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Pattern Analysis as Entrance to Algebraic Proof Situations at C-level

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47. Lotte Nørtoft: The Trigonometric Functions - The transition from geometric tools to functions (2016)
48. **Aske Henriksen: Pattern Analysis as Entrance to Algebraic Proof Situations at C-level (2016)**

Abstract

Algebraic work often appears as isolated problems in High School mathematics, for which reason students find it difficult to comprehend the different meanings of algebraic symbols, as these appear as unknowns, variables and parameters. This affects the student's ability to use algebra as a language to explain a mathematical phenomenon. This master's thesis seeks to examine danish gymnasium student's opportunities of applying algebra in proving situations. Based on the Theory of Didactical Situations and an analysis of six selected didactical variables, a course of study was constructed and implemented in December 2015 in a first year STX class. The desired effect of the didactical variables was to make the students examine patterns in number tables in order for them to formulate a conjecture and use this as a catalyst in a proving situation. This was sought fulfilled through an a priori analysis of the exercises, which was handed to the students during the course of study.

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University of Copenhagen

PATTERN ANALYSIS AS ENTRANCE TO ALGEBRAIC PROOF SITUATIONS AT C-LEVEL

Aske Henriksen

A Master's Thesis

Department of Science Education, Univeristy of Copenhagen

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ABSTRACT

Algebraic work often appears as isolated problems in High School mathematics, for which reason students find it difficult to comprehend the different meanings of algebraic symbols, as these appear as unknowns, variables and parameters. This affects the students' ability to use algebra as a language to explain a mathematical phenomenon. This master's thesis seeks to examine danish gymnasium students' opportunities of applying algebra in proving situations. Based on the Theory of Didactical Situations and an analysis of six selected didactical variables, a course of study was constructed and implemented in December 2015 in a first year STX class. The desired effect of the didactical variables was to make the students examine patterns in number tables in order for them to formulate a conjecture and use this as a catalyst in a proving situation. This was sought fulfilled through an a priori analysis of the exercises, which was handed to the students during the course of study.

The a posteriori analysis focused on the affect of the didactical variables on the students' work and the students' transition between three intended phases: examination of pattern, formulation of conjecture and proving of conjecture. The analysis was based on audio recordings, pictures taken during the course of study and the students' written work.
The a posteriori analysis showed the importance of completing the three phases successfully in chronological order, as an incomplete phase hampers the work in the next. The analysis also showed that the didactical variable *the Algebraic Prerequisites* needs much attention, as this has the ability to hamper the work of formulating a proof. Lastly the a posteriori analysis showed the importance of creating a pattern, that diminishes the use of verbal proofs and that the didactical variables *the Calculations Needed* and *the Number Table* play an important role in this phase. The Sum Table undermined the use of algebraic proofs, while the Calendar and the Multiplication Table fostered the need for algebraic proofs, especially when the calculations involved multiplication.

The teaching experiment showed that if conjecture was based on numerical examination and the use of verbal arguments were diminished, the students engaged in meaningful proving situations by the use of algebra.

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CONTENTS

LIST OF FIGURES

Part I

INTRODUCTION AND STRUCTURE

1 INTRODUCTION AND STRUCTURE

1.1 INTRODUCTION

What is a proof? What is algebra? These questions are difficult to answer, especially for a first year high school student. Healy & Hoyles (2000) found in a survey, that the majority of their 14-15 year old students were unable to construct a valid proof by the use of algebra. The students rarely used algebra as they did not see algebra as a language usable to explain a mathematical phenomenon and they reverted to using empirical arguments, even though these were recognized by the students to have low status (Healy and Hoyles, 2000, p. 425 ff.). This raised questions about students' motivation of expressing themselves algebraically in the construction of a proof, when they were successful in forming a informal argument.

By the above and through my interest in mathematics teaching, I find it interesting to examine the possibility of making high school students apply algebra in proving situations.

This master's thesis therefore examines how six selected didactical variables affects students' work, in order for them to engage in such situations. The teaching experiment, which lays the foundation for this master's thesis, formed a course of study in pattern analysis which was implemented in a first year danish gymnasium class and is inspired by Mara V. Martinez' qualifying paper: *Integrating Algebra and Proof in High School: Students' Work With Algebraic Expression Involving Variables when Proving*. The research questions the thesis seeks to answer is:

1.2 RESEARCH QUESTIONS

This master's thesis seeks to investigate the potential of an inquiry-based teaching design, concerning patterns in number tables. The focus will be on the students' opportunities of applying algebra as a tool to prove conjectures, based on examination of these patterns.

The master's thesis' aim is to answer the following questions:

1. How can a course of study be designed, in order for the students to use algebra based on their own curiosity, in situations of formulation and validation.

2. How do the didactical variables affect the students' works, in order for the students to engage in formulation and validation situations?
3. What may hinder the students' transition between the three phases: examination of pattern, formulation of conjecture and proving of conjecture, and how does the work in one phase influence the next?

1.3 STRUCTURE OF THE THESIS

My working process with this master's thesis is demonstrated in the figure below, which reflects the structure of the thesis. The methodology will be elaborated further in Chapter 3.

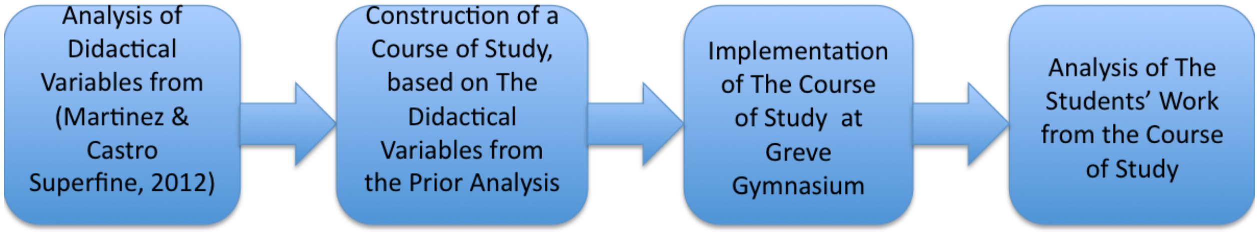


Figure 1: Structure of The Thesis

The didactical variables, which is central to the teaching experiment, was singled out through an analysis of Martinez' teaching experiment from (Martinez and Castro Superfine, 2012) and (Martinez, 2008). These didactical variables was then analyzed and on the basis of this analysis, a course of study was designed and an a priori analysis of the course of study was made. This course of study was then conducted in a danish first year STX class where audio recordings were recorded and written work collected. This data where then analyzed with the above research questions in mind.

The structure of the thesis is then as follows:

Chapter one contains this introduction to the thesis and the statement of the research questions.

Chapter two concerns the theoretical framework. This chapter contains a description of The Theory of Didactical Situations and the relevant concepts related to this theory and aspects of algebra which is of importance to the thesis. The chapter closes with a clarification of concepts and an analysis of the didactical variables.

Chapter three describes the methodology and the design of the course of study, including a description of the class and the collection of data.

Chapter four forms the analysis and is divided into two parts: An a priori analysis of the exercises and an a posteriori analysis of selected situations from the course of study.

Chapter five closes the thesis with a discussion and a conclusion. In this chapter, the research questions and relevant aspects of the teaching design, including weaknesses and improvements, are discussed.

Part II

THEORY

2 THEORY

This chapter contains an introduction to the Theory of Didactical Situations[1] (TDS) and a description of relevant aspects of algebra. In this section 'he' will be used to denote a teacher and 'she' will denote a student.

The section starts by introducing central ideas and concepts of TDS, which will be used in the design and analysis of the teaching experiment.
Then follows a review of some of the difficulties that students face when learning algebra and the advantages of integrating proofs in the teaching of algebra. These advantages will be exemplified by the use of exercises from the article by Mara V. Martinez and Alison Castro Superfine (Martinez and Castro Superfine, 2012).
The section closes with a presentation of the main didactical variables, their influence on the teaching design and how they must be incorporated.

2.1 THE THEORY OF DIDACTICAL SITUATIONS

The design of my teaching experiment and analysis of the empiricism is based on *The Theory of Didactical Situations*. TDS is a didactical theory developed by Guy Brousseau in the 1970s and has since been developed further (Måsøval, 2011, p. 32). The theory is both a tool for analysis of scientific teaching and a frame for the design of teaching situations.

This section explains those parts of the theory, which is relevant to this thesis, but before that three concepts, which is helpful in order to understand TDS is presented. These are: Didactiques, The Didactical Triangle and The Didactical Transposition.

2.1.1 *Didactiques, The Didactical Triangle and The Didactical Transposition*

TDS is a theory about the dissemination of knowledge, why it has been described as an epistemological programme to didactiques. Here didactiques should be understood as the study of transformations of mathematical knowledge under some certain conditions and constrains of education (Måsøval, 2011, p. 26). This transformation is a relationship between three components: Some mathematical knowl-

1 TDS from now on

edge to be taught and learned, the knowings that emerge when the students work in a mathematical milieu[2] and the social, educational project between student and teacher, which requires the reproduction and decontextualization of old knowledge. Knowledge is hence not transferred form one individual to another, but is constructed in the work with a specific milieu.

When new official knowledge is created, it undergoes several transformations, before a student "encounters" this knowledge in a given milieu. This is what Yves Chevallard calls **The Didactical Transposition**. This transposition contains two main steps (Winsløw, 2006, p. 19):

- *The external didactical transposition*: The movement and deformation of knowledge from sources to the officially determined educational knowledge
- *The internal didactial transposition*: The movement and deformation of the officially determined educational knowledge to teaching situations

As this transposition happens, all traces of how the official knowledge was created disappears. If the student are to acquire the knowledge, the teacher has to recontextualize the official knowledge to create a situation in where the student can construct the desired target knowledge. As the student personalizes the target knowledge it is attached to this specific situation, why the teacher must secure that the student is able to use this knowledge in other situations. The teacher must therefore help the student to redepersonalize and redecontextualize the knowledge in order to give it a universal character (Måsøval, 2011, p. 33) (Winsløw, 2006, p. 13).

The production of knowledge is hence an interplay between three parts: The teacher, the student and the knowledge in play, which makes up a *didactic system*. This didactic system is called the **The Didactical Triangle**(See Figure 2). The interplay between these three parts, is central in TDS: The *teacher* arranges a situation, he devolves a milieu so to speak, which requires that the *student* personalizes the target *knowledge* in order to solve the problems in the situation(Måsøval, 2011, p. 34).

With the main idea of TDS in place, it is possible to define **learning**, **teaching** and **knowledge** in terms of TDS. **Learning** is here understood as sense making of situations in a milieu and developing ways of coping with them. **Teaching** is organizing a mathematical, didactical milieu which requires the acquisition of some intended knowledge. **Knowledge** is the outcome of the interaction between the student and the milieu organized and devolved by the teacher(Ibid.; p. 34)

2 To be explained later

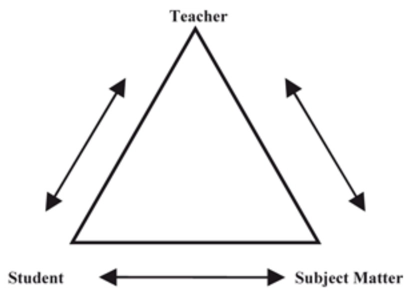


Figure 2: The Didactical Triangle (Winsløw, 2006)

The term *milieu* has been frequently used and plays a significant role in TDS, why it now will be explained.

2.1.2 *Milieu*

Brousseau describes the milieu as *"Within a situation of action, everything that acts on the student or that she acts on is called the 'milieu' "* (Brousseau, 1997, p. 9). The milieu is the didactical environment (french: milieu) that is relevant with respect to some given piece of knowledge. It is constituted by textbooks, materials, aids, teacher, other students, mathematical problems etc. The student's work in the didactical milieu can be seen as a "game" that has to be won and the student must adapt to the milieu in order to win the game by constructing a "winner strategy". The winner strategy is in this sense the target knowledge (Måsøval, 2011, p. 55)(Winsløw, 2006, p. 135).
It is the teachers job to create and devolve this milieu to the student, and to change the milieu if required. The student's work with the milieu, can in some sense be seen as the work that the researcher carries out when working with an open problem. In this way, the student creates new knowledge by taking the same paths as the researcher.

It is essential, that the milieu requires the personalization of the target knowledge, in order for the participants to survive in it (Sierpinska, 1999, p. 2). When the teacher designs the milieu, he or she must conceal the target knowledge so much, that the student is able to acquire this by producing answers to the milieu based on the student's existing knowledge. This design is made through an *a priori* analysis of the learning situations and milieu, which in this thesis is done in Chapter 4. If the milieu is created in this way, and the student acquires the target knowledge by adaptation to the situation, then this situation is called and *fundamental situation*(Winsløw, 2006, p. 144). It is a hypothesis in TDS that all knowledge originates from such a situation and the concept of *fundamental situations* will now be elaborated further:

2.1.2.1 *Fundamental Situations*

A fundamental situation is a didactical situation in which the target knowledge is crucial in solving the mathematical problem. A fundamental situation is therefore fundamental for some piece of knowledge, as the students "survival" in the situation depends on the acquisition of this target knowledge. As stated above, it is a hypothesis that all mathematical knowledge can be characterized by one or more adidactical situations (Winsløw, 2006, p. 144)(Måsøval, 2011, p. 47). It shall be emphasized that it is not necessarily just one single situation that is fundamental, but can be a set of situations in which the mathematical problem is representative of the aspects of the target knowledge, why it provokes the learning of this knowledge (Perrin-Glorian, 2008, p. 3).

The exercises in this teaching experiment is therefore created with the progression in mind, that at some point, the student engages in a situation that is fundamental for applying algebra in the formulation of a proof.

2.1.3 *Didactical- and adidactical situations*

Guy Brousseau makes a clear distinction between what he calls *didactical situations* and *adidactical situations*. I will first define what an adidactical situation is, as this makes it easier to understand Guy Brosseaus definition of a didactical situation.

As described above, learning and teaching is not a direct relation between student and teacher. The teaching of students takes place in an interaction with a milieu, that has a clear target knowledge encapsulated, and hence learning happens first and foremost in this interaction. The student therefore needs to work actively and independently of the teacher with the milieu to personalize the target knowledge and this happens when the student accepts the problem as a her own and solves it on the basis of her prior knowledge and internal logic. This situation where the teacher withdraws and the student work autonomously with a given didactical milieu, is what Guy Brousseau calls an **adidactical situation** (Sierpinska, 1999, p. 2)(Winsløw, 2006, p. 139)(Måsøval, 2011, p. 47).

When the teacher interferes the situation becomes a **didactical situation**. These two terms are important when discussing TDS and the design of a teaching experiment by the basis of TDS.

2.1.4 *Five phases of TDS*

The didactical "game" which the teacher creates can be divided into five situations, that ideally but not necessarily occurs chronologically. These are: Devolution, action, formulation, validation and institutionalization and will here be presented:

2.1.4.1 *Devolution*

This is the situation where the teacher hands over (devolves) the didactical milieu to the student. The teacher introduces the game: tasks, groups, settings, rules etc. to the student. This is a didactical situation as it is the teachers job to assure, that the milieu is created and devolved such that the game becomes winnable to the student, but at the same time the student must accept the responsibility of the learning situation. The student must ensure that she understands the rules of the game, why the student carries a responsibility for the devolution (Måsøval, 2011, p. 47) (Winsløw, 2006, p. 138).

2.1.4.2 *The situation of action*

As the teacher has devolved the milieu to the students, he now completely withdraws from the situation(Sierpinska, 1999, p. 3). The students now engages and examines the milieu by concrete actions relating to the milieu, without the teacher interfering. This situation is therefore an adidactical situation, but the teacher can choose to devolve a modified milieu, if the obstacles seems insurmountable the students (Winsløw, 2006, p. 138). The situation of action is characterized by the students examining the milieu in order for them to determine a "final" strategy, that will allow them to win the game (Måsøval, 2011, p. 51). Knowledge appears in this situation, as a way of solving a problem (Sierpinska, 1999, p. 3).

2.1.4.3 *The situation of formulation*

After the students has examined and finalized their strategy by rejecting other fallible strategies, the students now formulates possible hypothesis' surrounding the given problem (Winsløw, 2006, p. 138). In the situation of formulation, the students must develop a common language in order for them to discuss their findings from the prior situation and agree on some common meaning. The teacher can re-enter in this situation, to assure that the different hypothesis are made visible and to make the students clarify their hypothesis (Måsøval, 2011, p. 52) (Winsløw, 2006, p. 138). If the teacher does so, the situation becomes a didactical situation. In this situation, knowledge appears as personal findings, which needs to be formulated and communicated and therefore slightly de-personalized and de-contextualized (Sierpinska, 1999, p. 3)

2.1.4.4 *The situation of validation*

The students have by now formulated and discussed different possible hypothesis concerning solutions to the given problem, so the students now need to validate or refuse their hypothesis' (Måsøval, 2011, p. 53). This takes place in the situation of validation as a hypothesis needs argumentation in order to be qualified as a winning strategy, according to Brousseau (1997) (Måsøval, 2011, p. 52).

The element of establishing a truth-value is central in the didactical game, as it is through convincing arguments that mathematical knowledge is generated. The students must be ready to support or reject a proposition and the "truth" about a proposition must originate from within the student, from a personal conviction, which can not be learned by reference to the authority (Brousseau, 1997, p. 15). Hence the vital aspect is not only to validate a hypothesis, but just as much to convince one self about the truth. The argumentation and validation can be done by making a formal proof or by making numerous experiments (Winsløw, 2006, p. 139), but ultimately the convincing takes form as a formal proof. This ability to produce a formal proof must be developed gradually in the student. Hence the student must move from the use of repetition and numerical verification towards the use of logic, symbols and clear thinking (Warfield, 2006, p. 23). This is an aspect this master's thesis seeks to examine: The student's use of algebra in a formal proof, that convinces them self and others about the truth-value of a selfmade conjecture.

The teacher works in this situation as a theoretician who evaluates other theoreticians' (the student's) work. The teacher can therefore organize a discussion and a more thorough test of the different hypothesis' or devolve a milieu wherein it is possible for the students to validate their hypothesis (Sierpinska, 1999, p. 3)(Winsløw, 2006, p. 139).

The knowledge in the situation of validation has the status as a theory in the making and not as an institutionalized theory (Måsøval, 2011, p. 53)

2.1.4.5 *The situation of institutionalization*

As the knowledge still has the status as a theory in the making, the teacher takes on the role as a representative of the official curriculum in the situation of institutionalization, in order to validate the students' theories. The teacher therefore presents the official formulations, definitions, laws etc. to the students and the milieu now has its validity rooted in the larger, official mathematical community. The teacher institutionalizes the knowledge in order for the student to decontextualize it, to make the knowledge a part of the cultural knowledge(Winsløw, 2006, p. 139).

In the situation of institutionalization, knowledge has the status as an official law, rather than just an answer to a single, isolated mathematical problem (Måsøval, 2011, p. 54)

2.1.5 *The didactical contract*

When the teacher devolves the milieu and creates a didactical situation, or didactical game, it is a premise that it is winnable and this happens as the student learns a piece of intended knowledge. In order for the student to win this game, she must follow the rules and strategies of the game, which are specific to the knowledge in play. These rules are not simple or explicit and can be described as an implicite *contract* between the student and the teacher, which models the mutual expectations (Winsløw, 2006, p. 145)(Måsøval, 2011, 48). These rules are implicit created, as the student must accept the didactical milieu and take the responsibility for the adidactical learning situation, and at the same time the teacher are responsible for the student's success through the creation of the milieu. It is these rules and expectation between the teacher and student that forms the *didactical contract* (Winsløw, 2006, p. 145 f.) (Hersant and Marie-Jeanne, 2005, p. 116). Hersant & Perrin-Glorian (2005) distinguishe between four dimensions of the didactic contract, which are not independent: The mathematical domain, the didactic status of the knowledge, the nature and characteristics of the ongoing didactic situation and the distribution of responsibility between the teacher and the students (Hersant and Marie-Jeanne, 2005, p. 118). As the didactical contracts are not central to my thesis, I will not go into details about this.

The authors also distinguish between three level in the structure of the didactic contract. There are: the macro-, the meso- and the micro contract. The macro contracts relates to the teaching object, the memo contract concerns the realization of the activity and the micro contract concerns the concrete episode.

Even though the didactical contract is an inevitable element of a didactical situation and can be used to regulate to mutual expectations between the students and the teacher, it can entail some unwanted consequences. These consequences will be described next.

2.1.6 *Consequences of The Didactical Contract*

As written above, the didactical contract is made up of implicit expectations between the students and the teacher, and this creates different paradoxes. One paradox occurs as the contract is implicit and it is only through the breaking of a contract, that it can be made visible and secondly, the contract can not be fulfilled unless it disappears

(Warfield, 2006, p. 33). Another paradox occurs as the teacher poses a problem to the student, which the teacher already knows the answer to. The teacher then wants the student to find the answer without asking him what to do, because of the risk of ruining her chances to find it her self (Winsløw, 2006, p. 146). These paradoxes can result in some unwanted consequences, which Brousseau (1997) divides in to seven types.

As these consequences are not central part the the master's thesis, only the ones observed will be described here:

2.1.6.1 *The Topaze Effect*

The Topaze effect arises as the teacher, in an attempt to obtain the correct answer from the student, poses easier and easier question and in the end gives away the answer. In this way, the teacher takes the full responsibility for the determination of the correct solution to a problem, he already knows the answer to. As the teacher gradually changes the questions, so does the knowledge necessary to produce an answer to the question and hence do the meaning (Winsløw, 2006, p. 148) (Måsøval, 2011, p. 37). This will in the end lead to total disappearing of the target knowledge.

2.1.6.2 *The Jourdain Effect*

The Jourdain effect arises when the teacher is eager to recognize a specifik knowledge in a student's answer, as the student just follow the teachers instructions. As the student answers the teachers guiding questions, the teacher can convince himself and others of, that the student have done some independent, scientific work (Winsløw, 2006, p. 148f.) (Måsøval, 2011, p. 38f.).

2.1.7 *Obstacles in a Learning Situation*

As described earlier, learning is defined as adaptation and sense making to a given milieu, but a common misapprehension is that, if the teaching is done right then this learning will not lead to any misapprehensions (Warfield, 2006, p. 28). According to Brousseau this is not true, as students always face obstacles when new learning occurs, as new learning happens both on the basis of old knowledge, but also against old knowledge (Ibid, p. 28). Brousseau (1997) distinguishes between three types of obstacles: **epistemological-**, **ontogenic-** and **didactic obstacles**.

2.1.7.1 *Epistomological Obstacles*

An epistemological obstacles occurs when old knowledge is in opposition with the new knowledge, but remains a fabric part of this new knowledge. The obstacle occurs as a learning situation is created so old knowledge initially can be used and later replaced by new knowledge, but becomes a part of this new concept. An example of a epistemological obstacle is found in (Warfield, 2006, p. 28), where a child has internalized that multiplication makes whole numbers bigger. This old knowledge is then in struggle with the notion of multiplying with fractions. The old knowledge is not false, but it is in opposition with the new knowledge.

2.1.7.2 *Ontogenic Obstacles*

Ontogenic obstacles are related to cognitive and neurophysical limitations in the student. These obstacles occurs when the teacher tries to teach a subject, that is beyond the age-related, mental abilities of the student. An example of an ontogenic obstacle, is when trying to teach a child of age five, to subitise sets containing more than 6 objects, which is more or less impossible because of a missing development of the brain (Warfield, 2006, p. 29).

2.1.7.3 *Didactical Obstacles*

Didactical obstacles are related to the teaching because of choices made by the teaching system or the teacher. Warfield (2006) illustrates this with an example in teaching of decimals: The students learn by computational examples, that decimals beyond the hundredths place have very little influence on the result. They therefore learn that $\pi = 3,14$ and this remains, as the didactical obstacle hinders the effort of teaching approximation (Warfield, 2006, p. 29).

2.1.8 *Didactical Variables*

As one of the main elements in this master's thesis are the didactical variables and the affect they have on the students' work in the study course, I will here briefly describe what a didactical variable is.

A didactical variable can be variations in the didactical milieu, variations in the devolution and the institutionalization (Winsløw, 2006, p. 143), hence they are variables that influence the learning and which the teacher can choose the value of. This could for example be to complicate the numerical calculations, which the students are asked to do in a given exercise.

A didactical variable concerns a learning situation and the properties of the didactical variable either helps or hinders a specific piece of knowledge to be achievable. When the didactical variables and their values are determined, it is possible to study these in causing a piece of knowledge to evolve (Brousseau, 1997, p. 66). When planning and evaluating the design of teaching situations, it is therefore essential, that these variable are made explicit (Winsløw, 2006, p. 143).

An example could be to let the student determine the sum of the diagonal numbers in the matrix M(below) and then find the difference between these. This would give the following result: $(1+9)-(2+8)=0$.

If we change the didactical variable and ask the student to determine the difference between the product of the diagonal numbers we get a different result: $(1 \cdot 9)-(2 \cdot 8)=-7$

M= $\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$

The didactical variables can be used to change the design of an exercise, in order to make the target knowledge achievable or to change the target knowledge in an exercise.

2.2 TEACHING AND LEARNING ALGEBRA

The following section contains a description of the aspects of algebra, which is of main interest to this master's thesis. The section starts by describing some of the difficulties, students faces when working with algebra in high school. This section is mainly a recap of the difficulties found in (Bosch, 2015) and (Martinez and Castro Superfine, 2012), but it points to some central aspects in the process of learning and using algebra.

Then follows a section describing some of the affordances algebra have and which this thesis hopes to illustrate and make use of.

Before closing the chapter with a description of the didactical variables relevant to the thesis, I will describe and discuss the integration of algebra and proof, my definition of algebra and the characterization of proof.

2.2.1 *Difficulties with learning algebra and algebraic notation*

Previous studies has shown the students' difficulties when learning the use and techniques of algebra (Martinez, 2008, p. 23). As Bosch (2015) argues, algebra is largely identified as equation solving of first and second degree equations in secondary school (Bosch, 2015, p. 6). This approach has reduced algebra to a formal frame of terminology: *"algebraic expression; evaluation; terms, members and coefficients; similar terms; equations, equalities and identities; etc."* (ibid, p. 6), where the students learn to *"simplify"*, *"develop"* and *"factorise"* as goals in itself. This way of learning algebra is insufficient, if the pupils are to learn the great variety of manipulations that is needed to use algebra in a functional way (ibid, p. 6).

In (Ruiz et al., 2007, p. 1) the authors points out that in secondary school, letters generally play the role of unknowns in equations or as variables in functions. Letters rarely appears as the result of algebraic work or as unknowns, variables or parameters in algebraic models.

Måsøval (2011) argues that it in order to obtain a broader and deeper understanding of algebra, it is necessary to use and see algebra than more than primarily a *"...syntactically-guided, symbolic manipulation..."* (Måsøval, 2011, p. 81). It is here argued that this deeper and broader view of algebra, is essential in order to support the integration of *algebraic thinking*. Algebraic thinking is a complex composite of related and intertwined forms of reasoning, which can be divided into five subcategories. Two of these are *Algebra as Generalising and Formalising Relationships and Constraints* and *Algebra as Syntactically-Guided Manipulation of Formalism* (Måsøval, 2011, p. 82), where the first covers the rules of generalized arithmetics, with focus on the field axioms and generalizations established about number properties or relationships, e.g. finding and determining regularities in the times table, calendar and so on. It is this subcategory which is of main interest for this thesis. The other category covers among other things, factoring and simplifying algebraic expressions (ibid.). If we want the students to master algebraic thinking, it is not sufficient to teach algebra as symbolic manipulation and reduction, as this way of teaching algebra only supports *Algebra as Syntactically-Guided Manipulation of Formalism* (Måsøval, 2011, p. 83).

By teaching algebra as the study of generalization of numerical patterns, the focus shifts from dealing with letters as unknowns to letters as variables, and determining relations between numbers (Måsøval, 2011, p. 84).

2.2.2 *Algebra: A Tool to Solve Mathematical Problems*

By purely teaching algebra as a set of techniques one of the affordances of algebra is neglected. Martinez & Castro (2012) argues that algebra has the potential to *"...make explicit what previously was implicit..."*(Martinez and Castro Superfine, 2012, p. 123), which the authors rates as one of the most important features of algebra. For instance, by the use of algebra it is possible to show, that the sum of three consecutive integer numbers, always will be a multiple of three:

$$a + (a + 1) + (a + 2) = 3a + 3$$

By factoring 3 from each term, we can rearrange the expression and see, that the sum is always three times the second number: $3(a + 1)$. A fact that might not be clear to all students. By making the intermediate results explicit, the students will be acquainted with the laws of distribution, commutativity and associativity, as they rearrange the terms:

$$a + (a + 1) + (a + 2) = (a + a + a) + (1 + 2) = 3a + 3 = 3(a + 1)$$

So by transforming an algebraic expression into another equivalent expression, it is possible to reveal information that was hidden in the first expression. It is important to emphasize, that it is not necessarily solely the new expression, which produces new information, but also the process by which this new expression is attained (Martinez and Castro Superfine, 2012, p. 124). By using algebra, it is then possible to generalize patterns and determine relationships in number systems and thus capture all statements in a single mathematical expression (ibid, p. 124).
Bosch (2015) names this advantage of algebra as the *"universal arithmetics"* (Bosch, 2015, p. 10): Algebra gives the possibility to study relationships between arithmetic patterns independently of the related objects. This makes the student able to make a generalized solution to a whole type of problems, instead of just a single answer, which is usually the case with arithmetic (Bosch, 2015, p. 10). This feature of algebra is a central component in my teaching design: The students' work with patterns to discover a built-in principle, in order to introduce algebra to prove the existence of the pattern that characterizes is.

When algebra is being introduced through equation solving, factorizing, reduction etc. the mathematical problems are of an algebraic form. Students must solve an algebraic problem, which in many cases are isolated problems and the tools to solve the problems are, more or less, meaningless procedures concerning manipulation of symbols (Bell, 1995, p. 44) (Arcavi, 1994, p. 33). In contrast to this, as the students work with algebra as a tool to prove the existence of patterns or relationships, the use of algebra shifts focus from being the problem in itself, as in the case with equation solving, factorizing, reduction etc., to be a tool to enlighten and solve the problems involved. In this way algebra does not appear as an isolated domain of the mathematical curriculum on the same level as geometry, arithmetics and so on, but as a general tool to model and solve problems across the mathematical curriculum. In fact, algebra becomes the key tool to approach theoretical questions, which can not be solved within these domains (Bosch, 2015, p. 10).

To sum up, the use of algebra to prove a certain pattern or relationship in a number system, has two big advantages: First of all, it is possible to capture all cases in a single expression, which is necessary if the student is to make and prove a general mathematical statement. Secondly, in using algebra to manipulate and transform a mathematical expression into an equivalent expression, it is possible to explicate what we wanted to prove, and through the intermediate expressions reveal information that else would not be easy to see (Martinez and Castro Superfine, 2012, p. 124).

2.2.3 *The integration of algebra and proofs in this teaching experiment*

By the above, it is necessary to teach students in different aspects of algebra, in order to promote the their algebraic thinking. By this I mean, that it is not enough to teach algebra as *Syntactically-Guided Manipulation of Formalism* (Måsøval, 2011, p. 83). In order to let the students work with algebra as a tool to prove conjectures and make generalization of arithmetic patterns, it is necessary that the students are given the possibility of formulating a conjecture about a possible result, when examining a given pattern. For now, *pattern* should be understood as a result that systematically appears, when making some prescribed calculations. An example of a pattern is the outcome that appears when calculating the determinant of a 2×2 matrix, located somewhere in a Calendar (see figure 3). Invariant of the placement of the matrix, the outcome when calculating the determinant will always be -7[3] $((7 \cdot 15) - (8 \cdot 14) = 105 - 112 = -7)$ and hence the pattern here is -7. The outcome need not be a fixed number, but there need to be some systematic in the outcome in order to call it a pattern.

		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Figure 3: An example of a 2×2 matrix containing the numbers $\{7, 8, 14, 15\}$

According to Martinez & Castro Superfine (2012), this way of teaching algebra can be done by using algebra as a modeling tool, in order to integrate algebra and proofs and I have therefore adopted three design principles from (Martinez and Castro Superfine, 2012, p.124), which I seek to integrate in my teaching design:

- In order for the student to prove that a universally quantified statement is true, it is not enough, that the student exhausts all numerical possibilities. The teaching design therefore focuses on, that the students move away from using empirical evidence to using deductive reasoning, when proving their conjectures
- The role of the proofs in the teaching design is not just to validate a statement, but also to reveal and explain why a specific pattern or relationship in a number table[4] occurs. Hence the

3 Examining the determinant and algebraically show, that the result is always -7, will from now on be denoted The Calendar Problem
4 Relationship in a number system will be explained later

proofs must foster the students' understanding of why a specific phenomenon happens.

- As conjecturing and proving are interrelated and essential in the construction of mathematical knowledge, a crucial part of the design is that the student have to construct and produce their own conjectures, before proving them.

The reason for basing my design on the above principles, is to enable the students to make proofs based on their deductive reasoning. These proofs must reveal information about a particular pattern or relationship, in order to show why a specific mathematical phenomenon happens. The general imagined trajectory of the students' work in this study course is then: Through numeric examples based on investigation of a problem concerning a number table, the students formulate a conjecture about a problem and engage in a "proving situation" (Martinez and Castro Superfine, 2012, p. 125) in order to come up with evidence to show that the conjecture is always true, and why this is so.

As proving situations are central to this thesis, a deeper discussion and definition of proofs is presented next

2.2.4 *The Definition of Proof in this Master's Thesis*

As the students' work in proving situations is central to the thesis, it is relevant to ask what is a proof? Balacheff (as cited in Martinez, 2008, p. 9) differentiates between two types of proofs: The *pragmatic* proof and the *intellectual* proof.

The pragmatic proof is characterized by being attached to the particular assignment that constitutes it, and are depended on the contingent material conditions. In this sense, a "proof" in the Calendar Problem (see figure 3) could be to use non-exhaustive numeric examples to show, that the outcome is always -7.

The intellectual proof are detached from the particular assignment. It is by the use of intellectual proofs, that it is possible to proof a universal statement, ex. that the outcome of the Calendar Problem is always -7. Here algebraic notation plays a major role, as it is a tool, to which the student can show a particular pattern.

The two different types of proofs are worth mentioning, as it is possible to exhaust all possible combinations of square matrices in the calendar problem. Hence it is possible for the student to proof a conjecture by pragmatic proofs, but this will not be accepted as a valid proof [5], as it relies on empirical evidence and not on mathematical

5 A proof by exhaustion would in another context be a perfectly fine proof, to prove a conjecture about the calendar

properties, like algebra. The numerical method does also not show why the outcome is -7.

This aspect of showing why a specific mathematical phenomenon happens, is emphasized in (Knuth, 2002). Knuth (2002, p. 486f.), building on (Hanna, 1990), distinguishes between proofs that only shows that a theorem is true, and proofs that also *explains* why a theorem is true. The author attach importance to the fact, that proofs must be used *"... as a vehicle to promote mathematical understanding"* (ibid., p.486) and that this should be the primary function of proofs in secondary school (ibid., p. 487). Hanna (1990) also emphasizes that an explanatory proof is not less valid than a formal proof (Hanna, 1990, p. 12).

Lastly Miyakawa (2002) also uses Balacheff (Balacheff, 1987) as theoretical frame of reference, when defining the concept of proof (Miyakawa, 2002, p. 353). Miyakawa (2002) operates with five types of proofs, which ranges from "pragmatic proof" to "intellectual proof". What distinguishes a pragmatic proof from a intellectual proof is that the pragmatic proof is based on numerical examples, where the intellectual proof is universal and based on logic, algebra or a method of exhaustion (Miyakawa, 2002). Besides the tools used (logic, algebra etc.), an intellectual proof is also based on the underlying rationality, the language level and the nature and status of knowledge (Miyakawa, 2002, p. 354). The intermediate types of proofs are: **naive empiricism**, **crucial experiment**, **generic example**, **thought experiment** and **mathematical proof**, where the first four types rely on numerical evidence or oral formulations, the last type of proof is based on mathematical properties.

In summary, I have adopted the view that a proof is an explanation that is accepted within the classroom, it shows why a mathematical phenomenon happens and it is based on mathematical properties. The students were during the study course encouraged to use algebra as a mathematical property.

The term 'algebra' has been used a lot, without making the definition clear. The next section aims to formulate a clear definition of what the term 'algebra' covers in this master's thesis

2.2.5 *What is Algebra?*

By the above, it is evident that there are many aspects to algebra, why it can be difficult to make a clear definition of what algebra is. Historically, the word *algebra* dates back to the book *Al-kitāb al muẖtaṣar fī ḥisāb al-jabr wa-l-muqābala* written about 825 AD by Muhammed ibn Mūsā al-Khwārizmī and is a corrupted form of the word *al-jabr*(Katz, 2009, p. 273). Drijvers, Goddijn & Kindt (2011) discusses the question *"What is algebra?"* and some of the many connotations that relates to the word algebra ranges from *"... everything (...) in which a letter appears..."* over equations solving, reduction, finding derivatives, tables and graphs to abstract study of number systems including topics as rings, fields, group theory etc. (Drijvers et al., 2011, p. 7 f.).

As many of the above mathematical operations and topics far exceeds what is expected from a first year high school student, the term 'algebra' in this master's thesis will not include all of these elements.

In the book by al-Khwārizmī, algebra is primarily understood as verbal formulations of *restoring and comparing*(Katz, 2009, p. 271), meaning the laws for equation solving. As my definition of proof is based on mathematical properties, a solely verbal argument of a kind seen in *al-jabr* will not be seen as algebra, but merely as pre-algebra as it does contain some algebraic elements.

When algebra is used in this master's thesis and when the students are asked to introduce algebra, the term primarily means the work with symbols as generalized arithmetics. Hence, the focus will be on the introduction of symbols or letters as variables, the mathematical work with these and modeling using algebra. This includes setting up parentheses, setting up relations between variables, creating formulas and expressions containing symbols or letters.

2.3 DEFINITIONS AND CLARIFICATION OF CONCEPTS

Before I present and describe the didactical variables important to this master's thesis, I will clarify and define some concepts which will be frequently used from here on and which normally have another meaning.

Result: The last numerical result that appears, when going through the described steps in an exercise. If the students are asked to add the largest and smallest number in a given rectangle containing the numbers $\{3, 4, 5, 6\}$ and subtract the second largest number from this sum, the result will be 4 $((3+6)-5=4)$. The result is always dependent on the numbers involved.

Pattern: The pattern that appears, when the student makes some predefined calculations. For example, the pattern in Exercise 1 is, that the result will always be the second smallest number in the rectangle. The pattern is therefore not dependent on the numbers involved, but an expression for the general outcome. In some cases the result and the pattern are the same, as in the Calendar Problem (figure 3) where the result is always -7 and therefore the pattern is -7.

Pattern analysis: The mathematical analysis of patterns, ranging from numerical examination to algebraic proofs.

Number Table: The main number system, that is being used in the particular exercise. This could be the Sum table, the Calendar, the Multiplication table and so forth. These can be found in Appendix B

Relation: Or relation of the Number Table. The particular relation that appears between neighbouring numbers in the number table. For example, the relation in the Sum table is, that a number increases by one, as the row- or column number increases by one.

Initial expression: An initial, algebraic expression which often needs to be reduced or rewritten by the use of the arithmetic axioms, in order to reach a final expression. This final expression often explicitly shows the desired pattern or needs to be interpret, in order to reach the desired conclusion.

Generic model: Or generic diagram. This is a model involving algebra, showing a "general" scheme of the numbers involved in the exercise. In the above example with the rectangle containing four numbers, a generic diagram could be:

$$\begin{bmatrix} x & x+1 & x+2 & x+3 \end{bmatrix}$$

A generic model therefore requires one or more independent variables and a formulation of the relationships between the other variables in the generic model.

Independent/dependent variable: In the above diagram, the x will be denoted the independent variable and the other entrances will be called dependent variables. Hence, the entrance of the diagram from which the students generates the relationships to the other entrances, will be denoted the independent variable.

Figure: The geometric figure that is central in the exercise in question. In the above example, the figure would be a 1×4 rectangle and in the Calendar Problem (figure 3) it is a 2×2-square.

Course of Study: The three days of lessons, which formed the empiricism of this master's thesis

Teaching Design: The design of the study course, including exercises, composition, theory etc.

Procedural Calculations: The pre-defined calculations the students are asked to do in a chronological order. This could for an example be to: First add the largest and smallest number in a rectangle and then secondly subtract the second largest number.

2.4 ANALYSIS OF DIDACTICAL VARIABLES

As one of the research questions addresses how the didactical variables affects the students' work, I have singled out six didactical variables which constitutes the main focus of attention in this master's thesis. The didactical variables are chosen in order to create a milieu, in which it is possible for the students to integrate algebra and proof. These didactical variables are divided into two groups: (1) The first group has to do with the form of the exercises and (2) the second group concerns possible solution to these exercises. They can of course not be seen as isolated variables, as they have influence on each other, but I have tried to make a clear distinction.

The didactical variables will for future reference be denoted: DV1, DV2, DV3, DV4, DV5 and DV6 and these are:

1. a) DV1: The Number Table
   b) DV2: Pattern to Investigate
   c) DV3: Calculations Needed
2. a) DV4: The Number of Variables
   b) DV5: The Choice of Variable
   c) DV6: The Algebraic Prerequisites

2.4.1 *Didactical variables concerning the form of the exercises*

2.4.1.1 *DV1: The Number Table*

One of the main variables when discussing proofs of patterns in number table, is the choice of number table. Martinez (2008) uses the Calendar as the number system, that forms the basis of her teaching experiment (Figure 4).

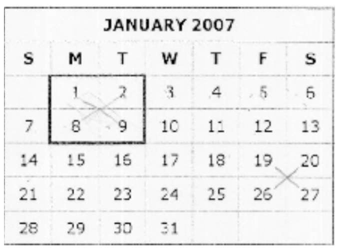


Figure 4: The calendar from (Martinez, 2008)

This number table has a certain relation, which the student is expected to unravel in order to create a generic model: If we fix any number, then a number to the right is always one larger and the number below is always 7 larger. A 2×2-matrix showing the relation in the Calendar, could look like:

$$\begin{bmatrix} x & x+1 \\ x+7 & x+8 \end{bmatrix}$$

If we instead consider the Multiplication Table (Figure 5), we obtain a number table with a different and more complex relation. It is more complex, as every number in the Multiplication Table is a product of two independent numbers and the increase in the numerical values are therefore dependent of the row- and column number. This is different from the Sum Table and The Calendar where the numerical increase is independent of the row- and column number.

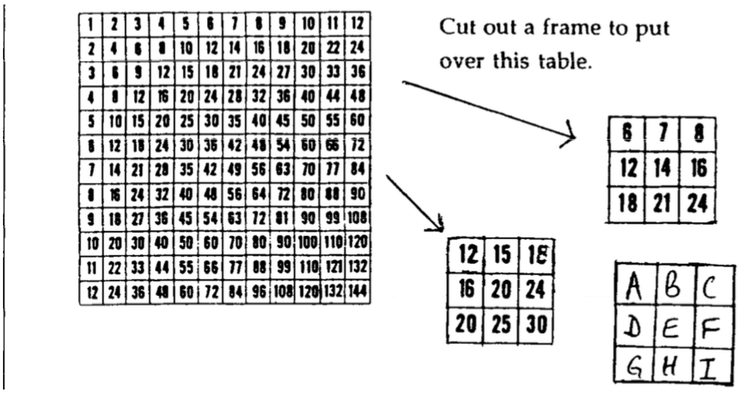


Figure 5: The Multiplication Table from (Bell, 1995)

If we use use a 3×3-matrix in this number table as an example, then the numerical increase in the value of the entry is depended on the location of the matrix. In figure 5, two 3×3 sub matrices are shown, if we discount the matrix containing letters. If we fix a number in the matrix to the right, then a number increases with $1, 2$ or 3 to the right, depending on the row number and with $6, 7$ or 8 going down, depending on the column number.

If we examine the 3×3 matrix to the left, then the increase is $3, 4$ or 5 going right and $4, 5$ or 6 going one down. A 2×2-matrix showing the relation in the Multiplication Table, could look like:

$$\begin{bmatrix} xy & x(y+1) \\ y(x+1) & (y+1)(x+1) \end{bmatrix}$$

where x denotes the number of the first row and y denotes the number of the first column, why $(x, y) \in \mathbb{N}$. This is another characteristic of the Multiplication Table that differs from the Calendar: It is possible to make the Multiplication Table infinite[6] large and this makes it impossible for the student to exhaust all possible combinations of

6 Or as large as the teacher desires

2×2 square matrices.

The two examples shows, that one way to increase the complexity of the exercises is to introduce a number table with more complex patterns. In order to make a progression in the difficulties in the exercises, the introduction of a new number table forces the students to create new generic models as the relation changes. One of the reasons that the relation of the Multiplication Table is harder to describe by the use of algebra, is because it requires the introduction of two independent variables.

2.4.1.2 *DV2: Pattern to investigate*

The two prior examples, both dealt with generic models of 2×2 matrices. But as each number table contains a unique relation in its construction, the design of the exercises makes it possible to twist these patterns and make the students work with different calculations. To explicit this, I will here list some different ways to make the students work with patterns and analyze, what kind of mathematical abilities it takes to solve the exercises:

One way to change the pattern, is to change the procedural calculations or the size of the sub matrix. If the students are asked to determine the difference between the <u>sum</u> of the opposite corner numbers in a 3×3 (see figure 6), this makes way for new calculations.

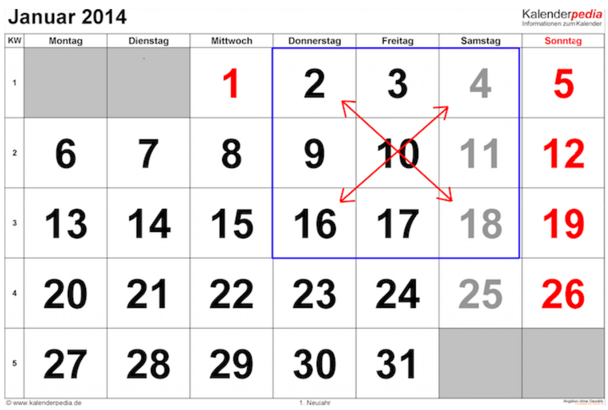


Figure 6: The sum of the corner numbers

This difference of the sums always equal 0, as

$$(x + (x + 16)) - ((x + 2) + (x + 14)) = 2x + 16 - 2x - 16 = 0$$

when x is the number in the upper left corner. A problem like this one, only requires addition, subtraction and the associative and commutative law to solve, though the biggest obstacle for some students, could be to distribute -1[7]. If the students are able to make these cal-

7 (Martinez and Castro Superfine, 2012, p. 129) reports, that one of challenges the students faced, was how to distribute -1

culations, then the difficult part would be to introduce a letter as the independent variable, in this case x, as a generalization of the number in the upper left corner and to make the relations to the dependent variables.

A way to let the students progress in this work, is to let them calculate the difference between the product of the the diagonal corner numbers in the same matrices. This would end up, with the following desired calculations

$$\begin{aligned} x(x+16)-((x+2)(x+14)) &= \\ x^2+16x-(x^2+2x+14x+28) &= \\ x^2+16-x^2-16x-28 &= -28 \end{aligned}$$

One interesting and maybe surprising thing is, that the result is not -14. Martinez & Castro (2012) reports, that the students quickly realizeed that the outcome of the generic diagram for the 2×2 matrix was -7, as there are 7 days in a week. Based on this result, they reasoned that if the week had *d* days, then the result would be -*d*. As the 3×3 matrix "adds an extra week", some student might make the initial conjecture, that the outcome of the 3×3 matrix would be -14. If we made the same procedural calculations for the 4×4-matrix the outcome would be -63.

An algebraic expression showing the relationship between the result and the size of the matrix, requires that the student introduces a parameter (n) to describe the size of the matrix. Without going into further details, the final expression resulting from these calculations would be:

$$-7n^2+14n-7$$

from which it is seen that the final expression only depends on the size of the matrix. The introduction of a parameter in the calculations and proof, is also a way to increase the degree of difficulty in the exercises.

Another aspect of letting the students examine other figures or matrices larger than 2×2 is the aspect of surprise, which works as a way to generate intrigue among the students, as the feedback from the exercises contradicts the students' anticipation (Martinez, 2008, p. 92). Martinez (2008) tries to trigger the curiosity of her students by letting the them examine which matrix that produces the biggest result, when making the above calculations. The author hypothesis that, the students will initially believe, that the matrix with the numerical biggest entries also generates the biggest output. It is this curiosity, that is supposed to intrigue the intellectual need to show why a certain supposition is wrong and in order to prove why this is so, the optimal solution to the posed problem requires the use of algebra. Hence, the problems are designed to promote an experience that raises an intellectual need of using algebraic tools to solve the problems (Martinez, 2008, p. 92).

Bell (1995) also underlines the strengths of using curiosity as a motive force in students work with algebra. Bell (1995) uses exercises where the answer is ambiguous, as he emphasizes exercises where relations can be true always, sometimes or never (Bell, 1995, p. 52). An example of this, can be seen in figure 5, where the relation $E = 2A$ is sometimes true (an example is when $A = 6$ and $E = 12$). In this way, the students curiosity paves the way for the need of introducing algebra as a mean to satisfy an intellectual need, or as Bell writes it:

> *"...illustrate how algebraic symbolism becomes advantageous for more complex problems. They also remind us of the motivating power of a good puzzle, which we should aim to retain in school work."* (Bell, 1995, p. 59).

Another pragmatic advantage is just to renew the exercises in order to maintain the students' interest and concentration, by changing the figures in the exercises. Martinez (2008) changes the figures from square matrices to rectangles ($1 \times n$ matrices and $n \times 1$ matrices) and also changes the procedural calculations, from the difference of the product of the corner numbers to the difference of the sum of the same numbers (Both examples are described above). Bell (1995) lets the students work with L-shaped diagrams, pyramid-shaped diagrams and also lets the students examine their own diagrams.

Hence it is possible to change the patterns by changing the figures and the procedural calculations, as this changes the intermediate algebraic calculations the students needs to make, in order to go from an initial expression to a final expression.

To summarize, there are two main ways to change the pattern to investigate: Change the figure and change the procedural calculations. This leads to the last didactical variable concerning the form of the exercises: The Calculations Needed.

2.4.1.3 *DV3: Calculations Needed*

As the students make numerical examinations of the pattern and number table in a given exercise, they are directly and indirectly asked to make some numerical calculations. All exercises contains some procedural calculations, which I now will exemplify by the use of figure 7[8]:

In the horizontal (black) rectangle, the students are asked to add the largest and smallest number and from this sum subtract the second

8 These examples are made up by me

M	T	O	T	F	L	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

Figure 7: Example of Different Calculations

largest number: $(3+6)-5=4$. These calculations involves addition and subtractions and should be relatively easy to most students in the first year of high school. To set up a joint numerical expression would only require a sign of addition and subtraction, though some student might use parentheses like above.

In the three vertical boxes (two blue and a red), the students are asked to find the product of the numbers in each of the blue boxes and the sum in the red box. They are then asked to subtract the product from the first blue box from the last blue box and from this difference, subtract the sum of the numbers from the red box: $(12 \cdot 19) - (10 \cdot 17) - (11+18) = 228 - 170 - 29 = 29$. These calculations involves addition, subtractions and multiplication. The joint numerical expression requires that the students sets up parentheses and uses the distributive law in order to distribute -1.

With out working it out in further details, the complexity of the exercises can be changed by involving different calculations like division, powers and setting up parenthesis. This didactical variable also involves a conflicting influence on the exercises: It is important that the possibility of determining a pattern does not drown in complicated numerical calculations, but at the same time the calculations must not be so easy that it eliminates the need for an algebraic proof.

Next, the didactical variables concerning the students' solutions to the problems, will be described

2.4.2 *Didactical variables concerning the solution to the exercises*

It shall be mentioned that the didactical variable 'The number of variables' also concerns the construction of the exercises, but are included in this section as it is essential in solving the exercises properly.

2.4.2.1 *DV4: The Number of Variables*

There are two aspects to this didactical variable: The increase in the complexity of the solutions the students need to make, by introducing exercises that requires several independent variables and the numbers of 'independent variables' the students chooses to use in their solution. The latter aspect will first be described:

As reported in (Martinez, 2008, p. 110), some students initially introduced several "independent" variables to create a generic diagram, as a solution to the Calendar Problem. One student (Brian) created a generic diagram (α) with the use of four variables: A, B, C and D, while another student (Cory) found a relation between the numbers in the upper row and the lower row (β):

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad \beta = \begin{bmatrix} a & b \\ a+7 & b+7 \end{bmatrix}$$

The students then later realized, that there were a relation between all of the numbers, which made it possible to write up a generic diagram with just one independent variable. This element of introducing several variables and then later working out the relations between the variables is a natural step, in solving exercises of these types. This is outlined in (Martinez and Castro Superfine, 2012, p. 123), where the authors describes the modeling process using algebra from (Chevallard, 1989).

Hence, this exercise only needs one variable to create a generic model, but it can be fruitful for the students to initially use several variables. Other exercises demands the introduction of multiple variables in an algebraic solution. This will be described next.

A way to increase the complexity and pattern of the assignments is to introduce number tables, with relations that requires more than one variable to prove. As outlined in (Martinez and Castro Superfine, 2012, p. 124), the need of more variables increases the difficulty of the assignment, as the students need to hold track of different variables and the relations among them.
Martinez & Castro (2012) also introduces the notion of parameter, as some of the problems also contains weeks where the length of the weeks varies. An example of the introduction of a parameter was shown above, where a parameter was needed to describe the size of the square matrix. This forces the student to: (1) identify the variables and parameters, (2) to determine the relationship between these and (3) to analyze the behavior of the outcome as a function of the parameters and variables (Martinez, 2008, p. 94).

If we compare the Calendar with the Multiplication Table, then the

Calendar only needs one variable to describe the relation, as the difference between two neighboring numbers is invariant of the placement of the figure.

This is considerable different when the exercises are based on the Multiplication Table, as it requires two variables, as described in *DV1: The Number Table*. The Number of Variables are therefore closely related to the didactical variable The Number Table

If we focus on the determinant problems from the Calendar and Multiplication Table, then an algebraic proof to both problems requires that the student is capable of setting up a generic model and formulating an initial expression, but also that the students know how to multiply two parentheses.

If the determinant problem concerned the Calendar an algebraic proof could be:

$$\begin{aligned} x(x+8)-(x+1)(x+7) &= \\ x^2+8x-(x^2+7x+x+7) &= \\ x^2+8x-(x^2+8x+7) &= \\ x^2+8x-x^2-8x-7 &= -7 \end{aligned}$$

If we instead calculate the determinant of the generic diagram in the Multiplication table, it gets even more complex:

$$\begin{aligned} xy(y+1)(x+1)-x(y+1)y(x+1) &= \\ xy(yx+y+x+1)-(xy+x)(yx+y) &= \\ x^2y^2+xy^2+x^2y+xy-(x^2y^2+xy^2+yx^2+xy) &= \\ x^2y^2+xy^2+x^2y+xy-x^2y^2-xy^2-yx^2-xy &= 0 \end{aligned}$$

The above algebraic calculations requires, that the student adds, subtracts and multiply numbers, and uses the distributive, associative and commutative laws. This is something that can pose a challenge for first year high school students.
A problem that also emerges when calculating determinants in the two described number systems, is that the student must know how to power numbers to reach the result that $x \cdot x = x^2$, $x \cdot y = xy$ and even more complex that $y \cdot xy = xy^2$.

By analyzing the Multiplication Table closer (figure 8), it is also seen that it contains a relation, which is describable by the use of one variable.

As seen on figure 8, the Multiplication Table comes with a natural diagonal[9] which divides the timetable in two symmetric parts. As described above, the Multiplication Table requires two variables to

9 The white/grey numbers

x	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Figure 8: The diagonal in the Multiplication Table

describe the relation and therefore to set up a generic model, but if the generic model is centered around the diagonal it is possible to set it up with the use of only one variable. If we pick a generic model, where the first entry is a white/grey number, then we obtain the following generic model for a 2×2 matrix:

$$\begin{bmatrix} x^2 & (x+1) \cdot x \\ (x+1) \cdot x & (x+1)^2 \end{bmatrix}$$

By letting the students work with the diagonal in the Multiplication Table, it could ease the transition from the Calendar to the more complex number system in the Multiplication Table. By letting the complexity in the number systems follow a natural progression, the students will work with exercises and patterns which is describable with the use of one variable and hopefully this would make a natural introduction to the Multiplication Table, and still make the students work in the zone of proximal development (Dolin, 2006, p. 340).

The next didactical variable do not concern the number of independent variables, but the placement of the independent variable

2.4.2.2 *DV5: The Choice of Variable*

As long as the students make a numerical analysis of a pattern, the work is strongly directed by the exercise description and the procedural calculations. In the Calendar Problem (figure 4) the students are asked first to calculate the product between the number in the upper left corner and the lower right corner. They are next asked to calculate the product between the number in the upper right corner and the lower left corner, and finally they are asked to subtract the second number from the first. This is a prescription most students can follow.

But as the students are asked to make a proof of their conjecture, the work happens on the students' own initiative. They no longer have a detailed description to slavishly follow, as they must base their work on their own conjectures and ideas on how to make their generic model. This makes room for deviations in the students' work, as there are different ways to create a generic model.

An example of this, occurs in (Martinez and Castro Superfine, 2012, p. 130) as the students create their generic model in groups to problem 9. The exercise is similar to the Calendar Problem, but Martinez & Castro (2012) in problem 9 introduces a week of 9-days. As the students constructs a generic model, two different models occurs[10]:

$$A = \begin{bmatrix} x & x+1 \\ x+9 & x+10 \end{bmatrix} \qquad B = \begin{bmatrix} a-10 & a-9 \\ a-1 & a \end{bmatrix}$$

Depending on where to place the independent variable, the relationships of the depending variable in the model changes. In matrix A, the independent variable is placed in the upper left corner, and hence the students need to use addition in order to generate the depending variables. In matrix B, the independent variable is located in the lower right entry, and the students therefor need to use subtraction to generate the depending variables. Already at this point, these different ways to generate the generic diagram, can result in disturbances in a first-year high school student, when represented with an alternative way to generate a generic model. This disturbance is important for the learning of proofs in high school mathematics, which I will return to.

The two different generic diagrams also produces different intermediate calculations, that both ends up with the same answer: -9. This also gives the problems an aspect as an *"open problem"*(Danish: Åben opgave), as there are many ways to reach the conclusion and none of the above two generic diagram is better than the other. The conclusion here is the same, namely that the answer is -9 when there are 9 days in a week, but the essence lies in the different ways to reach this conclusion. These two, seemingly different, ways to solve this task, can seem divergent for some students, as they often are puzzled by different ways to solve a task.

The algebraic proof based on diagram A would be:

$$x(x+10)-(x+9)(x+1) =$$
$$x^2+10x-(x^2+10x+9) =$$
$$x^2+10x-x^2-10x-9 = -9$$

And for diagram B:

10 Martinez & Castro do not use the labels A and B. These are used here for later references

$$a(a-10)-(a-9)(a-1) =$$
$$a^2-10a-(a^2-10a+9) =$$
$$a^2-10a-a^2+10a-9 = -9$$

This disturbance in the learning trajectory of the student, can be used in a constructive way, if handed correctly by the teacher, which will be discussed in Chapter 3.

What is important from this, is that there are many ways to generate a generic model depending on where to place the independent variable and that these different diagram makes "different" proofs.

Depending on the pattern, relation of the number table and the procedural calculations, the proof requires some *algebraic prerequisites*, which the students must be able to handle in order make a proof and this will be discussed next.

2.4.2.3 *DV6: The Algebraic Prerequisites*

The last didactical variable I will give a detailed account for, is one of very big importance: The algebraic abilities needed to solve the problems. Before describing this variable, it shall be made clear, that the term *algebraic prerequisites* both covers the algebraic operations that is needed in order to formulate a written algebraic proof for a given exercise, but also the algebraic skills or abilities the students need to have in order to formulate a proof and interpret the result. Hence, one thing is to make the algebraic operations or calculations, another thing is to understand what the algebraic operations and the result means. This is described earlier, where it was denoted *algebraic thinking*. To avoid misunderstandings it shall be made clear, that the didactical variable primary concerns properties of the situations and not the student, though these can be said to have a coherence.

When the students formulate an algebraic proof for a given exercise there will naturally occur algebraic obstacles that they need to overcome in order to obtain the desired answer. First of all, the students need some algebraic knowledge in order, to introduce a variable as a *generalized number*, generate a generic model and formulate an initial equation. These obstacles relates to the students ability to spot the pattern and describe this pattern with one or more variables. This requires, that the students has an understanding of algebra in order to generalize a number by the use of a symbol. A generic model for the Calendar Problem could be:

$$A = \begin{bmatrix} x & x+1 \\ x+7 & x+8 \end{bmatrix}$$

The creation of the generic model requires, that the students make a relationship between the independent variable and the depended variables, in this case by the use of adding a number. This could in another exercise require multiplying with a number or another variable. If the student are to take the step from the generic model to a initial expression, it requires that the student combines the procedural calculation with the generic model in order to formulate:

$$x(x+8) - (x+7)(x+1) = x^2 + 8x - x^2 - 8x - 7$$

which again requires that the student uses parentheses, distribute -1, the associative and commutative laws. Depending on the pattern, the number table and the calculations needed, the algebraic prerequisites requires more or less of the student.

If we take one step further and look at a problem from the Multiplication Table (see Section 2.4.2.1), then it is clear that the algebraic prerequisites needed to solve the determinant problem from the Multiplication Table requires that the students know how to multiply two parentheses together. It also requires that the student is familiar with multiplying with letters, as described in the same section. The students must also know how to use the distributive, associative and commutative laws:

$$\begin{aligned} xy(y+1)(x+1) - x(y+1)y(x+1) &= \\ xy(yx+y+x+1) - (xy+x)(yx+y) &= \\ x^2y^2 + xy^2 + x^2y + xy - (x^2y^2 + xy^2 + yx^2 + xy) &= \\ x^2y^2 + xy^2 + x^2y + xy - x^2y^2 - xy^2 - yx^2 - xy &= 0 \end{aligned}$$

This is something than can be an algebraic obstacle to many first year high school students.

As written above, the design of the exercises determines which algebraic tools the students need to solve the exercises, by changing the pattern, number table and/or calculations. By letting the students work with problems that has to do with addition instead of multiplication, it could leave space for the students to introduce and work with an algebraic proof, as I think that less misconception will occur when adding variables and parentheses compared to multiplying with variables and parentheses.

As the students reach an initial expression, the problem also relates to simple reduction, which contains many of the above obstacles. This is a mathematical activity, that requires manipulation with symbols and is something that often poses problems for a first year student. Bell (1995) addresses this problem, as he points out that *"...students must experience the full activity of: beginning with a problem, forming the equation, and then solving it and interpreting the result"* (Bell, 1995, p. 61).

In this way, reduction is rooted in a practical and intellectual problem originating from determining a pattern.

As mentioned above, the algebraic prerequisites also covers an understanding of algebra. This is what Måsøval (2011) calls *algebraic thinking* and Arcavi (1994) denotes *symbol sense*. Symbol sense contains many aspects and one of them is:

> *"An ability to manipulate and to "read" symbolic expressions as two complimentary aspects of solving algebraic problems. On the one hand, the detachment of meaning necessary for manipulation coupled with a global "gestalt" view of symbolic expressions makes symbolhandling relatively quick and efficient. On the other hand, the reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results."* (Arcavi, 1994, p. 31).

Hence, it is not enough that the students have the algebraic skills to just make the correct calculations, they must also be able to read the symbolic expression in order to interpret their result and intermediate calculations.

To sum up, there are big algebraic obstacles when working with pattern problems and these concerns both the algebraic calculations, but also the ability to interpret/"read" an algebraic expression. A large obstacle has to do with multiplication of brackets and multiplications with variables. Some of these might be overcome as the students work with meaningful and complete mathematical activities and other might be overcome by designing appropriate problems. As I hypothesis that the biggest obstacles have to do with multiplication, the course of study starts with exercises that concerns addition, subtraction or distributing -1 and then progressing from this.

Part III

METHODOLOGY AND DESIGN

3 METHODOLOGY AND DESIGN

The following section contains a description of the methodology and design of the teaching experiment. I will start by shortly outline the ideas behind the exercises and the purpose of the design, which will be exemplified by the Calendar Problem found in (Martinez, 2008) and (Martinez and Castro Superfine, 2012). The exercises for the course of study can be found in Appendix A

When the ideas behind the creation of the exercises and the purpose of these has been outlined, the methodology will be explained. This includes a description of the course of study, the class and the group division. Then follows a description of the data collection and finally the chapter closes by outlining the method of the data analysis, which ends with a statement of the research question for this master's thesis.

3.1 THE DESIGN

In order to exemplify the principles of the design, I have included an exercise, taken from (Martinez, 2008). This exercise has been used before and many of the ideas behind this exercise has also been outlined, why this will only be a short summary.

The main purpose of my teaching design, is to make the students engage in proving situations by the use of algebra. The general structure of the exercises is adopted from (Martinez and Castro Superfine, 2012) and one the main ideas is that, the student must be allowed to make their own numerical exploration of a certain pattern in order to formulate a conjecture. The process of formulating a conjecture is extremely important to the proving situation, as described in (Martinez and Li, 2010). Here the authors concludes that conjecturing is

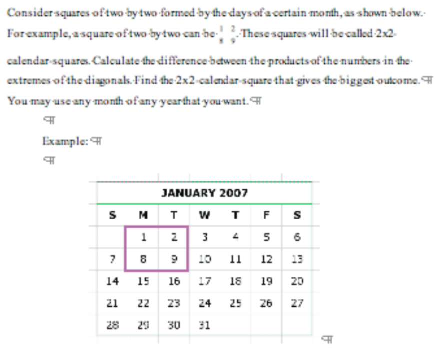
"*... the process through which a person produces a mathematical statement and becomes* confident *about its* plausibility" (Martinez and Li, 2010, p. 271).

and that an important product of a conjecturing process is the "*... construction of mathematical relations that are central to the construction of a proof*" (Martinez and Li, 2010, p. 272). It is therefore important that the students formulate their own conjectures, as this has positive effect on the students work in a proving situation.

All exercises should, more or less, lead to this trajectory: The students examine the pattern and through examples and counter-examples formulates one conjecture. Next they engage in a proving situation. Here they construct a proof which validates their conjecture, but also explains why this is so and this proof is ideally based on algebra.

I will now exemplify this trajectory with an exercise from (Martinez, 2008):

The purpose of the exercise is for the student find the location that produces the biggest outcome, when calculating determinants of 2×2-square matrix in a 7-day week calendar. Martinez (2008) do not use the word determinant, but explicitly describes what calculations the students are to make.


Consider squares of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be $\begin{smallmatrix}1 & 2\\ 8 & 9\end{smallmatrix}$. These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals. Find the 2x2 calendar square that gives the biggest outcome.
You may use any month of any year that you want.

Example:

JANUARY 2007

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Figure 9: Problem 1 from the Calendar Sequence (Martinez, 2008)

By numerical examination, the students are made aware that the result is always -7:

$$\begin{bmatrix} 1 & 2 \\ 8 & 9 \end{bmatrix}$$

With the determinant:

$$1 \cdot 9 - 2 \cdot 8 = -7$$

The examination phase, therefore leads to the conjecturing phase where they determine that the result is always -7 invariant of the placement of the figure.
This findings should arise curiosity and in order to satisfy this curiosity, the students are encouraged to prove the conjecture. As describe in Martinez (2008), the curiosity also worked as a catalyst, that oriented the students work towards an understanding of *why* this conjecture is true. This proved effective in giving the students experience in the use of proofs as a tool for explaining and answering (Martinez, 2008, p. 56). For an algebraic proof of this pattern, see Section 2.4.2.1.

As described earlier, the different possibilities of constructing a generic diagram, leads to seemingly different proofs which proofs the same.

If the teacher is successful in using this in a rational discussion of the different methods to prove a conjecture, then it can have a positive affect in the student (Winsløw, 2006, p. 130). This requires much of the teacher during the institutionalization phase, as the teacher must make it clear to the student, that the task can be solved through different paths. This is an important outcome of this teaching design: There are many ways to prove a conjecture and to shatter the *"one way - one result"* way of thinking, that many students have.

The course of study was therefore a combined design of the principles of TDS, with the above purpose of bringing the students in proving situations

3.2 THE COURSE OF STUDY

The course of study took place at Greve Gymnasium in December 2015 in a first year Gymnasium (High School) class. The course of study was composed of three days, denoted day 1, 2 and 3 in this master's thesis. Each day of teaching lasted 2×50 minutes. The students were divided into 8 groups, with 3 to 4 students in each group. The groups were fixed for the entire week. The group were then paired with another group, which they continuous were going to discuss their findings with.

I tried to incorporate classroom discussions based on the students' different type of proofs. One of the reasons for doing this, is described in (Knuth, 2002, p. 489) where the author emphasizes, that a discussion of students' different proofs is a way to incorporate a variety of proofs into the classroom. This discussion is supposed to furnish a forum for discussing, what constitutes a proof. This is important as many students have a inadequate understanding of the notion 'proof' (Ibid.).

The three days were structured in the following way:

Day 1 (Tuesday the 8/12-2015)

Time	Content	Exercises	Focus
10.10-10.20	Intro + notation of absence	-	Explanation of the composition of the exercise books
10.20-10.30	Devolution	-	
10.30-10.50	Groupwork	Exercise 1, 2 and 3	The students work on their own in groups
10.50-11.15	Institutionalization	-	Pattern determination and formulation of a conjecture
11.15-11.35	Groupwork	Exercise 4 + 5	The students work on their own in groups
11.35-11.55	Institutionalization	-	Formulation of a conjecture and argumentation. Maybe introduce algebra
If any more time:	Finish the exercises Use algebra to proof their conjectures from exercise 1, 2 and 3		

Day 2 (Thursday 10/12-2015)

Time	Content	Exercises	Focus
10.10-10.20	Intro + notation of absence	-	Repetition of the prior day of teaching
10.20-10.30	Groupwork	Exercise 1, 2 and 3	The students are supposed to: - Formulate one sentence(conjecture) describing the pattern - An argument/proof for their conjecture
10.30-10.40	Presentation in paired groups.	Exercise 1, 2 and 3	The groups shall on the basis of their argument/proof, convince the other group about the truth-value of their conjecture The best conjetcure + argument/proofs are chosen. They get a couple of minutes to improve these.
10.40-11.00	Institutionalization	Exercise 1 - 3	The blackboard are divided into 4 and each group writes up their conjecture + argument/proof
11.05-11.40	Groupwork	Exercise 4 + 5	The students work on their own
11.40-11.55	Institutionalization	-	Each group shall in the same google-document, write their conjecture + proof → summerization by the use of the projector (This was postponed till friday because of lack of time)

Figure 10: Day 1 and 2

Day 3 (Friday 11/12-2015)

Time	Content	Exercise	Focus
12.25-12.35	Intro + notation of absence	-	
12.35-12.55	Repetition from last time – Teacher-directed	Opgave 4.1	- To create a scheme (generic model) by the use of - Why is the 'proof' a proof for the conjecture - "Different" proofs, proves the same thing -
12.55-13.30	Groupwork	Gruppe 1 + 2: Opgave 6 (hele opgave 6) og fremlæg **opgave 6A** Gruppe 3 + 7: Opgave 7 (Hele opgaven) og fremlæg **opgave 7.1** Gruppe 4 + 8: Opgave 8 (Hele opgaven) og fremlæg **opgave 8A.1** Gruppe 5 + 6 Opgave 4.2 og opgave 10 – og fremlæg **opgave 4.2**	Try to prove the pattern by the use of algebra
13.30-14.00	Institutionalization	-	Gruppe 1 + 2: Fremlæg **opgave 6A** Gruppe 3 + 7: Fremlæg **opgave 7.1** Gruppe 4 + 8: Fremlæg **opgave 8A.1** Gruppe 5 + 6 Fremlæg **opgave 4.2**

Figure 11: Day 3

3.2.1 *The Class, The Teacher and Group Division*

The class was a first year high school class at Greve Gymnasium. The class was a basic training class (Dansk: Grundforløbsklasse), why there were a large spread in the mathematical abilities of the students. The class consisted of 28 students: 18 girls and 10 boys.

The course of study was an isolated course, placed in the middle of a course in statistics. The students had not prior had a course in algebra, but algebraic symbols and algebraic calculations had been covered when needed in other courses.

The teacher is an experienced teacher who is also a course director (dansk: Kursusleder), with focus on the pedagogically and didactically education of new teachers, internally at the gymnasium.

The students were divided into 8 fixed groups for the entire week. This division was based on the students' mathematical level by the teacher. Students were grouped together with students, who the teacher thought, was on the same mathematical level. This is discussed in Chapter 6. Especially three groups were of interest, as they were given a dictaphone for the entire course of study: Group 2 were a group of students on low level, Group 4 were a group of mid-level students and Group 6 were high-level students.

3.3 DATA COLLECTION

During the entire course of study I had 5 dictaphones recording. I chose to record the same three groups, hence they were each handed a dictaphone for the week. These three groups were 2, 4 and 6. Then I had one dictaphone at the teachers desk to record any talk in plenum and then I carried a dictaphone on me. This would have measured up to 25 hours of recordings, but unfortunately I had problems the first two days, with the dictaphone I carried on me, why it only recorded half an hour each of these two days.

During the course of study I took pictures of the students' work and the blackboard after institutionalizations, and I collected all the written work I could get my hands on. These three components constitutes the empiricism for this master's thesis.

3.4 ANALYSIS OF DATA

Before the course of study took place, I made an a priori analysis of the exercises. This can be found in Chapter 4. In the a priori analysis I focused on the purpose/the target knowledge of the exercise, possible strategies to solve the exercise and the didactical variables. The importance of doing an a priori analysis is outlined in (Måsøval, 2011, p. 28 & p. 46), where the author emphasizes that a didactical situation can not be developed spontaneously, but must be created in order to study the effects of the didactical situation. The a priori analysis is therefore a mean to which, the teacher can examine the properties of the milieu, in order to provoke the interaction and knowledge aimed at.

After the course of study an a posteriori analysis was made of the collected empiricism. I here listened to the recorded data in order to examine the didactical variables' affect on the students' work, in order for the students to engage in situation of formulation and situation of validation. I singled out three situations, which I think enlightens my research questions, but there were a lot of other situations I had to leave out, even though I think they contained a lot of interesting material. The situations all contain aspects, in which the students apply algebra in their attempt to solve the exercise. I chose to pick situations from each group in order to answer the research questions, with the work of three different groups. I also integrated the students' written work in this analysis, though not all group handed in their written work.

Part IV

ANALYSIS

This chapter is divided into two parts, which together forms the analysis of the thesis: The first part treats the a priori analysis of the exercises handed to the students during the course of study, while the second part concerns the a posteriori analysis of selected situations.

4 A PRIORI ANALYSIS

Making an a priori analysis of the milieu is of great importance, as it makes it possible to set up a teaching situation that contains an opportunity for a student to learn some specific target knowledge. According to Brousseau, this teaching situation does not emerge and develop spontaneously, but requires a detailed analysis and caring to blossom, as a florist nurses his flowers (Måsøval, 2011, p. 28). By synthesizing a teaching situation through an a priori analysis, the situation will evolve in a controlled manner and to use Brousseaus own words:

> *"A system delivers more information when it reacts to well chosen stimulation"* (Brousseau, cited in (Måsøval, 2011, p. 28))

The following a priori analysis therefore includes the didactical variables, which is described in Section 2.4, the target knowledge, solutions, problems and purpose of the exercises. The exercises can be found in Appendix A

4.1 EXERCISE 1

Number table:
Exercise 1, 2 and 3 concerns the *Sum Table* (danish: sumtabel). The relation in the Sum Table, is made up of the sum of two independent variables: Each entry in the sum table is the sum of the row- and column number, why the neighbouring numbers to the right and below are 1 bigger. The row- and column number is highlighted in the sum table (see figure 41 in Appendix B). This constant increase in the entry values, makes it possible to describe the relation in the Sum Table with one variable.
A generic 2×2 table could be:

$$\begin{bmatrix} x & x+1 \\ x+1 & x+2 \end{bmatrix}$$

Target Knowledge:
The purpose of the exercise, is for the student to be acquainted with discovering a pattern when doing some pre-defined calculations and to formulate a conjecture, but not necessarily to prove the pattern.
The students are asked to come up with the location of a 1×4-rectangle, which produces the largest result.

Exercise 1B contains the possibility for the students to introduce and work with a parameter for the length of the rectangle. This is by no mean expected and I do no require it for a satisfactorily solution.

Problems:
A problem that could occur is, that the students initially try the rectangle that contains the largest numbers and in order to prove their premonition, they just try one more calculation. This would result in the students not noticing any pattern in contradiction to the purpose of the problem. The result in Exercise 1B is affected by the length of the rectangle, if the largest number in the rectangle is held firm, and the rectangle is expanded to the left.
The two questions in **Exercise 1B.1** is supposed to address these problems, as the students are asked to make explicit if there seems to appear a pattern and explain why this is so.

Solution:
In this problem it is possible to determine the correct location of a 1×4-rectangle, which will contain the numbers: $\{40, 41, 42, 43\}$, and produce the result: $(43 + 40) - 42 = 41$. In Exercise 1B the length of the rectangle does not affect the result as long as the largest number is not held firm. A verbal argument could be: as the length of the rectangle is increased, the largest and second largest number is increased in the same manner and hence the result will always be the second smallest number.

Choice of independent variable and the algebraic solutions:
There are different ways to solve the exercise algebraically, but if the first number in the rectangle is chosen as the independent variable, x, then a proof is:

$$x + (x + 3) - (x + 2) = x + 1$$

from which it is seen that the result is always the second smallest number.
If the students use the largest number as independent variable, then a correct proof is

$$(x - 3) + x - (x - 1) = x - 2$$

The results are algebraically different, so the students needs to make an algebraic expression for the second smallest number in order to compare the results, which could provide breeding ground for a discussion about proofs, but as it is the first exercise this is not intended.

In Exercise 1B an algebraic solution requires the introduction of a parameter for the length of the rectangle. If we call this parameter n

and use the smallest number as independent variable, the proof will be :

$$x + (x + (n - 1)) - (x + (n - 2)) = x + x + n - 1 - x - n + 2 = x + 1$$

Prerequisites and calculations needed:
The students only need to add and subtract numbers in order to come up with the correct result:
$(43 + 40) - 42 = 41$. An algebraic proof also requires the distribution of -1.

Number of variables:
Exercise 1A requires the use of one independent variable and three dependent variables, in order to make an algebraic proof and Exercise 1B requires at least four dependent variables.

A proof in Exercise 1A involving two independent variables, one for the smallest number (x) and one for the largest number (y) could be:

$$x + y - (x + 2) = x + 1$$

which requires the the students set up a relation between the intermediate numbers.

4.2 EXERCISE 2

Target Knowledge
The purpose of this exercise, is again to let the students work with, and discover, a pattern and to formulate a conjecture.

Problems:
If some students tries to solve the exercise by using algebra, it is thinkable, that some students would use two variables: one for each rectangle, which would complicate the process of introducing and using algebra.

Solutions:
A verbal solution to Exercise 2 could be: No matter where the two rectangles are placed the difference between the numbers will always be the same, and therefore the result will always be -12.

Number of variables:
This exercise requires the use of one independent and seven depending variables to describe the pattern, but it is possible that the students will use two variables(x, y): one for each rectangle. This could result in the proof:

$$x + (x + 1) + (x + 2) + (x + 3) - (y + (y - 1) + (y - 2) + (y - 3))$$

where x denotes the smallest number and y denotes the largest number.

Choice of independent variable and the algebraic Solutions:
If the smallest number is used as independent variable:

$$x+(x+1)+(x+2)+(x+3)-((x+3)+(x+4)+(x+5)+(x+6)) = -12$$

It is also possible to use the common number, as the independent variable. This would change the proof a little, but the difficult part would still be to distribute -1:

$$(x-3)+(x-2)+(x-1)+x-(x+(x+1)+(x+2)+(x+3)) = 4x-6-4x-6 = -12$$

Prerequisites and calculations needed:
To solve the initial exercise, the students needs to add and subtract numbers: $5+6+7+8-(8+9+10+11) = -12$ and depending on how they do their calculations, they need to distribute -1, hence they need to use the distributive law and the associative law of addition

4.3 EXERCISE 3

Target Knowledge:
The purpose of this exercise is to make the students formulate and argue for a conjecture. Exercise 3A and 3B concerns pattern determination and making the students argue for their conjectures. Exercise 3B.2 contains the possibility of introducing a parameter for the size of the square
The calculations are fairly easy, which should leave space for the student to focus on conjecture and argumentation. it can be difficult to describe the difference in outcome when multiplying compared to addition, without algebra.

Problems:
It can be difficult for a student to prove a conjecture based on the difference between patterns.
The algebraic abilities needed to make a proof could also cause problems. These are described below.

Solutions:
3A:

$$30+32-(31+31) = 0$$

$$24 \cdot 26-(25 \cdot 25) = -1$$

3B:

$$25+29-(27+27) = 0$$

$$28 \cdot 32-(30 \cdot 30) = -4$$

Number of variables
This exercise requires the use of one independent variable, three depended variables and one parameter, though it is not expected that the students use a parameter. If they do, the proof is:

$$x(x+2n-2)-(x+n-1)(x+n-1) =$$
$$x^2+2xn-2n-(x^2+xn-x+xn+n^2-n-x-n+1 =$$
$$x^2+2xn-2x-x^2-2xn-n^2-2x+2n-1 = -n^2+2n-1$$

From which it is seen, that the result is only dependent on the size of the square (n)

Choice of independent variable and the algebraic solutions:
By denoting the smallest number x, an algebraic solution could be:

$$x\cdot(x+2)-(x+1)^2 = x^2+2x-(x^2+2x+1) = -1$$

By using the last entry as independent variable, the proof would be:

$$x\cdot(x-2)-(x-1)(x-1) = x^2-2x-(x^2+1-2x) = x^2-2x-x^2-1+2x = -1$$

Prerequisites and calculations needed:
The students needs to be able to add, subtract and multiply in order to reach the correct conclusion. To make the algebraic prove, the students needs to make use of the associative-, distributive- and commutative laws. They also needs to lift a square bracket/multiply out brackets, which could cause problems for some students. As in the other problems, distributing -1 could cause problems, while the calculations: $x\cdot x = x^2$ and $x\cdot 2 = 2x$ are often confused by the students.

4.4 EXERCISE 4

Number table:
Exercise 4, 5, 6 and 7 concerns the *Calendar* (see figure 42 in Appendix B), which contains a more complex relation than the Sum Table. The relation between the entries can be described with one variable, as the value of an entry increases with one, as the column number increases with one, and the value of an entry increases with seven, as the row number increases with one. A generic 2×2 table for the Calendar could be:

$$\begin{bmatrix} x & x+1 \\ x+7 & x+8 \end{bmatrix}$$

Target Knowledge:
The purpose of this exercise is once again to make the students work with a pattern, but this time the number table changes. The pattern

in Exercise 4 is, that the result always will be the second smallest number in the staircase, hence the result changes depending on the place of the staircase, why it is possible to find such a staircase. The pattern is similar to Exercise 1, but the difference in the relation of the number table and the numbers, makes the pattern more difficult, but still explainable by the use of a verbal argument.

Solutions:
The largest result: $13 + 31 - 25 = 19$
The smallest result: $22 + 4 - 16 = 10$

Number of variables
The Exercise requires one independent variable and three independent variables in order to make an algebraic proof. It could be the case, that the students uses two variables to create a generic model: One variable for the largest number and one variable for the smallest number.

Choice of independent variable and the algebraic solutions:
By denoting the smallest number x an algebraic proof would be:

$$x + (x + 18) - (x + 12) = 2x + 18 - x - 12 = x + 6$$

If the students use the largest number in the staircase as the place of the independent variable a proof is:

$$x + (x - 18) - (x - 6) = 2x - 18 - x + 6 = x - 12$$

Prerequisites and calculations needed:
In order to reach the conclusion, the students only needs to add and subtract numbers.
To make an algebraic proof, the students need to use the distributive, associative- and commutative law. They also need to be able to distribute -1 and calculate $x + x - x = x$

4.4.1 *Exercise 4.2*

Target Knowledge:
The purpose of this problem, is to use algebra to explicate a relation between two variables. The students should use algebra to reach the conclusion that, if they add the largest and smallest number they get: $a + a + 18 = 2(a + 9) = 2x$, as x denotes half the sum of the largest and smallest number.
From this conclusion they reach that $x = a + 9$, if a is the smallest number in the staircase.

Solutions:
If the students can relate x to the average of the smallest and largest

number, they can verbally formulate the relation, as the difference between these two numbers is always 18.

Problems:
The pitfall of Exercise 4.2 is that the students may reach the conclusion from the fact that the x is the average of the sum of a and $a + 18$, without using algebra.
It is also the first exercise that requires the introduction of two variables, it can be difficult for the students to formulate an initial algebraic expression.

Number of variables
As the students need to set up a relation between two variables, the exercise requires the use of two variables to formulate the expression:

$$x = a + 9$$

where a denotes the smallest number in a staircase.

Choice of independent variable and the algebraic solutions:
As x is described in the exercise description and is defined by the use of the smallest number, these is foreseen to be used as the choice of independent variable, giving the algebraic solution:

$$x = \frac{a + (a + 18)}{2} = \frac{2a + 18}{2} = a + 9$$

Prerequisites and calculations needed:
The same numerical calculations is needed as Exercise 4, but this exercise puts more weight on the algebraic calculations.
The students need to use the associative- and commutative law, and to divide with a factor 2 in order to reach the final algebraic expression. The students then need to be able to interpret this result, to conclude that x is always 9 larger than a.

4.5 EXERCISE 5

Target Knowledge:
The purpose of this exercise is to make the students work with patterns that increases the need for introducing algebra.
The *Infinite Calendar* (see figure 43 in Appendix B) is introduced in order to make it impossible for the student to exhaust all possible placements of the square and make the students argue for their result.
The *9-day Calendar* (see figure 44 in Appendix B) is introduced in order to make the students work with another relation, that looks a little like the the one in the calendar. This should make the students formulate an algebraic proof in Exercise 5C, in order to compare the

different results.

Prerequisites and calculations needed:
In order to obtain the numerical results, the students needs to add, subtract and multiply with 2.
If the students are to make an algebraic proof, the students need to use the distributive-, associative- and commutative laws for addition. They also needs to distribute -1 and use the distributive law for multiplication in order to obtain that $2 \cdot (x-1) = 2x-2$, depending on the location of the independent variable.

Solutions:
5A:
$9+16+23+24-(10+11+18+25)=8$
5B:
$2 \cdot 15+22+29+30-(16+17+24+31)=23$

Number of variables
It is possible to make an algebraic proof of the pattern by the use of one independent variable and seven dependent variables, but it is thinkable that some students will use a variable for each figure (The 7 and the L). If we denote the independent variable in the 7 by x and the independent variable in the L by y, the proof could be:

$$\begin{aligned}&(x+(x+7)+(x+14)+(x+15))- \quad (y+(y+1)+(y+8)+(y+15)) = \\ &4x+36-4y-24= \\ &4x-4y+12\end{aligned}$$

where $y = x+1$

Choice of independent variable and the algebraic Solutions:
5A:
If the students uses the smallest number in the **L** as independent variable, an algebraic proof could be:

$$\begin{aligned}&x+(x+7)+(x+14)+(x+15)-((x+1)+(x+2)+(x+9)+(x+16) = \\ &4x+36-4x-28= \\ &8\end{aligned}$$

By using the smallest number in the **7** as independent variable, the proof becomes:

$$\begin{aligned}&(x-1)+(x+6)+(x+13)+(x+14)-(x+(x+1)+(x+8)+(x+15)) = \\ &4x+32-4x-24= \\ &8\end{aligned}$$

5B: As in the first case above, we get:

$$2 \cdot x + (x+7) + (x+14) + (x+15) - ((x+1) + (x+2) + (x+9) + (x+16) =$$
$$5x + 36 - 4x - 28 =$$
$$x + 8$$

5C: And by using the 9-day calendar, the proofs are:

$$x + (x+9) + (x+18) + (x+19) - ((x+1) + (x+2) + (x+11) + (x+20) =$$
$$4x + 46 - 4x - 34 =$$
$$12$$

and

$$2 \cdot x + (x+9) + (x+18) + (x+19) - ((x+1) + (x+2) + (x+11) + (x+20) =$$
$$5x + 46 - 4x - 34 =$$
$$x + 12$$

By which it is seen, that the number is no longer the middler number, but a number lying just outside the square to the left, in the middle row.

Prerequisites and calculations needed:
In order to obtain the numerical results, the students needs to add, subtract and multiply with 2.
If the students are to make an algebraic proof, the students need to use the distributive-, associative- and commutative laws for addition. They also needs to distribute -1 and use the distributive law for multiplication in order to obtain that $2 \cdot (x-1) = 2x - 2$, depending on what they use for an independent variable.

4.6 EXERCISE 6

Target Knowledge:
The purpose of Exercise 6 is to introduce calculations more complex than in the previous exercises and at the same time work with a different pattern. The calculations in this exercise corresponds to determining the determinant of a 2×2-square and as I do not think that it is explainable that the outcome is always -7, I think that algebra is needed in proving this pattern.

Problems:
Setting up an algebraic expression and describing the coherence between the independent variable and rest of the numbers could cause problems. Especially as the algebraic proof now involve multiplication.

Solutions:
6A:

$$\begin{bmatrix} 12 & 13 \\ 19 & 20 \end{bmatrix}$$

with the calculations: $12 \cdot 20 - 13 \cdot 19 = -7$

6B:

$$\begin{bmatrix} 12 & 13 & 14 \\ 19 & 20 & 21 \\ 26 & 27 & 28 \end{bmatrix}$$

with the calculations: $12 \cdot 28 - 26 \cdot 14 = -28$

By doing the above calculations, the students are enable to exhaust all possible placements of the matrices, and use this as a proof of a conjecture. Exercise 6B.1 tries to get around this problem, by introducing the Infinite Calendar.

6B.1:
As this exercise contains a variety of calculations, I will only show the calculations for a $9-$day calendar in a 2×2-matrix:

With a $9-$day calendar:

$$\begin{bmatrix} 13 & 14 \\ 22 & 23 \end{bmatrix}$$

with the calculations: $13 \cdot 23 - 14 \cdot 22 = -9$

The case with a $d-$day calendar is shown below

Number of variables
It is possible to make an algebraic proof of a conjecture in Exercise 6A by using one independent variable and three depending variables. Exercise 6B needs one independent variable and 8 dependent variables in order to make a generic diagram. Exercise 6B.1 also needs a parameter to describe a calendar with d-day weeks.
It is thinkable though, that the students will use two independent variables. One for the upper row and one for the bottom row. This would result in a generic diagram:

$$\begin{bmatrix} x & x+1 \\ y & y+1 \end{bmatrix}$$

Choice of independent variable and the algebraic Solutions:
6A:

If the students choose to locate the independent variable in the upper left corner, the generic diagram will be:

$$\begin{bmatrix} x & x+1 \\ x+7 & x+8 \end{bmatrix}$$

with the calculations: $x \cdot (x+8) - (x+1) \cdot (x+7) = x^2 + 8x - (x^2 + 8x + 7) = -7$

If the students rather chooses the lower left corner:

$$\begin{bmatrix} x-8 & x-7 \\ x-1 & x \end{bmatrix}$$

with the calculations: $(x-8) \cdot x - (x-1) \cdot (x-7) = x^2 - 8x - (x^2 - 8x + 7) = -7$

6B:
If the students choose to locate the independent variable in the upper left corner, the generic diagram will be:

$$\begin{bmatrix} x & x+1 & x+2 \\ x+7 & x+8 & x+9 \\ x+14 & x+15 & x+16 \end{bmatrix}$$

with the calculations: $x \cdot (x+16) - (x+2) \cdot (x+14) = x^2 + 16x - (x^2 + 16x + 28) = -28$

6B.1:
A 2×2 generic diagram for 9-day with the independent variable located in the upper left corner could be:

$$\begin{bmatrix} x & x+1 \\ x+9 & x+10 \end{bmatrix}$$

with the calculations: $x \cdot (x+10) - (x+1) \cdot (x+9) = x^2 + 10x - (x^2 + 10x + 9) = -9$

A 2×2 generic diagram for d-day with the independent variable located in the upper left corner could be:

$$\begin{bmatrix} x & x+1 \\ x+d & x+d+1 \end{bmatrix}$$

with the calculations: $x \cdot (x+d+1) - (x+1) \cdot (x+d) = x^2 + dx + x - (x^2 + dx + x + d) = -d$

Prerequisites and calculations needed:
In order to obtain the numerical results, the students need to add,

subtract and multiply with numbers.
If the students are to make an algebraic proof, the students need to use the distributive-, associative- and commutative laws for addition. They also need to distribute -1 and use the distributive law for multiplication in order to obtain that $x(x+8) = x^2 + 8x$ depending on where they locate their independent variable. They also needs to multiply brackets and arrange terms like $x(-2) + (-14)x = -16x$.

4.7 EXERCISE 7

Target Knowledge:
This exercise introduces a pattern, that uses multiplication and at the same time is difficult to unravel why the purpose is, that the students uses algebra in the proving situation, as it is difficult to prove the pattern without.

Solutions:
The three boxes:

$$\begin{bmatrix}1\\8\end{bmatrix}\begin{bmatrix}2\\9\end{bmatrix}\begin{bmatrix}3\\10\end{bmatrix}$$

is the correct placement, as the result will be -11

Number of variables
An algebraic proof of the pattern requires one independent variable and five dependent variables. As the exercise concerns three boxes, some student may be inclined to use three independent variables, which could result in a generic diagram:

$$\begin{bmatrix}x\\x+7\end{bmatrix}\begin{bmatrix}y\\y+7\end{bmatrix}\begin{bmatrix}z\\z+7\end{bmatrix}$$

With the algebraic proof:

$$x \cdot (x+7) - (z(z+7)) + (y + (y+7)) = x^2 + 7x - z^2 - 7z + 2y + 7$$

Where $z = y + 1 = x + 2$

Choice of independent variable and the algebraic Solutions:
If the students chooses to locate the independent variable in the first entry in the first box, an algebraic proof could be:

$$\begin{aligned} x(x+7) - (x+2)(x+9) + ((x+1) + (x+8)) &= \\ x^2 + 7x - x^2 - 11x - 11 + 2x + 9 &= \\ -2x - 9 &= \quad -(2x+9) \end{aligned}$$

If the students chooses to locate the independent variable in the last entry in the last box, an algebraic proof could be:

$$\begin{aligned}(x-9)(x-2)-(x(x-7))+((x-8)+(x-1)) &= \\ x^2-11x+18-x^2+7x+2x-9 &= \\ -2x+9 &= \quad -(2x-9)\end{aligned}$$

Prerequisites and calculations needed:
In order to obtain the numerical results, the students needs to add, subtract and multiply with numbers.
If the students are to make an algebraic proof, the students need to use the distributive-, associative- and commutative laws for addition. They also needs to distribute -1 and use the distributive law for multiplication in order to obtain that $x(x+7) = x^2+7x$ depending on where they locate their independent variable. They also needs to multiply brackets and arrange terms like $(x+2)(x+9) = x^2+x9+2x+18 = x^2+11x+18$.

4.8 EXERCISE 8

Number Table:
Exercise 8, 9 and 10 concerns the *Multiplication Table* (see figure 45 in Appendix B). Each entry is the product of the row and column number and therefore depended on the location of the figure, hence the multiplication table requires two independent variables to describe the relation.
A generic 2×2 diagram could be:

$$\begin{bmatrix} xy & y(x+1) \\ x(y+1) & (x+1)(y+1) \end{bmatrix}$$

Where x denotes the column number of the first entry and y denotes the row number of the first entry

The Diagonal

The Multiplication Table contains a relation, which I called *The Diagonal*. These numbers are also highlighted in the exercises. The diagonal is composed of square numbers, as $x = y$ in the diagonal. This makes it possible to describe the relation in the diagonal with just one variable, and is here used as an entrance to the Multiplication Table. A generic 2×2 diagram could be:

$$\begin{bmatrix} x^2 & x(x+1) \\ (x+1)x & (x+1)^2 \end{bmatrix}$$

Where x denotes the column- and row number of the first entry.

Target Knowledge:
The purpose of Exercise 8 is to make the students acquainted with the Multiplication Table, by letting them work with patterns and calculations in the diagonal.
In Exercise 8B the students work with squares in and out of the diagonal. This is made, as an attempt to make the students work with pattern that requires two variables to describe, with out complicating the calculations.

Problems:
The problems in Exercise 8 could revolve around describing the relation of the number table, which requires the use of product of one or two variables. Hence it is foreseen, that the introduction of two variables will be the major obstacle.

Solutions:
By expanding the size of the squares, the result stays the same if they use multiplication, but changes when using addition. This should provide a breeding ground for the students to prove this change in their results.

Below are listed some of the calculations necessary to conclude the above:

8A and 8A.1:

$$\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

with the calculations: $(9+16)-(12+12) = 25-24 = 1$ in 8A and $(9 \cdot 16)-(12 \cdot 12) = 144-144 = 0$ in 8A.1

8A.2:

$$\begin{bmatrix} 25 & 30 & 35 \\ 30 & 36 & 42 \\ 35 & 42 & 49 \end{bmatrix}$$

with the calculations: $(25+49)-(35+35) = 74-40 = 4$ and $(25 \cdot 49)-(35 \cdot 35) = 1225-1225 = 0$

8B:

$$\begin{bmatrix} 81 & 90 & 90 & 108 \\ 90 & 100 & 110 & 120 \\ 99 & 110 & 121 & 132 \\ 108 & 120 & 132 & 144 \end{bmatrix}$$

with the calculations: $(81+144)-(100+121) = 225-221 = 4$

$$\begin{bmatrix} 28 & 30 & 32 & 134 \\ 42 & 45 & 48 & 51 \\ 56 & 60 & 64 & 68 \\ 70 & 75 & 80 & 85 \end{bmatrix}$$

with the calculations: $(28 + 85) - (45 + 64) = 113 - 109 = 4$

Number of variables

It is possible to make an algebraic proof of a conjecture in Exercise 8A and 8A.1 by using one independent variable and three depending variables. Exercise 8A.2 needs one independent variable and 8 dependent variables in order to make a generic diagram.

Exercise 8B requires two independent variables to describe an entry in a square outside the diagonal, but only one independent variable to describe an entry in a square located in the diagonal.

Choice of independent variable and the algebraic Solutions:

8A and 8A.1:

If the students choose to locate the independent variable in the upper left corner, a generic diagram could be:

$$\begin{bmatrix} x^2 & (x+1)x \\ x(x+1) & (x+1)^2 \end{bmatrix}$$

with the calculations in 8A:

$$\begin{aligned} x^2 + (x+1)^2 - ((x+1)x + x(x+1)) &= \\ x^2 + x^2 + 2x + 1 - (x^2 + x + x^2 + x) &= \\ 2x^2 + 2x + 1 - 2x^2 - 2x &= \quad 1 \end{aligned}$$

And the calculation in 8A.1:

$$\begin{aligned} x^2 \cdot (x+1)^2 - ((x+1)x \cdot x(x+1)) &= \\ (x(x+1))^2 - (x(x+1))^2 &= \quad 0 \end{aligned}$$

If the students choose to locate the independent variable in the lower right corner, a generic diagram could be:

$$\begin{bmatrix} (x-1)^2 & (x-1)x \\ x(x-1) & x^2 \end{bmatrix}$$

More or less the same calculations would be done in exercise 8A.2, so these are omitted here. A generic diagram could be:

$$\begin{bmatrix} x^2 & (x+1)x & (x+2)x \\ x(x+1) & (x+1)^2 & (x+2)(x+1) \\ x(x+2) & (x+1)(x+2) & (x+2)^2 \end{bmatrix}$$

8B:
If the students choose to locate the independent variables in the upper left corner, a generic diagram for a 4×4 square in the diagonal could be:

$$\begin{bmatrix} x^2 & (x+1)x & (x+2)x & (x+3)x \\ x(x+1) & (x+1)^2 & (x+2)(x+1) & (x+3)(x+1) \\ x(x+2) & (x+1)(x+2) & (x+2)^2 & (x+3)(x+2) \\ x(x+3) & (x+1)(x+3) & (x+2)(x+3) & (x+3)^2 \end{bmatrix}$$

with the algebraic proof:

$$\begin{aligned} x^2 + (x+3)^2 - ((x+1)^2 + (x+2)^2) &= \\ x^2 + x^2 + 9 + 6x - (x^2 + 1 + 2x + x^2 + 4x + 4) &= \\ 2x^2 + 9 + 6x - 2x^2 - 6x - 5 &= 4 \end{aligned}$$

And for a square outside the diagonal a generic diagram could be:

$$\begin{bmatrix} xy & (x+1)y & (x+2)y & (x+3)y \\ x(y+1) & (x+1)(y+1) & (x+2)(y+1) & (x+3)(y+1) \\ x(y+2) & (x+1)(y+2) & (x+2)(y+2) & (x+3)(y+2) \\ x(y+3) & (x+1)(y+3) & (x+2)(y+3) & (x+3)(y+3) \end{bmatrix}$$

with the algebraic proof:

$$\begin{aligned} xy + (x+3)(y+3) - ((x+1)(y+1) + (x+2)(y+2)) &= \\ xy + xy + 3x + 3y + 9 - (xy + 2x + 2y + 4 + xy + x + y + 1) &= \\ 2xy + 3x + 3y + 9 - 2xy - 3x - 3y - 5 &= 4 \end{aligned}$$

Prerequisites and calculations needed:
In order to obtain the numerical results, the students needs to add, subtract and multiply with numbers.
If the students are to make an algebraic proof, the students need to use the distributive-, associative- and commutative laws for addition. They also needs to distribute -1 and use the distributive law for multiplication in order to obtain that $x(x+1) = x^2 + x$, depending on where they locate their independent variable. They also needs to multiply brackets containing different variables and arrange terms like $(x+3)(y+3) = xy + 3x + 3y + 9$.
It benefits the students if they know the laws of exponents in order to make the expression: $x^2(x+1)^2 = (x(x+1))^2$, but is not required.

4.9 EXERCISE 9

Target Knowledge:
The purpose is for the student to determine how the result changes in accordance with the placement of the 3×3-square. The result depends lineary of the placement of the square, which is a fact that can be difficult to reveal without the use of algebra. The students therefore needs to be able to relate the difference with an algebraic expression, in which a formal expression requires two variables.

Problems:
As this problem, to some degree, requires that the students know how to divide and cancel out with letters, it can be difficult for some students to make this step. By *to some degree* I mean, that it is possible to see from the Exercise formulation, that we multiply each term by the number in the upper left corner, and then divide by the same number again, hence this can be omitted from the calculations. I included this calculation in order to camouflage the pattern and anticipate that most students will not realize this in their calculations.

Solutions:
A verbal solution could be: Every time the 3×3-square is moved one place, the result changes with -3. As the smallest result possible is -5, we can describe the result by a linear function $f(x) = -2x - 3$, where x denotes the row- or column number.
Below are listed some of the calculations needed in order to make this conclusion:

$$\begin{bmatrix} 25 & 30 & 35 \\ 30 & 36 & 42 \\ 35 & 42 & 49 \end{bmatrix}$$

with the calculations: $\frac{(25 \cdot 36)-(25 \cdot 49)}{25} = -13$

Number of variables
It is possible to make an algebraic proof of a conjecture by using one independent variable and two depending variables. The generic diagram requires one independent variable and 8 dependent variables.

Choice of independent variable and the algebraic Solutions:
As the number in the upper left corner is used twice in the calculations, I foresee that the students will use this location for an independent variable. This could result in the following generic diagram:

$$\begin{bmatrix} x^2 & (x+1)x & (x+2)x \\ x(x+1) & (x+1)^2 & (x+2)(x+1) \\ x(x+2) & (x+1)(x+2) & (x+2)^2 \end{bmatrix}$$

with the algebraic proof:

$$\frac{x^2(x+1)^2 - x^2(x+2)^2}{x^2} =$$
$$\frac{x^2((x+1)^2 - (x+2)^2)}{x^2} =$$

$$(x+1)^2 - (x+2)^2 =$$
$$x^2 + 2x + 1 - x^2 - 4x - 4 = \quad -2x - 3$$

From these calculations, it is seen that the result linearly depends on the location of figure. The students then must be able to interpret this result, to reach this conclusion.

Prerequisites and calculations needed:
In order to obtain the numerical solutions, the students must be enable to multiply, subtract and divide with numbers.
The students must know how to use the laws of exponents and cancel out factors in order to make the above proof. They must also know how to multiply brackets or the rules of the square of a binomial (dansk: Kvadratsætningerne) and distribute -1. If the students chooses to remove the brackets, they must use the distributive law for multiplication and the commutative law for addition.

4.10 EXERCISE 10

Target Knowledge:
The purpose of this exercise is to make the students work with patterns outside the diagonal. In order to make the students able to work out the pattern and proof a conjecture, I kept the calculations simple and concerning a 1×4 rectangle. As the exercise calculations requires division, multiplication and subtraction, the pattern is difficult to prove without the use of algebra.

Problems:
A problem that could arise, is that the student will incline to work with the product of the row- and column number as one variable, instead of as two separate variables. This will be explained in *Number of variables*

Solutions:
Exercise 10:
The result is always the diagonal number, from the used row.

An example of a 1×4-rectangle in row 8 and the calculations belonging to this rectangle:

$$\begin{bmatrix} 144 & 152 & 160 & 168 \end{bmatrix}$$

With the calculations: $\frac{(144 \cdot 168)-(152 \cdot 160)}{-2} = \frac{-128}{-2} = 64 (= 8^2)$

Number of variables
As the values of the entries depends of the location of the rectangle, it requires the use of two independent variables.
As described above, it is thinkable, that the students will use just one variable to make a proof of their conjecture. This would result in a proof that is fixed to the specific row number.

To make an example:
If row number 5 is chosen, a generic diagram could be:

$$\begin{bmatrix} z & z+5 & z+10 & z+15 \end{bmatrix}$$

Which would result in the algebraic proof:

$$\frac{(z \cdot (z+15)) - (z+5)(z+10)}{-2} = \frac{z^2 + 15z - z^2 - 15z - 50}{-2} = \frac{-50}{-2} = 25$$

Which is not a wrong proof, but only a proof of that single, limited example. As the purpose of this exercise is to make a proof for the pattern related to the entire Number Table, such a proof will not be sufficient.

Choice of independent variable and the algebraic Solutions:
Exercise 10:

If the students uses the row- and column number for the first entry, a generic diagram could be:

$$\begin{bmatrix} xy & (x+1)y & (x+2)y & (x+3)y \end{bmatrix}$$

With the algebraic proof:

$$\begin{aligned}\frac{xy(x+3)y-(x+1)y(x+2)y}{-2} &= \\ \frac{y^2(x(x+3)-(x+1)(x+2))}{-2} &= \\ \frac{y^2(x^2+3x-x^2-3x-2)}{-2} &= \\ \frac{-2y^2}{-2} &= \quad y^2\end{aligned}$$

If instead the students uses the row- and column number for the last entry, a generic diagram could be:

$$\begin{bmatrix}(x-3)y & (x-2)y & (x-1)y & xy\end{bmatrix}$$

Resulting in a algebraic proof:

$$\begin{aligned}\frac{(x-3)y\cdot xy-(x-2)y(x-1)y}{-2} &= \\ \frac{y^2((x-3)x-(x-2)(x-1))}{-2} &= \\ \frac{y^2(x^2-3x-x^2+3x-2)}{-2} &= \\ \frac{-2y^2}{-2} &= \quad y^2\end{aligned}$$

10.1:

As the calculations is pretty similar to the one above, only one case is shown. This is the case when the row- and column number is chosen for the first entry in the 4×1-rectangle:

$$\begin{bmatrix}xy \\ x(y+1) \\ x(y+2) \\ x(y+3)\end{bmatrix}$$

Resulting in a algebraic proof:

$$\begin{aligned}\frac{xy\cdot x(y+3)-(x(y+1)x(y+2))}{-2} &= \\ \frac{x^2(y(y+3)-(y+1)(y+2))}{-2} &= \\ \frac{x^2(y^2+3y-y^2-3y-2)}{-2} &= \\ \frac{-2x^2}{-2} &= \quad x^2\end{aligned}$$

Prerequisites and calculations needed:
In order to compute the numerical solutions, the students must be enable to multiply, subtract and divide with numbers.
The students must know how to use the laws of exponents and cancel out factors in order to make the above proof. They must also know how to multiply brackets or the rules of the square of a binomial and distribute -1. This is for example the case when reducing the expression: $-(y+1)(y+2) = -(y^2+2y+y+2) = -y^2-3y-2$. If the students chooses to remove the brackets, they must also use the distributive law for multiplication and the commutative law for addition.

4.11 EXERCISE 11

Number Table:
Exercise 11 and 12 concerns the number table *The S-Table* (see figure 46 in Appendix B). This number table contains a relation which is difficult to describe with two independent variables in a small scale, as a number in the table is a sum of two previous numbers. Below is an illustration of this:

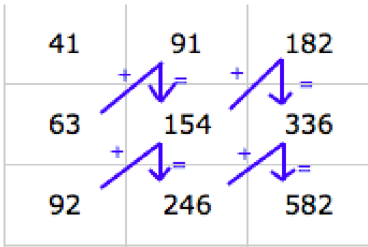


Figure 12: An illustration of the relation in The S-Table

It is possible to make a generic diagram, as the numbers in a small scale can be described with three variables. A 2×2 generic diagram could be:

$$\begin{bmatrix} x & y \\ z & z+y \end{bmatrix}$$

The S-Table also contains a symmetry, as it is symmetric around the diagonal containing the numbers $\{1, 4, 14, 50, 182, \ldots\}$. Hence, a diagram containing this diagonal can be described by fewer variables. This will be explained in the analysis of each exercise concerning the S-Table. The work with the diagonal could make the introduction of a third variable easier.

Target Knowledge:
The purpose of this exercise is to make the students examine the S-Table and use algebra to describe a complex pattern. Exercise 11 can however be solved by a verbal formulation, if the students unravel

the relation of the number table.

Problems:
It can be difficult for some students to introduce and work with three variables in an algebraic proof.

Solutions:
The correct solution is the triangle containing the numbers:

		32318
	35750	68068
32318	68068	136136

With the desired calculations:

$$32318 + 35750 + 32318 - 136136 = -35750$$

A verbal solution can therefore be based on the fact that:

$$136136 = 68068 + 68068 = 32318 + 2 \cdot 35750 + 32318$$

Why the result is always the negative of the middle number in the "hypothenuse"

Number of variables
It is possible to describe the pattern in a triangle by the use of three independent variables, if the triangle is located outside the diagonal. A generic diagram containing the diagonal is describable by the use of two variables (See below):

Choice of independent variable and the algebraic Solutions:
If the three numbers in the hypothenuse is used as the location of the independent variable, a universal generic diagram could be:

		z
	y	$y+z$
x	$x+y$	$x+z+2y$

With an algebraic proof:

$$x + y + z - (x + z + 2y) = -y$$

I do not foresee, that the three numbers not lying in the hypothenuse, would be used as the location of the independent variables, as this will hamper the calculations severely.

If the students choose to make a generic diagram that contains the diagonal, such an diagram could be:

		x
	y	$y+x$
x	$x+y$	$2x+2y$

With an algebraic proof:

$$x + y + x - (2x + 2y) = -y$$

Prerequisites and calculations needed:
The students needs to add and subtract numbers, in order to reach the desired numerical result.
If the students makes an algebraic proof, they must use the commutative- and associative law as well as distribute -1.
The difficult part concerns the introduction of two or three variables and working with multiple variables.

4.12 EXERCISE 12

Target Knowledge:
The purpose of this exercise is to force the students to make an algebraic proof, as it is difficult to explain why the result appears, without an algebraic argument.

Problems:
The problems can appear when the students tries to make an algebraic proof, as they need to factorize and divide with a difference.

Solutions:
As the result is always the number in the lower right corner, the correct location of the square, is the one that contains the number:

$$\begin{bmatrix} 35750 & 68068 \\ 68068 & 136136 \end{bmatrix}$$

With the desired calculations:

$$\frac{35750 \cdot 136136 - (68068 + 68068)}{35750 - 1} = 136136$$

Number of variables
As in Exercise 11, it is possible to make a universal statement by the use of three independent variables.
An algebraic proof concerning a 2×2 square containing numbers from the diagonal, is possible by the use of two independent variables.

Choice of independent variable and the algebraic Solutions:
An universal proof of the pattern, where the independent variables

are located in the upper left-, upper right and down left corner of a generic diagram could be:

$$\begin{bmatrix} x & y \\ z & z+y \end{bmatrix}$$

With an algebraic proof:

$$\frac{x(y+z)-(y+z)}{x-1} = \frac{(y+z)(x-1)}{x-1} = y+z$$

A generic diagram around the diagonal could be:

$$\begin{bmatrix} x & y \\ y & 2y \end{bmatrix}$$

And an algebraic proof:

$$\frac{x \cdot 2y - 2y}{x-1} = \frac{(2y)(x-1)}{x-1} = 2y$$

Prerequisites and calculations needed:
The students needs to add, subtract and divide numbers, in order to reach the desired numerical result.
To make an universal proof of the pattern, the students must use the distributive law of multiplication. They must also know how to factorize in order to deduce that

$$x(y+z)-(y+z) = (y+z)(x-1)$$

and to divide with a difference in order to reach the desired conclusion:

$$\frac{(y+z)(x-1)}{x-1} = y+z$$

4.13 EXERCISE 13

Number Table:
The last number table used in this thesis is the *Power Table*$(\mathrm{Danish}: \mathrm{Potenstabel})$ (see figure 47 in Appendix B), which I regard to contain the most difficult relation. This is due to the fact, that many students find it difficult to calculate with exponents.
Exercise 13, 14 and 15 all involve calculations with numbers from the Power Table.

The Power Table is constructed with the use of two variables, where x is the base number and y is the exponent and at the same time,

x is the column number and y is the row number. A generic 2×2 diagram with x and y defined as this would look like:

$$\begin{bmatrix} x^y & (x+1)^y \\ x^{y+1} & (x+1)^{y+1} \end{bmatrix}$$

As in the Multiplication- and Sum Table, the Power Table also contains a diagonal with a certain relation: The numbers in this diagonal is made up of numbers of the form x^x. A generic diagram in the diagonal could be:

$$\begin{bmatrix} x^x & (x+1)^x \\ x^{x+1} & (x+1)^{x+1} \end{bmatrix}$$

Target Knowledge:
The purpose of this exercise, is to let the students make algebraic proofs which involves base numbers, exponents and therefore the laws of exponents.

Problems:
As many students are not acquainted with the rules of exponents, this can hinder the production of an algebraic proof.

Solutions:
The result is always 1

An example of a 4×1 rectangle and the numerical calculation, that supports this statement:

$$\begin{bmatrix} 125 \\ 625 \\ 3125 \\ 15625 \end{bmatrix}$$

The calculations:

$$\frac{125 \cdot 15626}{625 \cdot 3125} = \frac{1953125}{1953125} = 1$$

An example of a 6×1 rectangle and the numerical calculation in Exercise 13C

$$\begin{bmatrix} 1296 \\ 7776 \\ 46656 \\ 279936 \\ 1679616 \\ 10077696 \end{bmatrix}$$

The calculations:

$$\frac{1296 \cdot 10077696}{46656 \cdot 279936} = \frac{13060694016}{13060694016} = 1$$

The same digram can be used in Exercise 13D, but another calculation is needed:

$$\frac{1296 \cdot 10077696}{1679616} = \frac{13060694016}{1679616} = 7776$$

Number of variables
Exercise 13 requires two independent variables to describe an entry in the rectangle. Exercise 13A and 13B therefore requires two independent variables and six dependent variables to create a generic diagram, exercise 13C and 13D requires two independent variables and eight dependent variables in order to make the generic diagram.

Choice of independent variable and the algebraic Solutions:
If the students use the first row number and first column number as independent variables, then a generic diagram in Exercise 13A and 13B could be:

$$\begin{bmatrix} x^y \\ x^{y+1} \\ x^{y+2} \\ x^{y+3} \end{bmatrix}$$

And an algebraic proof:

$$\frac{x^y \cdot x^{y+3}}{x^{y+1} \cdot x^{y+2}} = \frac{x^{y+y+3}}{x^{y+1+y+2}} = \frac{x^{2y+3}}{x^{2y+3}} = 1$$

If the students use the last row number and first column number as independent variables, then a generic diagram in Exercise 13A and 13B could be:

$$\begin{bmatrix} x^{y-3} \\ x^{y-2} \\ x^{y-1} \\ x^y \end{bmatrix}$$

And an algebraic proof:

$$\frac{x^{y-3} \cdot x^y}{x^{y-2} \cdot x^{y-1}} = \frac{x^{y-3+y}}{x^{y-1+y-2}} = \frac{x^{2y-3}}{x^{2y-3}} = 1$$

A generic 1×6-rectangle, where the first row- and the first column number is chosen as independent variables, in Exercise 13C and 13D could be:

$$\begin{bmatrix} x^y \\ x^{y+1} \\ x^{y+2} \\ x^{y+3} \\ x^{y+4} \\ x^{y+5} \end{bmatrix}$$

And an algebraic proof in Exercise 13C:

$$\frac{x^y \cdot x^{y+5}}{x^{y+2} \cdot x^{y+3}} = \frac{x^{y+y+5}}{x^{y+2+y+3}} = \frac{x^{2y+5}}{x^{2y+5}} = 1$$

The same diagram is used in Exercise 13D, but another proof is needed here:

$$\frac{x^y \cdot x^{y+5}}{x^{y+4}} = \frac{x^{y+y+5}}{x^{y+4}} = \frac{x^{2y+5}}{x^{y+4}} = x^{2y+5-y-4} = x^{y+1}$$

Prerequisites and calculations needed:
The students needs to multiply and divide numbers, in order to reach the desired numerical result.
To make an universal proof of the pattern, the students must use the rules of exponents in order to reach the desired conclusion. Here they must know how to multiply and divide with numbers that have the same base number, in order to deduce that

$$\frac{x^{2y+5}}{x^{y+4}} = x^{2y+5-y-4} = x^{y+1}$$

4.14 EXERCISE 14

Target Knowledge:
The purpose of this exercise, is to force the students to make an algebraic proof by camouflaging the pattern in calculations. Through this work with the algebraic proof the students are being acquainted with the rules of exponents. Especially the rule of the product of numbers with the same exponents but different base number: $a^b \cdot c^b = (ac)^b$

Problems:
The students incapability in working with the rules of exponents, can be the biggest hindrance in making an algebraic proof.

Solutions:
The result is alway 0

An example of a 2×2 square and a numerical calculations needed to reach this conclusion:

$$\begin{bmatrix} 32 & 243 \\ 64 & 729 \end{bmatrix}$$

The calculations:

$$(32 \cdot 243) \cdot (2 \cdot 3) - (64 \cdot 729) = 46656 - 46656 = 0$$

Number of variables
Exercise 14 requires the use of two independent variables and six dependent variables in order to create a generic diagram.

Choice of independent variable and the algebraic Solutions:
If the first row number and the first column number of the generic diagram is chosen, then such a diagram could be:

$$\begin{bmatrix} x^y & (x+1)^y \\ x^{y+1} & (x+1)^{y+1} \end{bmatrix}$$

Where x and $x+1$ are the two **highlighted** numbers (see the exercise description of Exercise 14). x denotes the column number and y denotes the row number.

And an algebraic proof:

$$\begin{aligned} (x^y(x+1)^y) \cdot (x(x+1)) - (x^{y+1} \cdot (x+1)^{y+1)} &= \\ (x(x+1))^y \cdot (x(x+1)) - (x(x+1))^{y+1} &= \\ (x(x+1))^{y+1} - (x(x+1))^{y+1} &= \quad 0 \end{aligned}$$

Prerequisites and calculations needed:
The students need to multiply and subtract numbers, in order to reach the desired numerical result.
To make an universal proof of the pattern, the students must use the rules of exponents to make a sufficient proof. Here they must know both how to multiply and divide with numbers that have different base numbers and to multiply with numbers that have the same exponents. In order to make the above calculation the students must also know that $(x-1)x = ((x-1)x)^1$

4.15 EXERCISE 15

Target Knowledge:
This last exercise contains the possibility of introducing a parameter

for the length of the rectangle. The exercise therefore contains the potential for the students to make an algebraic proof that uses two independent variables and one parameter.

Solutions:
The result is the product of the two base numbers.

An example of a 2×2 square and a numerical calculation needed to reach this conclusion:

$$\begin{bmatrix} 49 & 64 \\ 343 & 512 \end{bmatrix}$$

The calculations:

$$\frac{343 \cdot 512}{49 \cdot 64} = \frac{175616}{3136} = 56$$

Number of variables
Exercise 15A requires two independent variables and six dependent variables to make a universal 2×2 generic diagram: One independent variable for the base number, one for the exponent and six dependent variable for the rest of the entries in the generic diagram. Hence such a generic diagram could look like:

$$\begin{bmatrix} x^y & (x+1)^y \\ x^{y+1} & (x+1)^{y+1} \end{bmatrix}$$

x denotes the independent variable describing the column number and y denotes the independent variable representing the row number.
If the students chooses to place the generic diagram in the diagonal, then it is possible to create such a diagram by the use of one independent variable. Such a diagram can be found in Section 4.13

Exercise 15B contains the possibility of introducing a parameter describing the width of the rectangle.

Choice of independent variable and the algebraic Solutions:
If the independent variable is located in the upper left corner the diagram will be as above and the algebraic proof could be:

$$\frac{x^{y+1}(x+1)^{y+1}}{x^y(x+1)^y} =$$

$$\frac{(x(x+1))^{y+1}}{(x(x+1))^y} =$$

$$(x(x+1))^{y+1-y} = \quad x(x+1)$$

An algebraic proof for Exercise 15B involving a parameter n for the width of the rectangle could be:

$$\frac{x^{y+(n-1)}(x+1)^{y+(n-1)}}{x^y(x+1)^y} =$$

$$\frac{(x(x+1))^{y+(n-1)}}{(x(x+1))^y} =$$

$$(x(x+1))^{y+(n-1)-y} =$$

$$(x(x+1))^{y+n-1-y} = \quad (x(x+1))^{n-1}$$

Prerequisites and calculations needed:
The students needs to multiply and divide numbers, in order to reach the desired numerical result.
To make an universal proof of the pattern, the students must use the rules of exponents. Here they must know both how to multiply and divide with numbers that have different base numbers and to multiply with numbers that have the same exponents, in order to deduce that:

$$\frac{x^{y+1}(x+1)^{y+1}}{x^y(x+1)^y} =$$

$$\frac{(x(x+1))^{y+1}}{(x(x+1))^y} =$$

$$(x(x+1))^{y+1-y} = \quad x(x+1)$$

5 A POSTERIORI ANALYSIS

The following section contains the a posteriori analysis of three selected situations. The situations are selected in order to illustrate some of the obstacles the students faced in their work with the exercises. In order to focus and limit the analysis, I have chosen to center my analysis around two central questions:

Questions of interest

- How do the didactical variables affect the students' works, in order for the students to engage in formulation and validation situations?
- What may hinder the students' transition between the three phases: examination of pattern, formulation of conjecture and proving of conjecture, and how does the work in one phase influence the next?

The analysis integrates the students' milieu, the a priori analysis of the exercises, the written work produced by the students and the audio recordings.

As mentioned in Section 3.3, three groups were permanently handed a recorder for the whole duration of the course. These groups were put together on the basis of the teachers view on the students mathematical performance level. It will in Chapter 6 be discussed, what classifies a student's performance level and if it is possible at all, to make such a classification. For now, I hope the reader have some personal comprehension of what a below average, average and above average performing student is.

I have tried to maintain the danish formulations in my translation of the transcription, why some of the sentences might seem a bit odd. I have done this, in order to prevent my self from altering the meaning of the dialogues by rewording them.

5.1 A POSTERIORI ANALYSIS OF SELECTED SITUATIONS

5.1.1 *Situation 1*

In this situation, we are following Group 4 in their work with Exercise 1 during the first day of the course. The group consists of three students, two girls and one boy. These students will be denoted A, B and

C, and the teacher will be denoted by T. Prior to this work, the students have been introduced to the exercises, but are not told to use algebra or to make conjectures. The following analysis will contain fragments of the transcript from the situation. The entire transcription can be found in the appendix. See Section C.1

The Target Knowledge

The target knowledge of this exercise, was to make the students acquainted with pattern determination and formulating a conjecture, when making the procedural calculations. It was not a purpose of the exercise, to make the students prove their conjectures by the use of algebra.

The Milieu

The objective milieu is made up of Exercise 1 (see Section A.1) and the Sum Table (see Figure 41) in which the students were encouraged to make drawings. The students were asked to make individual examinations of the problem, before discussing them in groups.

Exercise 1A concerns 1×4-rectangles, while Exercise 1B asks the students to vary the length, in order to examine the result, when making the same calculations as in 1A:

The students are asked to draw 1×4-rectangles in the Sum Table and then make the following calculations in order to obtain the largest result:

1. Add the largest and smallest number
2. Subtract the second largest number from this sum
3. Repeat this by "moving" the figure

Such a rectangle could contain the numbers: $\{20, 21, 22, 23\}$, which would give the following result:

$$20 + 23 - 22 = 21$$

And by repeating the process, the students should notice, that the result is always the second smallest number and from this conclude that the largest result is 41, obtained with the rectangle $\{40, 41, 42, 43\}$. Some students would naturally think, that the largest result is obtained by the rectangle containing the largest numbers, but the students must verify this, by making other numerical examples.

It is not expected, that the students proves this result. This is due to three things: Firstly, it is the first exercise of the course of study and the exercise is therefore meant as an introduction to these type of exercises, where the focus is on determining a pattern. Secondly, there exists exactly one, numerical result. Thirdly, the relation of the

Sum Table and the Calculations Needed diminishes the need for using algebra: The number you subtract, is always one smaller than the largest number and therefore you add one to the smallest number. The result must therefore always be the second smallest number. The situation is therefore not a fundamental situation for using algebra, as there are other possible and maybe easier way, to solve the problem. As a verbally formulated proof could be enough to convince the students about the truth-value of a conjecture, the fundamental need to introduce algebra is removed

Nevertheless, if the students should use algebra, a generic model could look like:

$$\begin{bmatrix} x & x+1 & x+2 & x+3 \end{bmatrix}$$

Where x denotes the smallest number in the rectangle. With the following initial expression:

$$x + (x+3) - (x+2)$$

And the following proof:

$$x + (x+3) - (x+2) = x + x + 3 - x - 2 = 2x + 3 - x - 2 = x + 1$$

By which it is seen, that the result is always the second smallest number. A deeper analysis of Exercise 1 can be found in Section 4.1

Exercise 1B asks the students to examine if the pattern changes, as they enlarge the rectangle in the horizontal dimension. They are here asked to formulate a conjecture and an argument for this pattern, if such a pattern emerges. The purpose of Exercise 1B is therefore to make the students formulate an argument, for a phenomenon they have examined, though this is not expected to be of algebraic kind.

To formulate an universal algebraic proof of this pattern, would require the introduction of a parameter representing the length of the rectangle. If we denote this parameter n, an algebraic proof could be:

$$x + (x+n-1) - (x+n-2) = 2x + n - 1 - x - n + 2 = x + 1$$

From which it is seen, that the result is independent of the parameter n.

The numerical solution to the exercises is relatively easy to obtain, as it requires addition and subtraction of numbers. A generic model requires, that the students sets up relations between the variables and the relation of the Sum Table eases this work, as a the difference between two neighbouring variables is always one. An algebraic proof requires that the students formulates an initial expression, by the use of parentheses and to obtain the desired final expression, the students must master the associative, distributive and commutative laws.

5.1.1.1 *The situation*

In the beginning of the situation, the three students have been trying to figure out which rectangle produces the largest result. They have been trying different rectangles and some of the students are left frustrated, as they fear they need to check all possible rectangles. The teacher happens to pass by and one student asks the teacher:

B: "(...) But how do you find the rectangle, that gives the largest result? We all agree, that is probably not there [points to a drawn rectangle on the paper]. It's that one, down there [points to a rectangle, in the lower right corner]
T: "Is it? Then you have to try"
C: "No it is not necessarily that one"
A: "Isn't it?"
T: " [Student A] argues. She says it is the last one"
C: "It is..."
B: "What does this give? It gives..."
T: "Then write it down and see what it gives"
C: "Okay. It is 43 + 40, thats 83 minus 42. That is 41
T: "Okay"
C "I can't see how any could give a larger result, as there aren't large enough values in any other"

As described, the three students have been examining the Sum Table and even though they are frustrated by the prospect of checking all possible combinations, they do seem to have some initial idea of the pattern. Student C does not initially have a comprehension of the pattern, but is through the numerical example convinced about the truth value of Student B's statement (*"It's than one, down there"*). Even though its not stated or outspoken, it seems as if the students through examples and counter-examples have produced a conjecture.

If we start by looking at the didactical variables concerning the design of the exercise, it looks like the relatively easy relation in the number table, makes it easier for the student to work out the pattern. Student C makes the right conclusion, that the largest result must be obtained from the rectangle that contains the largest numbers, based on his calculations. A conclusion that Student B also made pretty quickly, but without forming an argument. Student A's wondering about Student C's first wrong conclusion also shows, that Student A has some idea about the pattern.

It is relatively intelligibly that the largest result, must be obtained from the rectangle containing the largest number, but at this point the students have not unraveled the pattern. The group have reached an initial orally conjecture through examination of the number table, which enables the group to start a formulation phase. Both the easy calculations, addition and subtraction, combined with the pattern of a 1×4 rectangle contributes to Groups 2's transition from the examination phase to a phase of conjecturing.

At this point, the students are still examining the pattern and have not formulated a formal conjecture about the pattern. This shall of course be analyzed in the light of the short time the students have been working with these types of exercises. It is noticeable, that the group on the basis of their own wondering, starts a conjecturing phase, as we will see next.

After the above discussion about the 1×4 rectangle, follows a discussion between the group members and the teacher, on where and how to find the largest result. During this discussion, Student C says:

C: "In that rectangle, the result will always be the second smallest number"

This is the first time, a student in the group formulates a precise conjecture about the pattern. It seems as if Student C on the basis of his own wondering and different numerical examples reaches this conclusion. This is not something the students are asked to do in the exercise, but the oral formulation of a conjecture seems to have arisen from Student C's own reasoning. In this part of the situation, the knowledge about the pattern seems to be personal knowledge to the group members and the problem is then to make this public by formulating an argument or proof. Especially Student C tries to do this later in the situation.

The three didactical variables concerning the design of the exercise are all in play here: The easily comprehensible relation of the sum table together with the simple pattern of the 1×4-rectangle makes the students enable to engage in a formulation situation. Student C orally formulates a conjecture, which the group later writes down (See figure 13 below). It is not clear if the rest of the group members have reached the same conclusion as Student C, but they quickly accepts this conjecture as being true and this is partly due to the palpable pattern. At this point, Student C's conjecture is formed from an pragmatic approach and can be classified as a conjecture based on naive empiricism (Miyakawa, 2002, p.2), as it is concluded on basis of trial and error.

It can be seen by figure 13, that the group have written *"The result is the second smallest"* (Danish: Resultatet er det næstmindste). It is unclear at which point, the group formulates this conjecture in writing, but the conjecture is clear and precise. Figure 13 also shows, that the group have made a solid argument, for their conjecture: *"You add*

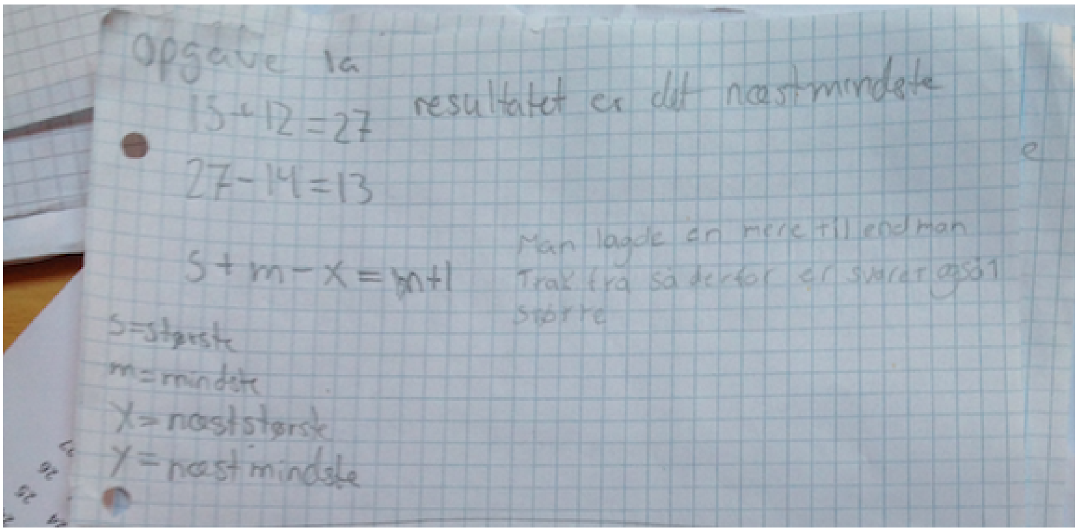


Figure 13: Exercise 1 - Group 4

one more than you subtract, therefore the answer is also 1 bigger"(Danish: Man lagde en mere til end man trak fra så derfor er svaret også 1 større). This is an argument Student C comes up with, later in this ongoing discussion. To compare this argument to the definition of proof from Section 2.2.5, this argument is accepted within the group and it, to some degree, explains why the mathematical phenomenon happens. It do however not contain mathematical properties, though it does contain elements of pre-algebra: Student C introduces a verbal formulation, that just needs to be formalized in order to fulfill the definition of proof.

After Student C orally formulates a conjecture about the pattern, follows a quick discussion with the teacher about the second smallest number, which ends with the teacher admitting to have looked at the wrong rectangle. What is more interesting is that Student C in this discussion realizes - without help from the teacher - that there are some kind of coherence between the rectangle, the calculations and the result, which could be described by the use of letters:

C: "It's like... It's just a formula, where you put in some numbers"
T: "Okay, so it can be a formula. Then you try, if you can write up that formula. That was a good idea"
The teacher leaves
A: "Then, then what's the formula...?"
C: $a + b - c = d$...?
Student A laughes
C: "But isn't it just that?"
A: "Now, what is it again...? The biggest... $s + m$... The biggest plus the smallest... Hmm.."

C: "Then what will you denote the middle numbers?"

A: "What?"

C: "What will you denote the two numbers in the middle?"

A: "We'll figure that out"

C: "It doesn't matter what the number is... Or letter (Corrects himself)"

A: "Maybe. It's definitely s and m"

C: "Then, do you just call the two other x and y or...?"

A: "Yes, yes. Minus... No... Yes... You can say it directly like that, right? Minus... And that was..."

C: "Minus x, and that equals y"

The dialogue above shows, that Student C and Student A is entering a proving situation. They both have some idea on how to implement algebra in there formula, but they still face some challenges on how to formulate a precise algebraic proof. Figure 13 shows, that the students started with four variables: s, m, x and y, where s denotes the largest (Danish: Største) number, m denotes the smallest (Danish: Mindste), x denotes the second largest and y denotes the second smallest. Hence the students, do have the algebraic prerequisites to engage in a formulation phase, but still face problems with identifying relations among the variables. From the written work (figure 13) it becomes apparent, that the students somewhere along the way, derived a relation between y and m, by concluding that $s + m - x = m + 1$. In other words $y = m + 1$. But the truth value of the algebraic statement $s + m - x = m + 1$ is difficult to determine, when a relation between s, m and x has not been stated. A fact that also frustrates Student C, as he in the discussion frustratedly utters:

C: "Yes, but that's probably just the way it is, because it is four different numbers that is needed... *swears*"

At the seemingly meaningless expression $a + b - c = d$

The calculations needed to solve this exercise (Add two numbers and subtract one) contributes to the students possibility of engaging in a formulation phase, which was the purpose described in Section 4.1.

In this situation, the two students introduce letters for each number, which is used in the rectangle: The smallest(m), the largest(s), the second largest(x) and the second smallest(y). Hence, they are trying to take the step from a pre-algebraic formulation, to an expression containing algebra and their pre-algebraic formulation makes it easier for them to arrange an initial algebraic expression. The problems occur, when they need to set up relations among the variables.

It therefore seems, as if the biggest obstacle in order for the students to engage in a validation situation, is the algebraic prerequisites needed to set up precise relations between the variables. The students' final algebraic expression contains three variables, which is an expression of the procedural calculations the exercise tells them to do and not an algebraic proof of the outcome. As described in Section 2.4.2.1, the students are missing the last two steps of the Chevellards (1989) modeling proces: To set up relations between the variables and then to "work" the model, with the goal of producing knowledge of the mathematical problem. In this case, why the result is always the second smallest number.

It can also be the case, that the didactical variables DV1, DV2 and DV3 prevents the students from engaging in a validation situation, as a proof is easily stated by the use of reason and words: As written above, figure 13 shows that the group have made an argument by the use of words: "*You add one more that you subtract, therefore the answer is always one 1 bigger*". This argument is perfectly reasonable and easy to follow, as the second smallest number is one larger than the smallest number, and as the students correctly states, you add one more than you subtract, and hence the result is always the second smallest number. This argument diminishes the use of an algebraic expression and why the students do not need to introduce letters. The validation therefore becomes an trial and error that confirms the written argument. This becomes evident during the groups work with Exercise 1B, as Student C expresses that the formula covers all rectangles:

> **C:** “It will always be the second smallest number, with that formula there"

As neither Student A or B seem to be convinced by that argument, they test the formula with some numerical examples and as this confirms Student C's formula, they see this as a proof of his conjecture. As the formula is validated by testing it with arbitrary numbers, the proof now has the status as a *crucial experiment* (Miyakawa, 2002, p. 355).

Something which indicates, that the group are not convinced about there own algebraic formula is, that they present the verbal argument as their "proof" at the blackboard, during a summarization. As the formula does not convince themselves about the truth-value of their conjecture, this formula do not fulfill the definition of proof, even though it now contains a mathematical property.

In the described situation, the students follow the intentional trajectory: The students start an examination phase by trying different placements of the 1×4-rectangle.

Through this examination of the number table and results, the students discover that the result is always the second smallest number.

They state this during a discussion with the teacher, but it seems as if it is a fact that is obvious to all group members. Hence, they make a conjecture, which originates from the students own wonder, based on their experiences with the exercise.

The group ends up making an algebraic expression, which describes the procedural calculations in the exercise. Its not clear, whether the group accepts this expression as a proof, but when Student C proclaims:

C: “Because, you add 43 to that number, right? 37, right? Then as you subtract 42 again, then it is just one less you subtract than added before. So it is plus 1. So its 38"

They accepts this as a proof. This argument originates from the desire to explain why the result is always the second smallest number, hence the proving phase naturally proceeds from the conjecture phase, as Student C tries to convince the other group members of the truth value of his conjecture.

As written above, the situation revolves around Exercise 1. This exercise was planned as an entrance for the students, to work with pattern determination and from this, conclude that the correct answer will be the rectangle that contains the numbers $[40, 41, 42, 43]$. I did not think, that the students would necessarily end up with formulating a precise conjecture like *"... the result will always be the second smallest number"*. In the light of this, I find it surprising, that a student (Student C), so quickly realizes that there exists a pattern, and also takes it a step further, as he declares that it is *"... just a formula"*. This student, was the only recorded-student in the class, who suggested that it was a formula and got the idea of using letters to describe the relation between the placement of the rectangle and the result.

All in all, the didactical variables, DV1 and DV2, contributes to the students chance of engaging in a formulation phase, as the students quickly spots the pattern and the relation of the number table, but the two didactical variables, including DV3, also diminishes the need of initiating a validation phase by the use of algebra, as the calculations are to simple. As the students tried to set up an algebraic expression, they faced difficulties concerning their algebraic prerequisites since they had problems identifying relations between the variables. The lack of algebraic arguments is a consequence of the situation not being fundamental to introducing algebra as the students formulates other convincing arguments. The students use algebra in their formulation of a conjecture, as they combine the procedural calculations with algebraic symbolism. Unfortunately, they do not use algebra as a tool in a deductive proof and this is also because of the pre-algebraic argument being sufficient, as the pattern is easy to grasp.

The conjecturing- and the proving phase happened in interaction with the teacher and in the transition from one phase to the next, Student Cs activity was helpful as they ended up forming a conjecture and introducing letters.

Lastly the situation will be analyzed further, by the use of the situation of institutionalization, which followed after.

Just prior to the institutionalization, the groups have discussed their findings with their paired group, before presenting their findings at the black board. At the blackboard Group 1 + 2, presented an argument based on *crucial experiment*: By doing the calculations four times with arbitrary placed rectangles, the result was always the second smallest number. They then verified this conjecture by varying the length of the rectangle and making the same calculations, seeing that the pattern did not change.

Group 3 + 7 made an argument listing the procedural calculations: If you add the smallest and largest number, and from this sum subtract the second largest number, you obtain the second smallest number.

Group 4 + 5 argued by the use of Group 4's verbal argument:

C: "... If you add the smallest and largest number and subtract the second largest number, then the difference of the largest and second largest is 1. So this corresponds to just adding one to the smallest number and then you get the second smallest number"

The teacher then interrupts the groups' presentation

T: "Did you hear that? There was more mathematics in this, could you hear that, compared to the other two? (...) It becomes more and more general, compared to the first, that was concrete - we did this and this - and number two then says. There we get the mathematical calculations set up and now (Student C) says, that there are a system and no matter what, then we can see how big the difference is between the two numbers. That's why you end up there. Yes, what about the last two groups?"

The last two groups then formulates an argument that is similar to group 4 and 5. This formulation is long and a little difficult to keep up with, but the content is more or less the same as Group 4's argument. The teacher then continues by asking, what could be done in order to make it *"... Even more mathematical? (...) Do you have any offers on, what you could do, if you wanted to generalize this?"*. A student then answers:

Student(S): "Make a proof?"
T: "Yes. Maybe. Maybe a proof. Or maybe introduce some symbols"
S: "Symbols?"
T: "Yes. What do I think of here, when I say symbols?"
S: "x and y?"
T: "Yes! X and y for example! You could for example generalize it, so it is not a concrete row (dansk: række). (Indistinct talk). It could be x. X was maybe 13, then the next number in the row, that would be $x + 1$. Do you get it? (...) If you look at it as we normally... Proves something in math, then it is general. Then it is rarely that we say something like a long sentence (dansk: en lang række) like (Group 6) just did, because then it is difficult to keep up. There, this algebra, which is to make something with letters, can something completely different than... The example every time. Because then we have an example one place in the sum table and then something in another place. So if you want it even higher up, then you can try to make it general by introducing some symbolism and that is what you call algebra. Yes"

The last part of the situation, shows that none of the groups used algebra to prove their conjectures. The proofs ranged from *crucial experiment* to *thought experiment*, but none of them can be classified as a *mathematical proof* (Miyakawa, 2002, p. 355). From the data I have collected, it is not evident if any other group introduced algebra before the institutionalization, but in the written work Group 5 and 6 have both introduced algebra somewhere along the way: $A + D - C = B$ (Group 5) and $(2x + 4) - (x + 3) = x + 1$ (Group 6). Nevertheless, none of the groups use an algebraic argument in their presentation, which indicates that: (1) they did not see the advantage of such an approach, (2) that it would not be an advantage for them or (3) that they did not succeed in making an algebraic proof. I think that it is a due too all three things: To use algebra in an argument would be unnecessary and complicated, as a verbal argument worked just fine, while at the same time, a lot of the students were not confident in introducing algebra in a mathematical expression.

The teachers monolog shows, that the institutionalization of algebra as a tool to prove a conjecture happens on the basis of the teachers initiative and not on the basis of the work done by the students, contrary to what was described in Section 3.2 . At this point, the argument for using algebra is down to *"... as we normally... Proves something in math..."* and *"...if you want it even higher up, then you can try to make it general by introducing some symbolism..."*. The students are therefore told that the correct way to prove a mathematical statement, requires some symbolism, even though it is questionable if algebra is a better tool than a verbal argument in this situation. The teacher therefore cuts the double work of recontextualisation and redecontextualisation in order to *"... make the students learn a text of knowledge directly"* (Måsøval, 2011, p. 54). It could have helped the institutionalization, if the students prior have worked with exercises where algebra was a more effective tool to prove a conjecture and the institutionalization was based on a classroom discussion of the students' different arguments or proofs, as reported in (Knuth, 2002, p. 489).

5.1.2 *Situation 2*

In this situation Group 6 is working on Exercise 4.2 at the beginning of day 3 of the course of study. The situation ends just before the class summarization as Group 6 was not called to the blackboard and nor did they discuss their findings with their paired group.

Just prior to this situation, there had been an institutionalization of Exercise 4.1, where the use of variables were said to *"strengthen the argument"* in a proof. The method of constructing an algebraic proof by combining a generic model with the procedural calculations was also institutionalized. The teacher here focused on using the smallest number as placement of the independent variable and from this formulate relations to the rest of the variables in the generic model.

The full exercise description can be found in Section A.2 and the a priori analysis can be found in Section 4.4.1. Group 6 consists of four students, three girls and one boy. These will be denoted E, F, G and H. The group had earlier solved Exercise 1 to 6, but had skipped Exercise 4.2 as it was optional. The group solved the prior exercises with large succes and used algebra in more or less all of their solutions, where they seemed capable of applying algebra as a method of formulation and validation. The group were told to do Exercise 4.2 in order for them to present it at the blackboard.

The following analysis only contains fragments of the transcript of the situation. The entire transcript can be found in the appendix. See Section C.2

The Target Knowledge

The target knowledge for Exercise 4.2 is, to make the students use algebra to explicate that the number x, is always 9 larger than the smallest number in the staircase, where x is defined as half the sum of the largest and smallest number. The intended final algebraic expression, the students are to formulate is:

$$x = a + 9$$

The Milieu

The objective milieu in situation 2 is made up of Exercise 4.2 and the Calendar, which can be found in the appendix (see Figure 42). The group had the number table, the work sheet and blank paper available, which they were encouraged to use in order to visualize their figures and findings.

The pattern of Exercise 4.2 is an intermediate calculation in Exercise 4 and 4.1, so to clarify the purpose of Exercise 4.2, the desired calculations of Exercise 4 will here be shown:

The students are in Exercise 4 asked to add the smallest and largest number in a staircase in the Calendar and from this sum subtract the second largest number. An example of a staircase is shown in figure 14

M	T	O	T	F	L	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Figure 14: An example of a staircase in Exercise 4

This example would result in the following calculations:

$$5 + 23 = 28 \quad (1)$$
$$28 - 17 = 11 \quad (2)$$

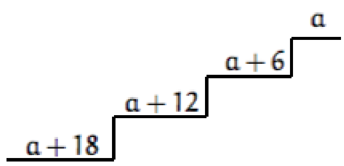
Where Exercise 4.2 focuses on calculation (1). In this example $x = 14$, as $2 \cdot 14 = 28$, from which it is also seen that $x = 5 + 9 = 14$.

Exercise 4.2 seeks to make the students formulate an algebraic relation between two variables, a and x, which indirectly had been a part of the calculations in Exercise 4. The pitfall with Exercise 4.2, is that the pattern can be fairly easy to spot, when the relation between the smallest and largest number is revealed, as x is then the average between these two numbers: The difference between the two numbers is always 18, why the relation between x and a will always be 9 and this can undermine the need for using algebra. This will be developed later.

Exercise 4.2 starts by defining the variable x as: *The sum of the largest and smallest number, will always be the double of an integer. Let us call this integer* x. The students are then asked to determine the relation between the smallest number in the staircase and this x. This approach is somewhat different, than in the other exercises as the definition of

x and the desired pattern are not directly connected. The pattern can be determined by simple numerical examination, but as the students are not asked to make any calculations and as the pattern can be numerically difficult to spot, the purpose of the exercise is to make the students use algebra to explicate this pattern. By numerical difficult I mean, that by pure calculation it is not evident, that the smallest number in the staircase is 9 larger than x. This could become clear to some students, if they think of x as the average of the two numbers, but as the exercise description does not directly describes the average, the purpose is, that the students sets up the algebraic expression in order to reveal the relation.

To solve Exercise 4.2, the students first of all need to determine the relation in the Calendar in order to successfully create a generic diagram: From figure 14 it can be seen, that if we start from the top, the next number in the staircase is always 6 larger than the prior, as the week-number increases and the day-number decreases. As the smallest number is contained in the exercise description, it is foreseen that this placement will be used as the independent variable. A generic diagram could then look like:



To set up a relation between the smallest number a and x, the students need to find an expression of half the sum of the largest and smallest number:

$$x = \frac{a + (a + 18)}{2} = \frac{2a + 18}{2} = \frac{2(a + 9)}{2} = a + 9$$

Alternatively the the students can determine x by relating the average of a and $a + 18$ to x and hence $x = a + 9$. As this approach would diminish the use of algebra, I will focus on the solution containing the above algebraic calculation to the exercise.

For the students to successfully solve the exercise, they firstly have to transform the exercise description into mathematical operations, apply algebra and set up an initial expression. The exercise description sounds:

> *The sum of the largest and smallest number, will always be double that of an integer. Let us call this integer x. (...). Determine the relation between the smallest number in the staircase and x.*

The above description concerns *double that of an integer* ($2x = a + a + 18$), but the students need to determine the relation between this integer and the smallest number ($x = a + 9$), hence there are a semantic gap between the exercise description and the final expression. In other words, the students needs to focus on the sum of the largest and smallest number: $a + (a + 18)$ and combine this with $2x$ to set up the initial expression: $2x = a + (a + 18)$. From this expression, they need to deduce that $x = a + 9$ in order to interpret this final expression.

The students need to make algebraic reductions and rewritings, which concerns factoring out and/or dividing different terms, in order to reach the final expression. Some of the major obstacles with this algebraic expression, could concern the use of two variables in an algebraic relation. Exercise 4.2 is the first exercise, that requires the introduction of two variables and this could cause problems. The exercise therefore requires, that the students understands the symbols as variables and not just as an unknown, that needs to be determined.

The situation

In the beginning of the situation, the students have been trying to figure out what the exercise is about. Especially the difference between the definition of x and the relation between the smallest number and x cause problems for the group.

As we enter the situation, the group are trying different numerical examples based on the definition of x. They have drawn a staircase containing the numbers $\{18, 24, 30, 36\}$ and from these numbers they calculated the result 27. They are though still not certain about what to do with this result.

F: "Then 27 are x. Wasn't it like that, the exercise should be understood?"

E: "That isn't true..."

H: "Yes"

//

G: "What are you supposed to do?"

//

F: "So, what you are supposed to do. We are doing 4.2, where you are told to make a staircase, where we shall prove that when you add the largest and the smallest number together, it will always be double that of an integer. But no... This is not an integer..."

Then follows a discussion about what an integer is, which results in Student F admitting that she confused an integer with even numbers. Nevertheless, it is clear from the dialogue, that Student F formulates a wrong conjecture. The conjecture Student F formulates is based on the definition of x and the group now thinks, that the task is to prove, that this number, x, is always an integer. Student G then formulates an oral proof of Student F's non-relevant conjecture:

G: "So if you add the uppermost and lowest number together, then either it is because both... It's because, either are they both uneven or they are both even, and if you... subtracts an uneven and an even number, then they are both even (...) If you subtracts two even numbers, then it becomes even. If you subtracts two uneven numbers, then it also becomes even"

The students' confusion about the definition of x and the relation between the smallest number and x, seems to withhold them from entering a proving situation, as they have troubles formulating one conjecture. The importance of the students' process in determining a conjecture to prove is emphasized in (Martinez and Li, 2010) and will be elaborated later. The conjecture Student F comes up with, is more or less just the exercise description and this conjecture is easily verified, which Student G also does. The group do however quickly unravel the relation of the number table and this leads the group in the correct direction:

G: "It's the 6-tables all the way around"
E: "Yes. 52, 58, 64, 70"
//
H: "I got it! Girls! Ehm... There are always 18... Ehm... There are always 18 between the uppermost and lowest number!"
E: "Yes"
H: "That is something we can put to use!"
E: "Can we?"
H: "Yes, because if we then say... Yes, look... x"
G: "What's the name of the exercise?"
H: "x. So you start by dividing it by two, right? That gives something unknown, let us call that y, right?"
F: "No, no, for x has to be the number you obtain, when you divide by 2"
H: "Yes, exactly! Oh no, then you multiply, sorry. That's right. You multiply. You multiply by 2 and that... and that number you obtain here, right? That you have to divide, so there are 18 between them"

The group are still somewhat confused about what they are supposed to prove, but especially Student H uses the relation of the number table to investigate the relation between the largest and smallest number. She concludes that if you multiply x by 2, then there must be 18 between the two numbers. Based on the numerical difference student H engages in an incipient formulation situation, where she is close to orally formulating the initial expression

$$2x = (y + (y + 18))$$

This expression can also be found in Student H's written work, see figure 15. It seems as if the relation of the number table and the pattern of the staircase helps Student H in engaging in a formulation situation. From the written work, it can be seen that Student H successfully relates the smallest number in the staircase(y) with the largest number $(y + 18)$ and makes the relation with another variable x. It shall be mentioned, that it is not clear at which point the student writes down this expression, but the above dialogue shows that she is close to formulating an initial algebraic expression. Even though Student

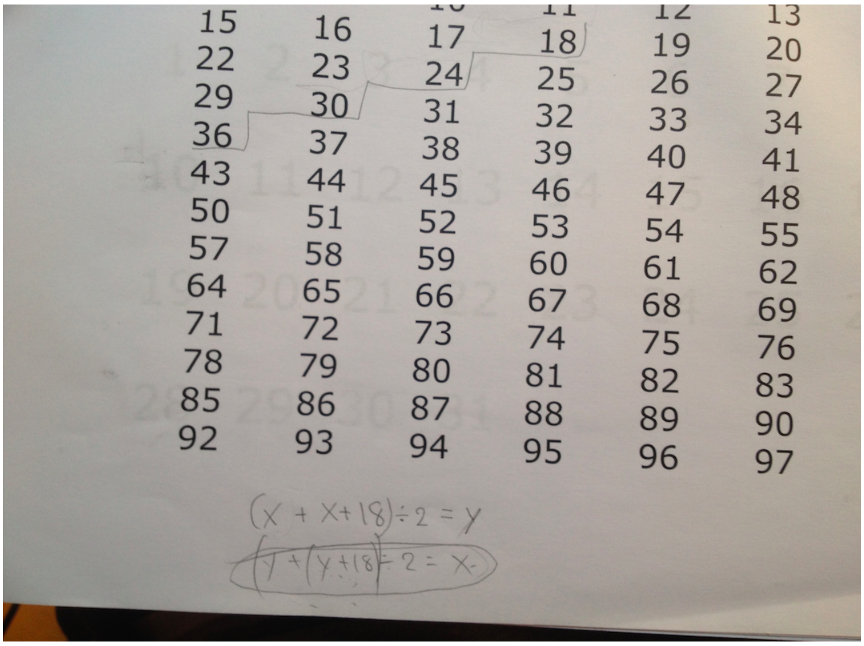


Figure 15: Student H's written work - Exercise 4.2

H seems to be engaging in a formulation situation, the missing conjecture seems to confuse the rest group. Student E, F and G discusses the use of the variable x on both sides of the equality sign, as Student H intervenes. Student H seems to have understood, that x is always 9 smaller than the largest number, without the use of algebra, which was the pitfall of this exercise, as described in *The Milieu*:

H: "What is it... x, that is also... It's 9 less than that one there. The uppermost number"
F: "But, is it always that?"
G: "That it's obvious, if there is always 18 between"
H: "No, no. Yes, yes. I mean... Yes, but there are also 9 there"
E: "Between which one?"
H: "Between the answer and the largest(Dansk: Højeste) number.
E: "Is it always like that?"
H: "Yes"
G: "But that's obvious. The difference does not change"
E: "No, it's not!"

H: “Yes, it’s true. The difference between it all will always be the same. So we just need to use the difference for that, after we have done that one"

F: “I don’t understand how we are supposed to show it"

H: “No..."

F: “Okay. We get this sum between the largest and smallest number. The largest and smallest number... If we just take it all... All the way back. Largest and smallest number, right? If we make a sequence of numbers: $x, x+1...$, No, $x+6$, right?"

What is interesting from the above dialogue, is that Student H and Student G seems to be on the right path: Student H is convinced about her conjecture: *"x is always 9 less than that one there..."*, which Student G thinks is obvious. To student H and G, the pattern now has the status as personal knowledge they need to depersonalize in order to share it and through this, make it official knowledge, which could be done through an algebraic proof or verbal argument, but this seems to cause problems. It is not clear, if Student G thinks of x as the average of the two numbers, but is seems as if he is aware of the fact that x is 9 less than the largest number. By now they could formulate a verbal relation between the smallest number and x, but they do not do this. This could be due to a missing formulation of a conjecture or because Student G thinks the relation is *"obvious"*, but what is obvious to Student G is not obvious to Student E (*"No, it is not!"*). Student G does not use the fact that x is 9 less, in a formulation or validation of a conjecture, which could indicate that he has not yet related this fact to the relation between the smallest number and x.

As Student H is not confident in convincing the other group members with either an explanation or a proof, the whole group engages in a proving situation, here shown by Student F’s initial algebraic work.

The dialogue shows, that even though there do not seem to be consensus about the relation between *"the uppermost number"* and x described by Student H, the rest of the students engage in a proving situation, based on the intellectual need to examine if and why, this specific phenomenon happens. This is something that is also emphasized in both (Martinez and Castro Superfine, 2012) and (Brousseau, 1997), and was one of the design principles, see section Section 2.2.3: A proof, beyond establishing a validity of a mathematical statement, must also show why this phenomenon happens in order for the proof to convince oneself or someone else about the validity. As the students have not yet formulated a conjecture about why this relation occurs, they engage in a proving situation in order to seek this explanation.

Brousseau (1997; p. 15) argues that doing mathematics is a social activity, which contains the possibility for a student to establish a truth value about a certain theorem or mathematical statement. This truth value of a statement must origin from the students own conviction and cannot be learned by reference to the teacher. It is therefore essential that the student engage in a formulation and validation situation, where the incentive is a need for a personal conviction about the truth value of a specific mathematical phenomenon.

With this in mind, the dialogue shows that the students are engaging in a validation situation based on their own curiosity about the relation between the two variables x and y. This curiosity seems to have been intensified by the appearance of two possible conjectures: One stated earlier by Student F (*"... when you add the largest and the smallest number together, it will always be the double that of an integer"*) and one stated by Student H (*"x (...) It's 9 less than that one there. The uppermost number"*). The group are now formulating an algebraic proof in order for them to validate the truth value about these two possible conjectures.

The above dialogue also shows, that the group have transited from a phase of examination, through a phase of conjecture to a phase of proof. The phase of examination was short and overcome as Student H unraveled the relation of the number table in order for her to set up the relation between an independent variable, y, which denotes the smallest number and the dependent variable $y + 18$. This short examination phase, could be one of the reasons why the group have troubles forming a conjecture, as the conjectures are not based on numerical findings.

The situation continues with the students trying to formulate an initial algebraic expression. This work is hindered by two obstacles: (1) Determining if it should be $\frac{1}{2}x$ or $2x$ and (2) the prospect of setting up an initial expression involving two variables.

G: "Why do you want it to give a half x?"

F: "Because then it is the double. It is this number here (...). And that is the half of this number here"

E: "But what is that up there then?"

F: "That's just the number sequence"

E: "Yes, but that is also the number sequence. (..) That is why I won't denote it the same as that number in the number sequence."

//

F: "You just can't get two unknown values!"

E: "No, that's true, but it's just because then x becomes this value. Then it's different"

//

H: "What exactly is it, the answer is supposed to give? (...) If you have. If we have x. Then what is it exactly, the answer is supposed to give?"
//
F: "The answer is supposed to be... The double of an integer"
E:"Can you just say... Just wait a little. I'll just try to write it down. Ehm.. If we have... What we have to do is... We'll say... $x + x + 18$. (...). Divided by two. Equals..."
F: "But no, no.... Because we don't know how we are going to solve it"
E: "Yes, I also wanted to write that. It is the very thing we are doing"
//
E: "Equals y. No, because x equals this one. This is 18. Then we write here"
F: "But then there are two values again! I hate it when there are two values!"
E: "But, that is what we need to. Then we reduce this, maybe. But, it is this that applies"
F: "Yes, but then the problem is, that you can never work out what y is".

From the above discussion, it is clear that the students' problems concerns the two mentioned obstacles. The first obstacle[1] do not seem to pose such big a problem, but it confuses the group members and this initially keeps Student F and Student E from formulating a written expression. This could relate to what is emphasized in (Martinez, 2008, p 53), where the author, on the basis of a study in (Healy and Hoyles, 2000), argues that students faced large difficulties with constructing and completing a proof, when the statement was unfamiliar. If we analyse the above situation with this in mind, the difficulties concerning this obstacle may relate more to the fact that the conjecture is uncertain, than the problem of multiplying or dividing with the number 2. This is seen from Student H's question: *"... what is it exactly, the answer is supposed to give?"*.

The other obstacle, which is explicitly stated by Student F (*"You just can't get two unknown values!"*) concerns the use of two variables in the initial expression. This obstacle could relate to several misconceptions by the students, which all relates to the use of letters in an algebraic expression.

One misconception is pointed out in (Bardini et al., 2005), wherein the author describes students' lack of understanding concerning the semiotic affinities and differences between unknowns, variables and

1 Much of the dialogue concerning this problem has been omitted, as the major obstacle is the introduction of two variables

parameters (Bardini et al., 2005, p. 129). Bardini et al (2005) argues, that many students see variables and parameters as temporally indeterminate numbers, whose fate it is to become determinate at some point during the students' work. The difficulties is due to the students' lack of understanding that letters, in this case y, can have the status as something that varies. This obstacle is clearly expressed by Student F as she in the last line in the above dialogue says: *"Yes, but then the problem is, that you can never work out what* y *is"* and is an example of a epistemological obstacle. The student's faulty understanding of a variable hinders her from determining a personal meaningful relation between the smallest number in the staircase and x, and it keeps her from entering a validation situation, as she does not accept an expression containing two variables as a meaningful answer. Even though the students do make various attempts in writing an initial expression, this expression is still seen as a something indeterminatet, that in the end must provide a numerical result. This is also emphasized in (Bardini et al., 2005, p. 134), where the authors discovered, that some students found it strange that the teacher could ask a question, which did not result in an actual number (Ibid, p. 135). As Exercise 4.2 is the first exercise, where the result is not numerical or only contains one variable, this could also play a part in Student F's difficulties with solving the exercise.

Another misconception relates to the problem of distinguishing between two variables and, maybe more important, understanding the relationship between these two. These problems are emphasized in (Binns, 1994, p. 335), where the authors found, that the literal similarity between two variables confused the students and made their understanding of an algebraic expression containing two variables vague. Though the students, in the experiment conducted by Binns (1994), had a clear understanding of the role of a letter as a variable, the introduction of another variable confused them, as they sought to evaluate the letter x (Ibid., p. 336). It is clear that Student F faces problems with an algebraic expression involving two variables or *"... unknown values"*, which could relate to the observations made by Binns (1994). This also indicates, that Student F sees the two variables as unknowns she is supposed to determine the value of.

Then follows a long discussion with a member from another group, who faces the same problems in her work on the same exercise. This student, who from now on will be denoted Student K, and our group discusses how the solution should look like and on what form.

K: "[The teacher] just said, that this formula looks reasonable"
E: "Yes, but that's just how you calculate it. It isn't a proof"
K: "It is a proof"

H: "No, you can't really calculate it, so you say, that this... Ehm.. That this gives x"

E: "But we can never get it to be the same!"

//

H: "Can you just say, that the number... What can you..."

F: "It is already that, because if you add these two, divide with 2, then it is...

H: "Yes, but then we could use that as a sentence (Dansk: Sætning)"

F: "Yes, but then you just find... Then you just find the average. That is actually what we do"

H: "Yes, but then the answer is the average of the number sequence"

The first part of the dialogue, shows that the students do seem to have found the right formula, but they do not see a formula containing two variables as a result. In the other part of the dialogue Student F mentions that *"Then you just find the average..."*. and from this Student H formulates a conjecture *"...then the answer is the average of the number sequence"*. The students have now formulated a clear conjecture and could from this fact, explain the relation between the smallest number and x, but it does not seem as if they successfully relates their conjecture to their written work.

From the written work, see figure 16, it can be seen that the students have found the correct formula:

$$y + 9 = x$$

where y is the upper most number in a staircase and below have written two conjectures. The first of these two conjectures: *The result is the average of the number sequence - or the largest and the smallest* emerges above and the other conjecture was stated earlier. The two conjectures are both right, but only one of them is relevant to the exercise and this seems to confuse the students.

The discussion continues as the students still believe, that they need to prove why the result is an integer. During this discussion the teacher intervenes, as the group seems to be stuck.

T: "Yes. What is the result? What is the relationship between the smallest number and x"?

//

H: "The result is the average of the number sequence"

//

T: "Yes. Then what is the relationship between the smallest number... If the smallest number is 126? What will x be?"

G: "42"

F: "Then it will be... Then it well be... $126 + 9$

T: "Yes, exactly!"

G: "Oh..."

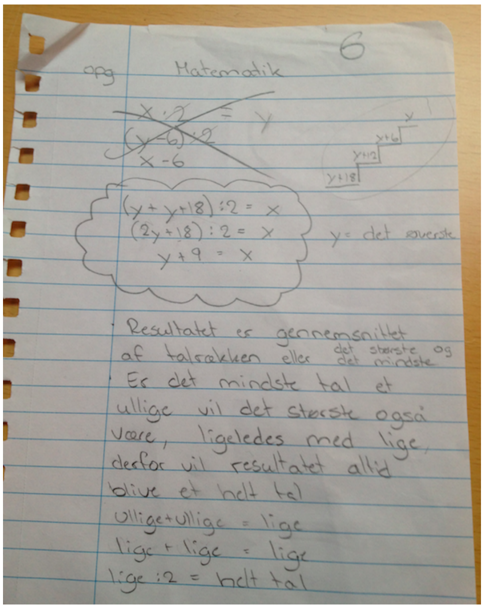


Figure 16: Witten work from Exercise 4.2 - Group 6

T: "Why?"
F: "Because it equals..."
E: "Because it is the difference..."
T: "Because it is the relationship you have found!"
E + F: "Yes"

During the class summarization, the group continues their discussion in a low voice, as another group presents their results from another exercise to the class:

F: "You could also say, that we have found out, that the relationship is that it is always $y + 9$, right?"
H: "Yes, or the average of..."
F: "Yes, but I think it is better to say the other one, as that is what we have proved"

To sum up on the last part of the situation, the students are still confused about what they are supposed to prove or what relation that they need to show. They come up with the conjecture that *"The result is the average of the number sequence"*, which is true but do not directly state the relation between x and the smallest number. Not even the confirmation from the teacher makes the students focus on the algebraic expression containing two variables, as the correct result. The students are still searching for a numerical result or a pattern they can unravel. This is expressed as Student E says *"But we can never get it, to be the same!"*. It seems as if the students do not see a relationship between two variables as a meaningful result, as described above.

This problem could also be due to a break in the didactical contract, as the students in the prior exercises have found a numerical result or a clear pattern, which they could prove by the use of algebra. This exercise is constructed another way around: The students were supposed to set up an initial algebraic expression on the basis of the definition of x and then through algebraic calculations reach an algebraic result, which they were to interpret. Hence the examination phase is scaled down and the focus is on the interpretation of an algebraic expression. As the examination phase is scaled down, it becomes difficult for the students to formulate a clear conjecture and the exercise therefore lays more importance to the students ability of analyzing an algebraic expression. This ability is emphasized in (Martinez, 2008, p. 34), where the author draws on the definition of *symbol sense* from (Arcavi, 1994). Martinez (2008) argues that the students, in order to have symbol sense, must be able to "read" information that was hidden in the original algebraic expression. This seems to be difficult to the students, as Student E expresses: *"Yes, but that's just how you calculate it. It isn't a proof!"*. The student has written the correct formula, but only sees this as a way to calculate a result and not as an expression from which they can determine the relationship between two variables.

The problem could also relate to the missing formulation of a clear *conjecture to prove* as stated in (Martinez and Li, 2010, p. 271), where the author points out that the process of formulating one, true conjecture involves the production of a mathematical statement and becoming confident about its plausibility. This process is missing in the students' work, why they formulates two competing conjectures, which is both true but not relevant and this makes if difficult to falsify the non-relevant conjecture. The students therefore have problems with relating their correct algebraic work together with their conjecture, as it is unclear what the algebraic expression should proof.

From the discussion with the teacher and from the written work (figure 15 and 16) it is seen, that the students have made the correct initial algebraic expression and reached the desired expression. This is especially seen as Student F expresses: *"Then it will be 126+9"* and *"... we have found out, that the relationship is that it is always* $y + 9$. That the students do not reach the desired conclusion, could be due to the above stated reasons.

To sum up on the entire situation, the didactical variables both hinders and helps the student in engaging in a formulation- and a validation situation. As described earlier, the relation in the number table combined with the pattern of the staircase seems to help the students to formulate an initial algebraic expression, as the students write up a generic diagram. The use of a generic diagram is emphasized in

(Martinez, 2008, p. 108 ff.), where the author argues that this diagram helps the students in identifying variables, the relations among the variables and to express them using algebraic notation. I therefore argue, that the relation in the number table and the pattern of the staircase, helped the students in creating a generic diagram and as this is the first step in engaging in a formulation situation, these variables assisted the students in their work.
The students also seems to have the algebraic prerequisites to both formulate an algebraic proof and validate this, as figure 16 shows, that they have correctly formulated an algebraic expression. The problems occurs when the students need to interpret their algebraic expression and as Martinez (2008), Binns (1994) and Bardini et al. (2005) argues, this is due to a lack of algebraic understanding. Therefore the students' algebraic prerequisites hinders them from validating their findings, which also can be seen from the different conjectures that emerges and which the students can not accept or refute.

The last didactical variables that is worth mentioning, is the number of variables which has been described earlier. The students are enable to formulate an expression containing two variables, but as the students do not see this formula as an answer, the students are left frustrated. This keeps the students from engaging in a validation situation and the teachers intervention do not seem to convince them about the truth value of their algebraic expression.

Lastly, the students do somehow seem to pass through the three phases: examination, conjecturing and proof, as described earlier. But as the examination phase is extremely short and scaled-down, this hampers the phase of conjecturing as the students come up with several conjectures, which they can not refute or accept. The problems seems to revolve around finding the correct conjecture, which could also be due to the exercise description as it is not stated here explicitly what the students should do. As the students conjecturing phase is not succeeded by a validation situation, in which the students are enable to validate their conjecture, the students seem unable to engage meaningfully in a proving phase.

This last part underlines the importance of the three phases, which are in interplay. As the students do not base their conjecture on a numerical examination, their transition between the phases seems to be hampered and this frustrates the students.

5.1.3 *Situation 3*

This situation is a part of Group 2's work with Exercise 6 (see Section A.3) during the third day of the course. The a priori analysis can be found in Section 4.6. The group consists of four members, two girls and two boys. They will be denoted S1, S2, S3 and S4. The days before, the students have worked and gone through Exercise 1 to 4. During the institutionalizations, the students have been introduced to the possibility of applying algebra as a mean of formulating and proving a conjecture. Prior to this situation, the students have been told to do Exercise 6, which they are to present at the blackboard in cooperation with another group. Before they present it at the blackboard, they are supposed to discuss it with this group, Group 1.

The following analysis will contain fragments from the transcript of the situation. The entire transcript can be found in the appendix. See Section C.3

The Target Knowledge

The target knowledge for Exercise 6 is divided in to two parts: First the construction of an initial algebraic expression, that contains multiplication in the calendar. This construction requires, that the students understand the relation of the number table, which is fairly new to them. The group has worked with the Calendar in Exercise 4[2], but as the pattern has changed so has the relation between the variables.

Secondly the students needs to work out the intermediate algebraic calculations, that is needed in order for them to obtain a final expression which supports their conjecture.

In the work of formulating an algebraic expression, that contains multiplication with variables, a vital aid is the creation of a generic model. It is therefore a purpose of the exercise, that the students successfully creates a generic model with correct relations between the variables. The students need to use parentheses to algebraically describe an expression like:

$$x \cdot (x+8) - (x+1)(x+7)$$

The important part is formulating a generic model, setting up an initial expression and working the expression in order to obtain a meaningful final expression.

The Milieu

The milieu in this situation is made up of Exercise 6 and the devolution made by the teacher. By this I mean, that the students are supposed to solve the exercise, with a group discussion and blackboard presentation in mind. The group are therefore obliged to solve the exercise in a way that makes them able to present and discuss it with another group.

The group have the number table, blank paper and the work sheet available on paper. This makes the group enable to make drawings in

2 This exercise is described in the milieu of situation 2

order for them to visualize the figures concerning the exercise.

In Exercise 6, the students are asked to work with 2×2 squares in the Calendar [3]. The students have to find the placement of the square that produces the largest result. This they have to find, by subtracting the product of the numbers in the upper right corner and lower left corner from the product of the numbers in the upper left corner and lower right. Or in other words, the students are asked to find the determinant of a 2×2-square. See figure 17.

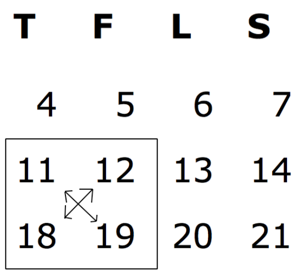


Figure 17: An example from Exercise 6

One way to solve the exercise, starts by doing the desired calculations. If we use the example shown in figure 17, the calculations will be:

$$(11 \cdot 19) - (12 \cdot 18) = 209 - 216 = -7$$

Repeating the same calculations with different squares, numerically shows that the result is alway -7 and is independent of the placement of the figure. One way to prove this algebraically, is by denoting the smallest number in a generic square x and setting up relations to the other three variables. This would result in the following generic square:

$$\begin{bmatrix} x & x+1 \\ x+7 & x+8 \end{bmatrix}$$

By combining this generic square with the above numerical calculations, one solution to the problem could be:

$$\begin{aligned} x \cdot (x+8) - (x+1) \cdot (x+7) &= \\ x^2 + 8x - (x^2 + x + 7x + 7) &= \\ x^2 + 8x - x^2 - 8x - 7 &= -7 \end{aligned}$$

This would be a perfectly sound proof, that consists of setting up the initial expression: $x \cdot (x+8) - (x+1) \cdot (x+7)$ and making the intermediate calculations, arriving at the final result -7.

3 Exercise 6A is strongly inspired by (Martinez, 2008)

Hence, in order to arrive at this conclusion, the students need to introduce a letter, in order to make a generalized statement about the result. As the generic diagram contains three dependent variables, the students also need to set up relations between the independent variable and the dependent variables. In the above example, this consists of determining the relations between the different entrances in the matrix.

When this is done, the students need to formulate an initial expression. This expression is made by combining the generic diagram with the procedural calculations, described earlier. At last, the students need the algebraic prerequisites that makes them able to work out the intermediate calculations, that involves reducing an expression containing parentheses: Distribution of -1 and x, multiplying parenthesis and multiplying variables

Some of the prominent problems, that could obstruct the students work with Exercise 6 and therefore hinder them from obtaining the target knowledge, is the algebraic work the exercise requires. This is especially a correct use of parentheses when setting up an initial expression, but also when doing the intermediate calculations, including distribution of -1 and reducing an algebraic expression by collecting like terms.

The pattern in this exercise is difficult to explain by the use of a verbal argument, why I think, that the use of algebra is fundamental to this situation. It is not obvious why the result is always -7 or why it depends on the length of the week. The difference between the numbers stay the same invariant of the placement of the figure, but this does not explain why the result is -7.

The Situation

Group 2 was the only group who mainly used the computer and therefore made very few drawings. The group members were encouraged to make their individual explorations, before discussing the exercise in the group.

As we enter the situation, the four students have been trying to figure out what *the product* means, what the exercise is all about and that they probably has to find a result bigger than -7 [4].

S3: "No, this doesn't make sense. Oh, maybe it does"
S1: "The product of these numbers are 9"
S3: "Oh yeah, this is really easy"
S1: "And then this is 28 times 20"

4 The exercise contains an example of a calculation, which gives the result -7

S2: "I have found one that gives plus 7"
S3: "Plus seven? Which one is that"
S2: "It's this one"
S3: "Okay"
S2: "I'll try another one"
[Loose talk]
S2: "This one also gives −7! God knows, if all of them gives −7...?"

In the beginning of this situation, the students are in an examination phase and Student 2's incorrect result in one of the numerical calculations, retains the group in this phase. The students could on the basis of their findings conclude, that the largest possible result is 7, but maybe in the hunt for a larger result, Student 2 makes another calculation with another square and this gives the result −7. This new result, makes the students consider the correctness of their prior results.

The group's last calculation, which produces the result −7, initiates a phase of conjecture, as especially Student 2 orally forms a conjecture, which is seen in the last line in the above transcription. This formulation is catalyzed, as the two different results initiates a wonder and this enables the students to create an incipient conjecture.

In the above dialogue, the students are still having problems determining the pattern. The students should be acquainted with solving these types of exercises and producing conjectures, but they still do not seem confident in this work. Even though the students have made several calculations that all resulted in the same answer, the one-time result 7 puts them on shaky ground. This is highlighted in the dialogue below, where the teacher has entered the group's discussion and asks if they have found a pattern. This makes Student 2 answer:

S2: "No, I found one that gave −7 and one that gave 7, but it could be, that I subtracted the reversed number"

The teacher then encourages the students to make some more calculations, in order to examine their conjecture. The students tries and obtains yet again −7. The students could now try to formulate an argument for their conjecture, but instead the teacher modifies the situation, to ensure the students introduce algebra. This modification strongly scaffolds the situation, which is now a didactical situation:

T: "So, it does look like it will be −7 every time"
S1 and S2:: "Yes, it could look like that"
T: "Is it possible to somehow show that? Like we did with the staircase[5]?"

5 Exercise 4 was called the staircase

S1: "Uuhhh... (baffled)"
T: "Could we introduce an x? Could we use an x somewhere?
S1: "Hmm..."
S2: "Most certainly. We could always put an x there" (Laughs)
T: "Is that possible? What if you draw a square, like you... Okay, you haven't drawn anything... But could you throw in an x somewhere? Could you make x represent some number?
S1: "7!"

Its not entirely clear, what Student 2 means by "7", but as the number 7 is placed in the upper right corner in the Calendar (See figure 42), its probably not the smallest number in a 2×2-square. It could be, that the student would use x to represent the second smallest number in a square or it could be, that the student just have an idea of the number 7 being important, but it is not clear from either the audiofiles or the written work.

What can be deduced from the above transcription is that the group has problems introducing algebra as a mean to examine and prove their conjecture. The students have been introduced several times to the possibility of applying algebra and should by now be acquainted with this method, but this does not seem to be the case. This is partly due to the students, but also due to the teacher. The students do not come up with the idea of using algebra, before the teacher imposes the idea in her question. This is an example of a Topaze effect (see Chapter 2) and this could hinder the students possibility of engaging in a situation of validation. The formulation of both the conjecture *"So, it does look like it will be -7 every time"* and the idea for validation by using algebra *"Could we introduce an x? Could we use an x somewhere?"* comes from the teacher. Because of this, the students do not solve the problem by adaptation of their own knowledge to the milieu, which is in contrast to the concept of learning in the Theory of Didactical Situations (Måsøval, 2011, p. 47). This could be one factor, which hinders the student from engaging successfully in a situation of formulation and therefore also a situation of validation.

If we leave the teachers role out of the account, the above transcription shows that the students do have some idea of a conjecture, even though they are not precisely formulating one. The pattern that appears as the students makes the desired calculations, does seem to allow them to formulate an incipient conjecture. They are not completely confident, that the result is always -7, but there does seem to be consensus about this being true.

The students' difficulty with engaging in situations of formulation and validation concerns one small, but still important, variable: the calculations that are required in order to solve the exercise. The students subtract the numbers in different order and this confuses them.

The uncertainty about the result being either 7 or -7, initially hinders them from entering a formulation phase. They probably would have formulated a conjecture about this themselves if the teacher had not interfered, but their hesitation makes the teacher intervene.

The teachers attempt to make the students use algebra could indicate that her belief was, that the students lacked the algebraic prerequisites that is needed in order to engage in a situation of validation. The students do not seem confident with the prospect of using algebra, which Student 1 and Student 2's answers indicates.

As we will see next, the students try to engage in a situation of validation, wherein they try to formulate an algebraic proof.

T: "Now, I don't know if you listened when the staircase (Exercise 4) was summarized, but what did x represent there?"

S2: "The smallest number...?"

T: "Yes, that was the smallest number and so, could we construct the rest after that? Could you do that? Could you introduce an x somewhere?"

S2: "Could we do it on the smallest number?"

//

T: "So, if you call the smallest number x, how could you represent... What would that number be named?"

S2:"x... What did they write in the other one? $x + 13$...."

T:"Yeah... How far is it from that number to that number?"

S2: "$x + 1$"

T: "Yes. Exactly. And how could you name that number down there?"

S4: "$x + 7$"

T: "Yes. Exactly! So, no matter where we place it, then that will always be 7 larger. 18 is 7 larger than 11. And what would that number be named?"

S2: "$x + 8$"

With strong influence from the teacher and on the basis of their experience from Exercise 4, Student 2 introduces an x to represent the smallest number in a 2×2 generic square. The students' work is still heavily guided by the teacher, but they do succeed in formulating correct relations between the variables. The students seem to be confident in setting up these relations, but it is uncertain if the students understand the coherence between these relations and the possibility of making a proof of their conjecture. The students' successful work with setting up the relations between the variables is partly due to the simple relations in the Calendar: The familiar relation of a 7-day week seems to help the students and teacher in setting up the algebraic relations, even though we once again observe a strong Topaze effect in the teachers help.

After the students have created a generic model by denoting the smallest number x, the students try to link the generic model with the procedural calculations, in order to formulate an initial expression. This happens as the teacher asks *'What kind of calculations would you then be asked to make?"*.

S4: "x plus..."
S2:" No, it's x times $x+8$..."
T: "Because, we say this number times this number"
S4: "x times $x+8$"
S2: "Now you have to say: $x+1$ times $x+7$"
T: "Yes. Exactly. And what should we do with these two numbers?"
S2: "Subtract them. You know, subtract that one, with that one. With that. No. With that. No. You know what I mean... So it gives -7"

By the help of the teacher, the students are now engaging in a proving situation, where the students combine their generic model and the procedural calculations with relatively large success. This can be seen from their written work (see figure 18) and the above dialogue. In the figure, the groups generic model can be seen and this model could have made their transition from a conjecture to the proving phase easier. The advantages of a generic model is described in Section 5.1.2.

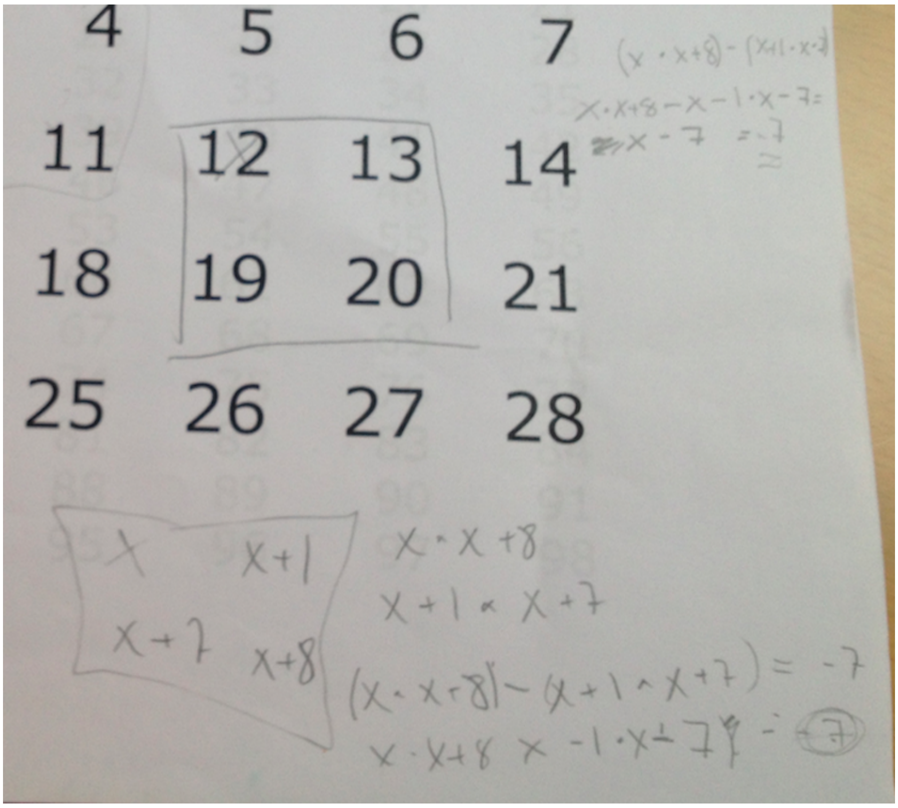


Figure 18: Written work from Group 6 - Exercise 6

Something that is noteworthy, is the last line in the above dialogue: Student 2's comment on the teachers question indicates, that the student sees the -7 as their hypothesis. Or in other words, they write up their initial expression on the left side of the equality sign and their expected outcome on the right side. The expected outcome, -7, is based on their numerical examples and discussion with the teacher. It therefore seems as if the equality sign in this situation, has the status as something that needs to be computed or as a representation of their conjecture. This incorrect understanding and use of the equality sign is outlined in (Martinez and Castro Superfine, 2012) & (Måsøval, 2011) and is a indication of an epistemological obstacle, which the student need to overcome. By this I mean, that the equality sign represents an - for the student - equality between their expression and conjecture, which needs to be proved. The students has no idea about the correctness of the equality sign, but believe that their initial expression should equal -7 and hence they use an equality sign to represent this.

As seen in figure 18 the students did not succeed in setting up a correct initial expression, as their use of parenthesis is wrong. The students writes[6]:

$$(x \cdot x + 8) - (x + 1 \cdot x + 7) = \tag{1}$$

$$x \cdot x + 8 - x - 1 \cdot x - 7 = \tag{2}$$

$$x - 7 = -7 \tag{3}$$

First of all, if the students had used this initial expression (1) and made the correct intermediate calculations, they would have ended up with the result:

$$x^2 - 2x + 1 = -7 \qquad (\forall x \in \mathbb{N})$$

which is easily falsified. But the students' lack of algebraic understanding and prerequisites, and their confused use of the equality sign locks the students' focus on the 'correct' answer -7. By lack of algebraic prerequisites, I mean, that the students first of all makes some algebraic miscalculations, though they do successfully distribute -1, as they deduce the expression:

$$(x \cdot x + 8) - (x + 1 \cdot x + 7) = x \cdot x + 8 - x - 1 \cdot x - 7$$

As seen in (2) and (3) above, the students make the following wrongful reduction:

$$x \cdot x + 8 - x - 1 \cdot x - 7 = x - 7$$

Student 2 and Student 4 discusses these calculations:

6 In the written work, it looks like the student has written $x \cdot 7$ in the first line, but as he uses the $\cdot$ as a + and have written + further down, I think it is just a mistype and I have therefore written $x + 7$

S4: [Indistinctly] “x like in x, isn’t that $2x$?"
S2: “$2x$. Like that. Minus x, that equals x. And x times x equals... Then it is x. 8 minus 1. No, 8 minus 1, that equals -7. And x minus 7, that equals -7, or what?"
S4: “I think so"

From the dialogue it is not certain where the mistake happens, but it looks like the two students deduces that

$$x \cdot x - x = 2x - x = x$$

and then that

$$x - 1 \cdot x = x \cdot x = x$$

This leads the students to the final expression

$$x - 7 = -7$$

It should be clear from the above miscalculations, that the students lack the algebraic prerequisites, which are needed in order to set up the correct initial expression:

$$x \cdot (x + 8) - (x + 1) \cdot (x + 7)$$

And then make the correct intermediate calculations that includes[7]:

- $x \cdot x = x^2$
- $(x + 1) \cdot (x + 7) = x^2 + 8x + 7$

As the students through their (incorrect) reductions, reaches the expression $x - 7$, they conclude that $x - 7 = -7$, based on their numerical analysis of the result. It should be apparent to the students, that the expression $x - 7 = -7$ is problematic, as they have denoted the smallest number in a square x. The expression could of course be true in the case when $x = 0$, but as 0 is not a number in the Calendar, the students should be aware of the incorrectness of their algebraic "proof". The expression contains some potential for feedback, as they could have numerically checked their algebraic expression, but this is ignored by the students. This could be due to a missing understanding of x as a generalized number in the calendar or lack of understanding of their algebraic expression.

If we examine the students comprehension of their "proof", it is clear that the students’ algebraic work do not fulfill this thesis’ definition of proof. As neither Student 2 nor Student 4 seems convinced about the truth value of their own algebraic work and as the students’ proof do not show <u>why</u> the result is always -7, I do not think the students algebraic work fulfill the definition of a proof. The students seems far from relating the algebraic work to the need of explaining

7 For detailed analysis of Exercise 6 see Chapter 4

the mathematical phenomenon.

As the definition of an *intellectual proof* incorporates the underlying rationality, the language level and the nature and status of knowledge (Miyakawa, 2002, p. 354), the students do not fulfill this definition of proof. The underlying rationality of the students proof do not seem to be, that they are about to prove and show why their conjecture is true. The students are doing the algebraic calculations as a result of the guided help from the teacher. From the prior lessons and summarizations the students know, that they need to use algebra in order to solve the exercise in accordance with the teachers expectations. The algebraic work therefore seems to be guided by the didactical contract between the students and the teacher

All in all, the students engages in a situation of validation as the they try to validate their conjecture through an algebraic proof. This work is guided heavily by the didactical contract and their knowledge about the result and therefore, the algebraic work does not have the status as a proof, which is also partly due to of the students' lack of algebraic prerequisites.

The didactical variables both help and hinder the students in the process of engaging in a formulation- and validation situation. The pattern of interest paves the way for the students to formulate a conjecture, as calculating a determinant for a 2×2 matrix seems to be manageable to the students. Even though the pattern seems manageable, the students have some troubles formulating a conjecture, as the calculations they are required to make results in two different answers: -7 and 7. This could be due to lack of concentration among the students, but nevertheless, the students are not confident with their own conjecture.

As the students engages in a validation phase they introduce algebra in order for them to proof their conjecture. They seem confident in the process of choosing an independent variable and setting up the relations between the independent and dependent variables. This process is catalyzed by the relation in the number table, which seems comprehensible to the students. Hence the number table assists the students in the early steps of a validation phase, but as they lack the algebraic prerequisites required in order to complete an algebraic proof, the students are obstructed in the process of formulating a sound proof. As described above, the algebraic work do not even seem to convince them self about the truth value of their conjecture and therefore this is a big obstacle that needs to be attended to, if the students are to engage in an adidactical proving situation.

Prior to this work, the group was told to discuss their findings and solution with another group, Group 1. Unfortunately, this was not done, so the group did not have the possibility to share and clarify their solutions in cooperation with another group who had made the same exercise. The two groups therefore selected two representatives, one from each group, to present their solution at the blackboard. The following dialogue is from this summarization at the blackboard, where the two students, Student 2 from the group above and Student 5 (S5) from Group 1, present their findings. It is mostly Student 5 who does the talking, but as Student 5 is unsure of her work, Student 2 often corrects her. I will skip most of the dialogue between Student 5 and the teacher, as it is Student 2 who is of main interest.

The summarization starts with the two students presenting the exercise and their conjecture based on their numerical exploration of the pattern and results.

(...)
S2: "No, we corrected it. We were supposed to examine if there were a possibility of finding a larger result"
T: "Okay"
S2: "The largest result. But then we found out, that it always gave -7"
T: "Okay. So the exercise was about, placing a square in the calendar. How do you get the largest"
S2: "Yes"
//
S2: "And then we found out it always gave -7 and now we will show why it always gave -7

By now Student 2 has a clear conjecture, namely that the result is always -7. The uncertainty about the 'larger' result 7, has vanished and the student clearly states that the result is always -7. The initial examination phase has by now resulted in this clear conjecture, which could have been supported by the prior prove situation, which was made in the individual groups. The student also clearly states that the following algebraic work will show exactly why, the result is always -7, hence she links the conjecture together with the algebraic work. This was not something that seemed certain to the student during the group work.

T: "Yes. What will you do now?"
S2: "Then... We will make the same calculations, but with these x's. So we can say something general"
T: "So that recipe on what to do. You will now do that with x's?"
S2: "Yes"
S5: "We actually just have to do that one over here. And then we will say x times $x + 8$ minus $x + 7$ times $x + 1$. And that will give -7

T: "That is your assumption?"
S5: "Yes (...)"

Student 2 orally formulates the coherence between the procedural calculations and the arithmetic generalization which is made by the introduction of an x. It seems as if Student 2 has a good idea of how the algebraic work will show why the result is always -7 and therefore prove their conjecture. It seems as if Student 2 correctly links the proving situation together with the phase of conjecturing.

What is also noteworthy, is Student 5's expression: *"And that will give −7"*. This indicates that her use of the equality sign, is similar to the one used during the group work described above, where the equality sign was used to represent something that was to be proved. She writes her initial algebraic expression on the left side of the equality sign and her conjecture on the right side. As before, the equality sign represents a question mark or something that has to be examined. What differs Student 5's use of the equality sign and the one described earlier, is that Student 5's expression is correct. It is still uncertain if her understanding of the equality sign is correct.

This use of the equality sign is also observed in (Martinez and Pedemonte, 2014), where the authors argues that this happens as the students links *"... two objects of different natures in regards to the field in which the generalisation is carried out (i.e., arithmetic versus arithmetic/algebra) and the type of inductive generalisation (i.e., result versus process)* (Martinez and Pedemonte, 2014, p. 136). The students are in other words trying to link their inductively obtained result from arithmetic cases together with the algebraic expression obtained via induction on the process of the pattern. The algebraic expression is obtained in an interplay between arithmetic and algebra, as the students generalizes the arithmetic properties to the rules of equation solving (algebra). The authors argue that this is an obstacle that needs to be addressed in order for the students to construct a proof (Martinez and Pedemonte, 2014, p. 146f.) and this could be one factor that hinders the students from correctly connecting their result together with their initial algebraic expression.

Student 5 then starts to reduce the left side of the equality sign in the expression

$$x \cdot (x+8) - (x+7) \cdot (x+1) = -7$$ [8]

in order for her to show that this will give -7.

During this work, Student 5 stumbles a few times, why Student 2 corrects her:

8 The student wrote this expression on the blackboard. Unfortunately I did not take a picture of this

S2: "Can't we just write x to the second and then minus instead of plus?"
T: "Yes!"
S2: "And then it's just the same, but then just minus where it says plus. It's just what's in the parenthesis you need to change"
S5: "minus x to the second and then plus x here?"
S2: "No, minus x, because this is placed outside. Now you need to write minus
S5: "Minus..."
S2: "Yes and then you just write one x, minus $7x+7$"

In the end Student 5 has troubles reaching the desired result, -7 and Student 2 steps in to help Student 5 finish the algebraic work.

What is interesting in this last part of the situation is, that Student 2 seems much more confident in making the algebraic calculations. If we look at the first line in the above dialogue, Student 2 mentions the calculation $-x^2$ (*"... just write x to the second")* and then *"... it's just the same, but then just minus where it says plus. It's just what's in the parenthesis you need to change"*. This refers to the step in the mathematical proof:

$$x^2 + 8x - (x^2 + 8x + 7)$$

by which we can see, that the she is aware of the fact that $x \cdot x = x^2$, something that was not evident during the group work. Group 2 never mentions x^2 or writes it (see figure 18) and during this group work, Student 2 comments on her encounter with this algebraic expression with *"And x times x equals... Then it's x"*.

It seems as Student 2 during the institutionalization becomes aware of the algebraic convention $x \cdot x = x^2$ and at the same time she shows a greater confidence in her algebraic expression. This could indicate that Student 2's cognitive distance(Martinez and Pedemonte, 2014, p. 126) between the arithmetic founded conjecture and the algebraic founded proof has diminished, which enables Student 2 to help Student 5 during the institutionalization. By this I mean, that the cognitive gap between the inductive conjecture based on arithmetic and the deductively proof based on algebra has been reduced (ibid, p. 132). The students now seem to grasp the relation between the conjecture and the algebraic work.

If we compare the students' resolution to the a priori analysis (see Section 4.6), the students more or less followed the imagined trajectory, though the troubles with the numerical calculations was not foreseen. The students faced bigger challenges with introducing algebra than intended, as this step required a lot of scaffolding from the teacher. The students rightly used only one independent variable in setting up a generic model, which could be due to teachers instructions and the prior summarizations, where this was institutionalized. Nevertheless, the students were comfortable at setting up the relation between the variables, as this was overcome without posing a problem. The students seemed comfortable in creating an initial algebraic expression, by combining their generic model and the procedural arithmetic calculations, though their lack of algebraic prerequisites initially posed an insurmountable obstacle in setting up a correct expression and making a solid algebraic proof. It is important to mention, that the students' algebraic work was better at the end of the situation, which questions the rigid definitions of 'algebraic prerequisites' in this master's thesis.

The students faced problems in a correct use of the distributive law as they failed to set up correct parenthesis and this hampered their proving situation. They did though correctly distribute -1, which was foreseen to be one of the biggest obstacle in the proving situation.

To sum up, the above situation showed that the algebraic prerequisites constitutes an important didactical variable, as it has the potential of hindering the students transition from a phase of conjecturing to a proving situation. The algebraic prerequisites thereby also initially hinders the students from validating their conjecture, but at the same time it contains the potential to link the students inductively based argumentation to a deductively based proof. This happens as the students translates their arithmetic rules into an algebraic expression and through intermediate algebraic calculations, ends up at the arithmetic founded result -7.

This situation shows how the students initially can not link these to different objects together, the arithmetic conjecture -7 and the algebraic initial expression $x \cdot (x+8) - (x+1)(x+7)$, but during the institutionalization signs of this linkage emerges. This is seen from Student 2's confident work in formulating a conjecture, setting up the algebraic expression and making the correct algebraic calculations in order to end up at the correct result -7.

Part V

DISCUSSION

6 DISCUSSION

In this section, I will discuss different aspects that all relates to the main subject of this thesis: The integration of algebra in proving situations through pattern analysis. I will start by discussing my research questions and try to answer these. This discussion will involve the theoretical analysis of the three situations. After this, other subjects will be discussed including the didactical variables, how to improve the design and a comparison with a master's thesis made by Asger Brix Jensen (Jensen, 2015).

6.1 DISCUSSION OF ANSWER TO RESEARCH QUESTIONS

The research questions can be found in Section 1.2.

6.1.1 *The design of the course of study*

One of the specific aims of the teaching experiment, was to design a course of study that appeals to the students' curiosity in order to use this as a catalyst in the work of applying algebra in proving situations.

If we look at **situation 1** and **situation 3**, then for both cases it seems as if, the work is guided by the result and pattern being manageable to the students. By this I mean, that the students are enable to recognize a certain pattern, which is essential if they are to formulate a conjecture, as they falsify wrong conjectures in the process of determining one conjecture. In order for the students to use algebra in situations of formulation and validation, it is essential that they have formulated a clear conjecture, as it is difficult for the students to produce a proof of an unfamiliar statement as described in (Martinez, 2008, p. 53). This is seen in **situation 2** where the students do engage in situations of formulation and validation by the use algebra. The problem occurs, as the students do not have a clear idea about what they are to prove. This makes them circle around different conjectures, which are all true, but not relevant to the exercise and this hinders them from selecting one conjecture. The importance of the conjecturing process is described in Situation 2 and Section 3.1.

It is therefore essential, that a design that seeks to let the students engage in a proving situation by the use of algebra, contains a milieu which makes the students able to produce their own conjectures, based on personal experience. It is also essential that if the milieu in which the students act results in several conjectures, the wrong conjectures can be falsified. For the students to meaningfully employ algebra, I therefore argue that the conjecture must be based in an examination phase, which leaves room for the students to construct and deconstruct conjectures.

This is something that is often missing in math classes in the Gymnasium, as the conjectures are given to students before they are to prove this conjecture.

As the students' use of algebra in proving situations is a central issue in this master's thesis, the design must also strongly encourage or somehow force the students to use algebra in these situations. This is something which the course of study lacked at times, as these situations first appeared on the last day. It was surprising that the students in **situation 1** used algebra in an attempt to prove their conjecture, as this was not expected of the students that early in the study course.

In this situation it becomes relevant to discuss my definition of algebra and to ask what is algebra, as the verbal proof made by the students in **situation 1** seems to contain pre-algebraic elements:

As written in section Section 2.2.5 the word *algebra* dates back to 825 AD and is a corrupted form of the word *al-jabr*. In al-jabr al-Khwārizmī among other things, verbally formulates solutions to first and second-degree equations, which today would be classified as algebraic problems.

With this in mind, it is interesting to discuss Group 4's work in **situation 1**. The group formulates a solution to the problem: *"Because, you add 43 to that number, right? 37, right? Then as you subtract 42 again, then it is just one less you subtract than added before. So it is +1. So its 38"*. The verbal formulation contains aspects of solutions found in al-jabr, but lacks the introduction of symbols as generalized numbers to be classified as a modern algebraic proof, but it indicates that the students' action with the milieu is fostering algebraic work. It is therefore of less importance, that the need for an introduction of symbols diminishes as the pre-algebraic formulation explains the group's hypothesis and the situation is therefore not a fundamental situation for the introduction of algebraic symbols. The application of symbols to formulate their argument seems to happen, as the students realize that the pattern is describable by the use of algebra.

If we return to **Situation 3**, then the opposite is more or less seen, as Group 2 does not employ algebra at all, before the teacher strongly encourages them to do so. There are several reasons discussed in Situation 3, that hinders them from doing so, but one of the main problems seems to be that the group initially have difficulties with the understanding of a letter as a representation of a generalized number when creating a generic model. These problems are also seen during the summarization of Exercise 1A, where group 3 and 7 relies solely on a *crucial experiment*(Miyakawa, 2002): *If I take an arbitrary rectangle, then the result will be the second smallest number* (figure 19).

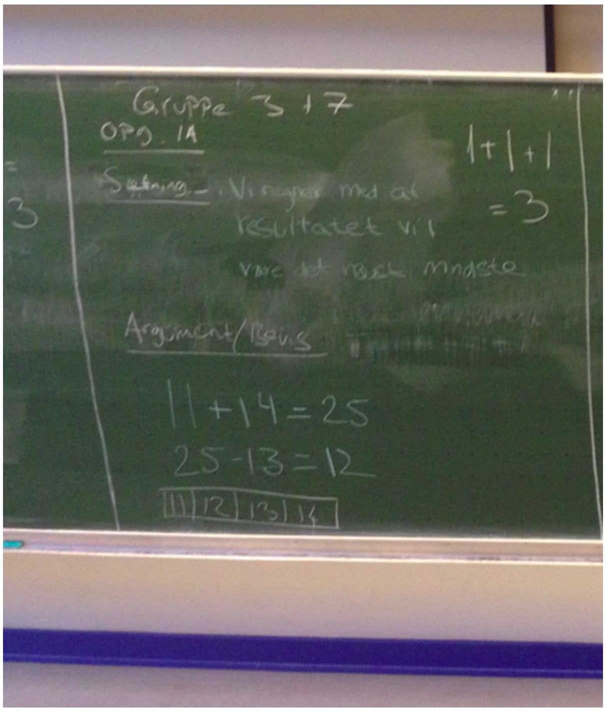


Figure 19: Summarization of Exercise 1A

In order for a study course to result in the students' own introduction of algebra, the design must include an intellectual need to prove why a specific outcome happens. It can be difficult to ensure that the students introduce algebra solely based on their own curiosity, but a way to enhance this possibility is by making the pattern adequately complex in order to make the situations fundamental for applying algebra in the proving situation, so the use of verbally formulated proofs diminishs. An example of a adequate complex pattern will be given later. The work of the teacher will then be, to institutionalize the limitations of numerical and pragmatic proofs. The teacher tried to do this during the teaching experiment, but the work resulted in the students using algebra because of the didactical contract and not because of the students' own personal need to convince themselves or others. As described in Section 2.1.2.1, it need not be one situation, but a set of situations that leads to the target knowledge being necessary for solving an exercise. The exercises in the teaching experiment, must be included in a set of milieus that fosters the students use of algebra as a tool to prove a conjecture. **Situation 1** and **situation 2** show that these situations are not fundamental for using algebra in proving situations, but they foster the students possibility of applying algebra: Even though the students in situation 1 eventually chose to apply algebra, they firstly resorted to a verbal proof and in situation 2 the students formulates both verbal and algebraic proofs.

A slow progression in the complexity of the patterns, resulted in exercises where algebra is a necessity in proving the pattern, first were introduced on the last day of the course of study. An example of this is Group 5's work with Exercise 10 during the third day of the Study Course. This situation will not be thoroughly analyzed, but used here to show an example where algebra is fundamental for proving a conjecture:

Exercise 10 concerns the Multiplication Table and the figure is a 1×4-rectangle where they have to find the difference between the product of the largest and smallest number, and the product of the two middle numbers. This difference they have to divide by -2. The students used a rectangle containing the numbers:

$$\begin{bmatrix} 120 & 132 & 144 & 156 \end{bmatrix}$$

They made the correct numerical calculations

$$\frac{120 \cdot 156 - 132 \cdot 144}{-2} = \frac{18720 - 19008}{-2} = \frac{-288}{-2} = 144$$

and formulated the initial expression:

$$(x \cdot (x+36) - (x+12)(x+24)) : -2$$

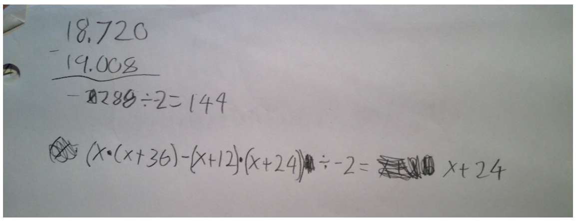


Figure 20: Group 4 - Exercise 8

The group have wrongly reduced the initial expression to $x + 24$, but if they had made the correct algebraic calculations, they would have obtained the answer x^2, which shows that the result is always the diagonal number from the concerned row.

As written in the a priori analysis, Section 4.10, the calculations for this exercise requires multiplication, division and subtraction, why this pattern is so complex that it is difficult to make a verbal argument and that algebra therefore are necessary for this didactical situation, i.e. it is fundamental for the algebraic proof. A general algebraic proof could be:

$$\begin{aligned}\frac{xy(x+3)y-(x+1)y(x+2)y}{-2} &= \\ \frac{y^2(x(x+3)-(x+1)(x+2))}{-2} &= \\ \frac{y^2(x^2+3x-x^2-3x-2)}{-2} &= \\ \frac{-2y^2}{-2} &= \quad y^2\end{aligned}$$

Where x denotes the column number and y denotes the row number.

The lack of proving situations that required algebra is to some degree missing in this master's thesis, as the students reached these complex patterns and number systems during the last day, which is due to the planning of the study course.

To shortly answer the first research question, the design must include a possibility of making the students formulate their own conjecture based on numerical examination. The design must secure, that the students can falsify and confine the possible conjectures, in order for them to have a clear idea about what to prove. The design must also ensure, that there is a need, through fundamental situations, to introduce algebra in order to exclude pragmatic and verbally created proofs.

6.1.2 *The didactical variables and phases in the teaching experiment*

The didactical variables have been the core of this master's thesis and their influence on the students work will now be discussed. As they have been discussed in the Analysis Section, this discussion will be short.

One of the didactical variables, that in it self has not been so decisive in this teaching experiment is the **Numerical Calculations** the students has to do, in order to obtain the desired result. In situation 1 the fairly easy calculations help the students to engage in a formulation- and validation situation, as the group quickly passes from an examination phase to a conjecturing phase. But as the calculations seems easy to the students, they formulate a verbal proof instead of an algebraic proof, which is seen on figure 13.

The Numerical Calculations are closely related to the **Pattern to Investigate** and **the Algebraic Prerequisites**, and this is one of the important affects of The Numerical Calculations. As described above, the pattern in **situation 1** seems so easy for the students to determine, that it undermines the need for a formal algebraic proof, but in **situation 2** the pattern keeps the students from forming a conjecture to prove, which impedes the students from engaging in a meaningful proving situation. In **situation 3** the pattern helps the students to engage in a formulation situation, but the students' **Algebraic Prerequisites** hampers them from engaging in a meaningful situation of validation. If we compare **situation 3** with **situation 2** the students in situation 2 do seem to have the algebraic prerequisites to create a generic model, formulate an initial expression and make the intermediate algebraic calculations in order to reach a final expression. What the students lack, is the ability to interpret a mathematical expression containing two variables as described in (Bardini et al., 2005) and (Binns, 1994), but it could also relate to their problems formulating a conjecture.

What has been clear from this teaching experiment, is that the Algebraic Prerequisites is a didactical variable, which needs much attention if the students are to successfully engage in situations of formulation and validation. If the students lack the understanding of a letter as a generalized number, it prevents them creating a generic diagram and this keeps them from formulating an initial algebraic expression. During a validation situation, the students also need the algebraic prerequisites in order to make the intermediate calculations, that will prove their conjecture. This is something that is outlined in **situation 2**.

Unfortunately the students did not reach exercises concerning the S-table or the Power Table and group 4 and group 5 was the only groups who reached exercises concerning the Multiplication Table, hence it is difficult to discuss the influence of **The Number Table**. Nevertheless, all of the groups who reached the Calendar seemed quite comfortable in setting up the relations between the variables in the exercises concerning this number table and, as **Situation 3** shows, the formulation of algebraic proofs still posed problems for the groups.

Group 4's written work, see figure 20, shows that the group succeeded in determining the relation in the Multiplication Table and related this to the pattern by the use of algebra. I therefore hypothesise, that this number table contains a large potential for letting students work with algebraic expression.

The didactical variable **The Number of Variables** is closely related to the Number Table and hence, the only exercise containing two variables which the students reached, was Exercise 4.2. It was clear from the work made by Group 5 and Group 6, that they struggled with formulating an initial algebraic expression containing two variables.

These problems has been covered in the Analysis section, but what is worth mentioning is that, this element made the students work with various algebraic expressions and this forced the students to link their algebraic expressions together with the different conjectures. I therefore argue that The Number of Variables contains a large didactical potential, as the students in a formulation situation needs to single out two variables and through their relation formulate an algebraic expression. Through the formulation of conjectures and relating this to algebraic expressions, the students obtain the possibility of 'reading' an algebraic expression as described by Arcavi (1994), which I see as a strength of these type of exercises.

An expression containing two variables also raises the possibility of institutionalizing that different algebraic expressions, may be able to prove the same conjecture. This possibility was in this teaching experiment sought to be fulfilled through a classroom discussion based on generic models and initial algebraic expressions created by different **Choices of Variables**. Unfortunately, it was pretty early in the study course institutionalized that the choice of variable should be the smallest number in a generic model. Almost all of the algebraic expressions during the study course, was formulated with the smallest number as placement of the independent variables, this can be seen in figure 15 and 18, and this hindered the possibility of discussing algebraic expressions that looks different, but proves the same thing.

To answer the second research question, it is clear that the didactical variables affects the students work with exercises concerning pattern analysis. This influence seemed to either hinder or help the students in engaging in situations of formulation and validation depending on their level of mathematical prerequisites. This is a very diffuse statement and to illustrate it, I will use the groups' work: The easy calculations initially hindered Group 6 from formulating an algebraic proof, but helped Group 2 in engaging in a situation of formulation. The algebraic prerequisites helped Group 4 in formulating a valid proof and engaging in a validation situation, but hindered Group 2 from engaging in a formulation situation, without the teachers help. I also hypothesis, that Group 2 would have faced problems in formulating an initial algebraic expression containing two variables, where as Group 6 in some aspects seemed to gain from it.

What also became evident through the a posteriori analysis of the situations is that, the didactical variable *the Algebraic Prerequisites* might have been to coarse. I argue, that this didactical variable rightly could be split into more narrow variables. This is evident, as Group 6 in **Situation 2** shows the algebraic ability to set up the algebraic expression, but lack the ability to interpret their expression. A possibility could be to divide the didactical variable into two variables: One concerning the techniques required and one concerning the understanding and ability to "read" an algebraic expression.

To answer the third research question, **situation 1** and **situation 3** shows that, the students passes through the three phases of examination, conjecturing and proof, in this chronological order. The situations also shows, that if one phase is completed successfully, then it helps the transition and work in the next phase. This is seen in **situation 3** where the students' examination phase helps them form two conjectures and where the students successfully falsifies one of them. They are then left alone with one clear conjecture, which helps them in the transition to a proving situation.

The opposite is seen in **situation 2**, where a short, maybe absent, examination phase hinders the students in forming a clear conjecture. As the students do not formulate one clear conjecture, their work during the proving phase is unstructured and the students seem to grope in blind as they do not have a clear aim.

I therefore argue, that it is extremely important, that the students transit chronologically through the three phases, as an incomplete phase hampers the work in the next phase. One phase can be seen as complete, when the desired outcome is obtained and the time aspect of this differs from student to student. The desired aim for the examination phase is to examine the number table, the pattern and the result. This work enables the students to formulate different conjectures, during the phase of conjecturing. This phase also contains a possibility of falsification of the wrong or non-relevant conjectures, which then makes the students able to make a proof of their conjecture. This happens during the proving phase, which is complete when the students have made a proof, that convinces themselves and others about the truth of their conjecture. I also argue, that **Situation 2** shows, that when the students pass successfully through the first two phases and the usability of verbal arguments are diminished, it creates a situation in which the students can meaningfully apply algebra in the proving situation. This is seen as Student 2 relates the conjecture together with the algebraic proof in order to show the validity of the conjecture.

What hampers the completion of one phase and the transition to the next phase depends on the phase and the students involved. These problems are described above, but some prominent problems relates to the didactical variables that are in play during the particular phase. **The Calculations Needed** is for an example an important didactical variable during the examination phase and **the Algebraic Prerequisites** is important during the proving phase.

6.2 STRENGTHS, WEAKNESSES AND IMPROVEMENTS

This section focuses on the strengths and weaknesses of the teaching experiment and will end with a discussion of improvements.

By letting the students examine patterns in various number tables, the students took control of the learning situation. By this I mean, that the students made their own explorations and from this constructed their own conjectures which guided their algebraic work. The meaning of the symbols therefore originated from the students' own understanding and not from and external sources like the teacher or a textbook. An algebraic proof only makes sense, if the student understands the meaning of the symbols involved and it seemed as if the students gained an understanding of letters as generalized numbers from the mathematical work done in the teaching experiment. This eased the students work in a proving situation, which else is something that troubles a lot of students. This is emphasized in the introduction to (Martinez, 2008), but also something I have experienced as a teacher: The students find it difficult to engage in a proving situation using algebra as they see letters as unknowns, and not as a generalized numbers. It seemed as if the students in this teaching experiment found it easier to prove a statement, when the conjecture was formulated by themselves and based on their own experience. This is also outlined in (Martinez and Castro Superfine, 2012) and (Martinez, 2008).

Another aspect that is worth mentioning is the levels of differentiation, that the exercises contain. This was something that I did not foresee before the study course, but something that very quickly appeared in practice. The objective of this teaching experiment was to make the students use algebra in proving situations, but what stood clear was, that even those students who faced serious problems in algebraic work, seemed to gain from the exercises: These students had an experience of success, as they were able to make the desired calculations and determine a pattern and this success, seemed to help them in the proving situation. Initially most of these students relied on pragmatic proofs (Miyakawa, 2002) and found confidence in proofs that convinced themselves about the truth of their conjecture. Even though one of the main purposes of the teaching experiment was to institutionalize the advantages of algebraic proofs, it seemed as a big step for these students to produce a proof by making a *Crucial experiment* or from numerical verification and the students seemed to blossom from this.

It then remains to be discussed, if the students gain anything if they are not capable of applying algebra or complete an algebraic proof. I think that the students gain a lot from forming conjectures, introducing algebra to describe a mathematical statement and create a generic model. To link a mathematical conjecture to a generic model, requires that the students introduce symbols and sets up relations between these variables and even though this do not end in an algebraic proof, the students do gain an algebraic understanding from this.

Different weaknesses of the teaching design occurred during the study course. The institutionalization of applying algebra in a proving situation was institutionalized early in the study course, which happened in consultation with me, but was not done on the basis of a classroom discussion. The result was, that some students applied algebra because of this institutionalization and not because algebra in some situations would be a good tool to construct a proof. An improvement of this teaching exercise could be to discuss the different 'types' of proofs that emerged, ranging from *naive empiricism* to *intellectual* algebraic proofs (Miyakawa, 2002), but this discussion was only superficially done at the blackboard and mainly by the teacher. As the students used algebra because of this, the algebraic work sometimes seemed meaningless and unnecessary difficult to the students, hence another improvement could be to target some of the algebraic difficulties the students faced: distributing -1, setting up parentheses, distribute a variable into a parenthesis etc.

Another reason why these discussion was mainly done by the teacher, was also due to bad planning: The division between didactical and adidactical work was skewed as the students had to much time working in groups without summarizations. Many of the main points for discussion and institutionalization was therefore not made in involvement with the students, but by the teacher telling the students what to do. An improvement to the teaching experiment would therefore be, to make the time between adidactical group work situations and institutionalizations at the black board shorter, with greater focus on classroom discussions.

6.3 THE CHOICE OF GROUP DIVISION

If we instead turn to another variable relating to the planning of the study course, then the division of the groups was made according to the teacher's personal view of the students' mathematical level. The students were divided into groups with students on the same mathematical level. This division of groups, makes it relevant to discuss two issues: (1) What does "mathematical level" mean and does it make sense to make a distinction based on this and (2) what effect does this division have on the students' work?

If we start with (1), the the division was done after a discussion between me and the teacher, in order to prevent some "high level" (high level students from now on) students from doing all the work and from this promote group discussions.

The teacher had a prior understanding of the students' mathematical level, which also was reflected in the students' individual grades. I did not discuss with the teacher, what she based her distinction between the students' mathematical level on, but I believe that the students were paired together on how they usually did in math class and this reflects their "mathematical level". The teacher must have had experience with some students performing "better" than other students and it is my belief that the teacher divided the students into groups based on their individual grades .

What may be more important is do discuss, if this division makes sense. The students who are rated on a "low" mathematical level by their teacher (From now on denoted low level students), may be more inclined to perform on a low level because of this and are therefore retained on this level. It can be difficult for students to change this self perception and the students in the study course were aware of this. They knew they were put in groups based on their mathematical level and in which end of the scale they were, and the low level students also expressed themselves negatively about it. This might have been a factor in the students' work and it remains to be examined how a low level student, would perform when put in a group with high level students and vice versa.

I think it is extremely difficult to classify between high level/low level students in this specific part of mathematics, based on their *mathematical level*, as this is a very diffuse term. Some students may have low self-esteem in math because of prior experiences or might just be low achievers in some area of mathematics, but are being rated as a whole on this. I think that it is difficult to make a clear division, but as a teacher you are forced to rate the students individually in order to grade them. Whether this rating is fair, is another discussion and not the one we should do here.

If we try to answer question (2), then it shall be mentioned, that during the study course, there were a tendency towards the high level students performing better than the low level students. The groups with high level students seemed to discuss their findings more than other groups and were more inclined to examine and test their results, where the groups of low level students were mostly inclined to discuss this with the teacher and not with each other. The low level students were more inclined to work individually or not at all. The low level students were also more inclined to stop working when they met obstacles which seemed insurmountable, instead of searching for methods to solve the problems.

It could have been interesting to mix the students, in order to see if the high level students could engage the low level students in their algebraic work. It seemed as if many of the low level students missed the confidence when confronted with obstacles they could not solve and a high level student might just have been the support the low level students needed.

To conclude on this discussion, then this is something I would have changed if I were to make this teaching experiment again. I think it would have been very interesting to see, how a low level student would solve the exercises, if they were paired with a high level student. I would also have encouraged the students more to work individually during the examination phases.

6.4 OTHER VARIABLES

The last two subjects I will discuss, is the structure of the study course and the level of scaffolding in the exercises. The last discussion will include a comparison with a master's thesis made by Asger Brix Jensen, as Jensen (2015) treats some of the same issues as my master's thesis.

6.4.1 *The Structure*

There are different aspects in the analysis, that indicates that the structure or organization of the study course could have been done differently, in order to promote the students' learning.

If we look at situation 3, then especially the difference in the work done by Student 2 during the group work and the summarization, indicates that some of the students could have benefitted from shorter time between the different elements: During the group work, Student 2 seems to struggle during the formulation and validation situations, as especially the algebraic work seems to be an obstacle. The same student, does however show a far greater understanding of the algebraic proof during the summarization, where she supports and corrects Student 5, who staggers at the blackboard.

This could indicate, that this group would have benefitted from more fixed and shorter bounds of the different situations: The students in Situation 3 looses their focus, as the adidactical sequence seems to long and this hampers their work. If there have been a quicker shift from the adidactical work to a situation of institutionalization, the students would have validated and institutionalized their findings quicker and it seems as Student 2 benefitted from this shift in situation. The drawback from a more firm steering, is that the students would have shorter time to make examples and counterexamples in order to examine and produce a conjecture. This is a central element of the design (Martinez and Castro Superfine, 2012, p. 125) and the study course should still contain a significant time for the students to work on their own, but as in Situation 2, then the students might have benefitted from having their conjectures validated quicker.

All in all I think, that a quicker shift between the situations would have benefitted the students and the teacher, as this would have left time for more validation and institutionalization, which would force the students to produce conjectures quicker. This should not happen on the expense of the adidactical potential, but I think that the adidactical situations were to long.

6.4.2 *Level of scaffolding*

During the study course, some students were either left helpless in algebraic proving situations or did not introduce algebra at all before they were told so, and this suggests two possible flaws of the teaching design: (1) The lack of feedback from the milieu and (2) a missing fundamental need of algebra to solve the exercises. These two flaws could be the result of a missing scaffolding in the exercises, which some students seemed to need.

I will start by briefly describing these two flaws, before I discuss possible solutions to these problems by the use of Asger Brix Jensens master's thesis (Jensen, 2015).

During the study course it became obvious that the objective milieu lacked feedback, as the students made wrongful conjectures or proofs as they solved the exercises, without getting aware of this. Ideally, the milieu should contain feedback, that make the students aware of their wrong strategies in order for them to reshape it to a *winning strategy* (Brousseau, 1997, p. 11). This feedback was not given in the proving phases, why several of the students made algebraic proofs which were obviously wrong, which is seen in situation 3.

If we turn to problem (2), then it was a design principle that the students should use algebra to prove their numerical based conjectures, but as seen in situation 1 and 2 a lot of the students resorted to verbally formulated proofs. As described earlier, the course of study lacked fundamental situations for applying algebra in proving situations, as these situations mainly appeared on the last day of the course, which I think shows that the progression in the exercises was to slow. It is of course difficult to ensure that a situation is fundamental for applying algebra for all students, entirely based on their own need to prove their conjecture. This would first of all require, that the students is capable of applying algebraic symbols and secondly that they can not solve the exercise with out the use of this. To make an exercise, which fulfills this is difficult, but (Jensen, 2015) contains some elements, that could be a help in the construction of such an exercise.

The thesis made by Asger Brix Jensen seeks to *"... examine the adidactical potentials of using number tricks as frame for teaching elementary algebra to low level math students at STX."* (Jensen, 2015, p. 1). The students were in this teaching experiment, given various *number tricks* or *"mind reading"*-math tricks, as the author calls it.

These exercises contains a chronological list of "mind"-operations, which the students are supposed to make on some individually chosen number and no matter which number they initially think of, the result will always be the same. Hence the name "mind reading"-trick. See figure 21 for an example of a number trick from (Jensen, 2015)

Step:	Word description:	Algebraic formulation:
1	Think of a number.	a
2	Add 2 to your number.	$a+2$
3	Multiply the resulting number by 2.	$(a+2)\cdot 2 = 2a+4$
4	Subtract 4 from the resulting number.	$2a+4-4=2a$
5	Divide the resulting number with your original number.	$\frac{2a}{a}=2$
6	You now have the number 2.	Confirmed.

Figure 21: Number Trick 1 from Asger Brix Jensens master's thesis

The idea is then, that the students are supposed to write down the algebraic operations next to the numerical operations in order for the students to see the coherence between the arithmetic operations and the algebraic operations. This should help the students realize the algebraic mistakes they make and from this become aware of the correct algebraic operations. Jensen wants to examine if this way of teaching elementary algebra, can help the students overcome some of the common mistakes students make when working with elementary algebra. The author concludes, that number tricks contain a large adidactical potential, which *"... allowed the students to personalize the target knowledge with very little to no interaction with the teacher"*. (Jensen, 2015, p. 101).

If we compare the setup from (Jensen, 2015) to this master thesis, then the feedback is clearer in the thesis of Jensen, as the students have a column with the algebraic calculations next to the arithmetic calculations. The students are in this way able to constantly check their algebraic findings with their arithmetic findings and if a missing coherence suddenly emerges, then the students are able re-evaluate their algebraic work. The author argues highly for the adidactical potential in these types of exercises and it seems as one of the big strengths of Jensens' teaching experiment.

If we compare this feedback potential with my teaching experiment, then it becomes obvious that the adidactical feedback potential could have been bigger. As described above, the teaching design lacked adidactical feedback as the students made some wrong algebraic work, without getting aware of this. It could strengthen this teaching experiment, if the students were forced to compare the algebraic work with their arithmetic work, as they made their algebraic proof. If every step in their algebraic proof was numerically tested, the students would be made aware of possible mistakes in their algebraic work. The disadvantage of this approach, is that the algebraic work becomes extremely guided or scaffolded. The students do not introduce algebra based on their own desire in (Jensen, 2015) and hence the students apply and work with algebra because of the limited milieu.

As one of the design principles in my master's thesis is to raise the students own desire to make a proof, by determining patterns or discovering unexpected results, this strongly scaffolded approach would have a significant influence on this principle. I think that this design principle is so important to this teaching design and the personalization of the target knowledge, that such a scaffolded approach would not benefit the teaching design. If the students are to personalize the target knowledge, it is vital that this knowledge becomes necessary if the students are to "survive" in the milieu (Måsøval, 2011, p. 34). If the students are forced to write down the algebraic steps as a consequence of a column in the exercise, then this work is not based on *"... its internal logic without the teacher's guidance..."*, but as a consequence of the limited milieu.

If the students were directly asked to formulate their algebraic steps and verify these numerically, then it would force the students to introduce algebra and as (2) above emphasizes, this is missing in this teaching design. The problem of forcing the students to do so, is that the students do not prove their conjectures on the basis of personal conviction, but by *"... reference to the authority of the adult"* (Brousseau, 1997, p. 15), and this would undermine another principle of the design: The use of algebra as a modeling tool and as a tool to make explicit what is implicit. If the students were directly asked to make the algebraic calculations, this use and understanding of algebra or symbols could be lost. The algebraic work could be diminished to meaningless symbol manipulation forced upon the students due to the teachers and the milieus demand and this is one of the obstacles this master's thesis seeks to avoid.

I therefore argue that the approach taken by Jensen contains elements, that would be beneficial for this teaching design, as the introduction of algebra would be an integrated part of the exercise and the algebraic work would be scaffolded, which could benefit a lot of students. The pitfall would be to much scaffolding, as this could neglect some of the objectives for this teaching design, as described above if this was integrated as a whole.

7 CONCLUSION

This thesis sought to examine first year high school students' opportunities of engaging in proving situations, by the application of algebra. In this examination a central component was the six didactical variables, which formed the exercises for the course of study.

The course of study was designed to make the students' examination foster curiosity and a need to explain a specific pattern. The a posteriori analysis showed that all groups formulated conjectures and sought to determine the truth-value of their conjecture by the use of different types of arguments. This work was guided by the process of formulating a conjecture to prove. The groups were at first more inclined towards verbal arguments and pragmatic proofs as these explained why a given conjecture was true. Some groups applied algebra before this was institutionalized, but algebra was in this situation used to validate a conjecture and not to explain why the conjecture was true. All students seemed at the end of the course of study confident in formulating a conjecture, though some met difficulties in constructing an algebraic proof and the proofs did not seem to be explanatory to these students, in opposition to the design principles.

Another design principle was to let students pass through three phases: examination, conjecture and proof, in order for them to formulate an argument or proof of their conjecture. The a posterior analysis showed the importance of completing the three phases in chronological order, as an unsuccessful work in one phase hampers the transition and work in the next phase. It is here essential that the students are given the possibility to construct and deconstruct conjectures, in order to formulate a conjecture to prove. The appearance of several competing conjectures, was a consequence of a short examination phase and obstructed the work in the proving situation, as the students had problems relating their proof to their conjecture.

The a priori analysis was done with the purpose of forming a set of milieus that should lead to situations being fundamental for applying algebra in the formulation of a proof. An important part of this work, was to change the value of the didactical variables to increase the complexity of the patterns in order to diminish the usability of verbal proofs. The didactical variables had a big affect on the work of the students, which was explained in Chapter 5 and discussed in Chapter 6. The Calculation Needed, The Number Table and the Algebraic Prerequisites had the ability to both diminish and enhance the need of using algebra in proving situations. The a posteriori analysis showed that when the Calculations Needed and relation of the Number Table was (to) simple, the use of algebra in an explanatory proof diminished as a verbal argument was preferred. At the same time, if algebraic prerequisites seemed unsurmountable to the students, the possibility of formulating and interpret an algebraic proof was also diminished. This showed one important influence of the didactical variables: The pattern must be both comprehensible and complex to require an algebraic proof. Situation 3 showed, that the students initially had problems with formulating an algebraic proof, but during the situation of institutionalization showed the Algebraic Prerequisites needed to complete the algebraic proof, why this situation is said to be fundamental for using algebra in the proving situation. The Number of Variables showed the ability to pose problems for the students, but at the same time contains didactical potential for letting students work meaningfully with algebraic expressions.

The teaching experiment showed that when the students constructed and de-constructed conjectures in order to obtain a conjecture to prove and the usability of a verbal proof was diminished, they engaged meaningfully in algebraic proof situations. Even though all students did not succeed in making an algebraic proof, they gained from the algebraic work: Creating a generic model, formulating relationships between the variables and setting up an initial expression.

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Part VI

APPENDIX

A

EXERCISES

I dette opgavehæfte kommer I til at arbejde med forskellige tabeller. Disse vil være: *Sumtabellen*, *Kalenderen*, *Multiplikationstabellen*, *S-Tabellen* og *Potenstabellen*. I vil få hver tabel udleveret i papirformat, men der vil også være mange udsnit af de forskellige tabeller undervejs i opgaverne. Disse udsnit er kun til illustration af eksempler på forskellige figurer og opgaver I skal lave.

I skal altså lave jeres egne undersøgelser af figurer og resultater ud fra de tabeller I har fået udleveret.

Alle opgaverne er stort set ens bygget op, dvs. at hver opgave starter med at I skal undersøge hvordan I får det største eller mindste resultat, eller undersøge hvordan resultatet ændres/ikke ændres når I laver nogle forskellige udregninger. Hvilke udregninger og figurer I skal lave, vil være beskrevet under hver opgave, hvor der ofte også vil være et eksempel på en udregning. Resultatet af en udregning vil være understreget i alle eksemplerne.

Jeg håber at I får det lige så sjovt med opgaverne, som jeg har haft med at lave dem :)

God arbejdslyst!

I skal i de følgende opgaver arbejde med sumtabellen. Denne er vedlagt som papir.
I skal i disse opgaver undersøge, om I kan beskrive forskellige mønstre i denne tabel.

Der vil til hver opgave følge en instruktion til hvad I skal gøre, efterfulgt af et eksempel. Eksemplet virker altså kun som en illustration af hvad I kan gøre, men I skal selv lave jeres egne udregninger og undersøgelser. Husk at skrive jeres mellemregninger ned på papir.

A.1 OPGAVE 1A

I skal i sumtabellen indtegne et vandret 1×4-rektangel. Altså et vandret rektangel der indeholder fire tal på række. I skal nu finde det rektangel der giver det største resultat, ved at lave følgende udregninger:

1. Læg det største og det mindste tal sammen
2. Træk det næststørste tal fra dette tal
3. Gentag dette ved at "flytte" figuren.

Eksempel:

1. $18 + 21 = 39$
2. $39 - 20 = 19$
3. Gentag dette, ved at "flytte" figuren.

12	13	14	15	16	17	18	19	20
13	14	15	16	17	18	19	20	21
14	15	16	17	18	19	20	21	22
15	16	17	18	19	20	21	22	23
16	17	18	19	20	21	22	23	24
17	18	19	20	21	22	23	24	25
18	19	20	21	22	23	24	25	26

Figure 22: Udsnit af sumtabellen - Opgave 1A

Opgave 1B

I skal i denne opgave gøre næsten det samme som i Opgave 1A, men I skal nu variere længden af rektangelet, uden at gøre rektanglet højere. I bestemmer selv hvor langt dette skal være, dette kunne f.eks. være 1×5, 1×6, $1 \times n$ osv. I skal nu forsøge at finde den længde at et rektangel, der giver det største resultat, ved at lave følgende udregninger:

Eksempel:

1. Tegn et længere rektangel, f.eks. et 1×6 rektangel: F.eks. et rektangel der indeholder tallene {9, 10, 11, 12 ,13, 14}
2. Læg det største og mindste tal sammen: $9 + 14 = 23$
3. Træk det næststørste tal fra: $23 - 13 = \underline{\underline{10}}$

Opgave 1B.1:

- Viser der sig et mønster, når I varierer længden af rektanglerne?
- Hvis der viser sig et mønster: Kan I forklare hvorfor dette mønster fremkommer?

OPGAVE 2

I denne opgave skal I arbejde med en "L"-formet figur (se figur 23), som består af to rektangler. I skal vælge et lodret rektangel, som indeholder fire tal og et vandret rektangel, som indeholder fire tal. Det vandrette rektangel, skal tegnes i forlængelse af det sidste tal i det lodrette rektangel.
Det er altså et 4×1-rektangel og et 1×4-rektangel, som skal ligge i forlængelse af hinanden.
NB: Rektanglerne må ikke være placeret sådan, at de indeholder et eller flere af de tal, som er fremhævet med **fed** i tabellen

I skal nu forsøge at finde den L-formet figur der giver det største resultat, ved at gøre følgende:

1. Læg tallene i det i lodrette rektangel sammen
2. Læg tallene i det vandrette rektangel sammen
3. Træk tallet fra 2. fra tallet fra 1.
4. Gentag dette ved at "flytte" figuren.

Eksempel:

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15

Figure 23: Eksempel på et "L" i sumtabellen

1. $5 + 6 + 7 + 8 = 26$
2. $8 + 9 + 10 + 11 = 38$
3. $26 - 38 = \underline{\underline{-12}}$
4. Gentag dette ved at "flytte" figuren. Hvad er det mindste/største mulige resultat? Argumenter for dit svar!

OPGAVE 3A

I denne opgave skal I arbejde med et 2×2-kvadrat. Dette kunne f.eks. være et kvadratet der indeholder tallene:

23	24
24	25

I skal her forsøge at finde det største resultat, ved at gøre følgende:

1. Læg tallet i øvre venstre hjørne sammen med tallet i nedre højre hjørne
2. Læg tallet i nedre venstre hjørne sammen med tallet i øvre højre hjørne
3. Træk tallet i 2. fra tallet i 1.
4. Gentag dette ved at "flytte" figuren.

Eksempel:

19	20	21	22	23	24
20	21	22	23	24	25
21	22	23	24	25	26
22	23	24	25	26	27
23	24	25	26	27	28
24	25	26	27	28	29

Figure 24: Udsnit af sumtabellen - Opgave 3

1. $23 + 25 = 48$
2. $24 + 24 = 48$
3. $48 - 48 = \underline{\underline{0}}$
4. Gentag dette ved at "flytte" figuren.

- Hvad tror I der vil ske med resultatet, hvis I ganger tallene sammen i 1. og 2. udregning og stadig trækker tallene fra hinanden i 3. udregning? Skriv en sætning ned, hvor I formulere hvad I tror der vil ske, **UDEN** at regne efter!

Opgave 3A.1

Forsøg nu at finde det største resultat ved at:

- Gange tallene i 1. sammen
- Gange tallene i 2. sammen
- Træk 2. fra 1.

- Hvilken forskel gør dette?

Når I har fundet det kvadrat der giver det største resultat, svar da på følgende:

- Passede jeres hypotese? Hvorfor/hvorfor ikke?
- Sammenlign resultatet fra Opgave 3A og Opgave 3A.1. Hvorfor får vi 0 i Opgave 3A og -1 i Opgave 3A.1?

Opgave 3B

I denne opgave skal I gøre næsten det samme som i opgave 3A, men i stedet for at arbejde med 2×2-kvadrater, skal I forsøge at finde det største resultat, ved at lave de samme udregninger med 3×3-kvadrater. Dette kunne f.eks. være et kvadrat der indeholder tallene:

9	10	11
10	11	12
11	12	13

Eksempel:

1. $9 + 13 = 22$
2. $11 + 11 = 22$
3. $22 - 22 = \underline{\underline{0}}$
4. Gentag dette ved at "flytte" figuren.

Opgave 3B.1

1. Ændres resultatet hvis I ganger tallene sammen, istedet for at lægge dem sammen?
2. Er det muligt at få et større resultat end i opgave 3A? Et mindre? Hvorfor/hvorfor ikke? - Argumenter for jeres svar

Opgave 3B.2

- Undersøg nu, hvad der sker hvis I øger størrelsen på kvadratet. Hvad sker der med resultatet hvis I bruger et 4×4-kvadrat? 5×5? Bestem selv størrelsen på jeres kvadrater. Til denne undersøgelse skal I kun bruge gange som regneoperation i 1. og 2. udregning.

I de følgende opgaver skal I arbejde med kalenderen. Denne er vedlagt som papir. Der vil dog være udsnit af kalenderen i opgaverne. Disse udsnit er kun til illustrationer af eksempler på udregninger.

A.2 OPGAVE 4 - TRAPPEN

I denne opgave skal I arbejde med det, vi vil kalde en 4—trappe (se figur 25). Denne trappe skal altså indeholde fire trin, hvor der kun må stå ét tal på hvert trin.
NB: Trappen skal gå op ad til højre

Her skal I lave følgende udregninger:

1. Læg det største tal og det mindste tal sammen
2. Træk det næststørste tal fra denne sum

I skal nu forsøge at finde den trappe, der giver det største resultat.

Eksempel:

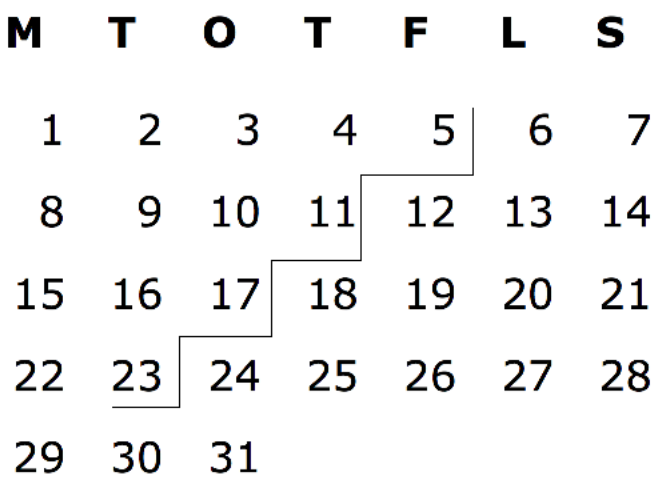


Figure 25: Et eksempel på en trappe - Opgave 4

1. $23 + 5 = 28$
2. $28 - 17 = \underline{\underline{11}}$

Opgave 4.1

1. Hvilken trappe giver det mindste resultat?
2. Forklar sammenhængen mellem placering af trappen og resultatet

Opgave 4.2 - Valgfri (Svær)

Summen af det største og mindste tal vil altid være det dobbelte af ét helt tal. Lad os kalde dette tal x. I ovenstående eksempel er summen af det største og mindste tal 28, hvilket er det dobbelte af 14 $(28 = 2 \cdot 14)$. Dvs. at i dette eksempel er $x = 14$

- Find sammenhængen mellem det mindste tal i trappen og x

Lad os nu 'udvide' kalenderen, så en måned ikke kun indeholder 31 dage, men indeholder uendeligt mange dage (se figur 26)

M	T	O	T	F	L	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56
57	58	59	60	61	62	63
64	65	66	67	68	69	70
71	72	73	74	75	76	77
78	79	80	81	82	83	84
85	86	87	88	89	90	91
92	93	94	95	96	97	98

Figure 26: Et udsnit af den 'uendelige' kalender :)

- Angiv på baggrund af din fundende sammenhæng hvad x er, hvis det mindste tal i trappen er 60, uden at tegne trappen? Hvad nu hvis det mindste tal er 127? 14?

OPGAVE 5A - L'ET OG 7-TALLET

I denne opgave skal I arbejde med *Et L og Et 7-tal*. I skal her tegne L'et og 7-tallet som på nedenstående figur (se figur 27).
L'et og 7-tallet skal altså altid være placeret sådan, at der er et "frit" tal i midten.

I skal nu finde den placering af et L og et 7-tal der giver det største resultat, ved at lave følgende udregninger:

1. Læg tallene i L'et sammen
2. Læg tallene i 7-tallet sammen
3. Træk summen fra 7-tallet fra summen fra L'et

Eksempel:
Jeg har valgt at placere L'et og 7-tallet på følgende måde:

1. $12 + 19 + 26 + 27 = 84$

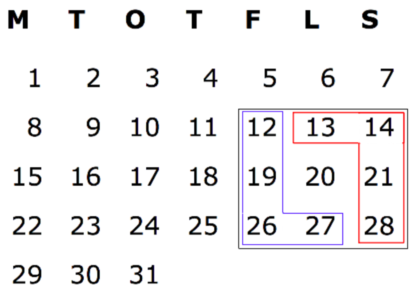


Figure 27: Et eksempel på L'et og 7-tallet

2. $13 + 14 + 21 + 28 = 76$
3. $84 - 76 = \underline{\underline{8}}$

- Vil I kunne få et større/mindre resultat hvis vi arbejdede med *Den Uendelige Kalender* (se figur 26)? Hvorfor/hvorfor ikke?

Opgave 5B - L'et og 7-tallet

I skal i denne opgave bruge samme type figur som i Opgave 5, men lave nogle andre udregninger.

I skal i denne opgave finde det størst mulige resultat, ved at lave følgende udregninger:

1. Gang det mindste tal i L'et med 2
2. Læg dette tal sammen med de resterende tal i L'et
3. Læg tallene i 7-tallet sammen
4. Træk summen fra 7-tallet fra summen fra L'et

I skal nu finde den placering et L og et 7-tal, der giver det største resultat.

Eksempel:

Med ovenstående figur (figur 27) som eksempel, vil vi få følgende beregninger:

1. $2 \cdot 12 = 24$
2. $24 + 19 + 26 + 27 = 96$
3. $13 + 14 + 21 + 28 = 76$
4. $96 - 76 = \underline{\underline{20}}$

- Vil I kunne få et større/mindre resultat hvis vi arbejdede med *Den Uendelige Kalender* (se figur 26)? Hvorfor/hvorfor ikke?

Opgave 5C - L'et og 7-tallet

Lad os endnu engang 'udvide' kalenderne, så en uge indeholder 9-dage i stedet for 7-dage (se figur 28) Vil dette ændre på resultatet,

M	T	O	T	F	Weekend			
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31					

Figure 28: Et eksempel en kalender med en 9-dags uge

hvis du:

1. Lavede samme udregninger som i Opgave 5A? Hvorfor/hvorfor ikke?
2. Lavede samme udregninger som i Opgave 5B? Hvorfor/hvorfor ikke?

A.3 OPGAVE 6A - KVADRATER

I denne opgave skal I arbejde med 2×2-kvadrater i kalenderen. Dette kunne f.eks. være et kvadrat der inderholder tallene (se figur 29):

$$\begin{matrix} 10 & 11 \\ 17 & 18 \end{matrix}$$

I skal nu finde det 2×2-kvadrat der giver det største resultat, ved at lave følgende udregninger:

1. Find produktet af tallet i øvre venstre hjørne og tallet i nedre højre hjørne
2. Find produktet af tallet i øvre højre hjørne og nedre venstre hjørne
3. Træk produktet i 2. fra produktet i 1.

Eksempel:

1. $10 \cdot 18 = 180$
2. $17 \cdot 11 = 187$

M	T	O	T	F	L	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Figure 29: Et eksempel på et 2×2-kvadrat

3. $180 - 187 = \underline{\underline{-7}}$

- Er det muligt at finde et 2×2-kvadrat der givet et større/mindre resultat? Hvorfor/hvorfor ikke?

Opgave 6B - Kvadrater

I denne opgave skal I arbejde med 3×3-kvadrater. Dette kunne f.eks være et kvadrat der indeholder tallene:

8	9	10
15	16	17
22	23	24

I skal nu finde det 3×3-kvadrat der giver det mindste resultat, ved at lave følgende udregninger:

1. Find produktet af tallet i øvre venstre hjørne og tallet i nedre højre hjørne
2. Find produktet af tallet i øvre højre hjørne og nedre venstre hjørne
3. Træk produktet i 2. fra produktet i 1.

Eksempel:

M	T	O	T	F	L	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Figure 30: Et eksempel på et 3×3-kvadrat

1. $8 \cdot 24 = 192$
2. $22 \cdot 10 = 220$
3. $192 - 220 = \underline{\underline{-28}}$

Opgave 6B.1

I skal nu lave de samme udregninger som i ovenstående opgave, men variere på:

- Kalender længden
- Uge længden
- Størrelsen af kvadratet

I skal lave jeres egen undersøgelse af resultatet, ved at variere på de tre ovenstående ting:

1. Hvis I har et 2×2-kvadrat, ændres resultatet hvis I bruger:
   a) *Den Uendelige Kalender* (se figur 26)?
   b) En kalender med en 9-dags uge (se figur 28)?
   c) En kalender med en d-dags uge?
2. Hvad sker der, hvis I ændrer på størrelsen af kvadratet og bruger:
   a) *Den Uendelige kalender*?
   b) En kalender med en 9-dags uge?
   c) En kalender med en d-dags uge?
3. Lav en samlet konklusion ud fra jeres undersøgelser. Denne skal indeholde eksempler fra jeres undersøgelse. Kan I beskrive de mønstre der evt. viser sig?

OPGAVE 7 - DE TRE BOKSE

I denne opgave skal I arbejde med *De Tre Bokse*. I skal lave tre bokse (en boks er et 2×1-rektangel), som skal placeres i forlængelse af hinanden (Se figur 31).
I skal nu finde frem til det mindste resultat ved at lave følgende udregninger:

1. Find produktet af tallene i den første boks
2. Find produktet af tallene i den sidste boks
3. Find summen af tallene i den midterste boks
4. Træk produktet fra den sidste boks fra produktet fra den første boks
5. Læg summen fra den midterste boks, til det tal du er kommet frem til i 4.

M	T	O	T	F	L	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Figure 31: Et eksempel på tre bokse

Eksempel:

1. $4 \cdot 11 = 44$
2. $6 \cdot 13 = 78$
3. $5 + 12 = 17$
4. $44 - 78 = -34$
5. $-34 + 17 = \underline{\underline{-17}}$

Opgave 7.1

- Hvad er det størst mulige resultat?
- Er det muligt at to forskellige placeringer af de tre bokse, kan give samme resultat? Hvorfor/hvorfor ikke?
- Hvor skal de tre bokse placeres, for at resultatet bliver -39?
- Kan I beskrive sammenhængen mellem resultatet og placeringen af de tre bokse?

I de følgende opgave skal I arbejde med multiplikationstabellen. Denne er vedlagt som papir, men der vil være udsnit af tabellen til hver opgave. Disse udsnit er kun til illustrationer af eksempler på udregninger. Brug det vedlagte papir til jeres egne undersøgelser

DIAGONALEN

Vi starter med at koncentrere os om en bestemt del af multiplikationstabellen, som vi vil kalde *Diagonalen*.
Diagonalen udgøres af tallene: $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, osv.\}$.

Diagonalen er fremhævet med gråt i figur 32:

x	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Figure 32: Diagonalen

- Hvad kendetegner tallene i diagonalen?

I skal i de følgende opgaver arbejde med forskellige størrelser kvadrater i multiplikationstabellen. Det er vigtigt, at I i opgave 8 og opgave 9 bruger kvadrater, hvor tallet i øvre venstre hjørne ligger i diagonalen. Et eksempel på sådan et 2×2-kvadrat kunne være nedenstående:

	1	2	3	4	5	6	7	8
1	**1**	2	3	4	5	6	7	8
2	2	**4**	6	8	10	12	14	16
3	3	6	**9**	12	15	18	21	24
4	4	8	12	**16**	20	24	28	32
5	5	10	15	20	**25**	30	35	40
6	6	12	18	24	30	**36**	42	48
7	7	14	21	28	35	42	**49**	56
8	8	16	24	32	40	48	56	**64**

Figure 33: Et 2×2-kvadrat i diagonalen

OPGAVE 8A - KVADRATER I DIAGONAL

I skal i denne opgave arbejde med 2×2-kvadrater i diagonalen. I skal her lave finde det kvadrat der giver det største resultat, ved at lave følgende udregninger:

1. Find summen af tallet i øvre venstre hjørne og tallet i nedre højre hjørne

2. Find summen af tallet i nedre venstre hjørne og tallet øvre højre hjørne

3. Træk det sidste tal, fra det første tal.

Opgave 8A.1:

1. Lav nu samme udregninger, men hvor I istedet finder produktet af tallene i udregningerne 1. og 2.

2. Er det muligt at finde et større/mindre resultat ved brug af denne regnemetode?

3. Beskriv hvorfor det er muligt at finde et større/mindre resultat

Opgave 8A.2:

1. Undersøg om resultatet ændres, hvis I istedet bruger et 3×3-kvadrat. I skal altså både lave udregningerne fra 1. og 2., men med et større kvadrat

2. Forklar hvorfor/hvorfor ikke der er en forskel i resultaterne.

Opgave 8B

I skal I denne opgave undersøge sammenhængen mellem tallene i diagonalen i et 4×4-kvadrater. Sådan et følgende 4×4-kvadrat kunne være:

$$\begin{bmatrix} \mathbf{81} & 90 & 99 & 108 \\ 90 & \mathbf{100} & 110 & 120 \\ 99 & 110 & \mathbf{121} & 132 \\ 108 & 120 & 132 & \mathbf{144} \end{bmatrix}$$

I skal kun beskæftige jer med de tal, som ligger i diagonalen (de tal som er fremhævet). I skal nu undersøge hvilket 4×4-kvadrat der giver det mindste resultat. Dette skal I gøre ved at lave følgende udregninger:

1. Find summen af tallet i øvre venstre hjørne og tallet i nedre højre hjørne

2. Find summen af de to midterste tal i diagonalen

3. Træk tallet fra 2. fra tallet fundet i 1.

Eksempel:

1. $81 + 144 = 225$

42	48	54	60	66	72	78	84
49	56	63	70	77	84	91	98
56	**64**	72	80	88	96	104	112
63	72	**81**	90	99	108	117	126
70	80	90	**100**	110	120	130	140
77	88	99	110	**121**	132	143	154
84	96	108	120	132	**144**	156	168
91	104	117	130	143	156	**169**	182
98	112	126	140	154	168	182	**196**

Figure 34: Et 4×4-kvadrat i diagonalen

2. $100 + 121 = 221$
3. $225 - 221 = \underline{\underline{4}}$

- Kan du få et større/mindre resultat ved at bruge 4×4-kvadrater der ikke ligger i diagonalen?
- Forklar sammenhængen mellem resultaterne, afhængig af om kvadratet ligger i diagonalen eller uden for diagonalen

OPGAVE 9

I skal I denne opgave undersøge hvilket 3×3-kvadrat der giver det mindste resultat. I skal stadig arbejde med 3×3-kvadrater som ligger i diagonalen. I skal undersøge 3×3-kvadraterne ved at lave følgende udregninger:

1. Find produktet af tallet i øvre venstre hjørne og tallet i midten af kvadraten
2. Find produktet af tallet i øvre venstre hjørne og tallet i nedre højre hjørne
3. Træk tallet fra 2. fra tallet fundet i 1.
4. Divider dette tal med tallet i øvre venstre hjørne

Eksempel:

	1	**2**	**3**	**4**	**5**	**6**
1	**1**	2	3	4	5	6
2	2	**4**	6	8	10	12
3	3	6	**9**	12	15	18
4	4	8	12	**16**	20	24
5	5	10	15	20	**25**	30
6	6	12	18	24	30	**36**

Figure 35: Et 3×3-kvadrat i diagonalen

1. $9 \cdot 16 = 144$
2. $9 \cdot 25 = 225$
3. $144 - 225 = -81$
4. $\frac{-81}{9} = \underline{\underline{-9}}$

- Forklar sammenhængen mellem placeringen af kvadratet og resultatet
- Forklar hvordan resultatet ændres, når placeringen af kvadratet ændres

OPGAVE 10

I skal nu ikke længere arbejde med diagonalen, men med alle tallene i multiplikationstabellen. I skal her arbejde med 1×4-rektangler. I skal finde og beskrive det mønster der fremkommer når I laver følgende udregninger:

1. Find produktet af det største og det mindste tal
2. Find produktet af de to midterste tal
3. Træk tallet fra 2. fra tallet fundet i 1.
4. Divider dette tal med -2

Eksempel:

6	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
6	7	8	9	10	11	12	13	14	15
12	14	16	18	20	22	24	26	28	30
18	21	24	27	30	33	36	39	42	45
24	28	32	36	40	44	48	52	56	60
30	35	40	45	50	55	60	65	70	75
36	42	48	54	60	66	72	78	84	90
42	**49**	56	63	70	77	84	91	98	105
48	56	**64**	72	80	88	96	104	112	120
54	63	72	**81**	90	99	108	117	126	135
60	70	80	90	**100**	110	120	130	140	150

Figure 36: Et 1×4-rektangel i multiplikationstabellen

1. $77 \cdot 98 = 7546$
2. $84 \cdot 91 = 7644$
3. $7546 - 7644 = -98$
4. $\frac{-98}{-2} = \underline{\underline{49}}$

Opgave 10.1:

- Undersøg hvordan resultatet ændres, hvis I istedet for 1×4-rektangler arbejder med 4×1-rektangler

S-TABELLEN OG POTENSTABELLEN

I har indtil videre arbejdet med *Sumtabellen*, *Kalenderen* og *Multiplikationstabellen*. Her til sidst skal vi arbejde med *S-Tabellen* og *Potenstabellen*.

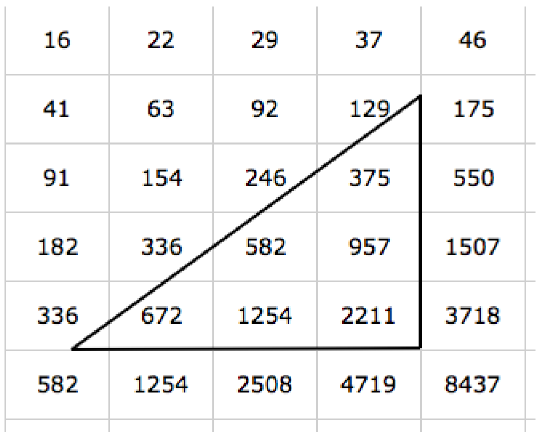
OPGAVE 11A

I denne opgave skal I arbejde med *S-Tabellen*. Denne er vedlagt som papir, men der vil være udsnit af tabellen til hver opgave. Disse udsnit er kun til illustrationer af eksempler på udregninger. Brug det vedlagte papir til jeres egne undersøgelser.

I skal i S-Tabellen lave retvinklede trekanter, som indeholder seks tal. Trekanterne skal vende på samme måde som nedenstående trekant (se figur 37). I skal her undersøge hvilken trekant der giver det mindste resultat ved at lave følgende udregninger:

1. Find summen af de tre tal, der ligger langs hypotenusen
2. Træk tallet i nedre højre hjørne fra dette tal

Eksempel:



16	22	29	37	46
41	63	92	129	175
91	154	246	375	550
182	336	582	957	1507
336	672	1254	2211	3718
582	1254	2508	4719	8437

Figure 37: Et eksempel på en trekant i S-Tabellen

1. $627 + 582 + 375 = 1629$
2. $1629 - 2211 = \underline{\underline{-582}}$

Opgave 11B

- Hvilken trekant giver det største resultat?
- Hvilken trekant giver resultatet -182?

- Hvilken trekant giver resultatet -9438?
- Beskriv sammenhængen af placeringen af trekanten og resultatet

OPGAVE 12

I skal nu arbejde med 2×2-kvadrater i S-Tabellen. I skal nu undersøge hvilket kvadrat der giver det største resultat, ved at lave følgende udregninger:

1. Find produktet af tallet i øvre venstre hjørne og tallet i nedre højre hjørne
2. Find summen af tallet i øvre højre hjørne og nedre venstre hjørne
3. Træk tallet i 2. fra det fundene tal i 1.
4. Træk 1 fra tallet i øvre venstre hjørne, og divider dette tal op i tallet fundet i 3.

Eksempel:

154	336	672	1254
246	582	1254	2508
375	957	2211	4719
550	1507	3718	8437
781	2288	6006	14443

Figure 38: Et eksempel på et 2×2-kvadrat i S-Tabellen

1. $957 \cdot 3718 = 3558126$
2. $1507 + 2211 = 3718$
3. $3558126 - 3718 = 3554408$
4. $\frac{3554408}{957-1} = \underline{\underline{3718}}$

Opgave 12.1

- Viser der sig en sammenhæng mellem resultatet og placeringen af kvadratet? Forklar denne sammenhæng, hvis der viser sig et mønster.

POTENSTABELLEN

Vi springer nu videre til næste tabel, som vi vil kalde *Potenstabellen*. Denne er vedlagt som papir, men der vil være udsnit af tabellen til hver opgave. Disse udsnit er kun til illustrationer af eksempler på udregninger. Brug det vedlagte papir til jeres egne undersøgelser.

OPGAVE 13A

I opgave 13 skal I arbejde med 4×1-rektangler i Potenstabellen. Dette kunne f.eks. være et rektangel der indeholder tallene (se figur 39):

$$\begin{bmatrix} 7776 \\ 46656 \\ 279936 \\ 1679616 \end{bmatrix}$$

I skal nu finde det 4×1-rektangel der giver det største resultat ved at lave følgende udregninger:

1. Find produktet af det øverste og nederste tal
2. Find produktet af de to midterste tal
3. Divider tallet fra 2. op tallet fundet i 1.

Eksempel:

256	625	1296	2401
1024	3125	7776	16807
4096	15625	46656	117649
16384	78125	279936	823543
65536	390625	1679616	5764801
262144	1953125	10077696	40353607

Figure 39: Et eksempel på et 4×1-rektangel i potenstabellen

1. $7776 \cdot 1679616 = 13060694016$
2. $46656 \cdot 279936 = 13060694016$
3. $\frac{13060694016}{13060694016} = \underline{\underline{1}}$

Opgave 13B

- Er det muligt at finde et større eller mindre resultat? Hvorfor/hvorfor ikke?

Opgave 13C

Udvid nu jeres rektangel til et 1×6-rektangel

- Lav samme udregninger som i opgave 13A (Dvs. I skal stadig finde produktet af det øverste og nederste tal, samt de to midterste tal). Hvordan ændres resultatet i forhold til opgave 13A?
- Forklar hvorfor/hvorfor ikke der sker en ændring

Opgave 13D

Forsøg nu at finde det største resultat ved at lave følgende udregninger i 1×6-rektanget:

1. Find produktet af det øverste og nederste tal
2. Divider det fundene tal med det næst største tal i rektanglet.
3. Forklar sammenhængen mellem resultatet og placeringen af rektanglet

OPGAVE 14

I denne opgave skal I arbejde med 2×2-kvadrater, men med et lille twist. Når I tegner et 2×2-kvadrat, skal I holde øje med de to øverste tal i de to kolonner (de tal der er skrevet med fed i øverst tabellen), hvor I placerer jeres kvadrat (se figur 40). I skal nu undersøge resultatet af et 2×2-kvadrat, når I laver følgende udregninger:

1. Find produktet af de to øverste tal i kvadratet
2. Find produktet af de to **fede tal**
3. Find produktet af det fundene tal i 1. og det fundende tal i 2.
4. Find produktet af de to nederste tal i kvadratet
5. Træk tallet i 4. fra tallet fundet i 3.

Eksempel:

1. $4096 \cdot 15625 = 64000000$
2. $4 \cdot 5 = 20$
3. $64000000 \cdot 20 = 1280000000$

Tallene med fed

3	**4**	**5**	**6**
3	4	5	6
9	16	25	36
27	64	125	216
81	256	625	1296
243	1024	3125	7776
729	4096	15625	46656
2187	16384	78125	279936
6561	65536	390625	1679616

Figure 40: Et eksempel på et 2×2-kvadrat + de to tal med fed i potenstabellen

4. $16384 \cdot 78125 = 1280000000$

5. $1280000000 - 1280000000 = \underline{\underline{0}}$

- Viser der sig et mønster? Forklar hvorfor dette mønster fremkommer.

OPGAVE 15A

I skal i denne opgave arbejde med rektangler af varierende størrelser. Vi starter dog med 2×2-kvadrater. I skal her undersøge resultat når I laver følgende udregninger:

1. Find produktet af de to øverste tal i kvadratet

2. Find produktet af de to nederste tal i kvadratet

3. Divider tallet i 1. op i det fundene tal i 2.

- Viser der sig et mønster?

- Sammenlign resultatet i opgave 15 med resultatet i opgave 14. Forklar sammenhængen mellem disse to resultatet.

Opgave 15B

Undersøg hvordan resultatet ændre sig, hvis I laver samme udregninger (Dvs. stadig finder produktet af de to øverste tal og produktet af de to nederste tal) men bruger et:

- 3×2-rektangel
- 4×2-rektangel
- $n \times 2$-rektangel
- Forklar sammenhængen mellem længden af rektangel og resultatet

B

NUMBER TABLES

Sumtabellen

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
5	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
7	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
8	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
9	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
10	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
11	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
12	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
13	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
14	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
16	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
17	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
18	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
19	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
20	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
21	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43

Figure 41: The Sum Table

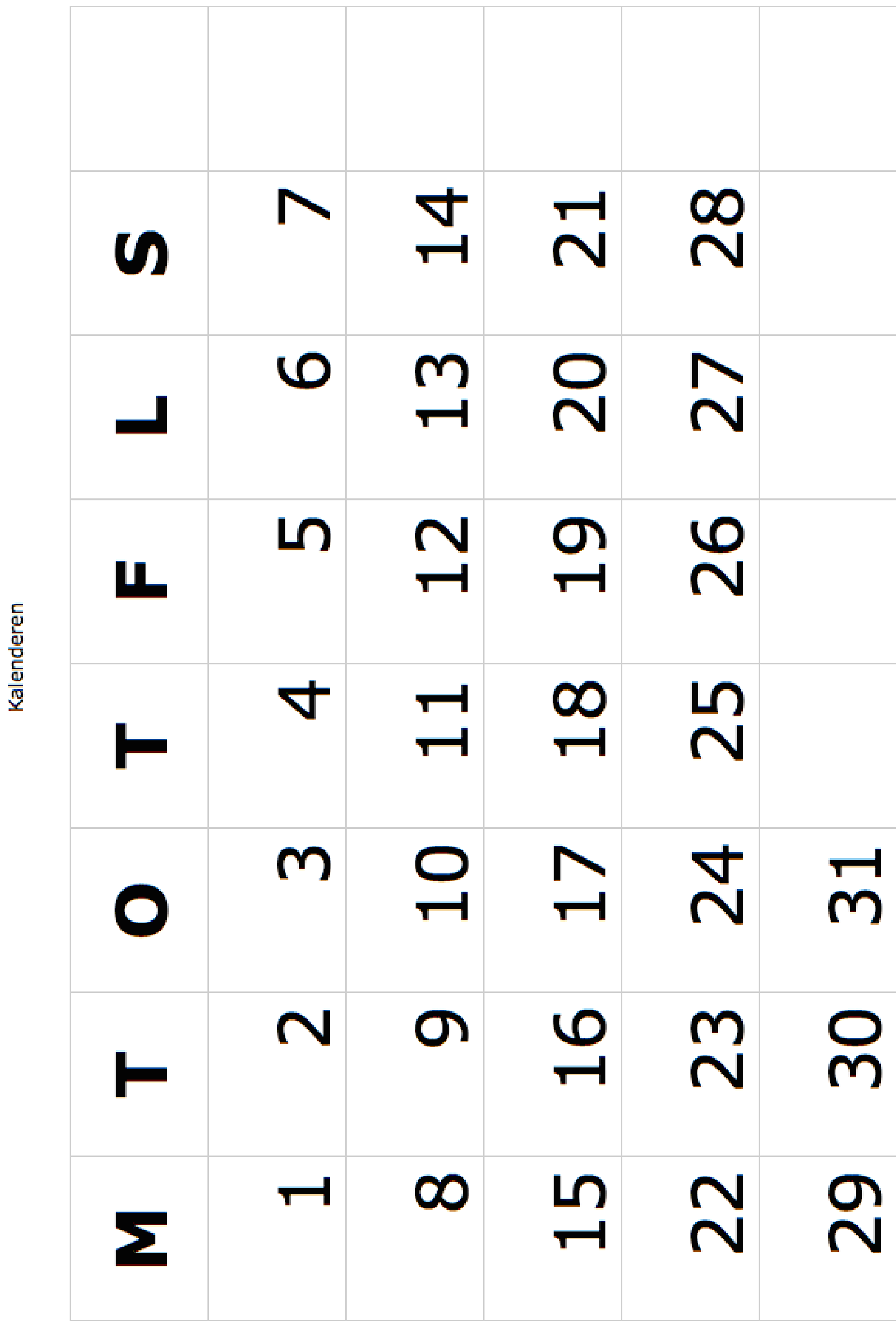
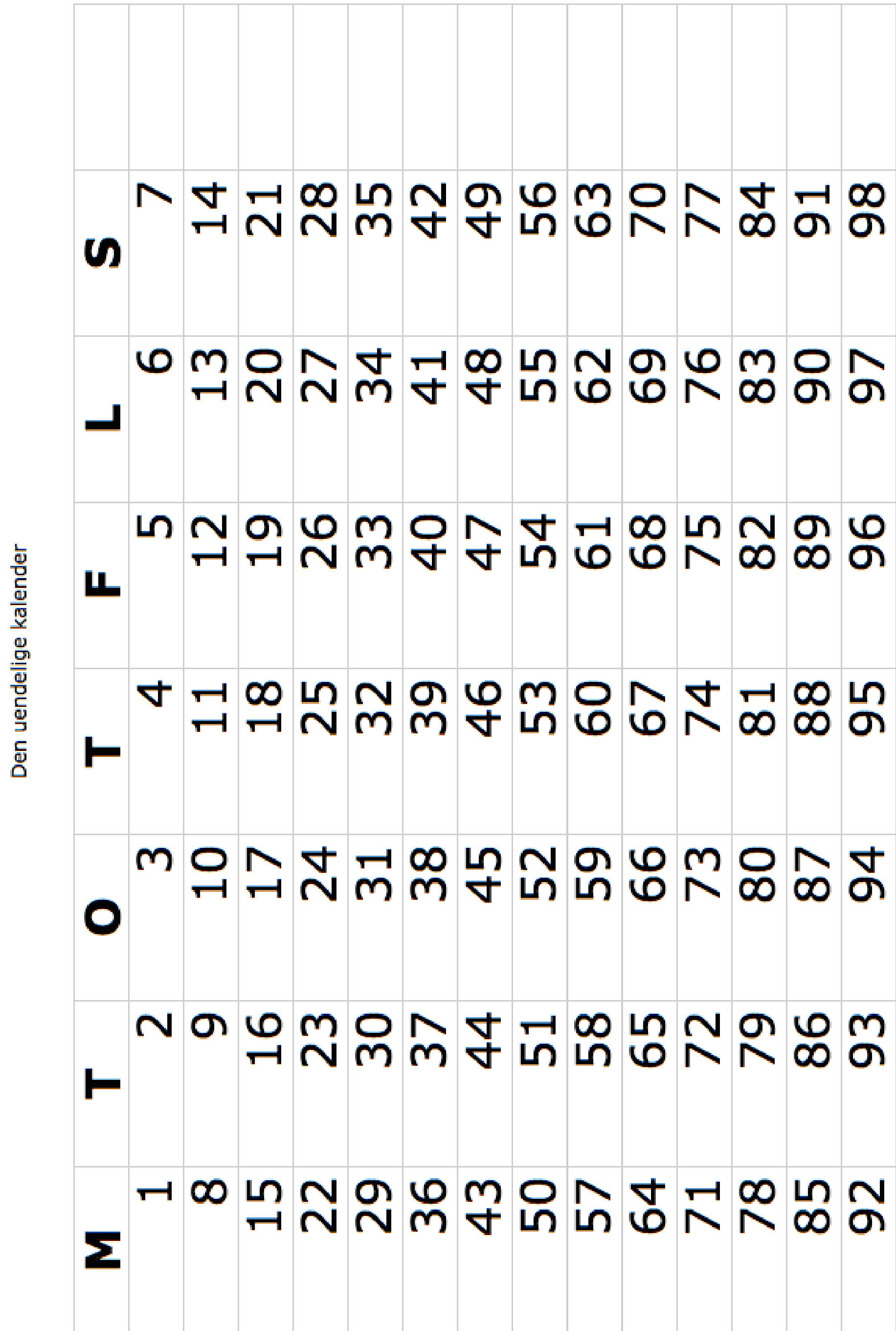


Figure 42: The Calendar



Den uendelige kalender

M	T	O	T	F	L	S	
1	2	3	4	5	6	7	
8	9	10	11	12	13	14	
15	16	17	18	19	20	21	
22	23	24	25	26	27	28	
29	30	31	32	33	34	35	
36	37	38	39	40	41	42	
43	44	45	46	47	48	49	
50	51	52	53	54	55	56	
57	58	59	60	61	62	63	
64	65	66	67	68	69	70	
71	72	73	74	75	76	77	
78	79	80	81	82	83	84	
85	86	87	88	89	90	91	
92	93	94	95	96	97	98	

Figure 43: The Infinite Calendar

9-dags kalenderen

M	T	O	T	F	Weekend			
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31					

Figure 44: The 9-Day Calendar

Multiplikationstabellen

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**
1	**1**	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	2	**4**	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44
3	3	6	**9**	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66
4	4	8	12	**16**	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88
5	5	10	15	20	**25**	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110
6	6	12	18	24	30	**36**	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132
7	7	14	21	28	35	42	**49**	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154
8	8	16	24	32	40	48	56	**64**	72	80	88	96	104	112	120	128	136	144	152	160	168	176
9	9	18	27	36	45	54	63	72	**81**	90	99	108	117	126	135	144	153	162	171	180	189	198
10	10	20	30	40	50	60	70	80	90	**100**	110	120	130	140	150	160	170	180	190	200	210	220
11	11	22	33	44	55	66	77	88	99	110	**121**	132	143	154	165	176	187	198	209	220	231	242
12	12	24	36	48	60	72	84	96	108	120	132	**144**	156	168	180	192	204	216	228	240	252	264
13	13	26	39	52	65	78	91	104	117	130	143	156	**169**	182	195	208	221	234	247	260	273	286
14	14	28	42	56	70	84	98	112	126	140	154	168	182	**196**	210	224	238	252	266	280	294	308
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	**225**	240	255	270	285	300	315	330
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	**256**	272	288	304	320	336	352
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	**289**	306	323	340	357	374
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	**324**	342	360	378	396
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	**361**	380	399	418
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	**400**	420	440
21	21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336	357	378	399	420	**441**	462
22	22	44	66	88	110	132	154	176	198	220	242	264	286	308	330	352	374	396	418	440	462	**484**

Figure 45: The Multiplication Table

S-tabellen

1	2	3	4	5	6	7	8	9	10
2	4	7	11	16	22	29	37	46	56
3	7	14	25	41	63	92	129	175	231
4	11	25	50	91	154	246	375	550	781
5	16	41	91	182	336	582	957	1507	2288
6	22	63	154	336	672	1254	2211	3718	6006
7	29	92	246	582	1254	2508	4719	8437	14443
8	37	129	375	957	2211	4719	9438	17875	32318
9	46	175	550	1507	3718	8437	17875	35750	68068
10	56	231	781	2288	6006	14443	32318	68068	136136

Figure 46: The S-Table

Potenstabellen

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	1	4	9	16	25	36	49	64	81
3	1	8	27	64	125	216	343	512	729
4	1	16	81	256	625	1296	2401	4096	6561
5	1	32	243	1024	3125	7776	16807	32768	59049
6	1	64	729	4096	15625	46656	117649	262144	531441
7	1	128	2187	16384	78125	279936	823543	2097152	4782969
8	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
11	1	2048	177147	4194304	48828125	362797056	1977326743	8589934592	31381059609

Figure 47: The Power Table

C TRANSCRIPTIONS

C.1 SITUATION 1

B: "Isn't it just to move it around until you find..."

C: "Yes"

B: "It doesn't make any sense"

A: "If we then say 15 minus 12, right? Ehm, that's 3, and then we take the second largest"

B: "No, and then minus 14, right?"

A: "No, no, we were supposed to add them. Then it gives 27"

B: "Yes"

C: "No, we were supposed to subtract the second largest number from the number we just got"

B: "But 27 minus 14"

C: "Yeah, yeah. That gives 27 minus 14"

Teacher interrupts"

B: "I don't understand the thing with, that you have to find the largest"

T: "Yes, you make a box, like that one [points at the paper]: 12, 13, 14, 15 and then you do as it says. What does it say? What is point 1?

B: "I understand that. But how do you find the rectangle, that gives the largest result? We all agree, that is probably not there [points to a drawn rectangle on the paper]. It's that one, down there [points to a rectangle, in the lower right corner]

T: "Is it? Then you have to try"

C: "No it is not necessarily that one"

A: "Isn't it?"

T: " [Student A] argues. She says it is the last one"

C: "It is..."

B: "What does this give? It gives..."

T: "Then write it down and see what it gives"

C: "Okay. It is 43 + 40, thats 83 minus 42. That is 41

T: "Okay"

C "I can't see how any could give a larger result, as there aren't large enough values in any other"

T: "Then write it down. Try to... Argue. You are supposed to argue for you explanation. To the other. And then try to see if there are some system. Now you have tried with one, the one at the bottom. You have tried this. How much did it give with this?"

B: “It gives..."
C: “Which one, did you say?"
T: “This one. The first one we framed"
C: “It gave 13"
T: “There it gave 13"
C: “No. It gave 12. No, 13... 13, 13, 13"
T: “Okay, 13. The next one gave 41. Are there some system in it?"
A: “Are we supposed to make it in nspire (CAS) ?"
T: “No, you are supposed to do it in hand. But, you are allowed to make it in nspire"
C: “In that rectangle, will it always give the second smallest number in the rectangle"
T: “Okay, but 41 wasn't the second smallest number"
A: “It is!"
C: “Besides this one here, then this can also be the largest. No"
T: “41? That isn't the second smallest number"
A: “Why isn't it that?"
C: “Yes, in the rectangle it is"
A: “It is"
T: “The second smallest, that will be 42"
A: “No, not when the smallest is there"
T: “Oh, the smallest are there... Yeah, the second smallest. Yes, thats just me talking"
C: “It's like... It's just a formula, where you put in some numbers"
T: “Okay, so it can be a formula. Then you try, if you can write up that formula. That was a good idea"
The teacher leaves
A: “Then, then what's the formula...?"
C: $a + b - c = d$...?
Student A laughes
C: “But isn't it just that?"
A: “Now, what is it again...? The biggest... $s + m$... The biggest plus the smallest... Hmm.."
C: “Then what will you denote the middle numbers?"
A: “What?"
C: “What will you denote the two numbers in the middle?"
A: “We'll figure that out"
C: “It doesn't matter what the number is... Or letter (Corrects himself)"
A: “Maybe. It's definitely s and m"
C: “Then, do you just call the two other x and y or...?"
A: “Yes, yes. Minus... No... Yes... You can say it directly like that, right? Minus... And that was..."
C: “Minus x, and that equals y"
A: “It's minus that one, right?
C: “Yes"

[The students work]

A: "That equals...

C: "y"

A: "No, that equals..."

C: "41"

A: "Yes, because we haven't used that before"

C: "Then we can denote that y"

A: "That's a really crappy formula..."

C: "Yes, but that's probably just the way it is, because it is four different numbers that is needed... *swears*"

A: "Why done we just write that? We have talked about it now

C: "Yeah, we do understand it"

[The group moves on to Exercise 1B.1. Student B reads the exercise description out loud]

B: "But again, does the biggest result not just give that whole row there?"

A: "Yes, but now it depends on how long it is. How long the rectangle is"

B: "Why don't we just try something?"

A: "Why don't we just see, if there are a difference..."

C: "It will always be the second smallest number, with that formula there"

A: "No matter...?"

C: "Because, you add 43 to that number, right? 37, right? Then as you subtract 42 again, then it is just one less you subtract than added before. So it is plus 1. So its 38"

B: "Yeah, yeah. Now we'll calculate it anyway"

C: "Or am I all wrong?"

A: "We'll try"

[The students tries different rectangle]

C: "It's the same with seven numbers"

A: "So it's just the same pattern as before

C + B: "Yes"

C: "Did you write, that it will always give the second smallest number in the rectangle?"

A: "Why do that pattern appear? Thats the same as before, right?"

C: "Yes, you subtract one less than you add, so it is one larger"

A: "Yes"

The students move on to Exercise 2 and the situation ends

C.2 SITUATION 2

The students have read the exercise description out loud and are left confused

F: "Should we just make an example?"

E: "Yes, I just need to know. We agree that it is a staircase we are to make, right?"

F: "Yes, we make staircases in this example"

H: "Also on four? Which one do we take first? 36?"

F: "Oh, but it is just a normal example, right? The sum of 36 and 18"

H: "Why I don't take my calculator"

F: "Its..."

E: "36 divided by 2 equals 18"

F: "No, no. 36 plus 18. What does 36 plus 18 gives?

E: "Oh, was it that?"

F: "Yes. What does it give?

E: "I don't know"

H: "Ehm, 54"

F: "Then it is an integer, as 54 divided by 2 gives..."

E: "36 plus 18...."

H: "27"

F: "Then 27 are x. Wasn't it like that, the exercise should be understood?"

E: "That isn't true..."

H: "Yes"

E: "But let us try this one"

H: "Yes, thats right

E: "Where is it. Their example"

F: "It doesn't say where it is. It just say that the answer is 28"

E: "But..."

F: "It's no where"

H: "27 or what?

E: "Its because its in the other calendar"

F: "No, 28 is here. It's this one here

H: "Oh"

G: "What are you supposed to do?"

F: "Its that one, they have used"

G: "Okay"

F: "So, what you are supposed to do. We are doing 4.2, where you are told to make a staircase, where we shall prove that when you add the largest and the smallest number together, it will always be twice that of an integer. But no... This is not an integer..."

E: "No"

F: "Yes, because its the double of an integer"

G: "It will be that, not matter what"

F: "No, that it isnt"

G: "The double of an integer will always be..."

E: "An integer can also be 27"

H: "Yes, yes. An integer is just a number without comma"

F: “Oh yeah, an integer is just a number without comma"

E: “Its because you think of even and uneven"

F: “Yes, I just thought about even and uneven"

G: “So if you add the uppermost and lowest number together, then either it is because both... It’s because, either are they both uneven or they are both even, and if you... subtracts an uneven and an even number, then they are both even"

E: “But this here, is the double of that"

G: “If you subtracts two even numbers, then it becomes even. If you subtracts two uneven numbers, then it also becomes even"

F: “But here we have to add"

G: “But, thats the same"

E: “Yes, thats right"

F: “In this case, it was the double of this"

[Indistinct talk]

E: “Its because it is these 6"

F: “Did you notice that? Its the 6-table"

H: “But, for example here, here et becomes 54 and not 72"

F: “No, no"

H: “So its not the double. Always"

F: “No, its not always the case"

F: “Ehm, but now we have to find... We have to set up a formula (Dansk: Ligning) for it. That’ll be fun"

E: “No, it’s not (As an answer to Student G)

G: “It’s the 6-tables all the way around"

E: “Yes. 52, 58, 64, 70"

F: “It will probably be: x. *Swears* it is difficult, this here."

F: “No, the hypothesis is..."

E: “What is our hypothesis?"

F: “It’s that, ehm... If you add the sum of the largest and smallest number together, then it will be the double of an integer"

E: “Yes"

[Unimportant talk]

F: “The proof is... but x, will always be"

H: “Thats what it gives, right?

E: “Can’t you just say"

F: “x is the double of..."

E: “Can’t we use x and y?"

H: “Oh"

F: “x, that corresponds to 27"

H: “Don’t you work the other way around? No, you can’t do that"

F: “But x, thats 27. In this example, right?

E: “Yes"

F: “Then we have to say

E: “x"

F: “Ehm... x times 2"

E: “Times.... Yes, x times 2. Thats right"

H: “I got it! Girls! Ehm... There are always 18... Ehm... There are always 18 between the uppermost and lowest number!"

E: “Yes"

H: “That is something we can put to use!"

E: “Can we?"

H: “Yes, because if we then say... Yes, look... x"

G: “What's the name of the exercise?"

H: “x. So you start by dividing it by two, right? That gives something unknown, let us call that y, right?"

F: “No, no, for x has to be the number you obtain, when you divide by 2"

H: “Yes, exactly! Oh no, then you multiply, sorry. That's right. You multiply. You multiply by 2 and that... and that number you obtain here, right? That you have to divide, so there are 18 between them"

E: “But we can't both call these two here x, if they are two different numbers"

F: “No, no, this here are the same, when you multiply with 2. Then it is the same as x. It has the same value"

E: “Yes, but you can't call it x, when there are x on both sides"

F: “Yes, yes. I can do that"

E: “No, if you replace that x, then it has to be the same as that one, when they are both called x"

H: “Yes, yes. Here I just wrote x"

E: “Can't we call the one x and the other y?

F: “No, no. Because its still the same"

E: “What?"

F: “This here, is still the same value"

E: “Okay, so if I wrote 4 equals 4 times 2, then it is not the same"

F: “But I'm not done yet"

E: “Okay, for that would give 8 and this would give 4"

F: “Yes, yes. I'm aware of that"

F: “Ehm"

E: “I'm not saying anything yet"

H: “What is it... x, that is also... It's 9 less than that one there. The uppermost number"

F: “But, is it always that?"

G: “That it's obvious, if there is always 18 between"

H: “No, no. Yes, yes. I mean... Yes, but there are also 9 there"

E: “Between which one?"

H: “Between the answer and the largest(Dansk: Højeste) number.

E: “Is it always like that?"

H: “Yes"

G: “But that's obvious. The difference does not change"

E: “No, it's not!"

H: "Yes, it's true. The difference between it all will always be the same. So we just need to use the difference for that, after we have done that one"

F: "I don't understand how we are supposed to show it"

H: "No..."

F: "Okay. We get this sum between the largest and smallest number. The largest and smallest number... If we just take it all... All the way back. Largest and smallest number, right? If we make a sequence of numbers: $x, x+1$..., No, $x+6$, right?"

H: "No, isn't it..."

F: "Yeah, $x+6$"

H: "Yes, yes, yes"

F: "$x+6$, $x+12$

E: "Why did you make a box around them?"

F: "Oh, thats just so I know that it is... $x+18$... So if we add x and... x plus... $x+18$"

E: "Yes, so... Now we use the same numbers as last, right?"

F: "What?"

E: "As in 4.1"

F: "Yes"

E: "Yes"

F: "And that equals the double. Equals the double of..."

E: "Oh, you just drew the sequence of numbers up there"

F: "Yes, yes. Equals the double of... Equals a half x"

E: "No. Yes, you can do that"

F: "Can't you sat that?"

E: "Yes. A half times x, which is also a half x, just..."

F: "I don't know if you can say that. Can you that?"

E: "No, because now it doesn't work"

F: "No, because now we have another value here, right?

E: "Yes. Then call this y. A half y"

F: "Then we have a third value. Because..."

E: "No, because, we can't both call this and that x. Because..."

F: "No, I know that. Yes, yes, you can have to x's on both sides of the equality sign

E: "Yes, but the, they have to give the same. It has to be the same x. x has to be equal the same

F: "Yes, I know that. No, that one on that side, has to be the same as that one. It doesn't have to be, that the x's..."

E: "Yes, the x's has to be the same"

F: "Yes, they have to be the same. Yes, yes, but they do not have to give the same thing"

G: "There do not have to be the same number of x's on both sides

E: "Oh, no, no. I didn't mean that, but the value has to be the same, when you put in numbers.

G: "Yes, yes. Or then you can call it y"

F: “Yes, you can do that. I think you can write a half x. I’ll try to write a half x. Don’t you think you can write a half x?"

E: “Yes, I would say that"

H: “No, I would still not say that

F: “No, no, then they are sure of..."

G: “Why do you want it to give a half x?"

F: “Because then it is the double.

E: “Okay, if we say that this is x, right?"

E: “No, no"

G: “You just turn it around. If that one gives half a x"

F: “This is x. Its the double. It is this number here"

E: “Okay"

F: “And thats the half of this number"

E: “But what is that up there then?"

F: “That’s just the number sequence"

E: “Yes, but that is also the number sequence. (..) That is why I won’t denote it the same as that number in the number sequence."

F: “Oh yeah. Thats right"

E: “Thats why I won’t call it. I won’t call it the same number in the sequence"

F: “No, I know that. I’m well aware of that. I just don’t know how I’m supposed to arrange it. Because then you can’t reduce it"

E: “No"

F: “You just can’t get two unknown values!"

E: “No, that’s true, but it’s just because then x becomes this value. Then it’s different"

F: “Yes, then it becomes a half of 18"

E: “Yes. That’s why I want to call it y"

F: “Or then, you could say: A half x equals... Ehm... I really don’t know"

[Unimportant talk]

H: “What exactly is it, the answer is supposed to give? (...) If you have. If we have x. Then what is it exactly, the answer is supposed to give?"

F: “The answer is supposed to be... The twice of an integer"

E:“Can you just say... Just wait a little. I’ll just try to write it down. Ehm.. If we have... What we have to do is... We’ll say... $x + x + 18$. (...). Divided by two. Equals..."

F: “But no, no.... Because we don’t know how we are going to solve it"

E: “Yes, I also wanted to write that. It is the very thing we are doing"

F: “No, because then you have to say. Then it is equal... Ehm... y..."

E: “Yes"

F: “Or what? Or equals x??

E: "Equals y. No, because x equals this one. This is 18. Then we write here"

F: "But then there are two values again! I hate it when there are two values!"

E: "But, that is what we need to. Then we reduce this, maybe. But, it is this that applies"

F: "Yes, but then the problem is, that you can never work out what y is".

H: "But, can I just ask a question? Because, now we got these x here, right? But didn't it say, that the answer should be x?"

F: "Yes, that's what it says. The answer shall be x"

H: "Then, isn't it just that, we call these y-values?"

F: "Yes, yes. We can do that, but it doesn't make a difference in ours"

H: "No, no. But we show it"

F: "No, it just say... Let's call this x"

H: "Still..."

E: "Then you can do like this"

[Person from another group - K]: "The boys just said, that this is wrong"

E: "It isn't right"

F: "Its not right. We just don't know what to do else..."

E: "Oh my god"

F: "Else, if we just put in things, right? If we just wright up the things it can be. Then it is named..."

E: "But thats just how you calculate it"

F: "Yes, but now we just write the examples. Then you can always put in x's afterwards. $x + 36$"

E: "Divided by 2"

F: "No, equals..."

E: "Equals 54 divided by 2"

F: "Yes"

E: "No, you have to do that in the first row. You can't do that"

F: "I know that"

F: "Oh, so if... Then it can be 18 + 36. Divided by 2. Equals... It does look like this, right?"

E: "Yes"

F: "The bottom one"

K: "[The teacher] just said, that this formula looks reasonable"

E: "Yes, but that's just how you calculate it. It isn't a proof"

K: "It is a proof"

H: "No, you can't really calculate it, so you say, that this... Ehm.. That this gives x"

E: "But we can never get it to be the same!"

K: "Thats 6, right?

E: "Yes"

K: “And then plus 6, plus 18"

H: “But you don’t know what y is"

E: “No, but it could be 6, and then the next gives 12, 24. 18, 24"

F: “But there have to be..."

E: “Yes... Its says... isn’t it?"

F: “No, they have to be equal"

E: “Yes, that is right, but we have to calculate this first. But she will also have around these..."

K: “Its to find x, but we have to find the relation between something else"

E: “Yes, thats the problem"

K: “But she says its right. But that is just to find x"

F: “Do you know what? What is the answer?"

K: “Yes, then we can’t figure out any more. They say something about a half"

F: “Yes, we also used something with a half once. That didn’t go well"

H: “I think I found it"

E: “What is it?"

H: “Yes, look. It makes sense. Because. Now I reduced it, right? And thats correct, what it gives. Then we could use this sentence, because it gives what it should"

F: “Yes, yes, but it is still just to find x. Or the thing you said"

H: “Yes, yes, but now I just calculated through and found that it does"

E: “But we have to find, why x equals 27"

H: “Yes. Ehm... but then"

K: “Now we found it"

E: “We already found that. Just the other way around"

F: “We just called it the other way around

K: “Did you also find that?"

F: “Yes, we found that"

K: “Its just because, I like it the best when it says y equals... You wrote x equals"

F: “Its because it says here, that it is x. But it is fine. Its the same"

K: “I know that"

F: “The problem is"

K: “Its really easy, when you think about it"

F: “But does it apply? Is that the proof? Or is it, that again? We still have to find the relation"

K: “Isn’t it?"

F: “We still have to find why it gives, what it gives"

E: “Cant we just pretend we misunderstood the exercise?

H: “Sorry... isn’t it all this, that lies between?"

F: “Yes"

H: “Can you just say, that the number... What can you..."

F: "It is already that, because if you add these two, divide with 2, then it is...

H: "Yes, but then we could use that as a sentence (Dansk: Sætning)"

F: "Yes, but then you just find... Then you just find the average. That is actually what we do"

H: "Yes, but then the answer is the average of the number sequence"

F: "Yes"

E: "The average of...?"

F: "But the hypothesis is still that one up there"

E: "The average of x?"

F: "No, the average of the number sequence. No. Yes. The average of the number sequence. No"

E: "No"

F: "The average of the largest and smallest number, right?"

H: "Ehm... Yes"

F: "That would mean the number sequence. It doesn't change that you say integer"

E: "Why don't we try it?"

H: "Yes"

E: "What row do you want?"

H: "Try this one"

E: "Okay, what is it? 72 plus"

H: "63... 66. Sorry. Plus 60, plus 54

E: "Divided by 4"

H: "Yes, what does that give?"

E: "63"

K: "Mads (A student from Student K's group) says, that this is the relation. This is the relationship"

F: "Relationship? How?"

K "I don't know. He just say so"

E: "But, that doesn't give anything. It doesn't have anything to do with what, why we get, the largest and smallest, exactly this number"

K: "No, I just say what I understood"

F: "But the thing about an integer. What was that, we were supposed to find in that connection?

E: "An integer?

F: "Yes, the thing with, it should be an integer?"

E: "I don't get it. They are all integers"

F: "Thats obvious when you work with integers"

E: "Yes"

F: "I think. Or else it is because..."

H: "We figured it out, right?

E: "Yes, I would also say that this is the answer"

H: "I also wrote that down, that because that the largest number and the smallest number..."

[The Teacher Interrupts]

T: “Yes. What is the result? What is the relationship between the smallest number and x"?

E: “So..."

H: “The smallest number?"

T: “Whats the relationship? What did you find?"

E: “We found..."

H: “The result is the average of the number sequence"

T: “Is it?"

E: “Yes"

T: “That I can’t..."

H: “No, it is a little messy."

T: “What?"

H: “The thing above is not in it. Its only this"

T: “Yes, yes. What is the number sequence to you?"

E: “No, that one we also found"

F: “The number sequence?"

H: “Yes, yes. It is this that is a part of it"

T: “What is y in your..."

F: “y is... It’s like... Its one of those letters"

E: “Yes, like the one that was named x before"

F: “And x, is like the thing that gives the answer"

T: “Yes. Is y just a random? It don’t think it is"

E: “No, it is 6"

T: “Is y always that?"

F: “No"

H: “No, that depends on the number sequence"

F: “y, is random, right?

T: “No, I ask you that. Is y always random?"

F: “It depends on the number sequence"

T: “Yes, but what number in the staircase does it represent?

F: “The first"

T: “Yes, the first. Or the smallest.

E: “Yes the upper"

TT: “Yes. Then what is the relationship between the smallest number... If the smallest number is 126? What will x be?"

G: “42"

F: “Then it will be... Then it well be... $126 + 9$

T: “Yes, exactly!"

G: “Oh..."

T: “Why?"

F: “Because it equals..."

E: “Because it is the difference..."

T: “Because it is the relationship you have found!"

E + F: “Yes"

T: “So y is the smallest number. So x will always be the smallest number plus 9

F: "Yes

T: "I don't understand the thing with the average. That is, that you add the four numbers and divide with 4"

F: "Yes"

E: "It's because that was what it said you should do"

T: "But you just add two numbers"

F: "Oh, yeah, yeah, but it is the same thing. Because up here, we have this piece. The result will always be the average of the number sequence"

G: "The average of what?"

T: "You just add two numbers and divide by two"

E: "Its the average of these two numbers"

T: "Yes, it is the average of these two numbers. You can say that"

F: "But you can use both"

E: "Its the same thing"

F: "Those two numbers"

T: "But the average of those two numbers are not the same as the average of those four numbers?"

E: "No, but it is because there are the same difference"

F: "Difference... Yes"

T: "Yes, there are the same... You add the same every time"

[Teacher Leaves]

During the class summarization, the group continues their discussion:

F: "You could also say, that we have found out, that the relationship is that it is always $y + 9$, right?"

H: "Yes, or the average of..."

F: "Yes, but I think it is better to say the other one, as that is what we have proved"

H: "Yes, because she said the relationship between"

H: "So, that gives that, but how did we calculate it? How do you go from there to there?

F: "You have to lift a division-bracket"

H: "Just like that?"

E: "You just divide that with that, because it looks like that is what you do"

B: "But"

E: "Then the number 2 disappears and then that one, we divide with 2, so i becomes 9"

[End of situation]

C.3 SITUATION 3

The students have read the exercise description out loud, as we enter:

S3: “I don’t get this exercise"

S2: “How can you not understand it?"

S1: “What the fuck?"

S2: “Give me that. I’ll make it, but then I wouldn’t make any of the other"

S1: “No, I’ll like to try to make it"

S1: “Find the product of the numbers in the upper left corner and lower right corner. This is probably the easiest thing I’ve tried"

S3: “No, this doesn’t make sense. Oh, maybe it does"

S1: “The product of these numbers are 9"

S3: “Oh yeah, this is really easy"

S1: “And then this is 28 times 20"

S2: “I have found one that gives plus 7"

S3: “Plus seven? Which one is that"

S2: “It’s this one"

S3: “Okay"

S1: “That gives what?"

S2: “Plus seven, if in the other gives minus 7"

S1: “Okay"

S3: “Okay"

S2: “I’ll try another one"

[Loose talk]

S2: “This one also gives -7! God knows, if all of them gives -7...?"

[Teacher Interrupts]

T: “How are you doing?"

S1: “It’s okay"

S2: “It’s..."

T: “Did you found out what the result was?

S2: “Yes, in one of them it gives minus 7"

T: “In which one?"

S2: “In the first one"

T: “Yes, does it always do that? Doesn’t it matter where you put your square?

S2: “No. Yes. What do you want me to say?"

T: “I don’t know"

S2: “No, I found one that gave -7 and one that gave 7, but it could be, that I subtracted the reversed number"

T: “So, have you tried one more?"

S2: “Jonathan. Your turn!"

S3: “But I was about to start on Exercise 6B

S2: “Give me 4 numbers then"

S1: “Okay. I’ll give you 4 random numbers.

S2: “No"

S1: “In a square. Yes, yes. 12, 13, 19, 20.

S2: “Yes, but what should I multiply? 12 times what?

S1: “12 times 20"

S2: "12 times 20. Remember 240"

S1: "Okay"

S2: "Next"

S1: "13 times 19"

S2: "13 times 19. 247"

S1: "Then it gives 7. Or minus 7"

S2: "No, it gives minus 7. Because the other number has to be ind the middle

T: "So, it does look like it will be -7 every time"

S1 and S2:: "Yes, it could look like that"

T: "Is it possible to somehow show that? Like we did with the staircase[1]?"

S1: "Uuhhh... (baffled)"

T: "Could we introduce an x? Could we use an x somewhere?

S1: "Hmm..."

S2: "Most certainly. We could always put an x there" (Laughs)

T: "Is that possible? What if you draw a square, like you... Okay, you haven't drawn anything... But could you throw in an x somewhere? Could you make x represent some number?

S1: "7!"

T: "Now, I don't know if you listened when the staircase (Exercise 4) was summarized, but what did x represent there?"

S2: "The smallest number...?"

T: "Yes, that was the smallest number and so, could we construct the rest after that? Could you do that? Could you introduce an x somewhere?"

S2: "Could we do it on the smallest number?"

T: "You could try. What would the next number be then? If you tried to make a drawing of it, for example"

S2: "x... Then it becomes... Now I just quickly have to find out... Which one did I use?"

T: "You used the smallest number"

S1:"$12, 13, 19, 20$"

S2:"But, it was just which one... Eh.."

T: "Why dont you make a square"

S2: "Oh yes"

T: "So, if you call the smallest number x, how could you represent... What would that number be named?"

S2:"x... What did they write in the other one? $x + 13$...."

T:"Yeah... How far is it from that number to that number?"

S2: "$x + 1$"

T: "Yes. Exactly. And how could you name that number down there?"

S4: "$x + 7$"

1 Exercise 4 was called the staircase

T: "Yes. Exactly! So, no matter where we place it, then that will always be 7 larger. 18 is 7 larger than 11. And what would that number be named?"

S2: "$x + 8$"

T: "Yes. Quite right! No matter where you place the square, then that will always be one larger, that one will always be 7 larger and how big is that one? That one is always 8 larger, right?

S2: "Yes"

T: "Okay, so if we make a thing like that, where we name the smallest x, well then you can make a box that generalize all boxes"

S2: Yes

T: "And then what happens, if you do the calculations? What would the calculations be, if you called this x, $x + 1$, $x + 2$, $x + ...$, and so forth? You could try to draw a box. Try to draw it"

S2: "Then don't we have to say..?"

T: "You could call this one here x"

S2: "x time something"

T: "Try to draw it"

S2: "You draw it. I'll say it"

S4: "What should I draw?"

T: "What would that number be called, if this is called x?

S1: "x+1"

T: "Yes"

S1: "x+7, x+8

T: "Yes. Exactly. Which calculations would you then be asked to make?"

S2: "x times x+8"

T: "Yes. Write it up"

S2: "You do it girl"

S4: "x plus..."

S2:" No, it's x times $x + 8$..."

T: "Because, we say this number times this number"

S4: "x times $x + 8$"

S2: "Now you have to say: $x + 1$ times $x + 7$"

T: "Yes. Exactly. And what should we do with these two numbers?"

S2: "Subtract them. You know, subtract that one, with that one. With that. No. With that. No. You know what I mean... So it gives -7"

T: "So we have to subtract something from something else?"

S2: "Yes, we have to subtract that from that"

T: "Yes. Exactly. Can we write a line? Could we put the two thing on the same line?"

S4: "I suppose"

T: "Yes"

S4: "Ehm..."

S3: "Remember parentheses, right?"

S4: "Yes, yes. I'll just write this thing down"

S2: “Good work, Jona!
S4: “Yeah, yeah"

Loose talk

T: “Now it is starting to..."
S3: “We made it boys!"
T: “Nah, I wouldn’t say that"
S3: “Oh"
T: “Because there are a long way to go yet"
S3: “We didn’t make it at all"
T: “You made it a long way. You have started to use x’s"
S2: “Yes, that is right"
T: “Then you could say, what would this give? What would your hypothesis be? Equals the numbers? You have now multiplied these two numbers and subtracted these two... products... these two. What do you get every time?"
S2: “Minus 7"
T: “Yes. That might be equal to minus 7. Thats what you think, right?
S2 + 4: “Yes"
T: “How can we figure out, if that is true?"
S2: “We could calculate it?"
T: “You could do that!"
S2: “Then should we lift the brackets?"
T: “You could try?"
S2: “Lift the brackets!
S4: “I can’t do that"
S2: “I can’t remember how you do it. x minus... What is it? What did we write here?"
S4: “Ehm... x+1 times x+7"
S2: “Oh no! Now I can’t remember it... Times... x minus... No wait..."
S3: “No!"
S2: “Plus... No... Oh my god"
S3: “Plus 7, man!"
S2: “No, it writes the opposite, because we subtracted it. Its a minus-parenthesis"
S3: “Oh, så it is minus 7"
S2: “Yes, right?
S3: “Yes"
S4: “Then I suppose it is x-1
S2: “No, minus 7"
S4: “Oh"
S2: “Ups, there are not supposed to be parenthesis any more. There vi have to, equals minus 7. Them we say. Then there is x times x. 8. 19. Minus 1. Thats 9."

S1: “Oh, I feel so stupid"
S2: “Minus. Thats 8"
S4: “No, that doesn’t really give minus 7"
S2: “I can’t get it to fit"
S4: “Aren’t we just supposed to reduce it?
S2: “Yes, but this ere, thats not right"
S4: “Are you sure?"
S2: “Yes, but that doesn’t give minus 7"
S4: “It can be true. There are x in it!
S2: “So when you lift the parentheses, then it becomes divided, when it was multiplied before. No, you don’t do that. Its just still this 8. Then because it says minus in front, I have to write plus now, or what? No, it still has to be minus"
Noise because of a break

S4: [Indistinctly] “x like in x, isn’t that $2x$?"
S2: “$2x$. Like that. Minus x, that equals x. And x times x equals... Then it is x. 8 minus 1. No, 8 minus 1, that equals -7. And x minus 7, that equals -7, or what?"
S4: “I think so"

The summerization:

Student 2 and Student 5(from another group) is at the blackboard presenting their findings
S2: “We should in the calendar select four numbers in a 2×2-square, and then we should say that number in the upper left corner, it should be multiplied with the number i the lower right corner. And then the number in the lower left corner, we have multiplied up in the upper right corner. Those two numbers should we subtract from each other. We were then supposed to find, if it gave the same result. We should find that one that gave the largest result. We then found, that it gave the same result"
S5: “It always give -7"
T: “The last thing here. You say two different things. Should you find if it gave the same or should you..."
S2: “No, we corrected it. We were supposed to examine if there were a possibility of finding a larger result"
T: “Okay"
S2: “The largest result. But then we found out, that it always gave -7"
T: “Okay. So the exercise was about, placing a square in the calendar. How do you get the largest"
S2: “Yes"

T: "Okay"

S2: "And then we found out it always gave -7 and now we will show why it always gave -7

S: "Yes. Okay. Then vi should put up the formula. That we will do by calling this x. And we chose 20. So if we make the box again, but where there are x instead, then is is x and because we now there are 1 between 20 and 2, then we can call this for x+1. And there are 7 between these, so we can call this x+7 and there are 8 between these two, so x+8"

T: "Will it be like that in all squares you place?

S: "Yes"

T: "Yes it will. You get that, all of you?"

S5: "Then we have to put it up"

T: "Yes. What will you do now?"

S2: "Then... We will make the same calculations, but with these x's. So we can say something general"

T: "So that recipe on what to do. You will now do that with x's?"

S2: "Yes"

S5: "We actually just have to do that one over here. And then we will say x times $x+8$ minus $x+7$ times $x+1$. And that will give -7

T: "That is your assumption?"

S5: "Yes. And if you reduce it or calculate it, then you can say"

T: "Now we want to what we did in the first lessons, like, show that you get -7, by reducing left side. Isn't that right?"

S5: "Yes. Then we will start by saying x times x - and that will give x to the second

T: "Yes. Try to do it without your paper"

S5: "That is a little to much, Kirsten"

T: "Okay. Thats fine"

S5: "Plus these in here. Thats 8x"

T: "No, thats not what happens. No, where do the 8x come from? You said x times x, that gives x to the second"

S5: "Oh, over here

T: "No"

S5: "x times 8"

T: "Yes"

S5: "Ehm"

T: "Yes. Minus. Then I think it is a good idea to make a large parentheses"

S5: "Yes, like that"

T: "Yes, and then you just multiply into the parentheses. Or just or just. There are nothing just in that. Then you multiply the two parentheses together"

S5: "So we have to multiply. Yes. x times x"

T: "Yes"

S5: "And that gives again x to the second, plus x times 1, which gives x, plus x times 7"

T: "x times 7. yes

S5: "7x"

T: "And then the last part"

S5: "This number 7"

T: "Multiplied with...? Thats what happens. On your paper it just say 7. I can understand that. Do everybody follow? Freja and Papasha said x times x, x times 1, 7 times x, then we miss..."

S5: "7 times 1, is just 7. And ehm... Then we have to reduce it"

T: "Yes, then there just needs a parenthesis in the end, and then you reduce it. When you have to reduce, and you have a large brackets with a minus in front. Then what?"

S5: "Ehm, then we have to change sign"

T: "Yes!"

S5: "And then it will be... Ehm, x to the second,plus 8x, over here and then minus...

T: "Now you stop using your paper, because they confuses you. You said you had to change signs on all of it. So you do that now"

5: "Yes"

T: "You look at it, and write it up as you go"

S5: "Ehm"

T: "This one, you are done with"

S5: "Yes"

T: "And this parenthesis, that has to be, so it shifts sign"

S: "Oh okay, so that one we don't have to include, as we have minus here.

S: "Yes"

[Indistinct talk]

S: "Freja dictates"

S2: "Can't we just write x to the second and then minus instead of plus?"

T: "Yes!"

S2: "And then it's just the same, but then just minus where it says plus. It's just what's in the parenthesis you need to change"

S5: "minus x to the second and then plus x here?"

S2: "No, minus x, because this is placed outside. Now you need to write minus

S5: "Minus..."

S2: "Yes and then you just write one x, minus $7x + 7$"

S5: "Ehm..."

T: "And then something cancels out, you say? Because you were just a step ahead"

S5: "Yes. We have to x to the second, so we say x to the second, minus, x to the second. They cancels out. Then we have 8x"

T: "There are no arrow here! No, but it equals"

S5: "Then we have 8x"
T: "Yes, we have 8x"
S5: "Minus one x, that gives"
[Indistinct talk]
T: "Whats left?"
S2: "No
T: "Yes, say it Freja, instead of just saying no every time"
S2: "But she want to do it.
T: "Yes, and so do you
S5: "Okay. 8x minus 1x.... Ehm... 8x minus 1x, that gives 7x"
T: "Yes"
S5: "And minus 7x, that gives 7 and that equals minus 7"
T: "That's just what it does!"
[End of situation]