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Number tricks as a didactical tool for teaching elementary algebra

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Abstract
The master thesis examines the adidactical potentials of using number tricks as frame for teaching elementary algebra to low level math students at Danish STX. A diagnostic test was made and implemented, and from this was defined target knowledge. Using the theory of didactical situations, two lessons were designed to specifically induce the target mathematical knowledge through adidactical situations. The lessons were analyzed using the theory of didactical situations, and it was concluded, that high adida tical potentials exist when using number tricks as a frame for teaching elementary algebra. It was also concluded, that the test students in low level math classes in Danish STX exhibited both didactical and epistemological obstacles with elementary algebra leading to misconceptions such as $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. Through the analysis, it was concluded, that in cases, where the devolution leading to the adidactical phase was successful, the misconceptions were successfully dealt with, while in cases where the devolu-tion was lacking, the misconceptions were not dealt with at all. Instances of broken didactical contract, Jourdain effect and Topaz effect were observed. Post-lesson diagnostics revealed, that progress had been made for the students both regarding the specific misconceptions and the overall abilities to handle algebraic expressions.

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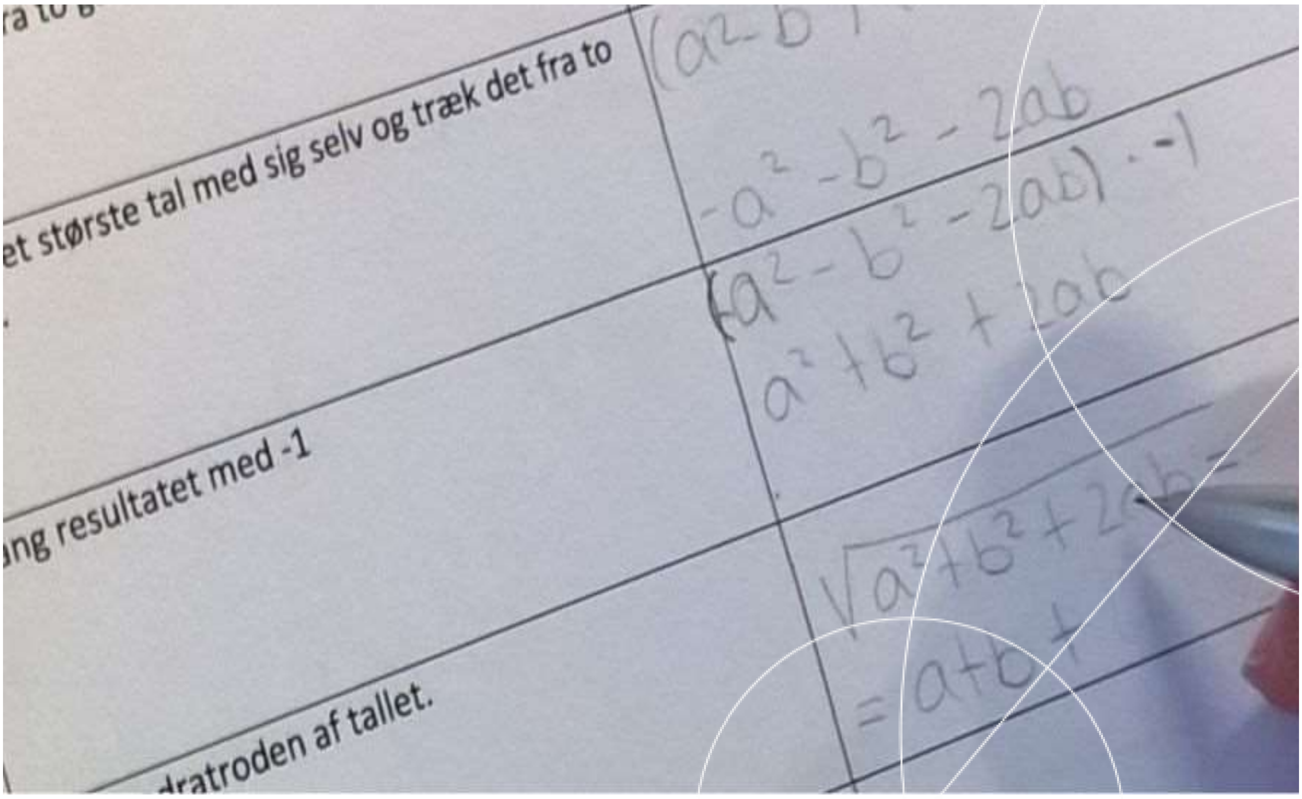
UNIVERSITY OF COPENHAGEN
DEPARTMENT OF SCIENCE EDUCATION

Master thesis, 30 ECTS

Asger Brix Jensen

Number tricks as a didactical tool for teaching elementary algebra



Advisor: Carl Winsløw

Submitted: 31-07-2015

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Resumé

Specialet undersøger de adidaktiske potentialer i brugen af taltricks som ramme for undervisning af elementær algebra til elever med lavt niveau af matematik på STX i Danmark. En diagnostisk test blev udarbejdet og implementeret, og ud fra denne blev defineret tilsigtet viden. Ved brug af teorien om didaktiske situationer blev to lektioner designet specifikt med henblik på at inducere den tilsigtede viden gennem adidaktiske situationer. Lektionerne blev analyseret med brug af teorien om didaktiske situationer, og det blev konkluderet, at der eksisterer et højt adidaktisk potentiale ved brug af taltricks som ramme for undervisning af elementær algebra. Det blev også konkluderet, testeleverne udviste både didaktiske og epistemologiske forhindringer ved elementær algebra medfølgende misforståelser såsom $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. Gennem analysen blev det konkluderet, at i tilfælde, hvor devolutionen var succesfuld, blev misforståelserne afviklet succesfuldt, mens i tilfælde hvor devolutionen var mangelfuld, blev misforståelserne slet ikke adresseret. Der blev observeret tilfælde af brudt didaktisk kontrakt, Jordain-effekt og Topaz-effekt. Post-lektionsdiagnostisk test afslørede, at eleverne havde gjort fremskridt både hvad angår de specifikke misforståelser og hvad angår generelle evner til at behandle algebraiske udtryk.

Contents

1 Introduction

My thesis is about the teaching of elementary algebra to students with a low interest in mathematics and or with poor previous experiences with learning of mathematics. My personal background, which is also to a large extent my motivation for choosing this particular subject, is as a teacher to exactly such mathematics students in the Danish STX through approximately 7 years, and one frustration, which has occurred repeatedly both in my own classes and in those of my colleges, is that of many students never quite acquiring the basic meaning of the simplest of algebraic operations. Even somewhat bright students again and again show signs of some rather basic but deeply rooted misconceptions with regards to elementary algebra, and it manifests itself throughout a wide range of mathematical subjects.

1.1 Danish STX levels of mathematics

I've restricted myself to dealing with student in the Danish STX, who have initially chosen the lowest possible level of math. In Danish STX math can be chosen at three levels named A, B and C respectively, with A being the highest level and C being the lowest. The level C is mandatory for students at STX in Denmark (Gymnasiereformen, 2003). Students, who have chosen level C or B can later choose to raise that level to a higher one. It is my personal experience as a teacher, that the level C to a large extent consists of students, who have suffered some personal defeats battling with mathematics throughout their previous school years. I've taught level C classes for 7 years, and in every such class, there have been predominately students with prior bad experiences with respect to mathematics. Some have been all round below average students, but most have generally been average or above average students with no outstanding problems in other subjects than mathematics. The students, who participated in this thesis, were all students, who had initially chosen level C but then after the first year chose to raise the level to B. I chose to use students, who had raised their level, in order to make sure not to include the students, who have a general apathy towards working with mathematics, as it is my experience, that such students exist to a small degree in the C classes, but not in the classes, who have raised from C to B.[1] Something could be said for including such students, but I have chosen to work with the general premise, that the students are willing to participate and have a personal interest in acquiring mathematical knowledge, since otherwise it would be a completely different project. As I have chosen to work with one of my own classes (one which I have taught for almost two years and thus know very well), I can say with some certainty, that this premise holds.

I find it particularly interesting to address the learning of algebra for these particular students, as they constitute a relatively new group of algebra learners in the Danish school system. Only recently (in 2005 after the implementation of the reform of Danish gymnasiums of 2003) did it become mandatory for every STX-student in Denmark to take mathematics at least at level C, and combined with the fact, that ever since this reform was implemented, there has been a year to year increase of students in the Danish STX (UNI-C, 2015). In fact there has been an increase from 23,015 students beginning an STX education in 2005 to 32,998 students in 2013. These facts imply, that a large amount of Danish students, who previously would most likely never have had to bother with algebra, all of a sudden was forced (well "voluntarily forced") to deal with mathematical subjects, which required some level of algebraic knowledge.

[1] Such students do to a relatively large degree seem to exist in the classes taking B level from the beginning, but that's another story.

1.2 Elementary algebra in the official STX curriculum

This is an issue, which in my personal experience as a teacher has not yet truly been resolved, and it's noteworthy, that explicit mentions of elementary algebra is barely to be found in the curriculum for the STX student taking level C of mathematics. The closest to an explicit demand for the teaching of elementary algebra in the official curriculum is the following line:

> The students must (...) be able to use a symbolic language to solve simple problems with mathematical contents. (UVM, 2015)

The word algebra is itself completely absent from the official curriculum. And this is despite the fact, that the students are demanded to work with a wide range of subjects, which are all very difficult to master without some basic knowledge of elementary algebra. The only proper interpretation of the curriculum must be, that the teaching of any and all algebra should be incorporated into the different subjects of the curriculum, however it almost seems as if it's expected, that the students either possess the appropriate knowledge of *elementary* algebra upon entering the STX, or that the subjects taught in the STX can stand on their own without some specific attention to elementary algebra. From my experience as a teacher both of these assumptions would be incorrect.

1.3 Number tricks as a catalyst

My thesis, however, is not about exploring whether or not the emphasis on elementary algebra in the STX curriculum is to small. It is my personal opinion, that it is, and I believe arguments can be made to support my opinion, and that (indirectly) they have been made (one can read, for examples, some of the material referred to by Martinez (Martinez, 2008, p.3). My thesis *is* on the other hand very much about what can be done to facilitate this transference of algebraic knowledge onto students, who fall into the demographic mentioned above.

The approach was suggested to me by my advisor, Carl Winsløw: Using so-called number tricks (more on those in a later section, but for now: Think of the "mind reading" math tricks, which children use) as a way to breathe life into an otherwise relatively dry and perhaps even seemingly (to the student) meaningless subject. I grabbed the suggestion instantly, as it right off hand seemed like the perfect angle on elementary algebraic operations, and it almost begs the use of the use of the theory of didactical situations to design and analyze lessons - a theory which I very much wanted to work with from the beginning.

1.4 Structure of the thesis

The structure of the thesis is as follows: First I go through the theory of didactical situations in section 2. Then in section 3, I elaborate on number tricks and elementary algebra: What constitutes number tricks? Why are they interesting, and what is their connection to elementary algebra? In section 4, I present my statement of problem, in section 5: My methodology. Then I describe the process of designing, implementing and analyzing both the diagnostic test and the main lessons i sections 6-9. I finish with a discussion and conclusion in sections 10 and 11.

2 The theory of didactical situations

The theory applied throughout my thesis is that known as the Theory of Didactical Situations (TDS) developed by Guy Brousseau (Brousseau, 2002). In this section, I'll explain the different components of the theory as well as provide an insight into some of it's possible applications.

It's important to note, that TDS is a scientific approach to a set of problems rather than a theory of learning (Måsøval, 2011, p.79). It is an epistemological theory (Winsløw, 2006, p.49) in the sense, that it treats the acquisition of knowledge by the students. In other words it is about the triadic relationship between teacher, student and knowledge (Måsøval, 2011, p.21).

TDS accepts the underlying, constructivistic hypothesis, that a mathematical notion is only understood, if it successfully solves some problem (Warfield, 2014, p.13, Hersant & Perrin-Glorian, 2005, p.113). In this hypothesis lies a division of any taught mathematical knowledge. On one hand there is the official knowledge, which is the target mathematical knowledge in it's pure form, and on the other hand there is the student's personal knowledge (Winsløw, 2006, p.51), which will be a composite knowledge based on the student's previous knowledge and his/her work with the problems to which the official knowledge is a solution. The important difference between the two thus lies in the integration of the knowledge into the individual - the personalization of the knowledge so to speak.
There is an important task for the teacher with respect to this status of knowledge for the students termed re-de-contextualizing and re-de-personalizing (Warfield, 2014, p.15, Herbst & Kilpatrick, 1999, p.6). This is the last step of a processing of some mathematical knowledge, when successfully transferring it to a student. Usually mathematical knowledge does not emerge as some pure mathematical

result. It will usually emerge in some personal context, from which it needs to be de-contextualized and de-personalized in order to become official knowledge. In a teaching situation, the aim is to first put that knowledge back into a context for the student and thus personalize the knowledge once again - termed re-contextualize and re-personalize. Then lastly, once the knowledge has a personal nature for the student, it must once again be de-contextualized and de-personalized in order to be used as official knowledge. This last step is what is termed re-de-contextualized and re-de-personalized, and it is at the heart of what is called institutionalization, which is described later.

2.1 What is meant by didactical?

The word didactical seems to be a somewhat loaded word with slightly different interpretations. So just to be clear, in this thesis the word didactical is to be understood simply as being about the conveying of knowledge. Thus it is implicitly about the entire triadic relationship between teacher, student and knowledge mentioned earlier. This relationship is illustrated by the so called didactical triangle (Andersen, 2006, p.72) in figure 1. The figure should be read as follows: The edge between two vertices represents the relation between these two vertices, and this relation draws it's meaning from the opposing vertex (Winsløw, 2009, p.18). So the relation between student and teacher, their interaction with each other, draws it's meaning from the knowledge to be conveyed.

2.2 Game metaphor

Brousseau metaphorically describes the teaching situation as a game played by the students:

> Modeling a teaching situation consists of producing a game specific to the target knowledge, among different subsystems: the educational

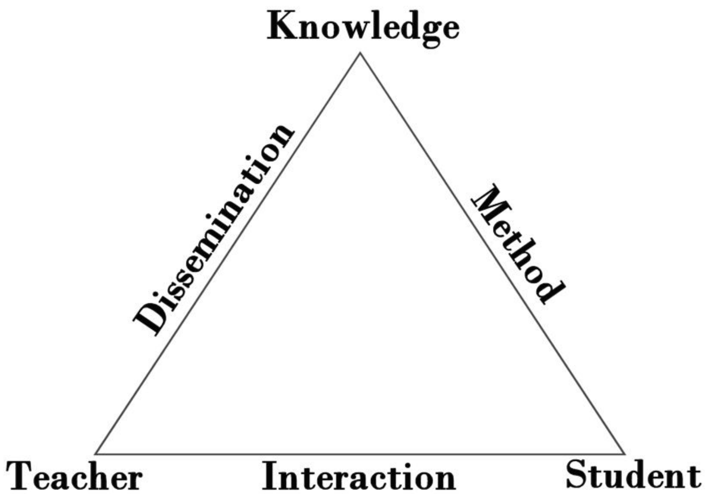

Figure 1: The didactical triangle as presented by Andersen, 2006. The edge between two vertices represents the relation between these two vertices, and this relation draws it's meaning from the opposing vertex (Winsløw, 2009, p.18).

> system, the student system, the milieu, etc. There is no question of precisely describing these subsystems except in terms of the relationships they have within the game. (...) The game must be such that the knowledge appears in the chosen form as the solution, or as the means of establishing the optimal strategy (...) (Brousseau, 2002, p.47.)

In general, Brousseau makes heavy use of metaphors. They are not to be taken literally or accurately. On the contrary they are productive because of their distance from the reality (Måsøval, 2011, p.32). The game as a metaphor works well because of similarities between the structures of games and teaching situations. A game is played by someone who apply strategies within some rules, and a price can be won, if the game if played successfully. In the metaphorical game of the teaching situation, the learners play against the milieu (a concept, which will be explained later) (Herbst & Kilpatrick, 1999, p.9) , the didactical contract (which will also be elaborated on later) constitutes the rules, and the prize of winning the game is the target mathematical knowledge (Herbst & Kilpatrick, 1999, p.8).

The game metaphor also aptly puts into perspective, that not only can you win the knowledge, but if the teaching situation does not go well, the knowledge can certainly as well be lost for the student. By that I don't just mean not acquired but lost in the sense, that it probably will prove even more difficult to acquire in the future, than it was before.

2.3 Didactical and adidactical situations

When speaking of a situation, Brousseau's definition varies from the one usually found in a dictionary. Everywhere in this thesis it is implied, when speaking of a situation, that it is a situation in which some element of transferring knowledge to a student occurs. It is thus implied, that a situation combines some mathematical knowledge, the meaning and possible uses of that knowledge and some connection(s) to the student's previous knowledge. It's tempting to name such a situation a didactical situation, but this term is used to separate a didactical situation from an adidactical situation - two terms, which I will elaborate on soon.

A prerequisite for any situation - be it didactical or adidactical - is, that the teacher wants to teach and students want to learn (Winsløw, 2006, p.54-55). This may seem obvious, but it's an important notion to make, as it points out, that TDS is not about the psychology of getting a student interested in receiving the knowledge. It's is in stead the tools with which to study the process of acquiring mathematical knowledge.

Next I'll be explaining the concepts didactical and adidactical situations, which will be followed by an overview of the standard situations of teaching - phases of teaching of which some or all seem to always occur when teaching mathematics is happening. First though, it should be stressed, that a situation should not be viewed as a discrete object. Often situations of different types will be seen embedded within or overlapping each other (Warfield, 2014, p.9).

The term didactical situation is reserved for the situations, where there is an interaction between the teacher and the student, and the intentions with these interactions are clear. In any situation, where the teacher's intentions are successfully hidden from the students, and the students can function without interaction with the teacher, the situation is termed adidactical (Warfield, 2014, p.12). As mentioned, situations can be embedded within each other, and obviously one can imagine didactical situations occurring in the middle of adidactical ones (the teacher intervenes in order to help a process along), so the division of situations into didactical and adidactical ones might seem strange, but indeed it is a matter of dividing the situations based on the level of degrees of freedom with respect to the control by the teacher (Andersen, 2006, p.71).

Brousseau defines the adidactical situation like so:

> (...) Between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. The student knows very well that the problem was chosen to help her acquire a new piece of knowledge, but she must also know that this knowledge is entirely justified by the internal logic of the situation and that she can construct it without appealing to didactical reasoning. Not only can she do it, but she must do it because she will have truly acquired this knowledge only when she is able to put it to use by herself in situations which she will come across outside any teaching context and in the absence of any intentional direction. Such a situation is called an adidactical situation. (Brousseau, 2002, p.30)

According to Brousseau, the act of teaching is the devolution of appropriate adidactical situations to the students (Brousseau, 2002, p.31). The word appropriate should be understood as follows: The student possesses the sufficient, prior knowledge in order to acquire the target mathematical knowledge, and the didactical milieu (a term which will be explained later) of the situation is set up properly.

2.4 Standard situations

According to TDS there are five different standard situations, which are all important ingredients when teaching mathematics. These will be explained below.

2.4.1 Devolution

Devolution is the (didactical) situation, in which the teacher prepares the students for some adidactical situation. This includes inducing the students to take responsibility for the adidactical situation (Måsøval, 2011, p.47, Warfield, 2014, p.16). It is thus a very important situation, since a failure here most likely will result in a failure in the following adidactical situation. The devolution entails devolving to the students the problems to be addressed adidactically as well as setting up the rules of the adidactical situation (staying in the game metaphor). Since the aim is to equip the students to take on the adidactical situation by themselves, it is of great importance, that the devolution is planned carefully. The teacher is faced with a paradox when devolving to the students: Devolving to much (i.e. giving away too much of the answers to the problems in the devolution) will deprive the students of the chance of actually acquiring the knowledge. Likewise the student - by accepting that the teacher gives too much away - also plays a role in removing the chances of learning (Warfield, 2014, p.18).

2.4.2 Action

In the devolution situation, the teacher has set the stage for an adidactical phase. Following this she withdraws, and the students take on the problems on their own - often in groups. They now interact with each other, the problems and their previous knowledge. This is called the situation of action. Previous knowledge is applied to the problems, and if the problems are well designed, the problems lead the students to concrete issues concerning the target mathematical knowledge. The teacher can choose to be completely absent during this phase, but it can certainly be prudent to stay as an observer and to intervene in cases, where the students are stuck. In case of intervention, it's important to keep in mind the paradox mentioned in the last section, as such an intervention not only can be the key to adjusting the milieu in a positive way, but can certainly as well cause any number of negative effects to the adidactical situation, that will follow.

2.4.3 Formulation

The (adidactical) situation of action will naturally be heavily mixed up with that of formulation, however the two are distincted between, as the essences of them are indeed very distinct. In the formulation situation, the students collectively develop the language necessary in order to express their observations, and through discussion an agreement on common meaning is achieved (Måsøval, 2011, p.52). This situation is sparked by that of action, as a need for a language with which to explain thoughts and hypotheses to each other arises in the groups of students. It is thus very much dependent on the situation of action being successful in the sense, that the students in their work with the problems have been forced to develop ideas/postulations, which they are now forced to formulate to each other in order to reach some sort of mathematical consensus.

2.4.4 Validation

The third type of adidactical situation is the situation of validation. In this phase we see the establishment of theorems (Måsøval, 2011, p.52). It is thus also interwoven with the previous situations, as it is in some sense the aim of the formulation phase and thus serves partly as a kind of feedback situation to the success of the formulation (and action) situation. This situation is all about the conveying of the individual student's convictions, and this shows the importance of this phase, as conveying of conviction generally plays a very important part of mathematics (Warfield, 2014, p.10).

The validation phase will often involve the necessity of some or all of the students to change their convictions and revise their hypotheses. Counterexamples can be very effective with this respect (Martinez, 2008, p.10), as it shows immediately and undeniably that something is wrong, so an incorporation of the possibility to easily (or naturally) create counterexamples can be an effective tool in keeping the adidactical nature of the situation alive.

2.4.5 Institutionalization

The last situation is one of didactical nature. It is that of institutionalization, which is where the teacher re-enters in order to make the students conscious about what knowledge they have developed, and how it relates to what they knew before. In other words it is the situation, in which meaning is given to the newly acquired knowledge. (Warfield, 2014, p.12).

Through the setting up of the adidactical situations, the teacher has attempted to re-contextualize and re-personalize the target knowledge. The institutionalization serves to perform the re-de-personalization and re-de-contextualization mentioned above. The knowledge is made official, and given conventional terminology (Måsøval, 2011, p.53).

2.5 Didactical milieu

A few times above, I've mentioned the term milieu. This is another one of Brousseau's metaphors (for a biological system (Winsløw, 2006, p.56), or society as the factor of the contradictions, difficulties and unbalances (Martinez, 2008, p.6)). It is to be understood as the subset of the student's environment with only features relevant to the target knowledge (Måsøval, 2011, p.55). It's is thus the students, their physical surroundings, the space in which they act, the teacher (if present), the materials they use, the problems at hand, the knowledge at hand, the difficulties that exist or arrive, etc.

According to the theory, the students learn by adapting to the milieu, in which they act, and the acquired knowledge from this learning will be the result of these adaptations (Martinez, 2008, p.6). Now since the milieu of the student is free from didactical intentions in any adidactical situation (Herbst & Kilpatrick, 1999, p.6), this implies something important concerning the desirability of the resulting knowledge: If the situation is arranged in a way, so that certain social skills are sufficient (rather than some target mathematical knowledge) in order to survive in the milieu, then these social skills are likely to be acquired in stead of the target knowledge (Måsøval, 2011, p.34). Likewise there is a danger of algorithms being the acquired knowledge in stead of the mathematical meaning, in any case where the memorizing and using of an algorithm makes it possible for the student to "survive" in the milieu. This is a very important notion to keep in mind when acting as the teacher, as teaching can be thought of as organizing the didactical milieu in a way, so that the target mathematical knowledge is a necessary acquisition in order to survive in it (Måsøval, 2011, p.34).

Accepting the premise, that the students construct their own knowledge as a response to the didactical milieu into which they have been placed by the teacher (as accepted by Herbst & Kilpatrick (Herbst & Kilpatrick, 1999, p.6)), the importance of careful considerations concerning the milieu are obvious. Especially it's the adidactical situations, which signal the importance of the didactical milieu (Herbst & Kilpatrick, 1999, p.6), and when constructing an adidactical situation, an a priori analysis of the situation is of great importance for establishment of the milieu (Måsøval, 2011, p.54) and the resulting knowledge acquired by the students.

2.5.1 Adidactical potential

The didactical milieu of any given situation may contain an adidactical potential. The existence of an adidactical potential means, that it is possible to have an adidactical situation with students working independently from the teacher, but the teacher may choose to ignore it (Hersant & Perrin-Glorian, 2005, p.117, Winsløw, 2006, p.58). Many situations have no room for any adidactical feedback, and so they contain no adidactical potential. Obviously for a situation to be classified as adidactical, an adidactical potential must be both present and explored. How the didactical milieu is set up is important with respect to how much adidactical potential exists. Different approaches to the target mathematical knowledge yields different levels of adidactical potential. A classical lecture for instance, as seen in most university courses, obviously contains no adidactical potential what so ever, and must be classified as purely didactical - a situation of institutionalization without any of the other types of situations present. Such a situation can easily be tweaked, though, in order to incorporate small situations with some adidactical potential: Small breaks in the classical lecture setup, where students interact with each other in attempts to solve problems posed by the lecturer opens up for the existence of adidactical potentials. Designing such situations requires attention to the milieu which is being set up - especially: Does surviving the milieu require the students to acquire the target knowledge, and can the milieu be survived in other ways than acquiring the target knowledge? The answers to these questions should be yes and no respectively, if the adidactical potential is to be the most prudent. I'll be visiting the subject of designing teaching situations further in a later section.

2.6 Didactical contract

In an earlier section, I mentioned that the rules of the metaphorical game played by the students against the milieu is called the didactical contract. I'll elaborate on that concept in this section. According to Måsøval, the didactical contract consists of the rules and strategies for the game between the teacher and the student-milieu which are all specific to the target knowledge (Måsøval, 2011, p.48).

The word contract often indicates some level of awareness of an agreement between two parties (student and teacher), however a didactical contract should not be considered an actual contract but rather as an interplay of obligations (Warfield, 2014, p17, Herbst & Kilpatrick, 1999, p.9). Also it should not be considered as a pedagogical contract (Måsøval, 2011, p.48). Thus a didactical contract in bullet form cannot be made available (Warfield, 2014, p18).

So why the word contract? This relates to what is really also the interesting part of the didactical contract: The breaking of it. Because it is the breaking of the contract, that is definitely the most revealing (Warfield, 2014, p.17, Måsøval, 2011, p.50). While it's not really possible to explicitly state, what constitutes the breaking of a contract beforehand, then whenever the contract is broken, everyone involved tend to act as if an actual contract has been broken (Warfield, 2014, p.17). This is seen for instance as students getting annoyed, because they feel, that the teacher is asking the impossible of them, or by the teacher getting annoyed because the students aren't carrying out the tasks assigned to them, even though they should be able to do so. An often used example of the breaking of a didactical contract is that of the contract behind the problem with finding "the captains age". In this example, the following problem was posed: "On a boat, there are 26 sheep and 10 goats. What is the age of the captain?" According to Måsøval, in a third grade class, 78% of the students gave the answer 36 (Måsøval, 2011, p.49). Obviously the answer stems from adding the only two numerical values given in the problem, but why do such a large percentage give this answer despite the fact, that it quite obviously to pretty much anyone makes absolutely no sense? The answer according to TDS is, that it is the result of a broken didactical contract. Inadvertently a contract has been established between the teacher and the students, which implicitly dictates, that when given a word problem by teacher, the students can solve the problem with the information given, and usually they are to use the already acquired knowledge (for a third grader this would be simple arithmetics) on some or all of the numerical values gives in the problem. So they simply do that regardless of the context of the problem. The didactical contract was broken, when the teacher posed a problem, which could not be solved. Note that the students may very well be aware of the fact, that their answer (or maybe even the problem) is nonsensical, however it is rooted deeply in them, that an answer should be provided, and rather than giving the answer, that the problem can't be solved, they get annoyed by the fact that the teacher has posed such a problem, and they give the answer, they think the teacher most likely expects.

The didactical contract is established mainly in the situation of devolution (Warfield, 2014, p.53), which underlines the importance of this situation. Likewise devolution and institutionalization are important ways of regulation the didactical contract (Hersant & Perrin-Glorian, 2005, p.116).

2.7 Obstacles

In this section, I'll be explaining the different types of obstacles, which students experience during their acquisition of new mathematical knowledge. By obstacle is meant a set of difficulties or constraints on the students with respect to the acquisition of the target mathematical knowledge. TDS distinguishes between three different types of obstacles: Epistemological, didactical and ontogenic.

2.7.1 Epistemological obstacles

An epistemological obstacle is when old learning occasions struggle with new learning (Warfield, 2014, p.14). An example is a student, who possesses the previous knowledge, that multiplying a number with another number makes the resulting number bigger than the original number. This previous knowledge has served the student successfully in the past, but with the introduction of multiplying with fractions or negative numbers, it serves as an epistemological obstacle, as the new knowledge invalidates his previous beliefs, which now must be regulated.

2.7.2 Didactical obstacles

Didactical obstacles are those, which stem from the choices made by the teaching institutions (Warfield, 2014, p.14). An example could be that a school chooses to use a mathematics teaching book, which teaches the students, that $\pi = 3.14$. This can potentially root so deep in the students understanding of π, that when the student later needs to modify his/her knowledge of π to now being the relationship between circumference and diameter of a circle, then the didactical obstacle in question occurs.

2.7.3 Ontogenic obstacles

An ontogenic obstacle is an obstacle, which occurs because of the students limitations (Warfield, 2014, p.14). As an example it will most likely prove difficult to teach a baby to count even before some sort of language is established. Likewise a three year old child will most likely experience ontogenic obstacles when trying to instantly count groups of objects with more than 5 objects contained, as the brain is yet not fully developed to that particular task.

2.8 Didactical phenomena

When analyzing a teaching situation using TDS, seven different didactical phenomena are sought. These are the counterproductive effects, which occur in cases, where the target knowledge is not acquired by the students. Identifying which phenomena is occurring in a given situation helps bringing clarity to what's going on. The seven different didactical phenomena will be described below.

2.8.1 Topaz effect

Topaz effect refers to giving away the answer in the question (Måsøval, 2011, p.37). By continually posing easier questions to the students, the target knowledge gradually disappears. I provide the following example: The target mathematical knowledge at stake is the justification (proof) for the fact that the function determining the area under a curve is an anti-derivative of the function determining the curve. The students know how to determine anti-derivatives beforehand, and the milieu is set up, so that the knowledge needed to survive in it is simply the ability to determine the area under a given curve. The student are supposed to be guided through a proof, that the area function is in fact an anti-derivative and is thus to personalize and appreciate, that their previous knowledge serves this "new" purpose. However in attempts to help the students along the way through difficulties, the teacher ends up reducing the entire task to simply in stead posing the problem of finding anti-derivatives and using these to compute the area. The target mathematical knowledge is lost, but the student has survived the milieu with only the superficial knowledge of how to use an algorithm.

2.8.2 Jourdain effect

Jourdain effect refers to the giving of a scientific name to trivial activities (Måsøval, 2011, p.38). Overeager to witness progress in the student, the teacher over interprets a product of the student to mean that the student has gotten a grasp of some larger mathematical meaning. In the cases, where this meaning is in fact not grasped by the student, it's a case of the Jourdain effect. It is a special case of the Topaz effect, and as an example, take the situation described above about anti-derivatives and area functions. At the end, the students have not acquired the target mathematical knowledge - the proof. However they have acquired the knowledge, that (and how) the area can be computed using the result. If the teacher finishes off by concluding, that since all the intermediate steps in the situation has been fulfilled be the students, they have now proved the result in question, then the Jourdain effect is in play, since the students have actually not acquired any knowledge about the proof - only the result. Their only product is the line of trivial exercises, which the teacher gradually has reduced the original problem to in an attempt to help the students along.

2.8.3 Improper use of analogy

Improper use of analogy is when the teacher replaces a question with a new one, with is identical to the old one only with some quantities changed (Måsøval, 2011, p.39). The skill learned is repetition and recollection - however the target mathematical knowledge is no longer necessary to survive in the milieu, and so it disappears. An example, that many teachers probably recognize, is when a student is having difficulties with a certain word problem and does not quite understand the reasoning made by the teacher in her illustration of the solution, then the teacher comes up with a new word problem with the same target knowledge and begins to walk the student through this in stead. The new problem might be intended to have the same target knowledge as the first, but the introduction of a problem of similar nature to the original one changes the milieu for the student, and now the student can be successful in the milieu simply by recognizing the similarities between the problems and remembering the trivial steps involved in solving them without ever acquiring any knowledge about the actual problem other than the algorithmic way of solving it.

2.8.4 Meta-cognitive shift

Meta-cognitive shift is when a process originally introduced in order to further some target knowledge becomes the actual object of learning (Måsøval, 2011, p.40). As an example consider the use of a scale metaphor as an aid to understand the logic behind equation solving (Sierpinska, 2011, p.4). The teacher uses the old fashioned scale with two compartments balancing out each other as a metaphor for an equation. The equality holds, when the metaphorical scale is balanced - when the two compartments contain equal mass. The two compartments are metaphors for the two sides of the equation, and the mass of a compartment is a metaphor for the value of the mathematical expression in the "compartment". Once established, this metaphor allows the teacher to aid the understanding of the logic behind the strategy of equation solving that consists of performing the same operations on both sides of the equation in order to maintain the truth value of the equation while in the process isolating which ever variable with respect to which the equation is to be solved. Using the metaphor of the scale, this corresponds to simply adding or subtracting equal amounts of weight to both of the compartments simultaneously. Appreciating that this procedure doesn't change the balance of the scale corresponds to appreciating, that performing the same operations to both sides of an equation doesn't change it's truth value. However: In order for this metaphor to have an effect, the students must be acquainted with old fashion scales. In the case, where students are not, the teacher will quickly find herself teaching the students about old fashioned scales in stead of the target mathematics, and thus a meta-cognitive shift has occurred.

2.8.5 Meta-mathematical shift

Meta-mathematical shift covers a substitution of some mathematical problem with a discussion about the logic behind the solution (Måsøval, 2011, p.41). The example given by Måsøval and Sierpinska (Måsøval, 2011, p.41, Sierpinska, 2011, p.6) is where the problem at hand is an equation needed to be solved. The student experience problems, and in an attempt to assist the student, the teacher begins talking about the theory of equation solving in stead of the actual equation at hand. She might address questions like: What is a mathematical expression? What constitutes an equation? What does it mean to solve an equation. The discussion thus turns from the mathematics at hand to the meta-mathematics of the same.

2.8.6 Dienes effect

The Dienes effect refers to the teacher's belief in the existence of what Måsøval refers to as an infallible genesis of some mathematical knowledge independent of the teacher (Måsøval, 2011, p.42). That is, according to Brousseau's theory, the knowledge is acquired trough the student's own adaptation to the didactical milieu. It does not automatically generate itself in the minds of the students as a natural consequence of the students working with some set of problems. The more confident the teacher is in the fact that the knowledge will be produced automatically without her involvement, the more like she will be to fail installing the knowledge in the minds of the students.

2.8.7 Aging of teaching

The aging of teaching effect refers to a scenario, where the teacher gradually has modified the teaching (through years of repetition perhaps) in a way so that important meaning of the target knowledge has been removed (Måsøval, 2011, p.45). Obviously it's not an effect, which I expect to witness anywhere in the data collected in this thesis, as this isn't based on any repetition of any kind. However

I do think it's worth including a description. So how does mathematical meaning disappear through gradual changes of the teaching, and how does it do this in an unapparent matter? Most likely by the gradual removal of the justification for whichever examples are used. On the surface it will seem like the students have been through the same mathematics, as the problems devolved haven't changed, but if the justification is gradually removed or underplayed, then the meaning follows the same path.

2.9 TDS as engineering tool

An integral part of this thesis is the design of a small set of lessons with high adidactical potential. So it's only natural to address TDS from an didactical engineering point of view.
Brousseau has written the following in an e-mail to Anna Sierpinska in 1999 (Måsøval, 2011, p.64):

> The theory of situations is aimed to serve both the study and the creation of all kinds of learning and teaching situations, whether they are "spontaneous", or the product of an experience or of a special didactical engineering project, and whether they are efficient or not. It is not a method of teaching. The theory can provide some methods of teaching, it can justify some methods and disqualify some other methods, as the case may be. The theory contains models that may support certain plans of action aiming at making the students (re)discover some mathematics. This way the theory can make suggestions for engineering.

According to Herbst and Kilpatrick, didactical engineering is the production of possible/available meanings of student's activity (Herbst & Kilpatrick, 1999, p.7). In other words it is the production of opportunity to learn (as opposed to simply production of say mathematical material), and focus is on the production of the meaning - not the activities themselves. So in didactical engineering one should try to induce specific obstacles in order to make the students realize the shortcomings of their previous knowledge and thus have a personal reason to acquire the target knowledge.

According to Brousseau, an a priori analysis of the target mathematical knowledge is of high importance (Måsøval, 2011, p.28). Questions like

- What precisely IS the target knowledge?
- Which context might it be placed in?
- What is the students' previous knowledge relating to the target knowledge?

should all be examined beforehand by the teacher, as this helps furthering along the setup of the appropriate didactical milieu.

Great emphasis should also be placed on the possibility for the students to construct the target knowledge in the didactical milieu (Winsløw, 2006, p.58). To this extent, the following three questions are helpful (Winsløw, 2006, p.60):

1. Are the students given the opportunity to get comfortable with the target knowledge?
2. In working with the milieu: Can the students see a relation to and expand and use their existing knowledge?
3. Does the milieu make it possible and necessary to construct the target knowledge (at the least in some specific cases)?

Finally there is a lot to be said for implementation - adjustment - and re-implementation of the teaching situations designed. However this has not been an option for me with my didactical design, since I've only been allocated just enough time with my test students to go through one single implementation.

3 Number tricks and elementary algebra

In this section I'll be elaborating on number tricks and their connection to elementary algebra. This entails explaining just what exactly I mean by both terms. First I'll be illustrating by example, what I consider to be a number trick. Secondly I'll deliver my own attempt at a definition of number tricks. Lastly I'll elaborate on what exactly I mean by "elementary" algebra. I'll incorporate examples of different number tricks and their specific connections to elementary algebra along the way.

3.1 What is a number trick?

So what do I mean by the term number trick? As mentioned in the introduction, you should think about the "mind reading" tricks, that children use. Below is a simple example (number trick 1, which is a simpler variation of number trick 2 in "Be A MatheMagician" (NCOM, 2008, p.1)):

Number trick 1

1. Think of a number.
2. Add two to your number.
3. Multiply the resulting number by 2.
4. Subtract four from the resulting number.
5. Divide the resulting number with your original number.
6. You now have the number 2.

Obviously the "trick" part of number trick 1 is fairly simple: A number of operations are performed to mask the fact, that the original number is canceled out in some manner, and so the resulting number is predetermined. The above number trick is thus equivalent to the following:

$$\frac{(a+2)\cdot 2-4}{a}=2,$$

where a, which represents the number chosen be the participant, eventually cancels out.

The strength of even such a simple number trick with respect to teaching elementary algebra lies in the nature of how it is unfolded: It consists of some number of operations, which are sequentially performed and with an intermediate result being calculated between every step. So by implementing only operations familiar to the participant, there is a fairly highly likelihood of success, i.e. that the participant will reach the predetermined number (the mind will be "read"). However it's one thing to master the arithmetics of simple addition, subtraction, multiplication and division between any two numbers - something that pretty much every STX-student does - but, as most mathematics teachers are aware, it's another matter entirely to be able to correctly reduce the fraction

$$\frac{(a+2)\cdot 2-4}{a}$$

- an exercise in algebra, which many (low level math) STX-students would have trouble performing correctly. To some extent one might think, that the two tasks are the same, but in the case of the algebraic exercise of reducing the fraction, the meaning behind the mathematics must come from within the student, and here in lie obstacles. Having the students work with even such a simple number trick as number trick 1 above provides the student with a didactical milieu, in which he/she can:

- Act with the target mathematical knowledge: Do the calculations both arithmetically and algebraically

- Formulate the target mathematical knowledge: Write up mathematical rules involved such as $(a + b) \cdot c = ac + bc$

- Validate his/her own work with the target mathematical knowledge: Compare at each step in the progress the arithmetical to the algebraic calculation

Thus a design based on number tricks should yield a high adidactical potential.

One way of highlighting the mathematics of a number trick is to explicitly formulate every step algebraically in a table. This is done for number trick 1 in table 1, and this is the way I'll be illustration number tricks' connection to elementary algebra in every example from here on out.

Table 1: Table showing the algebraic formulations of the steps involved in number trick 1.

Step:	**Word description:**	**Algebraic formulation:**
1	Think of a number.	a
2	Add 2 to your number.	$a + 2$
3	Multiply the resulting number by 2.	$(a + 2) \cdot 2 = 2a + 4$
4	Subtract 4 from the resulting number.	$2a + 4 - 4 = 2a$
5	Divide the resulting number with your original number.	$\frac{2a}{a} = 2$
6	You now have the number 2.	Confirmed.

3.1.1 An attempt at a definition

I'm now ready to attempt a definition of a number trick: I think it is important to restrict myself to be dealing with only number tricks, that are entirely mathematical in nature, so...

3.1.1.1 Definition: Number trick

I define a number trick to be a series of mathematical operations to be performed on one or more numbers of choice (from some predefined set), arranged in such a way, that the end result is predetermined regardless of the choice of number(s) along the way.

Note that this definition makes no mention of the masking of the pre-determinedness of the end result. So by my definition the following would constitute a number trick:

Number trick 2

1. Choose a number.
2. Subtract you number.
3. You now have the number 0.

This would admittedly be a very simple and seemingly uninteresting number trick (and I won't bother explicitly formulating the algebra in a table as with number trick 1), but in as much as it does entail the choosing of a number from some set and performing a mathematical operation on it in order to arrive at a predetermined number unaffected by the chosen number, it *is* by my definition a number trick. I believe it to be important, that even such a simple version should also be considered a number trick, as it serves to exemplify the very core of the nature and (my) purpose of number tricks. The fact that it's obvious what's going on in the trick, is what makes it a perfect example of the structure/nature behind the concept.
The reason that one might object to having such a simple series of operations be named a number trick of course lies in the word "trick". There is not much of a trick in number trick 2. And that is of course an interesting aspect of number

tricks - their status as tricks - however as my definition does not tamper in any way with the possibility of creating a number trick, which is very "tricky", I don't consider for it to be a problem.

Anyway, something should be said about a number trick's ability to trick the participant. A good trick is one with an effectful conclusion - a wow effect at the end. Number tricks 1 and 2 above are both very simple ones, and they are both unlikely to wow any STX-student, but one of the great things about number tricks is, that they can really be varied infinitely, and any and all operations can be applied. Below is number trick 3: An example of a more elaborate number trick, which masks the predetermination of the end result much more effectfully (borrowed from Sultan & Artzt, 2011, p.24):

Number trick 3

1. Chose a three-digit number.
2. Construct the six two-digit numbers, that can be made out of the three digits of your chosen number.
3. Add these six two-digit numbers together.
4. Now divide the resulting number with the sum of the three digits of your original number.
5. Your resulting number is 22.

This would, if not wow the participant, then at least make him/her curious. One way of satisfying such curiosity is to take a look at the algebraic formulation of the steps involved. This is done in table 2.

Table 2: Table showing the algebraic formulations of the steps involved in number trick 3.

Step:	**Word description:**	**Algebraic formulation:**
1	Chose a three-digit number.	abc as in: $100a + 10b + c$
2	Construct the six two-digit numbers, that can be made out of the three digits of your chosen number.	ab, ac, ba, bc, ca, cb as in: $10a + b, 10a + c, \ldots, 10c + b$
3	Add these six two-digit numbers together.	$ab + ac + \cdots + cb$ as in: $10a + b + \cdots + 10c + b$ $= 22(a + b + c)$
4	Now divide the resulting number with the sum of the three digits of your original number.	$\frac{22(a+b+c)}{a+b+c} = 22$
5	You now have the number 22.	Confirmed!

As seen in the algebraic formulations of the steps, this number trick makes use of the nature of the decimal system to mask the pre-determinedness of the resulting number. Basically the entire trick amounts to simply the following:

$$\frac{10a + b + 10a + c + 10b + a + 10b + c + 10c + a + 10c + b}{a + b + c} = 22$$

This is a fairly straight forward calculation, but the use of the nature of the decimal system to mask the choices of a, b and c creates the "mystery", and in order to solve the mystery, one has to familiarize one self with the nature of the decimal system. So number trick two would be a perfect aid, if the target mathematical knowledge at hand was the nature of the decimal system in so much as it provides a didactical milieu with adidactical potential, in which a personalization of the target mathematical knowledge is a necessity in order to survive.

As mentioned earlier, I will not place great emphasis in the part of teaching which is about the creation of motivation to study a particular subject. Thus I won't be all too concerned about the wow factor of the number tricks involved in my lesson design. What I will have a great focus on, are the specific obstacles induced by the algebraic formulation of the given number tricks. I do however think, that's is highly appropriate to demonstrate the level of diversity, by which number tricks can be formed, and I hope, that the very different nature of the two examples given has done that. Really the possibilities are endless. If a higher level of mystery is aimed for, the number tricks can even be coupled with non-mathematics as is the case with the example on page 564 of Koirala & Goodwin, 2000, where the numerical result is translated into letters of the alphabet and further into countries and animals. This example, though, would not by my definition be a number trick in itself but rather a combination of a number trick and something non-mathematical.

3.2 Elementary algebra

I've mentioned the term elementary algebra quite a few times so far - it's even in the title of the thesis. In this section I'll elaborate a bit on just what exactly I've been (and will be) talking about.

Most importantly, concerning my interpretation of the term elementary algebra, are the following two points:

1. It deals with the generalization of arithmetics.

2. It does not deal with any algebraic structures outside the realm of the real numbers.

The second limitation is imposed by the fact, that the level C of STX mathematics does not include any exploration of structures outside the realm of the real numbers.

I'll further be limiting the term to include only the following four concepts:

1. Variables (as a tool used in order to accomplish generalization of arithmetics).
2. Evaluation and simplification of algebraic expressions (what is often referred to in STX as "reduction of expressions" even though this in some cases is a bit misleading).
3. Solving of equations - in particular linear equations (including properties of equality).
4. Substitutions (such as substitution of variables with expressions or conventional (notational) substitutions).

An example of a combination of the first and the second concept is the evaluation and simplification of the expression $a + a + b - b$ This can be simplified to $2 \cdot a$. This can then further be simplified by using the notational convention, which allows one to write $2 \cdot a = 2a$.

An example of the third concept is the solving of the equation:

$$x + 3 = 2x + 1$$

The typical approach by a level C STX student to this exercise would be to perform identical, specific operations on the expressions on either side of the equality sign, such that eventually the variable, x is isolated on one side. For this particular example such operations could be $-1 - x$, yielding the result: $2 = x$. So obviously the first and the second concept is very much tethered to the third concept.

An example of the fourth concept is the exercise of combining the following two formulas into one:

$$a = b + c$$
$$b = 2 + c$$

This could be done in several ways resulting in formulas containing either a and b or a and c. The typical solution would be to substitute the b in the first formula with the expression of b in the second, resulting in the formula:

$$a = 2 + 2c$$

(In this last line, I've also made the conventional (notational) substitution of $2 \cdot c$ with $2c$. Other examples of conventional notations are: $a \cdot a = a^2$ and $\frac{a}{b^2} = ab^{-2}$.)

Finally my definition of the term elementary algebra also includes the operator precedence and the basic use of the following basic axioms of algebra on the real numbers $\mathbb{R}$:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. $\exists 0 \in \mathbb{R}, : 0 + a = a + 0 = a$
4. $\exists - a \in \mathbb{R} : a + (-a) = (-a) + a = 0$
5. $a \cdot b = b \cdot a$
6. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
7. $\exists 1 \in \mathbb{R} : 1 \cdot a = a \cdot 1 = a$

8. $a \cdot (b + c) = a \cdot b + a \cdot c$

9. $\exists a^{-1} \in \mathbb{R} : a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$

I include the term “basic use” only to point out, that I do not consider it a task of elementary algebra to work out complicated proofs of theorems based on the basic axioms listed above. I do however consider it to be a task of elementary algebra to master the axioms to the extent of solving problems fitting within the above mentioned four concepts of elementary algebra.

It should be noted, that I at no point in time has had the intention of including all aspects of the above definition of elementary algebra into my lesson design. The definition serves primarily as a limitation to what I allow myself to include in my diagnostic test, and the mathematical content of the final design very much depends on the specific outcome of the diagnostic test.

4 Statement of problem

I want to address the very basic obstacles/misconceptions in working with elementary algebra that occur especially in the low level math (C-level) classes in Danish STX.[2] A relatively small emphasis is placed on this problem in the official STX curriculum, however the failure to gain an acceptable knowledge of the basics of algebraic syntax and semantics have wide spread repercussions throughout the rest of the curriculum. As a teacher of low level math classes in Danish STX for approximately 7 years, I have personally felt both the difficulties of establishing such a basic algebraic knowledge in the students and the despair that follows, when students and teachers keep battling with the repercussions of the students never getting it.

I aim to design a set of lessons specifically towards facilitating such knowledge. I will be using Brousseau's Theory of Didactical Situations to design the lesson set, and I will be building the lesson set around number tricks, in a way to try to breath life into specific rules regarding manipulation of algebraic expressions including formulas from the official curriculum.

I will observe and record an implementation of the lesson set to a low level math class, and through analysis of the collected data, I aim to answer the following research questions:

1. What obstacles occur, when teaching elementary algebra to low level math students in Danish STX?

2. What are the adidactical potentials of using number tricks to facilitate knowledge about elementary algebra?

3. What effect did the lesson set have on the students' skills/knowledge?

[2]Note, that by this I include all students, who initially have chosen to take mathematics on level C, regardless of whether or not they've chosen to later raise that level.

4.1 Hypothesis

I hypothesize, that the obstacles are of a very basic, algebraic nature, i.e. I expect that for even otherwise mathematically gifted students, the obstacles are rooted in misconceptions concerning the simplest generalizations of arithmetic. One such misconception could be: $aa = 2a$.

I hypothesize also, that using number tricks to facilitate knowledge about elementary algebra yields high adidactical potential, as it seems intuitively obvious, that such exercises can contain all the necessary parts to that end (as argued in section 3).

Lastly, I hypothesize, that by attacking concrete misconceptions with respect to elementary algebra through number tricks in adidactical situations, a (positive) number of these misconceptions can be exposed of.

5 Methodology

In this section I'll be explaining the procedure by which my thesis has been made. I've included to a relatively large degree some reflecting thoughts along the process, since the methodology has been very much an evolving process along the way. There has been no one real "final choice"of methodology, but rather many "sub-final" choices, each following some attempts and circumstances. I feel that this way of describing the methodology provides the best insight into the process of especially producing and gathering the empirical data.

The basis of my thesis is the combination of three things:

- The use of the theory of didactical situations - both didactical engineering and analysis
- Obstacles/misconceptions occurring when teaching elementary algebra to low level math classes in Danish gymnasiums
- The use of number tricks as a way to breathe life to elementary algebra

The two main pillars of the thesis are: 1. the design and implementation of a small set of lessons and 2. An analysis of the collected data. The design process definitely proved to be somewhat more extensive than originally perceived. My original plan was to rather quickly wrap up the design, so that the lessons could be quickly implemented. However after numerous attempts at the design, I repeatedly ran into issues regarding the target mathematical knowledge. In addressing the three questions

- What precisely IS the target knowledge?
- Which context might it be placed in?
- What is the students' previous knowledge relating to the target knowledge?

especially the last one caused me trouble.

5.1 Diagnostic test

From the beginning, I had known, that some sort of diagnostic test should be performed - originally with the main purpose of measuring the effects of the lessons. During the attempts to design the lessons, it became more and more clear, that the diagnostic test would have to be an integral part of the design process itself. So eventually a diagnostic test was created and performed, following by an analysis of the results, which led to the final definition of the target mathematical knowledge, and then the main lessons were designed. This is very much a simplification of the process, both because a lot went wrong with my construction of the diagnostic test (more on this in section 6), and because the definition of the target mathematical knowledge and the diagnostic test proved to have some sort of chicken-egg-relationship. The process of defining the target mathematical knowledge can best be described by the following steps:

- Initial attempt at defining the target mathematical knowledge
- Design and implementation of diagnostic test
- Analysis of diagnostic test
- Redefinition/fine-tuning of definition of target mathematical knowledge

This strategy implies, that the diagnostic test was implemented before the main lessons were designed.

The diagnostic test consisted of a number of small problems, which were duplicated and administered to the students at the beginning of a mathematics class in the following manner: Each student answered the same number of problems, but each student did not answer the same problems as every other student - more problems were created than each student would answer. The purpose of this was to be able to administer the exact same diagnostic test after the implementation of the main lessons to the class as a whole, but without any individual student being given the same problems twice. A total of three groups of problems were created, and the students circulated these groups in the diagnostic testing. Admittedly only two groups would have enough to serve the practical purpose of administering the exact same diagnostic test twice, however I chose to add a third group in order to widen the pool of different problems. After the first implementation of the test, I've re-thought that reasoning, and I probably wouldn't bother make more than two groups, would I have to do it again.

5.2 The students

From the onset it was clear, that the test students were to be some of my own students. I taught two mathematics classes at the time, and both fit the target demographic. Arguments could certainly be made for the choosing of a different class than my own, but several practical reasons dictated, that it should be my own students, which were used:

1. At the time I taught two of the only four classes, that reasonably[3] fit the target demographic. Both of my classes were better fits than the rest, in the sense that their choice of subjects as a whole upon entering the gymnasium was collectively less mathematically orientated.

2. The bulk of my connection to teachers/gymnasiums was placed within my own workplace, so it seemed obvious not to go outside my own school.

3. Due to the timing of the project, it would have proven difficult to get another teacher to give up lessons for his/her class, considering that the

[3]By reasonably I mean classes consisting mainly of students, who had chosen mathematics at the lowest possible level upon entering the gymnasium.

final examinations of the students were eminent.

4. The unsure nature of the project (timing, number of lessons, contents of lessons) were unappealing to other teachers with respect to their personal planning.

At the time I taught one class consisting of students who were in the midst of the mandatory mathematics (level C) and one consisting of students, who had all chosen to raise the level C to level B. As explained earlier, I chose to use the latter.

5.3 The teacher

Originally I had planned to have someone else administer the lessons, so that I could focus on observing the class and collecting the data. I had arranged for a specific teacher to take on the job, but at the last moment something came up, and I was forced to administer the lessons myself. This proved to not be a major inconvenience, as the bulk of the lessons consisted of adidactical situations, and thus I was relatively free to observe and collect data anyway.

5.4 Grouping of students

I chose the group the students rather than to have them work individually. This was a fairly easy choice, since it certainly makes it a lot easier to collect data from observing say 5 groups rather than 26 students. Also there's a lot to said for the dynamics of the group, and in order to have a successful series of action-formulation-validation phases, grouping seem like the obvious route to take.
The students were all placed in groups of 4-5 students based on the outcome of their diagnostic tests. The grouping did not intentionally reflect the total level of the students in the sense that students who overall performed equally in the diagnostic test were grouped together. In stead they were grouped based on the

types of misconception they showed in the diagnostic test. I.e. students who made somewhat identical errors in the diagnostic test were grouped together. This resulted in a scenario, where every group had a somewhat diverse level of mathematical talent amongst the members, which was intentional, as I hoped it would lessen the need for teacher intervention in the adidactical situations by increasing the odds, that some student(s) in each group would be able to take a lead in cases, which could otherwise easily end in a kind of stall. It's important to note, that there were clear differences between the groups as wholes talent-wise. This seems to be a likely consequence from the way the groups were put together, however within each group, the talent was not spread out evenly.

5.5 Design of the lessons

The design was relatively straight forward, once the analysis of the diagnostic test was done. I did run into a few problems when analyzing the diagnostic test - mostly as a result from the test not being as diagnostic as I originally had thought. I elaborate on this in a later section. The main lessons had always been planned to consist of a few number tricks for the students to work with - translate into algebra - and with the target mathematical knowledge outlined very specifically by the diagnostic test, this went fairly straight forward and according to plan.

5.6 A priori analysis of design

The a priori analysis of the design didn't go according to plan. The reason is, that all of a sudden it was apparent to me, that the main lessons had to be implemented very soon (within days) if not to cause serious challenges with the schedule of the class involved. So the original plan to do an a priori analysis of the design, and then to adjust the design accordingly, was scrapped, and the a priori analysis was actually done after the implementation of the lessons. It was done before any kind of review of the collected data, so it is still in some sense a true a priori analysis, however some of the purpose was lost. I don't think the a priori analysis has thus been made irrelevant, as it is still quite purposeful to have an outlined view of the expectations when analyzing the actual events.

5.7 Data collection

Each group had all their dialog recorded using laptops/phones. In addition to this they all handed in every bit of material, they had produced during the lessons, and a few times during their work, I photographed their work at specific moments, where a group was at a point of particular interest. This served the purpose of saving some written moment before any erasing potentially removed interesting bits from the paper.

None of the collected data was reviewed until all lessons had been completed.

5.8 Analysis of collected data

The audio recordings were partly transcribed. I initially listened through the audio files at live speed noting only time stamps of any and all interesting passages. Afterwards I cut out the interesting bits and collected these in a single audio file. This file I once again listened to, and once again - this time more selectively - I noted down the time stamps of the interesting bits. These bits were transcribed, and the transcription was analyzed using the theory of didactic situations. The analysis of the transcript was supported by the photos taken and the handed in answers to the problems posed.

5.9 Post-lesson diagnostic test

As mentioned, the diagnostic test was created in way, so that it could be administered twice with the exact same problems used both times, yet without any student answering the same problem twice. Thus a second implementation would give a decent indication whether or not an improvement had occurred.

5.10 Summary of methodology

So in an attempt to sum up the methodology: The three research questions

1. What obstacles occur, when teaching elementary algebra to low level math students in Danish STX?
2. What are the adidactical potentials of using number tricks to facilitate knowledge about elementary algebra?
3. What effect did the lesson set have on the students' skills/knowledge?

were answered through the following respectively:

1. Through analysis of implementation of designed lessons using the theory of didactical situations
2. Through didactical engineering and implementation of lessons using number tricks as the context
3. Through implementation and analysis of pre- and post-diagnostic tests

6 Diagnostic test and the target mathematical knowledge

In this section, I'll be addressing the diagnostic test and defining the target mathematical knowledge. This includes describing the motivation for making the diagnostic test, the process of making it, issues which have occurred along the way, the implementation of it and the results from the pre-lesson implementation - leading to the definition of the target mathematical knowledge.

Before the designed lessons were held (even designed), the students were given a diagnostic test consisting of a number of small, simple problems all relating to elementary algebra, and all fitting within the completed curriculum of the class. The purpose of this diagnostic test was two fold:

1. By giving a diagnostic test before the lessons, the effects of the lessons could be measured by giving the same diagnostic test after the lessons. Of course giving "the same" test presented some challenges. I will touch upon that below in the subsection "Challenges of measuring the effect of the lessons".

2. In order to aptly design the lessons in such a way, that there is a balance between what skills is previously owned by the students and what knowledge is intended for the students to acquire, a diagnostic test must be held in order to uncover the previous. (I.e. the diagnostic test aids in determining the target mathematical knowledge of the main lessons.)

Elaboration on the second point: It's important to get a precise overview of what the different students can do, and what they can't do. This affects possible grouping of students, and it very much affects the possibilities of contents of the main lessons. It is of great importance to secure, that the target mathematical knowledge functions as an extension of the already acquired knowledge in order for everyone to be able to follow the lessons, i.e. carefulness must be applied to secure, that the mathematical content of the lessons are neither too hard nor too easy.

6.1 Challenges of measuring effect of lessons

In order to gain the best possible comparison between the pre-lessons diagnostic test and the post-lessons diagnostic test, the exact same diagnostic test was given both times. Of course this presented a small challenge in that the same student of course couldn't be given the exact same problems two times, since this obviously would affect the results. How this challenge was overcome is described in section 5.1.

6.2 Contents of the diagnostic test

As mentioned, the diagnostic test was based on small problems all relating to (my definition of) elementary algebra and all fitting within the completed curriculum of the class. For this particular class that basically boils down to the following two categories of problems:

1. Reducing (evaluating and simplifying) algebraic expressions

2. Solving linear equations

In order to best spread out the problems to the students, each of these two categories were split up into three levels of difficulty. Then in stead of all problems being assigned at total random, each student was assigned the same number of questions from each subcategory of problems. This made the individual tests the most diverse, and thus it prevented a possible skew from skilled students getting all the hard problems, or one skilled student getting only easy questions etc. Basically it ensured, that each student was tested thoroughly.

6.3 Epistemological reference model

In order to best construct the diagnostic test, I first outlined an epistemological reference model for the main lesson design.

An epistemological model is two things (Måsøval, 2011, p.28):

1. Some target mathematical knowledge
2. A process by with that target mathematical knowledge is learned

The diagnostic test had to of course reflect in some sense the target mathematical knowledge, as it besides being a tool helping with the design of the main lessons also serves as a tool in measuring the results of the main lessons. So even before designing the diagnostic test, it made sense to outline the epistemological reference model for the entire project, i.e. the following questions should be answered:

1. What is the target mathematical knowledge?
2. How can number tricks assist in the implementation of this target knowledge?

6.3.1 Target mathematical knowledge

As mentioned in section 5.1, the methodology gave a small challenge here: In order to have the same test be diagnostic and a measure of effect, it had to both precede the outline of the target mathematical knowledge and be designed with the target mathematical knowledge in mind. This of course is an impossibility, but in reality, it wasn't hard to overcome. Attacking the situation from a third direction solved the problem: I needed to first answer the following questions:

- Which are the prerequisites of working mathematically with number tricks?

- What is the mathematical range of number tricks?

Obviously the contents of the diagnostic test should contain elements of the answers to both questions. This differentiation was intended to lie in the separation into levels of difficulty. This was assumed to most likely result in students not being able to give a correct answer to all the problems in the diagnostic test at it's first run, but this was exactly the intention. I will go as far as saying, that if they were able to answer all problems correctly, then that would have posed a problem.

6.3.1.1 Prerequisites and range of working with number tricks

As I argued in section 3, the range of working with number tricks is really infinite within the field of elementary algebra, in the sense that number tricks can be made as simple or as complex as one desires. Take for instance number trick 4 below, which is an extremely simple number trick though still a bit more complex than number trick 2. It seems fair to assume, that every STX student should be able to see through the mathematics of number trick 4 (which is the reduction $x + 2 - x = 2$):

Number trick 4

1. Think of a number.
2. Add 2 to your number.
3. Subtract your original number from the result.
4. Your resulting number is 2.

The prerequisites of being able to work with a number trick of this level of difficulty is simply being able to add and subtract.

An example of a more elaborate number trick could be the one representing $a^2 + b^2 + 2ab - (a + b)^2 = 0$:

Number trick 5

1. Think of two numbers.

2. Square both numbers and add the two resulting numbers to each other.

3. Add to the result 2 times the product of the original 2 numbers.

4. Subtract from the resulting number the square of the sum of the original two numbers.

5. Your resulting number is 0.

Obviously the prerequisites for working with this number trick are a bit more than those of the former. The student must here be able to also square and multiply numbers. In general, obviously, the prerequisites of working with number tricks are familiarity with the mathematical operations included in the number tricks. Conversely, one could argue that the prerequisites determine the range, in which I can move around in my design, and then it all comes back to the fact, that the diagnostic test should simply include levels of difficulty which are varied enough so that:

1. Every student can answer something correctly, and

2. no student can answer everything correctly.

If this is achieved, then the main lessons can be designed with the purpose of moving the students from their respective initial levels to higher ones while still keeping within the pre-given range of the entire setup (which would then be considered the target mathematical knowledge). This last part is important in order to be able to use the diagnostic test as a post-lesson measure of effect.

So now I'm ready to address the question: "What is the target mathematical knowledge?":

The target mathematical knowledge was originally thought to be defined through the problems in the diagnostic test, varying for the individual student depending on their initial stance. As my phrasing suggests, this was eventually altered, however in order to provide the best insight into how the diagnostic test was formed, and why I eventually made the choices I made, I will describe my detour, in what I turned out to decide was a wrong direction, rather detailed. So from that particular stance, the only thing left was to choose the range of difficulty in which to operate.

I had decided to define three levels of difficulty in each of the two groups of problems: Reducing mathematical expressions of symbols and solving linear equations. The definitions of difficulty had to be concrete with respect to mathematical contents of the problems, as this would ultimately be the definition of the target mathematical knowledge. It seemed reasonable to let the lowest level of difficulty to some degree be defined by what students are expected to know, as they enter the STX. Silkeborg, 2010 contains a set of problems designed to be just so, so for the lowest level of difficulty, I chose to use problems from there as a basis. Silkeborg, 2010 presents a somewhat wide selection of problems with respect to difficulty, but seeing as my group of students all originally have chosen a composition of school subjects, which minimizes mathematics as much as legally possible for students at STX at the time (combined with my personal knowledge of the students as their teacher), I found it reasonable to pick problems from the least difficult portion of Silkeborg, 2010 as the basis for the lowest level of difficulty in both groups of problems. (A more precise definition of the lowest level of difficulty follows.)

For the medium level of difficulty, I chose to use problems reflecting, what the students were expected to be able to solve at their current state. As they were all very close to their final examination in mathematics, and seeing as they had by and large finished the official curriculum, it seemed reasonable to choose the problems of a medium level of difficulty from the most recent examination. These problems by definition reflect, what the students were expected to have learned at this point. (As with the lowest level of difficulty, a more precise definition of the medium level of difficulty follows.)

The highest level of difficulty should go beyond what the students were expected to be able to handle at their current state, as this will define the length of the direction in which the main lessons can try to push the students. In the next subsection I will go into details and define this and the previous two levels of difficulty.

Definition of the levels of difficulty - reduction of mathematical expressions of symbols

The lowest level of difficulty for the problems about reduction of algebraic expressions was chosen to consist of very simple problems of the simplest type from Silkeborg, 2010. Their purpose was thought to be to uncover whether or not the students were able to correctly use addition, subtraction, multiplication and division, and whether they were able to correctly multiply two parentheses, when reducing mathematical expressions. These two purposes were kept isolated from each other by a division into 4 sub-levels of problems: a) Addition/subtraction, b) Multiplication, c) Multiplying parentheses and d) Division. The list of problems in this category along with the rest of the problems in the diagnostic test can be found in table 3.

The medium level of difficulty (and this is the point, were I truly went off into the wrong direction - more on that later) was chosen to be (to some extent) combinations of the simpler problems from the lowest level. They were modeled after the exam problems from (Exam-8-14, 2014, Exam-5-14, 2014), and they represented the level of mathematical reduction, that the students are expected to be able to handle at the end of the year (UVM, 2015). Besides from the operations tested in the lowest level, the medium level also included the squaring of an expression, which I knew, that all the students had worked explicitly with before.

The highest level of difficulty was sought to be problems consisting of all the elements of the lower levels. Also they were made to contain two different letter-symbols rather than just one as in the previous levels, and they were also made to require knowledge about multiplying/dividing powers of symbols. The main difficulty in the problems of this level was in the hard mixture of the operations, which required the students to have a very firm grasp of the operator precedence and rules of association/distribution. The very things, that I had originally considered to be the target mathematical knowledge.

Definition of the levels of difficulty - linear equations

The levels of difficulty for the second type of problems, the linear equations, were designed in exactly the same way as above, except for the fact that the lowest level of difficulty was not divided into subgroups, and so I won't go into further detail except to say, that those problems likewise can be found in table 3.

Table 3: Table showing the entire set of problems in the diagnostical test. R=reduction, E=equation, L=low, M=medium, H=high, and the letters in parenthesis in the level column indicate the subtype: a=Addition/subtraction, b=Multiplication, c=Multiplying parentheses and d=Division.

Type:	Level:	Problem 1:	Problem 2:	Problem 3:
R	L(a)	$5+2a-2-a$	$8+4a-3+2a$	$3a-2+2a+4$
R	L(b)	$(5a)\cdot 2$	$(2a)\cdot 6$	$(6a)\cdot 4$
R	L(c)	$(a-1)(2-2a)$	$(3+a)(3-2a)$	$(2-3a)(1+a)$
R	L(d)	$\frac{2a-4}{2}$	$\frac{6+8a}{2}$	$\frac{9-3a}{3}$
R	M	$(a+b)^2-a^2-ab$	$(a-b)^2-b^2+ab$	$(a+b)^2-a^2+b^2$
R	H	$\frac{4(a+ab)(2a^2-(2a)^2)}{8a^3}$	$\frac{2(b-ab)(4b^3+(2b)^3)}{8b^4}$	$\frac{3(ab-a)(3a^3-(3a)^2)}{9a^3}$
E	L	$3x+1=7$	$2x-1=5$	$4x+2=10$
E	M	$2x+1=4x-3$	$3x-2=x+4$	$x+3=5x-5$
E	H	$\frac{(x+2)(3-x)}{1+x}=3-x$	$\frac{(4-x)(2x+1)}{2x-2}=1-x$	$\frac{(2-x)(4x-3)}{2x+1}=-2x-1$

So in my (failed) attempt to construct the target mathematical knowledge, I managed to also construct the contents of the diagnostic test itself, as I (mistakenly) went forward considering the problems comprising diagnostic test to be exactly the target mathematical knowledge. Feeling able to provide satisfying answers to the questions regarding target mathematical knowledge in section 2.9, I then naturally went on to answer the second question regarding the epistemological reference model, which is addressed in the next (sub)subsection.

6.3.2 How can number tricks assist in the implementation of the target knowledge?

As it should be quite clear by now, it has been my intention all along, that the design of the main lessons make a heavy use of the theory of didactical situations.

I postulate, that number tricks are especially well suited for this purpose, as they mask elementary algebra with a nice, fun and almost game-like structure, which promises a lot of adidactical potential. I've already made my arguments for this in section 3.1, so I won't go further into it here.

6.4 Implementation of diagnostic test

Having constructed what I believed to be a diagnostic test, the next natural step was of course implementation, and the diagnostic test was administered to the class in April 2015. It went off without any issues. The students were each given 9 problems (one column of table 3), they were not given a time limit, and the last student handed in the solutions after approximately 15 minutes.

Back on track

It wasn't until I evaluated the results of the diagnostic test, that I realized, that I had been going about it completely wrong. It was rather shortly into this process apparent, that:

1. My target knowledge was not cleverly chosen.

2. My diagnostic test was really not diagnostic at all (at least in a way that made sense).

After an appropriate amount of panic, I came to the realization, that the problems in the first four rows of table 3 could actually serve as a true diagnostic test, especially since I received everything written by the students in their attempts to solve the problems, and that the outcome of this "new" diagnostic test would eventually define a much more suitable target mathematical knowledge. I'll elaborate on this:

In my construction of the diagnostic test, I had made what I suspect to be a quite widespread error among teachers: I had constructed a test designed to diagnose the ability to solve mathematical problems rather than a test to diagnose misconceptions within a given mathematical knowledge. Reviewing the students' solutions to the test, it was clear, that it wasn't really interesting from a didactical point of view, whether or not the problems were answered correctly. The interesting part was the nature of any and all mistakes made in the attempt at the solution. Thus, in order for the test to be truly diagnostic with respect to misconceptions, the problems should be of a very simple nature, which the majority of my diagnostic test was not. I briefly considered creating a new test, but seeing as I had already administered the first test, and seeing as a subset of this test actually did serve it's true purpose, I opted to not. Note that the problems of type "R" level "L(b)" in table 3 were by far the most successful with respect to diagnosing misconceptions. Had I chosen to create a new diagnostic test, it would certainly contain only problems of such a simple and focused nature.

My changed view on the diagnostic test brought clarity to the question of what should be the target mathematical knowledge: It should naturally be the knowledge with which the students had the most outspoken and similar misconceptions, thus not to be precisely defined until after analyzing all the answers.

6.5 Results of the pre-lesson diagnostic test

Before recording any data based on the results, I had to choose just what exactly to record. This was a good point to recall the dual purpose of the test: Measuring effect of the lessons and determining the target mathematical knowledge. At this point it was clear, that the target mathematical knowledge should be defined by the specific misconceptions uncovered, so naturally I needed to record in some manner the misconceptions occurring. The question was then, how to measure effect. I chose to define "effect" as a change in the number of those specific misconceptions defining the target mathematical knowledge rather than the students' ability to generally answer the problems correctly. This however posed the following problem: When measuring the effect, how can I be sure, that I have accounted for all the students exhibiting a specific misconception and not just those, where the misconception in question was apparent? As a solution to this I decided to record two things:

- The specific misconceptions occurring (and how often they occur)
- The total number of correct answers

I argue, that the last category should indicate whether any effect measured in the first category is misleading or not.

Results of: Reduction - addition/subtraction

68% of the students answered this problem correctly showing no misconceptions what so ever. The remaining students' answers thus contained errors. The errors, which seemed appropriate to connect to a specific misconception, were all of the following type:

1. $\boldsymbol{a+b=ab}$

(I've given the misconception a number, in order to refer to it later.) There were other errors either due to other misconceptions or simply sloppiness, however type 1 was the predominant, and it was fairly obvious as can be seen by the examples provided in figure 2 below.

A side note: The examples in figure 2 could suggest, that a problem with the didactical contract might have occurred rather than a genuine belief that $a+b=ab$: Both students in question have actually answered the question correctly, before they move on to incorrectly try to simplify the expression further. An explanation for this could be, that the student feels, that it is expected of him/her to simplify

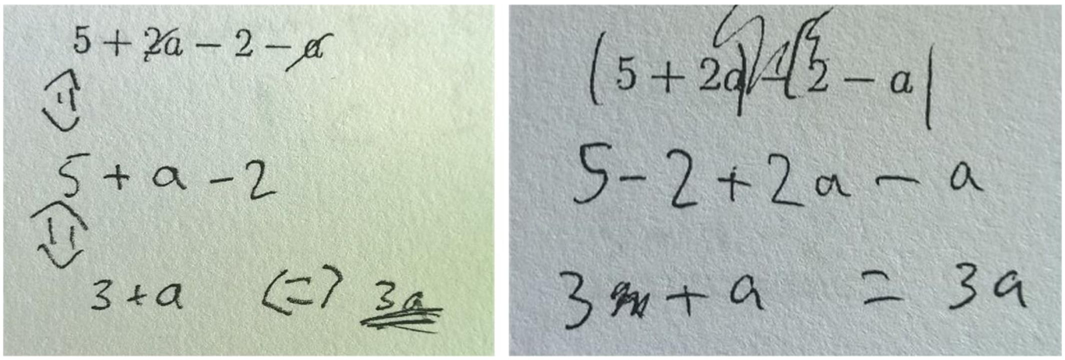

Figure 2: Two examples of the misconception: $a + b = ab$ (ignoring the incorrect notation).

at all costs - the aim for the student might be more to make the expression take up less space on the paper than to actually look at the mathematics.

Results of: Reduction - multiplication

73% of the students answered this problem correctly showing no misconceptions what so ever. The remaining students' answers corresponded primarily to the following misconception:

2. $\boldsymbol{(ab) \cdot c = ac \cdot cb}$

It seems reasonable to assume that this is a more genuine misconception than 1, and that the basis of it is a misconception of distribution (it would have been interesting to see, if the misconception would also show itself, had the parenthesis been removed). I've included an example in figure 3.

As with the previous problem, other wrong answers occurred, but as before I've only included the misconception, with which a relatively small amount of interpretation needed to be applied in order to characterize it.

Results of: Reduction - multiplying parentheses

Only 18% of the students answered this problem entirely correctly, showing no misconceptions what so ever. This drastic drop relative to the first two problems,

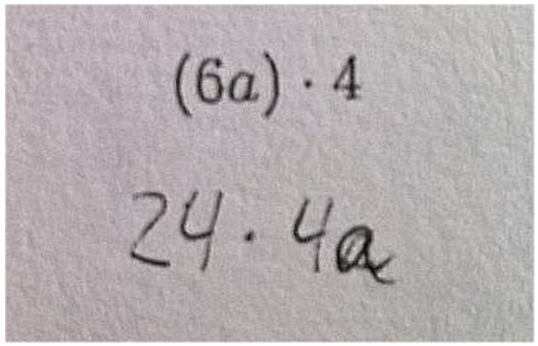

Figure 3: An example of the misconception: $(ab) \cdot c = ac \cdot cb$.

was not surprising, as the complexity of this problem is a lot higher. Thus it is also a much worse problem with respect to diagnostics than the previous two. However, since I have received all intermediate calculations by the students, I have been able to diagnose some misconceptions. These correspond to the following:

3. $\boldsymbol{ab \cdot b = ab}$

4. **No operation precedence**

5. General problems with $(a+b)(c+d)$

This one was harder to diagnose anything concrete from, since the problems se-emed more wide spread, however 3 and 4 were pretty clear in several answers, and I've included an example of each in figure 4. Number 5 isn't really a misconception as much as a lack of knowledge, hence I've distinguished it from the rest by having it not be in boldface like the rest. The last example of figure 4 shows an answer, which I feel clearly indicates, that the student has not acquired any knowledge of how to proceed with such an exercise. I decided to include 5 in the list, as it was fairly recurring, and that it therefor would be suitable as part of the target knowledge.

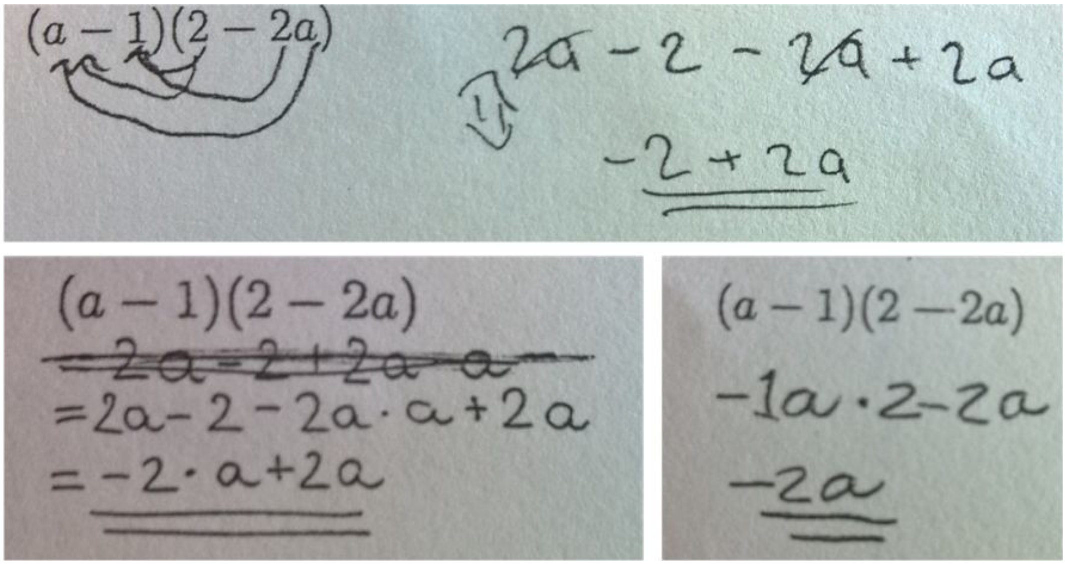

Figure 4: Examples of the misconceptions of type 3,4 and 5 respectively.

Results of: Reduction - division

32% of the students answered this problem correctly showing no misconceptions what so ever. The remaining students' answers corresponded entirely to misconceptions regarding operator precedence (already noted as 4) and/or distribution:

6. $\boldsymbol{\frac{a+b}{c} = \frac{a}{c} + b}$

I've included an example of both the disregard of operator precedence and the misconception regarding distribution in figure 5

I've collected all the results of the test in table 4.

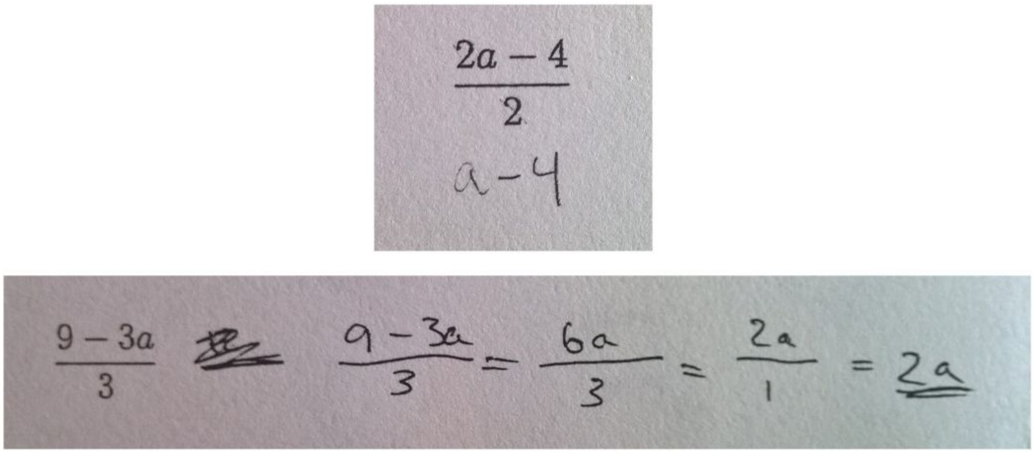

Figure 5: Examples of the misconceptions of type 6 and 4 respectively.

Problem:	**Correct:**	**Misconceptions (occurrence):**
Addition/subtraction	68%	1) $\boldsymbol{a+b=ab}$ (18%)
Multiplication	73%	2) $\boldsymbol{(ab)\cdot c = ac\cdot cb}$ (14%)
Mult. of parentheses	18%	3) $\boldsymbol{ab\cdot b = ab}$ (27% 4) **No operation precedence**(23%) 5) Problems with $(a+b)(c+d)$ (14%)
Division	32%	4) **No operation precedence**(32%) 6) $\boldsymbol{\frac{a+b}{c} = \frac{a}{c} + b}$ (32%)

Table 4: Overview of the results from the pre-lesson diagnostic test.

6.6 Final definition of the target mathematical knowledge

With a clear overview of precisely defined misconceptions occurring amongst the students in the class, I chose to define the target mathematical knowledge narrowly as exactly the parts of elementary algebra with which the misconceptions occurred. That is, the target mathematical knowledge is the following:

1. $a+b \neq ab$

2. $(ab) \cdot c \neq ac \cdot cb$

3. $ab \cdot b \neq ab$

4. The operation precedence

5. The Rules of distribution

One might say, that there is some overlap in those five point, as for instance 2 could arguably be called a problem with distribution, however I chose to include the entire list as the target mathematical knowledge, as the entirety of the list reflects the genesis of the target knowledge.

Since I initially set out to also include some specific formula from the official curriculum, I also included the quadratic formulas ($(a+b)^2 = a^2+b^2+2ab$, etc.):

6. The quadratic formulas

I do this without hesitating, as it doesn't in any way cause a problem with my methodology or anything else. On the contrary it opens up the opportunities for the development of the number tricks in my design of the main lessons.

It's worth to note, that 9% of the students answered all 4 questions entirely correct. Thus the target mathematical knowledge as I have defined it, could be argued not to allow for acquisition of knowledge for all the students. However, none of the students have answered all the problems in the diagnostic test (as it was administered) entirely correctly, and some of the misconceptions defining the target mathematical knowledge did appear in some answers for all students without exception. Therefore, I chose not to concern myself with this any further than to note down the total score of the test as it was administered, in order to better evaluate the total progress. I chose to treat this total scoring a little diffe-rent, as it really had nothing to do with diagnosing. In stead of simple giving 1 point for correct answers and 0 points for incorrect answers, I scored each answer with a rational number between 0 and 1. This procedure of course have it's build in errors, as a level of interpretation on my side must occur, however, I still feel, that this way of scoring provides a better number for a general comparison of the pre- and post-lesson diagnostic tests. **The pre-lesson score was 54% on average.**

Before moving on, the three questions by Brousseau in section 2.9 regarding target knowledge, should be addressed:

1. What precisely IS the target knowledge?
2. Which context might it be placed in?
3. What is the students' previous knowledge relating to the target knowledge?

1) The target knowledge has been very precisely defined above.
2) It can and will be placed in the context of number tricks.
3) The students' previous knowledge has been diagnosed by testing.

With the target mathematical knowledge defined, the next step is the design of the main lessons. This procedure is presented in section 7.

7 Main lessons

I was able to allocate two lessons of each 90 minutes with the students. The two lessons were about a week apart. I estimated, that 90 minutes should be enough to cover two number tricks, so a total of four number tricks were developed. I call these number tricks A,B,C and D respectively, in order to keep them separate from the ones earlier in the thesis. A and B were administered in the first lesson, and C and D were administered in the second lesson. C and D are more advanced/complicated than A and B. I decided, that this wouldn't cause a time issue, as I expected the devolution phase to be much shorter in the second lesson than the first.

7.1 Design

As it's probably clear by now, the layout of the lessons is the following:

- Devolution - The students were placed in groups, problems were handed out, and instructions on how to proceed were given.
- Adidactical phase - The students worked in groups with the problems.
- Institutionalization - A short discussion about problems and solutions having occurred resulting in a precise formulation of the acquired knowledge.

It was the middle part - the adidactical phase, which needed to be designed, as the devolution pretty much is laid out by this, and the institutionalization is dependent on the actual events during the adidactical phase.

So four number tricks were designed with one eye fixed on the list of target mathematical knowledge. The four number tricks are presented in tables 5-8 below. They are presented along with the algebraic formulations, and I've included a column with notes on which parts of the target knowledge is attempted induced by the step in question. The students did not receive the algebraic formulations. They received a table for each number trick with a blank column for filling in the algebraic formulations and for filling in an example along the way (feedback/validation).

Table 5: Number trick A along with algebraic formulation and target mathematical knowledge attempted induced (TMK).

Step:	**Instruction:**	**Algebraic formulation:**	**TMK**
1	Think of a number.	a	
2	Add this number to 4.	$4 + a$	1
3	Multiply the resulting number by 2.	$(4 + a) \cdot 2$ $= 8 + 2a$	5, 4
4	Subtract two from the resulting number.	$8 + 2a - 2$ $= 6 + 2a$	4
5	Subtract your original number from this - twice.	$6 + 2a - a - a$ $= 6$	4, 1
6	You now have the number 6.	Confirmed!	

Table 6: Number trick B along with algebraic formulation and target mathematical knowledge attempted induced (TMK).

Step:	**Instruction:**	**Algebraic formulation:**	**TMK**
1	Think of a number.	a	
2	Multiply 2 by this number.	$2a$	
3	Multiply the resulting number by your original number.	$2a \cdot a$ $= 2a^2$	2 or 3
4	Multiply the resulting number by 4.	$2a^2 \cdot 4$ $= 8a^2$	2 or 3
5	Subtract from the resulting number the result of multiplying your original number by 2.	$8a^2 - 2a$	
6	Divide the resulting number by your original number.	$\frac{8a^2-2a}{a}$ $= 8a - 2$	5
7	Add 2 to the resulting number.	$8a - 2 + 2$ $= 8a$	
8	Multiply the resulting number by 2.	$8a \cdot 2$ $= 16a$	2 or 3
9	Divide the resulting number by your original number.	$\frac{16a}{a}$ $= 16$	
10	You now have the number 16.	Confirmed!	

Table 7: Number trick C along with algebraic formulation and target mathematical knowledge attempted induced (TMK).

Step:	**Instruction:**	**Algebraic formulation:**	**TMK**
1	Choose two different numbers and add them together.	$a + b$	1
2	Multiply the difference between the two original numbers with the resulting number.	$(a - b)(a + b)$ $= a^2 - b^2$	5,6
3	Multiply the two original numbers together and subtract that from the resulting number - twice.	$a^2 - b^2 - ab - ab$ $= a^2 - b^2 - 2ab$	4
4	Multiply the largest of the original numbers by itself and subtract that from the resulting number - twice.	$a^2 - b^2 - 2ab - aa - aa$ $= -a^2 - b^2 - 2ab$	
5	Multiply the resulting number by -1.	$(-a^2 - b^2 - 2a) \cdot (-1)$ $= a^2 + b^2 + 2ab$	5,2,3
6	Take the square root of the resulting number.	$\sqrt{a^2 + b^2 + 2ab}$ $= (a + b)$	6
7	Subtract the sum of the original two numbers from the resulting number.	$(a + b) - (a + b)$ $= 0$	5,4
8	You now have the number 0.	Confirmed!	

Table 8: Number trick D along with algebraic formulation and target mathematical knowledge attempted induced (TMK).

Step:	**Instruction:**	**Algebraic formulation:**	**TMK**
1	Choose three numbers corresponding to the date, month and year of the birthday of a group member.	a, b, c	
2	Add 2 to the number representing the date.	$a + 2$	1
3	Multiply the resulting number by 200.	$(a+2) \cdot 200$ $= 200a + 400$	5
4	Subtract 400 from the resulting number.	$200a + 400 - 400$ $= 200a$	
5	Add the number representing the month - twice.	$200a + b + b$ $= 200a + 2b$	1
6	Multiply the resulting number by 5000.	$(200a + 2b) \cdot 5000$ $= 1,000,000a + 1,000b$	5
7	Add the number representing the year of birth.	$1,000,000a + 1,000b + c$	
8	The resulting number is the birthday of the chosen group member.	Confirmed?	

Number trick D arguably falls a little outside my definition of a number trick, in that it doesn't actually result in a predetermined number in the same sense as all the other examples trough out the thesis. I could argue, however, that it's really all about how to interpret "predetermined number", but basically I chose not to care (a view I regretted later, which will be addressed in section 10). One could also point out, that I in number trick D have compromised my earlier comment on not focusing on the trick part. Well, I couldn't resist. I likewise chose not to care about the fact, that it very much opens up for a discussion about something, that falls outside of the target knowledge. I basically thought it would be fun to include the trick.

7.1.1 Outline of the lessons

Table 9 and 10 show outlines of the different phases of the two lessons respectively.

Table 9: Outline of lesson 1.

Phase:	**Description:**	**Type:**	**Time:**
Devolution	- Work sheets are handed out. - Number trick A is run through with an example (arithmetically). - The first few steps are formulated algebraically together. - Instructions are given on how to proceed with the assignment. - The students are grouped. - It's checked that all understand the assignment.	Didactical	0-10m
Action/ formulation/ validation	- The students work in groups on the assignment. They work out an arithmetical example at the same time as they formulate the algebra, which provides feedback to the group along the way.	Adidactical	10-60m
Validation	- The groups take turns to present what they consider to be interesting aspect of their work. - The teacher directs a discussion of the findings.	Didactical	60-80m
Institutionalization	- The teacher highlights the dominating algebraic rules/formulas resulting from the discussion.	Didactical	80-90m

Table 10: Outline of lesson 2.

Phase:	**Description:**	**Type:**	**Time:**
Devolution	- Work sheets are handed out. - A brief recap of the last lesson is given. - Instructions are repeated on how to proceed with the assignment. - It's checked that all understand the assignment.	Didactical	0-5m
Action/ formulation/ validation	- The students work in groups on the assignment. They work out an arithmetical example at the same time as they formulate the algebra, which provides feedback to the group along the way.	Adidactical	5-65m
Validation	- The groups take turns to present what they consider to be interesting aspect of their work. - The teacher directs a discussion of the findings.	Didactical	65-85m
Institutionalization	- The teacher highlights the dominating algebraic rules/formulas resulting from the discussion.	Didactical	85-90m

It follows from the far right columns, that the adidactical phases take up considerably more than half the total time of the two lessons combined. This promises an effective exploitation of the adidactical potential of the lessons.

7.2 A priori analysis of lessons

As mentioned in section 5, the a priori analysis was actually conducted after the implementation. Well that is: Only partly, because the entire design was made with the inducement of specific obstacles through knowledge about the students' misconceptions about the target mathematical knowledge collected with the diagnostic test in mind. So I argue, that a level of a priori analysis was conducted before the implementation of the design - and the result of that "analysis" are the numbers in the far right columns of tables 5-8, namely the list of specific misconceptions expected to occur.

Part of the a priori analysis (which was conducted after the implementation) is the addressing of the following three questions from section 2.9:

1. *Are the students given the opportunity to get comfortable with the target knowledge?*

   I argue, that they are. The didactical milieu of the adidactical phases of the lessons has an incorporated feedback (a comparison of the arithmetical example of both the number trick's instructions and the reduced intermediate algebraic expressions), which at every step through the number trick tells the students (well indicates at least) whether or not they are on the right track. In case of a misconception about the algebraic formulations, the arithmetic example will demonstrate, that an error has occurred, and the group of students will be forced to discuss the issue - formulating, hypothesizing, validating the algebraic formulations.

2. *In working with the milieu: Can the students see a relation to and expand*

and use their existing knowledge?

I argue, that it's highly likely, that they can. The relevant existing knowledge in the milieu is the arithmetical knowledge corresponding to the generalizations in the algebra, that comprise the target knowledge. My expectation is, that the students, faced with a misconception causing a halt in their progress through the feedback mechanism, will eventually see a connection between the arithmetics and the algebra, and use this connection to expand their algebraic knowledge.

3. *Does the milieu make it possible and necessary to construct the target knowledge (at the least in some specific cases)?*

   Again, I argue that it does. Given that the design has been successful, and the intended misconceptions arise as planned, there will arise points, where existing knowledge can be expanded to the target mathematical knowledge, and furthermore an expansion of the existing knowledge (the generalization of the arithmetics) is necessary to surviving the milieu, i.e. to complete the assignment.

As mentioned in section 2.9, I did not apply the process of implementation - adjustment - re-implementation for practical reasons. I do feel justified to say however, that my design at this stage was ready for an implementation.

7.3 Implementation

The lessons were administered in may 2015 to 2a at Borupgaard Gymnasium in Ballerup, Denmark. The students were all willing participants, and everything went off without any major problems.

8 Analysis

I've chosen to focus on 3 specific episodes, in which some battle for the target knowledge occurs. The letter S indicates, that a student is talking, and T indicates teacher. I've translated all talk into English.

8.1 Episode 1: A successful overcoming of two misconceptions

Episode 1 happens in group 1, while they are working on number trick B. They have without any problems correctly arrived at the fraction $\frac{8a^2-2a}{a}$ in step 6. As we enter, they are attempting to simplify it. They have been working out an arithmetical example using 5 along the way.

1. S1: Isn't it just $8a^2 - a$, then?
2. S2: Yes.
3. S3: Shouldn't we check that?

In line 1, S1 demonstrates two misconceptions: One is that the denominator shouldn't be distributed to all the terms in the numerator, and the other is, that $\frac{2a}{a} = a$. It's immediately agreed upon by S2 in line 2, but then fortunately in line 3, S3 suggests to check it with the arithmetical example. They calculate the arithmetics using 5 (on calculator), and conclude, that it's incorrect:

4. S3: This one is not right. I don't think you can just remove the division sign.
5. S4: Yes, because this a cancels out this one. (Points to the denominator and the a in $2a$.)
6. S2: But then isn't there one a left..? Yes, there's one a left, yes.
7. S1: We could also just say $8a^2$ divided by a, because that's just $8a$. Isn't that easier?

8. S2: Yes, let's do that.
9. S1: But is it then $8a - a$ or $8a - 2a$?
10. S2: Then it is $2a$, right?
11. S1: Yes.
12. S3: (Calculates the arithmetical example.) It's still not right.

S3 questions the canceling out of the denominator, S4 reaffirms the misconception regarding distribution by pointing out, that the denominator cancels out with the last term of the numerator. S2 then in line 6 reaffirms the second misconception. Then in line 7, S1 suggests a way to not to deal with the second misconception, but her suggestion is yet again reaffirming the misconception regarding distribution, as she suggests, that they simply let the denominator cancel out the first term in stead. S1 shows the first sign of correcting the misconception in line 9, but it doesn't take with anyone - including herself. S3 checks with the example, and concludes, that it's still incorrect.
This bit of dialog is very important, as it very clearly shows, that the misconception regarding distribution is very deeply rooted. I argue, that this especially follows from line 7, as it is here directly used a means to move forward. I.e. it isn't just some rule, they are following without giving it any thought. It is a rule, that they explicitly bring up and use as a way to fix some problem. Fortunately the adidactical milieu is equipped with the feedback mechanism of the arithmetical example, and they conclude, that it's still incorrect. The feedback mechanism thus works properly giving the milieu a high adidactical potential, as well as underlying the importance of a counter example as a tool in the validation phase.
Following this bit of dialog, there is then a pause and some idle talking back and forth, and then the teacher, who have been listening in, intervenes:

13. T: As I understand it, you all agree, that this isn't entirely correct, and now you are stuck?
14. S4: Yes.

15. T: OK. Is there another way to write your fraction?
16. S3: You mean reduce it?
17. T: I'm talking purely notation. Is there another notation for division?
18. S4: We could use a slash. (They write: $8a^2 - 2a/a$.)
19. S1: But then you need a parenthesis.
20. S2: (Enthusiastically:) Yes, but then it has to be divided into both terms.

    The teacher leaves.
21. S1: Yes that's it... Then it's $8a - a$.
22. S2: Yes!
23. S4: Yes!

The teacher intervened to further along the situation, thus briefly turning the situation into a didactical one. He posed the leading question about changing the notation, and as soon as the question is understood by the students, He withdraws again, thus turning the situation back into an adidactical one. Note, that the comment by the teacher could potentially have led to a metamathematical shift, had the students not realized on their own, that the use of a slash required a parenthesis. This could have led to a discussion about notation and operator precedence in stead of the mathematical misconception at hand: Distribution. Fortunately this didn't occur though, and the students successfully rewrote the fraction to $(8a^2 - 2a)/a$. This very quickly prompted S2 to recognizing, that the denominator should be distributed amongst the terms in the numerator. Obviously the students possessed a previous knowledge about distribution, however they clearly hadn't previously attached that knowledge to a fraction. At the sight of the different notation, they expanded their previous knowledge about distribution to include fractions. (Side note: Here it would be quite interesting to see, whether this newly acquired, successful, rule of distribution of denominator into terms of numerator would cause the epistemological obstacle of thinking, that also a numerator shall be distributed amongst terms in the denominator.)

There was still the issue of the second misconception though, and the dialog continues:

24. S3: (After checking the arithmetic:) OK, but it's still not right.
25. S4: But...?

    (Pause.)
26. S3: (Referring to $2a/a$:) With 5, that's 2 times 5, that's 10. And 10 divided by 5 is 2. So it's a that cancels out.
27. S1: Yes, because it's 2 times a divided by a, so that's clear.
28. S4: Then it is $8a - 2$.
29. S2: Then we're left with $8a - 2$, that makes sense.
30. S3: That also fits with the example.
31. S2: Fantastic.

So once again the feedback mechanism provided a counterexample, and caused the students to re-evaluate. The breakthrough came, when S3 in line 26 addressed the arithmetical example and compared that with their generalized algebraic formulation. She recognized the pattern, and once the new hypothesis had been put forward, S1 saw and conveyed the logic behind it, and the rest of the group jumped on board. The feedback mechanism provided a high likelihood, that the answer was correct, and after the above dialog ended, they proceeded with the rest of the number trick without further problems. The overcoming of the second misconception happened as a result of examining the meaning behind the expression $2a/a$. First it was compared to an arithmetical example, and then some formulation of the logic behind the general simplification was made and validated. Thus the new knowledge was built on the arithmetical knowledge (line 26) and some previously owned knowledge of multiplication and division being each other's inverse operations (line 27).

Quite obviously the adidactical milieu did not allow the students to survive without correcting their two misconceptions. The feedback mechanism was crucial to that end. The interesting question now is, what caused the misconceptions? Was there a specific didactical obstacle behind them? This was briefly addressed later in the lesson - in the didactical phase of validation. S3 explained to the class, that they encountered this problem with the distribution in the fraction, and that they solved it by using a slash as notation. Then the teacher asked:

32. T: Why do you think that you all had trouble, when it was on the form of a fraction, but not when you used a slash?
33. S3: I don't know. I think it's because, that when we saw the parenthesis, then we just remembered it.
34. T: So you think, that you knew all along, but you just had to remember?
35. S3: Yes, because we did know eventually. We didn't look up any rules.
36. S4: I think maybe it's because, that with numbers you don't have to do it in all the terms.
37. S1: Well yes you do.
38. T: (To S4:) Explain that? What do you mean by that?
39. S4: When I calculate a fraction with numbers, I just type them in, and then it comes out correct. It also works with this example.

This last bit from S4 indicates, that at least for her, the obstacle was of a didactical nature: The consistent use of a calculator as a tool had rendered it unnecessary for her to acquire knowledge about what happens along the process. It's unclear whether she was talking about calculation a fraction in steps, i.e. $\frac{3+5}{2} = 3+5$ (enter, returning 8) /2 (enter - returning 4), or she was using a calculator that allows her to punch in a fraction, and then just returns the result, but based on my recollection of the calculator she held, I would say that it's the first, and in this case the use of the tool had rendered her unaware of the fact, that she was actually changing the operator precedence by the intermediate punching of enter - even though she had not punched in a parenthesis. It's tempting to argue, that this constitutes an epistemological obstacle, as she had had success with a practice in the past (as long as it's an arithmetical task, and the calculator is used correctly, no issues occur), but when her previous, arithmetical knowledge was expanded to the generalized case, then her knowledge was insufficient and needed to be adjusted. I argue though, that by that logic any didactical obstacle could be said to be epistemological. In stead, since this obstacle was not really a consequence of some generally incorrect, personal knowledge, it was not an epistemological obstacle. Rather: The unfortunate instructions on how to calculate fractions on a calculator could easily have caused the didactical obstacle leading to the misconception at hand.

8.2 Episode 2: An instance of insufficiency of the adidactical milieu and two instances of didactical phenomena

The second episode unfolded during the second lesson in group 2's work with number trick C. They'd reached step 6, where they were about to simplify the expression $\sqrt{a^2 + b^2 + 2ab}$. This proved to be very problematic. Along the way they had worked out an arithmetical example using $a = 5$ and $b = 4$.

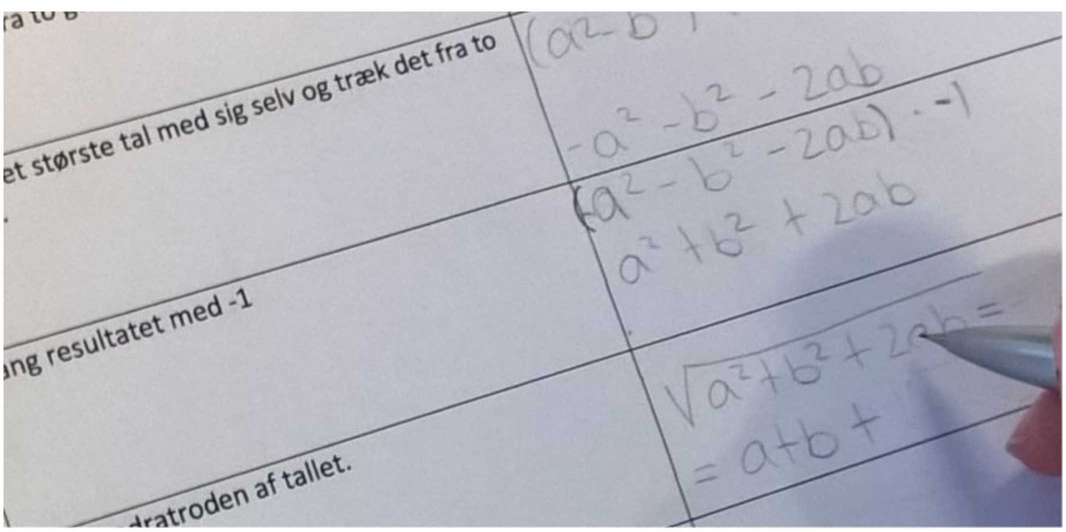

Figure 6: Photograph of a moment, where the students reach an impass when working with number trick C.

1. S4: Take the square root of the number. Hrrr!
2. S3: That must be 9 then (referring to the arithmetical example).
3. S4: (Loudly, playfully:) I is niiine! It's true. It's the sum of two original numbers, so we subtract that, and it's 0. Yeah, we've solved it.
4. S5: It makes sense.
5. S1: Then let's just add a root sign there, and we're done.

So here is seen a potentially hazardous aspect of this particular use of number tricks. The students were convinced, that they had solved the problem. They had to the extent of their knowledge completed the given task. They had survived the milieu. But they really hadn't dealt in any way with the specific target knowledge of that particular step of the number trick, which is the quadratic formula. In stead, they looked ahead in the steps and realized, that the simplification of the square root at hand *had* to be $a + b$ in order for everything to fit. However, they only really worked out the arithmetics - they never had any interaction with the algebra at stake. This example clearly shows the necessity of devolving to the students, that every step of the algebraic formulation must be verified algebraically.

Fortunately, in this episode, that *had* been properly devolved. The group continues:

6. S2: OK, so then that's just plus right?
7. S1: No we, can't say that it's nine, that's only...
8. S4: Didn't it give nine?
9. S1: Yes, but that's just the example.
10. S5: Yes, but we don't know that.
11. S2: Oh.
12. S4: Why?
13. S1: It was equal to $a + b$, but we don't know that.
14. S3: Aren't we supposed to simplify that?

15. S5: Yeah...? But that's what we did. No we didn't.
16. S1: Where did $a + b$ come from.
17. S4: They came from there (referring to the last steps of the number trick).

So the students realized on their own, that they were missing one crucial step. This is actually quite impressive - effective devolution or not. But that was far from the only issue making this particular episode interesting. The group was now faced with the task of simplifying the square root - they were engaging with the milieu to win the target knowledge at stake:

18. S3: But I think, this is wrong, because $a^2 + b^2$, that gives $a + b$, but then what about $+2ab$?
19. S4: It was the square root... It was the square root of this (referring to the arithmetical example), that gave 9.
20. S3: Yeah?
21. S4: Yeah yeah, but...
22. S1: Well we know that it's nine, but we should also say why, right?
23. S2: OK, so $a + b$ yields those two right, but then what about $2ab$?

Line 18 shows, that S3 had a specific misconception regarding the target knowledge at stake: That $\sqrt{a^2 + b^2} = a + b$. Most likely this was stemming from the misconception regarding distribution of operators stating: $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$, but line 18 doesn't specifically provide that information. The misconception was reaffirmed by S2 in line 23, but it's still not clear, whether or not it was a case of distributive misconception, and thus it is not yet possible to address the nature of the obstacle leading to the misconception.

The group continued to talk about the problem of simplifying $\sqrt{a^2 + b^2 + 2ab}$ to $a + b$ for almost 5 minutes without providing any useful information before finally giving up and calling for the teacher. The fact, that a group as insisting as this one (they did choose on their own not to bow out of the assignment in the beginning of the episode, and they have spent a long time discussing something intensely with no progress what so ever) chose to call it quits from the adidactical situation and call for the teacher, speaks to the fact, that there might be an insufficiency of the adidactical milieu with respect to allowing for the students to become intimate with the target knowledge at stake without interference by the teacher. It seems reasonable to read from the episode, that there was a gap between the previous knowledge (the arithmetical and algebraic skills previously acquired) and the target mathematical knowledge (an intimate knowledge of the quadratic formula), which the feedback mechanism of the adidactical milieu couldn't bridge.

The teacher entered:

24. S1: Our example is correct, but we don't know why the square root is $a+b$.
25. T: Your example is correct?
26. S3: Yes.
27. T: So what you need to do is to show, that those two are the same (pointing at $\sqrt{a^2+b^2+2ab}$ and $a+b$?
28. S1: Yes!
29. T: OK, how can you do that?
30. S1: Well we know that the square root of a^2 and b^2 are a and b, but then what about $2ab$?

In line 30, S1 reaffirmed the misconception, and she also provided information about it's nature. It's pretty clear from her statement, that it was indeed a case of faulty operator distribution, and thus it seems reasonable to diagnose it as an epistemological obstacle: The student has had success in the past with distributing operators on addition of terms, however while it's correct with respect to multiplication, it is not correct with respect to root taking, and her previous knowledge failed her in the given situation.

They continue:

31. S4: Is that just nothing then, or what?
32. S5: No it must give something.
33. S3: It gives the square root of $2ab$ (laughs).
    (...)
34. T: So you are saying, that the square root of $a^2 + b^2 + 2ab$ is the square root of a^2 plus the square root of b^2 and so on?
35. S1: Yes. Isn't it? It isn't.

Lines 34 and 35 makes it unquestionably sure, that the misconception was one of distribution.

36. S4: When he says it like that, it isn't.
37. T: (Smiling) Well... Is there a way, that you can check it?
38. S3: It's 9, and 4+5 is 9. And then $2 \cdot 4 \cdot 5$ that's... wait... that's 40, right. So that's not right.
39. T: OK, so now you've shown, that you can't do that. You can't just split up the square root like that.
40. S4: Why?
41. T: Your example doesn't fit to it.

Nudged along by the teacher, the students once again made use of the feedback mechanism of the milieu, and they provided a counterexample to their misconception. The teacher quickly grabbed that, and he concluded that the students have proven, that their misconception is false. However, here the teacher was a little too quick, which was made apparent by line 40. A Jourdain effect has occurred, as the teacher has concluded, that the students have "shown, that you (...) can't just split up the square root like that", whilst the only thing that has actually occurred is, that the students calculated an arithmetical example. There is no indication, that they grasped the connection between their example and the disproving of their misconception. As this was all regarding the very core of the specific target mathematical knowledge, that is actually a fairly big problem. The students might very possibly walk away from the situation with the knowledge, that you can't distribute every operator on added terms, but they haven't been given an opportunity to personalize that knowledge. The insufficiency of the adidactical milieu prompted a didactical situation, and there counterproductive effects are much more likely to occur.

The episode continues:

42. S1: OK but so then what?
43. T: But then what's the problem? You choose to take the terms in the square root and split those up and take the square root of that and that and that. And you can't do that, can you? So what *can* you do?
44. S4: We don't know.
45. S5: Uhhh!
46. T: It's not surprising, that you can't do that. It's the same as when you have $(a+b)^2$. That's not just a^2 plus b^2. There's something more. (Pause.) But is there another way to test, that something is the square root of something?
47. S4: What do you mean?

The teacher continued in lines 43 and 46 to almost institutionalize the target knowledge, still before the students had had a chance to personalize it, even though the comments by the students in no way indicate, that they had grasped the knowledge. "We don't know." "Uhhh!" "What do you mean?"
The teacher then went on to provide a hint as to how to proceed, now that the misconception is "removed":

48. T: Well in stead of going from there to there, is there another way? Can you move in a different direction?

49. S3: Can't you take the square of that one $(a + b)$?

    The teacher smiles and leaves.

50. S4: But were not supposed to square that?
51. S3: No but if you square that, then aren't you supposed to multiply it with... No.
52. S4: We HAVE to find out of it.

The teacher then posed the very leading question in line 48, and being the teacher in question myself, I can't add, that he also actually moved his index finger from $a+b$ to the square root, as he said "Can you move in a different direction?", making it a clear cut case of Topaz effect. The result was eminent, as S3 grabbed the ball and suggested exactly what she was suppose to suggest: To move from $a + b$ to $a^2 + b^2 + 2ab$ by squaring the first. The teacher left thinking, that they were on track, but as the following comments show, no real progress had actually been made.

The students discussed for a few minutes without getting closer. Then the teacher reappeared.

53. T: Have you reached anything?
54. S4: Nooo.
55. T: (At S3:) You said something very sensible wright before I left?
56. S1: We've done that. We squared it, and it gave 81.
57. S4: Well we squared $a + b$, and it gave 81.
58. T: OK, but what if you square $a + b$ without using numbers. What does that give?
59. S4: What?
60. T: a plus b squared. What's that? If you do it algebraically?
61. S1: Well that gives $a^2 + b^2$?
62. T: Does it? Take it from there.

    Teacher leaves.

First of all, line 56 clearly shows, that the question posed by S3 as a result of the Topaz effect earlier did not reflect any acquired knowledge at all. The students weren't even thinking algebraically but rather arithmetically. The teacher then continues the Topaz effect by delivering the target mathematical knowledge without any personalization (or maybe even apprehension) of the knowledge going on for the students. The specific target knowledge this time around is an intimate knowledge of the quadratic formula. I say intimate, because it is not just the knowledge of the formula, it is actually knowledge enough of it to recognize it backwards. That target knowledge was potentially lost in this situation.

After the teacher left, the students quickly decided to use arithmetical examples, and they wrote out the square, and they ended up concluding, that in general $(a+b)^2 = a^2 + b^2 + 2ab$. They also fairly quickly saw the connection to their problem, and they ended up writing as shown in figure 7. This indicates, that the damage by the Topaz effect wasn't as crucial, as it might could have been, and that the students despite all did acquire at least some of the target knowledge.

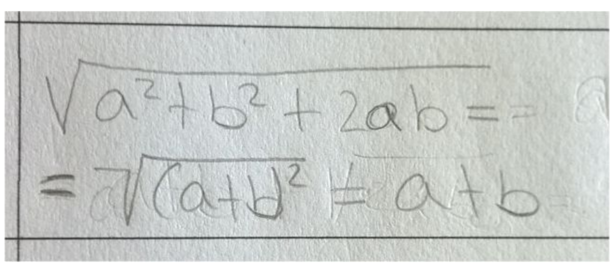

Figure 7: Final solution by the students to step 6 in number trick C.

8.3 Episode 3: An instance of failure of the feedback mechanism and a possible break of didactical contract.

The third and last episode described the work by group 3 with number trick A. They were using 2 in their arithmetical example, and the had reached step 3 after

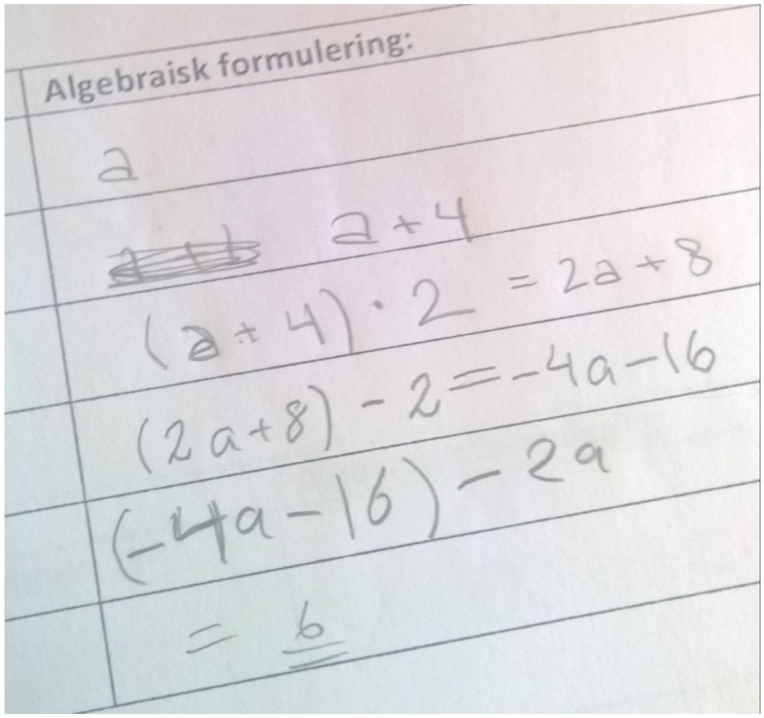

Figure 8: Final "solution" of number trick A by group 3.

having simplified $(a + 4) \cdot 2$ correctly to $2a + 8$ in step 2:

1. S1: So that's $2a + 8$.
2. S2: I just don't understand, why it's $2a + 8$?
3. S1: Because, if you have to multiply into the parenthesis, then it's 2 times a and 2 times 4. That's $2a + 8$.
4. S2: Oh yeah.

So far, so good. They seem to have possessed correct previous knowledge about distribution of multiplication on added terms.

5. S1: OK, so now we have $(2a + 8) - 2$.
6. S3: OK, so that's -2 times $2a$.
7. S1: Yes. -2 times $2a$.

Here they exhibited the following misconception: At the sight of a parenthesis and a number following it directly, they "knew" to multiply that last number into the parenthesis. They didn't seem to make a connection with the instruction of subtracting, that led them to write up their expression, and so they recognized the $(2a+8)-2$ as $(2a+8)(-2)$.

8. S2: But that's just $-4a$?
9. S3: Is it?
10. S1: I don't know?
11. S2: Yes because minus 2 times 2, that's 4, and then there is an a, you just... you just put that outside.
12. S3: Is that right?
13. S2: I would think?
    (Long pause.)
14. S3: Why isn't anyone else saying anything?
15. S2: (In an almost angry tone:) Or then just write it down. If it's wrong, then it's wrong.

From the above exchange, it's clear, that this group was very uncertain about the task at hand. Especially line 15 indicates, that the students weren't completely comfortable. In fact this entire exchange even indicates a dissatisfaction with the situation. The comment in line 15 could easily be interpreted as a sign of some didactical contract being broken. Perhaps S2 felt, that the teacher neglected his responsibility by not being there to check off the answers along the way.
The exchange continued:

16. S1: So what was it? $4a$? No $-4a$.
17. S2: And then it would be 8 times -2, that's just -16.
18. S4: Is a our number?
19. S2: Yes.
20. S1: What's the next?
21. S3: We have to subtract the original number... twice.
22. S4: Then it's just that one minus 4?

23. S3: Mmm.

In this bit of exchange two things were made apparent: 1) No one seemingly thought to calculate the arithmetical example, and so the milieu provided no feedback, and hence the misconception wasn't dealt with. 2) S4 had not yet grasped the concept of a as a variable. This is clear from lines 18 and 22. She didn't seem to think of a as a variable but rather as simply a placeholder for the number 2. Without displaying any signs of talking about the arithmetical check up, she still swaps a for 2.

24. S1: It's this part in a parenthesis, and then minus 2 times a.
25. (Long pause.)
26. S4: Isn't that then just a^2? Because it's two times.
27. S3: $2a$ and a^2 is that the same?
28. S4: Ehhm?
29. S4: No I don't think so?
30. (Long pause.)

In line 26 another misconception surfaced: That $2a = a^2$. The following lines show, that the other students were unsure of this fact, but yet again no one seemingly thought to address the arithmetical example.

31. S2: But I just don't understand this one: What's that? Minus 2?
32. S1: That's minus $4a$.
33. S2: Oh. And so that one is minus 16, but then why $-2a$?
34. S1: Because we subtract the original number twice.
35. S2: Oh yeah dammit.

These lines are actually quite interesting, because here it was made explicit, that the $-2a$ was a consequence of subtraction, and following this, the group did not proceed to treat $-2a$ as a factor to be multiplied with the parenthesis (see figure 8), as they did previously. This could indicate, that the group would actually have had a decent chance of overcoming that particular misconception earlier, had they just applied the feedback mechanism.

36. S3: But what's the original number?
37. S1: That's a.
38. S2: Oh.
39. (Long pause.)
40. S3: Shall we check, if it's correct?
41. S1: Yeah.
42. (Pause.)
43. S1: It's fine. It becomes 6, so we're done.

So having written up an (incorrect) expression in the final step without any attempt to simplify it, the group finally once again decided to check their arithmetical example. However, they had seemingly lost sight of the purpose of the example, as apparently all they did was to check, whether the number trick did in fact return 6 by the use of 2 as a chosen number. They concluded, that "it's fine", and they were "done".
After that, they moved on to the next number trick.

So obviously the feedback mechanism failed in episode 3, resulting in a situation, where none of the misconceptions were dealt with in any way what so ever. Clearly no new knowledge was acquired by anyone. The question then is: What went wrong? I argue, that this is a case of lacking devolution. Especially the very last bit of the devolution phase in table 9 must have bit neglected. Following episode 3 (in the group's work with number trick B), the teacher checked in with the group, and quickly realized the problem: That the feedback mechanism wasn't being used correctly, and after once again going through the purpose of arithmetical example, the group's work with number trick B was of a similar nature as that illustrated in episode 1, thus confirming, that the negative outcome of episode 3 was indeed a result of a lacking devolution (and thus underlining the importance of the proper attention to the devolution phase).

9 Post-lesson diagnostics

Approximately one week after the implementation of lesson 2, the students were given the diagnostic test once again. I administered the test in it's entirety (that is: the entire set of problems in table 1), in order to compare the total scores. The result of this test, regarding the occurrence of the specific misconceptions noted from the first run of the test, is displayed in table 11 below:

Problem:	**Correct:**	**Misconceptions (occurrence):**
Addition/subtraction	73%	1) $\boldsymbol{a+b=ab}$ (14%)
Multiplication	86%	2) $\boldsymbol{(ab)\cdot c=ac\cdot cb}$ (9%)
Mult. of parentheses	23%	3) $\boldsymbol{ab\cdot b=ab}$ (9% 4) **No operation precedence**(14%) 5) Problems with $(a+b)(c+d)$ (14%)
Division	36%	4) **No operation precedence**(14%) 6) $\boldsymbol{\frac{a+b}{c}=\frac{a}{c}+b}$ (27%)

Table 11: Overview of the results from the post-lesson diagnostic test.

The total score of the entire class for all the questions in the post-lesson diagnostic test was 69%.

9.1 Comparison of pre- and post-lesson tests

Below, in table 12, is a comparison between the pre- and post-lesson results:

Category:	**Pre:**	**Post:**
Correct answers in addition/subtraction	68%	73%
Correct answers in multiplication	73%	86%
Correct answers in mult. of parentheses	18%	23%
Correct answers in division	32%	36%
Occurrence of type 1	18%	14%
Occurrence of type 2	14%	9%
Occurrence of type 3	27%	9%
Occurrence of type 4	23%, 32%	14%, 14%
Occurrence of type 5	14%	14%
Occurrence of type 6	32%	27%
Overall score	54%	69%

Table 12: Comparison between pre- and post-diagnostic tests.

It's apparent, that the two lessons have had a positive effect on both the specific misconceptions of the target mathematical knowledge and on the general ability of the class to work with elementary algebra. Every number has improved except the one representing the occurrence of misconception number 5, which hasn't changed at all. It's worth noting here, that misconception 5 was not only the one with the widest definition, it also wasn't as much a misconception as simply a lack of knowledge.

10 Discussion/perspectives

In this section, I'll discuss the validity of some of procedures and results from my thesis, as well as reflect on the perspectives following thereof. What I will not discuss in this section, are the effects, misconceptions and obstacles, which I have uncovered in my analysis, as I feel, those have been properly discussed along the way.

10.1 The diagnostic test

The diagnostic test proved to consume a relatively large part of my time working on the thesis for what should now be quite obvious reasons. It's clear to me now, that my initial design of the diagnostic test was based on a misconception of my own regarding what constitutes diagnostics with respect to didactics of mathematics. Consequently the diagnosis of the misconceptions in the class could very likely have been improved considerably, had the test been constructed with much simpler problems each designed to diagnose very specific misconceptions. I do however think, that the outcome of the mess was acceptable, as a number of precisely defined misconceptions actually were diagnosed and consequently treated. Besides having been the single biggest pain of working with the thesis, it has also been the greatest epiphany for me personally, how powerful a tool a carefully constructed diagnostic test can be, and thus it's perhaps the single most important outcome of my work with the thesis for me personally.

10.2 The choice of teacher

I taught the class myself out of sheer necessity, and that is something, that could easily pose some problems. First of all, it could have made it difficult to properly observe the situations in the classroom. Second of all it makes the analysis very personal, as I then analyze to a large extent my own professional work. Even though (and this is generally important to note) the pointing out of negative effects of choices by a teacher in an analysis in no way is intended to reflect the teacher's abilities to teach - but rather just to bring focus to effects themselves, it does impose a level of self consciousness to the analyzer, as the object of analysis is himself. I feel, that I have been able to not take on the role of the defender of the teacher in my analysis, but clearly I would choose another person as the teacher, had I had the opportunity.

10.3 The specific choice of number tricks

I briefly touched upon, earlier, the nature of number trick D. Seeing as I had laid out a specific definition of number tricks and the premise, that I wouldn't concern myself with the trick part, number trick D was admittedly an odd choice, and after having reviewed the implementation of the lessons, I have to further admit, that I should have kept myself to my original plan. The conundrum of whether or not to include a trick of D's nature is interesting though, because it reflects on the (possibly counterproductive) impulse of the teacher to introduce material, that really isn't well suited for the specific situations., on the grounds that he/she finds it interesting. That really is outside the scope of my thesis though, so I'll only say this: Were I to do it over, I would have not included number trick D.

10.4 The effect of the lessons

Even though it is clear from the results, that some positive effect occurred as a result of the two lessons, it does not in any way reflect from the results, whether this effect was due to the specific contents of the lectures (working adidactically with number tricks) or simply a result from working with elementary algebra. Personally I suspect, that a large part of the progress measured was due the carefully designed content of the lesson. I base this on the fact, that this particular class had previously been working with elementary algebra in a more traditional way, and to my recollection, that didn't really cause any progress. I don't however have any numbers to support that suspicion. It would definitely be interesting to add a control group to the experiment, thus revealing how large a part of the progress was due to specifically working adidactically with number tricks.

11 Conclusion

In conclusion, I'll try to sum up some of the interesting points of the thesis, as well as address my three research questions.

11.1 Summation

The implementation of a diagnostic test revealed the following misconceptions regarding elementary algebra:

1. $\boldsymbol{a + b = ab}$
2. $\boldsymbol{(ab) \cdot c = ac \cdot cb}$
3. $\boldsymbol{ab \cdot b = ab}$
4. **No operation precedence**
5. Problems with $(a + b)(c + d)$
6. $\boldsymbol{\frac{a+b}{c} = \frac{a}{c} + b}$

The didactical design based on these and the use of number tricks, resulted in two lessons with a total of 61% of the time allocated to adidactical situations. The analysis of the implementation revealed, that in some cases, the adidactical potentials of these situations were exploited effect-fully, while in others, the failure to correctly use the built in feedback mechanism resulted in failure to properly exploit the adidactical potential.

The analysis revealed following points of interest:

- One possible case of a didactical obstacle
- One possible case of an epistemological obstacle

- One case of the Jourdain effect
- One case of the Topaz effect
- One possible breaking of didactical contract

The comparison of the pre- and post-diagnostic tests showed a general improvement in the students ability to handle elementary algebra, as well as a decrease in the occurrence of the specific misconceptions listed above.

11.2 Addressing the research questions

In this subsection I'll attempt to sum up some concise answers to my three research questions:

1. *What obstacles occur, when teaching elementary algebra to low level math students in Danish STX?*

   Both didactical and epistemological obstacles were observed, however more interestingly is the nature of the misconceptions related to those obstacles: They were all of a relatively simple nature. Some of the misconceptions could even be argued to actually just be misconceptions regarding conventions of notation.

2. *What are the adidactical potentials of using number tricks to facilitate knowledge about elementary algebra?*

   I argue, that I have demonstrated, that the adidactical potentials are high, as especially episode 1 shows, that following an effect-full devolution, the design clearly allowed the students to personalize the target knowledge with very little to no interaction with the teacher.

3. *What effect did the lesson set have on the students' skills/knowledge?*

   As the comparison of the two diagnostic tests show, a positive effect occurred in two parts: 1) There was a clear drop in the occurrence of the misconceptions of the target knowledge. 2) There was a generally increased ability to handle algebraic expression amongst the students after the lessons. The analysis indicates, that at least some of the students were allowed to personalize the target knowledge.

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