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Basic algebra in the transition from lower secondary school to high school

Caroline Sofie Poulsen
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Abstract
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In this thesis the problems related to algebra in the transition from lower secondary school to high school will be examined. This will be done by identifying the techniques in exercises related to algebra that are present in the two institutions and the challenges that can be associated with these techniques. Furthermore, it is discussed how those challenges can be met on the basis of the MathBridge project.

First material containing exercises from Danish 9th grade, high school, and a screening test from Silkeborg high school are analysed using the Anthropological Theory of the Didactic (ATD). Types of task and associated techniques are identified and on the basis of this an epistemological reference model (ERM) is developed.
A pilot test is then constructed from the ERM with the purpose to test the overlapping techniques in the ERM in order to identify the students' misconceptions and challenges with the techniques. The strategies from the MathBridge project are considered in relation to the found challenges.

The analysis of the exercises related to basic algebra showed that a large part of the identified techniques were present in both lower secondary school and high school. However, the increased use of IT-tools at the written examinations in mathematics in Danish 9th grade may remove the focus from the algebraic techniques.
Some of the students' answers to exercises in the constructed pilot test indicated that some severe misconceptions were present. In particular, collecting and reducing like terms were challenging for the majority of the students.
In the Mathbridge project teachers from lower secondary school and high school collaborate with the purpose of addressing the transition problems. Focus is on developing teaching sequences in algebra and modelling to be taught in lower secondary school which may prove to be beneficial for reducing the students challenges with mathematics in the transition.

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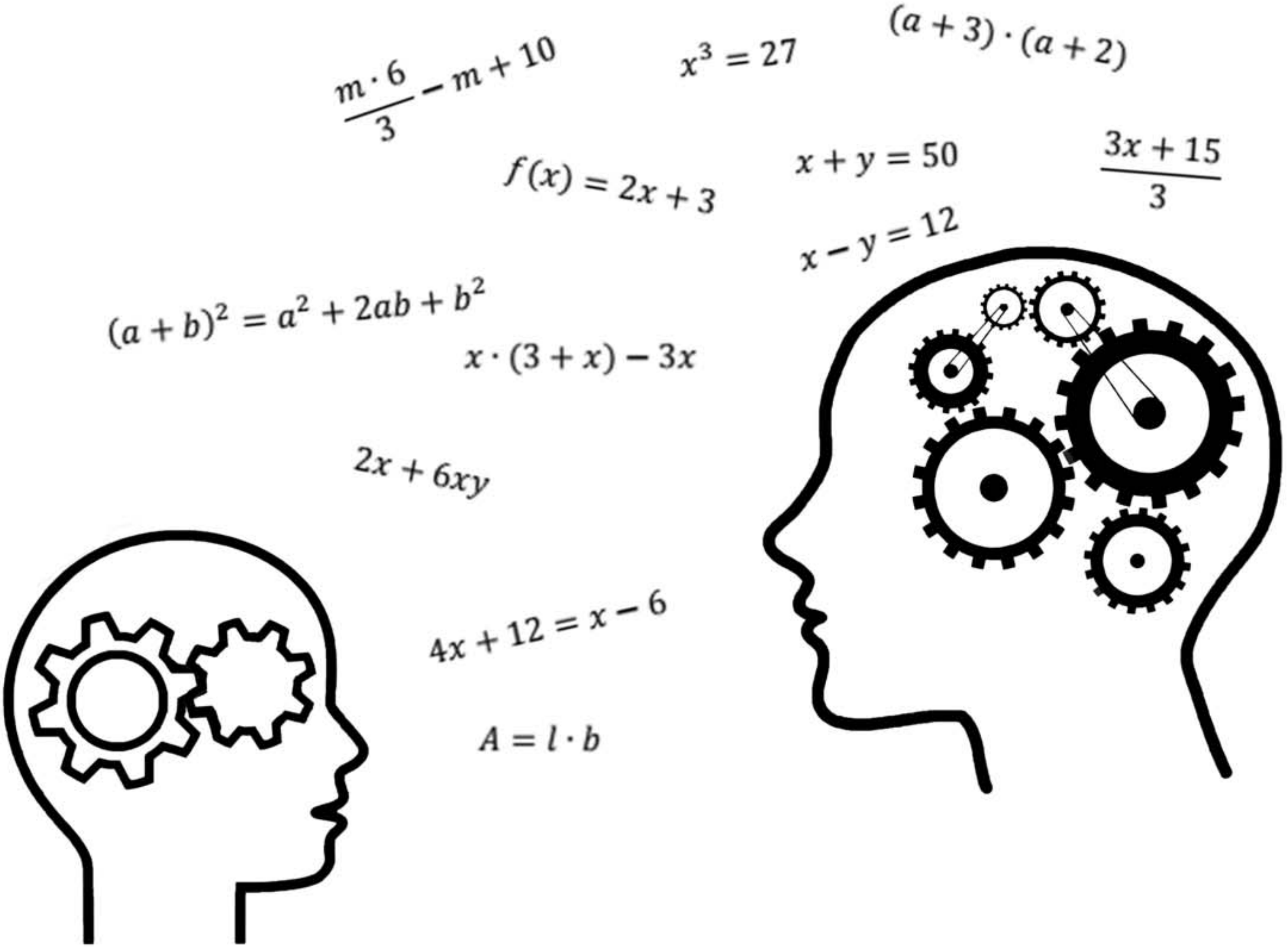
Basic algebra in the transition from lower secondary school to high school

Caroline Sofie Poulsen



Thesis for the Master degree in Mathematics. Department of Science Education, University of Copenhagen

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Basic algebra in the transition from lower secondary school to high school

NAME:
Caroline Sofie Poulsen

DEPARTMENT:
Department of Science Education, University of Copenhagen

SUPERVISORS:
Carl Winsløw and Britta Eyrich Jessen

ABSTRACT

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ACRONYMS

ATD The Anthropological Theory of the Didactic

MO Mathematical organisation

CP Calculation programmes

SRP Study and research paths

ERM Epistemological reference model

CAS Computer algebra system

stx Danish high school

FSA Leaving examination in Danish lower secondary school

1 INTRODUCTION

1.1 MOTIVATION AND RESEARCH QUESTIONS

Is it possible to completely avoid problems in the transitions between the institutions in the education system? Particularly the transition from lower secondary school to high school is interesting due to the increasing number of students in Danish high school and the more varied group of students in relation to academic qualifications and background (Ebbensgaard et al., 2014, p. 5). Several studies have investigated the transition in mathematics from lower secondary school to high school and one of the studies came to the conclusion that the students find the transition in mathematics particularly difficult partly because algebra is used more in mathematics in high school (Ebbensgaard et al., 2014, p. 95).

In the light of this the thesis will examine the students' difficulties with basic school algebra in the transition from lower secondary school to high school in mathematics. This will be done through three research questions:

1. What types of tasks and techniques related to basic algebra are central in Danish 9th grade and in Danish high school?

2. What challenges may be associated with the different kind of types of tasks and techniques in basic algebra and in the transition from lower secondary school to high school in relation to this?

3. How can experienced high school teachers try to meet those challenges? How can they support lower secondary school teachers to prepare the students better for the transition?

The first research question will be examined by building a epistemological reference model based on praxeologies. The model will be constructed from an analysis of material containing exercises from the written examinations in mathematics in 9th grade for the last few years, exercises from a screening test used at Silkeborg high school for several years, and tasks from two C-level math books for high school. In the mentioned material only tasks related to basic algebra will be analysed. The delimitation of exercises will be explained in the method section.
The second research question will be answered by looking at data and observations from the "Silkeborg" project, which has recently been extended by the Mathbridge project. The data include statistics of

answers to the screening test by first-year students from Silkeborg high school. In addition, statistics of the students' performance to the written examination in mathematics in 9th grade will be examined. Finally, a pilot test will be designed on the basis of the epistemological reference model. The test will be given to a 1st year class in high school and data of the students' answers will be collected and analysed.
The last research question will be answered by looking at the strategies from the MathBridge project through my participation in the workshops and the developed course material. Furthermore, ideas and initiatives from other projects examining the transition from lower secondary school to high school in mathematics will be included in the discussion on how the challenges can be met.

1.2 STRUCTURE OF THE THESIS

First of all, the design of the MathBridge project will be described in chapter 2. Then an overview of recent research in transition problems in mathematics from lower secondary school to high school will be given in chapter 3. This is followed by a description of the theoretical background for the thesis in chapter 4. This will include important notions from the Anthropological Theory of the Didactic as well as research studies about algebra.
Subsequently, the methodology for the thesis in relation to the delimitation of material and exercises along with an explanation of the analysis of the material and design of a pilot test are described in chapter 5. Then the analysis of the material and the ERM is presented in chapter 6 by explaining each of the discovered techniques using examples of exercises from the test. On the basis of the ERM problematic techniques in the screening test and the written examinations in mathematics in 9th grade (FSA) are identified in chapter 7 along with a description of the techniques which is in focus in the MathBridge material for the course.
Chapter 8 concerns the pilot test and it includes the a priori analysis of exercises in the test and the a posteriori analysis of the students' answers to the exercises. Finally, a discussion of the techniques, challenges, and initiatives to minimize problems in algebra in the transition from lower secondary school to high school will be given in chapter 9. The thesis ends with a conclusion regarding the research questions in chapter 10.

1.3 MATERIAL THAT ARE NOT OPEN TO THE PUBLIC

The written examinations in mathematics in 9th grade are a central part of the empirical material, but the material is not open to the public since it requires a code to acquire the examination papers through

a website. I have obtained admission through contacts in primary and lower secondary school which I will not take advantage of. Therefore, the material will only be included in the appendix in the paper version but not in the electronic version of the thesis which is problematic for the reader of the thesis. This is done in order to give the examiner and the external examiner access to the material whereas the electronic version which is made public does not include the FSA.
References to exercises in the written examinations in mathematics in 9th grade will be written in the following way:

Exercise number in FSA month year

For example the reference exercise 45 in FSA December 2014 indicates that it is exercise 45 from the FSA from December 2014. If the number of the exercise is an integer the exercise is from the first part of the written examination and if the number is $n.m$ for n and m integers then the exercise is from the second part of the written examination.

2 THE MATHBRIDGE PROJECT

The MathBridge project is about the transition from Danish lower secondary school to high school in mathematics. The reason for the project is because the transition involves some challenges and problems which will be outlined in chapter 3. The project will be described in the following section.

2.1 PROJECT DESCRIPTION

The purpose of the project is to reduce the problems that the students experience in the transition from lower secondary school to high school in mathematics (Styregruppe, 2014, p. 5). This will be done by developing and trying out specific initiatives that are supposed to build a bridge between mathematics in the two institutions (Styregruppe, 2014, p. 1).
The project involves several initiatives that will be described.
First of all two workshops that each last two days will be held for the project group in the spring and summer 2015 (Styregruppe, 2014, p. 5+14). The project group consist of mathematics teachers from lower secondary schools in Copenhagen and Silkeborg, mathematics teachers from Gefion and Silkeborg high school, and mathematics teachers from a teacher training school in Silkeborg and Copenhagen. In addition to that, Department of Science Education at University of Copenhagen contributes to the project by participating in the workshops and through out the project (Styregruppe, 2014, p. 11). The workshops are supposed to be used for the preparation of the subsequent courses for teachers in mathematics in lower secondary school. On these workshops the focuses will be known transition problems in mathematics, didactic design of teaching such as study and research courses, development of teaching materials, and the conditions for collaboration between the different institutions such as visits in the mathematics lessons in the two institutions (Styregruppe, 2014, p. 5-6).

The second initiative is the course for the mathematics teachers in lower secondary school in Silkeborg and Copenhagen. The course will be held by the members of the project group and consist of two days in week 40/41 2015, one day in January 2015, and a follow-up day in April 2016. The mathematics' content of the course will be 'Numbers and algebra' and 'Modelling' (Silkeborg Gymnasium, 2015). The elements in the first two course days will be (Styregruppe, 2014, p. 6-7), (Silkeborg Gymnasium, 2015):

* Introduction to known problems in mathematics in the transition from lower secondary school to high school in mathematics.

* Introduction to mathematics in high school through concrete examples of exercises and video clips together with study and research courses.

* Introduction to concrete teaching materials in 'Numbers and algebra' and 'Modelling'.

* The further collaboration between the different institutions though visits on the schools.

The third initiative in the project is implementation and continued collaboration. In this stage teaching sequences on the basis of the course are prepared and carried through by use of a website that acts as a shared resource platform. In addition to that, lower secondary teachers whom have been on the course share their experiences with members of the project group through visits at the schools (Styregruppe, 2014, p. 8).

A fourth initiative is follow-up research and a effect study. This involves observations of the developed sequences with focus on didactic challenges and potentials in the collaboration between the two institutions. This is carried out by means of methods from ATD. The effect of the study is measured the subsequent year. This is done by testing if the high school students, whose mathematics teacher from lower secondary school has participated in the course, experience an easier transition to high school in mathematics. It is examined via a quantitative test which this thesis helps develop. The results are compared by a similar test which have been tested on some first-year high school students the year before i.e. in the autumn 2015 (Styregruppe, 2014, p. 8).

The last initiative is continued development of the course and materials based on experiences from the project. After that the course is diffused to other municipalities through a public resource platform. The material is also supposed to be available for future mathematics teachers in lower secondary school through their teacher education (Styregruppe, 2014, p. 9).

To sum up, the project consist of workshops for the project group. In the workshops the project group develops a course for mathematics teachers in lower secondary school. After the teachers from lower secondary school has participated in the course they are supposed to try out some teaching sequences that are based on the course. During this period members of the project group carry out follow-up visits such that the teachers can share experiences. The effect of the course is measured through a test of the students in high school that have had a mathematics teacher in lower secondary school that have participated in the course. Hopefully, the test shows that the students experience fewer problems with the transition from lower secondary school to high school in mathematics. If this is the case the knowledge are shared with more mathematics teachers in lower secondary school.

3 TRANSITION PROBLEMS IN MATHEMATICS FROM LOWER SECONDARY SCHOOL TO HIGH SCHOOL

There are several recent research studies regarding transition problems in mathematics from Danish lower secondary school to Danish high school. Among these studies are:

* Transition problems between lower secondary school and high school in the subjects Danish, mathematics, and English (2015): Danish Ministry of Education project led by Silkeborg high school and Department of Science Education at University of Copenhagen.
* Project about high school strangers: Danish Ministry of Education project led by Department of Science Education at University of Copenhagen. Particularly: Reading mathematics texts in high school.
* The high school' boys - the mathematics subject' boys (2011): Danish Ministry of Education project.
* Transition problems as challenges in the education system (2009): Research project at University of Aarhus.

The first study mentioned above focused on examining differences between the three subjects Danish, mathematics, and English in lower secondary school and high school. Furthermore, the focus was on determining the teachers' perceptions of the subjects differences and problems in lower secondary school and high school. In addition, the students' experiences of what is difficult and different in the subjects in the transition from lower secondary school to high school was investigated (Ebbensgaard et al., 2014, p. 6).
Based on questionnaires and interviews of students and teachers as well as analyses of the regulatory documents for mathematics in lower secondary school and high school the differences between mathematics in the different institutions were outlined (Ebbensgaard et al., 2014, p. 93). The main difference is that mathematics is more abstract in high school due to the fact that algebra is a major part of mathematics in high school and since the understanding of mathematics is in focus through proofs and reasoning (Ebbensgaard et al., 2014, p. 95). From the regulatory documents it is seen that the topics are closely related in the institutions and that it is the organization of the teaching by the teacher which is different (Ebbensgaard et al., 2014, p. 96).

In this study the students' answers to what is difficult in mathematics in high school is illustrated in the diagram below. The diagram shows that a majority of the students think that mathematics is more difficult in high school compared to lower secondary school. Many students point out that specific topics are difficult, but many students emphasizes that almost everything is difficult in mathematics in high school. Also the teacher and teaching are mentioned by many students as being problematic (Ebbensgaard et al., 2014, p. 65-66).

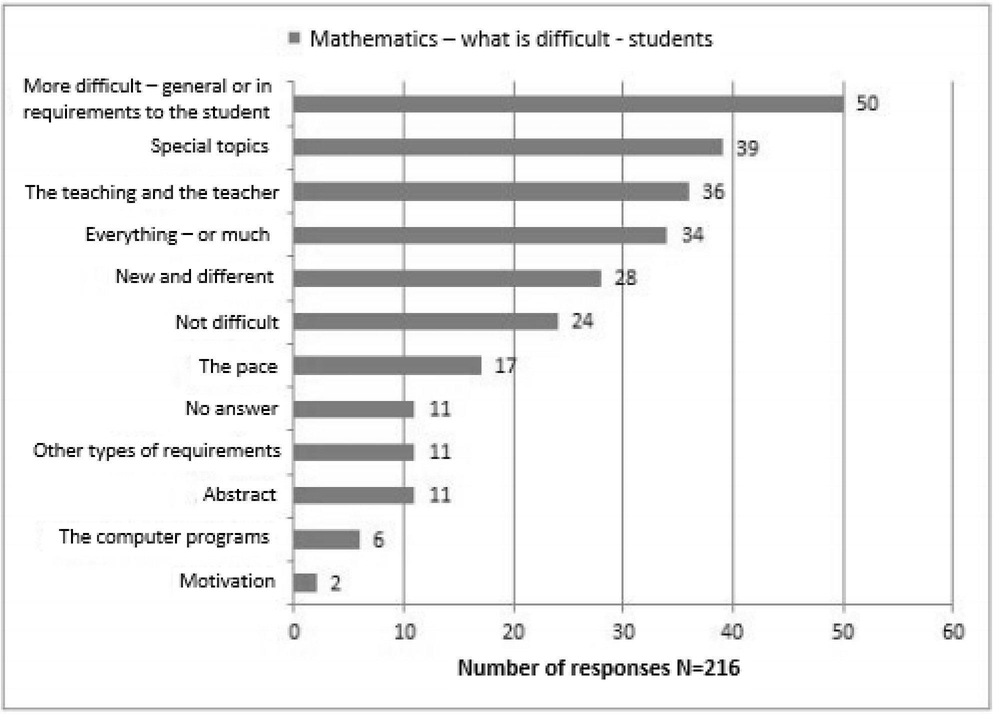


Figure 1: What students find difficult in mathematics in high school (Ebbensgaard et al., 2014, p. 65)

Many students characterize the transition problems in mathematics from lower secondary school to high school by i) everything is difficult, ii) that they always have had difficulty with mathematics, and iii) that the teacher is not very good at explaining the difficult parts. Thus a large part of the students begin in high school with the idea that they are bad at mathematics (Ebbensgaard et al., 2014, p. 66).

The students' experiences of what is different between mathematics in lower secondary school and high school are mainly that it is more difficult, which more than half of the surveyed students in the study answer. In addition to this, the students think the teaching that includes new teaching methods and pace, and the academic content that consists of proofs and deductions, differs from mathematics in lower secondary school. This indicates that the level of abstraction is increased since it is no longer sufficient only to be able to calculate. The students also have to explain (Ebbensgaard et al., 2014, p. 67-68).

The teachers' answers to what is difficult for students in mathematics in high school are mainly the notions and methods in mathematics that enables the students to use different mathematics skills in problem solving as well as basic mathematics. This includes basic calculation skills such as the four basic arithmetical operations, fractions, simple equation solving or the use of symbols (Ebbensgaard et al., 2014, p. 68-69). The high school teachers experience a big difference in the students' academic level and that mathematics is difficult for most students, so some teachers think it is limited how much the students can use their knowledge from lower secondary school (Ebbensgaard et al., 2014, p. 98). Furthermore, the teachers mention more advanced mathematics consisting of algebra, proofs, elementary functions, and theories as problematic for the students (Ebbensgaard et al., 2014, p. 68-69).
The teachers mention the level of abstraction as the biggest difference between mathematics in lower secondary school and high school. Many teachers also consider mathematics in high school as a subject that requires more precision and reasoning when it comes to problem solving and in construction of proofs compared to the subject in lower secondary school (Ebbensgaard et al., 2014, p. 69).
When comparing the students' understanding of the differences to the teachers' it is seen that the students experience a change in the teaching's methods and content whereas it is the teachers experience that the students are insufficiently prepared for the transition to high school (Ebbensgaard et al., 2014, p. 70).

In the project about high school strangers one of the examined areas was reading of mathematics texts in high school. The students' reading strategies in mathematics and lack of such were in focus, but also the development of methods to improve reading strategies in mathematics was developed (Hjorth et al., 2012, p. 46). The study was carried out in 2010 in htx (technical high school) and is a part of a project about high school strangers which is a project led by the Danish Ministry of Education. High school strangers are examined because studies have shown that they have problems in high school. The study included three htx classes where two of the classes had 60% students that were high school strangers and one of the classes had 80% students that were high school strangers (Hjorth et al., 2012, p. 47).
In this project high school strangers are defined as

> *students whose parents either not at all have a high school leaving examination or have a high school leaving examination but have not completed a further education of medium or long length, that have required a high school leaving examination. (Hjorth et al., 2012, p. 47)*

The challenges associated with reading mathematics texts are related to the mathematical language of symbols and the disconnected texts such as figures and graphs. Furthermore, the students do not have a reading strategy from lower secondary school because they do not read longer mathematics text in lower secondary school but often only read short texts or texts of exercises (Hjorth et al., 2012, p. 48-49). This became clear in the qualitative test in reading strategies in the study, where three students with low grades in mathematics and three students with high grades in mathematics were tested. The students with low grades in mathematics read the text from beginning to end without connecting the text to an important diagram and skipped several of the mathematical symbols (Hjorth et al., 2012, p. 53-54).
The faulty reading strategy in mathematics was addressed by several initiatives. The intention of the initiatives was to help the students to read mathematics texts by helping the students to understand mathematical symbols and words and to "jump" between different pieces of a text such as figures and the like were included when deemed appropriate (Hjorth et al., 2012, p. 54). Some of the initiatives that worked as intended were:

- Word-knowledge-cards: used for clarifying and getting a deeper understanding of an academic notion. It can be used before reading to activate the students previous knowledge or after reading a text to revise it (Hjorth et al., 2012, p. 56-57).
- Reading exercises in the teaching: used during reading to get the students to analyse the content and purpose of the text by reading the whole text while including relevant figures, graphs, tables an so on (Hjorth et al., 2012, p. 58).
- Computer-based training of notions: used to train the students fingertip knowledge by getting the students to formulate in writing the definition of a notion or choosing the correct definition of a notion among several others (Hjorth et al., 2012, p. 60).
- Rearrangement of homework: reading as a part of the teaching and not just as homework (Hjorth et al., 2012, p. 62).

In the years $2008-2010$ the Danish high school boys performed significantly worse than the girls in written exam in mathematics at B-level at stx as can be seen in figure 2 on the next page, even though the boys performed better than the girls at the exam in mathematics at the end of lower secondary school (Bacher et al., 2012, p. 23). Overall the girls performed better at all mathematics exams in stx year 2009, but especially at the written exam in 2009 in mathematics at B-level the girls performed much better than the boys. At the written and oral exam in the years $2000-2004$ in mathematics in lower secondary school the boys outdid the girls' performances (Bacher et al., 2012, p.

24-25). This resulted in a study that investigated the reasons for the two gender's differences in performance in the exams in mathematics at B-level, which was initiated by the Danish Ministry of Education (Bacher et al., 2012, p.24).

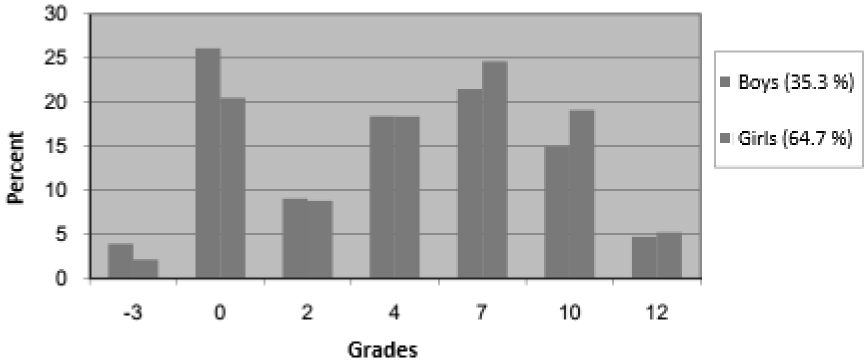


Figure 2: Frequency diagram showing the distribution of grades by gender at the written exams in stx mathematics B-level for students in the years 2008 − 2010

The study examined 1) if the differences could be explained by the types of exercises in the exam, 2) if a relationship between the boys' performances in mathematics in lower secondary school and in mathematics at B-level existed, and 3) if the boys' experiences of the teaching could explain the differences between boys' and girls' performances (Bacher et al., 2012, p. 25). The results of these three focal points will be explained.

In almost all of the exercises in the written exams in mathematics at B-level from 2008 to 2010 the girls had the best average points compared to the boys. Only in 12 out of 81 exercises the boys performed better than the girls. Out of the 12 exercises seven of them were exercises without aids, and six of them were exercises concerning geometry. It is not a general tendency that the boys performed better in geometry exercises since there were several cases where the girls got more points than the boys in geometry exercises where aids were allowed. Overall the boys and girls got points in the same types of exercises which indicates that none of the exercises favour girls over boys or vice versa (Bacher et al., 2012, p. 29).
There is no clear relationship between the boys' performances in mathematics in lower secondary school and in mathematics at B-level in high school. The boys perform marginally better at the mathematics exams in lower secondary school than the girls both if the entire population of students from lower secondary school is considered and if only the group of students that later had mathematics at B-level in high school is considered. However, if the boys got the grade 8 (old grading scale) or less at the exams in mathematics in lower secondary school it was very probable that they would fail the written

exam in mathematics B-level in high school (Bacher et al., 2012, p. 30).

The majority of the boys explain that they have mathematics at B-level because it is important for their further education (Bacher et al., 2012, p. 31). This connects with the changed admission requirements at the university. After the reform in Danish high school from 2005 a bigger proportion of the students have mathematics at B-level at stx. Furthermore, mathematics at B-level has become an admission requirements for the long social science studies at the university from 2008 (Bacher et al., 2012, p. 24).
The boys' experiences of the teaching concern their preparation for the lessons, their description of the daily teaching, their experience with mathematics in lower secondary school, and their experience with mathematics in high school in relation to gender. The majority of the interviewed boys tell that they use half an hour maximum to prepare for each mathematics lesson at B-level in high school, and that they find it especially difficult to prepare for the lessons because the books are difficult to read. They describe the daily teaching as theory explained by the teacher followed by problem solving done by the students. They emphasize the importance of the teacher for their benefit of the lessons. The majority of the boys describe their attitude towards mathematics in lower secondary school as positive. Many of the boys did not need to do homework in mathematics in lower secondary school. The majority of the interviewed boys think that the girls perform better in mathematics in high school than boys and explain it by the girls' seriousness in approach (Bacher et al., 2012, p.33-34).

The teachers' view of the teaching in mathematics B-level in high school and differences between gender is consistent with the boys'. In addition to that, the teachers mentions that the boys focus on getting the correct answer whereas the girls are better at writing down the methods they used to solve a given exercise (Bacher et al., 2012, p. 34-35).
One of the conclusions of the study in relation to likely explanations of the sudden shift in boys' and girls' performances in mathematics is that girls are better at planning and using the necessary time to do their homework compared to boys. It is worth considering if the structure of the teaching could be more varied in order to motivate the boys (Bacher et al., 2012, p. 41-42). It is recommended that the two school levels "build a bridge" between them to reduce the problems that especially the boys experience in the transition to mathematics at B-level in high school (Bacher et al., 2012, p. 44).

The study about transition problems in the education system examines many aspects of transition problems between different institutions. One of the aspects of case 2, which examines the use of academic qualifications in the transition from Danish lower secondary school to high school in mathematics and science (Søndergaard et al., 2009, p. 45), is the teachers' and students' assessment of the students' academic qualifications in mathematics. The students assess their academic qualifications higher than the teachers assess the students academic qualifications in all except for one of the categories, which is the use of IT. The teachers think that the students in the transition are ill prepared for using simple statistic models, using relationships between variables and expression of functions, making mathematical reasoning, and demonstrating knowledge about mathematics' development (Søndergaard et al., 2009, p. 76-77).

4 THEORETICAL BACKGROUND

The thesis will use the Anthropological Theory of the Didactic (ATD) as theoretical foundation. This is a didactic theory for research in mathematics education which is useful in the work of building a reference model for a given specific mathematical knowledge.

ATD is a relatively new research program in mathematics education. It was initiated by Yves Chevallard. He was a student of Brousseau, who developed the Theory of Didactic Situations (TDS) (Bosch and Gascón, 2006). In ATD the focus is on analyzing teaching and learning in mathematics and not on prescribing how or what to teach. The focus in TDS is mainly on designing and experimenting with teaching sequences. It is one of the reasons that ATD has been chosen as theoretical foundation since the thesis will concentrate on describing and analysing basic school algebra. Furthermore, school algebra has always been of great interest to ATD (Bosch, 2015).

In ATD it is assumed that persons' know-how and knowledge are acquired from the institutions where this know-how and knowledge exists. As Bosch (2015, p. 53) puts it,

> *...human practices and human knowledge are entities arising in institutional settings.*

Thus, as a researcher it is important to be aware of the institutions' dominant point of views and to try to detach ourselves from the institutional point of views that we are a part of (Bosch, 2015).

In the following sections praxeologies and the didactic transposition processes will be explained. These are important concepts in ATD. But before these concepts are introduced an explanation of the notions exercise, question, and task will be given.

4.1 EXERCISE, QUESTION, AND TASK

It is necessary to distinguish between the notions exercise, question, and task. I will illustrate the differences by an example taken from the screening test. Table 1 shows an example of an exercise. An exercise is given in math textbooks and may need to be translated to a task that can be described by a mathematical praxeology. Exercises may also contain several questions. In the example a) and b) are not formulated like questions but that could easily have been the case.

A question is explicit and depends on the theory or technology. This is not the case for a task. Furthermore, not all questions can be answered by a technique unlike tasks that always can be solved by a

technique or several techniques. A question can be identified with types of tasks which are possible to solve with a technique (Winsløw et al., 2013, p. 272). The exercise below could be translated to the following types of tasks:
T_1: Explains a's meaning in a linear relationship $y = ax + b$.
T_2: Explain b's meaning in a linear relationship $y = ax + b$.

The price for a ride in taxi is indicated by y (in DKK) and the length of the ride is indicated by x (in kilometres). For a particular taxi firm the relationship between price and the length of the ride can be described by the equation $y = 17x + 49$. a) Explain what the number 49 in the equation says about the ride in taxi. b) Explain what the number 17 in the equation says bout the ride in taxi.

Table 1: Exercise 25 in the screening test

4.2 PRAXEOLOGIES

In ATD mathematical activities are seen as any other human activity and can thus be described as such. In ATD any human activity can be described by a practical and a theoretical component which combined is called a praxeology. There is no praxis without theory or the other way around. The notion of praxeologies provides a useful tool for analysing activities.
Praxeologies consist of a practical block and a knowledge (logos) block. The practical block contains the know-how, whereas the knowledge block contains the discourse for the practice. The two blocks go hand in hand and constitute any mathematical praxeology, and even though one of the blocks can be more evident than the other they are still closely connected (Barbé et al., 2005, p. 236-237). Both the theoretical and practical part are equally important in human activities (Bosch and Gascón, 2006, p. 59) and the two blocks' development is based on each other. The practical block consists of various types of tasks T_i and the corresponding techniques τ_i used to solve a given task. Tasks are usually described by verbs such as "reduce the given algebraic expression". Techniques should be understood broadly as "ways of doing". The knowledge block includes the technology θ and theory Θ. The technology is the discourse in relation to the technique and thus it describes, justifies, and explains the used techniques and also produces new techniques (Kieran, 2008, p. 3). The theory is a

more general justification of the practice and it is often implicit for the students. It consist among other things of models, assumptions, and concepts (Bosch, 2015, p. 54).
An example of a mathematical praxeology in basic school algebra can be as follows.

T	To solve a first degree equation
τ	Adding, subtracting, multiplying, and/or dividing an element on both sides of the equality symbol
θ	Algebra of equations
Θ	Equations or algebraic manipulations

Table 2: A mathematical praxeology for a given task

The task of solving a first degree equation can be done by using the algebraic technique described in table 2. This technique has an associated technology about properties of the equality symbol and rules for arithmetic calculations. The theory could be called equations or algebraic manipulations. It consist of the axioms for a field. The theory will not be described in details since it is the techniques that are important in the a priori analysis in chapter 6.
But it is also possible to use other techniques to solve this task. For instance one could use a technique that relies on drawing straight lines and reading coordinates as described in table 3. The corresponding technology and theory depend on the used technique and are therefore not the same as for the algebraic technique. In this case the technology is properties of straight lines and the coordinate system. Therefore the theory concerns graphs, and thus depending on the used technique we can have several mathematical praxeologies for a given task.

T	To solve a first degree equation
τ	Draw the straight line in a coordinate system and read off the x value for the y value that is given on either the right or left side of $=$
θ	Properties of straight lines and the coordinate system
Θ	Graphs

Table 3: Another mathematical praxeology for the same task

4.2.1 *Mathematical organisations*

A mathematical praxeology is also called a mathematical organisation (MO). MOs can be organised into different types dependent on

their complexity. A *point* MO consists of one type of task and the corresponding technique (Bosch and Gascón, 2006, p. 59). An example of a point MO is 'to solve a first degree equation using algebraic properties' as described in table 2. A *local* MO contains several types of tasks and techniques that share the same technology. Hence a local MO consists of a number of point MOs which can be explained by the same technology. It is possible for a given point MO to be a part of different local MOs since the point MOs' techniques can be explained by some other technology. A *regional* MO is a set of local MOs that have the same theory. A local MO can be a part of different regional MOs (Barbé et al., 2005, p. 237-238).
Below is a model which clarifies the complexity of the different MOs.

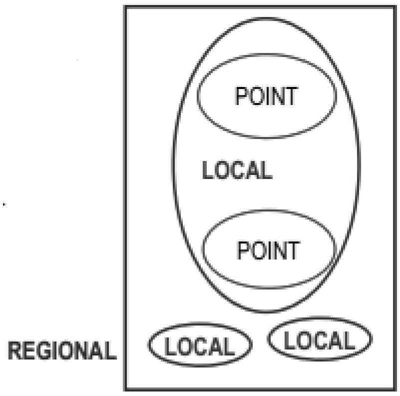


Figure 3: Mathematical organisations of increasing complexity

4.2.2 *Values of a technique*

For a given task different techniques may be available. The techniques can have a different value which can be described by the following two characteristics.

1. **Pragmatic value:** the efficiency for solving tasks in an institution. Focus is on the productive potential.

2. **Epistemic value:** the value is related to knowledge that is central in the praxeologies of an institution. How well does the technique link the knowledge? This concerns the techniques ability to give understanding to mathematical objects and theories.

(Artigue, 2002, p. 248) It is thus possible to evaluate a technique by examining its pragmatic and epistemic value. For example, solving an quadratic equation using SOLVE on a CAS-tool has a high pragmatic value, since it is very efficient. But it provides no understanding of why there are two solutions or how they have been reached. Hence

the epistemic value is low.
The epistemic value is very relevant in relation to learning and understanding some mathematical body of knowledge, but it is important to bear in mind that the institutions do not always value this. The reason for this is that the institutions wish that their students do well at the examinations and with regard to this techniques of high pragmatic value may be very useful.

4.3 didactic transposition

The didactic transposition concerns the transformation of some mathematical body of knowledge. The conditions and constraints for teaching are examined and it is thus a tool for analysing and describing mathematical knowledge. The mathematical knowledge taught in school is questioned. This is examined by looking at different institutions in different periods of time. The theory of the didactic transposition process assumes that there are tensions between the different institutions. From the examination of institutions, mathematical praxeologies are selected. These are transformed to "knowledge to be taught" and in school to "taught knowledge". During this transposition process many choices are taken based on various conditions and constraints (Bosch, 2015).
The focus of the didactic transposition analysis is that what is being taught in school is determined by factors outside the school. Hence it is stressed that what happens in class is not independent of what happens outside the class. The knowledge in school is built outside the school where bodies of knowledge is selected and then transposed to 'live' in the school. When the knowledge is transposed to the school it is rebuild by politicians, the association of mathematics teachers etc., called the 'noosphere', to make it teachable. The transpositon process is influenced by the historical and institutional conditions. This may lead to a lack of rationale for the teaching since the reasons and interesting questions for teaching a specific topic are decided upon far away from the teachers (Bosch and Gascón, 2006).

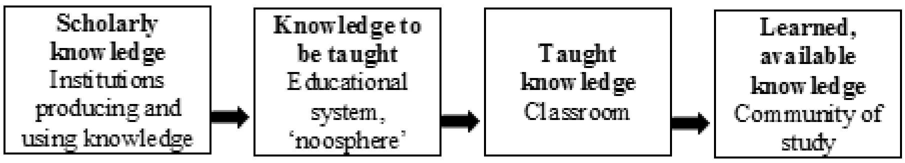


Figure 4: The didactic transposition process

The didactic transposition process is shown in the model above. The purpose of the model is to describe transposition processes and to examine what could be taught and why. The model makes it explicit that many factors may cause the teaching to fail. The scholarly mathematical knowledge is the original knowledge which is often produced by mathematicians for a long time ago. The knowledge to be taught is visible in the curricula and textbooks for mathematics teaching and is often influenced by politicians. The taught knowledge is the knowledge actually taught by a teacher in a given classroom in a school. The learned/available knowledge is the knowledge the students acquire (Bosch and Gascón, 2006). It is important to notice that purpose and context are different for the different types of knowledge.

The didactic transposition process points out that knowledge is relative to the institutions and that didactic problems are institutional. The didactic transposition process consists of three steps that describes the relations between the different types of knowledge. It is important to look at all the steps when one is analysing a didactic problem, since they all affect each other (Bosch and Gascón, 2006).
The first step is the process from scholarly knowledge to knowledge to be taught. This step concerns the study of the formation of knowledge to be taught which takes place via the noosphere that produce curricula, textbooks etc. for the educational system. In the study of this step the conditions and constraints of the knowledge to be taught become evident by asking critical questions such as why a giving teaching domain is relevant or why the curricula is organized in a certain way (Bosch and Gascón, 2006).
The second step is the transition from knowledge to be taught to taught knowledge. In this process the teachers use the 'products' that the noosphere has produced to transform the knowledge into the teaching in the classroom. This consist among other things of interpreting the curricula and selecting tasks from the textbooks to be solved by the students. Here lies the teacher's problem to decide how to teach a given topic (Barbé et al., 2005).
In the third and last step the process from the teaching to the students' achieved knowledge is studied. This can be studied by looking at empirical evidence such as test results.

The didactic transposition analysis makes it possible for researches to detach themselves from epistemological models determined by the institutions in education that the researchers are a part of. The researchers need to develop their own epistemological reference model (ERM), that is, a basic theoretical model based on empirical data coming from the mathematical community, the educational system, and the classroom (Bosch and Gascón, 2006). An ERM can be constructed in terms of local and regional praxeologies. It is important to emphasize that an ERM is preliminary in the sense that it is necessary to revise the ERM constantly (Ruiz-Munzón et al., 2013).
In the following section an example of an ERM for basic algebra developed by Bosch (2015) is presented.

4.4 AN EXAMPLE OF AN EPISTEMOLOGICAL REFERENCE MODEL FOR BASIC ALGEBRA

Bosch (2015) suggests an epistemological reference model for elementary algebra. Her ERM starts with elementary praxeologies for arithmetic with techniques based on calculation programmes (CP), which are arithmetic operations carried out in a specific order to some set of numbers or quantities that gives a final number or quantity as a result. It is a kind of "step by step" description. The reason for the focus on CP is due to the fact that the corpus of problems of classic elementary arithmetic can be solved by producing and carrying out a CP. This initial system will then lead to technical limitations and theoretical questions about the obtained result, and the link between the different kind of problems and techniques will arise. This will then lead to a necessary enlargement of the initial system where modelling is used to create a basis for the algebraization process (Bosch, 2015).
Bosch (2015) classifies the algebraization process in three stages. I will use an example from Ruiz-Munzón et al. (2013, p. 6) to make the characterization of the algebraization process explicit. The example is based on the following task:

> "Think of a number, multiply it by 4, add 10, divide the result by 2 and subtract the initial number."

The above task can be solved by a CP described by $P(n) = \frac{n \cdot 4+10}{2} - n$. The first stage of the algebraization process can be started if the problem $P(n) = 7$ has to be solved meaning what did we start with if we ended with the number seven. In order to solve this task the algebraic expression of $P(n)$ should be simplified. The simplified expression is $P(n) = n + 5$ and it is easy to see that only $n = 2$ solves the problem $P(n) = 7$. It is a bit difficult to solve the problem without the use of letters since the unknown number appears two times in the CP but it is possible.
To start the second stage of the algebraization process a problem of the form $P(n) = 3n - 7$ should be solved. It is possible to solve this problem in the first stage but it is far more difficult than using techniques from the second stage. The problem can be solved by solving the equation $n + 5 = 3n - 7$. If the CP was $P(n, a) = \frac{n \cdot 4+a}{2} - n$ where a is a parameter and the problem was $P(n, a) = 2n - a$ it could be solved by the same techniques and theory as before by solving the equation $n + \frac{a}{2} = 2n - a$ to get the relationship $n = \frac{3a}{2}$ between n and a.
The third stage of the algebraization process could be reached if the CP contains more than two arguments e.g. $P(n, a, b) = \frac{n \cdot 4+a}{2} - b$. To solve a problem related to this CP new techniques are required.

1. stage This first stage is about making the structure of the CP explicit. This can be done with or without the use of letters. When making the structure of the CP explicit one should be aware of the

hierarchy of arithmetic operations and rules for brackets. In this process new techniques are developed such as "simplifying" and "transposing" which are used for working with the symbolic model of a CP i.e. algebraic expressions. The theoretical block of these praxeologies are also developed to justify the new techniques.

2. stage The second stage arises when the problem concerns relationships between variables of a CP. The techniques from the first stage are no longer sufficient and hence more complex algebraic techniques to manipulate CP are necessary. Equations are new mathematical objects at this stage and techniques to solve equations are a part of it. The involved equations contain both unknowns and parameters, that is using letters to denote both known and unknown quantities.

3. stage At the last stage of the algebraization process the number of arguments in the CP is unlimited and the distinction between unknowns and parameters no longer exists. The work with creating, transforming and interpreting formulas is the important part of this stage.

The majority of current school algebra is about solving first and second degree equations in one variable and solving word problems with the use of such equations. However, the use of parameters in equations in school algebra is absent, since letters are only used as unknowns or variables in school. Besides solving first and second degree equations students are also solving simple inequalities and manipulating algebraic expressions. The third stage of the algebraization process is almost gone in high schools nowadays which is unfortunate because of the fact that the algebraic techniques used for dealing with formulae can come in handy when starting to work with functions and differential calculus (Bosch, 2015).

Bosch (2015) finds it necessary to change the existing epistemological but also pedagogical models in different institutions in order to integrate an ERM, where algebra is a modelling tool. But in order to make these changes one must understand the constraints that limit the possible praxeologies. Bosch (2015) mentions the cultural negativity towards written symbolism, the cultural preference for classical geometry rather than algebra, and the formulae disappearing from school mathematics as examples of constraints that affect school algebra. These constraints are beyond the reach of didactic research.
ATD handles this situation by analysing teaching processes with the purpose of proposing modifications and then study the result of the modifications. New ways of teaching have been proposed by using study and research paths (SRP) that set the stage for the students to work through the algebraization process. Examples of such SRP that

contain the algebraization process are SRP involving "Think of a number" games and SRP with questions related to economics and finance (Bosch, 2015).

4.5 ALGEBRA IN RESEARCH

Bosch (2015) treats the case of school algebra. It is mentioned that the structure of school mathematical curricula has been changed after the New Mathematics reform which has influenced the contents in school algebra. Before the reform the school mathematical curricula were divided into arithmetic, algebra, and geometry. After the reform the PISA commission has proposed to divide the school mathematical curricula into quantity, space and shape, change and relationship, and uncertainty. It is interesting to see that in this structure algebra is not one of the topics as it just falls within change and relationship (Bosch, 2015).

When looking in 'Common Objectives' (2009), which is a curricula for mathematics teaching in 9th grade in Denmark, the curricula is divided into numbers and algebra, geometry, and statistics and probability. By this classification algebra is not a topic in itself but connected to numbers. In addition to these, some of the teaching should be devoted to applied mathematics where it is possible that the three mathematical topics mentioned above may come into play (Undervisnings Ministeriet, 2009, p. 26-28).

With the New Mathematics reform the approach to algebra was changed. Previously the approach to algebra had been functional meaning that algebra was used to model systems and mathematical realities. Algebra was used to solve arithmetic problems and it was arithmetic that gave algebra its justification. After the reform algebra was given a formal approach in the sense that the algebraic language and manipulations with algebraic expressions are the core of algebra. Thus elementary algebra is seen as solving equations for then to apply this to problems that do not have any justification (Bosch, 2015). This change in the approach to algebra causes problems which is described in the following quotation.

> *This formal learning is unable to recreate the big variety of manipulations that are needed to use algebra in a functional way, which will be required when students arrive at higher secondary education and find "completely algebraized" mathematics (Bosch, 2015, p. 6).*

In the Western civilisation the oral expression has always been considered to precede the written. But in algebra the case is just the opposite since it is the written that is important and it precedes the oral. Written algebraic formulations can often be difficult to explain orally (Bosch, 2015). Thus there is an issue compared to the "normal" way of expressing thoughts. This affects the ecology of elementary algebra since there is a incomprehension towards algebra's written nature.

The Western civilisation also has a negative view on symbolism and formulae. This is because people associate symbolism and formulae with something without meaning and reasoning (Bosch, 2015). It has been shown in the 1980s that the majority of secondary school students think that algebra is something ugly, cold, superficial, and masculine. The society has this view on algebra which also leads to problems when carrying out proofs (Bosch, 2015).

In Bosch (2015)'s proposal for an ERM for school algebra described in section 4.4 it was evident that algebra was not a content/domain in itself like arithmetic and geometry but algebra was instead seen as a tool to model already existing school mathematical praxeologies. With the ERM that Bosch proposes for basic algebra it is possible to give an outline of the rationale of school algebra. First of all algebra is a tool when working with theoretical questions in domains such as arithmetic and geometry where the solution cannot be found inside that domain. Secondly algebra is "universal arithmetics" meaning that it is used to study relationships between numbers and give general solutions to a whole set of problems. Lastly algebra is used to organise mathematical tasks into different types and to generalize solutions (Bosch, 2015).

4.5.1 *CAS in algebra*

Kieran (2008) has examined the use of computer algebra system (CAS), a widely used technology, in teaching elementary algebra. Previous research has found that CAS can help students learn algebra since it has been shown that CAS has helped students understand equivalence, parameters and variables, and symbolic algebraic objects.
The teacher has an important role when students learn algebraic technique and theory in a CAS environment. Especially the teacher's handling of the class discussions is of great importance, since it is here the students' thoughts about conjectures, observations, and explanations appear (Kieran, 2008).

According to Kieran (2008) a conceptual understanding of algebraic technique is to be able to see a certain form in algebraic expressions and equations, to see relationships between algebraic expression that are expressed in different ways, and to see through algebraic transformations to the changes and explain the changes. A CAS-tool can provide exact answers that the students can use to examine the structure and form of algebraic expressions. This has been proven to be successful for weak algebra students who would otherwise have had difficulty validating their own answers (Kieran, 2008).

4.5.2 *5 perspectives on research in elementary algebra*

In a qualifying paper by Martinez (2007) an overview of different perspectives in research on teaching and learning of elementary algebra from 1977 to 2006 is given. The research studies in this topic are divided into five categories. These are:

1. Arithmetic and algebra
2. Algebra, generalization and patterns
3. Algebra and proof
4. Function perspective
5. The modelling perspective

Some of the categories will be outlined on the next pages.

arithmetic and algebra In studies looking at the relation between arithmetic and algebra two issues have been investigated. First of all, attempts have been made to characterize the two domains and their development over time as well as their similarities and differences. Secondly, designs of instructional activities that involves the transition from arithmetic to algebra and analysis of these activities have been examined (Martinez, 2007).

One of the findings of these studies is that students keep an arithmetic interpretation of mathematical objects. This is problematic for the students (Martinez, 2007). A study by Kieran (1981) looks at the students' understanding of the equality symbol. The students do not perceive the equality symbol as an equivalence relation which means that they get into difficulties when solving some algebraic tasks (Martinez, 2007).
In elementary school students perceive the equality symbol as an operator i.e. as a "do something signal". As a consequence of this, students have problems reading $3 = 3$ and $\square = 3 + 4$. They believe that the answer has to be presented after the equal sign instead of seeing the equal sign as symmetrical. High school and college students are confused by the use of the equality symbol. They are in a transition period from understanding the equal sign as providing the answer after $=$ to a point of perceiving it as an equivalence relation. A consequence of this is that they have a tendency to develop false identities such as $17 + 5 = 22 \cdot 3 = 66$ when solving a word problem. The procedures they use when solving equations also indicate a tendency to perceive the equality sign as an operator symbol. In a teaching experiment with students aged $12 - 14$ it was possible to expand the students' perception of the equality symbol to some extend by working with arithmetic equalities such as $2 \cdot 6 = 10 + 2$ first and then

constructing algebraic equations (Kieran, 1981).

Regarding equations a study by Filloy and Rojano (1989) categorize first degree equations with one unknown into two types. These are arithmetical and non-arithmetical. An arithmetical equation is on the form $Ax + B = C$ whereas a non-arithmetical equation is on the form $Ax + B = Cx + D$, $C \neq 0$. The study emphasize that arithmetical equations can be solved by pure arithmetic simply by "undoing" the operations but that non-arithmetical equations require algebra to be solved since it is necessary to operate on the unknown (Martinez, 2007).
Filloy and Rojano (1989) point out that in the 13th-15th century equations of the form $x^2 + c = 2bx$ and $x^2 = 2bx + c$ were solved by different strategies because operations on the unknowns were not carried out. They claim that the students' perceptions of operations on numbers have to change to develop a perception of operations on unknowns. This can be done by two different approaches. One can use the semantic approach which is to model in some concrete context or one can use a more syntactic approach where the algebraic rules are learnt and used to solve equations. Filloy and Rojano propose that non-arithmetical equations could be modelled by using different contexts such as the balance model and the geometrical model. The geometrical model consists of modelling $Ax + B = Cx$ by seeing Ax as the area of a rectangle with side lengths A and x, B as the area of another rectangle, and Cx as the area of a rectangle with side lengths C and x. The balance model looks at Ax, B, and Cx as objects with a known/unknown weight. Filloy and Rojano used these two models in their research and analyzed abstraction processes on operations on the unknowns. They found that students temporarily lose previous abilities and at the same time focus too much on the models which postpones algebraic abilities. Concrete models however, can be used to learn algebra as long as one is aware of the main components of modelling (Filloy and Rojano, 1989).

Bednarz and Janvier (1996) also looked at the arithmetic-algebra relation on the basis of equations. In their study Bednarz and Janvier identified major difficulties of students aged $12-13$ in equation solving. The difficulties the students had were to generate an equation in one unknown, substitution, avoid operating on the unknown, and the use of symbolism to represent a relationship by an equation (Martinez, 2007).

Balacheff (2000) states that there is a tendency to maintain an arithmetical interpretation in algebraic situations. This means that in an algebraic exercise such as solving a linear equation the techniques applied are arithmetical in the sense that operations on the unknown are not carried out. Therefore, it is important to produce a rupture between arithmetic and algebra and not a transition. To create such a rupture it is necessary to consider the problems that are used when teaching algebra. Arithmetic and algebra often compete for many of the same problems but it is important to choose problems where only algebra can solve the problem. This can for example be problems involving proving some property such as why a certain CP always gives the same answer (Balacheff, 2000).

FUNCTION PERSPECTIVE With this perspective the focus is on learning elementary algebra through the use of functions. Functions have different representations such as tabular forms, graphs, and algebraic expressions. It has been investigated how especially technology may help in the work with representations of functions and algebra.

MODELLING PERSPECTIVE The concept of modelling can be understood in different ways. Some understand it as creating a mathematical model to describe a non-mathematical reality, i.e. to solve a "real-world" problem. Others understand it as modelling in itself meaning that every mathematical activity is modelling where algebra can be said to be a modelling tool.
Chevallard, Bosch and Gascón (1997) view algebra as a modelling tool of both real-world problems and mathematical problems. They explain that every mathematical activity involves making a model of systems (Martinez, 2007).

5 METHODOLOGY

This chapter will first describe the choice of material for the analysis. Then it will be described and explained how the analysis of the material will be carried out including how the subject of this thesis is delimited in relation to selected exercises. A short description of the tables that represent the analysis will be given. Finally, the design of the pilot test and the evaluation of the students' test results will be explained and justified.

5.1 CHOICE OF MATERIAL FOR THE ANALYSIS

To investigate which types of tasks related to basic algebra that are central in Danish 9th grade and in Danish high school (stx) some materials were chosen for the analysis. Materials from both Danish 9th grade and stx were selected but also a screening test from Silkeborg high school has been included. The screening test is made by experienced high school teachers and shows what they expect the students are able to do in the transition from 9th grade to Danish high school. Thus it is of interest in the analysis. Data about the students' results for the last couple of years are available and are also a motivation for including the screening test in the analysis.
The material which will be analyzed from Danish 9th grade is the final written examination in mathematics for 9th grade (FSA). Only the FSAs from May and December 2013 and 2014, i.e. a total of four sets, have been analysed. The reason for this delimitation is that these are the latest available FSA.
Two books for teaching mathematics at level C in stx will be analyzed. The books are 'Gyldendals Gymnasiematematik C Arbejdsbog' by Clausen et al. (2005) and 'MAT C stx' by Carstensen et al. (2005). The basis for selecting 'Gyldendals Gymnasiematematik C Arbejdsbog' and 'MAT C stx' is that these were the first books to be published after the new reform in Danish high school in 2005. Therefore it is reasonable to assume that many high schools in Denmark are using these books.
The C-level mathematics books have been chosen instead of books for B- or A-level since all high school students begin their first year in high school with a basic training course that lasts about half a year. After the basic training course the students choose an area of study which means that they choose which subjects that they want to study at a high level (Minesteriet for Børn, Undervisning og Ligestilling, 2015). Because of this structure the mathematics which is taught in

the basic training course in the different classes have to be more or less the same such that it is possible for the students to change classes subsequently. The teachers can choose to emphasize areas of mathematics differently dependent on the expected level that the class will study mathematics on after the basic training course. Hence, some additional techniques related to basic algebra will be studied if the students were supposed to have mathematics on B- or A-level but the techniques may not be discovered in the analysis.
The reason for looking at the FSA in the analysis of basic algebra in Danish 9th grade instead of textbooks is that the FSA shows what students in Danish 9th grade are expected to be able do in the end of 9th grade just before they start in high school by the teachers in lower secondary school. Therefore, this is part of the transition. Furthermore, several teachers are teaching to the test, meaning that they teach their students what they expect the test/exam will contain (Svendsen, 2009, p. 74). Even though the textbooks may contain more types of algebraic exercises that does not necessarily mean that the teachers choose to teach them or study them as much as exercises that the FSA tests. This view appeared in workshop 2 of the 'MathBridge-project' when a teacher in Danish 9th grade mentioned that he thought that many teachers would skip some interesting chapters in the book if they were not a necessity for the FSA.
To sum up the material that will be used in the analysis of central types of tasks for basic algebra in Danish 9th grade and high school are four FSAs, the two high school books 'Gyldendals Gymnasiematematik C Arbejdsbog' and 'MAT C stx', as well as the screening test from Silkeborg high school.

5.2 ANALYSIS OF MATERIAL

In the above-mentioned material exercises related to basic school algebra will be selected and analyzed. This a priori analysis of the material will be used to build an epistemological reference model based on praxeologies.
To select exercises related to basic school algebra a definition of exercises concerning basic school algebra is necessary. My first idea of a delimitation of exercises related to basic school algebra is the following:

An exercise is associated with basic algebra if an unknown quantity appears in one way or another in the exercise or solution.

However, this definition of exercises related to basic school algebra turned out to be too broad since an exercise of the following type by this definition would be included in the analysis:

A quantity increases by 5% from the value $K_1 = 1000$ to the value K_2. Determine K_2. (Clausen et al., 2005, exercise 904, p. 53)

In this exercise the letters are placeholders and not necessary in order to solve the exercise since it can be solved by pure arithmetic by calculating $1.05 \cdot 1000 = 1050$. Therefore, a further delimitation of exercises concerning basic school algebra is required.
The second definition of exercises related to basic school algebra:

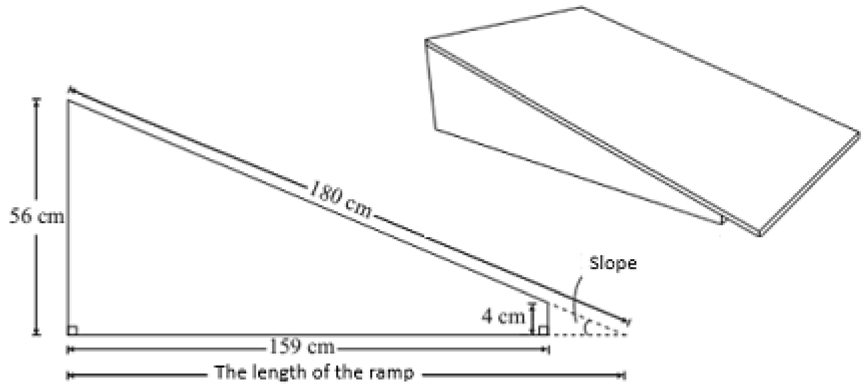
An exercise is associated with basic algebra if an unknown quantity appears in the exercise or solution ***and*** *if the unknown quantity has a function (not just placeholders) in the solution to the exercise.*

Besides this delimitation some techniques are excluded from the analysis. This concerns instrumented techniques such as using a CAS-tool to solve equations, draw graphs, calculate, and make regression or to use spreadsheets to extrapolate values, draw graphs and so on. The reason for these techniques to have been excluded from the analysis is that they are neither relevant nor important since instrumented techniques often removes focus from the algebraic techniques which is the focus of the analysis.

From the above delimitation exercises in the different materials from Danish 9th grade, the screening test, and stx will be selected. For some exercises it first becomes clear if they are related to basic school algebra when they are being analysed. It is thus not always transparent whether an exercise belongs to the examined area or not solely by looking at the statement of the exercise but it becomes clearer after the analysis of the exercise has been carried out.
The exercises are being analysed by examining the different ways an exercise can be solved. ATDs notion of praxeology is used in the analysis since it captures the specific techniques required for a given exercise. Techniques should be understood as ways of doing, such as to draw a line in a coordinate system. Thus, the analysis will be carried out by examining how an exercise can be solved so the focus is on making the techniques used to solve an exercise explicit.

The possible and realistic techniques that can solve an exercise from the analysis have been attempted to be identified. This has not been straightforward since some of the exercises were quite difficult to identify with regard to types of tasks and techniques. An example of this can be illustrated by the exercise in table 4.
From the statement of the exercise it is difficult to see what types of tasks and techniques it can be associated with. Partly because the exercise is not similar to any other exercise in the material and partly because the exercise is in a geometric context. One way to solve the exercise is to bear in mind that the triangles have similar angles which implies that corresponding sides have the same ratio. If this is used correctly the equation $\frac{159+x}{x} = \frac{56}{4}$ can be set up where $159 + x$ denotes the length of the ramp. The equation can be solved by algebraic techniques. If this approach is chosen the exercise can be identified with the type of task to solve an equation on the form $\frac{A+x}{x} = B$ with

The two sketches below show what a ramp for skating should look like when it is finished.



The students are allowed to put the ramp in the school yard if its length is less than 175 cm.

How much will the length of the ramp be?

Table 4: Exercise 2.3 in the FSA from May 2014

algebraic techniques. Another way to solve the exercise is to use the trigonometrical formulas for rectangular triangles. The slope of the ramp can first be determined by using the fact that sine to an angle is the opposite side divided by the hypotenuse. Subsequently the length of the ramp can be determined by dividing the opposite side in the big triangle with tangent to the found angle. This approach does not involve basic school algebra. It is of course also possible that the students use their IT-tool to solve the exercise but this will not be explained further as previously mentioned.
The exercise have been included in the analysis since it is estimated that the students will not choose the second approach. This can be substantiated by looking at the exercise that follows. In this exercise the students have to examine whether the slope of the ramp is less than or more than 20° which implies that the students have not found the slope of the ramp in the previous exercise.
In similar cases where it is not easy to decide if the exercise should be included in the analysis the most simple of the techniques have been selected as the one the students would probably use. On the basis of this it has been decided if the exercise should be included in the analysis or not.

From this a division into types of tasks and the corresponding techniques have been made. In that way the model relies on the analysed exercises i.e. data. An exercise such as solving the equation $5x + 7 = 13$ can be identified by different types of tasks dependent on whether algebraic, graphic, or trial-and-error techniques are used to solve the exercise.
The division into types of tasks and techniques will be structured in

tables as will be explained in the section below. The reason for using the table form is that with the chosen material, especially the text books, many exercises are included in the analysis (approximately 10 exercises from the screening test, 50 exercises from the FSA and more than 100 exercises in each of the text books for stx). Because of this large amount of exercises it was necessary to present the results from the analysis in a way that gave a survey of the types of tasks and techniques.
The different tables are shown in section A.1 in the appendix which are based on the analysis of the exercises. The tables consist of four columns that shows i) a categorization of the techniques into topics such as fractions, equation solving etc., ii) a list of the different techniques such as the distributive law, multiply two fractions and so on, iii) an example showing how the technique is used, and iv) a reference to the material that use the techniques (Gyldendals Gymnasiematematik C Arbejdsbog, MAT C stx, FSA and/or screening test). From the tables it is possible to get a quick overview of all the techniques associated with a given type of task but also to see if a given type of task is present in both Danish 9th grade and high school, or only one of the institutions from the indication of material. By looking at all the tables one can get an idea of the different types of tasks and techniques in basic school algebra in the transition from lower secondary school to high school.

From these tables a big table will be made which shows all the different techniques used to solve basic school algebra exercises in Danish high school and in lower secondary school. The big table will be made by listing all techniques that are present in one or more of the smaller tables and refer to the given material that uses these techniques. The big table will be divided in a way similar to the smaller tables. Therefore, the big schema will contain an overview of techniques and institutions. From this table it should be clear to see which techniques are present in both institutions and which techniques are absent in one of them, which answers the first research question since the central techniques related to basic algebra in Danish 9th grade and in Danish high school are identified.

5.3 THE PILOT TEST

The pilot test will be based on the ERM. The purpose of the test is to be used in the beginning of 1st year in high school to measure the students whose mathematics teachers in lower secondary school have participated in the Math-Bridge seminar. The hope is that these students will experience an easier transition to high school in relation to algebra which has been one of the areas of focus in the seminar. With this in mind, it makes sense that the test includes techniques which are present in both lower secondary school and high school. In addition to that it would be interesting to test the techniques that caused problems for the students in the screening test from Silkeborg. This can be determined from data showing the students success ratios for each exercise in the test.

The test is designed to be a diagnostic test. According to Brekke (2002) a diagnostic exercise tests the students misconceptions that is incomplete thoughts associated with a notion. It is important to distinguish between mistakes and misconceptions since a mistake is more random and may be a result of lacking attention whereas a misconception consists of a certain idea that is used consistently (Brekke, 2002, p. 10). A diagnostic test is used for identifying and highlighting misconceptions that the students have developed. The goal of a diagnostic test is to discover the students' thoughts about different notions and their difficulties associated with these notions (Brekke, 2002, p. 16). On the basis of this the students are asked to include all intermediate results in their answers since these provide useful information about the way the students think and about their misconceptions.

The pilot test is constructed by choosing a technique or several techniques from the ERM that satisfies the criteria that it is present in both institutions or problematic for the students in the screening test from Silkeborg. Then, a simple exercise that can test the techniques is constructed often with the help of exercises from the analysis. It is done by simplifying the exercise or changing it a bit so the exercise becomes simple enough to test the desired technique or techniques. Some of the exercises in the pilot test are unchanged exercises from the material and a few of the exercises in the test are constructed without the use of exercises from the material.
After this a praxeological analysis of each exercise in the first edition of the pilot test has been carried out, and in light of this the exercise has possibly been changed before the exercise is included in the final edition of the pilot test. The analysis of the exercises has focused on techniques used for solving the exercise and possible misconceptions with regard to the techniques that the students will encounter when solving the exercise.

5.4 ANALYSIS OF STUDENTS' ANSWERS

The pilot test has been given to 21 students. Their answers to the exercises will be analysed to see which techniques that cause them problems and which techniques they master. In the analysis the success ratio i.e. the number of correct answers divided by the total number of students participating in the test will be found for each exercise. For correct answers it will be examined if the students use the expected techniques or not. For wrong answers the corresponding misconceptions will be identified.
The a posteriori analysis of the students' answers to the exercises in

the pilot test can be used to clarify if they have some common misunderstandings and, furthermore, if some of the exercises do not work as intended. The identification of common misconceptions will assist in answering the second research question as challenges associated with different types of tasks and techniques in basic algebra in the transition from lower secondary school to high school will be evident.

6 ANALYSIS OF TASKS IN BASIC ALGEBRA

6.1 AN ERM FOR BASIC ALGEBRA

An ERM for elementary algebra can have different purposes. My purposes for building an ERM is in line with some of the purposes that Bosch (2015) mentions. First of all it is a tool to analyse what kind of algebraic praxeologies that already exist in Danish 9th grade and high school to make it explicit if they coincide. Secondly, it can be used to examine which aspects of algebra there are absent in lower secondary school or high school and the possible reasons for this can be studied. Furthermore, it can be used in a collaboration between research and teachers which is partly the case in this thesis since the ERM are supposed to help in the MathBridge project.

The ERM will be presented in the following sections. Each of the techniques that appeared in the analysis of the material have been categorized. Each category and its associated techniques will be elaborated through examples of exercises from the material.

6.1.1 *Laws*

There are three techniques in this category. I will define and describe the techniques by using examples from exercises from the analysis. The following exercise is taken from the screening test:

Reduce the following expression as much as possible:
c) $x \cdot (3 + x) - 3x$

Table 5: Exercise 16 c) in (Silkeborg Gymnasium, 2014, p. 4)

In order to solve this exercise the techniques τ_2 and τ_3 is needed. τ_3 is the distributive law which is used in the exercise to cancel the brackets, i.e. $x \cdot (3 + x) - 3x = x \cdot 3 + x \cdot x - 3x$. Then the commutative law for multiplication, τ_2, is used to make the transformation $x \cdot 3 = 3 \cdot x$. Then τ_{58} which will be explained in the subsection 6.1.11 is used to collect and reduce the similar terms. This gives the result x^2. In stx there are examples of the technique τ_3 being applied to a more complicated expression of the form $a \cdot (b + c) \cdot d$.
An example of an exercise where the technique τ_1 which is the associative law for multiplication is used is shown on the next page.
After using the technique for squares (which will be described later

Calculate using the square of a two-termed quantity:
c) $(3Tr + 9r)^2$

Table 6: Exercise 845 c) in (Clausen et al., 2005, p. 45)

in subsection 6.1.14) and the commutative law for multiplication the algebraic expression looks like $(3Tr)^2 + (9r)^2 + 2 \cdot 3 \cdot 9 \cdot Trr$. The associative law for multiplication is used to multiply 2, 3 and 9 together since the last term can be reduced in the two ways $2 \cdot 3 \cdot 9 \cdot Trr = 6 \cdot 9 \cdot Tr^2 = 54Tr^2$ or $2 \cdot 3 \cdot 9 \cdot Trr = 2 \cdot 27 \cdot Tr^2 = 54Tr^2$.

6.1.2 *Hierarchy*

The techniques associated with this category describe the order of arithmetic operations. This has been subdivided into three techniques with respect to the complexity of the involved arithmetic operations.

Calculate the value of each expression, when $a = 2$ and $b = -3$
45. $a^2 + 2ab + b^2 =$

Table 7: Exercise 45 in FSA December 2014

To solve the exercise above technique τ_5 is required after the numbers have been substituted into the algebraic expression. τ_5 is the technique that involves calculating roots and powers before the four basic arithmetic operations which are multiplication, division, subtraction, and addition. Technique τ_4 will also be used to multiply the numbers before adding and subtracting. The exercise will thus be solved by performing the calculations:

$$2^2 + 2 \cdot 2 \cdot (-3) + (-3)^2 = 4 + 2 \cdot 2 \cdot (-3) + 9 = 4 - 12 + 9 = 1.$$

Another technique in this category is calculating brackets before all other arithmetic operations. This technique is denoted τ_6. This technique is applied in Table 5 where the brackets are calculated first in $x \cdot (3 + x) - 3x$.

6.1.3 *Equation solving*

There have been found nine different techniques in the category of equation solving. Some of the techniques are involved in exercises with more than one equation and one unknown, whereas some techniques are also involved in exercises concerning only one equation in

Solve the system of equations a) $x + y = 50$ $x - y = 12$

Table 8: Exercise 858 in (Clausen et al., 2005, p. 47)

one unknown. Several of the techniques in this category can be defined by looking at the exercise on the top of the next page.
One way to solve the system of equations is to use technique τ_7 first which consists of adding, subtracting, multiplying or dividing by the same quantity on both sides of the equality sign, with the exception of multiplication and division by zero. In this case y is subtracted on both sides of the equality sign of the first equation and y is added on both sides of the equality sign in the second equation.

$$x + y = 50 \iff x = 50 - y$$
$$x - y = 12 \iff x = 12 + y$$

Then τ_9 can be applied which involves setting two algebraic expressions, both of which are equal to the same unknown, equal to each other. This can be done since both equations above have been transformed in such a way that they are both equal to x. This equation in one unknown can be solved by applying technique τ_7 again.

$$\begin{aligned} 50 - y &= 12 + y \\ 38 &= 2y \\ y &= 19 \end{aligned}$$

Eventually x can be found by substituting the value 19, which is the value of y, into one of the two equations. If the first equation is selected the calculation for finding x is $x = 50 - 19 = 31$.
It is also possible to solve the system of equations by using τ_{10}. This technique is to subtract two equations from each other. The technique is only useful when one of the unknowns have the same coefficient because in that case one of the unknowns will cancel out. Subsequently, the equation in one unknown can be solved by τ_7. In the chosen exercise the coefficient in front of x in both equations is 1. If this is not the case it is easy to get equal coefficients in front of one of the unknowns in both equations by multiplying each equation on both sides of the equality sign by a suitable quantity i.e. using τ_7.

$$\begin{aligned} x + y - (x - y) &= 50 - 12 \\ x + y - x + y &= 38 \\ 2y &= 38 \\ y &= 19 \end{aligned}$$

In the same way x can be found by substituting the value $y = 19$ into one of the two equations.
The system of equations can also be solved in a third way by substituting an expression for one of the unknowns into one of the equations. This is technique τ_{11}. In order to do this it is necessary to isolate one of the unknowns in one of the equations first. Again this is done by τ_7. In the exercise we isolate x in the first equation $x + y = 50$ which gives $x = 50 - y$. This is then substituted into the second equation and solved by τ_7.

$$\begin{aligned} x - y &= 12 \\ 50 - y - y &= 12 \\ -2y &= -38 \\ y &= 19 \end{aligned}$$

The x-value can be determined similarly as described above.
Technique τ_8 will be described by the following example:

Solve the equation $\frac{15}{7} = \frac{2}{x}$.

Table 9: Exercise 852 c) in (Clausen et al., 2005, p. 46)

The exercise can be solved by multiplying crosswise and subsequently using τ_7 to divide by 15 on both sides of the equality symbol.

$$\begin{aligned} \frac{15}{7} &= \frac{2}{x} \\ 15 \cdot x &= 2 \cdot 7 \\ x &= \frac{14}{15} \end{aligned}$$

The technique can also be used in the reverse direction which is technique τ_{14}.

Solve the equation $5,6 \cdot 1,2^x = 83,6 \cdot 0,75^x$.

Table 10: Exercise 865 b) in (Clausen et al., 2005, p. 48)

$$\begin{aligned}
5,6 \cdot 1,2^x &= 83,6 \cdot 0,75^x \\
\frac{1,2^x}{0,75^x} &= \frac{83,6}{5,6} \\
\left(\frac{1,2}{0,75}\right)^x &= \frac{83,6}{5,6} \\
\log\left(\frac{1,2}{0,75}^x\right) &= \log\left(\frac{83,6}{5,6}\right) \\
x \cdot \log\left(\frac{1,2}{0,75}\right) &= \log\left(\frac{83,6}{5,6}\right) \\
x &= \frac{\log\left(\frac{83,6}{5,6}\right)}{\log\left(\frac{1,2}{0,75}\right)}
\end{aligned}$$

An example in which τ_{14} is used can be seen by looking at the above exercise and its solution.

The first step in the solution of the equation above is technique τ_{14} which is multiplying crosswise in the opposite direction than usual. It is also possible to reach the second step without using technique τ_{14}. Technique τ_7 can be applied to first divide by $0,75^x$ on both sides of the equality sign and afterwards divide by $5,6$ on both sides of the equality sign. It is an example that illustrates that not all of the techniques are disjoint. In the second step a technique for exponents is used. This technique will be described later in the subsection 6.1.8. In the third step another technique from equation solving is applied. The technique is to take the logarithm on both sides of the equal sign corresponding to τ_{13}. Then a technique for logarithms is used and lastly τ_7 is used to divide by $\log\left(\frac{1,2}{0,75}\right)$ on both sides of the equality sign.

The last two techniques in the category equation solving will be defined on the basis of two examples different from those already mentioned.

Solve the equation $(x-2)(x-6) = 0$.

Table 11: Exercise 208 3) in (Carstensen et al., 2005, p. 321)

The method for solving this exercise is the zero-divisor law which states that if a product is zero then at least one of the factors is zero i.e. $ab = 0$ then $a = 0$ or $b = 0$. This is technique τ_{12}. Furthermore, τ_7 is once again used when respectively 2 and 6 is added on both sides of the equality sign in each of the equations to find x.

$$(x-2)(x-6) = 0 \iff x-2 = 0 \vee x-6 = 0 \iff x = 2 \vee x = 6$$

The last technique in equation solving is τ_{15}. The technique is to take the nth root of a number. Below is an example of an exercise that requires this technique to be solved.

Solve the equation $x^3 = 27$.

Table 12: Exercise 33 in FSA December 2013

$$x^3 = 27 \Longleftrightarrow x = \sqrt[3]{27} = 3$$

6.1.4 *Guess*

The techniques in this group consists of trial-and-error in various ways. It should be mentioned that the techniques in this category contains a small learning potential in the sense that these techniques are not fruitful in the long run. This is because the techniques have almost no pragmatic and epistemic value since the techniques are neither productive nor do they provide much understanding to the mathematical objects such as equations and equation solving.
τ_{16} is a technique that can be used in equation solving. The technique can be used in the exercise just above in Table 12. A strategy to solve the equation $x^3 = 27$ could be to try different values for x until the right value of x is found. If first the value $x = 1$ is tested it is seen that this is not the solution to the equation since $1^3 = 1 \neq 27$. If the value $x = 3$ is tested it is clear that this is the solution since $3^3 = 27$.
Another trial-and-error technique is τ_{17} which is used in tasks involving factorization. The technique is to guess a factorization and then check if it is correct by technique τ_3.

Rewrite the following expression by factorizing. Example: $3x + 6y + 3 = 3 \cdot (x + 2y + 1)$ b) $2x + 6xy$

Table 13: Exercise 18 in (Silkeborg Gymnasium, 2014)

In the above exercise τ_{17} can be applied. $2 \cdot (x + 4xy)$ could be a possible guess of a factorization. It can be checked that this is not correct by technique τ_3 which gives $2 \cdot (x + 4xy) = 2x + 8xy \neq 2x + 6xy$. Subsequently, a new guess of a factorization must be checked.

6.1.5 *Substitute*

This category concerns one technique. This technique, τ_{18}, is to substitute a specific number for a letter. The technique is used in various exercises. It was used in the exercise in Table 7 to substitute the values for a and b in the algebraic expression. It was also used in the exercise in Table 8 after the value for one of the unknowns was found this value was substituted in one of the equations to find the value of the second unknown. The technique is also used when τ_{16} is used to guess solutions to equations.

6.1.6 *Graphic*

15 graphic techniques have been identified.
τ_{19} concerns the technique to draw a straight line in a coordinate system. This technique can be used in the exercise in Table 8. If y is isolated in both $x + y = 50$ and $x - y = 12$ by τ_7 then the two straight lines $y = -x + 50$ and $y = x - 12$ can be drawn in a coordinate system. The solution to the system of equations can be interpreted as the intersection between two straight lines. Hence, the technique τ_{21} which is to read the intersection between lines can be applied to find the value for x and y.
It is also a technique to read off a x-value for a given y-value on a graph. This technique, τ_{20}, can be used in the following exercise.

Solve the equation $4x - 3 = 25$.

Table 14: Exercise 850 c) in (Clausen et al., 2005)

First of all the graph of $y = 4x - 3$ is drawn corresponding to τ_{19}. Then the x-value for $y = 25$ is read from the graph. This provides $x = 27$ as solution to the exercise.
Another graphic technique very similar to τ_{20} is τ_{22}. The technique is to read an y-value for a given x-value on a graph. In the exercise below technique τ_{22} can be applied to read the y-value for $x = 5$ on the graph after the straight line $y = 2x + 3$ has been drawn in a coordinate system.

There are the following relationship for y and x: $y = 2x + 3$. a) What is the value for y, when $x = 5$?

Table 15: Exercise 22 a) in (Silkeborg Gymnasium, 2014)

A fifth graphic technique is τ_{23} which is to plot points into a coordinate system and connect them to a straight line. This technique can be used to plot the points A and B from the exercise in Table 16 in a coordinate system and connect the points by a straight line. This line can be extended making it is possible to read the intersection with the y-axis which in this case is $(0, 3)$. This corresponds to the graphic technique τ_{27}. Furthermore, the slope of the line can be determined from the graph by choosing a point on the line and going 1 along the x-axis and then vertical up to a point on the line. The length of the vertical segment is the slope of the line and it is $\frac{1}{4}$. This technique is τ_{28}. From this the equation of the line is $y = \frac{1}{4}x + 3$.

Find an equation for the line passing through the points A and B when

1) $A = (4,4)$, $B = (8,5)$

Table 16: Exercise 314 1) in (Carstensen et al., 2005, p. 331)

τ_{24} is the technique to read the growth in y i.e. Δy on the y-axis by determining the difference between two y-values graphically. The ex-

The function determined by $N(t) = 3t + 5$ is a function of the time t.

a) Determine $\Delta N = N(14+3) - N(14)$.

Table 17: Exercise 248 a) in (Clausen et al., 2005, p. 74)

ercise in table 17 can be solved by τ_{24}.
First of all, the function $N(t) = 3t + 5$ should be drawn in a coordinate system. Then τ_{24} can be used to find the difference on the y-axis in the y-values for respectively $x = 14$ and $x = 14 + 3$. The difference in the y-values is $\Delta N = 9$.
Another graphic technique which is to draw the best straight line through some points can be used in the following exercise.

Describe from the following table how y depends on x, and determine, if possible, the relationship between the variables.

x	−10	−8	−6	−4	−2	2	4
y	−5,94	−5,28	−4,59	−3,98	−3,35	−1,95	−1,34

x	6	8	10
y	−0,65	−0,98	0,67

Table 18: Exercise 226 in (Clausen et al., 2005, p. 71)

One of the methods to solve this exercise could be to plot the points into the coordinate system and then use τ_{26} i.e. drawing the best straight line through the points. From the drawn line the equation of the line can be determined.
The technique, τ_{25}, to draw a line through a given point and with a given slope is very similar to τ_{19}. The technique can be used in the exercise in Table 19 to draw the line from which it is possible to find the equation of the line.
The graphic techniques to find the point of intersection with the x-axis, to read coordinates on a line, and to determine the minimum and maximum x- or y-value that a line segment can have are τ_{29},

A straight line passes through the point $A(4,5)$ and has the slope $\frac{1}{2}$. Determine an equation for the line.

Table 19: Exercise 320 in (Carstensen et al., 2005, p. 331)

τ_{30}, and τ_{31} respectively. The techniques can be explained further by looking at the next example of an exercise in Table 20.
To solve this exercise the line segment is first drawn corresponding to τ_{19}. To determine the interval that the x-values belong to technique τ_{31} is used to find the minimum x-value and the maximum x-value. The minimum x-value is -1 and the maximum x-value is 3. Hence the interval is $]-1,3]$. Subsequently, the intersection with the x-axis can be determined. Technique τ_{29} is employed to do this which gives $(\frac{1}{2},0)$ as the point of intersection with the x-axis. Technique τ_{30} overlaps with many of the graphic techniques since the technique is to read any set of coordinates on a line. For that reason it can also be said that the technique is used to solve this exercise.

A line segment lies on the line $y = -2x+1$ and the y-values belong to $[-5,3[$. Draw the line segment and determine the interval that the x-values belong to. Afterwards, determine the coordinates to the point of intersection with the x-axis.

Table 20: Exercise 338 in (Carstensen et al., 2005, p. 333)

The second last graphic technique is to determine whether a point is on a given straight line or not by drawing the point and the line in a coordinate system i.e. technique τ_{32}. The technique is applied in the following exercise.

The points A and B have the coordinates $A(-4,-2)$ and $B(2,6)$. Determine an equation for the line that passes through A and B. Does the point $C(27,40)$ lie on the line?

Table 21: Exercise 328 in (Carstensen et al., 2005, p. 332)

The equation for the line can be determined in the same way as in the exercise in Table 16. The equation for the line is $y = \frac{4}{3}x + \frac{10}{3}$. To determine if the point C is on the line technique τ_{32} is used. The technique is to plot the point in the coordinate system with the line. From this it can be seen that the point is not on the line since the point does not coincide with the line.
The last graphic technique is τ_{33} which is to determine if 3 points lie on the same straight line by plotting them in a coordinate system.
The exercise in Table 22 can be solved by technique τ_{33}. This is done by plotting the points in the same coordinate system and examin-

Show that the points $A(1,4)$, $B(3,7)$, and $C(11,19)$ lie on the same straight line.

Table 22: Exercise 326 in (Carstensen et al., 2005, p. 332)

ing if it is possible to draw a straight line passing through the three points. It is concluded that the three points lie on the same straight line because it is possible to connect them with a single straight line.

6.1.7 *Fractions*

This category concerns techniques associated with calculation rules for fractions.

Solve the equation $\frac{3x+6}{4} = 3$.

Table 23: Exercise 50 in FSA May 2013

The exercise in table 23 involving equation solving can be solved by using different techniques involving fractions. The technique τ_{42} is to split a fraction where the numerator contains a sum or a difference into two fractions can first be carried out.

$$\frac{3x+6}{4} = \frac{3x}{4} + \frac{6}{4}$$

Then the technique to shorten a fraction i.e. τ_{37} can be used on the fraction $\frac{6}{4} = \frac{6:2}{4:2} = \frac{3}{2}$. If the divisor goes up in the dividend so that the fraction is an integer e.g $\frac{6}{3} = 2$ it is not viewed as shortening a fraction. Subsequently, $\frac{3x}{4}$ can be rewritten to $\frac{3}{4} \cdot x$ by technique τ_{41} and the number 3 can be rewritten to the fraction $\frac{3 \cdot 2}{2}$ by technique τ_{60} after using τ_7 to subtract $\frac{3}{2}$ on both sides of the equality sign. Then the two fractions on the left side of the equality sign can be subtracted by subtracting the numerators from each other. This corresponds to technique τ_{35}. Finally the technique τ_{39} which is to multiply two fractions together by multiplying numerator with numerator and denominator with denominator is used.

$$\begin{aligned}
\frac{3}{4} \cdot x + \frac{3}{2} &= 3 \\
\frac{3}{4} \cdot x &= \frac{6}{2} - \frac{3}{2} \\
\frac{3}{4} \cdot x &= \frac{6-3}{2} \\
\frac{4}{3} \cdot \frac{3}{4} \cdot x &= \frac{4}{3} \cdot \frac{3}{2} \\
x &= \frac{4 \cdot 3}{3 \cdot 2} = 2
\end{aligned}$$

Other techniques involving calculation with fractions are τ_{45} which is to divide a fraction by a number by multiplying the number in the denominator and τ_{38} which is to multiply a fraction by a number by multiplying the number in the numerator. These techniques will be exemplified by the next exercise in Table 24.

For a particular cylinder with lid the radius in the bottom is r, and the height is h. For the cylinder's volume V and surface O holds $$V = \pi r^2 h \text{ and } O = 2\pi r h + 2\pi r^2.$$ Use the two formulas to find O expressed by V and r.

Table 24: Exercise 882 in (Clausen et al., 2005, p. 49)

To solve this exercise h is isolated in the expression for the volume of the cylinder. The technique to isolate h is τ_7 as it is done by first dividing by π on both sides of the equal sign and then dividing r^2 on both sides of the equal sign. This gives $\frac{\frac{V}{\pi}}{r^2}$ which can be simplified by τ_{45} so that it becomes $\frac{V}{\pi \cdot r^2}$.
This algebraic expression for h is substituted into the expression for O i.e. τ_{11} is used. Then τ_{38} may be applied to multiply the number $2\pi r$ with the fraction as shown below. The fraction can be simplified by cancelling the common factors π and r in the numerator and the denominator.

$$O = 2\pi r \cdot \frac{V}{\pi \cdot r^2} + 2\pi r^2$$
$$O = \frac{2\pi r V}{\pi \cdot r^2} + 2\pi r^2$$
$$O = \frac{2V}{r} + 2\pi r^2$$

The technique τ_{38} can also be in reverse direction meaning that a fraction where the numerator contains a product can be rewritten to a factor from the numerator multiplied by a fraction with only one factor in the numerator. This is τ_{41} and can be applied in the next exercise.

You should show by rewriting that the value of the calculation expression $\frac{m \cdot 6}{3} - m + 10$ is 10 greater than m.

Table 25: Exercise 5.4 in FSA problem solving May 2014

By using τ_{41} the expression can be rewritten to $m \cdot \frac{6}{3} - m + 10$. Then $\frac{6}{3} = 2$ and the commutative low for multiplication, τ_2, reduces the expression to $2m - m + 10$. This can be rewritten by collecting and reducing like terms to the expression $m + 10$.

The technique dividing by a fraction is carried out by interchanging the numerator and denominator and multiplying by this "new" fraction is divided into two techniques. The first one, τ_{43}, involves a number being divided by a fraction and the second, τ_{44}, involves a fraction being divided by a fraction.
Below is an example of an exercise which requires the technique τ_{44} to be solved.

Calculate $\frac{8T}{9U} : \frac{3G}{P}$.

Table 26: Exercise 814 d) in (Clausen et al., 2005, p. 40)

The division of the two fractions using τ_{44} and τ_2 gives us:

$$\frac{8T}{9U} : \frac{3G}{P} = \frac{8T}{9U} \cdot \frac{P}{3G} = \frac{8T \cdot P}{9U \cdot 3G} = \frac{8TP}{27UG}$$

Similar to τ_{41} there is a technique which is the reverse direction of τ_{39}. This technique is to split a fraction where both the numerator and denominator are products into the product of two fractions. This technique is named τ_{40}. It can be used in the exercise in Table 27.

Shorten the fraction $\frac{a \cdot m}{a \cdot n}$.

Table 27: Exercise 847 a) in (Clausen et al., 2005, p. 45)

To shorten the fraction by applying τ_{40} one gets $\frac{a \cdot m}{a \cdot n} = \frac{a}{a} \cdot \frac{m}{n}$. Since $\frac{a}{a} = 1$ it is just equal to $\frac{m}{n}$.
The next exercise exemplifies the techniques τ_{34} and τ_{36} which are to add two fractions with the same denominator by adding only the numerators and to extend a fraction by multiplying by the same number in the numerator and denominator respectively.

Reduce the following expression as much as possible: 1) $\frac{3}{2x} + \frac{5}{6x}$

Table 28: Exercise 147 1) in (Carstensen et al., 2005, p. 318)

τ_{36} is first used to extend the fractions so that they have the same denominator. The first fraction is extended by 3 i.e. $\frac{3}{2x} = \frac{3 \cdot 3}{2x \cdot 3} = \frac{9}{6x}$. Then the two fractions are added which gives $\frac{9}{6x} + \frac{5}{6x} = \frac{9+5}{6x} = \frac{14}{6x}$. This fraction can be shortened by 2, resulting in the fraction $\frac{7}{3x}$.
The last technique that needs to be mentioned regarding fractions is τ_{43}. This technique is to divide a number by a fraction by multiplying the number by the reverse fraction. The technique is used in the following exercise.
A possible method for solving this exercise is to set up the equation $\frac{3}{5}x + x = 120$ where x is the money Louise has to pay for the present.

Louise and her brother Frederik are buying a present for 120 DKK for Carina. Since Carina is Louise's school friend Frederik will only pay $\frac{3}{5}$ of what Louise is paying. How much are they paying each?

Table 29: Exercise 248 in (Carstensen et al., 2005, p. 326)

To solve this equation x is rewritten to $\frac{5}{5}x$ and the similar terms are reduced by technique τ_{58} and τ_{34}. This gives $\frac{8}{5}x = 120$. τ_7 is used to divide by $\frac{8}{5}$ on both sides of the equality sign. To calculate the value of x τ_{43} is used. Hence $x = 120 : \frac{8}{5} = 120 \cdot \frac{5}{8} = \frac{120 \cdot 5}{8} = \frac{600}{8} = 75$.

6.1.8 *Exponents*

There are seven techniques regarding exponents. Some of them can be exemplified by an exercise from the screening test.

Reduce the following expression as much as possible: d) $\frac{a^4 \cdot b^3}{a^2 \cdot b}$

Table 30: Exercise 16 d) in (Silkeborg Gymnasium, 2014, p. 4)

The exercise can be solved in various ways by techniques involving exponents. One way is to use τ_{50} yielding that $\frac{1}{a^n} = a^{-n}$. If the technique is used on the algebraic expression then

$$\frac{a^4 \cdot b^3}{a^2 \cdot b} = a^4 \cdot b^3 \cdot a^{-2} \cdot b^{-1}$$

From this the commutative law for multiplication, τ_2, gives $a^4 \cdot b^3 \cdot a^{-2} \cdot b^{-1} = a^4 \cdot a^{-2} \cdot b^3 \cdot b^{-1}$. Then another technique τ_{46} for exponents can be applied. The technique is that two powers, which have the same base in common, can be multiplied together by adding the exponents. When the technique is applied on the algebraic expression from the exercise the following result is obtained.

$$a^4 \cdot a^{-2} \cdot b^3 \cdot b^{-1} = a^{4+(-2)} \cdot b^{3+(-1)} = a^2 \cdot b^2$$

Another way to solve this exercise is by first applying τ_{40} such that $\frac{a^4 \cdot b^3}{a^2 \cdot b} = \frac{a^4}{a^2} \cdot \frac{b^3}{b}$. Subsequently, the technique τ_{47} which is that two powers with the same base is divided by subtracting the exponents can be used.

$$\frac{a^4}{a^2} \cdot \frac{b^3}{b} = a^{4-2} \cdot b^{3-1} = a^2 \cdot b^2$$

τ_{48} is the technique that the power where the base is a product is the power of the first factor multiplied by the power of the second factor i.e. $(a \cdot b)^n = a^n \cdot b^n$. This technique is used in both the screening test and in stx. In stx more complicated cases involving this technique are present. This concerns cases where the product consists of three factors i.e. $(a \cdot b \cdot c)^n = a^n \cdot b^n \cdot c^n$ and cases where the factors themselves are powers i.e. $(a^p \cdot b^q)^n = a^{pn} \cdot b^{qn}$.

Write on the form $a^n \cdot b^n$:
a) $a^{14} \cdot (ab)^7 \cdot (b^{11})^3 \cdot b^{30}$

Table 31: Exercise 820 a) in (Clausen et al., 2005, p. 41)

Above is an exercise that requires τ_{48} to be solved. The technique gives that $(ab)^7 = a^7 \cdot b^7$. In addition, the technique that $(a^n)^m = a^{n \cdot m}$ i.e. τ_{49} is used such that $(b^{11})^3 = b^{11 \cdot 3} = b^{33}$. The combination of these two techniques and the exponent technique τ_{46} solves the exercise.

$$a^{14} \cdot (ab)^7 \cdot (b^{11})^3 \cdot b^{30} = a^{14} \cdot a^7 \cdot b^7 \cdot b^{33} \cdot b^{30} = a^{14+7} \cdot b^{7+33+30} = a^{21} \cdot b^{70}$$

The quotient exponent technique, τ_{51}, states that a power divided by a power with the same exponent is the bases divided and raised to the exponent is used in the exercise in Table 10. The technique makes the necessary rewriting

$$\frac{1{,}2^x}{0{,}75^x} = \left(\frac{1{,}2}{0{,}75}\right)^x.$$

The last exponent technique is τ_{52} which is τ_{51} in the opposite direction i.e. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. The technique can be employed in the following exercise.

Determine T expressed by h when $h = \frac{\sqrt{3}}{2}a$ and $T = \frac{\sqrt{3}}{4}a^2$.

Table 32: Exercise 203 b) in (Clausen et al., 2005, p. 68)

First of all, a is isolated in the expression for h. This is done by the techniques τ_7 and τ_{39} so that $h = \frac{\sqrt{3}}{2}a \Leftrightarrow \frac{2}{\sqrt{3}} \cdot h = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} \cdot a = \frac{2 \cdot \sqrt{3}}{\sqrt{3} \cdot 2} \cdot a = a$. Then the expression for a is substituted into the expression for T i.e. τ_{11} is used. Then the expression for T is reduced by applying the techniques τ_{48}, τ_{52}, τ_{39}, and τ_{37}.

$$T = \frac{\sqrt{3}}{4} \cdot \left(\frac{2}{\sqrt{3}} \cdot h\right)^2 = \frac{\sqrt{3}}{4} \cdot \left(\frac{2}{\sqrt{3}}\right)^2 \cdot h^2 = \frac{\sqrt{3}}{4} \cdot \frac{2^2}{(\sqrt{3})^2} \cdot h^2 = \frac{\sqrt{3} \cdot 4}{4 \cdot (\sqrt{3})^2} \cdot h^2 = \frac{1}{\sqrt{3}} \cdot h^2$$

6.1.9 *Signs*

Techniques for signs have been subdivided in three techniques. The first technique states that multiplication or division of two quantities with opposite sign is equal to a negative quantity. The second

technique states that multiplication or division of two quantities with minus sign is equal to a positive quantity. These techniques are τ_{53} and τ_{54}. Both of them are used to solve the exercise in Table 7. τ_{53} is used when $2 \cdot 2 \cdot (-3)$ is computed since it is $4 \cdot (-3) = -12$. τ_{54} is used in the exercise to compute $(-3)^2 = (-3) \cdot (-3) = 9$.
The remaining technique for signs is useful in the types of tasks involving factorization. The technique τ_{55} tells that the product of a negative and positive quantity is equal to the product of the same quantities with opposite signs i.e. $-a \cdot b = a \cdot (-b)$. This will be illustrated by the next example.

Factorize as much as possible: a) $23xL - 23wL$

Table 33: Exercise 842 b) in (Clausen et al., 2005, p. 44)

In this exercise technique τ_{55} is needed along with τ_{56} and τ_2. By applying technique τ_{55} and the commutative law for multiplication the expression can be rewritten to $23xL - 23wL = 23Lx + 23L \cdot (-w)$. This can be factorized by τ_{56} to $23L(x - w)$.

6.1.10 *Factorize*

A technique often used in exercises involving factorization is τ_{56} which is the technique to put a factor outside a bracket, i.e. $ab + ac = a(b + c)$. The exercise in Table 13 can be solved by this technique since $2x + 6xy = x(2 + 6y)$.
It is possible to put yet another factor outside the bracket of the above expression by using the second factorization technique. This is τ_{57} which consists in resolving a number into factors. When this technique is applied to the number 6 it can be resolved into $2 \cdot 3$ enabling a further factorization of the expression $x(2 + 6y) = x(2 + 2 \cdot 3y) = 2x(1 + 3y)$.
As in the example above from the screening test there are also exercises in stx where more factors can be put outside the bracket. This more complicated case is not present in the FSA. In addition to this, some factorization exercises in stx and one exercise in the FSA involve an expression with more than two terms.

6.1.11 *Simplify*

This category has two associated techniques.
The first one is τ_{58}. This technique is to collect and reduce like terms and it is applied in a lot of exercises. For example the exercise in Table 5 uses the technique to collect and reduce similar terms with x in the

expression $3x + x^2 - 3x$ so that $3x$ and $-3x$ cancel out.
The second simplifying technique is to cancel common factors if they appear in all terms in a expression involving fractions which is τ_{59}. The technique is for example used in the previous exercise in Table 24 to cancel the common factors π and r in the expression $\frac{2\pi r V}{\pi \cdot r^2} = \frac{2V}{r}$. The technique was only present in one exercise in lower secondary school shown in table 25. However, the technique was not necessary in order to solve the exercise since other techniques could be applied. This is indicated by a bracket around the cross in table 48.

6.1.12 *Rewrite*

The technique to rewrite a number into a fraction, i.e. $a = \frac{a \cdot b}{b}$, can sometimes be useful. This is technique τ_{60}. It can be used in the exercise in Table 23 to rewrite $3 = \frac{3 \cdot 2}{2} = \frac{6}{2}$.

Solve the equation $\frac{x}{2} = 7$

Table 34: Exercise 851 a) in (Clausen et al., 2005, p. 46)

Another useful rewriting is to write a fraction $\frac{a}{c}$ as $\frac{1}{c} \cdot a$. This is technique τ_{62} and it is a special case of τ_{41} combined with τ_{61}. It can be applied in the exercise above to rewrite $\frac{x}{2}$ to $\frac{1}{2}x$. Then the equation can be solved by multiplying by 2 on each side of the equality sign.
The technique τ_{61} to write x as $1 \cdot x$ can be useful in some exercises.

Factorize as much as possible: a) $ax + x$

Table 35: Exercise 843 b) in (Clausen et al., 2005, p. 44)

The exercise above requires the techniques τ_{56} and τ_{61} to be solved. τ_{61} is used to make the rewriting $ax + x = ax + 1 \cdot x$. From this the expression can be factorized by τ_{56} to $x(a + 1)$.
The last technique in this category is τ_{63}. The technique is to rewrite an integer to a product that consists of the square root of the integer multiplied by the square root of the integer i.e. $a = \sqrt{a} \cdot \sqrt{a}$.

Determine O expressed by h when $O = 3a$ and $h = \frac{\sqrt{3}}{2}a$.

Table 36: Exercise 203 c) in (Clausen et al., 2005, p. 68)

The exercise in Table 36 can be solved by τ_{38} and τ_{63} after isolating a in the expression for h and substituting the expression for a in the expression for O. Thus, the expression for O becomes

$$O = 3 \cdot \frac{2}{\sqrt{3}} \cdot h = \frac{3 \cdot 2}{\sqrt{3}} \cdot h = \frac{\sqrt{3} \cdot \sqrt{3} \cdot 2}{\sqrt{3}} \cdot h = \sqrt{3} \cdot 2 \cdot h.$$

6.1.13 *Brackets*

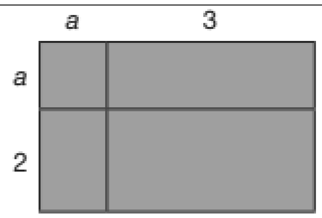
These techniques relates to removing brackets.
τ_{64} is the technique to remove a negative bracket, i.e. a bracket with a minus sign in front of it. This is done by by changing all the signs in front of the terms in the bracket. This technique is applied in the exercise in Table 8 when the technique τ_{10} for solving the system of equations is chosen. In this case the equation $x+y-(x-y)=50-12$ is obtained. The negative bracket is removed by τ_{65} in the following way:

$$x+y-(x-y)=x+y-x+y.$$

The technique to multiply two brackets with each other is τ_{65}. It can be used to find the two calculation expressions that describe the area of the figure in the exercise in Table 37. The expression $(a+3)\cdot(a+2)$ is multiplied together with technique τ_{65} and then τ_2 is used to change the order of a and 2 and τ_{58} is used to reduce the terms $2a$ and $3a$. I.e.

$$(a+3)\cdot(a+2)=a\cdot a+a\cdot 2+3\cdot a+3\cdot 2=a^2+2a+3a+6= a^2+5a+6$$

Hence, the two expressions describing the area of the figure is the expression a^2+5a+6 and $(a+3)\cdot(a+2)$.



Mark with a cross the two calculation expressions that describes the area of the blue figure.

a^2+5a+6	☐
$4a+10$	☐
$2a^2+3a+2a+6$	☐
$2a+5a+6$	☐
$(a+3)\cdot(a+2)$	☐
$6a+6$	☐

Table 37: Exercise 39 in the FSA from May 2013

The last technique in the category brackets is τ_{66}. The technique is to remove a bracket inside a bracket. The technique is only used in one of the analysed exercises shown below.

Reduce $x(x(x(x+1)+1)+1)+1$.

Table 38: Exercise 162 in (Carstensen et al., 2005, p. 320)

To solve this exercise τ_{66} is used twice to remove the inner bracket. In addition, the distributive law i.e. τ_3 is used. The reduction is as follows

$$x(x(x(x+1)+1)+1)+1 = x(x(x^2+x+1)+1)+1 = \\ x(x^3+x^2+x+1)+1 = x^4+x^3+x^2+x+1$$

6.1.14 *Squares*

These techniques deal with rewriting three expressions with symbols containing squares.
τ_{67} is the technique to rewrite expressions on the form $(a+b)^2$ to a^2+b^2+2ab. This technique can be used in the exercise in Table 6 and it gives that $(3Tr+9r)^2 = (3Tr)^2+(9r)^2+2\cdot 3Tr\cdot 9r$.
The technique to rewrite an expression on the form from τ_{67} in the opposite direction is called τ_{68}. The exercise in Table 7 can be solved in another way by applying τ_{68}. The value of the expression $a^2+2ab+b^2$ when $a=2$ and $b=-3$ can be calculated by first rewriting the expression to $(a+b)^2$. Then the values for a and b can be substituted to calculate the value which is $(2+(-3))^2 = (-1)^2 = 1$.
A third technique involving rewriting of symbols containing squares is τ_{69}. The technique is similar to τ_{67} apart from the fact that it is $(a-b)^2$ which can be rewritten to a^2+b^2-2ab.

$(a-2b)^2$

Table 39: Exercise 17 b) in (Silkeborg Gymnasium, 2014, p. 4)

The above exercise from the screening test can be solved by means of τ_{69}. The exercise however, is quite esoteric since there is no description of what the exercise is about. By applying τ_{69} followed by τ_{48} and τ_2 the expression can be expanded to

$$(a-2b) = a^2+(2b)^2-2\cdot a\cdot 2b = a^2+4b^2-4ab$$

τ_{70} is the last technique in the category of squares. The technique is to rewrite two numbers' sum multiplied by the same two numbers' difference to the square of the first number minus the square of the second number i.e. $(a+b)\cdot(a-b) = a^2-b^2$. This technique can be applied to multiply the brackets in the following exercise.
By applying technique τ_{70} the exercise is quickly solved since $(a+b)(a-b) = a^2-b^2$.

Multiply the brackets $(a+b)(a-b)$.

Table 40: Exercise 839 e) in (Clausen et al., 2005, p. 44)

6.1.15 *Rules for logarithms*

There are mainly three calculation rules for logarithms. The technique stating that the logarithm of a power is the exponent multiplied by the logarithm of the base is τ_{71}. It is useful in the exercise in Table 10, where it can be applied to the expression $\log\left(\frac{1{,}2}{0{,}75}^{x}\right)$ so that this is $x \cdot \log\left(\frac{1{,}2}{0{,}75}\right)$.
The other two techniques involving logarithms are τ_{72} and τ_{73} respectively. τ_{72} says that the logarithm of a product is the sum of the logarithm of the first factor and the logarithm of the second factor i.e. $\log(a \cdot b) = \log(a) + \log(b)$. τ_{73} states that $\log(a) - \log(b) = \log(\frac{a}{b})$. These techniques can be used in types of tasks where an equation of the form $b \cdot a^{x} = c \cdot d^{x}$ has to be solved by algebraic techniques. Thus, the exercise in Table 10 can also be solved by applying these techniques instead of τ_{14}. This requires first τ_{13} and then τ_{72}, τ_{71}, τ_{7}, τ_{73}, and τ_{56} is used. The calculations can be seen below.

$$\begin{aligned}
5,6 \cdot 1,2^{x} &= 83,6 \cdot 0,75^{x} \\
\log(5,6 \cdot 1,2^{x}) &= \log(83,6 \cdot 0,75^{x}) \\
\log(5,6) + \log(1,2^{x}) &= \log(83,6) + \log(0,75^{x}) \\
\log(5,6) + x(\log(1,2) &= \log(83,6) + x\log(0,75) \\
x\log(1,2) - x\log(0,75) &= \log(83,6) - \log(5,6) \\
x(\log(1,2) - \log(0,75)) &= \log\left(\frac{83,6}{5,6}\right) \\
x \cdot \log\left(\frac{1,2}{0,75}\right) &= \log\left(\frac{83,6}{5,6}\right) \\
x &= \frac{\log\left(\frac{83,6}{5,6}\right)}{\log\left(\frac{1,2}{0,75}\right)}
\end{aligned}$$

6.1.16 *Reading*

There are three more or less simple techniques involving reading. The first one is τ_{74} which is to read the values of a and b from the equation $y = ax + b$ of a line. The technique is for example used in the exercise in Table 15 where the line $y = 2x + 5$ can be drawn in a coordinate system to determine the y-value for $x = 5$. Before drawing the line the values for a and b must be determined from the equation of the line so that the slope of the line and the intersection with the y-axis is known.

Another technique is to read b as initial value i.e. τ_{75}. The technique is applied in the exercise in Table 1 in chapter 4. To give an interpretation of the number 49 in the equation $y = 17x + 49$ that describes the relationship between price and length of a taxi ride τ_{75} is used to interpret 49 as the initial value, i.e. the value for $x = 0$. This is then translated to the specific context so that 49 is interpreted as the initial fee for a taxi ride.
The last reading technique is τ_{76}. This technique concerns the reading of the y-value for $x = 0$ in a table. The technique was used in a single analysed exercise which is shown below.

> The table shows the coordinates for points on the straight line m.
>
> | | | | | | |
> |---|---|---|---|---|---|
> | x | 0 | 1 | 2 | 3 | 4 |
> | y | 0 | 0,5 | 1 | 1,5 | 2 |
>
> The equation for the straight line m is $y =$

Table 41: Exercise 45 in the FSA from May 2013

In order to determine the equation of the line the y-value for $x = 0$ is read from the table since this corresponds to the value of b in the equation $y = ax + b$ of the line. This gives $b = 0$. A way to determine the value of a is to use the formula for a that will be explained in the subsection 6.1.23. This is possible since the points lie on the same straight line. This gives the equation $y = 0,5x$ for the line m.

6.1.17 *Size factor*

This category contains one technique which is τ_{77}. The technique is to determine the 'size factor' between two fractions. This will be exemplified by the following example.

> The fraction below is extended or shortened. What must stand on the blank space?
>
> b) $\frac{12}{\quad} = \frac{6}{5a}$

Table 42: Exercise 130 b) in (Carstensen et al., 2005, p. 315)

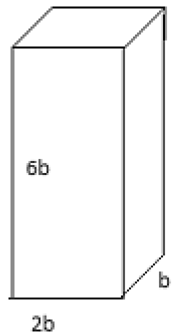
An approach to finding the solution to this exercise is by using τ_{77}. Then the size factor must be determined i.e. the number a such that $12 = a \cdot 6$. It is easy to see that $a = 2$. The expression that must stand on the blank space can be found by multiplying the size factor with the denominator $5a$. Hence, the expression is $2 \cdot 5a = 10a$.

6.1.18 *Sketch*

The only technique in this category is τ_{78} and it is to draw a sketch. It can be useful in exercises involving geometric figures to visualize the figure with the help of a sketch. It can be applied in the following exercise.

A closed box has a surface (including cover and bottom) of 1000 cm^2. It is twice as long as it is wide and six times as high as it is wide. What is the volume of the box?

Table 43: Exercise 254 in (Clausen et al., 2005, p. 327)



To solve the exercise τ_{78} is first used. This provides the figure shown to the right in the margin. From the figure and τ_{80} an equation expressing the surface area can be developed. The equation is $2 \cdot b \cdot 2b + 2 \cdot b \cdot 6b + 2 \cdot 6b \cdot 2b = 1000$.
By using the commutative and associative law for multiplication this can be reduced to $4b^2 + 12b^2 + 24b^2 = 1000$. The equation can be reduced further by τ_{58} to $40b^2 = 1000$. By using the techniques τ_7 and τ_{15} for equation solving the value for b is found to be $b = 5$. From this the volume of the box can be found by the use of τ_{80} again, which gives $V = 2 \cdot 5 \cdot 5 \cdot 6 \cdot 5 = 1500\ cm^3$.

6.1.19 *'Proportions calculus'*

The technique τ_{79} in this category concerns a specific type of task in one of the high school books. It has been difficult to classify what technique is used in this type of task and for that reason the technique is explained in broad terms as proportionality between quantities.

A quantity rises by 7% from the value K_1 to K_2. Determine K_2 expressed by K_1.

Table 44: Exercise 905 in (Clausen et al., 2005, p. 53)

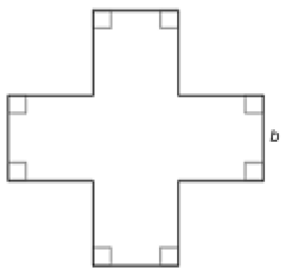
The above exercise can be solved by τ_{79}. The proportionality between the quantities K_1 and K_2 can be expressed by multiplying K_1 with the factor $1,07$ since this corresponds to a 7% rise in the value K_1. Hence, $K_2 = 1,07 \cdot K_1$.

6.1.20 *Standard formulas*

The technique in this category is τ_{80}. The technique is to use standard formulas such as area, circumference, volume, and surface of a figure or formulas for speed, average, and Pythagoras. It can be employed in the exercise in Table 37 to find the area of the figure which is a rectangle. The area of a rectangle is the length multiplied by the width. The length of the figure is $a+3$ and the width of the figure is $a+2$. Hence, the area of the figure can be expressed by $(a+3)\cdot(a+2)$.

6.1.21 *Figure*

This category contains two techniques which are only present in lower secondary school. The first on τ_{81} is to divide a figure into squares, rectangles, and triangles. This technique can be applied in the exercise in Table 37 to divide the figure into a square, two similar rectangles, and another rectangle. The area of each of these figures can be expressed by the product of their sides, so the total area of the figure can be found by adding the areas of each of these figures' areas.
The second technique is τ_{82} which is to count the sides of a figure. It is useful in the following exercise.



All the sides in the figure above have the length b. How big is the figure's circumference?

Table 45: Exercise 49 in the FSA from May 2014

The number of sides on the figure is counted so that an expression for the circumference can be determined. The number of sides is 12 and the length of all the sides is b, this implies that the circumference is $12b$. The techniques involved in solving the exercise were τ_{82} and τ_{80}.

6.1.22 *Rules for square roots*

There are two techniques involving rules for square roots that were identified in the analysis of the exercises. τ_{83} is that the square root of a product is the product of the square root of the first factor and the square root of the second factor i.e. $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$. τ_{84} is that the square root of a quotient is the quotient of the square root of the dividend divided by the square root of the divisor i.e. $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$. The two techniques are used in the exercise below.

Consider a figure pair consisting of a equilateral triangle and a square. Write up a relationship between the sides of the square and the triangle when the sum of the triangle's area and the sum of the square's area should be 100 cm^2.

Table 46: Exercise 207 in (Clausen et al., 2005, p. 68)

The side in the square is denoted a and the side in the equilateral triangle is denoted s. τ_{80} is used to determine an expression for the sum of the area of the two figures. The area of the square is a^2 and the area of the equilateral triangle is $\frac{1}{2}hs$ where h is the height in the equilateral triangle. It is possible to find an expression for h by using τ_{80} again. The equilateral triangle can be divided into two similar rectangular triangles with the length of the sides equal to h, s, and $\frac{1}{2}s$. Pythagoras' theorem gives that $h^2 + (\frac{1}{2}s)^2 = s^2$.

$$h^2 + (\tfrac{1}{2}s)^2 = s^2 \Leftrightarrow h^2 = s^2 - \tfrac{1}{4}s^2 \Leftrightarrow h^2 = \tfrac{3}{4}s^2 \Leftrightarrow h = \sqrt{\tfrac{3}{4}s^2} = \sqrt{\tfrac{3}{4}} \cdot \sqrt{s^2} = \tfrac{\sqrt{3}}{2}s$$

h can be isolated in the expression as shown above. First the techniques τ_7, τ_{48}, τ_{52}, τ_{58}, and τ_{15} are used. In the last rewritings τ_{83} is used to rewrite $\sqrt{\frac{3}{4}s^2}$ to $\sqrt{\frac{3}{4}} \cdot \sqrt{s^2}$ and τ_{84} is used to rewrite $\sqrt{\frac{3}{4}}$ to $\frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$.

The expression for h can be substituted into the expression for the area of the equilateral triangle. This gives $\frac{1}{2}hs = \frac{1}{2} \cdot \frac{\sqrt{3}}{2}s \cdot s = \frac{\sqrt{3}}{4}s^2$ where τ_{39} also has been used. The relationship between the square's and the triangle's sides can thus be described by the expression $100 = a^2 + \frac{\sqrt{3}}{4}s^2$.

6.1.23 *Formulas for the line equation*

Three techniques involving formulas for the equation of a line have been identified. τ_{85} is the technique to make use of the formula for the constant $a = \frac{y_2 - y_1}{x_2 - x_1}$ provided that two points (x_1, y_1) and (x_2, y_2) that lies on the line is known. τ_{86} is to use the formula $b = y_1 - a \cdot x_1$ to find the constant b when a point on the line and the constant a is known. The two techniques can be used to solve the exercise in Table 16 where the equation for the line passing through $(4, 4)$ and $(8, 5)$ should be found. First of all, a can be found by using τ_{85}. This gives

$$a = \frac{5-4}{8-4} = \frac{1}{4}$$

Then it is possible to find b by means of τ_{86} and τ_{38}. This gives

$$b = 4 - \frac{1}{4} \cdot 4 = 4 - \frac{4}{4} = 4 - 1 = 3$$

Thus, the equation of the line passing through $(4,4)$ and $(8,5)$ is $y = \frac{1}{4}x + 3$, as was the case when the exercise was solved by graphic techniques.
The third technique related to formulas for the line equation is τ_{87}. The technique is to use the formula $\Delta y = a \cdot \Delta x$ for linear growth. The formula can be rewritten to $y_2 = y_1 + a \cdot \Delta x$ and $x_2 = x_1 + \frac{\Delta y}{a}$ which is useful in some exercises in one of the analysed high school math books. The exercise in Table 17 can be solved by technique τ_{87} instead of the graphic techniques. ΔN can be determined by multiplying Δt which is 3 with $a = 3$. Thus, $\Delta N = 3 \cdot 3 = 9$.

6.1.24 *Compare*

The technique τ_{88} is about comparing values for slopes. In the exercise in Table 22 the technique can be employed instead of the graphic techniques. It can be shown that the three points $A(1,4)$, $B(3,7)$, and $C(11,19)$ lie on the same straight line by calculating the slopes for two pairs of points and deciding if the values are the same by comparison.
$a_{AB} = \frac{7-4}{3-1} = \frac{3}{2}$
$a_{BC} = \frac{19-7}{11-3} = \frac{12}{8} = \frac{3}{2}$
Since the slope between the points A and B is the same as the slope between the points B and C it can be concluded that the three points lie on the same straight line.

6.1.25 *Missing brackets*

The last technique is τ_{89}. This technique is to discover a missing bracket in an algebraic expression. Only one of the analysed exercises from lower secondary school requires this technique, and none of the analysed exercises from the screening test or high school books require the technique.
Before presenting the exercise that uses this technique the context of the exercise will be explained. A CP is described in word. The CP is:

1. Choose a number.
2. Add 10.
3. Multiply by 3.
4. Subtract the number that you chose in line 1.
5. Divide by 2.
6. Subtract 15.

The exercise is shown above.
To solve the exercise the missing brackets should be found. Miriam

Four students from 9.A use the letter n to write a calculation expression which is meant to show the calculations of the prescription. Anton: $\frac{(n+10)\cdot 3-n}{2} - 15$ Miriam: $\frac{n+10\cdot 3-n}{2} - 15$ Haider: $(n+10)\cdot 3 - n : 2 - 15$ Rune: $((n+10)\cdot 3 - n) : 2 - 15$ Two of the students' calculation expressions are not in accordance with the prescription. Which two of the students' calculation expressions are not in accordance with the prescription? You should justify your answer.

Table 47: Exercise 5.3 in the FSA from May 2014

lacks a bracket around $n + 10$, since it is $n + 10$ that should be multiplied by 3. Haider lacks a bracket around $(n + 10) \cdot 3 - n$, since all of it should be divided by 2.
It is interesting to see that an exercise of this type is included in the FSA. This was the type of exercises that Bosch (2015) proposed to start the ERM for basic algebra with.

The techniques in the overall table are not disjoint in the sense that some of the techniques are overlapping. The techniques are overlapping because it is meaningful with regard to the purpose of the analyse which is to be specific and explicit about techniques related to school algebra in lower secondary school and high school.
The techniques are supposed to be read from left to right in the sense that the equal sign should not be considered symmetrically. An example of this is technique τ_3 the distributive law $a \cdot (b + c) = a \cdot b + a \cdot c$ which should be understood as making the rewriting from $a \cdot (b + c)$ to $a \cdot b + a \cdot c$ and not the other way around.

6.1.26 *Schema of all the techniques*

Categorization	Techniques	FSA	ST	stx
Laws	τ_1: Associative law for multiplication	X	X	X
	τ_2: Commutative law for multiplication	X	X	X
	τ_3: Distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$	X	X	X
Hierarchy	τ_4: $\cdot$ and $:$ before $+$ and $-$	X	X	X
	τ_5: Roots and powers before $+, -, \cdot, :$	X	X	X
	τ_6: Brackets before all other arithmetic operations	X	X	X
Equation solving	τ_7: $+, -, \cdot, :$ on both sides of the equality sign	X	X	X
	τ_8: Multiply crosswise $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$			X
	τ_9: Set two expressions equal to each other	X		X
	τ_{10}: Subtract two equations from each other	X		X
	τ_{11}: Substitution of algebraic expression	X		X
	τ_{12}: Zero-divisor law $a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$			X
	τ_{13}: Take logarithm on both sides of $=$			X
	τ_{14}: Multiply crosswise reverse $ad = bc \Rightarrow \frac{a}{b} = \frac{c}{d}$			X
	τ_{15}: $x^n = c \Rightarrow x = \sqrt[n]{c}$, $n \in \mathbb{R}$	X		X
Guess	τ_{16}: Try different values for the unknowns in equations	X	X	X
	τ_{17}: Guess at different factorizations		X	X
Substitute	τ_{18}: Substitute number	X	X	X
Graphic	τ_{19}: Draw straight line in coordinate system	X	X	X
	τ_{20}: Read x-value for given y-value	X	X	X
	τ_{21}: Determine intersection between lines	X	X	X
	τ_{22}: Read y-value for given x-value		X	X
	τ_{23}: Plot and connect points to a straight line	X	X	X
	τ_{24}: Read Δy			X
	τ_{25}: Draw line through point			X
	τ_{26}: Draw best straight line			X
	τ_{27}: Intersection with the y-axis	X	X	X
	τ_{28}: Slope of a straight line	X	X	X
	τ_{29}: Intersection with the x-axis			X
	τ_{30}: Coordinates on line	X	X	X
	τ_{31}: Min/max in endpoints			X
	τ_{32}: If a point is on a straight line			X
	τ_{33}: If 3 points is on the same straight line			X

Table 48: ERM showing the techniques in exercises related to basic algebra in the transition

Categorization	Techniques	FSA	ST	stx
Fractions	τ_{34}: Addition $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$	X		X
	τ_{35}: Subtraction $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$	X		X
	τ_{36}: Extend $\frac{a}{b}=\frac{ak}{bk}$	X	X	X
	τ_{37}: Shorten	X	X	X
	τ_{38}: Multiplication $a\cdot\frac{b}{c}=\frac{ab}{c}$	X	X	X
	τ_{39}: Multiplication $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$	X		X
	τ_{40}: Multiplication $\frac{ac}{bd}=\frac{a}{b}\cdot\frac{c}{d}$	X	X	X
	τ_{41}: Multiplication $\frac{ab}{c}=a\cdot\frac{b}{c}$	X		
	τ_{42}: Addition/subtraction $\frac{a\pm b}{c}=\frac{a}{c}\pm\frac{b}{c}$	X		X
	τ_{43}: Division $a:\frac{b}{c}=a\cdot\frac{c}{b}=\frac{ac}{b}$			X
	τ_{44}: Division $\frac{a}{b}:\frac{c}{d}=\frac{ad}{bc}$			X
	τ_{45}: Division $\frac{a}{b}:c=\frac{a}{bc}$			X
Exponents	τ_{46}: $a^m\cdot a^n=a^{m+n}$		X	X
	τ_{47}: $\frac{a^n}{a^m}=a^{n-m}$		X	X
	τ_{48}: $(a\cdot b)^n=a^n\cdot b^n$		X	X
	τ_{49}: $(a^n)^m=a^{n\cdot m}$			X
	τ_{50}: $\frac{1}{a^n}=a^{-n}$		X	X
	τ_{51}: $\frac{a^n}{b^n}=(\frac{a}{b})^n$			X
	τ_{52}: $(\frac{a}{b})^n=\frac{a^n}{b^n}$			X
Signs	τ_{53}: $(-)\cdot(+)=(-)$, $(+)\cdot(-)=(-)$, $\frac{(-)}{(+)}=(-)$, $\frac{(+)}{(-)}=(-)$	X	X	X
	τ_{54}: $(-)\cdot(-)=(+)$, $-(-a)=a$, $\frac{(-)}{(-)}=(+)$	X	X	X
	τ_{55}: $(-)\cdot(+)=(+)\cdot(-)$			X
Factorize	τ_{56}: $ab+ac=a(b+c)$	X	X	X
	τ_{57}: Resolve number into factors		X	X
Simplify	τ_{58}: Collect and reduce like terms	X	X	X
	τ_{59}: Cancel common factors	(X)	X	X
Rewrite	τ_{60}: Number to fraction $a=\frac{ab}{b}$	X		X
	τ_{61}: $x=1\cdot x$		X	X
	τ_{62}: $\frac{a}{c}=\frac{1}{c}\cdot a$	X		X
	τ_{63}: $a=\sqrt{a}\cdot\sqrt{a}$			X
Brackets	τ_{64}: Remove negative bracket $-(a+b)=-a-b$	X	X	X
	τ_{65}: Multiply 2 brackets $(a+b)(c+d)=ac+ad+bc+bd$	X	X	X
	τ_{66}: Remove bracket in bracket			X
Squares	τ_{67}: $(a+b)^2=a^2+b^2+2ab$			X
	τ_{68}: $a^2+b^2+2ab=(a+b)^2$	X		
	τ_{69}: $(a-b)^2=a^2+b^2-2ab$		X	X
	τ_{70}: $(a+b)\cdot(a-b)=a^2-b^2$			X

Categorization	Techniques	FSA	ST	stx
Rules for logarithm	τ_{71}: $\log(a^x) = x \cdot \log(a)$			X
	τ_{72}: $\log(a \cdot b) = \log(a) + \log(b)$			X
	τ_{73}: $\log(a) - \log(b) = \log(\frac{a}{b})$			X
Reading	τ_{74}: a and b from $y = ax + b$	X	X	X
	τ_{75}: b as initial value		X	X
	τ_{76}: y-value for $x = 0$ in table	X		
Size factor	τ_{77}: Determine the size factor			X
Sketch	τ_{78}: Draw a sketch			X
'Proportions calculations'	τ_{79}: Proportionality between quantities			X
Standard formulas	τ_{80}: Use formulas for area, circumference, volume, surface, speed, average, Pythagoras	X		X
Figure	τ_{81}: Divide figure into squares, rectangles, and triangles	X		
	τ_{82}: Count sides on the figure	X		
Rules for square roots	τ_{83}: $\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$	X		X
	τ_{84}: $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$			X
Formulas for line equation	τ_{85}: $a = \frac{y_2 - y_1}{x_2 - x_1}$	X	X	X
	τ_{86}: $b = y_1 - a \cdot x_1$	X	X	X
	τ_{87}: $y_2 = y_1 + a \cdot \Delta x$, $x_2 = x_1 + \frac{\Delta y}{a}$, $\Delta y = a \cdot \Delta x$			X
Compare	τ_{88}: Compare values for slopes			X
Missing brackets	τ_{89}: Discover missing brackets	X		

7 PROBLEMATIC TECHNIQUES AND TECHNIQUES IN THE MATHBRIDGE PROJECT

7.1 OBSERVED CHALLENGES IN THE SCREENING TEST FROM SILKEBORG

Silkeborg high school has made statistics of the students' overall achievements in the screening test from the results from the screening test in 2014. The following section is based on this unpublished statistics. The statistics indicates the maximum score for each exercise, the average score for all students, and the quotient 'average/maximum'. The quotient is marked red if it is below 0.5, and it is marked yellow if it is 0.5 or between 0.5 and 0.75. A total of 14 exercises out of 26 have a the colour red or yellow in the quotient box. The challenges in the test that will be outlined in this section are exercises within the definition of basic school algebra that has a quotient value, which is marked red or yellow. In addition, the overview of the results contains two bar graphs that show the distribution of the students' total score in the test and in relation to which level they are going to have mathematics in high school.

The first problematic exercise containing algebra from the screening test is exercise 11, which consists in solving two linear equations. The quotient is 0.56 meaning that the students on average got 56% of the maximum points in the exercise. The techniques involved in the exercise are the distributive law τ_3, the techniques τ_4 and τ_6 which relate to the order of arithmetic operations, to add/subtract/divide/multiply by a number on both sides of the equality symbol τ_7, multiplication of a positive and a negative quantity τ_{53}, collection and reduction of similar terms τ_{58}, and multiplication of a fraction τ_{38}. In addition to that, the graphic techniques τ_{19} and τ_{21} may also come in to play, if the students choose a graphic approach to the exercise, or the trial-and-error technique τ_{16} if the students try to guess the solution.

Exercise 12 is also problematic with a quotient equal to 0.59. The exercise deals with explaining what it means that a specific linear equation has a given solution. The techniques that can be used in this exercise are substitution of a number τ_{18} and order of arithmetic operations τ_4, or to draw lines τ_{19} and read off the intersection τ_{21}, or to solve the equation by τ_7 and τ_{58}.

Another and even more problematic exercise for the students is number 16, which is four exercises concerning reduction. The quotient is only 0.45 i.e. the students have on average got less than half of the possible points. The techniques in the exercise are the distributive law τ_3 and the commutative law for multiplication τ_2, the order of arithmetic operations τ_4 and τ_6, simplifying terms τ_{58} and factors τ_{59}, and removing a negative bracket τ_{64}. The last reduction exercise has the potential that the students use various rules for exponents in interaction with extension τ_{36} and shortening of fractions τ_{37}. The techniques involving exponents will not be mentioned further, since they are not a part of the common techniques in lower secondary school and high school as seen in the schemas in subsection 6.1.26.

Exercise 17 is the most problematic in the test. A quotient of only 0.18 indicates that students have serious problems with the techniques associated with the exercise which is about expanding an algebraic expression. The exercise is somewhat esoteric, since there is no text explaining what the exercise is about. The techniques required for the exercise are the commutative law for multiplication τ_2, the three techniques for order of arithmetic operations i.e. τ_4, τ_5 and τ_6, calculation of signs τ_{53} and τ_{54}, simplifying like terms τ_{58}, multiplying two brackets τ_{65}, calculating the square of a two-termed quantity τ_{69}, and the exponent rule τ_{48}.

Yet another difficult algebraic exercise in the screening test is exercise 18 with the quotient 0.5. The exercise is to factorize two algebraic expressions. The techniques that is used in the exercise is the factorization techniques τ_{56} (also for more than one factor) and τ_{57}, the commutative law for multiplication τ_2, the guess of a factorization τ_{17}, and to rewrite x to $1 \cdot x$ i.e. τ_{61}.

Exercise 20 which is to calculate the value of an algebraic expression for given values for the unknowns was also a bit difficult for the students in the screening test, since the quotient is only 0.65. The techniques involved in this exercise is first of all substitution of a number τ_{18} and the order of arithmetic operations τ_4 and τ_5. In addition, the technique to multiply a positive and a negative number τ_{53} is in play.

Exercise 24 in the screening test is also problematic. The quotient is 0.58. The exercise consists of two partially different exercises, which are to read off the values a and b for a straight line $y = ax + b$ in a coordinate system and to draw a straight line given by $y = -x + 5$ in a coordinate system. The first part of the exercise can be solved by the graphic techniques to read off the intersection with the y-axis τ_{27} and to read off the slope from the graph τ_{28} and substitute the values τ_{18} into $y = ax + b$. The first part of the exercise can also be solved by reading off two coordinates on the line τ_{30} and using the formulas for a τ_{85} and for b τ_{86}. The second part of the exercise can be solved by reading a and b from the equation of the line τ_{74} and drawing the line τ_{19} or substituting values, τ_{18}, for x into the equation for the line to find y such that the points can be plotted and connected to a straight line i.e τ_{23}. It can be useful to use τ_{61} to rewrite $-x$ to $-1 \cdot x$ such that it is easier to see the value of a.

The last problematic exercise in the screening test, with a quotient equal to 0.67, is exercise 25. It is about giving an explanation of the constants a and b's meaning in a specific linear relationship in a real world context, which in this case is a ride in a taxi. The associated techniques are to read b as initial value τ_{75}, substitute number τ_{18}, order of arithmetic operations τ_4, and draw line τ_{19}.

A total of 8 of the 9 algebraic exercises in the screening test are problematic, of which especially two of the exercises are very problematic. Hence the tendency is that the algebraic exercises in the test in particular are difficult for the students compared to the rest of the exercises in the test.

7.2 OBSERVED CHALLENGES IN THE FSA

The data in this section concerns the success rates for exercises in the FSA from May 2013 and 2014. The data is based on an excerpt from an unpublished report of Niels Jacob Hansen, who is chairman of the examination board in leaving examinations in mathematics in lower secondary school.
First a description of the problematic exercises related to algebra from the FSA in May 2013 will be described. Then the same will be done for the exercises in the FSA from May 2014.
The FSA is divided into a one-hour test with only writing and drawing tools such as ruler and compass as aids and a three-hour test with all the aids that the students has used in the class. The first part is testing mathematical skills and the second part is testing the students abilities within problem solving (Kvalitets- og Tilsynsstyrelsen: Center for Prøver, Eksamen og Test, 2014, p. 5, 10, 12).

In May 2013 the average grade for the test in mathematical skills was $6,94$. The problematic exercises related to algebra in this test included number 21, which was a calculation involving Pythagoras. Only $35,1\%$ of the students managed to solve the exercise correctly. The exercise involve the techniques of using Pythagoras τ_{80}, substituting a number τ_{18}, subtracting the same quantity on both sides of the equality sign τ_7, and taking the square root of number τ_{15}. It is also possible to guess the solution by trial and error i.e. τ_{16}.

Also number 39, which is about recognizing two algebraic expressions that describe the area of a figure, had a low success rate of $17,6\%$. To solve the exercise the techniques of using the formula for the area of a rectangle τ_{80}, divide figure in squares τ_{81}, multiply two brackets together τ_{65}, commutative law for multiplication τ_2, and reduce like terms τ_{58} can be used. It is possible to avoid some of these techniques if the strategy to reduce the expressions is chosen instead of looking at the figure. Then only τ_{65}, τ_2, and τ_{58} are required.

Number 45 was answered correctly by only 35, 8% of the students. The exercise involves establishing a formula for a function given by a table. The techniques to read off the y-value for $x = 0$ in the table τ_{76}, to determine the slope of the line by using the formula for a τ_{85}, and to insert the values in the equation of the line τ_{18} are used or the techniques to plot and connect the points from the table in a coordinate system τ_{23}, to find the line's intersection with the y-axis τ_{27}, and read off the slope of the line τ_{28} can be used.

Number 47 has a low proportion of correct answers of 17, 9%. The exercise is to calculate the algebraic expression that has the biggest value for a given value of the variable. The exercise can be solved by substituting the value of the variable into the expression τ_{18}, using the rules for sign τ_{53} and τ_{54}, and multiplying before adding and subtracting τ_4.
The rest of the algebraic exercises in the test in mathematical skills were three exercises in equation solving and a exercise about calculating the value of a simple expression. They all had a success rate above 50%.

The students' use of digital tools at the written exam in mathematics in 9th grade in May 2015 will be described before given an overview of the problematic exercises in the problem solving part of the written exams from May 2013 and 2014. The data come from an unpublished report similar to the reports from 2013 and 2014 described in the introduction to section 7.2 and the data is collected by the external examiners who have registered each student's use of digital tools. This has been done by identifying which types of digital tools each student has used. The external examiners had the possibility of identifying the following types of digital tools:

* Writing tool such as Word
* Dynamic program for geometry such as GeoGebra
* CAS such as WordMat, MatematiKan, Smath, and (GeoGebra)
* Spreadsheets

It should be noted that the registration of CAS tools lacks consensus since not all external examiners have registered when GeoGebra has been used as a CAS tool.
From the registrations it is seen that 67, 2% of the students have used a digital writing tool. 39, 1% of the students have used a dynamic program for geometry, 7, 1% of the students have used a CAS tool, and 24, 2% of the students have used spreadsheets. The sum of the percentages are higher than 100% since several students have used more than one of the digital tools. The percentage of students that have not

used IT is about 27% so about 73% of the students have used some kind of IT tool. Thus, the majority of the students used IT-tools to the exam. When an IT tool is used to solve an exercise related to algebra it may remove the potential for using algebraic techniques depending on the type of IT tool that is used. In the unpublished report it is furthermore noted that the percentage of students using some kind of IT tool has increased by 10% compared to last year. This indicates that the use of IT tools at the written exam in mathematics in 9th grade is becoming more prevailing.

The average grade in the test from May 2013 in problem solving was $6,04$. The success rates for each exercise are calculated by adding the percentage of correct answers with $0,5$ multiplied by the percentage of partially correct answers. Exercise 3.5 was problematic with a success rate of only $16,1$%. The exercise is to find a formula for a linear relationship. From the bar graph it can be seen that slightly more than 40% of the students did not answer the exercise. The techniques are to use the formula for speed τ_{80}, shorten a fraction τ_{32}, and substitute a number into the formula τ_{18}.

Exercise 6.4 had a success rate of $23,3$%. The exercise concerns making a simple proof for a relationship in sum-triangles. Slightly above 40% of the students skipped the exercise. The exercise require two algebraic techniques which are to collect and reduce like terms τ_{58} and factorizing τ_{56}. In addition, the students have to write up the algebraic expression before they use the algebraic techniques.

The data for the FSA from May 2014 only shows the success rates for the test in mathematical problem solving. From this it can be seen that all the algebraic exercises have a problematic success rate, since it is below 50% in all the exercises related to basic school algebra. Exercise 1.4 had a success rate of 45% where 36% of the students got maximal points in this exercise and 18% solved the exercise partially. The exercise can be solved by solving two equations in two unknowns by substituting an algebraic expression τ_{11}, collecting and reducing like terms τ_{58}, dividing by the same number on both sides of the equality sign τ_7, and substituting the value for one unknown to find the value of the other unknown τ_{18}.

In exercise 2.3 the students achieved a success rate of 30%. 37% of the students got 0 points for their solution, which indicates that the students struggled with this exercise. The exercise can be solved by solving an equation in one unknown by using τ_7 and reducing like terms τ_{58}. The students have to establish the equation on the basis of their knowledge about equiangular triangles.

Exercise 4.3 is about finding a formula for a linear function. The success rate is very low since it is only $17,5$% and only 10% of the students have got the maximum number of points. The techniques involved in the exercise are the distributive law τ_3, rules for signs τ_{53} and τ_{54}, and reducing like terms τ_{58}.

The two subsequent exercises are also problematic. Exercise 4.4 has a success rate of 33%. The exercise concerns equation solving. The techniques involved are either setting two expressions equal to each other τ_9, using τ_7 to isolate the unknown, and reducing like terms τ_{58} or draw the lines in a coordinate system τ_{19} and read the intersection between the lines τ_{21}. It is worth noting that the students had a spreadsheet table at their disposal which they instead may have used to solve the exercise.
The success rate of exercise 4.5 is 46%. In the exercise a claim that concerns when a sale generates profit has to be justified. It can be done by solving an equation with two variables, such that it is rewritten to one of the variables expressed in terms of the other, which require the same techniques as in exercise 4.3, but also τ_7 and τ_{37}.

Exercise 5.3 has a success rate of 49% corresponding to that almost half of the points in this exercise is on average achieved. The exercise is about identifying and justifying two wrong calculation expressions on the basis of a description of the order of the calculations. The technique is to discover two missing brackets i.e. τ_{89}.

The last exercise that needs to be mentioned is 5.4. The success rate is 26, 6% but it is worth noting that only 7% of all the students got maximum points. The exercise involves rewriting a calculation expression i.e. reducing it. The techniques used in the solution can be rewriting a fraction τ_{41}, the commutative law τ_2, and collecting and reducing like terms τ_{58}. But it is also possible to use τ_{60} to rewrite a number to a fraction, and τ_{35} to subtract two fractions in combination with τ_2 and τ_{58}.

The results of the above problematic techniques can be summarized as follows. The techniques concern the laws τ_2 and τ_3, order of the four basic arithmetic operations τ_4, equation solving techniques including τ_7, τ_9, τ_{11}, and τ_{15}, substitution of numbers τ_{18}, graphic techniques including τ_{19}, τ_{21}, τ_{23}, τ_{27}, and τ_{28}, fractions including τ_{35}, τ_{37}, and τ_{41}, signs τ_{53} and τ_{54}, factorizing τ_{56}, collecting and reducing like terms τ_{58}, number to fraction τ_{60}, multiplying two brackets τ_{65}, read y-value for $x = 0$ in table τ_{76}, using standard formulas τ_{80}, diving figure τ_{81}, using the formula for the slope of a line τ_{85}, and discover missing brackets τ_{89}. The majority of the techniques were also problematic in the screening test with exception of τ_{37} and obviously the techniques that are not tested in the screening test.

7.3 MATERIAL FOR THE MATHBRIDGE COURSE

The teachers from lower secondary school that participate in the MathBridge course have received some material from the course. A part of the material is different teaching courses that they can adjust and try out with their own classes. An elaboration of the different techniques that the teaching courses addresses will be outlined.

On of the teaching courses in algebra is 'Think of a number'. The idea is that the students first carry out the instructions on various numbers until they see a pattern. Then they write the instructions using letters. Finally, they try to reduce the algebraic expression they have written to show that a general pattern holds.
In the teacher's guide an overview of the different techniques that can be trained in different 'Think of a number' games are described. The techniques are also listed on the page 'Algebraic calculation rules' which shows a schema of techniques. The techniques in this schema are all the techniques in the category 'Laws', τ_4 and τ_5 in the category 'Hierarchy', reducing like terms i.e. τ_{58}, techniques for sign τ_{53} and τ_{54}, techniques for brackets τ_{64} and τ_{65}, factorizing τ_{56}, the techniques for squares i.e. τ_{67}, τ_{69}, and τ_{70}, most of the techniques for fractions discovered in the analysis as existing in algebra in lower secondary school but also dividing a fraction by a number i.e. τ_{45}, and all the techniques except τ_{51} for exponents.
The most interesting is that techniques for exponents, dividing a fraction by a number, and most of the techniques for squares are included. These techniques were not found in the analysis of the material for lower secondary school.

The second teaching course in algebra is about 'Number-boards'. A number-board is a board which shows the numbers 1 to 10 in the first row, the numbers 11 to 20 in the second row and so on. A cross is placed at a random place in the number-board. The students are then supposed to discover different properties between the numbers in the cross. This is done first by calculating with the numbers in the cross and placing the cross somewhere else and calculating again until they make a discovery. From this the students are supposed to show their discovery by calculating with letters and reducing the expression.
The techniques the students have to use in the reductions are τ_{64}, τ_{58}, τ_{59}, τ_{34}, τ_3, τ_{65}, and τ_{70}. In addition, the material opens up to that the students themselves try to make some discoveries in the number-board by looking at properties for other figures such as triangles, rectangles etc.

The other two teaching courses relate to linear relationships. The first one, 'Barbie Bungee Jump', is focusing on shifts in representations i.e. tables, graphs, expression in formulas, and words. The focus is also on algebraic techniques. A mind map shows the possible ways to work with the problems, and the associated techniques is described on the following page. From these the following techniques overlap with the techniques found in the analysis of this thesis: τ_7 is used to solve first degree equations on the form $c = ax + b$, τ_{23} to plot points in a coordinate system, τ_{26} to draw the best straight line, τ_{85} is used to find slope from two given points on the line, τ_{20} and τ_{22} to find x- or y-value on the graph when y or x is known, and to determine a and b from the graph by using the techniques τ_{27} and τ_{28}.

The second teaching course in linear relationship is 'Paint irregulars stars'. The techniques in this teaching course is based on a mind map. The techniques that overlaps with the analysis of this thesis are similar to the teaching course 'Barbie Bungee Jump'. But in addition, the techniques to divide a figure into known geometric objects τ_{81}, and to use standard formulas to find volume of a cylinder and areas of rectangles τ_{80} also come into play in this teaching course.

9 DISCUSSION

9.1 TECHNIQUES IN THE ERM

Many of the discovered techniques are overlapping in the sense that the techniques are discovered in both the material from lower secondary school and high school. Despite of that some problems are identified. First of all, it is doubtful if the exercises in the problem solving part of the written examinations in 9th grade require that the students use algebraic techniques since many students use IT-tools as help in these exercises as mentioned previously. Hence, there is the possibility that the algebraic techniques may not come into play even though that an exercise has an algebraic potential because the IT-tool solves the exercise.
In the part of the written examinations in 9th grade without any aids besides writing and drawing instruments some of the techniques discovered in the analysis may be irrelevant for solving the exercise, since it is stated in the guidelines for the examinations in mathematics in lower secondary school, that in exercises that require a result where variables should be used it is not necessary to reduce the expression (Kvalitets- og Tilsynsstyrelsen: Center for Prøver, Eksamen og Test, 2014, p. 11-12). In addition to that, pilot experiments involving digital examinations of this part of the written examination in mathematics in 9th grade has been carried out (Minesteriet for Børn, Unge og Ligestilling, 2013). It can have consequences for the techniques that are required for the students to solve the exercises. For instance the students may change their strategies for solving an exercise if no calculations on paper are carried out. In solving a simple linear equation without paper and pencil it is likely that many students choose the undesirable trial-and-error approach.

Some of the techniques that are not identified in both lower secondary school and high school are surprising. An example is the zero-divisor law τ_{12} stating that if a product is zero then at least one of the factors is zero. The technique was only present in high school. This can be explained by the fact that the factorize technique τ_{56} was only present in a few of the exercises in lower secondary school. It was only in one of the exercises that the technique was absolutely necessary (Exercise 6.4 FSA from May 2013). This technique is connected to the zero-divisor law as it is often used for solving equations that are equal to zero after factorizing the expression on the other side of the equal sign.
It was also surprisingly to discover that none of the techniques in-

volving exponents appeared in lower secondary school. It would especially make sense that τ_{46} and τ_{48} were a part of the used algebraic techniques in lower secondary school since this could help the students in understanding that the expression $x^2 \cdot x^2$ is not equal to x^2 and that $(3T)^2$ is not $3T^2$.
The technique to cancel common factors only appeared in one of the exercises in the FSA but it was not necessary to apply the technique since τ_{41} could be used instead. It came as a surprise that this technique and the technique to rewrite x to $1 \cdot x$ were not practised in basic algebra exercises in lower secondary school. To rewrite x to $1 \cdot x$ can be very useful in several contexts. However, the technique was only found in high school and the screening test.

It is possible that the lacking techniques in lower secondary school mentioned above appear in exercises related to basic algebra in mathematics books for 9th grade. In addition to that, it should be kept in mind that a much larger part of the analysed exercises are from high school. This may explain some of the reasons for some techniques not being found in the lower secondary school.

9.2 CHALLENGES IN BASIC ALGEBRA IN THE TRANSITION

From the examination of the students' answers to the pilot test some common challenges related to basic algebra in the transition from lower secondary school to high school appeared. One of these challenges was the students problems with the equal sign which was very noticeably. Some students consider the equal sign in equation solving as an operator where the result is on the right hand side of the equal sign. An example is the student that used τ_7 to rewrite $2 = \frac{1}{4}x + 3$ to $\frac{1}{4}x + 3 = 2$. This agrees with Kieran (1981)'s findings.
The equal sign also caused other problems related to the different meanings that the sign can have. The equal sign acts differently dependent on the context. It can be defining such as $f(x) = 2x + 2$, be an identity such as $2x + 3 = 2(x + \frac{3}{2})$, or describe an equation $2x + 3 = -3x + 7$. In solving an equation a student arrived at the wrong identity $\frac{9}{4} = 3$. However, the student did not notice it since the student kept using the equation solving technique τ_7 at the identity. This points to that ambiguous symbols such as the equal sign are problematic for some students in the transition. This can be supported by the misconceptions that some students have when multiplying by a negative number such as in $2 \cdot 2 \cdot (-3)$. In the example the minus sign should be understood as a sign in front of 3 meaning the inverse with regard to $+$, but some students mistake it for an arithmetical operation such that they calculate $2 \cdot 2 \cdot (-3) = 4 - 3 = 1$.
The above observations indicate that some students find the syntax difficult in algebra since the notation, structure, and algebraic rules

for forming correct algebraic expressions, equations and so on do not make sense for them. The syntax in mathematics can be compared to learning a language where there are grammatical rules for construction of sentences and conjugate verbs. In order to learn and understand the syntax the form and meaning i.e. the semantics is an important part. The interaction between semantics and syntax is necessary in order to make the algebraic techniques in rewritings meaningful for the students.
In this connection some implicit conventions exist in algebra. An example is that $3x$ is written instead of $x3$ or $3 \cdot x$. People familiar with algebra know that this convention exist but this is not obvious for people that are not familiar with algebra. These conventions are hard to explain since they build on a common frame of understanding for people within the area of algebra and they are used implicit.

Another common characteristic in the students' answers to the pilot test is their difficulties with technique τ_{58}. A large part of the students collect and reduce terms that are not alike such as $3x + 9 = 9x$ and some students collect and reduce like terms in a very wrong way such as $5x - x = 5$. This is definitely something that causes problems for the students in the transition between the two educational institutions. In the same way, multiplication of variables or unknowns is very problematic for a large group of students who participated in the pilot test. The product of x and x is for some students x or $2x$. Likewise ax^n is for some students $a^n x$ or anx. It is necessary to focus on these misconceptions since they obstruct the way for the students' understanding of a great part of algebra.

It is interesting that some students made some algebraic rewritings in the exercise where they had to calculate the value of an algebraic expression for given values of the variables. The students just ignored the given values of the variables. This can be explained by the students in question have an understanding of the teacher's or institution's expectation that does not agree with the reality. Hence, the didactic contract is broken since the mutual expectations between the teacher and the student is different. Chevallard (1985) has an example of the same type.

> Example: to factorise $4x^2 - 36x$, a student writes:
>
> $$4x^2 - 36x = 4x^2 - 2 \cdot 2x \cdot 9 + 9^2 - 9^2$$
> $$(2x - 9)^2 - 9^2$$
> $$(2x - 9 + 9)(2x - 9 - 9)$$
> $$2x(2x - 18)$$ (Chevallard, 1985)

In the example the student makes an unnecessary rewriting as the expression is rewritten to a square. This is probably done because the student sees a separate value in making the rewritings. The teacher on the other hand will see it as misunderstood algebraic rewritings.

Thus, the student's and the teacher's expectations do not agree.

Some concluding remarks about the challenges in basic algebra in the transition in relation to the pilot test should be pointed out. First of all, some of the exercises in the pilot test turned out to show misconceptions about some of the techniques that were not supposed to be tested in the concerned exercise. Furthermore, the intermediate results provided useful information about the students' misconceptions but in some cases it was not enough because the ideas behind the intermediate results were difficult to follow. In these cases, interviews with the students could have provided information about the patterns of thought that the individual student has and thus contributed to discover the misconceptions.

9.3 THE MATHBRIGDE AND OTHER INTIATIVES

A collaboration between the educational institutions is necessary in order to address the transition problems in mathematics from lower secondary school to high school. In several of the previous research studies the high school teachers point out that the students are not prepared for mathematics in high school since the majority of the students lack basic skills. A report published recently about the role and need for further development of mathematics in high school supports this. One of the conclusions of the report is that the high school teachers experience that the lower secondary school students starting in high school lack skills in simple equation solving and reduction of expressions involving symbols (Jessen et al., 2015, p. 16).
The MathBridge project is an example of a collaboration between lower secondary school and high school teachers. The aim of the project is to minimize the challenges associated with the transition by preparing the students better for the transition. Unfortunately the project only addresses how the lower secondary school teachers in mathematics can prepare the students better for the transition and not how the high school teachers in mathematics can meet the students in a better way. In spite of this, the project has its advantages.
First of all, the Mathbrigde project involves teachers from both high school, lower secondary school, and teacher education in the development and the carrying out the course for the lower secondary mathematics teachers. They can provide useful information about the practices at the different educational institutions and the challenges that they experience. In this way, they gain insight into the reality at each institutions which may destroy prejudices about the different practices.
Furthermore, common teacher knowledge about the difficulties in the transition will be identified at the course and ideas for teaching sequences in algebra and modelling will be shown. On the basis of this, the lower secondary teacher will hopefully adjust the teaching

sequences and use them in their teaching. The visits in the teaching of these teaching sequences by members of the project group will contribute to that some of the teaching sequences will be tried out and that common knowledge about challenges are developed.
The course can then be improved on the basis of the common knowledge and a larger part of lower secondary school teachers can participate in an adjusted course if the course turns out to help addressing the transition problems. Thus, the MathBridge project is an initiative with the potential of helping many students in the transition to high school.

Experienced high school teachers can try to minimize the challenges in basic algebra in the transition from lower secondary school to high school by using strategies from projects other than the MathBridge project. These strategies include reading strategies for mathematical texts with the purpose of getting the students to understand the symbols in the text. Another study suggests that changing the teaching from the standard form of blackboard teaching by the teacher followed by exercises being solved by the students may motivative some of the boys. This could for example be done by inquiry based teaching such as to find a winning strategy for a particular game.

10 CONCLUSION

The transition in mathematics from lower secondary school to high school is for many students problematic. This is supported by several research studies where one of them points out that algebra is especially difficult for the students in the transition. It was evident that the students had problems with the exercises related to basic algebra in the screening test and FSA based on the success ratios. Furthermore, the students' answers to the exercises in the pilot test showed that a great part of the students lacked the necessary algebraic techniques. In 30 of the 41 exercises less than half of the students gave the correct answers and in 16 of the 41 exercises less than one third of the students solved the exercise correctly.

One of the purposes of this thesis was to identify the algebraic techniques in the transition from lower secondary school to high school. In order to determine the overlapping techniques an ERM was build. The ERM of the material showed that some of the algebraic techniques in the transition were present in both institutions which should lead to the expectation that the students master these techniques when they start in high school. However, statistics about the use of IT-tools in the written examinations in 9th grade indicate that an increasing number of students use IT-tools. This may result in some of the algebraic exercises being unnecessary. The majority of the remaining techniques appeared only in high school and thus, a few of the techniques were only present in lower secondary school.

The ERM was used to construct a pilot test with the purpose of testing the overlapping algebraic techniques in the transition. A few of the exercises turned out to be unsuitable for detecting misconceptions and are therefore recommended to be removed from the test. The greater part of the exercises had the desired effect and showed different misconceptions among the students. The misconceptions were of varying degree since some of them were more severe than others such as the misconceptions concerning τ_{58} and the use of ambiguous symbols.

The MathBridge project tries to address the transition problems from lower secondary school to high school in mathematics. In the project algebra and modelling are in focus in the course for the mathematics teachers in lower secondary school. The purpose of the course is that the lower secondary teachers make the transition for their students to high school easier by teaching in algebra and modelling. My findings showed that this is needed since many misconceptions regarding algebraic techniques were present. It could be valuable to bring into focus the lower secondary teachers use of IT-tools in their teaching to

see in which contexts the IT-tools is used.
In a further study it could be interesting to examine the use of IT-tools in algebraic exercises and identify the possible advantages that it can give.

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APPENDIX A

A.1 TABLES OVER TECHNIQUES

The tables in this section show the identifies types of tasks and techniques. The abbreviations that are used in the column 'Sources' are the following:

FSA: The part of the written examinations in Danish 9th grade where no aids are allowed.

PL: The part of the written examinations in Danish 9th grade where aids are allowed.

ST: The screening test from Silkeborg high school.

GG: The high school mathematics book 'Gyldendals Gymnasiematematik C Arbejdsbog.

MATC: The high school mathematics book 'MAT C stx'.

Two of the tables differ from the other tables. The two tables show techniques associated with modelling. The tables was used at workshop 2 in the MathBridge project to show the project group the techniques in exercises related to modelling which is one of the focuses in the MathBridge project. Here, modelling should be understood as creating a mathematical model to describe a non-mathematical reality, i.e. to solve a "real-world" problem. The reason for this understanding of the concept of modelling is because it was the understanding that appeared in workshop 1.

Table 53: Techniques in reduce tasks

	Techniques	Examples	Sources
Laws	Commutative law for multiplication	$x \cdot 3 = 3 \cdot x$	(PL), ST, GG, MATC
	Distributive law	$x \cdot (3 + x) = x \cdot 3 + x \cdot x$	ST, MATC
Hierarchy	Order of arithmetic operations	$x \cdot (3 + x) - 3x = x \cdot 3 + x \cdot x - 3x$	(Pl), ST, GG, MATC
Fractions	Addition	$\frac{9}{6x} + \frac{5}{6x} = \frac{9+5}{6x} = \frac{14}{6x}$	MATC
	Subtraction	$\frac{5}{10a} - \frac{2}{10a} = \frac{5-2}{10a} = \frac{3}{10a}$	MATC
	Extend fraction	$\frac{3}{2x} = \frac{3 \cdot 3}{2x \cdot 3} = \frac{9}{6x}$	ST, MATC
	Shorten fraction	$\frac{14}{6x} = \frac{14:2}{6x:2} = \frac{7}{3x}$	ST, GG, MATC
	$\frac{ab}{c} = a \cdot \frac{b}{c}$	$\frac{m \cdot 6}{3} = m \cdot \frac{6}{3}$	(PL)
	$\frac{a \cdot b}{c \cdot d} = \frac{a}{c} \cdot \frac{b}{d}$	$\frac{g \cdot f^2 \cdot b^3}{g^2 \cdot b^2 \cdot f} = \frac{g}{g^2} \cdot \frac{f^2}{f} \cdot \frac{b^3}{b^2}$	GG
Exponents	$a^m \cdot a^n = a^{m+n}$	$p^2 \cdot p^{-1} \cdot c \cdot q^2 \cdot q^{-1} = p^{2+(-1)} \cdot c \cdot q^{2+(-1)}$	ST, GG
	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{a^4 \cdot b^3}{a^2 \cdot b} = \frac{a^4}{a^2} \cdot \frac{b^3}{b} = a^{4-2} \cdot b^{3-1}$	ST, GG
	$a^{-n} = \frac{1}{a^n}$	$\frac{a^4 \cdot b^3}{a^2 \cdot b} = a^4 \cdot b^3 \cdot a^{-2} \cdot b^{-1}$	ST, GG
	$(a \cdot b)^n = a^n \cdot b^n$	$\frac{A^2 \cdot B \cdot C^3}{(A \cdot B \cdot C)^2} = \frac{A^2 \cdot B \cdot C^3}{A^2 \cdot B^2 \cdot C^2}$	GG, MATC
Sign	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$-4 \cdot 2s = -8s$	ST, MATC
	$(-) \cdot (-) = (+)$	$-4 \cdot (-5) = 20$	MATC
Factorize	Dissolve number in factors	$4 = 2 \cdot 2$	GG
	Factorize	$\frac{A \cdot b + A \cdot c}{b \cdot M + c \cdot M} = \frac{A(b+c)}{M(b+c)}$	GG
Simplify	Reduce like terms	$8 + 5x - 3x = 8 + 2x$	(PL), ST, GG, MATC
	Cancel common factors	$\frac{x \cdot y \cdot z}{y \cdot z} = x$	(PL), ST, GG
Brackets	Remove negative bracket	$-(x - 3x^2) = -x + 3x^2$	ST, MATC
	Remove bracket in bracket	$x(x(x(x + 1) + 1) + 1) + 1 = x(x(x^2 + x + 1) + 1) + 1$	MATC
	Multiply brackets	$(a - b)(a + b) = a \cdot a + a \cdot b - b \cdot a - b \cdot b$	MATC
Squares	Difference of squares	$(a - b)(a + b) = a^2 - b^2$	MATC

Table 54: Techniques in expand tasks

	Techniques	Examples	Sources
Laws	Associative law for multiplication	$a^2 + (2b)^2 + 2 \cdot 2b \cdot a = a^2 + (2b)^2 + (2 \cdot 2) \cdot ba$	ST, GG, MATC
	Commutative law for multiplication	$x \cdot 2 = 2 \cdot x$	(FSA), ST, GG, MATC
	Distributive law	$a(b + c) = ab + bc$	GG, MATC
Hierarchy	Order of arithmetic operations	$x \cdot x + x \cdot 2 - 1 \cdot x - 1 \cdot 2 = x^2 + x2 - x - 2$	(FSA), ST, GG, MATC
Exponents	$(a \cdot b)^n = a^n \cdot b^n$	$(2b)^2 = 2^2 \cdot b^2 = 4b^2$	ST, GG, MATC
Sign	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$-x \cdot x = -x^2$	ST, GG, MATC
	$(-) \cdot (-) = (+)$	$(-x) \cdot (-x) = x^2$	ST, GG, MATC
Simplify	Reduce like terms	$a^2 - ax - ax + x^2 = a^2 - 2ax + x^2$	(FSA), ST, GG, MATC
Brackets	Remove negative bracket	$-(ax + ay + x^2 + xy) = -ax - ay - x^2 - xy$	MATC
	$a(b + c)d = abd + acd$	$3(2 - a) \cdot 2 = 3 \cdot 2 \cdot 2 + 3 \cdot (-a) \cdot 2$	MATC
	Multiply brackets	$(x - 1)(x + 2) = x \cdot x + x \cdot 2 - 1 \cdot x - 1 \cdot 2$	(FSA), ST, GG, MATC
Squares	The square on a 2-termed quantity	$(10G + 7x)^2 = (10G)^2 + (7x)^2 + 2 \cdot 10G \cdot 7x$	GG
	The square on a 2-termed quantity	$(a - 2b)^2 = a^2 + (2b)^2 - 2 \cdot a \cdot 2b$	ST, GG, MATC
	Difference of squares	$(a - x)(a + x) = a^2 - x^2$	GG, MATC

Table 55: Techniques in factorize tasks

	Techniques	Examples	Sources
Laws	Commutative law for multiplication	$11Lp = 11pL$	ST, GG, MATC
	Distributive law	$3 \cdot (x + w) = 3x + 3w$	ST, GG, MATC
Guess	Trial-and-error	Show that a factorization is correct	ST, GG, MATC
Sign	$(-) \cdot (+) = (+) \cdot (-)$	$-11pK = 11p(-K)$	GG, MATC
	$(+) = (-) \cdot (-)$	$x \cdot y = (-x) \cdot (-y)$	GG, MATC
Factorize	Factorize *) More factors *) More terms	$3x + 3w = 3(x + w)$ $xaby + xabw = xab(y + w)$ $xa + xy + xw = x(a + y + w)$	ST, GG, MATC ST, GG, MATC GG
	Dissolve number in factors	$15 = 5 \cdot 3$	ST, GG, MATC
Rewrite	Rewrite	$x = 1 \cdot x$	ST, GG

Table 56: Techniques in expand tasks using the rules for squares

	Techniques	Examples	Sources
Laws	Associative law for multiplication	$2 \cdot 3 \cdot 9 \cdot T \cdot r \cdot r = (2 \cdot 3) \cdot 9 \cdot T \cdot (r \cdot r) = (6 \cdot 9) \cdot T \cdot r^2 = 54Tr^2$	GG
	Commutative law for multiplication	$-2a \cdot 3x = -2 \cdot 3 \cdot ax$	GG
Hierarchy	Order of arithmetic operations	$(4x)^2 + 1^2 - 2 \cdot 4x \cdot 1 = 16x^2 + 1 - 2 \cdot 4x \cdot 1 = 16x^2 + 1 - 8x$	GG
Exponents	$(a^n)^m = a^{n \cdot m}$	$(b^2)^2 = b^{2 \cdot 2} = b^4$	GG
	$(a \cdot b)^n = a^n \cdot b^n$ *) More factors	$(4w)^2 = 4^2 w^2 = 16w^2$ $(wPT)^2 = w^2 P^2 T^2$	GG GG
Sign	$(-) \cdot (+) = (-)$	$-2 \cdot 4w \cdot 9P = -72wP$	GG
Squares	The square on a 2-termed quantity	$(a + x)^2 = a^2 + x^2 + 2 \cdot a \cdot x$	GG
	The square on a 2-termed quantity	$(a - x)^2 = a^2 + x^2 - 2 \cdot a \cdot x$	GG

Table 57: Algebraic techniques for solving first degree equations tasks

	Techniques	Examples	Sources
Laws	Distributive law	$3(x+2) = 9 \Leftrightarrow 3x + 6 = 9$	GG, ST, FSA, MATC
	Associative law for multiplication	$7x = 2 \cdot 5 \cdot 14 = (2 \cdot 5) \cdot 14 = 10 \cdot 14 = 140$	MATC
	Commutative law for multiplication	$5 \cdot x \cdot (-2) = 5 \cdot (-2) \cdot x$	GG, MATC
Equations	$+, -, \cdot, :$ on both sides of $=$	$3x = 6 \Leftrightarrow 3x : 3 = 6 : 3$	ST, FSA, GG, MATC
	Multiply cross-wise	$\frac{15}{7} = \frac{2}{x} \Leftrightarrow 15x = 2 \cdot 7$	GG
Fractions	$a \cdot \frac{b}{c} = \frac{ab}{c}$	$3 \cdot \frac{1}{3} = \frac{3}{3} = 1$	FSA, GG, MATC
	Shorten fraction	$\frac{3}{6} = \frac{3:3}{6:3} = \frac{1}{2}$	GG
	Extend fraction	$\frac{x}{2} + \frac{x}{3} = \frac{x \cdot 3}{2 \cdot 3} + \frac{x \cdot 2}{3 \cdot 2}$	MATC
	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{4}{3} \cdot \frac{3x}{4} = \frac{4 \cdot 3x}{3 \cdot 4}$	FSA, GG, MATC
	Addition	$\frac{x}{2} + \frac{16}{2} = \frac{x+16}{2}$	GG, MATC
	Addition	$\frac{2x+3}{5} = \frac{2x}{5} + \frac{3}{5}$	FSA, GG
	Subtraction	$\frac{3x}{8} - \frac{32}{8} = \frac{3x-32}{8}$	FSA, GG, MATC
	Subtraction	$\frac{3-x}{-8} = \frac{3}{-8} - \frac{x}{-8}$	GG
	Rewrite $\frac{a}{c} = \frac{1}{c} \cdot a$	$\frac{x}{2} = \frac{1}{2}x$	FSA, GG, MATC
	Rewrite number to fraction $a = \frac{ab}{b}$	$\frac{x}{2} + 8 = \frac{x}{2} + \frac{8 \cdot 2}{2} = \frac{x+16}{2}$	FSA, GG, MATC
	$a : \frac{b}{c}$	$2 : \frac{4}{9} = \frac{2 \cdot 9}{4}$	GG
	$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$	$\frac{24}{5} : \frac{24}{5} = \frac{24 \cdot 5}{5 \cdot 24}$	GG
Signs	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$\frac{42}{-6} = -\frac{42}{6} = -7$	ST, GG, MATC
	$(-) \cdot (-) = (+)$	$\frac{-8}{-3} = \frac{8}{3}$	GG, MATC
Simplify	Collect and reduce like terms	$\frac{2x+3x}{6} = \frac{5x}{6}$	GG, ST, FSA, MATC
	Cancel common factors	$\frac{4 \cdot 3x}{3 \cdot 4} = x$	FSA, GG, MATC
Brackets	Remove negative bracket	$1 - (8 - q) = 1 - 8 + q$	GG, MATC
	Multiply two brackets	$(20 + x) \cdot (20 - x) = 20^2 - 20x + 20x - x^2$	MATC
Squares	$(a - b)(a + b) = a^2 - b^2$	$(20 - x) \cdot (20 + x) = 20^2 - x^2$	MATC

Table 58: Graphic techniques for solving first degree equations tasks

	Techniques	Examples	Sources
Laws	Distributive law	$3(x+2)=9 \Leftrightarrow 3x+6=9$	ST, FSA, MATC
Graphic	Draw line in coordinate system	Draw the line $3x+6=y$ in a coordinate system	ST, FSA, GG, MATC
	Read x-value for given y-value	Read the a-value for $y=9$	FSA, GG, MATC
	Intersection between lines	Read the intersection point between the two lines	GG, ST, FSA, MATC

Table 59: Trial-and-error techniques for solving first degree equations tasks

	Techniques	Examples	Sources
Guess	Trial-and-error	Insert number on x's place: $x=2$ for $3x=6$ since $3\cdot 2=6$	ST, FSA, GG, MATC
Hierarchy	Order of arithmetic operations	$2\cdot 2-4=4-4=0$	ST, FSA, GG, MATC

Table 60: Techniques in solving equations on form $b\cdot a^x = c\cdot d^x$ or $a^x+b = c$ for $a, c \neq 0$ tasks

	Techniques	Examples	Sources
Equations	$\cdot, :, +, -$ on both sides of $=$	$x\cdot \log(1,2)=\log(8,6) \Leftrightarrow x=\frac{\log(8,6)}{\log(1,2)}$	GG
	Take logarithm on both sides of $=$	$1,2^x = 8,6 \Leftrightarrow \log(1,2^x) = \log(8,6)$	GG
	Multiply crosswise reverse	$5,6\cdot 1,2^x = 83,6\cdot 0,75^x \Leftrightarrow \frac{1,2^x}{0,75^x}=\frac{83,6}{5,6}$	GG
Exponents	$\frac{a^x}{b^x}=\left(\frac{a}{b}\right)^x$	$\frac{1,2^x}{0,75^x}=\frac{83,6}{5,6} \Leftrightarrow \left(\frac{1,2}{0,75}\right)^x=\frac{83,6}{5,6}$	GG
Logarithms	$\log(a^x)=x\cdot\log(a)$	$\log(1,2^x)=\log(8,6) \Leftrightarrow x\cdot \log(1,2)=\log(8,6)$	GG
	$\log(a\cdot b)=\log(a)+\log(b)$	$\log(5\cdot 3^x)=\log(40) \Leftrightarrow \log(5)+\log(3^x)=\log(40)$	GG
	$\log(a)-\log(b)=\log(\frac{a}{b})$	$x\cdot\log(3)=\log(40)-\log(5) \Leftrightarrow x\cdot\log(3)=\log(\frac{40}{5})$	GG

Table 61: Techniques in solving equations on form $a^{bx+c} = d$ tasks

	Techniques	Examples	Sources
Laws	Distributive law	$(x-4)\cdot\log(2) = \log(10) \Leftrightarrow x\log(2) - 4\log(2) = \log(10)$	GG
Equations	$\cdot, :, +, -$ on both sides of $=$	$(x-4)\cdot\log(2) = \log(10) \Leftrightarrow x - 4 = \frac{\log(10)}{\log(2)}$	GG
	Take logarithm on both sides of $=$	$2^{x-4} = 10 \Leftrightarrow \log(2^{x-4}) = \log(10)$	GG
Fractions	$\frac{a}{b} : c = \frac{a}{b\cdot c}$	$x = \frac{\frac{\log(6)}{\log(3)}}{2} = \frac{\log(6)}{\log(3)\cdot 2}$	GG
	$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$	$x = \frac{\frac{\log(6)}{\log(3)}+7}{2} = \frac{\frac{\log(6)}{\log(3)}}{2} + \frac{7}{2}$	GG
Sign	$\frac{(-)}{(+)} = (-)$, $\frac{(+)}{(-)} = (-)$	$\frac{\log(9)}{-3\log(4)} = -\frac{\log(9)}{3\log(4)}$	GG
	$\frac{(-)}{(-)} = (+)$	$\frac{-12\log(4)}{-3\log(4)} = \frac{12\log(4)}{3\log(4)}$	GG
Logarithms	$\log(a^x) = x\cdot\log(a)$	$\log(2^{x-4}) = \log(10) \Leftrightarrow (x-4)\cdot\log(2) = \log(10)$	GG

Table 62: Algebraic techniques in solving equations on form $b\cdot x^a + c = d$, $a \in \mathbb{R}$ tasks

	Techniques	Examples	Sources
Equations	$\cdot, :, +, -$ on both sides of $=$	$5\cdot x^{1,74} + 8 = 16 \Leftrightarrow 5\cdot x^{1,74} + 8 - 8 = 16 - 8$	GG
	Take $\sqrt[a]{\ }$	$x^{1,74} = 16 \Leftrightarrow x = \sqrt[1,74]{16}$	FSA, GG, MATC

Table 63: Trial-and-error techniques in solving equations on form $b\cdot x^a + c = d$, $a \in \mathbb{R}$ tasks

	Techniques	Examples	Sources
Guess	Trial-and-error for $x^a = d$, $a \in \mathbb{N}$	$x^4 = 16$ then $x = 2$	FSA, GG

Table 64: Algebraic techniques in solving equations on form $ax^2 + b = cx^2 + d$, $a \neq 0$ tasks

	Techniques	**Examples**	**Sources**
Equations	$\cdot, :, +, -$ on both sides of $=$	$5 \cdot x^{1,74} + 8 = 16 \Leftrightarrow 5 \cdot x^{1,74} + 8 - 8 = 16 - 8$	GG
	Take $\pm\sqrt{}$		GG
Simplify	Reduce like terms	$96 = 25x^2 - x^2 = 24x^2$	GG

Table 65: Trial-and-error techniques in solving equations on form $ax^2 + b = cx^2 + d$, $a \neq 0$ tasks

	Techniques	**Examples**	**Sources**
Guess	Trial-and-error	$3x^2 = 12$ then $x = 2$	GG

Table 66: Algebraic techniques for solving a system of two equations tasks

	Techniques	Examples	Sources
Laws	Distributive law	$8m = 2(m+18) = 2m+36$	GG, MATC
	Associative law for multiplication	$6 \cdot 2 \cdot B = 12B$	MATC, PL
	Commutative law for multiplication	$y = x4 = 4x$	GG
Substitute	Substitute numbers	$x = 6$: $y = 5x = 5 \cdot 6 = 30$	GG, PL, MATC
Hierarchy	Order of arithmetic operations		GG, PL, MATC
Equations	$+, -, \cdot, :$ on both sides of $=$	$5x = 2x + 18 \Leftrightarrow 5x - 2x = 18$	GG, PL, MATC
	Multiply cross-wise	$\frac{-13+3b}{2} = \frac{2-4b}{5} \Leftrightarrow 5(-13+3b) = 2(2-4b)$	GG
	Substitute algebraic expression	$x = \frac{8-3y}{2}$: $3x - 4y = -5 \Leftrightarrow 3(\frac{8-3y}{2}) - 4y = -5$	GG, PL, MATC
	Subtract two equations from each other	$s = -3r$, $s = 8 + r$: $s - s = 8 + r - (-3r)$	GG, PL, MATC
	Set two expressions equal to each other	$y = 5x$, $y = 2x + 18$: $5x = 2x + 18$	GG, PL, MATC
Fractions	$a \cdot \frac{b}{c} = \frac{ab}{c}$	$3 \cdot \frac{8-3y}{2} = \frac{3(8-3y)}{2}$	GG, PL, MATC
	Subtraction	$\frac{8-3y}{2} = \frac{8}{2} + \frac{-3y}{2}$	GG
	Subtraction	$\frac{3x}{8} - \frac{32}{8} = \frac{3x-32}{8}$	PL
	Rewrite number to fraction	$3y = \frac{6y}{2}$	MATC
Signs	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$q = 3 \cdot (-5) = -15$	GG, MATC
	$(-) \cdot (-) = (+)$, $-(-a) = a$	$q = -2 \cdot (-1{,}5) = 3$	GG, MATC
Simplify	Collect and reduce like terms	$5x - 2x = 18 \Leftrightarrow 3x = 18$	GG, PL, MATC
Brackets	Remove negative bracket	$-(3a + b) = -3a - b$	GG, MATC
Squares	$(a - b)(a + b) = a^2 - b^2$	$(20 - x) \cdot (20 + x) = 20^2 - x^2$	MATC

Table 67: Graphic techniques for solving a system of two equations tasks

	Techniques	Examples	Sources
Graphic	Draw line in coordinate system	Draw the line $y = 5x$ and $y = 2x + 18$ in a coordinate system	GG, PL, MATC
	Intersection between lines	Read the intersection point between the two lines	GG, PL, MATC

Table 68: Trial-and-error techniques for solving first degree equations tasks

	Techniques	Examples	Sources
Guess	Trial-and-error	Insert numbers on the unknowns place	GG, MATC
Hierarchy	Order of arithmetic operations	$2 \cdot 2 - 4 = 4 - 4 = 0$	GG, MATC

Table 69: Techniques for isolating variables in formulas tasks

	Techniques	Examples	Sources
Laws	Distributive law	$P = ka(T_{inde} - T_{ude}) = kaT_{inde} - kaT_{ude}$	GG
	Associative law for multiplication	$3 \cdot \frac{1}{3}a = 1a$	GG
	Commutative law for multiplication	$A = \frac{1}{2}gh = \frac{1}{2}hg$	GG
Equations	$+, -, \cdot, :$ on both sides of $=$	$A = Lb \Leftrightarrow \frac{A}{L} = \frac{Lb}{L}$	GG
	Take $\pm\sqrt{}$	$\frac{O}{4\pi} = r^2 \Rightarrow r = \pm\sqrt{\frac{O}{4\pi}}$	GG
Fractions	$a \cdot \frac{b}{c} = \frac{ab}{c}$	$\frac{a}{\sqrt{3}}h = \frac{2h}{\sqrt{3}}$	GG
	Addition	$T_{inde} = \frac{p+kAT_{ude}}{kA} = \frac{p}{kA} + \frac{kAT_{ude}}{kA}$	GG
	$\frac{a}{b} : c = \frac{a}{bc}$	$b = \frac{V}{L} : h = \frac{V}{Lh}$	GG
	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{2 \cdot \sqrt{3}}{\sqrt{3} \cdot 2}$	GG
Signs	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$kA \cdot (-T_{ude}) = -kAT_{ude}$	GG
Roots	$\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$	$\sqrt{4T} = \sqrt{4} \cdot \sqrt{T}$	GG
	$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$	$\sqrt{\frac{4T}{\sqrt{3}}} = \frac{\sqrt{4T}}{\sqrt[4]{3}}$	GG

Table 70: Techniques for inserting a formula in another formula tasks

	Techniques	Examples	Sources
Laws	Associative law for multiplication	$T = M \cdot W \cdot W = M \cdot (W \cdot W) = M \cdot W^2$	GG
	Commutative law for multiplication	$O = \sqrt{3} \cdot 2h = 2\sqrt{3}h$	GG
Equations	$+, -, \cdot, :$ on both sides of $=$	$T = SW \Leftrightarrow \frac{T}{W} = \frac{SW}{W}$	GG
	Substitute algebraic expression	$S = MW, T = SW$: $T = MWW$	GG
Fractions	$a \cdot \frac{b}{c} = \frac{ab}{c}$	$O = 4 \cdot \frac{V}{a} + 2a^2 = \frac{4V}{a} + 2a^2$	GG
	$\frac{a}{b} : c = \frac{a}{bc}$	$\frac{V}{\pi} : r^2 = \frac{V}{\pi r^2}$	GG
	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{\sqrt{3}}{4} \cdot \frac{4}{3} = \frac{\sqrt{3} \cdot 4}{4 \cdot 3}$	GG
Simplify	Cancel common factors	$O = \frac{4aV}{a^2} + 2a^2 = \frac{4V}{a} + 2a^2$	GG
Exponents	$(ab)^n = a^n b^n$	$T = \frac{\sqrt{3}}{4} \cdot (\frac{2}{\sqrt{3}} \cdot h)^2 = \frac{\sqrt{3}}{4} \cdot (\frac{2}{\sqrt{3}})^2 \cdot h^2$	GG
	$(\frac{a}{b})^n = \frac{a^n}{b^n}$	$T = \frac{\sqrt{3}}{4} \cdot (\frac{2}{\sqrt{3}})^2 \cdot h^2 = \frac{\sqrt{3}}{4} \cdot \frac{2^2}{\sqrt{3}^2} \cdot h^2$	GG
Rewrite	$a = \sqrt{a} \cdot \sqrt{a}$	$O = \frac{3 \cdot 2}{\sqrt{3}} \cdot h = \frac{\sqrt{3} \cdot \sqrt{3} \cdot 2}{\sqrt{3}} \cdot h = 2\sqrt{3}h$	GG

Table 71: Techniques in determining what should stand on an empty place in the denominator or numerator of a fraction tasks

	Techniques	**Examples**	**Sources**
Laws	Associative law for $\cdot$	$24 \cdot 14 \cdot a \cdot a = (24 \cdot 14) \cdot (a \cdot a)$	MATC
	Commutative law for $\cdot$	$7a^2 \cdot x = 24a \cdot 14a \Leftrightarrow 7a^2 \cdot x = 24 \cdot 14 \cdot a \cdot a$	MATC
Equations	$+, -, \cdot, :$ on both sides of $=$	$60a = 6x \Leftrightarrow \frac{60a}{6} = \frac{6x}{6}$	MATC
	Multiply crosswise	$\frac{12}{x} = \frac{6}{5a} \Leftrightarrow 12 \cdot 5a = 6 \cdot x$	MATC
Fractions	Extend	$\frac{3}{7b} = \frac{}{28b^2}$: $\frac{3 \cdot 4b}{7b \cdot 4b} = \frac{12b}{28b^2}$	MATC
Simplify	Cancel common factors	$\frac{7a^2}{24a} = \frac{14a}{}$: $\frac{7a^2}{24a} = \frac{7a}{24}$	MATC
Size factor	Determine size factor	$\frac{12}{} = \frac{6}{5a}$ then $12 = 2 \cdot 6$ hence $2 \cdot 5a = 10a$	MATC
Exponents	$a^m \cdot a^n = a^{m+n}$	$7a^2 \cdot a^{-1} = 7a$	MATC
	$(a \cdot b)^n = a^n \cdot b^n$	$7a^2 \cdot (24a)^{-1} = 7a^2 \cdot 24^{-1} \cdot a^{-1}$	MATC

Table 72: Techniques in calculating/shortening fractions tasks

	Techniques	Examples	Sources
Laws	Associative law for $\cdot$	$10 \cdot 11 \cdot S \cdot P = 110SP$	GG
	Commutative law for $\cdot$	$\frac{pm}{nm} + \frac{qn}{mn} = \frac{pm}{nm} + \frac{qn}{nm}$	GG, MATC
Factorize	$ab + ac = a(b + c)$	$\frac{av+aw}{k(v+w)} = \frac{a(v+w)}{k(v+w)}$, $\frac{RL+Rw+Rv}{k(L+w+v)} = \frac{R(L+v+w)}{k(L+v+w)}$	GG, MATC
	Resolve number into factors		MATC
Simplify	Collect and reduce like terms	$\frac{3y-2{,}7y}{3{,}3y} = \frac{0{,}3y}{3{,}3y}$	MATC
	Cancel common factors	$\frac{am}{an} = \frac{m}{n}$	GG, MATC
Fractions	Extend	$\frac{p}{n} + \frac{q}{m} = \frac{pm}{nm} + \frac{qn}{mn}$	GG, MATC
	Shorten		GG, MATC
	Addition	$\frac{p}{n} + \frac{q}{n} = \frac{p+q}{n}$	GG
	Subtraction	$\frac{pn}{3k} - \frac{qn}{3k} = \frac{pn-qn}{3k}$	GG
	$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$	$\frac{a}{n} \cdot \frac{b}{m} = \frac{ab}{nm}$	GG
	$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$	$\frac{a}{b} : \frac{m}{n} = \frac{an}{bm}$	GG, MATC
	$\frac{a}{b} \cdot c = \frac{ac}{b}$		MATC
	$\frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}$	$\frac{am}{an} = \frac{a}{a} \cdot \frac{m}{n}$	GG, MATC
Exponents	$a^m \cdot a^n = a^{m+n}$	$a \cdot a^{-1} \cdot m \cdot n = a^{1-1}mn = mn$	GG, MATC
	$(a \cdot b)^n = a^n \cdot b^n$		MATC
Signs	$(+) \cdot (-) = (-)$		MATC

Table 73: Techniques in writing an algebraic expression on a certain form by powers tasks

	Techniques	Examples	Sources
Laws	Associative law for $\cdot$	$a^8 \cdot a^3 \cdot a^5 = a^8 \cdot (a^3 \cdot a^5) = a^8 \cdot a^8$	GG
	Commutative law for $\cdot$	$a^7 \cdot b^3 \cdot a^{15} \cdot b^5 = a^7 \cdot a^{15} \cdot b^3 \cdot b^5$	GG
Simplify	Cancel common factors	$\frac{a^{13}}{a^6} = \frac{a \cdot a \cdots a}{a \cdot a \cdots a} = a^7$	GG
Fractions	Extend	$a^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = a^{\frac{3}{2\cdot 3}+\frac{2}{3\cdot 2}+\frac{1}{6}}$	GG
	Shorten	$a^{\frac{18}{6}} = a^{\frac{18:6}{6:6}} = a^3$	GG
	Addition	$a^{\frac{3}{6}+\frac{2}{6}+\frac{1}{6}} = a^{\frac{3+2+1}{6}}$	GG
	$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$	$a^{\frac{4}{15} \cdot \frac{5}{8}} = a^{\frac{20}{120}}$	GG
	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$a^{\frac{3}{5} \cdot 15} = a^{\frac{3 \cdot 15}{5}}$	GG
Exponents	$a^m \cdot a^n = a^{m+n}$	$a^2 \cdot a^5 = a^{2+5} = a^7$	GG
	$\frac{a^n}{a^m} = a^{n-m}$	$\frac{a^{13}}{a^6} = a^{13-6} = a^7$	GG
	$(a \cdot b)^n = a^n \cdot b^n$		GG
	$(a^n)^m = a^{n \cdot m}$	$(a^6)^{10} = a^{6 \cdot 10} = a^{60}$	GG
	$\frac{1}{a^n} = a^{-n}$	$(ab)^5 = a^5 \cdot b^5$, $(abc)^{12} = a^{12}b^{12}c^{12}$, $(a^5b^2)^5 = a^{5 \cdot 5}b^{2 \cdot 5}$	GG
	$\frac{a^n}{b^n} = (\frac{a}{b})^n$	$\frac{a^{17}}{b^{17}} = (\frac{a}{b})^{17}$	GG

Table 74: Techniques in proportional calculus tasks

	Techniques	Examples	Sources
Proportional	Proportionality between quantities	$1,07 \cdot K_1 = K_2$	GG
Substitute	Substitute algebraic expression	$K_2 = 0,94K_1$: $K_3 = 0,94 \cdot K_2 = 0,94 \cdot 0,94K_1$	GG

Table 75: Techniques in calculating the value of an algebraic expression tasks

	Techniques	Examples	Sources
Laws	Distributive law	$2(-3-4) = 2 \cdot (-3) + 2 \cdot (-4)$	MATC, FSA
	Associative law for $\cdot$	$2 \cdot 2 \cdot (-3) = 4 \cdot (-3)$	FSA
Equations	$\cdot, :, +, -$ on both sides of $=$	$\frac{x}{y} = \frac{4}{5} \Leftrightarrow x = \frac{4}{5} \cdot y$	MATC
Hierarchy	Order of arithmetic calculations	$2 \cdot (-3) - 4 = -6 - 4 = -10$	MATC, ST, FSA
Fractions	Extend	$\frac{3}{4} + \frac{2}{3} = \frac{3 \cdot 3}{4 \cdot 3} + \frac{2 \cdot 4}{3 \cdot 4}$	MATC
	Shorten		MATC
	Subtraction	$\frac{9}{12} - \frac{8}{12} = \frac{9-8}{12}$	MATC
	Addition	$\frac{9}{12} + \frac{8}{12} = \frac{9+8}{12}$	MATC
	$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$	$\frac{x}{y} \cdot \frac{y}{z} = \frac{xy}{yz}$	MATC
	$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$	$\frac{17}{12} : \frac{1}{12} = \frac{17 \cdot 12}{12 \cdot 1}$	MATC
Substitute	Substitute numbers	$a = 2$, $b = -3$, $c = 4$: $a \cdot b - c = 2 \cdot (-3) - 4$	MATC, ST, FSA
	Substitute algebraic expression	$y = \frac{5}{4}x$, $z = \frac{10}{3}y = \frac{10}{3} \cdot \frac{5}{4}x$	MATC
Signs	$(+) \cdot (-) = (-)$, $(-) \cdot (+) = (-)$	$2 \cdot (-3) - 4 = -6 - 4$	MATC, ST, FSA
	$(-) \cdot (-) = (+)$, $-(-a) = a$	$-4 \cdot (-5) = 20$	MATC, FSA
Brackets	Remove negative bracket	$2 - (2 - 5) = 2 - 2 + 5$	FSA, MATC
Squares	$a^2 + b^2 + 2ab = (a + b)^2$	$a^2 + 2ab + b^2 = (a + b)^2$	FSA

Table 76: Techniques in solving equations by the zero-divisor law tasks

	Techniques	**Examples**	**Sources**
Laws	Distributive law	$5(x+3)x = (5 \cdot x + 5 \cdot 3)x$	MATC
Equations	$\cdot, :, +, -$ on both sides of $=$ Zero-divisor law	$x - 1 = 0 \Leftrightarrow x - 1 + 1 = 1$ $x(x-1) = 0 \Rightarrow x = 0 \lor x - 1 = 0$	MATC MATC
Factorize	$ab + ac = a(b+c)$ Resolve number into factors	$x^2 - 3x = 0 \Leftrightarrow x(x-3) = 0$ $2x^2 + 12x = 2x^2 + 2 \cdot 6x$	MATC MATC
Signs	$(-) \cdot (+) = (-)$	$-2 \cdot 2x - 2 \cdot 3 = -4x - 6$	MATC
Rewrite	Rewrite $x = 1 \cdot x$	$2x^2 - x = 0 \Leftrightarrow 2x^2 - 1 \cdot x = 0$	MATC

Table 77: Techniques in solving equations by multiplying crosswise tasks

	Techniques	**Examples**	**Sources**
Laws	Distributive law	$3(x+5) = 3x + 15$	MATC
Equations	$\cdot, :, +, -$ on both sides of $=$ Multiply crosswise	$3x + 15 = 10x - 34 \Leftrightarrow 15 = 10x - 34 - 3x$ $\frac{x+5}{2} = \frac{5x-17}{3} \Leftrightarrow 3 \cdot (x+5) = 2 \cdot (5x - 17)$	MATC MATC
Simplify	Collect and reduce like terms	$15 = 10x - 34 - 3x = 7x - 34$	MATC
Signs	$(+) \cdot (-) = (-)$	$2 \cdot 5x + 2 \cdot (-17) = 10x - 17$	MATC
Rewrite	Rewrite number to fraction	$\frac{8x}{3} + 11 = \frac{8x}{3} + \frac{11 \cdot 3}{3}$	MATC
Fractions	Addition	$\frac{8x}{3} + \frac{33}{3} = \frac{8x+33}{3}$	MATC

Table 78: Techniques in explaining what it means that a number is a solution to an equation by substitution tasks

	Techniques	**Examples**	**Sources**
Substitute	Substitute numbers	$x + 4 = 2x + 2$, $x = 2$: $2 + 4 = 2 \cdot 2 + 2$	ST
Hierarchy	Order of arithmetic calculations	$2 + 4 = 2 \cdot 2 + \Leftrightarrow 2 + 4 = 4 + 2$	ST

Table 79: Algebraic techniques in explaining what it means that a number is a solution to an equation tasks

	Techniques	Examples	Sources
Equations	$\cdot, :, +, -$ on both sides of $=$	$x+4=2x+2 \Leftrightarrow 4=2x+2-x$	ST

Table 80: Graphic techniques in explaining what it means that a number is a solution to an equation tasks

	Techniques	Examples	Sources
Graphic	Draw line	Draw the lines $y=x+4$ and $y=2x+2$ in a coordinate system	ST
	Intersection between lines	Read the intersection between the two lines	ST

Table 81: Techniques in determine if a number is a solution to a linear equation by substitution tasks

	Techniques	Examples	Sources
Substitute	Substitute numbers	$3x-1=5$, $x=2$: $3\cdot2-1=5$	MATC
Hierarchy	Order of arithmetic calculations	$3\cdot2-1=6-1=5$	MATC
Signs	$(+)\cdot(-)=(-)$, $(-)\cdot(+)=(-)$	$20-3\cdot2=20-6$	MATC
	$(-)\cdot(-)=(+)$	$1-3\cdot(-1)=1+3$	MATC

Table 82: Graphic techniques in determine if a number is a solution to a linear equation tasks

	Techniques	Examples	Sources
Graphic	Draw line	Draw a line $y=3-x$ in a coordinate system	MATC
	Read x-value given y	Read the x-value for the value $y=4$ on the line	MATC
	Intersection between lines	Read the intersection between two drawn lines	MATC

Table 83: Techniques in solving system of equations with three or more unknowns tasks

	Techniques	Examples	Sources
Laws	Commutative law for $\cdot$	$2b \cdot 2b = 2 \cdot 2 \cdot b \cdot b$	MATC
	Associative law for $\cdot$	$2 \cdot 2 \cdot b \cdot b = (2 \cdot 2) \cdot (b \cdot b) = 4 \cdot b^2$	MATC
Equations	$\cdot, :, +, -$ on both sides of $=$	$3v_2 + 15 = 180 \Leftrightarrow 3v_2 + 15 - 15 = 180 - 15$	MATC
	Substitute algebraic expression	$v_3 = v_1 + 35$: $v_1 = v_2 - 10$: $v_3 = v_2 - 10 + 35$	MATC
Substitute	Substitute numbers	$v_2 = 55, v_1 = v_2 - 10 = 55 - 10$	MATC
Hierarchy	Order of arithmetic calculations	$B = 2 \cdot 44 + 35 - 60 = 88 + 35 - 60$	MATC
Simplify	Collect and reduce like terms	$v_2 - 10 + v_2 + v_2 + 25 = 180 \Leftrightarrow 3v_2 + 25 = 180$	MATC
Sketch	Draw a sketch	Draw a sketch of the box to visualize it	MATC
Formulas	Use formula for area	Use the standard formula for the area of a rectangle	MATC
Fractions	$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$		MATC

Table 84: Techniques in modelling tasks

	Techniques	Examples	Sources
Laws	Distributive law	$50(350-B)+30B = 50\cdot 350+50\cdot(-B)+30B$	MATC, PL
	Commutative law for $\cdot$	$2b\cdot 2b = 2\cdot 2\cdot b\cdot b$	MATC, GG, PL
	Associative law for $\cdot$	$2\cdot 2\cdot b\cdot b = (2\cdot 2)\cdot(b\cdot b) = 4\cdot b^2$	MATC, GG, PL
Equations	$\cdot, :, +, -$ on both sides of $=$		MATC, ST, GG, PL, FSA
	Subtract equations from each other	$12V+9v=132$, $12V+8v=128$: $12V+9v-(12V+8v)=132-128$	MATC, PL
	Set expressions equal to each other		MATC, GG, PL
	$x^2=c \Rightarrow x=\pm\sqrt{c}$	$h^2=\frac{3}{4}s^2 \Rightarrow h=\pm\sqrt{\frac{3}{4}s^2}$	GG, FSA, PL
	Multiply crosswise		MATC
Guess	Trial-and-error		MATC, GG, FSA
Hierarchy	Order of arithmetic calculations		MATC, ST, GG, PL
Fractions	Extend	$45=(\frac{3}{4}-\frac{1}{8})x=(\frac{3\cdot 2}{4\cdot 2}-\frac{1}{8})x$	MATC, PL
	Shorten	$\frac{2}{24}x=\frac{2:2}{24:2}x=\frac{1}{12}x$	MATC, GG, PL
	Subtraction	$45=(\frac{6}{8}-\frac{1}{8})x=\frac{6-1}{8}x=\frac{5}{8}x$	MATC, PL
	Addition	$1=\frac{4}{12}+\frac{3}{12}+x=\frac{4+3}{12}+x$	MATC
	$a\cdot\frac{b}{c}=\frac{ab}{c}$	$x=45\cdot\frac{8}{5}=\frac{45\cdot 8}{5}=\frac{360}{5}$	MATC, PL
	$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$	$\frac{5}{8}\cdot\frac{8}{5}x=\frac{5\cdot 8}{8\cdot 5}x$	MATC, GG
	$a:\frac{b}{c}=\frac{ac}{b}$	$x=45:\frac{5}{8}=\frac{45\cdot 8}{5}$	MATC
	$\frac{a}{b}:\frac{c}{d}=\frac{ad}{bc}$	$\frac{5}{8}:\frac{5}{8}=\frac{5\cdot 8}{8\cdot 5}$	MATC
Rewrite	$\frac{a}{b}=\frac{1}{b}\cdot a$	$\frac{t}{5}=\frac{1}{5}t$	PL
Factorize	$ab+ac=a(b+c)$	$80=0,5\cdot 60+0,5v=0,5(60+v)$	MATC, PL
	Resolve number in factors	$\frac{45\cdot 8}{5}=\frac{9\cdot 5\cdot 8}{5}$	MATC
Substitute	Substitute numbers	$x_1+x_2+x_3+x_4=8$, $\frac{x_1+x_2+x_3+x_4+x_5}{5}=4$: $\frac{8+x_5}{5}=4$	MATC, ST, FSA, PL, GG
	Substitute algebraic expression	$B=2L-30$, $B+L=240$: $2L-30+L=240$	MATC, GG, PL

Table 85: Techniques in modelling tasks continued

	Techniques	Examples	Sources
Simplify	Collect and reduce like terms		MATC, GG, PL
	Cancel common factors		MATC, GG, (PL)
Standard formulas	Use formulas for speed, average, area, volume, circumference, surface and Pythagoras	$\frac{82+89+x}{3} = 90$	MATC, GG
Sign	$(+) \cdot (-) = (-)$, $(-) \cdot (+) = (-)$	$50 \cdot (-B) = -50B$	MATC, PL
	$-(-a9 = a$, $(-) \cdot (-) = (+)$	$-20 \cdot (-x) = 20x$	MATC, PL
Brackets	Remove negative bracket	$-12(12V + 8v) = -12V - 8v$	MATC
Sketch	Draw a sketch	Draw a sketch of a figure	MATC, GG
Graphic	Draw line	Draw a line $y = ax + b$ in a coordinate system	ST, GG, PL
	Read x-value given y	Read the x-value for the given y-value on the line	MATC, GG
	Read y-value given x	Read the y-value for the given x-value on the line	MATC, GG
	Plot and connect points	Plot points into a coordinate system and connect them with a straight line	MATC, PL
	Read slope of a line	Determine the slope of a drawn line	MATC, PL
	Intersection with the x-axis	Read the drawn line's intersection with the x-axis	GG
	Intersection with the y-axis	Read the drawn line's intersection with the y-axis	MATC
	Intersection between lines	Read the intersection between two drawn lines	MATC, GG, PL
Roots	$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$	$\sqrt{\frac{2}{4}s^2} = \sqrt{\frac{3}{4}} \cdot \sqrt{s^2}$	GG, PL
	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}}$	GG
Line equation	$a = \frac{y_2 - y_1}{x_2 - x_1}$	$a = \frac{860-750}{71-27} = \frac{5}{2}$	MATC
	$b = y_1 - ax_1$	$y = 750 - \frac{5}{2} \cdot 27$	MATC
Reading	a and b from $y = ax + b$	$y = 1,38x + 313$: $a = 1,38$ and $b = 313$	MATC, ST, GG, PL
	b as initial value		MATC, PL

Table 86: Algebraic techniques in determine y in $y = ax + b$ given x tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers for x	$x = 5$, $y = 2x + 3$: $y = 2 \cdot 5 + 3$	ST, MATC, GG
Hierarchy	$\cdot, :$ before $+, -$	$y = 2 \cdot 5 + 3 = 10 + 3 = 13$	ST, FSA, MATC
Formulas	Use formula $y_2 = y_1 - a \cdot \Delta x$	$g(7) = g(6) + a = 12 + 9 = 21$	GG

Table 87: Graphic techniques in determine y in $y = ax + b$ given x tasks

	Techniques	Examples	Sources
Graphic	Draw lines	The line $y = 2x + 3$ is drawn in a coordinate system	ST, MATC, GG
	Read y-value given x	The y-value for $x = 5$ is found on the drawn line	ST, MATC, GG

Table 88: Algebraic techniques in determine x-value in $y = ax + b$ given y tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers for y	$y = 7$, $y = 2x + 3$: $7 = 2 \cdot x + 3$	ST
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$7 = 2 \cdot x + 3 \Leftrightarrow 7 - 3 = 2x$	ST

Table 89: Graphic techniques in determine x-value in $y = ax + b$ given y tasks

	Techniques	Examples	Sources
Graphic	Draw lines	The line $y = 2x + 3$ is drawn in a coordinate system	ST
	Read x-value given y	The x-value for $y = 7$ is found on the drawn line	ST

Table 90: Tril-and-error techniques in determine x-value in $y = ax + b$ given y tasks

	Techniques	Examples	Sources
Guess	Trial-and-error	$7 = 2x + 3$ then $x = 2$ since $7 = 2 \cdot 2 + 3$	ST
Hierarchy	$\cdot, :$ before $+, -$	$2 \cdot 2 + 3 = 4 + 3 = 7$	ST

Table 91: Techniques in area/circumference of figure through variables tasks

	Techniques	**Examples**	**Sources**
Formulas	Formulas for area and circumference of rectangles/triangles	Circumference=length·number	FSA
Substitution	Substitute variables in the formulas	Circumference= $b \cdot 12$	FSA
Figure	Divide into triangles and rectangles	The figure in table 45 can be divided into 5 squares	FSA
	Count sides	The figure in table 45 has 12 sides of length b	FSA
Laws	Commutative law for ·	Circumference= $b \cdot 12 = 12b$	FSA
Simplify	Collect and reduce like terms	$A = b^2 + b^2 + b^2 + b^2 + b^2 = 5b^2$	FSA
Brackets	Multiply two brackets	$(a+3)(a+2) = a \cdot a + a \cdot 2 + 3 \cdot a + 3 \cdot 2$	FSA
Fractions	$a \cdot \frac{b}{c} = \frac{ab}{c}$	$\frac{1}{2} \cdot a \cdot 2 = \frac{a \cdot 2}{2}$	FSA

Table 92: Algebraic techniques in determine point of intersection between lines tasks

	Techniques	**Examples**	**Sources**
Equation solving	Put two expressions equal to each other	$y = -x+3$, $y = x-1$ then $-x+3 = x-1$	FSA, MATC
	$\cdot, :, +, -$ on both sides of $=$	$-x+3 = x-1 \Leftrightarrow -x+3+x = x-1+x$	FSA, MATC
Substitution	Substitute numbers for x	$x = 2$: $y = 2-1 = 1$	FSA, MATC
Simplify	Collect and reduce like terms	$3 = x-1+x = 2x-1$	FSA, MATC
Hierarchy	$\cdot, :$ before $+, -$	$y = 2\cdot(-2)-5 = -4-5 = -9$	FSA, MATC
Sign	$(+)\cdot(-) = (-)$, $(-)\cdot(+) = (-)$	$2\cdot(-2) = -4$	FSA, MATC
Fractions	$\frac{a}{b}\cdot c = \frac{ac}{b}$	$\frac{-3}{4}\cdot 4 = \frac{-3\cdot 4}{4} = \frac{-12}{4}$	FSA, MATC
	Addition	$\frac{5}{4}x + \frac{3}{4}x = \frac{5+3}{4}x = \frac{8}{4}x$	FSA, MATC
	Subtraction	$\frac{60}{10} - \frac{49}{10} = \frac{60-49}{10}$	FSA, MATC
Rewrite	Number to fraction	$6 = \frac{6\cdot 10}{10} = \frac{60}{10}$	FSA, MATC

Table 93: Graphic techniques in determine point of intersection between lines tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers for x	$x = 2$: $y = 2 - 1 = 1$	FSA, MATC
Hierarchy	$\cdot, :$ before $+, -$	$y = 2 \cdot (-2) - 5 = -4 - 5 = -9$	FSA, MATC
Graphic	Plot and connect points to a line	Points on the line given by $y = x - 1$ are plotted and connected in a coordinate system	FSA, MATC
	Draw lines	The line $y = x - 1$ is drawn in a coordinate system	PL, MATC
	Determine intersection between lines	The point of intersection between two drawn lines is determined	PL, FSA, MATC
	Intersection with y-axis	A drawn line's intersection with the y-axis is determined	FSA
	Slope of a straight line	The slope of a drawn line is determined	FSA
Reading	Read a and b from $y = ax + b$	$y = -2x + 3$, so $a = -2$ and $b = 3$	MATC, FSA, PL
Sign	$(+) \cdot (-) = (-)$, $(-) \cdot (+) = (-)$	$2 \cdot (-2) = -4$	FSA, MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{-3}{4} \cdot 4 = \frac{-3 \cdot 4}{4} = \frac{-12}{4}$	MATC
	Extend	$\frac{-3}{5} = \frac{-3 \cdot 2}{5 \cdot 2} = \frac{-6}{10}$	MATC
	Addition	$\frac{-6}{10} + \frac{17}{10} = \frac{-6+17}{10} = \frac{11}{10}$	MATC
	Subtraction	$\frac{60}{10} - \frac{49}{10} = \frac{60-49}{10}$	MATC
Rewrite	Number to fraction	$6 = \frac{6 \cdot 10}{10} = \frac{60}{10}$	MATC

Table 94: Techniques in determine a and b from graph tasks

	Techniques	Examples	Sources
Graphic	Read intersection with the y-axis	A drawn line's intersection with the y-axis is determined	ST
	Read slope of line	The slope of a drawn line is determined	ST

Table 95: Techniques in draw line $y = ax + b$ in a coordinate system tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers for x	$y = -x + 5$, $x = 1$: $y = -1 + 5 = 4$	ST, MATC
Hierarchy	$\cdot, :$ before $+, -$	$y = 2 \cdot 2 + 1 = 4 + 1 = 5$	MATC
Sign	$-(-a) = a$ $(-) \cdot (+) = (-)$	$y = -(-2) + 5 = 2 + 5 = 7$ $\frac{-2}{5} \cdot 1 = \frac{-2}{5}$	ST, MATC MATC
Graphic	Plot and connect points to a line Draw lines	Points on the line given by $y = -x + 5$ are plotted and connected in a coordinate system The line $y = -x + 5$ is drawn in a coordinate system	ST, MATC ST, MATC
Reading	Read a and b from $y = ax + b$	$y = -x + 5$, so $a = -1$ and $b = 5$	ST, MATC
Rewrite	$x = 1 \cdot x$	$-x = -1 \cdot x$	ST, MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$ Addition	$y = \frac{-2}{5} \cdot 2 + \frac{3}{5} = \frac{-4}{5} + \frac{3}{5}$ $y = \frac{-4}{5} + \frac{3}{5} = \frac{-4+3}{5} = \frac{-1}{5}$	MATC MATC

Table 96: Algebraic techniques in determine y-value given corresponding x-value, a, and another point on line tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers in the equation of a line	$a = 3$, $(3, 4)$: $4 = 3 \cdot 3 + b$	MATC
Hierarchy	$\cdot, :$ before $+, -$ Brackets before $\cdot, :, +, -$	$4 = 3 \cot 3 + b = 9 + b$ $4 + 3 \cdot (5 - 3) = 4 + 3 \cdot 2$	MATC MATC
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$4 = 9 + b \Leftrightarrow b = 4 - 9$	MATC
Formulas	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$ Use formula $y_2 = y_1 + a \cdot (x_2 - x_1)$	$3 = \frac{y_2 - 4}{5 - 3}$ $y_2 = 4 + 3 \cdot (5 - 3)$	MATC MATC

Table 97: Graphic techniques in determine y-value given corresponding x-value, a, and another point on line tasks

	Techniques	Examples	Sources
Graphic	Draw line through point with given slope	Draw the line through $(3, 4)$ and with slope 3	MATC
	Read y-value for given x-value	Read y-value for $x = 5$ on the graph	MATC

Table 98: Algebraic techniques in determine x-value given corresponding y-value, a, and another point on line tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers in the equation of a line	$a = \frac{-1}{2}$, $(-4, 5)$: $5 = \frac{-1}{2} \cdot (-4) + b$	MATC
Hierarchy	$\cdot, :$ before $+, -$	$5 = \frac{-1}{2} \cdot (-4) + b = 2 + b$	MATC
Laws	Distributive law	$-6 = \frac{-1}{2} \cdot (x_2 + 4) = \frac{-1}{2} x_2 - 2$	MATC
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$5 = 2 + b \Leftrightarrow b = 5 - 2$	MATC
Formulas	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$\frac{-1}{2} = \frac{-1-5}{x_2 - (-4)}$	MATC
	Use formula $x_2 = x_1 + \frac{y_2 - y_1}{a}$	$x_2 = -4 + \frac{-1-5}{\frac{-1}{2}}$	MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$5 = \frac{1}{2} \cdot 4 + b = \frac{4}{2} + b$	MATC
	$a : \frac{b}{c} = \frac{ac}{b}$	$\frac{-6}{\frac{-1}{2}} = \frac{-6 \cdot 2}{-1}$	MATC
Sign	$-(-a) = a$, $(-) \cdot (-) = (+)$	$-(-4) = 4$	MATC
	$(-) \cdot (+) = (-)$	$\frac{-1}{2} \cdot 4 = -2$	MATC

Table 99: Graphic techniques in determine x-value given corresponding y-value, a, and another point on line tasks

	Techniques	Examples	Sources
Graphic	Draw line through point with given slope	Draw the line through $(-4, 5)$ and with slope $\frac{-1}{2}$	MATC
	Read x-value for given y-value	Read x-value for $y = -1$ on the graph	MATC

Table 100: Algebraic techniques in determine equation of line passing through two points tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers in formulas	$(4,4)$, $(8,5)$: $a = \frac{5-4}{8-4}$	MATC, GG
Hierarchy	$\cdot, :$ before $+, -$	$b = 4 - \frac{1}{4} \cdot 4 = 4 - 1$	MATC, GG
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$4 = 1 + b \Leftrightarrow b = 4 - 1$	MATC, GG
Formulas	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$a = \frac{5-4}{8-4}$	MATC, GG
	Use formula $b = y_1 - ax_1$	$b = 4 - \frac{1}{4} \cdot 4 = 4 - 1 = 3$	MATC, GG
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{-1}{4} \cdot 4 = \frac{-4}{4}$	MATC, GG
	Shorten	$a = \frac{-5}{10} = \frac{-5:5}{10:5} = \frac{-1}{2}$	MATC
Rewrite	Number to fraction $a = \frac{ab}{b}$	$7 = \frac{7 \cdot 4}{4} = \frac{28}{4}$	MATC
Sign	$-(-a) = a$, $(-) \cdot (-) = (+)$	$-(-\frac{1}{2}) = \frac{1}{2}$, $\frac{-2}{5} \cdot (-1) = \frac{2}{5}$	MATC, GG
	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$-\frac{1}{4} \cdot 4 = -1$	MATC, GG

Table 101: Graphic techniques in determine equation of line passing through two points tasks

	Techniques	Examples	Sources
Graphic	Plot and connect points	Plot the points $(4,4)$, $(8,5)$ and connect them with a line	MATC, GG
	Intersection with the y-axis	Determine the drawn line's intersection with the y-axis	MATC, GG
	Slope of a drawn line	Determine the slope of the drawn line	MATC, GG

Table 102: Algebraic techniques in determine equation of line with a given slope and passing through a point tasks

	Techniques	**Examples**	**Sources**
Substitution	Substitute numbers in formulas	$(4,5)$, $a = \frac{1}{2}$: $b = y_1 - ax_1 = 5 - \frac{1}{2} \cdot 4$	MATC, GG
Hierarchy	$\cdot, :$ before $+, -$	$b = 5 - \frac{1}{2} \cdot 4 = 5 - 2 = 3$	MATC, GG
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$5 = 2 + b \Leftrightarrow b = 5 - 2$	MATC, GG
Formulas	Use formula $b = y_1 - ax_1$	$b = 5 - \frac{1}{2} \cdot 4$	MATC, GG
	Use formula $a = \frac{\Delta y}{\Delta x}$	$a = \frac{4}{12}$	MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{1}{2} \cdot 4 = \frac{4}{2}$	MATC
	Subtraction	$\frac{18}{3} - \frac{4}{3} = \frac{18-4}{3} = \frac{14}{3}$	MATC
Rewrite	Number to fraction $a = \frac{ab}{b}$	$6 = \frac{6 \cdot 3}{3} = \frac{18}{3}$	MATC
Sign	$-(-a) = a$, $(-) \cdot (-) = (+)$	$-(-\frac{2}{3}) = \frac{2}{3}$, $\frac{-2}{3} \cdot (-2) = \frac{4}{3}$	MATC, GG
	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$-\frac{1}{2} \cdot 4 = -2$	MATC, GG

Table 103: Graphic techniques in determine equation of line with a given slope and passing through a point tasks

	Techniques	**Examples**	**Sources**
Graphic	Draw line through point	Draw line with the slope $\frac{1}{2}$ and passing trough $(4,5)$	MATC, GG
	Intersection with the y-axis	Determine the drawn line's intersection with the y-axis	MATC, GG

Table 104: Techniques in determine equation $y = ax + b$ of line from the graph by formulas tasks

	Techniques	Examples	Sources
Graphic	Coordinates on line	Read sets of coordinates on the line	MATC, FSA
Substitution	Substitute numbers in formulas	$a = 4$, $(4, 0)$: $0 = 4 \cdot 4 + b$	MATC, FSA
Hierarchy	$\cdot, :$ before $+, -$	$b = 4 - \frac{-1}{2} \cdot (-4) = 4 - 2 = 2$	MATC, FSA
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$0 = 16 + b \Leftrightarrow b = 16 - 0$	MATC, FSA
Formulas	Use formula $b = y_1 - ax_1$	$b = 4 - \frac{-1}{2} \cdot (-4)$	MATC, FSA
	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$a = \frac{0-4}{4-(-4)} = \frac{-4}{8}$	MATC, FSA
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{1}{2} \cdot (-4) = \frac{-4}{2}$	MATC, FSA
	Shorten	$\frac{-4}{8} = \frac{-4:4}{8:4} = \frac{-1}{2}$	MATC
Sign	$-(-a) = a$	$-(-\frac{1}{2}) = \frac{1}{2}$	MATC
	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$\frac{1}{2} \cdot (-4) = -2$	MATC, FSA

Table 105: Graphic techniques in determine equation $y = ax + b$ of line from the graph tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers in $y = ax + b$	$b = 2$, $a = \frac{-1}{2}$: $y = \frac{-1}{2}x + 2$	MATC, FSA
Graphic	Intersection with the y-axis	Determine the drawn line's intersection with the y-axis	MATC, FSA
	Slope of a drawn line	Determine the slope of the drawn line	MATC, FSA

Table 106: Algebraic techniques in determining if points lie on a line $y = ax + b$ tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers into $y = ax + b$	$A(3,4)$, $y = 2x - 1$: $4 = 2 \cdot 3 - 1$	MATC
Hierarchy	$\cdot, :$ before $+, -$	$2 \cdot 3 - 1 = 6 - 1 = 5$	MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$ Addition	$\frac{2}{3} \cdot 6 + 4 = \frac{12}{3} + 4$ $\frac{12}{3} + \frac{12}{3} = \frac{12+12}{3} = \frac{24}{3}$	MATC MATC
Rewrite	Number to fraction	$4 = \frac{4 \cdot 3}{3} = \frac{12}{3}$	MATC
Sign	$(-) \cdot (-) = (+)$ $(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$-3 \cdot (-3) = 9$ $2 \cdot (-2) = -4$	MATC MATC

Table 107: Graphic techniques in determining if point/points lie on a line $y = ax + b$ tasks

	Techniques	Examples	Sources
Graphic	Draw lines	Draw line $y = 2x - 1$ in a coordinate system	MATC
	Plot points	Plot the point $A(3,4)$ in a coordinate system	MATC
	Point on line?	Read if the point lies on the drawn line	MATC

Table 108: Techniques in determining if three points lie on the same line by using formulas tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers into formulas	$(11,19)$, $y = \frac{3}{2}x + \frac{5}{2}$: $19 = \frac{3}{2} \cdot 11 + \frac{5}{2}$	MATC
Hierarchy	$\cdot, :$ before $+, -$	$\frac{3}{2} \cdot 11 + \frac{5}{2} = \frac{33}{2} + \frac{5}{2}$	MATC
Formulas	Use formula $b = y_1 - ax_1$ Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$b = 4 - \frac{3}{2} \cdot 1$ $a = \frac{7-4}{3-1} = \frac{3}{2}$	MATC MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$ Addition	$b = 4 - \frac{3}{2} \cdot 1 = 4 - \frac{3 \cdot 1}{2}$ $\frac{33}{2} + \frac{5}{2} = \frac{33+5}{2} = \frac{38}{2}$	MATC MATC
Sign	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$-\frac{3}{2} \cdot 1 = -\frac{3}{2}$	MATC

Table 109: Techniques in determining if three points lie on the same line by calculating slopes tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers into formula	$(1,4)$, $(3,7)$: $a = \frac{7-4}{3-1}$	MATC
Formulas	Use formula $a = \frac{y_2-y_1}{x_2-x_1}$	$a = \frac{7-4}{3-1} = \frac{3}{2}$	MATC
Fractions	Shorten	$\frac{12}{8} = \frac{12:4}{8:4} = \frac{3}{2}$	MATC
Compare	Compare values for slopes	$a_{AB} = \frac{3}{2}$, $a_{BC} = \frac{3}{2}$ so the three points lie on the same line	MATC

Table 110: Graphic techniques in determining if three points lie on the same line tasks

	Techniques	Examples	Sources
Graphic	Plot points	Plot the points $A(1,4)$, $B(3,7)$ and $C(11,19)$ in a coordinate system	MATC
	Three point on same straight line?	Examine if three points can be connected by a straight line	MATC

Table 111: Graphic techniques in finding a point (x_1, y_1) on a drawn line $y = ax + b$ such that $x_1, y_1 \in \mathbb{Z}$ tasks

Graphic	Coordinates on line	Determine a set of coordinates on the graph were both the x- and y-value are integers	MATC

Table 112: Algebraic techniques in finding a point (x_1, y_1) on a drawn line $y = ax + b$ such that $x_1, y_1 \in \mathbb{Z}$ tasks

	Techniques	Examples	Sources
Substitution	Substitute integers into $y = ax + b$	$x = 4$: $y = \frac{-2}{5} \cdot 4 + \frac{3}{5}$	MATC
Hierarchy	$\cdot, :$ before $+, -$	$\frac{-2}{5} \cdot 4 + \frac{3}{5} = \frac{-8}{5} + \frac{3}{5}$	MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{-2}{5} \cdot 4 = \frac{-2 \cdot 4}{5} = \frac{-8}{5}$	MATC
	Addition	$\frac{-8}{5} + \frac{3}{5} = \frac{-8+3}{5} = \frac{-5}{5} = -1$	MATC
Sign	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$\frac{-2}{5} \cdot 4 = -\frac{8}{5}$	MATC

Table 113: Algebraic techniques in determine x_{min}/y_{min} and x_{max}/y_{max} for a line segment restricted to $a \leqslant x \leqslant b$ or $c \leqslant y \leqslant d$ passing through two points tasks

	Techniques	**Examples**	**Sources**
Substitution	Substitute numbers in formulas	$(-2,-4)$, $(8,1)$: $a = \frac{1-(-4)}{8-(-2)}$	MATC
Hierarchy	$\cdot, :$ before $+, -$	$b = -4 - \frac{1}{2} \cdot (-2) = -4 + 1 = -3$	MATC
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$-6 = 2x + 3 \Leftrightarrow 2x = -6 - 3$	MATC
Formulas	Use formula $b = y_1 - ax_1$	$b = -4 - \frac{1}{2} \cdot (-2)$	MATC
	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$a = \frac{1-(-4)}{8-(-2)}$	MATC
	Use formulas $y_{min/max} = y_i + a \cdot \Delta x$ or $x_{min/max} = x_i + \frac{\Delta y}{a}$	$y_{min} = 1 + \frac{1}{2}(-4 - 8)$	MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{-1}{2} \cdot (-2) = \frac{-1 \cdot (-2)}{2}$	MATC
	Shorten	$\frac{5}{10} = \frac{5:5}{10:5} = \frac{1}{2}$	MATC
Sign	$-(-a) = a$, $(-) \cdot (-) = (+)$	$-(-4) = 4$	MATC
	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$\frac{-1 \cdot (-2)}{2} = \frac{2}{2} = 1$	MATC

Table 114: Graphic techniques in determine x_{min}/y_{min} and x_{max}/y_{max} for a line segment restricted to $a \leqslant x \leqslant b$ or $c \leqslant y \leqslant d$ passing through two points tasks

	Techniques	**Examples**	**Sources**
Graphic	Plot and connect points	Plot and connect the points $(8,1)$ and $(-2,-4)$ in a coordinate system	MATC
	Min/max in end points	Determine the smallest and biggest y-value by looking at the end points of the line segment	MATC

Table 115: Algebraic techniques in determining the intersection of a line passing through two points or given by an equation with axes of coordinates and lines $x = a$ tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers	$(2,5)$, $(5,-3)$: $a = \frac{-3-5}{5-2}$	MATC, GG
Hierarchy	$\cdot, :$ before $+, -$	$b = 5 - (\frac{-8}{3}) \cdot 2 = 5 + \frac{16}{3}$	MATC, GG
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$0 = -2x + 14 \Leftrightarrow -2x = -14$	MATC, GG
Formulas	Use formula $b = y_1 - ax_1$	$b = 5 - (\frac{-8}{3}) \cdot 2$	MATC
	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$a = \frac{-3-5}{5-2}$	MATC
Fractions	$\frac{a}{b} \cdot c = \frac{ac}{b}$	$\frac{8}{3} \cdot 2 = \frac{8 \cdot 2}{3}$	MATC
Sign	$-(-a) = a$, $(-) \cdot (-) = (+)$	$-(\frac{-8}{3}) = \frac{8}{3}$, $-2 \cdot (-1) = 2$	MATC, GG
	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$\frac{-8}{3} \cdot 9 = -24$	MATC, GG

Table 116: Graphic techniques in determining the intersection of a line passing through two points or given by an equation with axes of coordinates and lines $x = a$ tasks

	Techniques	Examples	Sources
Graphic	Plot and connect points	Plot and connect the points $(2, 10)$ and $(5, 4)$ in a coordinate system	MATC
	Draw lines	Draw the line $y = -2x + 1$ in a coordinate system	MATC, GG
	Read y-value for x-value	Determine y-value on line for $x = -1$	MATC
	Intersection with y-axis	Determine the drawn line's intersection with the y-axis	MATC, GG
	Intersection with x-axis	Determine the drawn line's intersection with the x-axis	MATC, GG

Table 117: Algebraic techniques in determine $y = ax + b$ from table tasks

	Techniques	Examples	Sources
Substitution	Substitute numbers	$(-6; 11,93)$, $(2; 5,81)$: $a = \frac{5,81-11,93}{2-(-6)}$	GG, FSA
Hierarchy	$\cdot, :$ before $+, -$	$b = 11,93 - (-1,53) \cdot (-6) = 11,93 - 9,18$	GG
Equation solving	$\cdot, :, +, -$ on both sides of $=$	$11,93 = 9,18 + b \Leftrightarrow b = 11,93 - 9,18$	GG
Formulas	Use formula $b = y_1 - ax_1$	$b = 11,93 - (-1,53) \cdot (-6)$	GG
	Use formula $a = \frac{y_2-y_1}{x_2-x_1}$	$a = \frac{5,81-11,93}{2-(-6)}$	GG, FSA
Sign	$-(-a) = a$, $(-) \cdot (-) = (+)$	$-(-6) = 6$	GG
	$(-) \cdot (+) = (-)$, $(+) \cdot (-) = (-)$	$\frac{-6,12}{4} = -1,53$	GG
Read	y-value for $x = 0$ in table	Read in table that $y = 0$ when $x = 0$	FSA

Table 118: Graphic techniques in determine $y = ax + b$ from table tasks

	Techniques	Examples	Sources
Graphic	Plot and connect points	Plot and connect the points from the table in a coordinate system	GG, FSA
	Draw best straight line	Draw the straight line that best describes the points	GG
	Slope of line	Determine the slope of the drawn line	FSA, GG
	Intersection with y-axis	Determine the drawn line's intersection with the y-axis	FSA, GG
Substitution	Substitute numbers into $y = ax + b$	$a = \frac{1}{2}$, $b = 0$ so $y = \frac{1}{2}x$	FSA, GG

Table 119: Techniques in draw line from table tasks

	Techniques	Examples	Sources
Graphic	Plot and connect points	Plot and connect the points from the table in a coordinate system	FSA
	Draw line	Draw the line $y = -x + 6$ in a coordinate system	FSA
Substitution	Substitute numbers into formula	$(0, 6)$, $(2, 4)$ so $a = \frac{4-6}{2-0}$	FSA
Formula	Use formula $a = \frac{y_2 - y_1}{x_2 - x_1}$	$a = \frac{4-6}{2-0}$	FSA
Read	y-value for $x = 0$ in table	Read in table that $y = 6$ when $x = 0$	FSA

Table 120: Techniques in determine a positive integer that satisfies a relation tasks

	Techniques	Examples	Sources
Laws	Distributive law	$a(a-b) = a^2 - ab$	MATC
Equation solving	$-,:$ on both sides of $=$	$a^2 - ab = 23 \Leftrightarrow -ab = 23 - a^2$	MATC
Fraction	$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$	$b = \frac{23-a^2}{a} = \frac{23}{a} - \frac{a^2}{a}$	FSA
Simplify	Cancel common factors	$b = \frac{23}{a} - \frac{a^2}{a} = \frac{23}{a} - a$	FSA

Table 121: Algebraic techniques in determine growth of a function tasks

	Techniques	Examples	Sources
Laws	Distributive law	$3(t + \Delta t) = 3t + 3\Delta t$	GG
Substitution	Substitute numbers	$N(t) = 3t + 5$, $t = 14$: $N(14) = 3 \cdot 14 + 5$	GG
	Substitute algebraic expression	$N(t) = 3t + 5$: $N(t + \Delta t) = 3(t + \Delta t) + 5$	GG
Hierarchy	$\cdot,:$ before $+,-$	$3 \cdot 14 + 5 = 42 + 5 = 47$	GG
Simplify	Collect and reduce like terms	$3t - 3t = 0$	GG
Formulas	Use formula $\Delta y = a \cdot \Delta x$	$\Delta y = 3 \cdot 3 = 9$	GG
Bracket	Remove negative bracket	$-(3t + 5) = -3t - 5$	GG

Table 122: Graphic techniques in determine growth of a function tasks

	Techniques	Examples	Sources
Graphic	Draw line	Draw line $N(t) = 3t + 5$ in a coordinate system	GG
	Read Δy	Read ΔN from drawn line by finding the difference between the two y-values of the function	GG

Table 123: Techniques in recognition of algebraic expression based on a calculation recipe tasks

Missing brackets	Discover missing brackets	Discover missing bracket around $n + 10$ in $n + 10 \cdot 3$	PL

Table 124: Techniques in complete sum-pyramid tasks

Simplify	Collect and reduce like terms	$8 + n + n = 8 + 2n$	PL

Table 125: Techniques in prove hypothesis about sum-triangles tasks

Simplify	Collect and reduce like terms	$a + b + a + c + b + c = 2a + 2b + 2c$	PL
Factorize	$ab + ac + ad = a(b + c + d)$	$2a + 2b + 2c = 2(a + b + c)$	PL

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