en introduktion til matematisk modellering og funktioner af én eller flere variable, som led i undervisningen i matematik på gymnasiets C-niveau. For at besvare dette spørgsmål udvikles et undervisningsforløb der omhandlende Fermi problemer, med det formål at teste forløbet i en dansk 1.g klasse.

Ved brug af teorien om didaktiske situationer udføres en á priori samt en á posteriori analyse af undervisningsforløbet. På baggrund af disse analyser, udforskes de didaktiske muligheder og udfordringer i forbindelse med brugen af Fermi problemer, som værktøj til introduktionen af matematisk modellering.

3. Introduction

What does it mean, when we talk about a *mathematical model*? In general, any model is a simplified representation of a real object, situation or system – a *mathematical model* being no exception, is created using objects with mathematical characteristics, for instance equations and functions.

Creating a mathematical model is essentially a transformation from a real-world situation into the abstract world that is the world of mathematics, wherein we can manipulate or “solve” the model through use of mathematical techniques. The usage is then when we re-enter the real world, bringing along the solution, which here can prove to be a solution to the real-world problem as well. An important characteristic of this process is that we both start and finish in the real-world; potentially leaving behind the mathematical model as an applicational tool that has done its part.

The role of mathematical models in today’s society is indispensable, and as a consequence, the teaching of mathematical modelling also hold great importance. This fact is evident from the evolving of the mathematics curricula in schools; the Danish high schools not being an exception.

But how do we effectively introduce mathematical modelling in today’s high schools?

Inspired by the teaching ideas of the renowned physicist, Enrico Fermi, we turn to *Fermi problems* for the answer. A Fermi problem is an estimation problem, in which estimates for various quantities are needed in order to solve the problem at hand. An example of such a problem could be “How many blades of grass are there in the park?” or “What is the population of the earth?” or “How many kernels of popcorn does it take to fill up this room?”

The exact answers to all of these problems are almost impossible to find, though through reasonable estimation of various parameters, a model that constitutes the answer is within reach. This model can be considered as a multivariate function in which the parameters act as variables.

Classically, the introduction of functions in Danish high-schools are through use of the formal definition. But is it possible to use Fermi problems in order to give students a realistic conceptual understanding of what constitutes a mathematical function?

In this thesis, I will attempt to give an answer to this question, and also highlight the challenges that arise when using Fermi problems as a part of a teaching sequence. This will be done through the use of a teaching sequence that I have designed.

During the process of writing this thesis, there are a few very important people who have been instrumental for me to bring this project to life. Most importantly, I would like to sincerely thank my advisor, professor Carl Winsløw, for his extremely competent guidance and understanding nature.

I would also like to thank the class of 1.mr at Rødovre Gymnasium and their teacher, Karen Mohr Pind. Without them, the realization of the teaching sequence would not have been possible.

4. The theoretical framework

In the following chapter, I will give a short description of the theoretical framework that is the foundation of this thesis. The framework in question is *the theory of didactical situations* (henceforth abbreviated TDS), developed by Guy Brousseau in the 1970's and 1980's, resulting in a publication in 1997: "Theory of Didactical Situations in Mathematics".

This work was in large developed as a result of the founding of thirty institutes: The Research Institutes on Mathematical Education (IREM) aswell as the research center *Centre d'Observation pour la Recherche sur l'Enseignement des Mathématiques* (COREM), which would function as a research laboratory on the observation of the teaching of mathematics.

The following sections will give insight into the theoretical foundation of the thesis, and provide the necessities with respect to an a priori analysis of the developed teaching sequence as well as an á posteriori analysis of the carried-out sequence.

TDS

The foundation of TDS as a tool is epistemological rather than psychological or pedagogical (Winsløw, 2006), and one of the greatest strengths of this theory as an analytic tool is the providing of various templates in examining the specific teaching situations with respect to the didactic triangle: teachers influence, the students' roles, and the mathematical knowledge to be taught. It is important to point out, that TDS does not provide teachers with a model for "good practice"; it is - however - an excellent tool for analyzing teaching (Hersant & Perrin-Glorian, 2005).

TDS is, in its nature, very diverse, and can be used both as practical analysis of teaching situations, in designing lesson plans and teaching situations; and it is also a research program that has evolved through the last 40 years in the didactics of mathematics (Winsløw, 2007).

A fundamental idea of TDS is that, that it is not sufficient for a teacher to just deliver knowledge to a student in order for the student to achieve what is desired. It is here we acknowledge that there are two different types of knowledge which we will call *personal knowledge* and *official knowledge*. These differ (as the names suggest) in the following way: *personal knowledge* is understood as how the individual perceives the knowledge at hand - often informal and implicit; and the *official knowledge* is the knowledge represented in scholarly texts, scientific articles etc. An important part of TDS is that new knowledge is attained by expanding personal knowledge through problems and exercises, followed by a formalization transitioning into official knowledge.

With this in mind, it is easy to draw parallels to the world of science. A scientist's work is (usually) a product in the form of new official knowledge; however, the process of creating this knowledge is based on the scientist's work and development of personal ideas and informal models and hypotheses. In mathematics, we often see the final product (official knowledge) in the form of a mathematical theorem followed by a proof, but this product has no information regarding the development of the knowledge at hand.

According to Winsløw (2007), it is natural to consider both a student's personalization of official knowledge as well as institutionalization of personal knowledge. In order for a student to attain official knowledge, she first has to personalize it. It is here the role of the teacher stands out; seeing as she must carry out the role of being mediator between the official knowledge at hand, and the student's personalization of it. In order for the teacher to effectively do this, she establishes a milieu in which the potential of the student attaining the objective knowledge is maximized. We call this environment the *didactic milieu*, and in general, these are situations where acquisition of official knowledge takes place. More specifically, these can take form of problems, exercises, lecturing etc. One could think of the student's performance in this milieu as playing a game; if the game is won, she will attain the personal knowledge at hand, provided that the milieu is designed accordingly.

An example

Let us consider a classic example courtesy of Guy Brousseau: *the puzzle exercise* (Winsløw, 2007). This exercise is part of a teaching sequence in which students (age 12-14) are to learn about proportionality. In this exercise, the students (in groups) are given a puzzle in which the pieces are triangles and rectangles (see Figure 3.1).

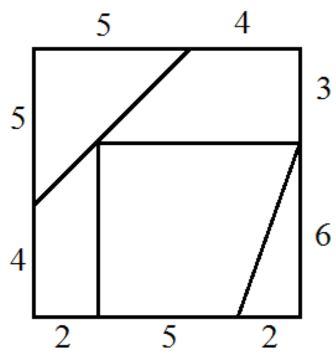

Figure 1: A puzzle (Winsløw, 2007).

Now the students are to create an enlarged model of the puzzle, where the pieces of size 4 cm are to measure 7 cm instead. Here it is expected that the students will attempt an additive approach, i.e. adding 3 cm to each side of the puzzle. By trial, the students will find that this attempt is faulty, and the milieu hereby provides direct feedback, forcing the students to rethink their approach. In order for the students to "win the game", they have to use the multiplicative approach.

It is reasonable to believe, that the students solving this exercise alone, doesn't provide them with an established knowledge of the general idea behind proportionality. It is, however, a step in which the teacher can establish the general idea that is to be taught. It is also very valuable to note the explicitness of the two strategies at hand (additive versus multiplicative): In order for the students to acknowledge the correct approach, they have to be aware of the difference between the two. This acknowledgement is crucial for the students to later gain the knowledge of the general principle.

In general, didactic milieus are created by the teacher through a *re-personalization* of the official knowledge (as in the example above). By interacting with this milieu, the students are to acquire the intended knowledge.

Didactic and adidactic situations

In TDS, there are five distinct types of situations - or phases - which doesn't necessarily occur in a given order in a teaching sequence. These situations are either *didactic* or *adidactic* (or a combination thereof) - the difference being that the teacher is directly interacting with the milieu.

- *Devolution.* The teacher establishes the milieu. This can be through an introduction of the problem at hand; here it is the student's task to understand the problem at hand, as well as the rules of the game. In the puzzle-example, the teacher might, for instance, clarify which tools are available. We consider this phase to be mainly didactic, as it more often than not, demands direct teacher-student interaction.

- *Action*. The students are working with the milieu without teacher interaction. This constitutes an adidactic situation, with the exception of adaptability of the milieu in the case that the task at hand is too difficult. In the example, the students are attempting to create the enlarged model of the puzzle.

- *Formulation.* The students formulate hypotheses about the problem at hand - this may be with or without teacher interaction, making this phase situationally didactic or adidactic.

- *Validation*. The students - often in conjunction with the teacher - assess the various hypotheses at hand. This is often in the form of a discussion between the students and/or the teacher, making the situation didactic in nature. In this situation, one or more new devolution situations may arise (i.e. if a given hypotheses demands further assessment through action and formulation).

- *Institutionalization*. In the final situation, the teacher presents the official knowledge at hand. This often as an extension of the validation situation - and this is (almost) always didactic in nature.

In this thesis, I will mainly focus on adidactic situations, which, according to Hersant and Perrin-Glorian (2005), can give rise to the construction of milieus with *adidactic potentials*. In the words of Hersant and Perrin-Glorian:

> […] In ordinary teaching, actual adidactic situations are rare, but one can observe situations that have some adidactic potential. This means that there is a milieu, which provides some feedback to the actions of the students, but the feedback alone may be insufficient for the students to produce new knowledge on their own. In this case, the teacher may have to intervene to modify the milieu, for example, so that the student becomes aware of an error. We say "potential" because the teacher may ignore this potential and manage the situation without using it, evaluating by himself the students' answers, instead of waiting for the students to react to a feedback of the milieu. But if the situation has no such potential the teacher can do nothing but react by himself to students' actions. (Hersant and Perrin-Glorian, 2005)

When we discuss didactical situations, we are bound to consider the so called *didactic variables* that are essential to the given situation. These are often considered as potential variations in the didactic milieu, which in turn does not affect the target knowledge at hand. As a teacher, it is crucial to identify the didactic variables in order to handle potential pitfalls that may arise when students work in the milieu. This can also function as a tool for potential modifications of the milieu during the unfolding of the situation, should the students run into (foreseen) trouble.

Didactic contracts

In order for the student (and the teacher) to win the game at hand, they must eventually learn the target knowledge at hand - and in order to play the game, the students must follow a set of rules set by the milieu and the teacher. These rules can be considered as informal *contracts* between the student and the teacher; in other words, they are based on the mutual expectations of the parties in the given situation. According to Hersant and Perrin-Glorian, the model of a didactic situation includes both an adidactic situation and such a *didactic contract*.

This - somewhat strict - notation is based on the work of Brousseau, where he specifically considers an experiment regarding the case of the schoolboy Gäel. Gäel was a young student who generally performed at a level below average in mathematics, and in the studies, Brousseau noticed that Gäel had a tendency to generally answer questions in a manner, as to how he expected the teacher would want it - thereby satisfying the *contract* at hand. An example of such a situation is given in Winsløw (2007, my own translation):

[…] In the beginning of the interview, Gäel is proposed the following problem: *On a parking lot, there are 57 cars. 24 of the cars are red. Find the number of cars on the parking lot that are not red.* Gäel thinks for a moment, and responds: "I will do, what my teacher has taught me." He then writes 57 followed by 24 below, and ultimately the answer 81. The quote is a direct appeal to the only authority he acknowledge in this situation; the teachers. […] In following analogous situations, Gäel again repeats the mistake, but through a series of games (…), in which he fulfills the contract, Gäel eventually realize that a problem at hand can have authoritative properties, which in turn make some answers more correct than others (Winsløw, 2007 p. 146-147)

It is easy to draw the conclusion, that this phenomenon only occur with smaller children, but further studies show that it also occur for older students, exemplified by a study conducted by the physics didactician C. Linders (Winsløw, 2007).

In order to give a good description of what exactly constitutes a didactic contract, I will use the definition as given by Hersant and Perrin-Glorian (2005):

Didactical contracts can be distinguished by four *dimensions*, namely

- *The mathematical domain* (i.e. the mathematical field relevant wrt. the knowledge at hand)
- *The didactic status* (i.e. the student's familiarity with the subject at hand; new, old or in between)
- *Nature and characteristics of the ongoing didactic situation*
- *The distribution of responsibility* (i.e. the amount of responsibility the teacher leaves with the student)

Furthermore, Hersant and Perrin-Glorian distinguish between three *levels* in the structure of didactical contracts; i.e.

- *Macro-contracts* (concerned with the main teaching objective)
- *Meso-contracts* (the realization of an activity, i.e. in the form of solving an exercise)
- *Micro-contracts* (corresponding to an episode wrt. an activity, i.e. answering a sub-question with regards to an exercise)

It is clear that the dimensions are highly mutually dependent. Let us, for instance, consider the dimension regarding the mathematical domain concerning algebra. Here the didactic status concerning the student's familiarity with this particular field may be a whole new knowledge, or it may be old knowledge (i.e. already institutionalized). There is an in-between, which is the knowledge in development. Here Hersant and Perrin-Glorian again distinguish between different states, namely *recently introduced knowledge, knowledge in the course of institutionalization* and *institutionalized knowledge, which must be consolidated*. This is exactly the dimension *distribution of responsibility*, seeing as the teacher gradually leaves the student with higher responsibility. The *nature and characteristics of the ongoing didactic situation* is self-explanatory, but still considered as a dimension in the sense that students are able to recognize the teacher's expectations, with respect to the situation at hand.

When considering the various levels of contracts, it is only when considering micro-contracts that the dimensions are fully stable. Contrarily, the dimensions are rarely stable when considering macro-contracts. It is therefore obvious to define a macro-contract as an implication of meso- and micro-contracts. An illustration of this principle as formulated by Hersant and Perrin-Glorian is shown in Figure 3.2 below.

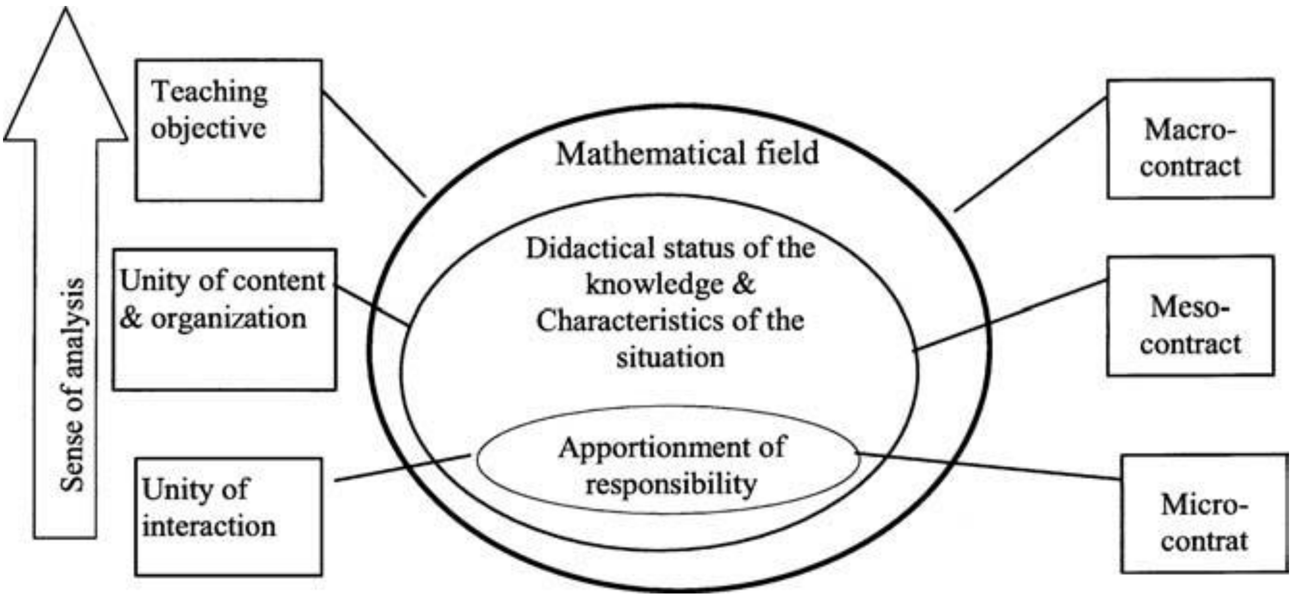

Figure 2: The structure of didactic contracts

The didactic obstacles

Consider again the situation of Gäel as discussed in the previous section. In this example it becomes clear that a fundamental paradox arises; the contract is implicit and informal - and it only becomes explicit the moment it is broken. Also, it cannot be fulfilled if it does not vanish. This paradox is clear in the case of Gäel, where his reaction to a given problem is dominated by his attempt to do what he expects is wanted from the environment - and thereby fulfilling the contract. If we consider didactic situations, it is essential that the didactic contract is not a dominating factor when achieving the winning strategy. It is therefore evident, that the contract must - in a sense - be suppressed for the student to achieve the intended knowledge.

It is evident that there potentially are consequences because of the properties of the didactic contract. Students can develop a contract-oriented behavior as a result; also, the teacher may be inclined to fulfil the part of her contract at all costs.

The Topaze effect

This effect arises when the teacher - in an attempt to avoid the student "losing" the game- gradually simplifies the problem at hand, until the solution is eventually delivered directly to the student. In this process, the intended knowledge necessary to provide the answer, changes.

The Jourdain effect

This effect is a form of the Topaze effect. In this case, it is the teacher who - be it intentional or unintentional - does not admit the student's lack of knowledge in the given situation. The teacher wrongly recognizes that the student has institutionalized the intended knowledge, perhaps because the student just follows trivial instructions from the teacher - or just coincidently answers correctly.

It is worth to note that there are also other potentially unfortunate effects due to the nature of didactic contracts (such as metacognitive shifts and improper use of analogies), but we will not go into further detail regarding those.

Didactic transposition

In 1980, the French didactician Yves Chevallard gave his first course on *the didactic transposition*. Heavily inspired by the work of Guy Brousseau, this would lay ground to a new theory in the didactics of mathematics. The basic idea is – evidently – based around the *transposition* of mathematical knowledge; how it transfers between different institutions, and specifically how it adapts when transferred from one institution to another. We are interested in what happens when a select piece of scholarly knowledge is edited and morphed to be applicable in everyday teaching. In the work of Bosch and Gascón (2006), the theory of didactic disposition handles four types of knowledge, namely

- The scholarly knowledge (the academic point of view)
- The knowledge to be taught (as determined by the educational board in curricula)
- The taught knowledge (as taught in the classroom); and
- The learned, available knowledge.

The process of the didactic transposition is also illustrated in Figure 3.3 (see below).

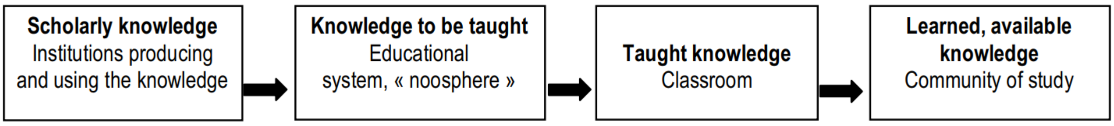

Figure 3: The didactic transposition process (Bosch and Gascón, 2006)

The scholarly knowledge is developed primarily by mathematicians, and is a part of the university and/or the scientific society. The knowledge to be taught is part of the educational system, i.e. in the form of a curricula. An example of this would be the new curricula describing the course of mathematics at C-level in Danish high schools[1]. In this, we find the following academic goals (own translation), which are relevant when working with Fermi problems:

- *Handling simple formulae, formulate simple variable dependencies and be able to use symbolic language to solve problems with a mathematical content.*
- *Translate between the four data-representations in the form of table, graph, formula and everyday language.*
- *Use simple functions for modelling purposes given sets of data,* (…) *and a developed critical sense regarding the scope and usefulness of the model in question.*

[1] https://www.uvm.dk/-/media/filer/uvm/gym-laereplaner-2017/stx/matematik-c-stx-august-2017.pdf (August 1, 2017)

- *Using mathematical programs for experiments and developing mathematical concepts, and also for handling symbols and solving problems*
- *Demonstrate and convey knowledge on mathematical application in select areas, including handling problems originating from everyday life and society.*

We also identify the central material as follows (own translation):

- *The concepts of functions, characteristics of linear functions* (…) *and their graphical representation*
- *Fundamental properties of mathematical models, simple mathematical modelling by use of the above-mentioned functions* (linear, power, exponential) *and combinations thereof.*

We also note the following regarding the didactic principles (own translation):

[…] A part of the course material in the basic course (…) is regarding linear models, including linear functions. […] In the everyday teaching situations, mathematical reasoning, problem solving and modelling is highly emphasized through independent work of the students, and formulating mathematical questions and problems is at the center of attention. When working with mathematical modelling, the students should gain insight into how the same mathematical theories and methods are applicable to widely different phenomena […]

It is worth to note, that the wording of the curricula enable the teacher to very freely decide *how* the central material is taught. In most cases, the teacher will make use of textbooks, old exam assignments, and the (with respect to the curriculum) associated teaching plan, when selecting the specific knowledge to be taught.

The knowledge taught is what is taught by the teacher in each teaching situation, and the corresponding *learned knowledge* is what the students themselves are able to formulate, apply and even teach to other students.

According to the theory of didactic transposition, it is of grave importance that all of the types of knowledge are taken into consideration when analyzing a didactic problem. It is consequently – in our case – important to understand the academic point of view when working with linear functions and modeling, in order to understand the same mathematical concepts subject to high school mathematics.

5. The scholarly knowledge to be taught

In the following chapter, I will give a brief presentation of the relevant academic subjects that I will be using throughout this thesis, and how these subjects are linked with the curriculum of mathematics at C-level in Danish high schools.

Mathematical models and modelling using Fermi problems

Modelling

Mathematical modelling as a learning and teaching subject has become a very prominent topic in the last decades, as the need for mathematical application in science and technology is ever growing (García, Gascón, Higueras and Bosch, 2006). This development has also greatly influenced the mathematics curricula of Danish high schools in recent years, and in 2002, the Danish department of education published the report *Kompetencer og matematiklæring – Idéer og inspiration til udvikling af matematikundervisning i Danmark* (Niss and Jensen, 2002). In this report, the authors present eight fundamental mathematical *competences*, that find its validity in all levels of the educational system (be it elementary school, high school, university etc.). Here the term *competence* is defined as:

[…] someone's insightful readiness to act in response to a certain kind of mathematical challenge of a given situation (Blomhøj and Jensen, 2007, p. 47).

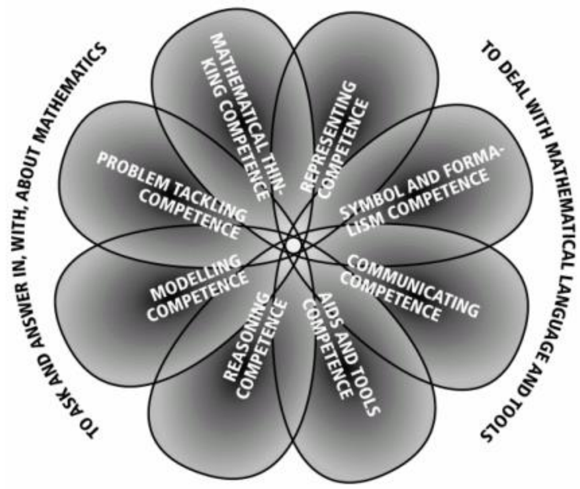

Figure 4: A representation of the eight mathematical competences presented in the KOM report (Blomhøj and Jensen, 2007)

One of the competences is that of mathematical modelling, consisting of both the analysis – and the creation of – mathematical models.

The *analysis* part is the student's ability to identify the properties and the foundation of a given mathematical model, as well as identifying the scope and validity of the model. This includes the ability to "de-mathematize" a presented mathematical model in the sense of deciphering and interpreting results given by the model, with respect to the situation. The *creation* part is the competence of actively creating a model based on a specific situation (i.e. mathematizing), thereby brining mathematical concepts to life, with the goal of applicational usage in said situation.

According to Blomhøj and Jensen (2007), the *process of mathematical modelling* can be divided into six sub-processes:

- *Formulation of task*
- *Systematization*
- *Mathematization*
- *Mathematical analysis*
- *Interpretation/evaluation*
- *Validation*

The *formulation of task* is regarding the real-world situation, where the student finds motivation in order to engage in the modelling process. This leads to the establishing of a domain of inquiry.

The *systematization* is the first step of translating the perceived reality into a model. Here the students aim to limit, structure and simplify the domain of inquiry. This process give rise to a *system* with respect to the situation.

The *mathematization* is the second step of the translation of the real-world situation into a mathematical model, in which the resulting systematization is mathematized.

The *mathematical analysis* is the analysis of the mathematical model, where, for instance, a mathematical solution to a given problem is found. This leads to a result of the mathematical model in question.

The *interpretation/evaluation* is the process of assessment: is the result reasonable with respect to the empirical data at hand? This process give rise to either *action*, where decisions are made based on the consequences of the result, or *insight* in sense that new knowledge of the real-world situation is acquired.

Finally, the model undergoes a process of *validation*, where extent and validity of the model undergo a questioning. This can be in the form of comparing the model to new empirical data.

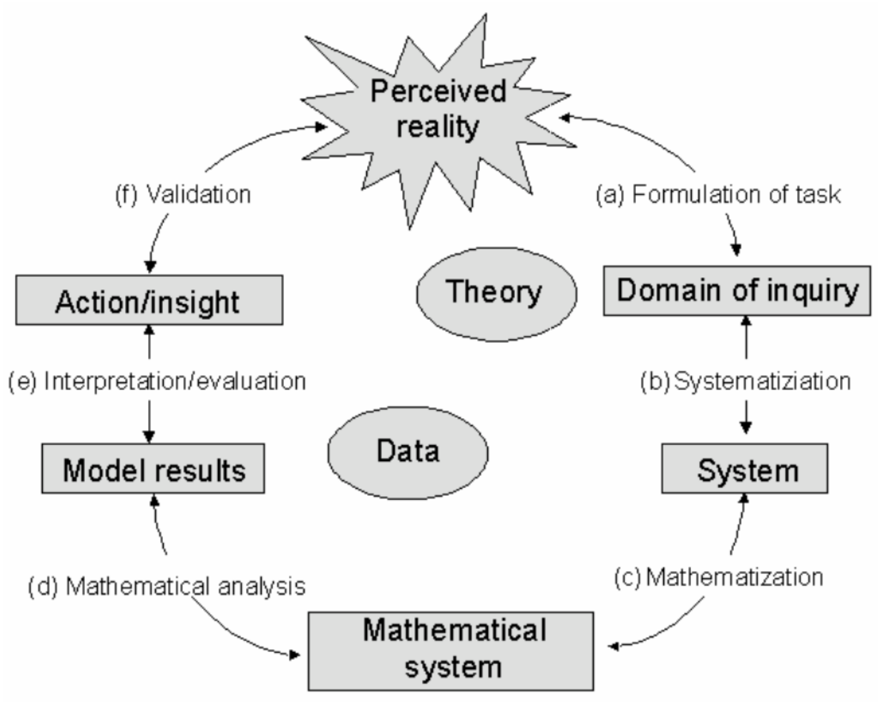

Figure 5: A representation of the mathematical modelling process (Blomhøj and Jensen, 2007)

According to Blomhøj and Jensen (2003), the modelling process is not linear, but rather cyclical (as illustrated by the double arrows in Figure 5) – in other words, when undergoing the overall process of modelling, it is not unusual to undergo some of the sub-processes several times. A simple example of such a modelling process is i.e. an inquiry regarding how much it costs to actively do fitness in a fitness center.

Fermi problems

The term *Fermi Problem* is named after the famous Italian physicist and Nobel Prize winner, Enrico Fermi (1901-1954), whose most notable scientific contribution was the creation of the world's first nuclear reactor, while he was a part of the Manhattan Project. Enrico Fermi was also a highly-appreciated teacher (Lan, 2002), and when teaching, he was very prone to stating problems like *How many railroad cars are there in the US?* or *How many piano tuners are there in the Chicago?* To answer these questions, he would use assumptions and estimates, which – often – would yield accurate and reasonable answers. These problems are examples of what we call *Fermi problems* (also known as back-of-envelope calculation problems), and Fermi himself firmly believed, that any "thinking person" (in the sense of physicists) could estimate any such quantity up to a factor 10 just using reasoning and intelligent estimates (Ärlebäck, 2009).

Fermi problems in mathematics education

The specific use of Fermi problems in mathematics education is not a subject that has been researched extensively, though they are mentioned in connection with the teaching of estimates and modelling.

According to Ross and Ross (1986), teachers use Fermi problems mainly for two reasons. The first, in the words of Ross and Ross,

[…] to make an educational point: problem-solving ability is often limited *not* by incomplete information but the inability to use information that are already available […] (p. 175)

The second, to show the students more aspects of what also constitutes mathematics; i.e. problem-solving in mathematics does not always yield an exact result. According to Sririman and Lesh (2006), Fermi problems also give rise to interdisciplinary work, and that the use of Fermi problems based on everyday situations hold higher meaning for the students while offering more pedagogical possibilities than traditional intellectual exercises:

[…] For instance, Fermi problems involving estimates of fresh water consumption, gasoline consumption, wastage of food, amount of trash produced, etc have the potential to lead to a growing awareness of ecological problems related to the environment we live in as well as provoke critical thought when checking the accuracy of computations with different governmental and corporate resources. Such activities also present the possibility for interdisciplinary activities with other areas of the elementary curriculum and cultivating critical literacy (Sririman and Lesh, 2006 p. 249).

Realistic Fermi problems

The definition of *realistic Fermi problems* I will use in this thesis, is that of Ärlebäck (2009, p. 339-340). These are characterized by the following five properties:

- they are highly *accessible*
- they are r*ealistic*
- they demand "a *specifying and structuring of the relevant information and relationships* needed to tackle the problem" (Ärlebäck 2009, p. 339)
- they demand *reasonable estimates* of the relevant quantities
- they *promote discussion*

The first property is in the sense that the problems are accessible to all individual students at all levels, and that the complexity is highly flexible in nature. The second is that they have a clear real-world connection, which – as previously mentioned (Sririman and Lesh, 2006) – is a great pedagogical advantage. The third property meaning that the stated problem is open in the sense that strategies previously known to the students are not analogously applicable in solving the problem. The fourth restricts the problem in the sense that there should not be any known numerical data associated with the problem stated. The fifth –

and last – is that they should be oriented towards group work, specifically promoting discussion regarding the relevance of the estimates in question. In the remainder of this thesis, a *Fermi problem* will refer to the above definition.

The characteristics mentioned above define a type of problem very different from traditional problems when working with mathematics at high school level. It is therefore expected that the students (given they have no initial experience with these sort of problems), will perhaps be a little lost when first encountering a Fermi problem with these properties. The fact that students will work in groups, however, is hopefully going to prevent them from being completely stuck.

How many piano tuners are there in Chicago?

As a final remark on Fermi problems, we will give an example through Enrico Fermi's own famous example:

How many piano tuners are there in Chicago?

First, it is a well-known fact, that the population of Chicago is about 3 million people. If we assume that an average family contains four members, we have approximately 750.000 families in Chicago. Seeing as Chicago is a city where music is of great cultural importance, we estimate that 20% of the families own a piano, such that there are approximately 150.000 pianos that demands tuning. If we assume that a piano must be tuned once a year, and that a tuner can service four pianos a day, with him working five days a week for 48 weeks a year, it would require a total of $150.000 \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{1}{50} \approx 157$ piano tuners to meet the requirements of the city. According to the Wikipedia entrance on Fermi problems[2], the actual number is about 290, showing that our simple calculation only is off with a factor of 2.

Fermi problems – why do they work?

In general, Fermi estimates are highly applicable seeing as the estimations of the individual quantities of the terms in question often are close to the correct number. As a result, overestimates and underestimates will often help cancelling each other out, hence a Fermi calculation that involves multiplication of several estimates will probably be more accurate than one could fear.

Since there is a natural correspondence between multiplying estimates and adding their logarithms, we can consider the overall over- or underestimation as following a random walk on the logarithmic scale, which will diffuse as $\sqrt{n}$, where n is the number of terms. Consequently, if one makes a Fermi estimate with n terms, where the standard deviation of σ (on the logarithmic scale), then the overall estimate will have

[2] https://en.wikipedia.org/wiki/Fermi_problem (last checked on the 1. of August 2017)

standard deviation of $\sigma^{\sqrt{n}}$ (Mahajan, 2008). Consider a Fermi problem with 16 terms with standard deviation 2, then the expected standard error will have grown as $2^4 = 16$. Therefore we expect the estimate to be within the range of $\frac{1}{16} \cdot p$ and $16 \cdot p$, where p is the correct value. When performing Fermi estimations, the margin of error is consequently highly dependent on the number of terms used as well as the margin of error of each estimated term.

Estimations

As already discussed, *estimation* represents a large part of the process when solving Fermi problems. As presented by Albaraccín and Gorgorió (2014), we define *estimation* as a rough calculation or judgment of the value, number or quantity at hand, where the value, which results from undergoing an estimation process, is dependent on the person who performs it. Albaraccín and Gorgorió identify four kinds of activities in relation to estimation, one of which is calculating values in predictive activities. This is the exact kind of estimation we will be working on in this thesis, and furthermore, the representations made do not allow for an exact answer to the problem at hand, but merely an approximation whose validity depends on how well the chosen model corresponds to the real-life situation.

Functions and their use when modelling with Fermi problems

Functions and their role in Danish high schools

As of August 2017, the curriculum of C-level mathematics in Danish high schools has undergone a great change; among other things, the reintroduction of the concept of functions now play a part in both academic goals and central material (see the section *didactic transposition*). Previously, the notion of *functions* did not appear at all in the curriculum, rather the more informal concept of relationships between variables was central.

The reintroduction of the concepts of functions give rise to making use of several representations of functions: formula representations, graphic representations, tables, computers (input/output) and even Venn diagrams.

But what is a function, when considering the formal definition, from an academic point of view? Kiming (2007, p. 104) uses the following definition:

Definition.

Let A and B be nonempty sets. A **function** f from set A to set B (denoted by $f: A \to B$) is a relation between A and B satisfying the following conditions:

1. For each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$, and
2. If (a, b) and (a, c) are in f, then $b = c$.

If $a \in A$, the unique element $b \in B$ for which $(a, b) \in f$ is denoted by $f(a)$.

This definition is, however, not very "user friendly" when teaching functions to students in high school, as the students aren't equipped with handling mathematical concepts at this level. As a result, functions are usually introduced using the *correspondence relational view of functions* (Slavit, 1997). Here the focus is on pairs of input- and output variables for which there for each input variable correspond exactly one output variable. With this view in mind, Danish high school students have traditionally been introduced to functions through use of the mathematical notions of *domain* and *image*, followed by a formal definition of functions, as well as compositions of functions and inverse functions (Carstensen, Frandsen and Studsgaard, 2013). In Carstensen, Frandsen and Studsgaard, the definition is as follows (own translation):

Definition.

A quantity y is called a *function* of a quantity x if there for each value of x correspond exactly one value, y, denoted *the function value of* x. This is written as $y = f(x)$. The set of numbers, for which the independent variable x can assume values, is called the *domain* of the function, $Dm(f)$. The set of numbers consisting of all function values is called the *image* of the function, $Vm(f)$.

For first-year high school students, this definition is – though a lot simpler than that of Kiming – very abstract and confusing, and it is mainly through the use of real-world examples and applications that students are able to understand the idea of this definition. In Laursen (2008), the author argues that a *covariance view of functions* (Slavit, 1997) can be used as an introduction to the concepts of functions (or in his words, variable relations) – specifically through the use of tables, graphs and equations. In the covariance view of functions, the focus is rather on the growth properties, and properties of the change of the variables (in this case, there are two variables) – i.e. what happens with one variable when we change another. This is usually introduced through use of tables, where calculations are performed and the resulting changes are observed.

Inspired by this work I am inclined to believe that the use of Fermi problems may prove rewarding as an introduction to the concepts of (linear) functions – as well as an introduction to mathematical modelling, specifically with the covariance view of functions as inspiration.

Using functions when modelling Fermi problems

Let us first define – in a mathematical sense – the elements that constitute a Fermi problem. Let $\{x_1, x_2, \dots, x_n\}$ be the set of estimates that are used in conjunction with solving the problem. Then we can formulate the solution P as a product of functions $f_1 \cdot f_2 \cdot \dots \cdot f_i$, with $1 \leq i \leq n$, as

$$P(x_1, \dots, x_n) = f_1(x_1, x_2, \dots, x_n) \cdot f_2(x_1, x_2, \dots, x_n) \cdot \dots \cdot f_i(x_1, x_2, \dots, x_n).$$

If we consider the set of estimates as variables, the solution P is a multivariate function of n variables. In the case of realistic Fermi problems, we have a function $P: \mathbb{Q}^n \to \mathbb{Q}$ (since, clearly, estimates are rational numbers).

Example.

Let P be a solution to the following Fermi problem:

How many drops of water is in a (rectangular) swimming pool?

The solution P is given as a product of the two functions $f_1(x_1, x_2, x_3, x_4)$ and $f_2(x_1, x_2, x_3, x_4)$, where f_1 is a function describing the volume of the swimming pool in cubic meters and f_2 is a function describing how many drops of water is required to occupy one cubic meter of space. In this case, the functions are trivially defined as

$$f_1: \mathbb{Q}^4 \to \mathbb{Q} \quad \text{defined by} \quad (x_1, x_2, x_3, x_4) \mapsto x_1 \cdot x_2 \cdot x_3$$
$$f_2: \mathbb{Q}^4 \to \mathbb{Q} \quad \text{defined by} \quad (x_1, x_2, x_3, x_4) \mapsto x_4$$

whence the solution is

$$P(x_1, x_2, x_3, x_4) = x_1 \cdot x_2 \cdot x_3 \cdot x_4.$$

This example illustrates that the functional aspects of Fermi problems are justified – in this case, we can define a multivariate function describing the number of drops of water in any rectangular swimming pool, in which we can estimate its dimensions and the number of drops of water occupying one cubic meter of space. It is easy to modify the example to handle any shape of swimming pool, however the number of relevant variable estimates will change, as it is directly dependent of the shape in question, also influencing the domain of f_1.

It is obvious that the complexity of a Fermi problem directly relates to the complexity of its functional solution, as also described in the above situation. The point is, that the variable relations when handling Fermi problems give rise to considerations of a functional nature.

Settings and representatives of functions

The work with functions allows the use of certain *settings* and *representatives*, as described by Schwarz and Dreyfus (1995) and Bloch (2003). When introducing the concept of functions, teachers usually make use of graphs in order to *represent* a function; they generally work as effective representatives of the function in question. The graphical setting is not the only way in which functions have representatives, and Bloch (2003) argue that the following settings are available:

- Numerical (i.e. tables of values)
- Algebraic (i.e. formulas and equations)
- Geometric (variable geometric magnitudes)
- Graphic (line segments, curves)
- Formal (symbols such as $f, f \circ g, f^{-1}, f(x)$ etc.)
- Analytic (used for heuristic purposes, we will not go into detail with this setting)

Clearly it is very different how the mathematical work with functions is carried out in the different settings; a problem concerning one representative in a given setting does not constitute the same mathematical work in a different setting. So how do these various settings present themselves when working with functions with respect to Fermi problems?

The numerical setting

Often illustrated by a table, the numerical setting with respect to functions in general, is an examination of the dependent variable as the independent variables change. Consider for instance the function

$$f(x_1, x_2, x_3) = 3x_1^2 - 6x_2^2 + x_3,$$

a numerical representative could be in the form of a table such as this:

f	x_1	x_2	x_3
0	0	0	0
3	1	1	6
-10	3	2	-13
⋮	⋮	⋮	⋮

In the context of Fermi problems, such a representative can be used when considering the estimation parameters as variables; i.e. what happens when we vary the different estimates.

The algebraic setting

In general, when working with functions, students working in the algebraic setting can conclude various properties of the specific function at hand (traditionally in high school settings, these are exponential functions, power functions, linear functions and polynomials). For instance, one can deduce that the 2nd degree polynomial

$$f(x) = 3x^2 - 6x + 2$$

has two distinct roots (as the discriminant $d = (-6)^2 - 4 \cdot 3 \cdot 2 = 12$ is greater than zero), the function has a global minimum at $x = 1$ and that the function is decreasing in the interval $(-\infty; 1)$ and increasing in the interval $(1, \infty)$.

It is also possible to algebraically manipulate the formula through various transformations; it is however not possible to see the curve.

Bloch (2003, p. 9) notes that students in general have great difficulties with algebraic manipulation; "(…) algebra is a setting where they have little knowledge".

When considering Fermi problems, algebraic manipulation is highly relevant, as the dependency of the quantities in play necessarily are related through various equations. In the traditional case of "How many piano tuners are there in Chicago", one can for instance algebraically manipulate (through a transformation) the relation

$$P = I \cdot H \cdot \Pi$$

where P is the number of pianos in Chicago, I is the number of inhabitants, H is the number of households per inhabitant and Π is the number of pianos per household. The difficulties the students have – as mentioned above (Bloch, 2003) – are potentially easier to overcome, seeing as it is reasonable to believe that the nature of a Fermi problem give rise to higher intuitive understanding, than algebraic manipulation of some formula or equation in the traditional sense.

The geometric setting

This setting is – when working with functions in general – not used much (Bloch, 2003). It is, however, a situation that naturally may arise when working with some Fermi problems – i.e. the swimming pool example as mentioned above. In this case, following the crude illlustraion in Figure 6, a geometric representative is in play.

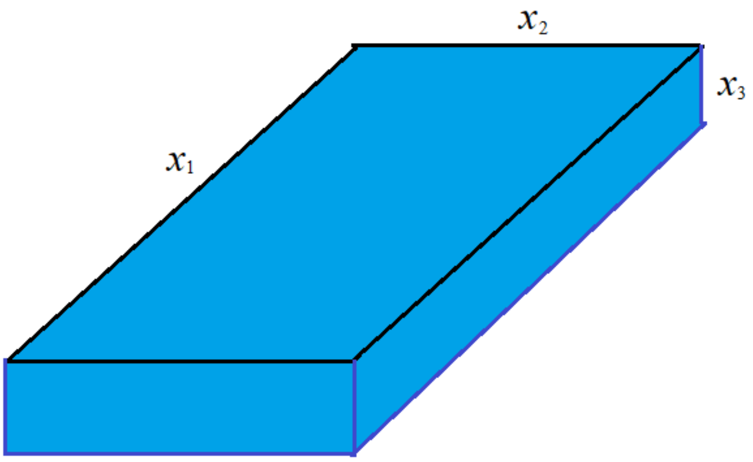

Figure 6: Crude illustration of rectangular swimmingpool with dimensions $x_1 \times x_2 \times x_3$.

The graphic setting

In the traditional sense, the graphic setting of functions is what "one can see in the window" (Bloch, 2003, p. 9). With respect to the polynomial

$$f(x) = 3x^2 - 6x + 2$$

(as used previously), it is the curve representative of this function (Figure 7).

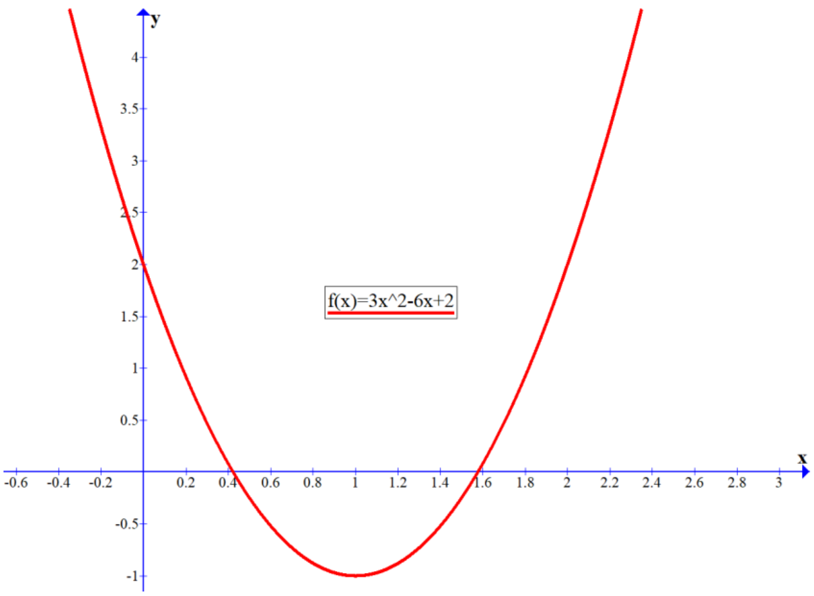

Figure 7: Curve representative of the function $f(x) = 3x^2 - 6x + 2$.

In this situation, one can for instance validate previously calculated roots of the polynomial, or validate that the global minimum is in fact in the point $(1, -1)$.

When working with Fermi problems, the graphic representative is somewhat difficult to handle in a high school setting, seeing as the functions often are of more than one variable. However, if a student would consider the case in which all of the estimates *except one*, are fixed, it is possible (and potentially rewarding) to graph the solution to the Fermi problem as a function of this variable estimate. In this case, the problem will – more often than not – be a representative in the form of a linear function.

The formal setting

When traditionally considering functions, the formal setting concerns the formal definitions, theorems, proofs, etc. It is, however, not a setting that has high impact when working with Fermi problems, as the algebraic setting sufficiently covers what one might perceive as a formal setting.

Ambiguity

Students often tend to treat graphs, formulas and tables as if they unambiguously characterize a function. This is obviously not the case, as, for instance, any straight line segment does not necessarily ambiguously represent one linear function, or similarly, a table with a constant rate of change won't necessarily represent one linear function. Consider for instance the table

x	-1	0	1
y	-1	0	1

which, at first glance, represent the function $f(x) = x$. It is, however, clearly not the only function satisfying the tabular values; all functions $f(x) = x^n$ with $n \geq 1$, and n odd also satisfy these values.

6. Defining the problem

In today's society, applied mathematical modelling serve as an indispensable tool within the fields of technology and science. This development has greatly influenced the various curricula in mathematics courses at high school level – and Denmark is no exception. As of august 2017, new curricula have taken effect, and the focus on mathematical modelling and applied mathematics is a big part of what constitutes the Danish courses of mathematics at A, B and C-level.

When teaching mathematical modelling – or applied mathematics – it is important to realize that learning to *apply* mathematics is a very different activity than that of learning mathematics in the traditional sense. It is a very different skill set that is at use than that of understanding theorems, doing proofs and solving equations.

The main purpose of this thesis is to investigate the hypothesis, that *Fermi problems can be used as an entrance to mathematical modelling with functions of one or severable variables with respect to teaching mathematics at C-level in Danish high schools*.

In order to examine this, I have designed a teaching sequence that will be tested in a junior year high school class. Specifically, I will analyze the adidactic situations that naturally arise when students work with Fermi problems.

7. How to overcome the problem

In order to investigate the hypothesis, I have developed a small course consisting of various Fermi problems. This course was tested on 1.mn at Rødovre Gymnasium, and was carried out in April 2017 over a period of four lessons.

The students, the teacher and the course

The claim that the students of 1.mn are overly eager to learn mathematics would be a great exaggeration – this is also expected when considering the study program that they have chosen. They are either following a study program with primary courses in music, English and drama (1.m) or English, social studies and geography (1.n). The mathematics teacher is experienced with more than 15 years as a teacher at Rødovre Gymnasium. In her opinion, the mathematical skills of the students are a bit below average, but there are a few students who excels. During the course, the students were divided into specified groups, in which the students were at a similar mathematical level. This was to, hopefully, gather data of a varying degree.

The 1st lesson gives a short introduction to Fermi problems, where the students also get to work with a simple case. In the 2nd lesson the students work with more complex problems, with algebraic notation as the focus of study. The 3rd lesson increase the complexity, and uncertainty intervals are also handled. In the 4th lesson, the focus is on the use of CAS and its applicability when modelling.

Data

The data used in this thesis is the course on Fermi problems, which will be described in further detail in the following sections, and observations made during the four lessons. In each lesson, several dictaphones were used to record what was going on. Seeing as a lot of the work was conducted in groups, dictaphones was assigned to three of those. There was also a dictaphone recording from the catheter in the didactic situations. This dictaphone was picked up by the teacher, who observed the students, in situations of adidactic nature. The students also handed in some written work, which was collected as part of the data used in the analysis.

My role in was that of an observer. During situations of a didactic nature, I would sit in the back, taking notes. When situations of an adidactic nature were in effect, I would sneak around in the class-room, observing the work of the various groups. In an ideal world, my presence would not have been noted by the students, but curiosity and playfulness of the students often resulted in me interacting with the students.

Method of data analysis

Through the use of the characteristics of realistic fermi problems as proposed by Ärlebäck (2009), a series of problems was designed. In order to identify what the students should estimate and model, an á priori analysis was also made of the problems (see the following section).

It is important to point out, that the amount of data is huge, and analyses of *all* the data is therefore not conducted (this would be far too extensive in a master thesis, and the time constraints do not allow such an analysis). The focus is therefore on three specific situations, which are all adidactic in nature. The first situation is regarding the students first encounter with a Fermi problem. The 2nd situation is regarding the students work within the algebraic system, and the final situation is concerning the functional aspects of Fermi problems.

The data relevant to the situations was afterwards handled in an á posteriori analysis, which is described in the subsequent chapters.

8. Design of a teaching sequence

The following teaching sequence is designed specifically for use in Danish junior year high schools, possibly as an introduction to mathematical modelling and functions. The sequence has been tested in cooperation with Karen Mohr Pind in a 1.g class at Rødovre Gymnasium. In this sequence, the focus of interest is mainly on adidactic situation; specifically, on students working with Fermi problems in predefined groups determined by the teacher. In total, the sequence is run over the course of four lectures.

The design of the sequence is heavily influenced by the characteristics of realistic Fermi problems as discussed earlier in the thesis (see chapter 5). I will in the following chapter give an overview of the lesson plan associated with each lesson as well as a presentation of the Fermi problems that are handled.

The 1st lesson

The first lesson is an introduction to the concepts of Fermi problems, where the focus is on establishing the ideas of the subject. Here the students will engage in solving the simple problem stated below. The focus here is more about understanding the concept, and we therefore allow the students to work informally with the problem. The duration of this lesson is in total 100 minutes (including a small break).

Material used

The first assignment the students are to solve is formulated as follows:

President Trump and the big numbers

In order to sweeten the life of newly elected American president, Donald Trump, his advisors make use of alternative means in order to describe the situation of the world. They specifically wish to make use of "alternative facts"[3] regarding Trump Tower, the Mexican wall and the presidential inauguration.

Exercise 1

In an attempt to advertise for Trump Tower, the president wants to give each visiting guest a letter that, among other things, describe the following:

A. **How many "5-kroner" (coins) one should stable to reach the top of the tower**
B. **How long it takes to reach the top floor when riding the elevator.**

---

[3] A phrase that was actually used during a "meet the press" interview on January 22, 2017. For further information, see https://en.wikipedia.org/wiki/Alternative_facts (active as of august 2017)

As a group, you are to give reasonable estimates for the questions stated above. **The answer is to be presented as a letter for the visiting guests.**

Lesson plan of the 1st lesson

A lesson plan of the first lesson is given in the following table.

Table 1

Activity	**Time**	**Role of teacher**	**Role of students**
Introduction (Niels)	7 min.	An introduction to the subject as well as an introduction of Niels and his role during the sequence. Methods for data collecting is presented, and the groups are established.	Listening; asking questions.
Introduction to realistic fermi problems	13 min.	Moderator	Debate; a discussion on how many Pinocchio beads (or M&M's) are in a jar, a discussion on the number of blades of grass are in the park. This debate is purely guessing; however strategies on how to come up with educated guesses should be considered.
Fermi-problem: Classic Fermi-problem on the number of piano tuners in Chicago	25 min.	Moderator; makes sure that the estimates used are reasonable; introduce relevant parameters that the students themselves do not think of.	Debate; the students discuss which parameters are relevant in order to decide how many piano tuners are in Chicago. Furthermore, they must agree on reasonable estimates.
Pause	5 min.	-	-
Introduction of the first Fermi-problem	5 min.	Explain the premises of the task at hand (the formulation of a letter)	Listening; asking questions
First Fermi-problem (**1A**)	10 min.	Observer; supervisor. Teacher influence is minimal.	The students work in groups on the Fermi problem formulated in exercise 1A.
Summary (**1A**)	7 min.	Moderator;	The groups present the different strategies (i.e. height of the tower, width of a 5-krone, etc.) Afterwards they propose a solution to the problem. Supposedly they have a critical mind regarding the estimates

			and question the validity of these. The teacher notes the estimates on the black board.
First Fermi-problem (**1B**)	15 min.	Moderator; supervisor. Teacher influence is, again, minimal; she may refer to the previous summary situation.	The students work in groups with the Fermi-problem given in exercise 1B.
Discussion (**1B**)	7 min.	Moderator/writer	The students present their strategies and estimates; a proposed solution is worked out; they may also ask questions, and should discuss the proposed solution.
Summary of the lesson and a short introduction to the next lesson	6 min.	Moderator	The students listen and ask questions.

The 2nd lesson

In the 2nd lesson, the focus is on formalization. The students are given a new problem in which formulas, variables and algebraic notation is used.

Material used

In the 2nd lesson, the students are given the assignment below. They are also handed a work-sheet in which variables, equations and their relations were to be written down. A Danish example of the work-sheet can be found in the appendix.

Exercise 2

In order to construct the Mexican wall, Donald Trump wants to investigate how much it will cost as well as the amount of building materials are to be used. In this exercise, you act as designers/engineers assigned to the project, and you are to answer the following question:

A. If the Mexican wall would be constructed using lego blocks, how much would it cost?

When answering this question, you may find it helpful to solve the following sub-questions:

i. Which variables are relevant when solving this problem? (i.e. h describing the height of the wall)
ii. What are the relations between the variables (i.e. price, length, width, height, etc.)?
iii. How many (regular) bricks does it take to construct the wall?
iv. How many regular (2×4) legos does it take to build a regular brick?

Since you are the main designers affiliated with this project, it is of great importance that all variables, estimates and equations are written down.

Lesson plan of the 2nd lesson

A lesson plan of the 2nd lesson is given in the following table.

Table 2

Activity	Time	Role of teacher	Role of student
Summary of the previous lesson	8 min.	Moderator	Active participants in the summary; listening/ asking questions
Introducing variable relations	17 min.	Lecturer. Using the examples on the black board, a formalization is carried out through use of variables and equations.	Active participants; the students note the most important aspects for further use.
Introduction **(episode 2)**	3 min.	Explaining the premises of the assignment	Listening; asking questions

Fermi-problem (**2**)	20 min.	Observer; supervisor in case the students stall. The teacher should, as much as possible, attempt to avoid guiding the students.	Group-work on exercise 2: The students share ideas and strategies. They should focus on relevant variables, estimates and equations, and explicitly write down which variables are used.
Pause	5 min.		
Status (variable)	5 min.	Moderator Writer; writes down relevant variables on the black board.	The students present their findings; specifically, which variables are needed to answer the question.
Further work on Fermi-problem (**2**)	30 min.	Observer; supervisor in case the students stall. The teacher should, as much as possible, attempt to avoid guiding the students. Creates a table on the black board in which the students can write down relevant variable relationships and estimates.	Through use of the variables they have identified, the students work out relevant relationships between them. When the students have, an educated guess regarding the sub-questions, they write it down on the black board. Finally, they work out a written formulation of their considerations; including formulas and variables that are in play.
Discussion **(2)**	10 min.	Moderator and writer	The students present their estimates and variable relationships as well as arguments surrounding those.
Final summary	5 min.	Moderator	Listening/asking questions

The 3rd lesson

In the 3rd lesson, the focus is on formulization as well as the use of linearity, intervals and geometric representatives.

Material used

In the 3rd lesson, the students are given the assignment below. They are also handed a work-sheet similarly to that of the 2nd lesson (see the appendix).

Exercise 3

In order to clarify the statements that were made regarding the attendance of the inauguration speech, Trumps advisors seek answers to the following questions:

A. If all of the inhabitants of Denmark were standing as closely together as possible, how much space would we occupy (area)?

B. What about the population of Copenhagen? The United States of America? The world?

Fill out the following tables ***after*** the classroom discussion of **A** and **B**.

	Number of people pr. m^2
Maximum value	
Minimum value	

C. How many attended president Obama's inauguration speech (see Figure 8**)? Use both the maximum and the minimum value of the table above in your analysis.**

D. How many attended president Trump's inauguration speech (see Figure 8**)? Use both the maximum and the minimum value of the table above in your analysis.**

E. Choose a president; Trump or Obama. Argue, through use of the previous analysis, that it is *your* president who had the greatest attendance.

Figure 8: National Mall when Trump was inaugurated in 2017 (left) and when Obama was re-inaugurated in 2009 (right).

Lesson plan of the 3rd lesson

A lesson plan of the 3rd lesson is given in the following table.

Table 3

Activity	Time	Role of teacher	Role of student
Summary of previous lesson	10 min.	Moderator	Active participants in the summary; listening/ asking questions
Introduction of exercise 3 **(3I)**	3 min.	Introducing the assignment and its premises.	Listening and/or asking questions
Fermi-problem nr. 3A+3B **(3I)**	25 min.	Observer/supervisor if needed (mainly if the students do not understand the questions).	Focus on identifying relevant variables and estimates concerning populations and areas.
Pause	5 min.		
Summary of 3A+3B **(3I)**	10 min.	Moderator; The class are to agree on an appropriate interval (max/min) of how many people can occupy a given area.	The students present their strategies, estimates and results. They also debate and ask questions regarding the validity of said results.
Fermi-problem nr. 3C+3D+3E **(3II)**	32 min.	Observer	The students decide which president they want to argue for, and work out arguments as to why it is *their* president who had the highest attendance at the inauguration.
Discussion of 3E **(3II)**	7 min.	Moderator	Debate on the problem at hand
Summary	8 min.	Moderator	Listening, asking questions

The 4th lesson

In the 4th lesson, the focus is on the use of CAS and how it is applicable when modelling, through the use of functions in Maple.

Material used

The assignments presented in lessons 1 through 3 are used again in this lesson. Furthermore, a Maple worksheet is introduced in which the students work out the problems stated.

Lesson plan of the 4th lesson

Table 4

Activity	Time	Role of teacher	Role of student
Summary of previous lesson	7 min.	Moderator	Active participants of the summary; asking questions.
Introduction to modellering using Maple **(8.4I)**	15 min.	Lecturer; Fermi problem nr. 1 is solved through using the Maple-document handed out.	Listening/Asking quesitons
Fermi problem nr. 2 (Maple) **(8.4II)**	20 min.	Observer; supervisor (Maple technician)	The students work with Fermi problem nr. 2, where they use the previously defined variables and relations. They are to write out a program that can solve the problem. Also the linear relation between price and length of the wall is examined as a function.
Pause	5 min.		
Student presentation and summary **(8.4II)**	7 min.	Observer/"technical assistent"	One (or more) of the students present their Maple-sheet that solves the problem (using the electronic black board).
Introducing problem 3 **(8.4III)**	3 min.	Introducing the premises of the third problem; specifically the linear relationship between the number of people and the minimum/maximum values of the previous lesson	Listening, asking questions
Fermi problem nr. 3 (Maple) **(8.4III)**	20 min.	Observer, Supervisor (maple-help)	Group-work on the third problem. Again they write out a program that solves the problem; furthermore they define functions for both minimum/maximum values of the previously mentioned values, such that the program outputs an interval (of the space occupied) as a

			function of the number of people (in a graphic representation).
Student presentation and summary (**8.4III**)	7 min.	Observer/"technical assistent"	One (or more) of the students present their Maple-sheet that solves the problem (using the electronic black board).
Summary	10 min.	Moderator	Listening/asking questions
End of sequence	6 min.	Niels' final remarks	Listening, asking questions

9. A priori analysis

In this section, I will perform an a priori analysis of select situations, which are smaller parts of the teaching sequence, as described in the previous section (see tables 1, 2, 3 and 4). In each of the situations, we will first describe the *phases* (see section 4), followed by an analysis of the prior knowledge of the students, the knowledge to be taught and the objective milieu.

A priori analysis of the 1st lesson (the 1st Fermi problem)

Context

We will now turn our attention towards specific situations in the teaching design of the first module. The overall goal of the lesson is to introduce a few, simple realistic Fermi problems (see Table 1). This lesson is the first time the students will get acquainted with these types of problems, but they do have some knowledge of linear functions and basic algebraic methods.

Specific situations

In this analysis, we wish to point out the knowledge to be taught in specific situations, and also to clarify what the objective milieu is, and what knowledge the students possess with regards to the situation. We will consider the situations (mainly the adidactic ones) regarding the episodes described as the first (1A) and second (1B) problem: "*How many 5-kroner coins should we stack in order to reach the top of Trump Tower?*" and "*How much time does it take to reach the top of Trump Tower, when riding an elevator?*"

Prior knowledge of the students

Prior to these above-mentioned episodes, the students have only had a short introduction to the concepts of Fermi problems. It is therefore not reasonable to already conceive the knowledge to be taught as well established; it is rather knowledge under development. It is worth noting that the students are - prior to this teaching sequence - acquainted with the concept of linear equations, and also an elementary idea of the concept of functions.

Phases of the episodes 1A and 1B

Let us first consider the situations regarding episode 1A. In accordance with the teaching sequence, there will first be a didactical situation in the form of devolution, where the teacher introduces the assignment at hand. Here the teacher explains the premises of the problem - the students are supposed to give a qualified answer to the problem in the form of a letter - and the students are allowed to ask questions. This didactic situation is followed by an adidactic situation where the students (in groups) work with the problem at hand. This situation is expected to consist of various phases of action, formulation and validation; i.e. the students will engage in the problem, formulate hypotheses and validate their work through group interactions. This situation is followed by a didactical validation phase, where the teacher and the students discuss the validity of the used estimates. This will lead to a didactical institutionalization phase, where the problem finally will be considered answered.

In order to introduce the episode 1B, a new didactical devolution phase will take place. Here the teacher will point to the estimates agreed upon with respect to episode 1A (i.e. height of the tower, which is relevant in answering 1B). This is again followed by an adidactic situation where the students (in the same groups) work with the problem at hand. This situation will similarly consist of phases of action, formulation and validation. This situation is analogously followed by a didactical validation phase, though this time a didactical institutionalization will not take place. Instead, the students will engage in an adidactic institutionalization phase in which they write out the letter that answers the questions at hand. A table with a focus on the phases of the situations is given below:

Table 5

Episode	**Phase**	**Situation**
Episode 1A	Devolution: Introduction and rules of the didactical game	Didactical
	Action/formulation/validation	Adidactical
	Classroom validation: Estimates are discussed	Didactical
	Classroom institutionalization: Agreement upon an answer to the problem	Didactical
Episode 1B	Devolution: Introduction and rules of the didactical game	Didactical
	Action/formulation/validation	Adidactical
	Classroom validation: Estimates are discussed	Didactical
	Institutionalization: Students write out letters	Adidactical

The knowledge to be taught

In the episodes 1A and 1B, the knowledge to be taught is very similar. The students are to use realistic estimates for various parameters, in order to implicitly set up a system of equations, wherein the solutions of the problems lie. Specifically, in 1A the students should be able to give sensible estimates for parameters such as the thickness of a coin and the height of one floor. By giving these estimates, they should be able to find expressions that describe the height of Trump Tower, and thereby also establishing how many coins one should stable in order to reach this magnitude. In 1B the students should be able to give sensible estimates of the speed of an elevator, and they should be able to discuss how many stops such an elevator on average would do on a bottom-to-top ride. This discussion should be sparked by the use of personal experience they themselves have of riding elevators. The use of systems of equations and methods of substitution should also appear in the process of solving this problem.

The Objective Milieu

In both problems, the objective milieu is defined by the assignment as shown in Figure 9 below. Furthermore, a stack of "5-kroners" is present in the classroom (for measuring if needed). There is a rich adidactic potential in this setting, since the possible paths to solving the problems all are based on common sense. Specifically, the adidactic validation phases in the group work situations are heavily based on this use of common sense, and the assignment will therefore give feedback to the students during the process of working out the solutions.

1.mr Matematik April, 2017

Introduktion til Fermi-problemer

Præsident Trump og de store tal.

I et forsøg på at forsøde tilværelsen for den nye amerikanske præsident, Donald Trump, benytter hans rådgivere bl.a. alternative midler til at beskrive verdenssituationen for ham. De ønsker i denne forbindelse at give nogle "alternative fakta" omkring Trump Tower, den mexikanske mur og præsidentindsættelsen.

Opgave 1

I forbindelse med promoveringen af Trump Tower ønsker præsidenten at der til enhver besøgende gæst udleveres et brev der blandt andet beskriver følgende:

A. Hvor mange 5-kroner man skal stable, for at nå toppen af tårnet
B. Hvor lang tid det tager med elevator at komme op til toppen af tårnet

I skal i grupperne komme med passende estimater for svarene på de to spørgsmål. Husk, at svaret skal udarbejdes som et brev til de besøgende gæster!

Figur 1 - Trump Tower (og Donald Trump der nyder udsigten)

Figure 9

In the process of solving the problem, the students are also asked to write relevant variables and estimates on the blackboard, thereby potentially inspiring other groups by giving hints/strategies. In the assignment, the groups are asked to do the following (last sentence):

"In the groups, you must find sensible estimates that answer the two questions above. Remember, that the answer must be in the form of a letter to the visiting guests."

The intents of letting the students work out a letter in order to answer the questions, are both such that they gain ownership of the problem, and thereby inspire the students to work out serious arguments and answers in the process. This phase of validation and institutionalization also hold a rich adidactic potential.

Expected student strategies

If the students are to solve the problem in Figure 9, certain quantities must be reasonably estimated and modeled. In this case, one can expect that the students give estimates of the following quantities:

- Height of the tower; possibly through use of
    - an estimate of the number of floors
    - an estimate of the average height of one floor
- Width of a "5-krone" (coin) – the students might measure this directly
- The speed of the elevator, including
    - an estimate of the average number of stops/starts during travel
    - an estimate of the average number of travelers using the elevator
    - an estimate of loading/offloading time

We will consider the second part of the problem, where a solution could be as follows:

Let P be the solution to the problem. I then estimate that the number of floors of Trump tower is at least 50 and at most 80. An educated guess would therefore be 65 floors. Through eye-measuring, I estimate the height of the class-room floor to be 3 m, so an estimate of the height of Trump Tower is approximately $65 \cdot 2{,}5\ \mathrm{m} = 195\ \mathrm{m}$. I will now estimate the speed of the elevator through use of personal experience from riding elevators. When living at "Grønjordskollegiet", it would take approximately 10 seconds to travel a height of 6 floors, hence I estimate that the average speed of an elevator is 2,4 m/s. In a skyscraper, such as Trump Tower, there are probably a lot of activity, whence many people use the elevators. Assuming that 15 people in average enter the elevator from the start, it is reasonable to assume that during the travel, the initial passengers will require at least 12 stops before reaching the top. People might also enter along the way, so I estimate that an additional 5 stops is required. I also estimate that the loading/offloading time is 16 seconds in average (people in New York are in a hurry), hence the total loading/offloading time is estimated to be $17\ \mathrm{stops} \cdot 15 \frac{\mathrm{s}}{\mathrm{stop}} = 255\ \mathrm{s}$. The estimate of the time it will take is therefore

$$P = \frac{195\ \mathrm{m}}{2{,}4 \frac{\mathrm{m}}{\mathrm{s}}} + 17\ \mathrm{stops} \cdot 15 \frac{\mathrm{s}}{\mathrm{stop}} \approx 80\ \mathrm{s} + 255\ \mathrm{s} = 335\ \mathrm{s} \approx 5{,}5\ \mathrm{minutes}$$

The actual time it takes is probably a lot lower than that, since elevators in skyscrapers, of course, are a lot faster than your average student hall elevator. However, reasoning along the lines of the above, is what I expect from the students.

A priori analysis of the 2nd lesson (the 2nd Fermi-problem)

Context

As a follow-up to the first lesson, the teacher will first use the results from this lesson, and describe the relevant quantities by use of algebraic symbols. This will be followed up by explicit use of equations to describe how the problem was solved. The use of algebraic notation will lead to the episode we wish to analyze in the lesson connected to the 2nd Fermi problem; specifically, episode 2 (see Table 2).

Specific situation

As in the previous analysis, we wish to do a thorough analysis of the episode described as Fermi problem number 2 as part of the 2nd lesson (see Table 2): "*In case the Mexican wall was built using Lego blocks, how much would the wall cost?*"

Prior knowledge of the students

When engaging the 2nd Fermi problem, the students now have a grasp of the concept of Fermi problems. It is, however, still knowledge under development - especially regarding the use of variables and the dependencies of the involved quantities. The use of basic algebraic manipulation will be of greater importance when engaging in problem 2 (compared to earlier), seeing as one of the main aspects of this problem is for the students to introduce variables and using those in order to describe various relations between those variables.

Phases of episode 2

In episode 2, the teacher introduces the assignment at hand. Here she explains the premises of the problem - this time the students are engineers doing calculations for the construction of the Mexican wall - and the students are allowed to ask questions. This didactical situation is followed by an adidactic situation where the students (in groups) work with the problem at hand. This situation is expected to consist of various phases of action, formulation and validation; i.e. the students will engage in the problem, formulate hypotheses and validate their work through group interactions. In an intermediate validation phase, the teacher will - in a didactical situation - ask for relevant variables which the students will present. After this validation, the students will once again (in groups) engage in adidactical action/formulation/validation phases, in which they work out the relevant variable relations and estimates. Finally, the lesson will be concluded with a didactical phase of combined validation and institutionalization, with the students handing in their supplementary work sheets.

Table 6

Episode	Phase	Situation
Episode 2	Devolution: Introduction and rules of the didactical game	Didactical
	Action/formulation/validation	Adidactical
	Classroom validation: Variables are discussed	Didactical
	Action/formulation/validation: Work with said variables in mind, work out equations (relations between variables) and estimates	Adidactical
	Validation/Institutionalization: Discussion of solutions to the problem. Groups hand in work sheets.	Didactical

The knowledge to be taught

In this episode, the focus is mainly on explicit algebraic modelling; i.e. formally defining relationships between symbolic representations of quantities. Giving good estimates for those quantities is still a focus point, and the students are in this sense still supposed to use their common sense. The complexity is, compared to problems 1A and 1B, a lot higher, though, and the explicit use of algebraic symbols is required, as a result.

The Objective Milieu

The objective milieu in this episode is provided by the assignment described in Figure 10 below, along with a worksheet, in which they are asked to fill out variables, estimates and connection to other variables (see Appendix 1). In this assignment, the students are supposed give a qualified solution to the Fermi problem of "the price of the Mexican wall, if this is built using Lego blocks". In order for the students to not drown in the complexity of this question, underlying questions are also provided; i.e. "*Which variables should we identify in order to solve this problem (i.e. h for the height of the wall)?*" and "*What are the relationships between those variables (i.e. relationship between price, length, height etc.)?*"

These questions along with the worksheet provide some degree of scaffolding in order to guide the students into answering the questions in a satisfying manner; however, it still keeps the question open and - hopefully - intriguing for the students to engage. The realistic nature of this problem also functions as a built-in validation tool, seeing as the students use of common sense will function as a validation in itself.

1.mr Matematik April, 2017

Opgave 2

I forbindelse med konstruktionen af den nye mexikanske mur, ønsker Donald Trump en undersøgelse af, hvad denne vil koste, samt hvor meget materiale man skal bruge i forbindelse med denne konstruktion. I skal i denne opgave agere bygningsingeniører for dette projekt, hvor der ønskes et kvalificeret svar på det følgende spørgsmål:

A. I tilfælde af, at muren skal bygges af legoklodser, hvad bliver prisen da?

I kan - med fordel - søge svar på nogle af disse underliggende spørgsmål i forbindelse med besvarelsen af den overordnede opgave:

Hvilke variable skal indgå i forhold til løsning af dette problem (f.eks h for murens højde)?

Hvad er sammenhængene imellem disse variable (f.eks. sammenhæng mellem pris, længde, højde osv.)?

Hvor mange mursten (af almindelig størrelse) skal der bruges til at bygge muren?

Hvor mange almindelige legoklodser (2x4) skal der bruges til at bygge en mursten?

Idet I fungerer som bygningsingeniører på dette projekt, er det nødvendigt at alle ligninger, variable og estimater er veldokumenterede. *I skal - med andre ord - skrive disse ned!*

Figur 2 - Donald Trump og den mexikanske mur

Figure 10

Similarly, to episodes 1A and 1B, the adidactic potential is high in this setting; also due to the fact that students are able to engage in the underlying questions without teacher interactions - also without giving too much away (i.e. in the sense of a Topaze effect).

Expected student strategy

In this Fermi problem, the students are expected to solve the problem with the explicit use of formulas – a clear difference to the more informal strategy proposed in the introduction. They are to write out formulas and use algebraic notation using the work-sheet (see Appendix 1), and the expected general strategy for solving the problem is as follows:

I start with estimating that the wall is 10 meters high, and denoting h_w as the height of the wall, I have

$$h_w = 10m$$

Furthermore, I estimate that the width of the wall, w_w, is approximately 2 meters, and that the length of the wall, l_w, is 3000 kilometers,

$$w_w = 2m \text{ and } l_w = 3 \cdot 10^6 m$$

Since the wall is to be built from legos, I need to estimate the size of such a block. As a kid, I played a lot with legos, and through this experience, I estimate that the dimensions are as follows:

$$l_b = 4cm = 4 \cdot 10^{-2} m, \ h_b = 1cm = 1 \cdot 10^{-2} m \text{ and } w_b = 1{,}5cm = 1{,}5 \cdot 10^{-2} m$$

Here, l_b denote the length of one block, and h_b and w_b denote height and width respectively. If I assume that the wall is completely solid, I can calculate an estimate for the number of blocks that are needed for the construction, n_b using the following equation:

$$n_b = \frac{l_w}{l_b} \cdot \frac{w_w}{w_b} \cdot \frac{h_w}{h_b} = 1 \cdot 10^{13} \text{ blocks}$$

Through personal experience, I know that legos are expensive, and a box of legos with approximately 500 pieces is about 400 kroner. Thus, the price for one block, p_b, is estimated to be

$$p_b = \frac{400 \text{ pieces}}{500 \text{ kroner}} = 0{,}8 \frac{\text{kr}}{\text{pc}}.$$

The wall does not build itself, and labor must also be considered. If I assume that a worker can apply one piece per second, the total number of work-hours, h_w, is

$$h_w = \frac{n_b}{3600 \frac{\text{blocks}}{\text{hour}}} \approx 2{,}8 \cdot 10^9 \text{hrs.}$$

Assuming a salary, s, of 120 kr/hr, we estimate that the total price P is

$$P = h_w \cdot s + n_b \cdot p_b = 8.3 \cdot 10^{13} \text{ kr.}$$

This price is, of course, completely ridiculous when compared to the actual estimate of the wall (which is estimated to be in the range of $12 billion to $15 billion)[4], however, this perspectivation I do not expect from the students at this level.

A priori analysis of the 3rd lesson (the 3rd Fermi-problem)

Context

While the previous sequences of Fermi problems have handled algebraic modelling in the sense of establishing linear relations between various quantities, they have not considered the potential of handling *intervals of estimation* of said quantities. Through my own personal experience, I find that this is conceptually difficult to grasp for students at this level, and an introduction should therefore be handled delicately. In order to examine the potential of a teaching sequence which handle these intervals of estimation, we consider a classic Fermi problem which is stated as follows: "*How much room would the entire world population take up, if standing as close as possible side by side?*"

Specific situations

In this problem, we consider a part of the 3rd lesson as described in Table 3; specifically, the situations connected to assignment 3 as given in Figure 11 below (see the objective milieu). This assignment gives rise to various interesting situations, and we will split this into two collections of situations (episodes): 3I and 3II respectively (see Table 3 for clarification). In episode 3I, the students are asked to solve variants of the classic Fermi problem (as described earlier; see the previous section and chapter 5); namely the questions "If the population of Denmark all stood next to each other, how much space would we occupy?*"* This question is then followed up by an alternate version with modified populations (i.e. Copenhagen, USA, the world). The final part of 3I, is asking for an expression describing the area as a function of the size of the population in question; and also, which kind of function is in question. In the episode 3II an interval of the people/area is given in the form of a minimum number of people/m^2 and a maximum number of people/m^2 (these numbers are estimated by the students themselves). This leads to questions 3C, 3D and 3E, where the students are to use the interval endpoints in order to estimate how many people were attending the inauguration speeches of Barack Obama (2009) and Donald Trump (2017) respectively, followed by an assignment in which they shall provide arguments for one or the other (regarding the question of which president had the most attendees).

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[4] https://www.cnbc.com/2015/10/09/this-is-what-trumps-border-wall-could-cost-us.html (active as of august 2017)

Prior knowledge of the students

When engaging this lesson, the students are expected to be familiar with the process of solving simple Fermi problems; specifically, in defining variables, setting up equations and providing reasonable estimates describing the given situation. They also have knowledge of linear functions from an earlier teaching sequence, and knowledge of intervals.

Phases of episodes 3I and 3II

In episode 3I, the teacher first introduces the assignment at hand. She explains the premises of the problem - the students are to answer questions for Donald Trump's advisors regarding the attendance at the inauguration speech - and the students are allowed to ask questions. This didactical situation is followed by an adidactic situation where the students (in groups) work with the problem at hand. This situation is expected to consist of various phases of action, formulation and validation; i.e. the students will engage in the problem, formulate hypotheses and validate their work through group interactions. In an intermediate validation phase, the teacher will - in a didactical situation - ask for relevant quantities regarding the maximum density (people/area).

The students will then engage in episode 3II; here they will once again (in groups) engage in adidactic action/formulation/validation phases, in which they work out the relevant variable relations and estimates. Finally, the lesson will be concluded with a didactical phase of combined validation and institutionalization, where the students argue their respective standpoints.

Table 7

Episode	**Phase**	**Situation**
Episode 3I	Devolution: Introduction and rules of the didactical game	Didactical
	Action/formulation/validation	Adidactical
	Classroom validation: Estimate interval is discussed	Didactical
Episode 3II	Action/formulation/validation	Adidactical
	Validation/Institutionalization: Debate; Who had the most attendees?	Adidactical/Didactical

The knowledge to be taught

In this episode, the focus is again on explicit algebraic modelling; i.e. formally defining relationships between symbolic representations of quantities. A new interaction, however, is in the use of intervals and how to apply a linear expression on an interval. The students are here, in groups, supposed to estimate a maximum of how many people can occupy a given area (say, 1 m^2). These answers will vary from group to group, and in order to establish an interval, the lowest and highest estimates respectively, will be used to define the interval, which the students are supposed to work with, when engaging episode 3II. Also, the students are intended to use geometric considerations and the concept of concentration measurement; i.e. in the sense of population density, in order to properly solve the problem.

The Objective Milieu

The objective milieu in these episodes is provided by the assignment described in Figure 11 below, along with a worksheet, in which they are asked to fill out variables, estimates and connection to other variables (see Appendix 2).

1.mr Matematik April, 2017

Opgave 3

I et forsøg på at rette op på udtalelserne omkring antallet af besøgende i forbindelse med præsidentindsættelsen (se Figur 3), ønsker Trumps rådgivere svar på følgende spørgsmål:

A. Hvis hele Danmarks befolkning stod samlet så tæt som muligt, hvor stort et areal ville vi så fylde?
B. Hvad med Københavns befolkning? USA's befolkning? Eller jordens befolkning?

Notér de værdier klassen har fundet; hhv. den laveste værdi af antal mennesker pr. m^2 og den største værdi for antallet af mennesker per m^2.

	Antal mennesker pr. m^2
Maksimale værdi	
Minimale værdi	

C. Hvor mange mennesker var der til præsident Obama's indsættelse (jvf. Figur 3)? Brug i analysen både den mindste og den største værdi fra ovenstående skema.
D. Hvor mange mennesker var der til præsident Trump's indsættelse (jvf. Figur 3)? Brug i analysen både den mindste og den største værdi fra ovenstående skema.
E. Vælg en præsident; enten Trump eller Obama. Argumentér ud fra svarene på ovenstående spørgsmål for, at netop *jeres* præsident havde flest tilhængere ved indsættelsestalen.

Figur 3 - National Mall da Trump blev indsat i 2017 (til venstre) og da Obama blev genindsat i 2009 (til højre)

Figure 11

In this assignment, the students are supposed give a qualified solution to the following Fermi problems:

3A: "If the population of Denmark all stood next to each other, how much space would we occupy?"

3B: "What about the population of Copenhagen? The United States of America? The world?"

3C: “How many people were at Obamas speech (according to Figur 3)? Use both the maximum and minimum value from the table above. “

3D: “How many people were at Trumps speech (according to Figur 3)? Use both the maximum and minimum value from the table above. “

3E: “Choose a president; Trump or Obama. Argue from the answers of **3.C** and **3.D** that it is *your* president who had the greatest attendance at the inauguration speech”

The students are here expected to be able to engage in answering the questions group-wise, this time without the use of scaffolding sub-questions. They will, however, get a worksheet (as in the 2nd Fermi problem), where they can fill out relevant information with respect to answering the questions. It is important to note here, that there will be a mid-way evaluation between questions **3B** and **3C**, where the interval of the number people per square meter is provided (see Figure 11).

As a validation, again we find that the realistic nature of this problem functions as a built-in validation tool, where the students’ use of their own rationale will function as validation.

Expected student strategy

A possible solution to this problem is as follows: Let $A(p)$ be the function describing the occupied area as a function of the population, p. Let A_1 be an estimate of how many people – on average – can occupy a space of $1\ \mathrm{m}^2$ (I assume that this is when standing shoulder to shoulder). I estimate that the value is $A_1 = 8\ \frac{\text{people}}{\mathrm{m}^2}$ (it is possible to do a sub Fermi problem here through use of average width and thickness estimates. I will leave this to the reader). Let D denote the population of Denmark. It is a well-known fact that the population of Denmark is roughly 5,5 million, and I will use this as my estimate. The general solution is then given as

$$A(p) = \frac{p}{A_1},$$

and in the case of the area for the Danish population, question 3.A, I find

$$A(5.5 \cdot 10^6) = \frac{5.5 \cdot 10^6 \text{ people}}{8 \frac{\text{people}}{\mathrm{m}^2}} \approx 7 \cdot 10^5\ \mathrm{m}^2.$$

Similar results may be found for the population of Copenhagen, The United States of America and the world, answering question 3.B. I now assume that an interval is given for the maximum and the minimum number of people occupying one square meter respectively. Using the image, I will estimate how many people are occupying the area of the National Mall, see Figure 12, answering exercise 3C.

Figure 12

First, I will estimate the area of the mall, by use of eye measure. I consider one of the trees in the right corner, and assume that it has a height of 20 meters. I then assume that the width of the mall at the bottom is approximately ten times the height of the tree, a total of 200 meters. Now the length is a bit more tricky. Here I assume that the length is about 2.5 times longer than the width of the mall. I now have an estimate of the total areal; $A_m = 2.5 \cdot 200\text{ m} \cdot 200\text{ m} = 100000\text{ m}^2$. I will now estimate the number of people occupying the space. It looks like a lot of the space is occupied by trees, and it is therefore difficult to see if there are people standing below those. A reasonable upper bound is 80% of the area, and a sensible lower bound is 50%. I will settle on 65%. I now use a modified version of the function from before, where I isolate the population p:

$$p = 8\frac{\text{people}}{\text{m}^2} \cdot 0.65 \cdot 10^5\text{ m}^2 \approx 5{,}2 \cdot 10^5\text{ people}$$

or about half a million. A similar analysis may be carried out for the Trump case, and I will leave this to the reader. In order to argue for which president had the most attendees – exercise 3.E – the aforementioned interval is used when calculating the estimation of the number; in this case in the form of an interval. If the resulting intervals overlap, it is possible to argue for either president as having the highest attendance.

A priori analysis of the 4th lesson (using CAS with Fermi problems)

Context

In this lesson, the students will be using CAS - specifically Maple - in order to algebraically model the Fermi problems discussed in the previous lessons. This includes describing some of the situations using functions of potentially multiple variables, which will be the main focus point. In order to secure a clear understanding of the objective, the students are asked to write Maple-code that is able to compute answers to the questions stated in problem two and three - in that way, they are comfortable with the assignment at hand.

Specific situations

We will again focus on the adidactic situations, where the students in groups work with the assignments at hand. Here they will rework their answers to the questions from the assignments shown in Figure 10 and Figure 11 (exercises 2 and 3). Before they engage in these, the teacher will introduce a pre-made Maple-sheet (see Appendix 3), in which the 1st Fermi problem is treated. This sheet will also work as the objective milieu, where the assignments are stated.

Prior knowledge of the students

In this lesson, it is expected that the students are familiar with the theory described in lessons one, two and three, as well as the fundamental tools of Maple. Among other things, they know how to assign variables, define functions, plot graphs and solve equations. These tools are all fundamental in order to design Maple-code that solves exercises 2 and 3.

Phases of episode 4

In episode 4I, the teacher first introduces Maple as a useful tool in solving the 1st Fermi problem. She also explains the premises of the 2nd problem: The students are to design Maple-code that is able to solve the 2nd Fermi problem - a problem which they are already very familiar with. This didactical situation is followed by an adidactical situation where the students (in groups) work with the problem at hand. This situation is expected to consist of various phases of action, formulation and validation; i.e. the students will engage in the problem, formulate hypotheses and validate their work through group interactions. Finally, a voluntary student is to present the Maple-code for the rest of the class on the e-board ("electronic blackboard").

The students will then engage in episode 4II; here they will once again (in groups) engage in adidactical action/formulation/validation phases, in which they work out the Maple-code that can solve the 3rd problem. Finally, the lesson will be concluded with a didactical phase of combined validation and institutionalization, where the students also are to present the solutions.

Table 8

Episode	Phase	Situation
Episode 4I	Devolution: Introduction and rules of the didactical game	Didactical
	Action/formulation/validation	Adidactical
	Classroom validation: Estimate interval is discussed	Didactical
Episode 4II	Action/formulation/validation	Adidactical
	Validation/Institutionalization	Adidactical/Didactical

The knowledge to be taught

The focus in this episode is on algebraic modelling using CAS, the modelling of functions – including graphic representation of a given problem. Potentially, the students may engage in multivariate linear functions; however, the graphic representation of those will be limited to single-variable ones.

The Objective Milieu

In order for the students to meaningfully engage this task, they are provided with the worksheets they have previously worked on, along with the aforementioned Maple-document, in which the 1st Fermi problem is treated. This document can be found in Appendix 3.

Expected student strategy

In this situation, I expect that the students more or less follow the guide as given in the appendix. The strategies are already well-documented, seeing as the assignments are already previously handled.

10. The realized teaching sequence

The teaching sequence that was in fact carried out differed quite a bit from what was planned in the first place. In general, all the situations regarding Fermi problems did in fact demand a lot more time than my initial plan had taken into account. As an unfortunate result, the final lesson was therefore modified quite a lot.

The 1st lesson

In the first lesson, the introduction using the example "*How many piano tuners are there in Chicago?*" took in fact a little over 35 minutes, which was way more than anticipated. The reason for this was not a bad one – quite the contrary. The students showed a very high interest when going through this example, and a lot of parameters that were not anticipated came to fruition. The students, for instance, were very focused on the musical culture of Chicago, and used a lot of time arguing whether or not it was relevant to include pianos of bars, educational institutions, churches, piano stores, etc.

It was also more time demanding to go through the summaries of problems **1A** and **1B**, and a result of this was that the students did not have the time to properly formulate the letter as intended in the assignment. They did, however, work with a high level of enthusiasm throughout the entire lesson, and in the following section I will present a short á posteriori analysis of the situation regarding problem **1B** as highlighted in Table 5.

Á posteriori analysis of episode 1B

In the following analysis, I have included a short select dialogue of a group (Group A) of students, who, according to their teacher, are considered to be at a lower mathematical level. The group consist of four students – the three girls Anna (A1), Alma (A2) and Aisha (A3), and the boy Alan (A4) (all of the student names are pseudonyms). The situation is initiated with the teacher stating the purpose of the exercise, as well as the time-frame for which they are to solve the problem. Immediately after, the students proceed:

1. **A1:** How much does an elevator weigh?
2. **A4 and A3:** No idea, but how what is the speed of an elevator?
3. **A2:** I think that we, sort of, just have to think about, how fast it is.
4. **A3:** Ehm, what about we use… doesn't it depend on how many times it stops on the way?
5. **A4:** I don't think it stops along the way.
6. **A1:** Yeah I also think it's without stops.
7. **A2:** Ehm, shall we count in, waiting in line in order to use the elevator?

(The teacher overhears this question, and immediately answers)

8. **T:** No! You consider from "I walk into the elevator and travel all the way up".
9. **A3:** What if others have to get off?
10. **T:** No, there isn't. You're all alone and going all the way to the top.
11. **A3:** (Frustrated) Arhhh.. but that could be the case..?

Here we see clear signs of the students engaging constructively in the problem at hand. The 1st quote is somewhat irrelevant, and A4 and A3 quickly discard it, moving along with stating a more relevant: *what is the speed of an elevator?* In quote 4, A3 states a relevant problem, but unfortunately it is also quickly discarded by A4 and A1. The immediate interaction followed by the teacher, is – in my opinion – a violation of the didactical contract of the situation, in the sense that the *solving* of the problem holds higher importance than the *process* of solving the problem. Further on, the students proceed where they use information from the previous exercise, namely that the height of the tower is approximately 200 meters, and that there an estimated 62 floors in the building.

12. **A1:** So each floor is approximately 3,2 meters.
13. **A3:** But that means that… it should actually… ehm… run a distance of 200 minus 3,2 meters… because it doesn't run all the way to the top.. you could say…
14. **A4:** yes, yes.
15. **A3:** So if we for instance consider the bottom of the elevator starting all the way down, it only reaches the floor, and not the top of the tower
16. **A4:** yes exactly.
17. **A3:** but are we just calculating with the 200, or shall we plus.. or minus…?
18. **A4:** Yes we calculate using 200.
19. **A2**: but how do we figure out how fast an elevator is?
20. **A4:** No idea!
21. **A3 and A1:** I have no idea how fast an elevator runs either.
22. **A1:** It's not like a roller-coaster.
23. **A3:** This is difficult!

After this, there is a long period where the students stall and instead starts paying attention to other things going on in the classroom (as well as the dictaphone), before the lesson eventually is ended. Unfortunately, following a very promising start, the students rather quickly give up on the assignment, which may be a direct consequence of the way in which they are used to act, when working with problems in mathematics, that they are not immediately familiar with. At the end – in a didactic summarizing situation – the students immediately expect that the teacher delivers "the correct answer" to the students, which also indicate that the delivery of the premises of the assignment was not handed accordingly (or the students may just have missed the fact that the exact correct answer is not known – even by their teacher). This is also an unfortunate effect of the didactic contract that is in play here; the students are apparently used to being given the answer at some point, thereby not having to procure it themselves.

In other words, the students violate the didactic contract in the sense, that they do not actively seek help when it is needed. It is worth noting, however, that there is a high potential in this assignment when working with students at this level; if the students would have sought the help they needed, they could perhaps have made a greater attempt at finding a solution.

Another group (Group B) consisting of three girls, Berit (B1), Bente (B2) and Bodil (B3) and a boy, Brian (B3), also worked with the problem (again the names are pseudonyms). This is a group of average students, and their work with the problem at hand was as transcribed:

1. **B3:** How tall is an elevator?
2. **B1:** Ehhhmmm…. as tall as a floor?
3. **B2:** I guess..
4. **B4:** Is it really a whole floor??
5. **B3:** No! I guess it's a little less.
6. **B4:** So it's at approximately one floor per second?
7. **B1:** I'm not sure about that… the thing with elevators is… if you run.. I mean if you go from one to two (floors, red.), its at one speed, but if you take from one to something more its another (speed red.). The point is, it will move faster with time… it will add more and more pace.. it accelerates.
8. **B4:** Yes, it just says "wrooom!!" and then its up there.
9. **B1:** exactly, so it accelerates, and then it slows down again.
10. **B3:** When they made this exercise, did they ask people to *not* press the buttons at all floors?
    (students laughing)
11. **B1:** (jokingly) Listen to this… I just wanted to go in/out on each floor
12. **B2:** But seriously, if it's from the bottom, aren't we supposed to subtract one floor??
13. **B1:** Yes of course
14. **B4:** There are 62 including the lower floor
15. **B3:** I don't know…. (pause)… How are we supposed to figure that out?
16. **B1:** How long do you think it takes to run one floor?

17. **B4:** One-and-a-half second, and then it decreases to one second when it has run for a while. I don't think it runs faster than one second (per floor, red.).. do you think?
18. **B1:** So lets agree one approximately one second per floor.
19. **B2:** It doesn't just take one second!!
20. **B3:** I don't think so either!
21. **B1:** Then one-and-a-half second per floor
22. **B3:** I just think it takes quite a long time to ride an elevator.. or is it just me?
23. **B2:** What about two seconds?
24. **B1:** (counting) One… two… yeah that might be the case
25. **B2:** So if we take from when we start moving, because there is a lot of time in the elevator spent on waiting for it to start, right?
26. **B4:** Approximately 10 seconds.
27. **B3:** 10 seconds of wait!!
28. **B1:** Try and start the clock (she is probably referring to a mobile phone) so we can see how much time two seconds take.
29. **B3:** The next time I ride the elevator, I will count this!!

After quite a while, the students settle on four seconds per floor:

30. **B2:** So it's just 4 times 62?
31. **B4:** Plus/minus
32. **B1:** I think with acceleration, we can cut maybe…. five seconds off.. so what about we say its 240 seconds?
33. **B2:** Yes, approximately
34. **B4:** That's not that long, is it?..... It's three minutes… No it's more.. It's four minutes… Five minutes… four minutes…
35. **B3:** Does it really take five minutes to reach the 62nd floor?
36. **B1:** Easily!!
37. **B4:** Four minutes.. that sounds good
38. **B2:** But there is also a big difference between those elevators (she refers to skyscrapers) and those in our old apartment.

In this sequence, we see a lot of both modelling and estimating, for instance the quotes 12-14 indicate that of a modelling task. The fact that acceleration and deceleration also is considered is furthermore examples of actual modelling (quotes 7-9 and 16-17). The students also make a lot of estimates, and they discuss very vividly how many seconds it takes to travel one floor (quotes 19-24), clearly making use of their own real-world perception in the process (quotes 28-29). They even make their own validations in the sense that they discuss whether or not their result seems reasonable (quotes 35-38), and they also doubt the validity of their estimates. This builds up to the students finally agreeing upon four minutes as their answer. The final remark shows again, that they are validating the answer, as B2 (correctly) points out that skyscraper elevators are faster than those in Danish apartment buildings. In this situation, the students seemingly have full ownership over the problem, as per the terms of the didactic contract. The target knowledge – that is, the process of modelling a Fermi problem – is also somewhat acquired.

The 2nd lesson

In the 2nd lesson, the planned summary of the previous lesson also took a lot longer than anticipated. As a consequence, the time the student had to work with the Fermi problem clearly was not enough. The teacher and I therefore agreed to use part of the 3rd lesson as well.

Á posteriori analysis of situations of episode 2

In this analysis, I consider a sequence that is part of episode 2; specifically, that highlighted in Table 6. In this situation, the group we follow consist of three girls and a boy; Celia (C1), Cecilie (C2), Caroline (C3) and Carl (C4) (again pseudonyms). This group consist of the students of the class that are most gifted when it comes to mathematics.

1. **C4:** Does anyone know, ehmmm… the standard size of a lego block?
2. **C3:** I think it depends on whether we use duplo (larger blocks, red.) or the other ones
3. **C4:** The normal ones, I think
4. **C3:** But duplo's are more effective because they are larger
5. **C4:** I don't think that we are allowed to use duplo's
6. **C1:** No.. We can ask?
7. **C4:** I think we should use 2 times 4 centimeters.
8. **C2:** It is 2 times 4!
9. **C1:** I don't think so…

The students decide to ask the teacher for help, and she advises that the students should use the small, regular 2 times 4 blocks.

10. **C2:** Its' because, what we say… I think we can agree that a 2 times 4 is a good, normal one.. but that doesn't mean that its 2 times 4 centimeters.

11. **T:** Yes exactly. But there are legos that are for small children that put it in their mouths, that's what we call duplos. And I don't think you are supposed to use those.
12. **C2:** Because we talked about duplos because it would become a bit larger, and maybe more effective.
13. **T:** You think it is easier that way.
14. **C3:** If I were to build it I would certainly prefer the bigger ones
    (students laughing)

When confronted with the problem, this group immediately considers what the definition of "legos" are. It is worth to note, that this strategy is not one I expected; when I think about legos, I personally always picture the regular 2x4 blocks. This is a clear example of making a model, where the group argue and negotiate on how to engage in solving the problem. The students also have an expectation that when in doubt, the teacher will give them the answer, but in this case (as opposed to the one in the 1st lesson), the teacher does not just give them a method; rather it is up to the students themselves to decide which path they choose. This also becomes apparent in the following situation:

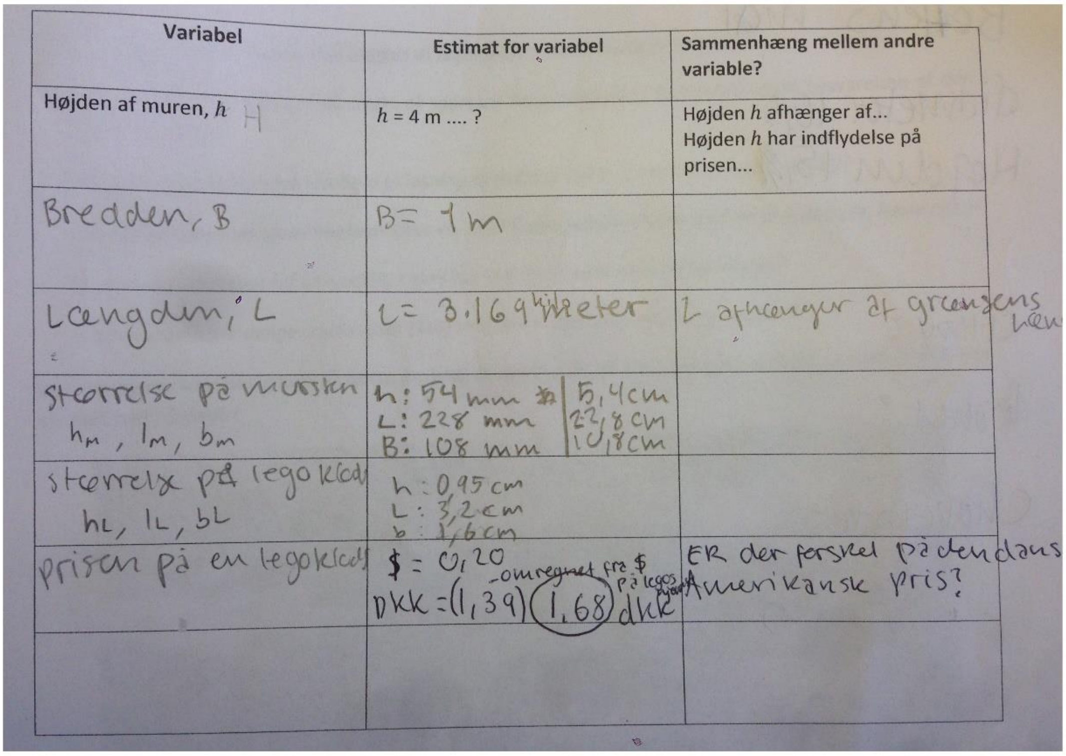
15. **C2:** (in discussing the height of the wall) 12 meters?!? Are you insane?! I don't believe that for a second. Karen, won't you say 12 meters is a bit too much??
16. **T:** I will not say anything! I want to hear what you have to say.
17. **C2:** (simulating crying) That's not fair!

This clearly shows a shift in the establishment of the didactic contract, compared to what the students are used to. The responsibility is entirely the students'; a situation they seemingly are not used to.

The group then carry out a discussion regarding the dimensions of the wall and the dimensions of the lego blocks, which is followed by an intermediate class-discussion regarding the findings so far:

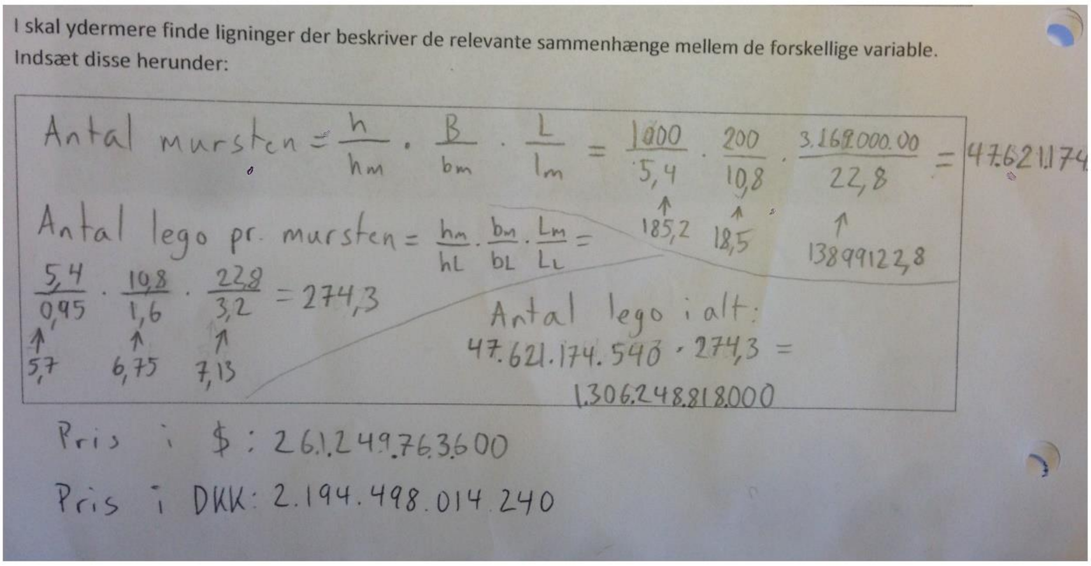
18. **(other student):** We also thought about.. maybe… that the wall should also be under the ground..
19. **T:** Why?
20. **(other student):** Because.. then they can't dig under the wall..
21. **T:** Yeah, we could imagine that a sneaky Mexican would decide to dig his way under the wall, right?
22. **C2:** (enthused) That's genius!! Why didn't we think of that?

Here, the group validate the work of another student, and also make use of this consideration, in order to calculate the number of legos needed to build the wall. They afterwards engage in a process of mass-calculations through use of the various estimates they have found. The transcript of this situation is, however, very vague, but given their written work it is clear, that they have the right idea, and have understood the intended use of algebraic notation, in accordance with what was the target knowledge:



Variabel	Estimat for variabel	Sammenhæng mellem andre variable?
Højden af muren, *h* H	*h* = 4 m .... ?	Højden *h* afhænger af... Højden *h* har indflydelse på prisen...
Bredden, B	B = 1 m	
Længden, L	L = 3.169 kilometer	L afhænger af grænsens læn
størrelse på mursten h_m, l_m, b_m	h: 54 mm 5,4 cm L: 228 mm 22,8 cm B: 108 mm 10,8 cm	
størrelse på legoklods h_L, l_L, b_L	h: 0,95 cm L: 3,2 cm b: 1,6 cm	
prisen på en legoklods	$ = 0,20 DKK = (1,39) (1,68) dkk omregnet fra $ på legos	ER der forskel på den dans Amerikansk pris?

Figure 13: Work-sheet of group C, part 1.



I skal ydermere finde ligninger der beskriver de relevante sammenhænge mellem de forskellige variable. Indsæt disse herunder:

$$\text{Antal mursten} = \frac{h}{h_m} \cdot \frac{B}{b_m} \cdot \frac{L}{l_m} = \frac{1000}{5,4} \cdot \frac{200}{10,8} \cdot \frac{3.169.000,00}{22,8} = 47.621.174$$

185,2   18,5   138991,2,8

$$\text{Antal lego pr. mursten} = \frac{h_m}{h_L} \cdot \frac{b_m}{b_L} \cdot \frac{L_m}{L_L} = \frac{5,4}{0,95} \cdot \frac{10,8}{1,6} \cdot \frac{22,8}{3,2} = 274,3$$

5,7   6,75   7,13

Antal lego i alt:
47.621.174.540 · 274,3 =
1.306.248.818.000

Pris i $: 261.249.763.600
Pris i DKK: 2.194.498.014.240

Figure 14: Work-sheet of group C, part 2.

The concept of estimation is, however, seemingly not something they have yet clearly understood, as demonstrated by the groups use of decimals and lack of significant figures. I would have expected to see the use of scientific notation, i.e. a total price of the wall of $2{,}2 \cdot 10^{12}$ kr.

Afterwards, the group engage in a discussion regarding labor:

23. **C1:** Well… Then we are supposed to find out how many workers we think are needed to build this… because… how much time will he (she probably refers to Trump) use?
24. **C4:** (jokingly) He can just get some slaves to do it!
25. **C2:** Well… now we are… we stand… if we take Trumps point of view… there should not be any pay for the workers…. The Mexicans are supposed to build it for free.. or mexico can pay for it… that's what he said…
26. **C1:** That's probably not realistic…
27. **C2:** But a lego wall is not realistic in the first place
28. **C1:** But now if there were to be some (workers)… I have no idea… it takes a long time to build lego if they should all be connected…
29. **C3:** How long is the Chinese wall?

The students have apparently lost their motivation, and find the discussion regarding labor very confusing and difficult. It doesn't help either, that they are abrupted by a student from another group, who finds a discussion regarding Roskilde festival a lot more interesting. The group seemingly does not study the problem any further afterwards.

The following didactic situation takes place after the class have found the number of pieces of legos that are needed to build the wall, as well as the cost of those pieces. The setting of the situation is in the form of a discussion, where the subject is the price of the labor. The teacher's role is that of mediator rather than a lecturer.

1. **T:** But what about the wages?
2. **Boy student 1:** We assumed that the wages of Mexicans are 30 kr. per hour.
3. **T:** OK, we assume that Mexicans work for 30 kr. per hour.
4. **Boy student 2:** That sounds reasonable
5. **Girl student 1:** That is a really low payout.
6. **Boy student 1:** Yeah but Mexicans don't make that much money.

7. **T:** So we have a variable we will call "løn"… right? because we already have a variable called l (she refers to the length of the wall).. So a Mexican worker is paid 30 kr./hour, but it takes more than an hour to build this wall. What else do we need to find the total payout?
8. **Girl student 2:** We also need to know how many workers there are
9. **Girl student 3:** And their age, because they are not paid the same amount (she refers to age-different wages)
10. **Girl student 4:** I think the age doesn't matter.
11. **T:** I think we need to use an average wage at this point, if we are to ever get the result… How long does it take to build the wall?
12. **Boy student 2:** 2 years.. or one year with 100.000 workers.
13. **T:** Now you are just guessing randomly. We have to find a reasonable estimate. How can we estimate the time it will take?

    (students murmuring)

In quote 1, the teacher asks the question that sets the stage for the situation at hand. Associated with this situation, she also attempts to establish a contract in which the students are to take responsibility for establishing an algebraic expression of the estimate of the wages. The students do seem to have difficulties understanding what is required in order to answer the question. This is clear from the triadic dialogue (Winsløw, 2007 p. 164) that occur in this situation. In quotes 7, 11 and 13, the teacher both evaluate the previously attempted answers, as well as establishes new micro contracts in the form of new questions. The rules are, however, apparently unclear to the students, evidenced by the final, completely random guess (quote 12).

14. **C1:** We have to figure out how many workers there are, how many hours they work each day, and how many days they work.. right?
15. **T:** That is a good guess, but we can simplify by figuring out how many man-hours it takes to construct the wall… so we have to figure out how fast we can assemble legos in order to calculate how much time is required.. how much time is required to assemble $13{,}2 \cdot 10^{12}$ lego bricks?

    (students are murmuring again)
16. **T:** How many bricks can we assemble in an hour?
17. **C1:** In one second we can assemble two legos.
18. **T:** How many bricks is that per hour?
19. **C1:** 60… no wait.. a lot more than that… no it's 60 times 60.. 3600 lego bricks per hour.
20. **T:** Do we agree on 3600 lego bricks per hour?

21. **C1:** If we work really fast.
22. **Girl student 5:** That is for one person, right?
23. **T:** Yes.. so that is how many lego bricks we assemble per hour.. how long does it take to assemble $13{,}2 \cdot 10^{12}$ lego bricks? Come on…. calculate calculate calculate….
24. **Boy student 2:** I get 28712
25. **T:** I'm not sure about that result…
26. **C3:** I get $3.6 \cdot 10^{9}$ hours
27. **T:** That I believe. Now we have the number of hours. How much does it cost to assemble the wall? **(students are murmuring)**
28. **C2:** I got $1{,}08 \cdot 10^{11}$ …?
29. **T:** That I believe.
30. **Girl student 4:** That's great, Karen!
31. **T:** We don't have much time left, but as a final remark, are there any parameters we need to include? Have we forgotten anything? We don't have the time to do it
32. **Various students:** Cement… Transport… Dig a hole…
33. **T:** Yes, those are parameters that could be relevant too.

The rest of the situation is also in the form of a triadic dialogue, but it also bears distinct resemblance to that of a Topaze effect situation. This comes to show in the on-going establishment of micro-contracts in which it is unclear whether or not the students understand the evaluations. The guessing nature of the students' contributions also bear resemblance to the case of Gäel: they simply guess in a manner, in which they expect, it is what the teacher wants to hear.

The 3rd lesson

Due to the nature of the lessons carried out earlier, the teacher had to use a bit of time in order to tie up some loose ends regarding the 2nd Fermi problem. The introduction of the 3rd Fermi problem was delivered approximately 30 minutes into the lesson (20 minutes later than anticipated), and the teacher and I further went on to agree to use some time of the 4th lesson, such that the plan could be carried out as smoothly as possible. As a result, the first 25 minutes of the 4th lesson actually correspond to the final situations as planned in accordance with the 3rd lesson (see Table 3).

Á posteriori analysis of episode 3II

In the following analysis, we will consider the highlighted situations of episode 3I, as illustrated in Table 7. Again, we follow Group B (Berit (B1), Bente (B2), Bodil (B3) and Brian (B4)) in the adidactic situation where they are to solve exercises 3C, 3D and 3E. In a previous situation (episode 3I), they have established estimates for how much space is occupied by various populations.

1. **B2:** How many people were there at president Obama's inauguration? Use both the minimum and the maximum value from the table above.
2. **B4:** Here's a paper… we can write on that.
3. **B1:** So we are…. We must figure out how big the place is (National Mall, red.), so that is.. yeah.. that is what we need… so the estimate for the variable.. is that something we google?
4. **B3:** We are supposed to limit our use of google..
5. **B4:** The place? Isn't it okay to find out how big it is?

Immediately, the group formulate the task at hand, but they decide to use a search engine in order to establish the area of National Mall. Here it is clear, that the didactic contract is not clearly specified, as the students are not completely sure as to whether or not they are allowed to do this. In this case, they were in fact supposed to use the geometric representation and measuring abilities.

6. **B4:** 59 acres.
7. **B2:** Can we get that as a number we can use?

   (students laughing)
8. **B4:** 59 acres!
9. **B2:** Okay…?
10. **B3:** Okay…?
11. **B4:** Don't be so angry!

The group suddenly lose focus, and a few minutes is used on a non-related debate. Finally, B2 asks the question:

12. **B2:** How much is one acre?
13. **B1:** I don't know.. look it up
14. **B3:** Another excellent question
15. **B4:** But that's all of it (the whole area).. That doesn't work… It's not just this (probably points to the figure), it's all of it! (frustrated) Why don't they give us an idea of how big this is?? Now I'm mad!
16. **B2:** Ehm…. How many parts is it divided into? One.. two… three.. four.. five.. approximately even sized.
17. **B1:** I have figured out why it is like that. It's from two different times of the day, or something.. It was something about one of the pictures was taken in the end, and the other at the beginning.

Here they identify the actual problem at hand; using a search engine to solve the problem is not optimal. However they do not identify how the solution could be made. The afterwards group goes into a debate regarding time and place, as well as a discussion on the next questions of the exercise. This goes on for a bit, along with some irrelevant situations.

18. **B2:** Okay let's do some maths… Ehmmm… yeah… Okay… Brian you said the 59 acres was for the whole area of the park, not just the lawn..
19. **B4:** Yeah, I think it was for the whole park.
20. **B2:** Okay… then it won't help us that much, but let us just say that if we cut in half, then I think it fits approximately.
21. **B1:** But 59 acres… We are supposed to figure out how much one acre is in square meters.
22. **B2:** Okay.. it's actually.. we are supposed to figure out how many people on average can fit in one square kilometer. That is easier than square meters.
23. **B3:** Can't we just convert it?
24. **B2:** Yeah…. But… so can you!
25. **B4:** Isn't it just by moving a zero?
26. **B3:** I'm not sure about that
27. **B2:** An acre is an area of 100 m times 100 m. That's 10.000 square meters…. So Its 59 times 10.000.. That's quite a lot!
28. **B1:** We can just apply zeros…
29. **B3:** (Calculating) Okay… So it's… 59… yeah… 59…. Divided by…. How do we do this?

30. **B2:** So the idea is, we have this much space here (probably points at the figure)..
31. **B3:** Yeah, so I have done it wrong.. I should multiply instead of dividing.. Okay…
32. **B2:** So… My guess is… they take up two thirds of the space…

Unfortunately, the group completely stalls after this, even though they were in fact heading in the right direction. The intended use of geometry was in this case completely missed, and the responsibility as to who violates the didactic contract in this case is perhaps on the students, as they do not seek out help from their supervisor, or perhaps on the teacher, as she has not established the terms accordingly.

In the following sequence, we consider the didactic situation in which the problems of episode 3II are evaluated. The various groups have beforehand written their proposed solutions to the problems on the black board.

1. **T:** So, one of the groups have estimated that the mall is 390.000 square meters… Do we agree on that?
   (Students complain)
2. **T:** Seemingly not.. How did you estimate the size of the mall?
3. **Girl 1:** What we did was… whats on the black board… we took… we googled how big the mall was.. it said 59 acres… or something… so we assumed that's the whole area including trees, lawn, and so on…
4. **Girl 2:** But then its 590.000 square meters?
5. **Girl 1:** Yes that's what it says the whole park is.. but the point is we use… you know… the whole park is including the park, lawn, trees, and all that stuff, and that's not on the picture.. we cant see all of it on the picture.. so therefore, we estimate that we can see approximately two thirds of the picture…
6. **T:** OK, so using this, you have estimated that the minimum is 1 million and 560 thousand and maximum 5 million and 70 thousand… So if we look down here, a group has used the whole area… but then you have divided by 5.. why is that?
7. **Girl 3:** Because we could see on the picture that there are some occupied spaces and some that are not.. we estimated approximately one fifth on the Trump picture…
8. **T:** So what you are saying is that the people in the Trump picture take up about one fifth of the area.. Is that what you are saying?
9. **Girl 3:** Yes.
10. **T:** Good. And then you said that if its 4 people per square meter we end up here (points at black board), and if its 13 people per square meter we end up here (points at black board).

11. **Girl 3:** Yes.

Again, we find a situation in which a triadic dialogue is present. In this case – conversely to the previous didactic situation – the students seemingly have a higher responsibility towards their contractual obligations. We also see a situation in quote 5, where a student to student validation takes place. In quote 6, the teacher establishes a micro contract in which the student of quote 7 contribute with mathematical knowledge through use of geometric arguments.

12. **T:** Okeydokey. Then we have Obama down here (points at black board)…. Approximately 80 percent of the mall is occupied… some of you have used the total area… so that's 472.000 square meters… And then you have calculated with 4 per square meter and 13 per square meter… So that leads us to the last question… Let me read it up… Choose a president, Trump or Obama.. you have already done that… Argue from the analysis that it's exactly your president who had the highest attendance.
13. **Girl 2:** Clearly Trump had a higher attendance, since on this day it was rainy so actually most of the people are crowded up under the tents… and the people are sitting on each other's shoulders…
14. **T:** So how many attendees are there?
15. **Girl 2:** More than Obama…
16. **T:** That's might just be the best answer in this case…
    (classroom laughing)
17. **T:** What do you think (points at another student)?
18. **Girl 4:** Obama! You can see it from the calculations… Even if there were 4 per square meter at Obama's inauguration and 13 per square meter at Trump's there would still be more attendees at Obama's inauguration…
19. **T:** Yes! As you can see there's not a big difference between these two numbers (points at black board), so this is very dependent on the estimate we use, right? With another estimate we can argue that there are a lot more or a lot less.. right? And they actually say that sometimes in the news… that they have fitted an estimate to get an appropriate answer…

In this short transcript, we find both a situation in which a student completely misunderstands the purpose of the exercise (quote 13, 15). She does not use mathematical arguments as was intended, but the teacher later makes up for this with help from another student (quotes 18-19). In this case, we have a situation of validation (18) as well as institutionalization (19).

The 4th lesson

As previously mentioned, this lesson underwent quite a dramatic series of changes, when compared to that of the initial lesson plan. This heavily influenced the amount of time the students had to solve the problems stated (see Appendix 3). As a result, none of the groups managed to do the third assignment, which in light of the intended target knowledge was the interesting one. The students did, however, work a bit with assignment two, and an example hereof will be given below.

Á posteriori analysis of episode 4I.

We will consider the episode 4I, in which the students use Maple in order to algebraically model the solution to the 2nd Fermi problem. Specifically, we will focus on the situation highlighted in Table 8. The group we follow is again Group C, who are considered to be the students of the class with the highest mathematical skillset. Following a situation in which the teacher goes through the first assignment using Maple (see Appendix 3), the students engage in solving assignment 2 in a similar fashion. C2 is writing on the computer with guidance from the rest of the group:

1. **C2:** Let's take height, width and length of the wall…
2. **C3:** I have no idea what we're doing right now, cause I can't see the screen!
3. **C1:** We are solving this exercise (probably points at the paper)!
4. **C2:** So the height was 10 meters, right?
5. **C4, C3 and C1 simultaneously:** Yes.
6. **C2:** Then we have the width of the wall… ehmm…. The width of the wall… it was..
7. **C4:** Was it one? Or two?
8. **C1:** I think it was 2 meters wide, when we did it in class
9. **C3:** I think so too
10. **C2:** I don't know what to write… Are we just writing the length of the wall we found?
11. **C3:** We should find the volume of the wall.. and divide it with the volume of a lego block… right?
12. **C4:** I didn't get that, sorry..
13. **C3:** In order to find the number of legos, we should find the volume of the wall and divide it with the volume of a lego block.. And we should use millimeters…

    (C2 is writing on the computer, in an inaudible situation.. It appears as if the others have stalled a bit, as discuss a lot of non-relevant topics)

The above situation clearly shows that the students are engaged in the problem at hand, but when working with CAS in groups, it is almost impossible to keep the whole group focused simultaneously, as – for obvious reasons – only one can write at a time. The students demonstrate clear use of modelling skills as well as discussions aimed at estimating the relevant parameters.

14. **C2:** Guys, we have the number of blocks now.. but we need to count in workers and wages… Did we write anything on this?
15. **C3 and C1:** No, because we didn't take this into account ourselves…
16. **C4:** Let's just say they are paid 30 kroner per hour.
17. **C2:** But it was that thing about how fast they could assemble…
18. **C4:** Wasn't it something like 3500 an hour? Wasn't it?
19. **C1:** How many legos could we assemble per second..
20. **C4:** Per second?!
21. **C1:** Yes, how many per second…
22. **(simultaneously) C4:** One

    C1: Two
23. **C2:** (counting) that's doable… okay so now we calculate… so 60… one per second... that's 60 per minute… in an hour that's…. 3600 blocks that can be assembled… and how many legos should be assembled in total?
24. **C4:** ******* many!
25. **C2:** Is it this number (assumably points to the screen)?
26. **C3:** I think so… but we need to write this down…
27. **C1:** So we call this n.. a… Who works (number of "arbejdere", red.)… Not length workers, cause we have used length before…
28. **C4**: Ah, okay..
29. **C2**: Should I figure out how many hours is needed? It was 3600 per hour, right? Wait a sec… one… two... three…
30. **C3**: Shouldn't we find out, per person, how many he can assemble per hour, then we can figure out how many hours one person should do… and then we can distribute among them…
31. **C1:** We just need to find out how long it takes to build the wall and therefore how much we should pay for that… then it doesn't matter how much they work each day…
32. **C2:** So, we just need the number of legos… and that was 60 times 60…. Right?
33. **C3:** This is more than a million-million legos!! Is it trillion??
34. **C2:** Billion I think…

The students discuss the numbers for a while before ending up with a solution. In the situation above they also demonstrate modelling skills, and their use of a real-world rationale is very much there (quote 23 and 29). They also demonstrate knowledge regarding use of algebra and variables, specifically symbolic representation of the variables in play (quote 27).

35. **C2:** I will call this t_n the number of legos... Then it takes a total of 3,6195 times 10^9 hours… Then I should multiply this with the hourly wage, right?... That's 1,0858 times 10^{11} kroner…
36. **C3:** I also think there should be someone to supervise…
37. **C4:** like guards…?
38. **C3:** Yes, but they should be paid 60 kroner per hour… we don't need that many guards… I think there should be one guard per 100 workers…
39. **C2:** So if we say there is a worker per 50 meters of the wall… then there are this many workers simultaneously (points at the screen)
40. **C1:** I think we should say a worker per 10 meters… Trump wants this wall built!
41. **C2:** Then I think there should be one guard per 200 workers…

Here the students add flavor to the problem at hand, even though they know this may not be necessary. This is a very strong property of working with such a problem; the enthusiasm and engagement of the students is clearly present – even after working with this for many days. After this situation, the lesson is over, and the students hand in their Maple-worksheet associated with the assignment. Below is an excerpt of the Maple-document handed in by Group C.

In Figure 15**Fejl! Henvisningskilde ikke fundet.**, the group demonstrate an impressive display of using symbolic notation, which likely is derived from the nature of the variables in play, and the general realistic nature of the problem. They have clearly gained an ownership relation towards the assignment, which also was the intention of the initially established didactic contract.

Nedenfor er disse defineret i Maple - ret selv estimaterne (nedenfor er de 0, hvilket naturligvis *ikke* er tilfældet!).

$h_k := 0.0095$

$$0.0095 \quad (2.1)$$

$b_k := 0.016$

$$0.016 \quad (2.2)$$

$l_k := 0.032$

$$0.032 \quad (2.3)$$

Rumfanget af en legoklods er således givet (i meter) ved V_k (V for volumen på engelsk) :

$V_k := h_k \cdot b_k \cdot l_k$

$$0.0000048640 \quad (2.4)$$

Find selv de resterende variable, og brug sammenhængene vi fandt i lektion 2 til at give et endeligt estimat for prisen...

Mål på mur:

$h_m := 10$

$$10 \quad (2.5)$$

$b_m := 2$

$$2 \quad (2.6)$$

$l_m := 3169000$

$$3169000 \quad (2.7)$$

Rumfanget af mur

$V_m := h_m \cdot b_m \cdot l_m$

$$63380000 \quad (2.8)$$

Antal klodser til mur

$k_m := \frac{V_m}{V_k}$

$$1.303042763\ 10^{13} \quad (2.9)$$

Pris pr. klods

$p_k := 1.68$

$$1.68 \quad (2.10)$$

Pris på mur

$p_m := k_m \cdot p_k$

$$2.189111842\ 10^{13} \quad (2.11)$$

Figure 15

11. Discussion

In this chapter, I will attempt to answer the research question, that *Fermi problems can be used as an entrance to mathematical modelling with functions of one or severable variables with respect to teaching mathematics at C-level in Danish high schools*, through use of the observations in the previous section.

The initial problem (of finding the number of 5-kroners needed to reach the top of Trump tower) was quite successful, which was also to be expected, as the complexity of the problem was very low. The problem is also quite similar to that characterized by an ordinary linear equation – one could perhaps even argue that it is not a Fermi problem at all (in the sense of Ärlebäck). Transcripts of this problem have been omitted, and the more interesting case of estimating the time it will take for an elevator to reach the top of Trump tower was analyzed instead.

In this episode – Fermi problem number one, part B – the students were at first glance very confused as to how to proceed with answering the questions at all. This even though the Piano-tuner example was provided before they went on with the task.

A possible reason for this, is that the class is used to mathematics being performed in an absolute certain way; that a mathematical problem is always defined absolutely, and with it is associated one – and only one – correct solution. This characteristic is definitely challenged when working with Fermi problems, since "the correct" answer to such a problem (using Ärlebäck's definition) is (almost always) impossible to find.

In the analysis, the students show signs of engaging in actual mathematical modelling; they do formulate the task, they do systematize through identifying relevant parameters (to a certain degree) – but the only mathematization taking place is stating basic estimates that were established in the previous exercise. A possibly unfortunate situation also arises, when the teacher applies a restriction to the exercise at hand, thereby modifying (one could argue breaking) the didactic contract as well as the milieu. The student (Aisha, A3) indeed has a great point in considering intermediate stops as a factor. I am not certain that this was an intentional move on behalf of the teacher (as the mathematical skill of the group is quite a lot lower than average), but I did point out to her after the lesson, that this was not the intention of the assignment.

Group B handle the similar situation with a seemingly higher enthusiasm, motivation and expertize, they immediately show clear-cut signs of actual modelling, and even consider acceleration and deceleration as a relevant factor. The students of Group B also make use of *extra-mathematical* knowledge (Ärlebäck, 2009) in the sense that they draw from personal experience as to how fast an elevator is. Motivation and interest is also present, and a student even wants to examine the speed of an elevator the next time she rides one!

The target knowledge was certainly acquired in this situation, where the informal approach to modelling a Fermi problem was carried out in an acceptable way. The students in this situation demonstrated processes of interpretation, evaluation and validation in accordance with Blomhøj and Jensen (2007), albeit the processes of mathematization and mathematical analysis was somewhat implicit. This was, however, also expected, due to the informal nature of the problem.

In the 2nd episode, the target knowledge is the algebraic formulization of a Fermi problem – in this case the problem of estimating the price of the Mexican wall if it's built using lego blocks. In this episode, the students demonstrate that they undergo all the processes that constitutes a modelling problem, as described by Blomhøj and Jensen (2007). There is still evidence that the students are not entirely secure in this type of didactic environment, which is demonstrated by their frustration, that the teacher refuses to give absolute answers, when they run into trouble. This is evident in the sense that the responsibility of the didactical contract mainly lies with the students when in an adidactic situation such as this.

The written work of Group C shows that they have in fact introduced variables, variable dependencies and algebraic notation, which was the main objective with respect to target knowledge. The lack of scientific notation when using estimates is however a big issue, and should have been dealt with explicitly. It is, following this observation, very clear, that the concept of an estimate is not yet properly established.

In the classroom validation situation of episode 2, a triadic dialogue with traces of a Topaze effect take place. Though it indicates that a lot of the students have not fully found an answer to the problem, this does not necessarily mean that the objectives of the lesson have failed; a partial solution to the problem may still prove fruitful in this regard, as observed above.

In the 3rd episode, a part of the target knowledge was unfortunately completely missed by the students. The target knowledge in question here is the geometric modelling potential that was present in the assignment. This was definitely something both the teacher and I could have prevented through a modification of the objective milieu (maybe by banning search engines, or by a limitation of the problem), and it is surely something that one should take into consideration when using this teaching sequence.

In the given situation, the students took to search engines in order to find results for the relevant parameters regarding area, rather than estimating these through geometric argumentation. In the observation of Group B, they lose focus when they are unable to make use of the result of Google, and the process of establishing a mathematical model in the form of an estimate is eventually abandoned. The students in general do, however, again demonstrate use of algebraic notation along with variable dependencies in a similar manner to that of episode 2. This could indicate that a consolidation of the target knowledge of the previous episode. In the associated class-room didactic validation situation, the students do demonstrate that the intended target knowledge regarding intervals is acquired. It is reasonable to believe, that the realistic nature of the problem at hand has helped with this understanding; the students can relate to the fact that there were between 1,5 million and 5,6 million people present at Obama's inauguration – i.e. an answer in the form of an interval. The work with intervals – albeit implicit – is also demonstrated in the validation situation, where a mathematical argument on which president had the highest attendance was presented. A problem does arise as well in this situation. A student attempts at an argument solely through the use of non-mathematical perspectives. This is a real danger when working with problems of this kind; there is evidently a risk that the mathematization process is lost if the milieu is not designed properly.

The biggest inconvenience – or maybe even failure – in this thesis, was the fact, that the anticipated sequence of the final lesson was never carried out as intended. The students should have explicitly worked with functions as a part of this episode, but time constraints (maybe as a result of poor planning) interfered with this goal. Evidently, this was a big part of the actual research question of the thesis, which specifically mentions functions as part of the mathematical modelling, so one can only speculate as to whether or not Fermi problems in fact do constitute a good tool with respect to introducing functions in C-level mathematics.

The episode was not a complete disaster, however, as the students demonstrated a remarkable ability to model the 2nd problem through use of variable dependencies and variable declarations in Maple. This is not a far step from actually defining a function – one could even argue that this is the case – however the graphical representation (which is a huge part of the function concept) was not at all present. This aspect is therefore still an interest of study – one I definitely will examine further when teaching my own classes.

The main motivation of using Fermi problems was the adidactic potential that they naturally possess. When students work in groups, problems such as this evidently spark discussion with clear-cut properties of mathematical modelling, and this feature seemed to work out in most cases presented here. Of course, there are exceptions; specifically, there is a danger that the students lose focus if they stall, and don't seek out help autonomously.

When one decides to do Fermi problems in a teaching situation, one should also be very considerate of the didactic variables that are in play. For instance, Ärlebäck (2009) states that "group dynamics are essential for the evolution of and activation of the different sub-activities during the problem-solving process (…) Group behavior is strongly influenced by individual preferences and group composition (…)" (p. 354) This " double-edged sword" was also evident in this teaching sequence; for instance Groups B and C worked really well with decent dynamics, whereas Group A and an unspecified group presented with having severe difficulties engaging in the problems of the sequence.

After the teaching sequence was carried out, an evaluation of the process took place with the teacher. In her opinion, the sequence "was an unqualified success", where the assignments, among other things, had promoted student activity from otherwise dormant students. She would even go as far as saying, that "the sequence had had a positive influence on the student evaluations".

But let us again turn our attention towards the research question. *Can* Fermi problems be used as an entrance to mathematical modelling with functions of one or severable variables? Through the writing of this thesis, along with the experience of observing the designed teaching sequence in action, I personally firmly believe that this is the case!

The line between modelling a Fermi problem, and making use of one of the representatives of functions as described in chapter five is seemingly very subtle; this combined with possible transitions between those representatives using a Fermi problem could very well inspire further studying on the subject.

12. Conclusion

The aim of this thesis was to examine whether Fermi problems could be used as an introduction to mathematical modelling and functions of one or more variables in Danish high schools at C-level. To answer this question, a teaching sequence dealing with Fermi problems was developed and tested in a junior year high school class at Rødovre Gymnasium. The design of the sequence was heavily inspired by the ideas of Jonas Ärlebäck (2009) on the use of realistic Fermi problems as an introduction to mathematical modelling.

Using the theory of didactical situations (Winsløw, 2007), both an á priori analysis and an á posteriori analysis of the teaching sequence was conducted, and following this analysis, it was evident that the intended knowledge regarding explicit algebraic modelling, estimating quantities, using intervals, establishing variable dependencies, and performing algebraic modelling in a CAS environment was acquired. The intended knowledge regarding geometric modelling and functions was, however, not acquired – and seeing as the latter was due to time constraints, this still remains an interesting potential topic of study.

The thesis furthermore illustrates a series of didactical challenges concerning the use of Fermi problems as a tool for introducing mathematical modelling. Specifically, the adidactic nature in which Fermi problems are conducted, should be handled with care – especially concerning the design of the associated milieu as well as the design of the groups who are to act on it.

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14. Appendix 1 (Worksheet used in exercise 2)

I skal i nedenstående tabel udfylde med relevante variable, estimater og sammenhænge i benytter i forbindelse med svaret på opgave 2.

Variabel	**Estimat for variabel**	**Sammenhæng mellem andre variable?**
Højden af muren, h	h = 4 m .... ?	Højden h afhænger af... Højden h har indflydelse på prisen...

I skal ydermere finde ligninger der beskriver de relevante sammenhænge mellem de forskellige variable. Indsæt disse herunder:

15. Appendix 2 (Worksheet used in exercise 3)

I skal i nedenstående tabel udfylde med relevante variable, estimater og sammenhænge i benytter i forbindelse med svaret på opgave 3.

Variabel	**Estimat for variabel**	**Sammenhæng mellem andre variable?**
Areal af pladsen, A	A = ... ?	Arealet A afhænger af... Arealet A har indflydelse på...

I skal ydermere finde ligninger der beskriver de relevante sammenhænge mellem de forskellige variable. Indsæt disse herunder:

16. Appendix 3 (Maple document used in the 4th lesson)

Maple-arbejdsark i forbindelse med Fermi-problemer

Udarbejdet af Niels Hvitved i samarbejde med Karen Mohr Pind & 1. mr, Rødovre Gymnasium

I dette arbejdsark skal I i grupper løse opgave 2 og 3 ved hjælp af Maple. Som et eksempel er en løsning på opgave 1 gennemgået i dette ark.

Opgave 1 (eksempel)

I et forsøg på at reklamere for Trump Tower ønsker præsidenten, at der til enhver besøgende gæst udleveres et brev der blandt andet beskriver følgende:

A. Hvor mange 5-kroner man skal stable, for at nå toppen af tårnet
B. Hvor lang tid det tager med elevator at nå toppen af tårnet

For at løse denne opgave benytter vi følgende strategi:

1. At Opstille passende variable, der beskriver problemet

2. At give passende estimater for disse variable

3. At opstille formler der giver en sammenhæng mellem disse variable

4. At udregne det endelige estimat på baggrund af ovenstående.

Opgave A.

restart : *with*(*Gym*) :

Vi opstiller først passende variable, der beskriver problem A. Til disse giver vi tilhørende estimater.

Lad S være svaret på opgaven (antal 5-kroner). Lad desuden h være højden af tårnet, og lad d være tykkelsen af en 5-krone.

Efter en måling i klassen af en stak med fem 5-kroner fandt vi, at én 5-krone i gennemsnit havde tykkelsen 2 millimeter, eller målt i meter:

$d := 2 \cdot 10^{-3};$

$$0.0020000 \qquad \textbf{(1.1.1)}$$

Antallet af etager kalder vi f, og denne var givet ved

$f := 62;$

$$62 \qquad \textbf{(1.1.2)}$$

For at finde højden af tårnet skulle vi først finde et estimat for højden af en etage, h_f, og denne satte vi til 3.2 meter:

$h_f := 3.2;$

$$3.2 \qquad \textbf{(1.1.3)}$$

Den totale højde (formel) af tårnet i meter er således givet ved

$h := f \cdot h_f;$

$$198.4 \qquad \textbf{(1.1.4)}$$

For at besvare spørgsmålet opstiller vi den sidste formel; nemlig antallet af 5-kroner der skal stables for at nå toppen:

$S := \frac{h}{d};$

$$99200.0 \qquad \textbf{(1.1.5)}$$

Svaret S er således, at vi skal stable ca. 100.000 5-kroner for at nå toppen af Trump Tower.

Bonus:

Ønsker vi nu at finde antallet af 5-kroner for en 100 etagers bygning, kan vi nøjes med at udskifte f med 100 og køre "dokumentet" igennem igen:

$d := 2 \cdot 10^{-3};$

$$\frac{1}{500} \qquad \textbf{(1.1.6)}$$

$f := 100;$

$$100 \qquad \textbf{(1.1.7)}$$

$h_f := 3.2;$

$$3.2 \qquad \textbf{(1.1.8)}$$

$h := f \cdot h_f;$

$$320.0 \qquad \textbf{(1.1.9)}$$

$S := \frac{h}{d};$

$$1.600000\ 10^{5} \qquad \textbf{(1.1.10)}$$

Ud fra denne "korte" udregning kan vi se, at vi skal bruge ca. 160.000 5-kroner til at stable højden af en 100-etagers bygning.

På samme måde kan vi finde svaret for 1-kroner eller 20-kroner ved at ændre på d.

Opgave B.

restart : *with*(*Gym*) :

Vi opstiller først passende variable, der beskriver problem B. Til disse giver vi tilhørende estimater.
Lad igen f være højden af tårnet i etager, og lad v være hastigheden, når elevatoren kører ved maksimal hastighed, målt i etager/sekund.
Som mange i klassen diskuterede, går der måske et par sekunder ekstra i starten/slutningen, idet elevatoren accelererer hhv. decelererer. Vi kalder denne tid for a.
Vi antog desuden, at elevatoren ikke gjorde stop undervejs (er dette en rimelig antagelse?).

$f := 62;$

$$62 \qquad \textbf{(1.2.1)}$$

$v := 0.75;$

$$0.75 \qquad \textbf{(1.2.2)}$$

$a := 1;$

$$1 \qquad \textbf{(1.2.3)}$$

Den samlede tid det således tager, er givet ved formlen T:

$T := \frac{f}{v} + a + a;$

$$84.66666667 \qquad \textbf{(1.2.4)}$$

altså kan vi estimere, at det tager cirka 85 sekunder at nå toppen af Trump Tower, under antagelse af, at der ikke er stop undervejs.

Opgave 2

restart : *with*(*Gym*) :

I forbindelse med opførelsen af den nye mexikanske mur, ønsker Donald Trump en undersøgelse af, hvad denne vil koste, samt hvor mange byggematerialer man skal bruge i forbindelse med denne konstruktion.
I skal i denne opgave lade som om, at I er designere/ingeniører på dette projekt, hvor der ønskes svar på det følgende spørgsmål:

A. I tilfælde af, at muren skal bygges af legoklodser, hvad bliver prisen da?

Opgave.

Opstil passende variable der beskriver problemet, og skriv et "program" i Maple, der kan udregne prisen på muren.

Hint:
Regn alle længde/bredde/højde-mål ud i meter. Kald f.eks. højden af en legoklods for h_k, bredden b_k og længden for l_k
Nedenfor er disse defineret i Maple - ret selv estimaterne (nedenfor er de 0, hvilket naturligvis *ikke* er tilfældet!).

$h_k := 0;$

$$0 \quad \textbf{(2.1)}$$

$b_k := 0;$

$$0 \quad \textbf{(2.2)}$$

$l_k := 0;$

$$0 \quad \textbf{(2.3)}$$

Rumfanget af en legoklods er således givet (i meter) ved V_k (V for volumen på engelsk)
:

$V_k := h_k \cdot b_k \cdot l_k;$

$$0 \quad \textbf{(2.4)}$$

Find selv de resterende variable, og brug sammenhængene vi fandt i lektion 2 til at give et endeligt estimat for prisen...

Opgave 3 (A+B)

restart : *with*(*Gym*) :
I et forsøg på at redegøre for udtalelserne omkring antallet af besøgende i forbindelse med præsidentindsættelsen, ønsker Trumps rådgivere svar på følgende spørgsmål:

A. Hvis hele Danmarks befolkning stod samlet så tæt som muligt, hvor stort et areal ville vi så fylde?

B. Hvad med Københavns befolkning? USA's befolkning? Eller jordens befolkning?

I skal i denne opgave gøre ligesom før; nemlig opstille passende variable der er relevante i forhold til at løse dette problem, samt opstille ligninger der giver sammenhængene imellem disse variable.

A.

Skriv et "program" der løser denne opgave. :-)

B.

Skriv et "program" der løser denne opgave. :-)

I skal til slut finde en *funktion*, der beskriver arealet af en befolkningsgruppe, *målt i kvadratmeter*, hvor vi lader variablen *x* beskrive antallet af mennesker.

Hvordan ser denne funktion ud, hvis vi antager at der kan være 4 mennesker/m^2?

Hvordan ser den ud, hvis vi antager at der kan være 13 mennesker/m^2?

Indtegn begge funktioner i et koordinatsystem.

Kan vi give et interval-gæt for, hvor mange kvadratmeter en befolkning på 60 milioner mennesker fylder?

Brug funktionen til at svare på spørgsmål C, D og E fra opgavearket (også angivet herunder).

C. Hvor mange mennesker var der til præsident Obama's indsættelse (se figuren i opgavearket)? Brug i analysen både den mindste og den største værdi fra skemaet i opgavearket.

D. Hvor mange mennesker var der til præsident Trump's indsættelse (se figuren i opgavearket)? Brug i analysen både den mindste og den største værdi fra skemaet i opgavearket.

E. Vælg en præsident; enten Trump eller Obama. Argumentér ud fra svarene på ovenstående spørgsmål for, at netop jeres præsident havde flest tilhængere ved indsættelsestalen.