INSTITUT FOR NATURFAGENES DIDAKTIK
KØBENHAVNS UNIVERSITET

Teaching infinitesimal calculus in high school

with infinitesimals

Mikkel Mathias Lindahl og Jonas Kyhnæb

Kandidatspeciale

September 2016

IND's studenterserie nr. 53

INSTITUT FOR NATURFAGENES DIDAKTIK, www.ind.ku.dk
Alle publikationer fra IND er tilgængelige via hjemmesiden.

IND's studenterserie

43. Lene Eriksen: Studie og forskningsforløb om modellering med variabelsammenhænge (2015)
44. Caroline Sofie Poulsen: Basic Algebra in the transition from lower secondary school to high school
45. Rasmus Olsen Svensson: Komparativ undersøgelse af deduktiv og induktiv matematikundervisning
46. Leonora Simony: Teaching authentic cutting-edge science to high school students(2016)
47. Lotte Nørtoft: The Trigonometric Functions - The transition from geometric tools to functions (2016)
48. Aske Henriksen: Pattern Analysis as Entrance to Algebraic Proof Situations at C-level (2016)
49. Maria Hørlyk Møller Kongshavn: Gymnasieelevers og Lærerstuderendes Viden Om Rationale Tal (2016)
50. Anne Kathrine Wellendorf Knudsen and Line Steckhahn Sørensen: The Themes of Trigonometry and Power Functions in Relation to the CAS Tool GeoGebra (2016)
51. Camilla Margrethe Mattson: A Study on Teacher Knowledge Employing Hypothetical Teacher Tasks - Based on the Principles of the Anthropological Theory of Didactics (2016)
52. Tanja Rosenberg Nielsen: Logical aspects of equations and equation solving - Upper secondary school students' practices with equations (2016)
53. Mikkel Mathias Lindahl and Jonas Kyhnæb: Teaching infinitesimal calculus in high school - with infinitesimals (2016)

Se tideligere serier på: www.ind.ku.dk/publikationer/studenterserien/

Abstract

In high school students have their first encounter with the notion of infinitesimal calculus. A common problem encountered when introducing this subject is for the students to understand the theory behind the techniques they develop when being taught this. In this thesis the use of infinitesimals will be reinstated to teach this particular theme. They were used to teach infinitesimal calculus even after their abolishment, when the real numbers were constructed. With the dawn of the nonstandard analysis, through the construction of the hyperreal numbers, the infinitesimals were reinstalled as a mathematical sound quantity. This thesis will give reasons for or against the use of infinitesimals in the form of nonstandard analysis by employing the anthropological theory of didactics. The thesis will give a short introduction of how the hyperreal numbers are constructed and how the definition of the differential quotient and integral is equivalent statements to the definitions usually encountered in high school.

A teaching course was done in order to survey if the didactical reasons suggested by the theories are noticeable when introducing infinitesimal calculus, with infinitesimals, in high school. As such a textbook material had to be developed in order for the students to prepare for the exam(s). This textbook material is void of the construction of the hyperreal numbers, as introduced in the thesis, but hinges on an intuitive construction of infinitesimal quantities to generate them.

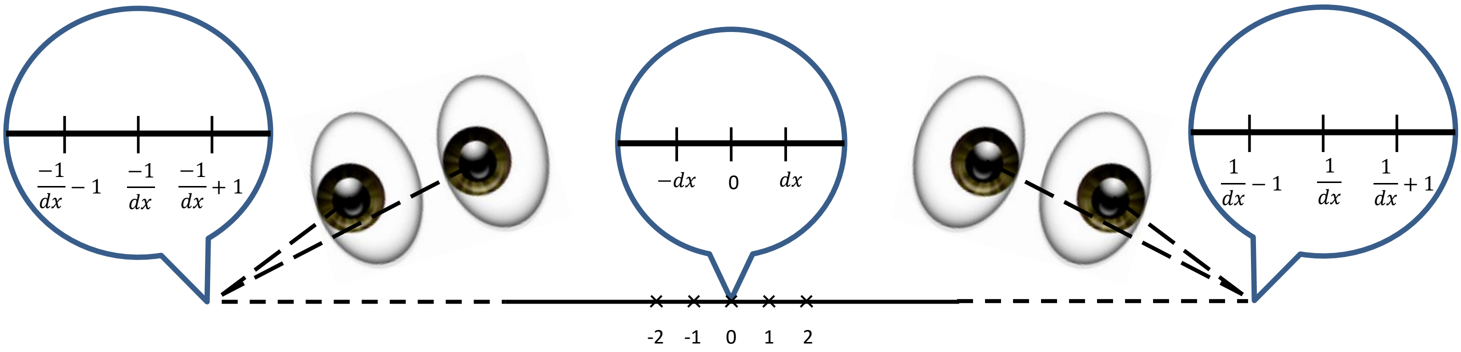
An analysis of the teaching employing the anthropological theory of didactics was done in order to determine if the merits of the infinitesimals are on par with the theoretical reasons. The thesis concludes that teaching based on infinitesimals are realistic and helps engage the students by letting their intuition guide them to a great degree..

IND's studenterserie *består af kandidatspecialer og bachelorprojekter skrevet ved eller i tilknytning til Institut for Naturfagenes Didaktik. Disse drejer sig ofte om uddannelsesfaglige problemstillinger, der har interesse også uden for universitetets mure. De publiceres derfor i elektronisk form, naturligvis under forudsætning af samtykke fra forfatterne. Det er tale om studenterarbejder, og ikke endelige forskningspublikationer.*
Se hele serien på: www.ind.ku.dk/publikationer/studenterserien/

UNIVERSITY OF COPENHAGEN

Teaching infinitesimal calculus in high school

with infinitesimals



Master's Thesis

Mikkel Mathias Lindahl and Jonas Kyhnæb

Supervisor: Carl Winsløw

Department of Science Education

Date of submission: 08.08.2016

Names: Mikkel Mathias Lindahl & Jonas Kyhnæb

Title: Natsværmer

Department: Department of Science Education

Supervisor: Carl Winsløw

Time frame: February – August, 2016

Abstract

In high school students have their first encounter with the notion of infinitesimal calculus. A common problem encountered when introducing this subject is for the students to understand the theory behind the techniques they develop when being taught this. In this thesis the use of infinitesimals will be reinstated to teach this particular theme. They were used to teach infinitesimal calculus even after their abolishment, when the real numbers were constructed. With the dawn of the nonstandard analysis, through the construction of the hyperreal numbers, the infinitesimals were reinstalled as a mathematical sound quantity. This thesis will give reasons for or against the use of infinitesimals in the form of nonstandard analysis by employing the anthropological theory of didactics. The thesis will give a short introduction of how the hyperreal numbers are constructed and how the definition of the differential quotient and integral is equivalent statements to the definitions usually encountered in high school.

A teaching course was done in order to survey if the didactical reasons suggested by the theories are noticeable when introducing infinitesimal calculus, with infinitesimals, in high school. As such a textbook material had to be developed in order for the students to prepare for the exam(s). This textbook material is void of the construction of the hyperreal numbers, as introduced in the thesis, but hinges on an intuitive construction of infinitesimal quantities to generate them.

An analysis of the teaching employing the anthropological theory of didactics was done in order to determine if the merits of the infinitesimals are on par with the theoretical reasons. The thesis concludes that teaching based on infinitesimals are realistic and helps engage the students by letting their intuition guide them to a great degree.

Acknowledgements

A great big thanks goes to Carl Winsløw for his help in shaping the thesis.

The school, Roskilde Katedralskole, deserves a lot of credit for letting the study take place. An infinitely amount of gratitude and respect goes to the wonderfully sweet and awesome 1u, who took opened their arms wide and made the study all the more fun; a better class does not exist. Thanks for the constant surprises and for tolerating all the shenanigans.

The biggest of thanks to family and girlfriend for being so patient and understanding. Thanks for the tremendous amount of support.

The girls who sat in the other cubicle in the office for about 4 months, Anne and Line, thank you. To the new office, particularly Nine, thanks for making the final month or so bearable; from the breakfast table to the midday pokéhunt, all the way to the afterhours beer.

$\frac{1}{dx}$ thanks to one another. How on Earth this thesis would have come to be without the lucrative cartalks, the mutual respect, The Trilogy, and the fantastic friendship that came with it all, is beyond infinity. The late nights paid off, let there be more of them.

Contents

Acronyms

NSA Nonstandard analysis

ATD Anthropological theory of didactics

DO Didactical organization

MO Mathematical organization

HMO Hyperreal mathematical organization

DMO Differential mathematical organization

IMO Integral mathematical organization

$\mathbb{R}_+$ The set of strictly negative real numbers, i.e. $\mathbb{R}_+ = \{a \in \mathbb{R} \,|a > 0\}$

$\mathbb{R}_-$ The set of strictly positive real numbers, i.e. $\mathbb{R}_- = \{a \in \mathbb{R} \,|a < 0\}$

$\mathbb{Q}_+$ The set of strictly positive rational numbers, i.e. $\mathbb{Q}_+ = \{a \in \mathbb{R} \,|a > 0\}$

$\mathbb{Q}_-$ The set of strictly negative rational numbers, i.e. $\mathbb{Q}_- = \{a \in \mathbb{R} \,|a < 0\}$

${}^*\mathbb{R}$ The set of hyperreal numbers

$\mathbb{I}$ The set of all infinitesimals

1 Introduction

Going down memory lane to what mathematics was learned in high school can produce headaches for most people. If asking a former student of high school what the hardest part in high school mathematics was, the common answer is infinitesimal calculus. By prodding with a few more questions as to why it was this part which was so difficult, another common answer is that it was abstract, i.e. non-intuitive. As such an interest of imbuing infinitesimal calculus with intuition came to be. The interest was further piqued when discovering that this interest was shared with a(nother) great mathematician, Leibniz. Leibniz' differential calculus seemed so much more intuitive than the one remembered from high school, but he used infinitesimals which did not seem as a rigorous mathematical object. The "project" of imbuing infinitesimal calculus with intuition was set on hold but was resumed when discovering that the infinitesimals were reinstated as a rigorous mathematical object by Robinson in 1966(Robinson, 1996). In the following section the motivation for the thesis is elaborated.

1.1 Motivation

When reading the texts "Didactic restrictions on teachers practice – the case of limits of functions in Spanish high schools" (BarbÉ, Bosch, Espinoza, & GascÓN, 2005) and "Mathematical analysis in high school – A fundamental dilemma (Winsløw, 2013), it becomes clear that teaching analysis in high school contains some dilemmas, one of which is the use of the limit operation. With this in mind, it became very interesting to try to circumvent the use of these limits, which can be done using nonstandard analysis. Nonstandard analysis (NSA) is an approach to mathematical analysis, as introduced by Robinson in 1966 (Robinson, 1996), where infinitely small and large quantities are well-defined, through the use of the axiom of choice. I.e. nonstandard analysis operates with a different set of numbers, called the hyperreal numbers, as its base for mathematical analysis. The set of hyperreal numbers contains both the real numbers and the infinitely small, called infinitesimals, and the infinitely large quantities. Robinson's creation of the hyperreal numbers shows they are consistent if and only if the real numbers are. The conundrum of proofs based on infinitesimals and infinite numbers being sound or not, was finally given a mathematically satisfactory answer. With this in mind, the historical development of the infinitesimal calculus has been scrutinized, to see if some of the proofs done by the great mathematicians in history could be saved. The discoveries, in this field of mathematical history, found few theorems which proofs were saved by the rigorous foundation of the infinitesimals. The most frequent reason for this was that they lacked rigorous definitions.

In 1976 H. Jerome Keisler published a textbook called "Elementary calculus: An Infinitesimal Approach" (Keisler, 2012). This textbook contains, as the title suggests, an approach to elementary calculus, where infinitesimals are used. When reading various texts about NSA, the reason for using this approach can more or less be boiled down to one primary reason: it is a more intuitive way to introduce infinitesimal calculus, than the (standard) limit operation approach. Although this statement has been questioned by a rather big population of mathematicians over the years as seen in ("Criticism of non-standard analysis," 2015), no solid argument can say that it is not more intuitive. On the contrary as seen in (O'Donovan & Kimber, 2006), the fact that the development of the infinitesimal calculus, as done by Leibniz and Newton in 1670-1680, was based on the use of infinitesimals supports the suggestion of infinitesimal calculus being more intuitive, when introduced by infinitesimals. One could ask why infinitesimals disappeared from the mathematical discipline, when it makes analysis more intuitive; the answer to this question is found in the historical development of the notion of quantity. In the introduction to Cauchy's Cours d'Analyse (Bradley

& Sandifer, 2009) from 1821 can be found a general description, of what quantities are. This is one of the first places, where the understanding of quantity is described in greater depth, than what Euclid did almost 2000 years in book five of (Heiberg, Fitzpatrick, & Euclid, 2008). The notion of quantity was scrutinized by various mathematicians Weierstrass, Dedekind and cantor etc. (Gray, 2015) in the nineteenth century, ending with the construction of the real numbers around 1872. Regardless of what construction of the real numbers was used, the construction made the notion of infinitesimals into something, which no longer made sense to use in the discipline of mathematics. This led to the introduction of epsilon delta argumentation and limit operation, which was then to deal with the problems that previously had been solved using infinitesimals.

Even with the construction of the real numbers, the notion of infinitesimals persisted in the school system, for a longer time than most would believe, which can be seen in "The tension between intuitive infinitesimals and formal mathematical analysis" (Katz & Tall, 2011) where one of the introductory passages is:

"Infinitesimal calculus is a dead metaphor. In countless courses of instruction around the globe, students register for courses in "infinitesimal calculus" only to find themselves being trained to perform epsilontic multiple-quantifier logical stunts, or else being told briefly about "the rigorous approach" to limits, promptly followed by instructions not to worry about it"

This illustrates quite well the irony which can appear in mathematics when mathematicians stick to the names from the days of yore while using the rigorous standard approach to analysis. The last part of the conclusion from the same text is:

"Most modern mathematicians now admit the axiom of choice, in the knowledge that it offers theoretical power without introducing contradictions that did not exist before. Is it not time to allow infinitesimal conceptions to be acknowledged in their rightful place, both in our fertile mathematical imagination and in the power of formal mathematics, enriched by the axiom of choice?"

Another passage that motivated the use of NSA is in Richard O'Donovan and John Kimber's "Non standard analysis at pre-university level – naive magnitude analysis –" (O'Donovan & Kimber, 2006)

"There is a heated debate whether non standard analysis should be introduced at pre-university level. It has been demonstrated herein that it is possible, emphasising that in mathematics, simplicity rhymes with beauty"

With the interest in using NSA already piqued, the reasons listed above further motivated the pursuit of a NSA approach to infinitesimal calculus in high school.

On a side note, it is called infinitesimal calculus and not ε-δ argument calculus or limit-operation calculus, so why not use the quantities which gave rise to the name?

1.2 Problematique

In order to teach infinitesimal calculus with NSA in Danish high school (from here on out, high school is assumed Danish), some kind of textbook material in Danish is required by the Ministry of Education. There is no Danish textbook using NSA that rigorously describes this, which poses a problem when teaching infinitesimal calculus in high school, for the first time. Since infinitesimal calculus using NSA has not been taught in high school in a long time, with the exception of a school in Geneva (O'Donovan & Kimber, 2006), there exists little up-to-date previously used teaching material. Since the material found in "Nonstandard analysis at pre-university level: Naive magnitude analysis" is not as rigorous as waned it left two options: Either producing teaching material from the knowledge obtained by studying the works of Robinson (Robinson, 1996), Stroyan (Stroyan, 1997) or Ponstein (Ponstein, 2001) etc. or using a transposition of the teaching material already available for universities e.g. "elementary calculus: an infinitesimal approach" (Keisler, 2012). Acquiring students to teach posed less of a problem than the aftermath of the teaching process, in that the original teacher (the teacher who taught the class before the course on infinitesimal calculus) needed to obtain knowledge of NSA as would an external censor for a possible oral exam.

1.3 Research questions

From the problematique the following research questions were developed.

1. What are the reasons for or against the nonstandard approach, to infinitesimal calculus in high school?
2. How can a nonstandard introduction to infinitesimal calculus in high school be developed, in particular how to create textbook material suited for high school students?
3. What results can be observed from a first experiment, implementing such a material?

The first research question is to be answered by employing a didactical theory in order to generate well founded reasons for or against the nonstandard approach. As such the reasons also depend on the didactical literature which can be found on the subject.

The second research question is to be answered by composing textbook material for the students to be taught.

The third research question is to be answered by a didactical analysis of the course of teaching infinitesimal calculus.

Since NSA is a rather uncommon approach to analysis, the first thing to do was to obtain a sufficient understanding of the NSA, which is presented in section 2.1, in order to both teach and analyze using a didactical theory.

The choice of *anthropological theory of didactics* (ATD) as the didactical theory was based on previous encounters with the theory. A short introduction to this theory can be found in section 2.2.

With NSA and ATD in place, the first research question can be answered in theory see section 2.2.3.

With the hypothesis from section 2.2.3, an answer to the second research question can be found in section 3.

In section 4 a presentation of the observed teaching course can be found which generates an answer for the third research question.

The concluding remarks regarding the research questions and the study in general will be presented in section 5.

The individually written sections are 4.3 Differential calculus and 4.4 Integral calculus. Section 4.3 was written by Jonas Kyhnæb and Section 4.4 was written by Mikkel Mathias Lindahl.

2 Theory

This section includes both the mathematical and didactical theory used in the thesis.

2.1 Mathematical theory – NSA

This introduction to the Hyperreal numbers is heavily influenced by the chapter written by Tom Lindström called "an Invitation to nonstandard analysis" in the book "Nonstandard analysis and its applications" (Cutland, 1988). An intuitive introduction can be found in appendix 3 or in the "Elementary calculus: an infinitesimal approach" (Keisler, 2012).

The hyperreal line is a line of numbers in the same way as the real line, it contains "more" numbers than the real line though. Clearly it cannot abide by the same axioms as the real line, in this case it does not abide by the Archimedean principle, i.e. it contains elements, a and b where $a < b$ such that no finite number n makes $na > b$. Since the hyperreal numbers are going to be constructed in a way which is similar to one of the ways the real numbers can be constructed, a short description of how to construct of the real numbers will appear as the first thing in this introduction.

2.1.1 Real numbers

The real numbers can be constructed by the use of Cauchy sequences. Let the set $\mathbb{Q}_C$ be the set of all the Cauchy sequences of rational numbers. Let N_0 be the set of all null-sequences (the sequences that tend to zero).

2.1.1.1 Definition – Equivalence relation for constructing the real numbers, ≡

Define a equivalence relation $\equiv$ on $\mathbb{Q}_C$ in the following way:

Let $\{a_n\}, \{b_n\} \in \mathbb{Q}_C$ then $\{a_n\} \equiv \{b_n\}$ if the term wise difference$\{(a_n - b_n)\} \in N_0$.

The real numbers can then be defined as the Cauchy-sequences with rational numbers under the above defined equivalence relation.

$$\mathbb{R} \coloneqq \mathbb{Q}_C/\equiv$$

The additive and multiplicative structure of $\mathbb{R}$ is defined through term wise addition and multiplication, i.e. if $\langle a_n \rangle$ and $\langle b_n \rangle$ are the equivalence classes of $\{a_n\}, \{b_n\} \in \mathbb{Q}_C$ then

$$\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle \qquad \text{and} \qquad \langle a_n \rangle \cdot \langle b_n \rangle = \langle a_n \cdot b_n \rangle.$$

The order relation is defined as; $\langle a_n \rangle < \langle b_n \rangle$ if and only if, for some $\varepsilon \in \mathbb{Q}_+$ and sufficiently large N, $a_n < b_n - \varepsilon$ for all $n \geq N$.

The rational numbers are seen as a subset of the real numbers through the inclusion map $\phi: \mathbb{Q} \to \mathbb{R}$ which is defined as taking a rational element a to $\phi(a) = \langle a, a, \dots \rangle$.

2.1.2 The hyperreal numbers

This section will present a construction of the hyperreal numbers.

To define the hyperreal numbers the same procedure, as when construction the real numbers, will be used but with a different starting set and a different equivalence relation.

Let $\mathcal{R}$ be the set of all sequences of real numbers, then in order to introduce the hyperreal numbers, a specific subset of the null-sequences in $\mathcal{R}$ is needed to define the equivalence relation. When constructing the real numbers a coarse equivalence relation is used, since there is no distinguishing between two sequences that only differs by a sequence tending to zero. For example the sequences $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\left\{\frac{1}{n^2}\right\}_{n\in\mathbb{N}}$ both correspond to the real number zero. When introducing the hyperreal numbers an equivalence relation that makes a difference between the two sequences is needed. If the finest equivalence relation is used, as in "every sequence of real numbers defines a hyperreal number", then the constructed set, will include zero-divisors, since

$$\{0,1,0,1,0,\dots\}\cdot\{1,0,1,0,1,\dots\}=\{0,0,0,\dots\}=0,$$

where both the sequences in the product are nonzero. This is unwanted, and therefore the finest equivalence relation $\sim$ such that $\mathcal{R}/\sim$ has no zero divisors is the preferable one. To define such an equivalence relation $\sim$, a free ultrafilter is needed. In order to explain what this is consider the definition of a filter and its specific classification of a free ultrafilter seen below.

2.1.2.1 Definition – Filter

A filter over a set I is a collection of subsets $\mathbb{F}$ of I such that $\emptyset \notin \mathbb{F}$ and

- If $A, B \in \mathbb{F}$, then $A \cap B \in \mathbb{F}$.
- If $A \in \mathbb{F}$ and $A \subset B$, then $B \in \mathbb{F}$.

A free filter is a filter $\mathbb{F}$ of an infinite set I where the set I/A is finite for all $A \in \mathbb{F}$, i.e. none of the elements $A \in \mathbb{F}$ are finite.

An ultrafilter $\mathbb{U}$ of I is a filter where, for all $A \subset I$, either $A \in \mathbb{U}$ or $A^C \in \mathbb{U}$, but not both.

Thereby a free ultrafilter is a filter $\mathbb{U}$ of a set I where both the above statements are true. Thus if $\mathbb{U}$ is a free ultrafilter then every element $A \in \mathbb{U}$ is a subset $A \subset I$ ($\mathbb{U}$ is a filter), the set I/A is finite for all $A \in \mathbb{U}$, even though I is an infinite set ($\mathbb{U}$ is a free filter), and for all subsets $A \subset I$ either $A \in \mathbb{U}$ or $A^C \in \mathbb{U}$ but not both ($\mathbb{U}$ is an ultrafilter).

2.1.2.2 Proposition – Extending a free filter to a free ultrafilter

Any free filter $\mathbb{F}$ over I can be extended to a free ultrafilter, when I is an infinite set.

Proof. Let $\mathbb{J}$ be the set of free filters over I, hence $\mathbb{F} \in \mathbb{J}$. Order $\mathbb{J}$ by inclusion, then by Zorn's Lemma there exists a maximal element in $\mathbb{J}$, denote it $\mathbb{U}$. If $E \subset I$ then either $F \cap E$ or $F \cap E^C$ is infinite for all $F \in \mathbb{U}$. If not then there exists $F_1, F_2 \in \mathbb{U}$ such that both $F_1 \cap E$ and $F_2 \cap E^C$ are finite. Assume for contradiction the latter, and consider that $F_1 \subseteq E^C \cup (F_1 \cap E)$ and $F_2 \subseteq E \cup (F_2 \cap E^C)$, and

$$F_1 \cap F_2 \subseteq \left(E^C \cup (F_1 \cap E)\right) \cap \left(E \cup (F_2 \cap E^C)\right) = (F_1 \cap E) \cup (F_2 \cap E^C).$$

Since the union of two finite sets is finite, then $F_1 \cap F_2 \in \mathbb{U}$ is finite which contradicts the free part. If $F \cap E$ is infinite for all $F \in \mathbb{U}$, then $E \in \mathbb{U}$, otherwise $\mathbb{U}$ is not an maximal element, the same argument can be used if $F \cap E^C$ infinite for all $F \in \mathbb{U}$, making either E or E^C an element of $\mathbb{U}$ but not both, hence $\mathbb{U}$ is an ultrafilter.▪

To make an ultrafilter on the natural numbers $\mathbb{N}$, take the Fréchet filter $\mathcal{F}$ (which is a free filter)on $\mathbb{N}$.

2.1.2.3 Definition – Fréchet filter, $\mathcal{F}$

Denote the Fréchet filter $\mathcal{F}$ and define it as.

$$\mathcal{F} = \{A \subset \mathbb{N} | A^C \text{ finite}\}$$

and extend $\mathcal{F}$ with proposition2.1.2.2 to a free ultrafilter $\mathbb{M}$.

2.1.2.4 Definition – The finitely additive measure, m

Let m be the finitely additive measure on $\mathbb{N}$ defined by,

$$m(A) = \begin{cases} 1 & \text{if } A \in \mathbb{M} \\ 0 & \text{otherwise} \end{cases}.$$

Now since $\emptyset \notin \mathbb{M}$ then it is clear that $\mathbb{N} \in \mathbb{M}$, hence $m(\mathbb{N}) = 1$.

2.1.2.5 Definition – Equivalence relation to construct the hyperreal numbers, ~

Let $\sim$ be the equivalence relation on the set of all real valued sequences $\mathcal{R}$ defined by

$$\{a_n\} \sim \{b_n\} \qquad \text{if and only if } m(\{n \mid a_n = b_n\}) = 1.$$

The hyperreal numbers can then be defined as

2.1.2.6 Definition – The hyperreal numbers, $^*\mathbb{R}$

Define $^*\mathbb{R}$ as

$$^*\mathbb{R} \coloneqq \mathcal{R}/\sim.$$

The additive and multiplicative structure of $^*\mathbb{R}$ is defined through component wise addition and multiplication, as when defining them for the real numbers. I.e. if $\langle a_n \rangle$ and $\langle b_n \rangle$ are the equivalence classes of $\{a_n\}, \{b_n\} \in \mathcal{R}$ then

$$\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle \qquad \text{and} \qquad \langle a_n \rangle \cdot \langle b_n \rangle = \langle a_n \cdot b_n \rangle.$$

The order relation is defined as; $\langle a_n \rangle < \langle b_n \rangle$ if and only if $m(\{n \mid a_n < b_n\}) = 1$.

If another representative $\{a'_n\}_{n \in \mathbb{N}}$ for the equivalence class $\langle a_n \rangle$ is used then $m(\{n \mid a_n = a'_n\}) = 1$, hence

$$m(\{n \mid a'_n < b_n\}) = m(\{n \mid a_n < b_n\} \cap \{n \mid a_n = a'_n\}) = 1,$$

since both $\{n \mid a_n < b_n\}$ and $\{n \mid a_n = a'_n\}$ are in $\mathbb{M}$. Thus the order relation is independent of the choice of representatives of the equivalence classes. In order to check that there are no zero divisors in $^*\mathbb{R}$, assume that

$$\langle a_n\rangle \cdot \langle b_n\rangle = \langle 0,0,\dots\rangle, \text{ i.e. } m(\{n \mid a_n \cdot b_n = 0\}) = 1,$$

hence

$$m(\{n \mid a_n \cdot b_n = 0\}) = m(\{n \mid a_n = 0\} \cup \{n \mid b_n = 0\}) = m(\{n \mid a_n = 0\}) + m(\{n \mid b_n = 0\}) = 1.$$

Then either $\langle a_n\rangle = \langle 0,0,\dots\rangle$ or $\langle b_n\rangle = \langle 0,0,\dots\rangle$.

The hyper real numbers ${}^*\mathbb{R}$ are then an ordered field with multiplicative and additive identity, $1 = \langle 1,1,\dots\rangle$ and $0 = \langle 0,0,\dots\rangle$ respectively.

2.1.2.7 *Example – Use of order relation, <*

Let $a, b, c \in {}^*\mathbb{R}$ such that $a > 0$ and $b < c$. How to prove $ab < ac$?

If $a = \langle a_n\rangle,\ b = \langle b_n\rangle$ and $c = \langle c_n\rangle$, from the definition of the order relation, then $m(\{n \mid 0 < a_n\}) = 1$ and $m(\{n \mid b_n < c_n\}) = 1$, hence $m(\{n \mid a_n \cdot b_n < a_n \cdot c_n\}) = m(\{n \mid 0 < a_n\} \cap \{n \mid b_n < c_n\}) = 1$. Proving that $ab < ac$ as wanted.

As an analogue to the map $\phi: \mathbb{Q} \to \mathbb{R}$, there is an injective order preserving map $\psi: \mathbb{R} \to {}^*\mathbb{R}$, taking a real number, a, to a sequence of the same number $a \to \langle a, a, \dots\rangle$. Thus the real numbers form a subset of the hyperreal numbers, $\mathbb{R} \subset {}^*\mathbb{R}$.

To identify the numbers, which are not real numbers, consider the next definition as a way of describing different kinds of hyperreal numbers.

2.1.2.8 *Definition – Infinitesimal, finite and infinite*

1. A hyperreal number $\beta \in {}^*\mathbb{R}$ is an infinitesimal if $-a < \beta < a$ for all $a \in \mathbb{R}_+$.
2. A hyperreal number $\beta \in {}^*\mathbb{R}$ is finite if $-a < \beta < a$ for some $a \in \mathbb{R}_+$.
3. A hyperreal number $\beta \in {}^*\mathbb{R}$ is infinite if no $a \in \mathbb{R}_+$ exists such that $-a < \beta < a$.

2.1.2.9 *Examples – Operations with infinitesimals and infinite numbers*

Describe the hyperreal numbers, $\delta_1, \delta_2, \Omega_1$ and Ω_2 as shown below:

$$\delta_1 = \langle \tfrac{1}{n}\rangle \qquad \delta_2 = -\delta_1^2 = \langle \tfrac{-1}{n^2}\rangle \qquad \Omega_1 = \langle n\rangle \qquad \Omega_2 = -\sqrt{\Omega_1} = \langle -\sqrt{n}\rangle$$

For δ_1 and δ_2 every positive real number $a \in \mathbb{R}$ will make $m(\{n \mid -a < \frac{1}{n} < a\}) = 1$ and

$m(\{n \mid -a < \frac{-1}{n^2} < a\}) = 1$, hence both δ_1 and δ_2 are infinitesimals. It is clear that 0 is the only real infinitesimal.

For Ω_1 and Ω_2 then every real number $a \in \mathbb{R}$ will make $\{n \in \mathbb{N} \mid -a < n < a\}$ and $\{n \in \mathbb{N} \mid -a < -\sqrt{n} < a\}$ finite sets, making Ω_1 and Ω_2 two infinite numbers.

To understand the arithmetic on the infinitesimals consider first the infinitesimals, δ and δ', then the sum of these should again be an infinitesimal, i.e. $\forall\ a \in \mathbb{R}_+$

$$-a < \delta + \delta' < a.$$

Since δ and δ' are both infinitesimal then

$$-a'' < \delta < a'' \quad \text{and} \quad -a' < \delta' < a',$$

for all $a', a'' \in \mathbb{R}_+$.

For every $a \in \mathbb{R}_+$ let $a' = a'' = a/2$ then the sum; $\delta + \delta'$ has the property

$$-a' - a'' < \delta + \delta' < a' + a''$$

thus

$$-a < \delta + \delta' < a.$$

Consider the product of a finite number β and an infinitesimal δ, $\beta \cdot \delta$. Let $a_\beta \in \mathbb{R}_+$ be such that $-a_\beta < \beta < a_\beta$ then

$$-aa_\beta < -a_\beta\delta < \beta\delta < a_\beta\delta < a_\beta a$$

for all $a \in \mathbb{R}_+$, i.e. this is again an infinitesimal.

Moreover if δ is an infinitesimal, then δ^{-1} is an infinite number and vice versa. Furthermore any infinite number Ω multiplied by a number β which is not infinitesimal, will again be an infinite number.

2.1.2.10 Definition – The set of infinitesimals, $\mathbb{I}$
Let $\mathbb{I} \subset {}^*\mathbb{R}$, denote the set of all infinitesimals in ${}^*\mathbb{R}$.

It is clear to see that the set $\mathbb{I}$ is an ideal for the finite hyper real numbers, since any finite number multiplied by an infinitesimal is again an infinitesimal.

The finite numbers in ${}^*\mathbb{R}$ behave in a most appreciative way, namely

2.1.2.11 Proposition – Finite hyperreal numbers is a sum of a real number and an infinitesimal
Any finite number $\beta \in {}^*\mathbb{R}$, can be written as a uniquely determined sum, $\beta = a + \delta$, where $a \in \mathbb{R}$ and $\delta \in \mathbb{I}$.

Proof: Uniqueness; if $\beta = a_1 + \delta_1 = a_2 + \delta_2$, then $a_1 - a_2 = \delta_2 - \delta_1$, now since the right side of the equation is an infinitesimal and the left side is a real number it must be a real infinitesimal, hence $0 = a_1 - a_2 = \delta_2 - \delta_1$, asserting the uniqueness. Existence; Let $a = \sup\{b \in \mathbb{R} \,|b < \beta\}$, if $(\beta - a) \in \mathbb{I}$ then the proof is done. Assume $(\beta - a) \notin \mathbb{I}$, then there exists some number, $a_1 \in \mathbb{R}$, such that $0 < a_1 < |\beta - a|$, contradicting the choice of a.▪

2.1.2.12 Definition – Infinitesimal difference, $\approx$
For $\alpha, \beta \in {}^*\mathbb{R}$ let $\alpha \approx \beta$ if $(\alpha - \beta) \in \mathbb{I}$.

2.1.3 Internal sets and functions, and the connection between $\mathbb{R}$ and ${}^*\mathbb{R}$.
In order to use the hyperreal numbers to analyze real functions, a connection between functions and sets in $\mathbb{R}$ and ${}^*\mathbb{R}$ is needed, the following definition will help establish that connection.

2.1.3.1 Definition – Standard part and monad

For every finite $\beta \in {}^*\mathbb{R}$, let the unique real number $a \approx \beta$, be the standard part of beta and denote it $st(\beta)$. Furthermore denote the set $\{\beta \in {}^*\mathbb{R} \mid st(\beta) = a\}$ the monad of a.

If A is a subset of ${}^*\mathbb{R}$, then the standard part of A, is defined by

$$st(\mathrm{A}) \coloneqq \{st(\alpha) \mid \alpha \in \mathrm{A}\}$$

2.1.3.2 Example – Standard part of a hyperreal number

- Let $\langle b_n \rangle = \beta \in {}^*\mathbb{R}$ and $\langle a_n \rangle = \alpha \in {}^*\mathbb{R}$ be a finite number and let $a = \inf(\{a \in \mathbb{R}_+ \mid -a < \beta < a\})$. If $a = 0$ then $\beta \in \mathbb{I}$, making $st(\beta) = 0$. If $a \neq 0$, then either $\beta - a$ or $\beta + a$ is infinitesimal, if $(\beta - a) \in \mathbb{I}$, then
$$st(\beta) = st(\langle b_n \rangle) = st(\langle a + (b_n - a) \rangle) = st(\langle a \rangle + \langle (b_n - a) \rangle) = a$$
and if $(\beta + a) \in \mathbb{I}$ then
$$st(\beta) = st(\langle b_n \rangle) = st(\langle -a + (b_n + a) \rangle) = st(\langle -a \rangle + \langle (b_n - a) \rangle) = -a$$

2.1.3.3 Proposition – Rules for the standard part

- If α and β two finite hyperreal numbers such that $st(\alpha) = a$ and $st(\beta) = a$, then $st(\alpha + \beta) = a + b$, since the sum of two infinitesimals is again an infinitesimal. I.e. $st(\alpha + \beta) = st(\alpha) + st(\beta)$ if α and β are both finite.
- If α and β two finite hyperreal numbers such that $st(\alpha) = a$ and $st(\beta) = b$, then $st(\alpha\beta) = ab = st(\alpha)st(\beta)$. I.e. $st(\alpha\beta) = st(\alpha)st(\beta)$ if α and β are both finite.

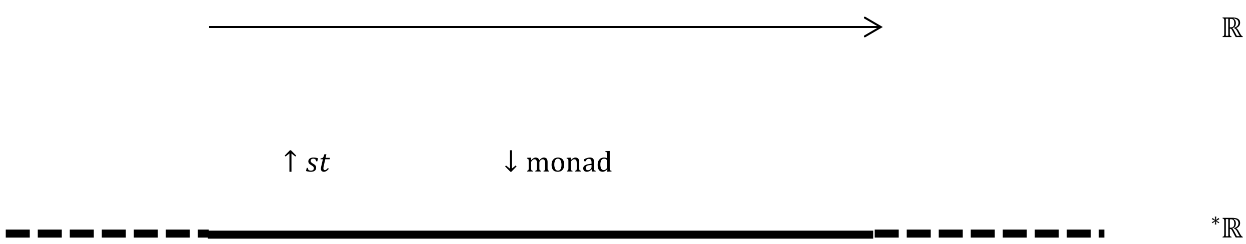
2.1.3.4 Definition – Star operation, *

- For every subset $A \subset \mathbb{R}$ denote ${}^*A = \langle A, A, \dots \rangle$ the nonstandard version of A.
- For every function $f \colon \mathbb{R} \to \mathbb{R}$ denote ${}^*f = \langle f, f, \dots \rangle$ the nonstandard version of f.

2.1.3.5 Examples – Star operation

- The interval $(a, b) \in \mathbb{R}$ will be extended to the interval ${}^*(a, b) = \langle (a, b), (a, b), \dots \rangle = \{\beta \in {}^*\mathbb{R} \mid a < \beta < b\}$.
- Consider now the standard part of ${}^*(a, b) \subset {}^*\mathbb{R}$, since there exists some $\delta \in \mathbb{I}$, such that $\alpha = a + \delta$ and $\beta = b - \delta$ are both in the mentioned set, then
$$st({}^*(a, b)) = [a, b].$$

Thus it is possible to go back and forward between the two number lines but the inequalities when going from the hyperreal line to the real line, will go from strict to non-strict $(st(<) = \leq, st(>) = \geq)$.



The relationship between the real and hyperreal numbers can be described by the figure above, the thickness of the hyperreal line is a way of showing the infinitesimals around every real number, and the dotted lines are where the infinite numbers are situated.

2.1.3.6 Definition – Internal sets and internal functions

1. A sequence $\{A_n\}_{n\in\mathbb{N}}$, where $A_n \subset \mathbb{R}$ for all n, defines a subset $\langle A_n \rangle \subset {}^*\mathbb{R}$ by
$$\langle \alpha_n \rangle \in \langle A_n \rangle \text{ if and only if } m(\{n \mid \alpha_n \in A_n\}) = 1.$$
A subset of ${}^*\mathbb{R}$ which can be obtained in this way is called internal.
2. A sequence of functions $\{f_n\}$ where $f_n: \mathbb{R} \to \mathbb{R}$ for all n, defines a function $\langle f_n \rangle: {}^*\mathbb{R} \to {}^*\mathbb{R}$ by
$$\langle f_n \rangle(\langle x_n \rangle) = \langle f_n(x_n) \rangle.$$
Any function obtained in this way is called internal.

Two internal sets $\langle A_n \rangle, \langle B_n \rangle$ or functions $\langle f_n \rangle, \langle g_n \rangle$ are equal if $m(\{n \mid f_n = g_n\}) = m(\{n \mid A_n = B_n\}) = 1$.

2.1.3.7 Examples – Internal set and internal function

- If $\alpha = \langle a_n \rangle$ and $\beta = \langle b_n \rangle$ are two elements in ${}^*\mathbb{R}$, then the interval $[\alpha, \beta] = \{x \mid \alpha \leq x \leq \beta\}$, is an internal set since, it is obtained as $\langle [a_n, b_n] \rangle$.
- If $\zeta = \langle c_n \rangle$ is an element of ${}^*\mathbb{R}$, then the function $e^{\zeta X}$ is an internal function defined by $e^{\zeta x} = \langle e^{c_n x_n} \rangle$.

With this definition of internal sets and functions many of the results and principles known about the real numbers $\mathbb{R}$ can be carried over to the internal sets. One of the more interesting things is the least upper bound of the completeness axiom.

2.1.3.8 Proposition – Least upper bound for internal sets

An internal nonempty set, $\mathrm{A} = \langle A_n \rangle \subset {}^*\mathbb{R}$, which is bounded above has a least upper bound.

Proof: If A is bounded above by $\alpha = \langle a_n \rangle$, then $m(\{n \mid \sup A_n \leq a_n\}) = 1$, hence the set of unbounded A_n's has measure zero. Without loss of generality the A_n's are all bounded above, hence $\beta = \langle \sup A_n \rangle$ is the least upper bound of A.▪

2.1.3.9 Corollary – Overspill and underspill

Let A be an internal subset of ${}^*\mathbb{R}$.

1. If A contains arbitrarily large finite elements, then A contains an infinite element.
2. If A contains arbitrarily small positive infinite elements, then A contains a finite element.

Proof:

1. If there is a least upper bound α of A, α must be infinite and there must be a $\chi \in \mathrm{A}$ such that $\frac{\alpha}{2} \leq \chi \leq \alpha$, i.e. A contains an infinite element. ▪
2. If β is the greatest lower bound of A_+, the set of positive elements in A, then β must be finite and there must exist a $\chi \in \mathrm{A}$ such that $\beta \leq \chi \leq 2\beta$.▪

2.1.3.10 Proposition – Standard part of an internal set is closed

If $\mathrm{A} \in {}^*\mathbb{R}$ is internal then $st(\mathrm{A})$ is a closed set in $\mathbb{R}$.

Proof: Let $a \in \overline{st(\mathrm{A})}$ and consider the sequence of internal sets

$$\mathrm{A}_n = \mathrm{A} \cap \left\{\beta \in {}^*\mathbb{R} \mid |\beta - a| < \frac{1}{n}\right\}.$$

Since $a \in \mathrm{A}_n$ for all n, then $\bigcap_{n\in\mathbb{N}} \mathrm{A}_n \neq \emptyset$. Pick $\zeta \in \bigcap_{n\in\mathbb{N}} \mathrm{A}_n$, then $\zeta \in \mathrm{A}$. The definition of the sequence ensures that $\zeta \approx a$, i.e. $a \in st(\mathrm{A})$.▪

With this in mind, consider the following proposition

2.1.3.11 Proposition – Inclusion or equality when using star operation

For all $A \in \mathbb{R}$, then $A \subseteq {}^*A$ with $A = {}^*A$ if and only if A has only finitely many elements.

Proof: For the inclusion, let $a \in A$, then $a = \langle a, a, \dots \rangle \in \langle A, A, \dots \rangle = {}^*A$, hence $A \subseteq {}^*A$. Now assume A is infinite, then show that there exists an element $\alpha \in {}^*A$ such that $\alpha \notin A$. Construct a sequence of distinct elements from A, $\{a_1, a_2, a_3, \dots\}$, then $\alpha = \langle a_1, a_2, a_3, \dots \rangle \in {}^*A$, but $\alpha \notin A$. Now assume A is finite, i.e. $A = \{a_1, a_2, a_3, \dots, a_k\}$, by the finite additive measure m, then for any sequence $\{b_n\}_{n\in\mathbb{N}}$, where $b_n \in A$ for all n,

$$1 = m(\{n \mid b_n \in A\}) = m(\{n \mid b_n = a_1) + m(\{n \mid b_n = a_2) + \cdots + m(\{n \mid b_n = a_k).$$

This makes exactly one of the measures on the right equal one, hence $\langle b_1, b_2, b_3 \dots \rangle = \langle a_i, a_i, a_i, \dots \rangle = a_i \in A$.▪

Considering a function $f: \mathbb{R} \to \mathbb{R}$, the nonstandard version of f, ${}^*f = \langle f, f, \dots \rangle$, is an extension of the original function, by the fact that for any $a \in \mathbb{R}$, then

$${}^*f(a) = \langle f, f, \dots \rangle(\langle a, a, \dots \rangle) = \langle f(a), f(a), \dots \rangle = f(a).$$

I.e. if the domain of the function f, is not finite, then the domain of the function is a larger set, making ${}^*f$ a proper extension.

With the mathematics described above it is possible to extend some of the generally used subsets of the real numbers to the nonstandard counterparts, ${}^*\mathbb{N}, {}^*\mathbb{Z}, {}^*\mathbb{Q}$. One of the more interesting ones of these are the nonstandard integers, ${}^*\mathbb{N}$.

The set of nonstandard integers ${}^*\mathbb{N}$, includes what is called infinite integers, i.e. the number $\langle 1,2,3, \dots \rangle \in {}^*\mathbb{N}$. These infinite integers abide by the same rules as the normal integers, so if $\alpha = \langle a_1, a_2, \dots \rangle$ and $\beta = \langle b_1, b_2, \dots \rangle$ are two infinite integers, then $\frac{\alpha}{\beta} \in {}^*\mathbb{N}$, if an only if $m(\{n \mid \frac{a_n}{b_n} \in \mathbb{N}\}) = 1$.

In order to define and use some of the more interesting parts of nonstandard analysis, another new notion, namely a hyperfinite set is needed. These are sets of infinite order, but with a combinatorial structure like the finite sets, which it inherits from the nonstandard integers.

2.1.3.12 Definition – Hyperfinite set

An internal set $\mathrm{A} = \langle A_n \rangle \subset {}^*\mathbb{R}$ is called hyperfinite if $m(\{n \mid A_n \text{ is a finite set}\}) = 1$. In this case, the cardinality of the set A will be the infinite integer $|\mathrm{A}| = \langle |A_n| \rangle$, where $|\mathrm{A_n}|$ is the number of elements in $\mathrm{A_n}$.

Since (almost) all the A_n's are finite, all the combinatorics for finite sets can be used on hyperfinite sets, this property enables the use of induction over hyperfinite sets, just as with finite sets.

2.1.3.13 Example – Cardinality of a hyperfinite set

Let $\mathrm{N} \in {}^*\mathbb{N}$, then the set

$$\mathrm{T} = \{0, \frac{1}{\mathrm{N}}, \frac{2}{\mathrm{N}}, \frac{3}{\mathrm{N}}, \dots, \frac{\mathrm{N}-1}{\mathrm{N}}, 1\}$$

is hyperfinite, with cardinality $\mathrm{N}+1$. If $\mathrm{N} \in \mathbb{N}$, then it is a finite set, and it is known that $|\mathrm{T}| = \mathrm{N}+1$. In order to convince oneself of this being true when N is an infinite integer consider, for $\mathrm{N} = \langle N_n \rangle$ and $\mathrm{T} = \langle T_n \rangle$ where

$$T_n = \left\{0, \frac{1}{N_n}, \frac{2}{N_n}, \dots, \frac{N_n - 1}{N_n}, 1\right\},$$

that $|\mathrm{T}| = \langle |T_n| \rangle = \langle N_n + 1 \rangle = \mathrm{N} + 1$.

With the above definition of a hyperfinite set another thing is able to be constructed, a hyperfinite sum, which is also an infinite sum.

2.1.3.14 Definition – Hyperfinite sum

Consider the hyperfinite set $\mathrm{A} = \langle A_n \rangle \subset {}^*\mathbb{R}$, then the sum of all elements in A can be written as

$$\sum_{a \in A} a = \langle \sum_{a_n \in A_n} a_n \rangle.$$

This sum though infinite is subject to the rules established for finite sums, thus making operating with these particular infinite sums much more appreciable than the infinite sum over other infinite sets. This can be seen by the fact that it is a sequence of finite sets. As such the hyperfinite sum allows the use of induction on an infinite set, but in a broader sense than the regular infinite induction done on a countable set.

2.1.4 Analysis using the hyperreal numbers

With these notions in mind the definitions of continuity, differentiability and integration for functions defined on the real line can now be made without the use of ε-δ's, in each case these new definitions, will be proved to be equivalent statements to the standard way of defining them, by limit operation.

2.1.4.1 Definition – Continuity

- The standard definition, $f: \mathbb{R} \to \mathbb{R}$ is continuous in a point x in the domain of f, if
$$\forall\, \varepsilon \in \mathbb{R}_+\ \exists \delta \in \mathbb{R}_+ : |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$
- And the nonstandard definition, $f: \mathbb{R} \to \mathbb{R}$ is continuous in a point x in the domain of f, if
$$\forall\, y \in {}^*\mathbb{R} : y \approx x \Rightarrow {}^*f(y) \approx {}^*f(x) = f(x).$$

Proof: Assume (standard) continuous at x; let $\langle a_n \rangle = \alpha \approx x$. If $|{}^*f(x) - {}^*f(\alpha)| < \varepsilon$ for all $\epsilon \in \mathbb{R}_+$, then $|{}^*f(x) - {}^*f(\alpha)| \in \mathbb{I}$, by definition of an infinitesimal. Now for any $\epsilon \in \mathbb{R}_+$ pick $\delta \in \mathbb{R}_+$ such that
$$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$
Since, $|x - \alpha| < \delta$ for all $\delta \in \mathbb{R}_+$, $|{}^*f(x) - {}^*f(\alpha)| < \varepsilon$ for all $\varepsilon \in \mathbb{R}_+$. Equivalently consider for any $\epsilon \in \mathbb{R}_+$ pick $\delta \in \mathbb{R}_+$ such that
$$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,$$
then
$$\{n \mid |x - a_n| < \delta\} \subseteq \{n \mid |f(x) - f(a_n)| < \epsilon\},$$
which makes
$$m(\{n \mid |x - a_n| < \delta\}) = m(\{n \mid |f(x) - f(a_n)| < \epsilon\}) = 1.$$
Since the measures are both one, then for any $\langle a_n \rangle = \alpha \approx x$ then $f(x) \approx f(\alpha)$, as wanted.
Now assume f is not continuous at x, then there exists an $\varepsilon \in \mathbb{R}_+$ and $\langle a_n \rangle = \alpha \approx x$, with $a_n \neq x$ for all n, such that $|f(x) - f(a_n)| > \varepsilon$ for all n, which makes $|{}^*f(x) - {}^*f(\alpha)| > \varepsilon$, contradicting the nonstandard definition of continuity.▪

2.1.4.2 Definition – Differential quotient

- The standard way: A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x \in \mathbb{R}$ if and only if,
$$\lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$
exists and is the same both when 0 is a maximum and a minimum for dx. In this case
$$\lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} = f'(x).$$
- The nonstandard way: A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x \in \mathbb{R}$ if and only if, for every nonzero infinitesimal dx,
$$st\left(\frac{{}^*f(x+dx) - {}^*f(x)}{dx}\right) = b$$
exists and is equal to the same value $b \in \mathbb{R}$. In this case $st\left(\frac{{}^*f(x+dx) - {}^*f(x)}{dx}\right) = f'(x) = b$.

Proof: Assume standard differentiability, then

$$\lim_{dx\to 0}\left(\frac{f(x+dx)-f(x)}{dx}\right)=\lim_{y\to x}\left(\frac{f(y)-f(x)}{y-x}\right)=f'(x)$$

By definition of the limit

$$\forall\, \varepsilon \in \mathbb{R}_+\ \exists\delta \in \mathbb{R}_+ : 0<|x-y|<\delta \Rightarrow \left|\frac{f(y)-f(x)}{y-x}-f'(x)\right|<\varepsilon.$$

By the same argument as for continuity and for $0 \neq x-y=dx \in \mathbb{I}$, then

$$\left(\frac{{}^*f(y)-{}^*f(x)}{y-x}-f'(x)\right)\in \mathbb{I}$$

i.e. $st\left(\frac{{}^*f(y)-{}^*f(x)}{y-x}\right)=st\left(\frac{{}^*f(x+dx)-{}^*f(x)}{dx}\right)=f'(x)$.

Now assume it is not differential at a point x, then the limit does not exist.

Assume the limit does not exist, then for all $b \in \mathbb{R}$

$$\exists\, \varepsilon \in \mathbb{R}_+\ \forall\, \delta \in \mathbb{R}_+\ \exists\, y \in \mathbb{R} : 0<|x-y|<\delta \Rightarrow \left|\frac{f(y)-f(x)}{y-x}-b\right|\geq\varepsilon$$

Now assume that $st\left(\frac{{}^*f(x+dx)-{}^*f(x)}{dx}\right)=b\in\mathbb{R}$ for some infinitesimal $dx=\langle\Delta x_n\rangle$, then

$$\frac{{}^*f(x+dx)-{}^*f(x)}{dx}=\frac{\langle f(x+\Delta x_n)\rangle-\langle f(x)\rangle}{\langle\Delta x_n\rangle}=\langle\frac{f(x+\Delta x_n)-f(x)}{\Delta x_n}\rangle.$$

Consider now $\frac{f(x+\Delta x_n)-f(x)}{\Delta x_n}$ and pick y_n such that $|x-y_n|<\ \Delta x_n \Rightarrow \left|\frac{f(y_n)-f(x)}{y_n-x}-b\right|\geq\varepsilon$ for each $n\in\mathbb{N}$. Let $dy=\langle\Delta y_n\rangle=\langle x-y_n\rangle$, then $0\neq dy\in\mathbb{I}$ since $dy<dx$ and $0<\Delta y_n$ for all $n\in\mathbb{N}$. Thus

$$st\left(\frac{{}^*f(x+dy)-{}^*f(x)}{dy}-b\right)=st\left(\frac{{}^*f(x+dy)-{}^*f(x)}{dy}\right)-b\geq\varepsilon$$

i.e. $st\left(\frac{{}^*f(x+dx)-{}^*f(x)}{dx}\right)\neq st\left(\frac{{}^*f(x+dy)-{}^*f(x)}{dy}\right)$, contradicting the nonstandard definition.

2.1.4.2.1 Corollary – A differentiable function is continuous

If $f:\mathbb{R}\to\mathbb{R}$ is differentiable in $x\in\mathbb{R}$, then f is continuous in x.

Proof: If f is differentiable then $\frac{{}^*f(x+dx)-f(x)}{dx}=\beta$ is a finite number, since otherwise the standard part does not exist but then ${}^*f(x+dx)-f(x)=\beta dx$ is an infinitesimal, making $f(x+dx)\approx f(x)$, i.e. f is continuous in x.▪

2.1.4.2.2 Rules of differentiation

Let $f(x)$ and $g(x)$ be differentiable functions, then the following rules apply.

1. For $h(x)=f(x)+g(x)$ then $h'(x)=f'(x)+g'(x)$.
2. For $h(x)=f(x)g(x)$ then $h'(x)=f(x)g'(x)+g(x)f'(x)$
3. For $h(x)=f(g(x))$ then $h'(x)=f'(g(x))g'(x)$

Proof:

1. $h'(x) = st\left(\frac{^*h(x+dx)-h(x)}{dx}\right) = st\left(\frac{^*f(x+dx)-f(x)+^*g(x+dx)-g(x)}{dx}\right) = st\left(\frac{^*f(x+dx)-f(x)}{dx}\right) + st\left(\frac{^*g(x+dx)-g(x)}{dx}\right) = f'(x) + g'(x)$, since a finite sum of standard parts are equal to the standard part of the sum, when the arguments are both finite, which they are because they are differentiable.
2. $h'(x) = st\left(\frac{^*h(x+dx)-h(x)}{dx}\right) = st\left(\frac{^*f(x+dx)^*g(x+dx)-^*f(x)^*g(x)}{dx}\right) = st\left(\frac{(^*f(x)+^*f(x+dx)-f(x))(^*g(x)+^*g(x+dx)-g(x))-^*f(x)^*g(x)}{dx}\right)$
$= st\left(\frac{^*f(x)(^*g(x+dx)-g(x))+^*g(x+dx)(^*f(x+dx)-f(x))}{dx}\right) = st\left(\frac{^*f(x)(^*g(x+dx)-g(x))}{dx}\right) + st\left(\frac{^*g(x+dx)(^*f(x+dx)-f(x))}{dx}\right) = f(x)g'(x) + st\left(^*g(x+dx)\right)f'(x) = f(x)g'(x) + g(x)f'(x)$. here the same thing as in the first prove are employed together with the rule of the standard part of a product is equal to the product of the standard part when the arguments are both finite. In the end since g is differentiable it is also continuous, which makes $st\left(^*g(x+dx)\right) = g(x)$.
3. Firstly, if $^*g(x) = {^*g}(x+dx)$, then $st\left(\frac{^*h(x+dx)-h(x)}{dx}\right) = 0 = g'(x)f'\left(g(x)\right)$.
Secondly, if $^*g(x) \neq {^*g}(x+dx)$, let $^*g(x+dx) - g(x) = dy$, then
$$h'(x) = st\left(\frac{^*h(x+dx)-h(x)}{dx}\right) = st\left(\frac{^*f\left(^*g(x+dx)\right) - f\left(g(x)\right)}{dx}\right)$$
$$= st\left(\frac{^*f\left(g(x) + {^*g}(x+dx) - g(x)\right) - {^*f}\left(^*g(x)\right)}{dy} \cdot \frac{^*g(x+dx) - g(x)}{dx}\right)$$
$$= st\left(\frac{^*f(g(x)+dy) - {^*f}\left(g(x)\right)}{dy} \cdot \frac{^*g(x+dx)-g(x)}{dx}\right) = f'\left(g(x)\right)g'(x) \blacksquare$$
Again the rules for how the standard part can be taken on a product are employed. And the fact that $st\left(^*g(x+dx)\right) = g(x)$ for any continuous or differentiable function.

2.1.4.3 Definition: (real) integral

Let the function $f: \mathbb{R} \to \mathbb{R}$.

- The definition of the standard (Riemann) integral is, for a partition of $[a, b] \in \mathbb{R}$,
$$a = x_0 \le x_1 \le x_2 \le \dots \le x_{\tau-1} \le x_\tau = b,$$
and any $x'_n \in [x_i, x_{i+1}]$, where $\Delta x_i = x_{i+1} - x_i$:
$$\int_a^b f(x)dx = \lim_{\max\{\Delta x_i\}\to 0} \left(\sum_{n=0}^{N-1} f(x'_n)\Delta x_n \right).$$
Note that for $\max\{\Delta x_i\} \to 0$ then $N \to \infty$, since $\sum_{n=0}^{N} \Delta x_n = [a, b]$.
- The definition of the (Riemann) integral of f, based on the infinite sum in 2.1.3.14, is for $\langle T_i \rangle = \tau \in {}^*\mathbb{N}$ an infinite integer and the infinite partition of the interval $[a, b] \in \mathbb{R}$:
$$a = x_0 \le x_1 \le x_2 \le \dots \le x_{\tau-1} \le x_\tau = b,$$
with infinitesimals $dx_n = \langle \Delta x_{n,i} \rangle = \langle x_{n+1,i} - x_{n,i} \rangle$ and $x'_n \in [x_n, x_{n+1}]$ then
$$\int_a^b f(x)dx = st\left(\sum_{n=0}^{\tau-1} {}^*f(x'_n)dx_n \right) = st\left(\langle \sum_{n=0}^{T_i-1} {}^*f(x'_{n,i})\Delta x_{n,i} \rangle \right).$$

Proof: Let f be as above, then the definition of the standard (Riemann) integral translates into the following ε-δ definition:

For all $\forall\varepsilon \in \mathbb{R}_+$ there exists $\delta \in \mathbb{R}_+$ such that for any partition $x_0, x_1, \dots, x_N$ of the interval $[a, b]$ with $\max\{\Delta x_i\} < \delta$ then
$$\left| \sum_{n=0}^{N-1} f(x'_n)\Delta x_n - \int_a^b f(x)dx \right| < \varepsilon.$$

Now consider
$$\sum_{n=0}^{\tau-1} {}^*f(x'_n)dx_n = \sum_{n=0}^{\langle T_i \rangle -1} \langle f \rangle (\langle x'_{i,n} \rangle) \langle \Delta x_{i,n} \rangle = \sum_{n=0}^{\langle T_i \rangle -1} \langle f(x'_{i,n}) \rangle \langle \Delta x_{i,n} \rangle$$
$$= \sum_{n=0}^{\langle T_i -1 \rangle} \langle f(x'_{i,n})\Delta x_{i,n} \rangle = \langle \sum_{n=0}^{T_i-1} f(x'_{i,n})\Delta x_{i,n} \rangle.$$

Thus for all $\varepsilon \in \mathbb{R}_+$
$$m\left(\left\{ i : \left| \sum_{n=0}^{T_i-1} f(x'_{n,i})\Delta x_{n,i} - \int_a^b f(x)dx \right| < \varepsilon \right\} \right) = 1.$$

Thereby the integral, $\int_a^b f(x)dx$, and the hyperfinite sum, $\sum_{n=0}^{\tau-1} {}^*f(x'_n)dx_n$, differs at most by an infinitesimal. If the standard integral exists there exists a partition for which the limit of the Riemann sum exists. This partition can be described by every integer extending the description of the partition to an infinite integer, will make the partition an infinite partition and can be used in the infinite sum which defines the non-standard integral, thus making the nonstandard integral exists. If the nonstandard integral does not exist, the standard version cannot either, since there is no partition, not even infinite, which can be used so that the function value in the partition interval varies with an infinitesimal. This entails that the limit of a finite partition will not be well defined, since the choice of variable in any of the intervals will make the function vary. I.e. the limit of the upper- and lower sums (sometimes used to define the integral) will not coincide, hence the limit, and integral, does not exists. ▪

2.1.4.3.1 Definition – Internal integral

Define the integral over an internal set $\langle A_n \rangle = A \in {}^*\mathbb{R}$ of an internal function $\langle f_n \rangle = f: {}^*\mathbb{R} \to {}^*\mathbb{R}$ as:

$$\int_A f(x)dx = \langle \int_{A_n} f_n(x)dx \rangle,$$

Where each $f_n: \mathbb{R} \to \mathbb{R}$.

2.1.4.3.2 Example

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function around $x \in \mathbb{R}$ and let $\langle \Delta x_m \rangle = dx$ be an infinitesimal then

$$\int_x^{x+dx} {}^*f(y)dy = \langle \int_x^{x+\Delta x_m} f(y)dy \rangle.$$

Now let τ_m be the hyperfinite integer such that $x = y_{m,0} \leq y_{m,1} \leq y_{m,2} \leq \cdots \leq y_{m,\tau_m - 1} \leq y_{m,\tau_m} = x + \Delta x_m$ is an infinite partition, i.e. the difference $y_{n+1} - y_n$ is infinitesimal. Let $y'_{m,n} \in [y_{m,n}, y_{m,n+1}]$ then

$$\int_x^{x+dx} {}^*f(y)dy = \langle \int_x^{x+\Delta x_m} f(y)dy \rangle = \langle st\left(\sum_{n=0}^{\tau_m - 1} {}^*f(y'_{m,n})\Delta y_{m,n} \right) \rangle = \langle st\left(\langle \sum_{n=0}^{T_{m,i}-1} f(y'_{m,i,n})\Delta y_{m,i,n} \rangle \right) \rangle.$$

2.1.4.3.3 Rules of integration

1. For any function $f: \mathbb{R} \to \mathbb{R}$ and any real number $a \in \mathbb{R}$ the one point integral is zero, i.e.
$$\int_a^a f(x)dx = 0.$$
This ensures that the integral over the closed interval $[a, b]$ is the same as the integral over the upon interval (a, b).

2. Let $f: \mathbb{R} \to \mathbb{R}$ and $k \in \mathbb{R}$ a constant then
$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$
3. Let $f: \mathbb{R} \to \mathbb{R}$ and (a, b) and (b, c) be real intervals, then
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
4. Let $f: \mathbb{R} \to \mathbb{R}$ and (a, b) a real interval then
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Proof:

1. consider the definition of the integral,
$$st\left(\sum_{n=0}^{\tau-1} {}^*f(x_n')dx_n\right)$$
since the partition can only consist of the single value a then the $dx_n = a_{n+1} - a_n = a - a = 0$, making the sum equal to zero. i.e. it does not depend on the function value $f(a)$.▪
2. For a hyperfinite integer $\tau = \langle T_i \rangle$, then
$$m\left(\left\{i : \sum_{n=0}^{T_i-1} k \cdot f(x_{n,i}')\Delta x_{n,i} = k \cdot \sum_{n=0}^{T_i-1} f(x_{n,i}')\Delta x_{n,i}\right\}\right) = 1$$
and since $k \cdot st({}^*f(x)) = st(k \cdot {}^*f(x))$, the desired result follows. If the integral does not exist then both sides of the equation does not exist. ▪
3. The partition in the left integral, $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{\tau-1} \leq x_\tau = c$, must contain some $x_i \approx b$, making $b \approx x_i' \in [x_i, x_{i+1}]$ and thus
$$\int_a^c f(x)dx = st\left(\sum_{n=0}^{\tau-1} {}^*f(x_n')dx_n\right) = st\left(\sum_{n=0}^{i-1} {}^*f(x_n')dx_n + \sum_{n=i}^{\tau-1} {}^*f(x_n')dx_n\right)$$
$$= \int_a^b f(x)dx + \int_b^c f(x)dx \text{▪}$$
4. Consider the first rule of integration and use the third rule for integration then
$$\int_a^b f(x)dx = \int_a^b f(x)dx + \int_b^a f(x)dx - \int_b^a f(x)dx = \int_a^a f(x)dx - \int_b^a f(x)dx = -\int_b^a f(x)dx$$
asserting the desired result. ▪

2.1.4.4 Theorem – Fundamental theorem of calculus part 1

Let $f: \mathbb{R} \to \mathbb{R}$ be continuously differentiable on the interval $[a, b]$, then the following holds for $x \in [a, b] \subset \mathbb{R}$:

$$\int_a^x f'(y)dy = f(x) - f(a).$$

Proof: By definition $f'(x) = st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$, now let $\delta(x) \coloneqq st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right) - \frac{{}^*f(x+dx)-f(x)}{dx}$ then $f'(x) = st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right) = \frac{{}^*f(x+dx)-f(x)}{dx} + \delta(x)$ replacing this expression with the integrand leads to the following

$$\int_a^x f'(y)dy = st\left(\sum_{n=0}^{\tau-1}\left(\frac{{}^*f(x_n' + dx) - {}^*f(x_n')}{dx} + {}^*\delta(x_n')\right)dx_n\right).$$

The definition of the differential quotient makes it possible to use any infinitesimal to describe it and still get the same result, thus

$$\int_a^x f'(y)dy = st\left(\sum_{n=0}^{\tau-1}\left(\frac{{}^*f(x'_n+dx_n)-{}^*f(x'_n)}{dx_n}+{}^*\delta(x'_n)\right)dx_n\right)$$
$$= st\left(\sum_{n=0}^{\tau-1}{}^*f(x'_n+dx_n)-{}^*f(x'_n)+{}^*\delta(x'_n)dx_n\right)$$
$$= st\left(\sum_{n=0}^{\tau-1}{}^*f(x'_{n+1})-{}^*f(x'_n)+{}^*\delta(x'_n)dx_n\right) = st\left({}^*f(x'_\tau)-{}^*f(x'_0)+\sum_{n=0}^{\tau-1}{}^*\delta(x'_n)dx_n\right)$$
$$= f(x)-f(a)+st\left(\sum_{n=0}^{\tau-1}{}^*\delta(x'_n)dx_n\right)$$

By induction over the hyperfinite set $X_\tau = \{x'_n \in {}^*\mathbb{R} \mid n \in \{1,2,3,\dots,\tau-1\}\}$ a maximal element, δ_m, can be chosen for ${}^*\delta(x'_n)$. This maximal element is an infinitesimal by definition of $\delta(x)$, thus

$$st\left(\sum_{n=0}^{\tau-1}{}^*\delta(x'_n)dx_n\right) \le st\left(\sum_{n=0}^{\tau-1}|{}^*\delta(x'_n)|dx_n\right) \le st\left(\sum_{n=0}^{\tau-1}\delta_m dx_n\right) = st\big(\delta_m\cdot(a,x)\big) = 0.$$

With this the first part of the fundamental theorem is proved.▪

In the following some of the more complicated proofs in analysis can be found, which enables the proof of the second part of the fundamental theorem.

2.1.4.5 Theorem – Bolzano's theorem

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on the interval $[a,b]\subset\mathbb{R}$ and $(a)<0<f(b)$, then there exists $c\in[a,b]$, such that $f(c)=0$.

Proof:

Let $N\in{}^*\mathbb{N}$ be an infinite integer and let

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

be the infinite partition, where the difference between two consecutive partition points are of equal length, i.e. $x_{n+1}-x_n = dx = \frac{(b-a)}{N}$ for all n. Let $\mathbb{J}$ be the set partition points, x_j, for which $0 < {}^*f(x_j)$, by definition the set $\mathbb{J}$ is a hyperfinite set and thus an x_{j_0} for which ${}^*f(x_{j_0}) \le {}^*f(x_j)$ for all $x_j\in\mathbb{J}$ can be chosen by induction. Since f is continuous then

$$st\left({}^*f(x_{j_0})\right) = f\left(st(x_{j_0})\right) = 0.$$

Thereby the sought $c\in\mathbb{R}$ can be found by letting $c = st(x_{j_0})$. ▪

2.1.4.6 Theorem – Extreme value theorem

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on the interval $[a,b]\subset\mathbb{R}$, then the function $f(x)$ attains both a (local)maximum, $Max = f(c_M)$, and a (local)minimum $min = f(c_m)$ value, for some $c_m, c_M\in[a,b]$. I.e. for every $x\in[a,b]$ the following holds:

$$min \le f(x) \le Max$$

Proof:

Let $N \in {}^*\mathbb{N}$ be an infinite integer and let

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

be the infinite partition of ${}^*[a,b]$, where the partition intervals are of equal length, i.e. $x_{i+1} - x_i = dx = \frac{(b-a)}{N}$ for all i. Now let A be the hyperfinite set of all the partition points. i.e.

$$A = \{x_0, x_1, x_2, \dots, x_{N-1}, x_N\}$$

Then by induction there exists x_m and x_M such that for all $x_i \in A$

$$f(x_m) \le f(x_i) \le f(x_M).$$

For every real number $x \in [a,b]$ there exists a partition interval $[x_i, x_{i+1}]$ such that $st(x_i) = x$ and since the partition intervals are of infinitesimal length, no interval can contain more than one real number. Let $st(x_m) = c_m$ and $st(x_M) = c_M$, then by continuity the standard part of the inequality translates to; for all $x \in st(A) = [a,b]$, then

$$st\big(f(x_m)\big) \le f(x) \le st\big(f(x_M)\big)$$

$$f\big(st(x_m)\big) \le f(x) \le f\big(st(x_M)\big)$$

$$f(c_m) \le f(x) \le f(c_M).$$

By definition $f(c_m)$ is the minimum and $f(c_M)$ the maximum of the function over the interval $[a,b]$. Thereby asserting the desired result

$$min = f(c_m) \le f(x) \le f(c_M) = Max.▪$$

2.1.4.7 Theorem – Intermediate value theorem

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on the interval $[a,b] \subset \mathbb{R}$, then the function $f(x)$ attains every real value between its minimum, $f(c_m) = min$, and maximum, $f(c_M) = Max$, for some $x \in [a,b]$. I.e.

$$f([a,b]) = [f(c_m), f(c_M)] = [min, Max].$$

Proof:

If $m = M$, then the function is constant over the interval and there is nothing to prove. Now let $f(c_m) = min < Max = f(c_M)$ and let $d \in \mathbb{R}$ be such that $f(c_m) < d < f(c_M)$. By definition the function $g(x) = f(x) - d$ will be continuous in the same interval as $f(x)$. Now subtract d from the inequality, i.e.

$$f(c_m) < d < f(c_M)$$

$$f(c_m) - d < d - d < f(c_M) - d$$

$$g(a) < 0 < g(b)$$

By Bolzano's theorem there exist $c \in [a,b]$ such that $g(c) = f(c) - d = 0$, thus $f(c) = d$. Since d was chosen arbitrarily in $\big(f(c_m), f(c_M)\big)$, the desired result follows.▪

2.1.4.8 *Definition - Average*

Given a finite set $A = \{x_0, x_1, x_2, \dots, x_n\}$ for $n \in \mathbb{N}$ the average of the values, $x_0, x_1, x_2, \dots, x_n$ are given by

$$x_{ave} = \frac{1}{n+1}\sum_{i=0}^{n} x_i.$$

2.1.4.9 *Definition - Mean of a function*

Given a function $f: \mathbb{R} \to \mathbb{R}$, the mean of the function over the interval $[a, b]$ is given by

$$f_{mean}([a,b]) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Explanation:

Take an infinite integer $N \in {}^*\mathbb{N}$ and make an infinite partition of the interval ${}^*[a, b]$, where the length of the partition intervals are equal. I.e.

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b,$$

where $x_{i+1} - x_i = dx = \frac{(b-a)}{N}$ for all $i \leq N - 1$, which makes $\frac{1}{N} = \frac{dx}{b-a}$. Following the definition of the average the mean of f should be

$$f_{ave}([a,b]) = \frac{1}{N+1}\sum_{i=0}^{N} f(x_i) = \sum_{i=0}^{N} f(x_i)\frac{1}{N+1}$$

Consider the difference $\frac{1}{N} - \frac{1}{N+1} = \frac{N+1-N}{N^2+N} = \frac{1}{N^2+N} < \frac{1}{N^2} = \frac{dx^2}{(b-a)^2}$. Thus $f_{ave}([a,b])$ is:

$$\begin{aligned} f_{ave}([a,b]) &= \sum_{i=0}^{N-1} f(x_i)\frac{1}{N+1} = \sum_{i=0}^{N-1} f(x_i)\left(\frac{1}{N} + \frac{1}{N^2+N}\right) = \sum_{i=0}^{N-1} f(x_i)\frac{dx}{b-a} + \sum_{i=0}^{N-1} f(x_i)\frac{1}{N^2+N} \\ &< \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)dx + \sum_{i=0}^{N-1} f(x_i)\frac{1}{N^2} = \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)dx + \sum_{i=0}^{N-1} f(x_i)\frac{dx^2}{(b-a)^2} \\ &= \frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)dx + \frac{dx}{(b-a)^2}\sum_{i=0}^{N-1} f(x_i)dx. \end{aligned}$$

Taking the standard part of this makes

$$\begin{aligned} st(f_{ave}) &= st\left(\frac{1}{b-a}\sum_{i=0}^{N-1} f(x_i)dx + \frac{dx}{(b-a)^2}\sum_{i=0}^{N-1} f(x_i)dx\right) \\ &= \frac{1}{b-a} st\left(\sum_{i=0}^{N-1} f(x_i)dx\right) + st\left(\frac{dx}{(b-a)^2}\right) st\left(\sum_{i=0}^{N-1} f(x_i)dx\right) \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx + 0 \cdot \int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

This makes the standard part of the average of the function into the definition of the mean of the function over the interval as wanted.

2.1.4.9.1 Corollary: Mean of a continuous function

Given a continuous function $f: \mathbb{R} \to \mathbb{R}$, the mean of this function over the interval $[a, b]$ is equal to a function value evaluated in $d \in [a, b]$.

$$f_{mean}([a,b]) = \frac{1}{b-a}\int_a^b f(x)\,dx = f(d).$$

Proof:

By definition of the mean of a function then f_{mean} is a value between the extreme values of the function over the interval. So by the intermediate value theorem the function f attains this value for some $d \in [a, b]$.

2.1.4.10 Theorem – Fundamental theorem of calculus part 2

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on the interval $[a, b]$, and let $F(x) = \int_a^x f(y)dy$ then the following holds

$$F'(x) = f(x)$$

Proof:

Let $dx = \langle \Delta x_n \rangle$, then by definition of the integral and the star operation

$$F'(x) = st\left(\frac{{}^*F(x+dx) - {}^*F(x)}{dx}\right) = st\left(\frac{{}^*\int_a^{x+\mathrm{d}x} f(y)dy - {}^*\int_a^x f(y)dy}{dx}\right) = st\left(\frac{{}^*\int_x^{x+\mathrm{d}x} f(y)dy}{dx}\right)$$

$$= st\left(\frac{1}{dx}\int_x^{{}^*x+\mathrm{d}x} f(y)\,dy\right) = st\left(\langle\frac{1}{\Delta x_n}\int_x^{x+\Delta x_n} f(x)\,dx\rangle\right) = st(\langle f_{mean}([x, x+\Delta x_n])\rangle)$$

By corollary 2.1.4.9.1, the mean of the function can be written as $f_{mean}([x, x+\Delta x_n]) = f(d_n)$, where $d_n \in [x, x+\Delta x_n]$ for every n. I.e. $\langle d_n \rangle = d \in [x, x+dx]$ making

$$F'(x) = st(\langle f_{mean}([x, x+\Delta x_n])\rangle) = st(\langle f(d_n)\rangle) = st({}^*f(d)) = f(x)$$

where the last equality is true since the function is continuous.

With this the first part of the theorem can also be proved, but as seen it required a great deal of intermediate theorems to get to the second part. As such both proofs can be useful.

2.1.4.10.1 Corollary - Fundamental theorem of calculus part 1

Let $f: \mathbb{R} \to \mathbb{R}$ be continuously differentiable on the interval $[a, b]$, then the following holds:

$$\int_a^x f'(y)dy = f(x) - f(a)$$

for $x \in [a, b] \subset \mathbb{R}$.

Proof: Taking the differential on both sides (using theorem 2.1.4.10 on the left side), shows that it is true up to a constant term. The only thing to prove is that the constant term, c, which $G(x) = \int_a^x f'(y)dy$ differs from $f(x)$ by, is $-f(a)$. Now consider

$$f(a) + c = G(a) = \int_a^a f'(y)dy = f(a) - f(a) = 0.$$

Thus $c = -f(a)$, asserting the desired result.▪

2.1.4.11 Proposition – Integration by parts

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions on the interval $[a, b]$, then the following holds:

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$

Proof: Let $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$. Using the rules for differentiation and for integration the following can establish:

$$\int_a^b f'(x)g(x)dx = \int_a^b f'(x)g(x) + f(x)g'(x) - f(x)g'(x)dx = \int_a^b h'(x) - f(x)g'(x)dx$$
$$= \int_a^b h'(x)dx - \int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$▪

2.1.4.12 Definition – Antiderivative

Define the antiderivative of a function $f: \mathbb{R} \to \mathbb{R}$, as a function $F(x): \mathbb{R} \to \mathbb{R}$, for which the following holds:

$$F'(x) = f(x)$$

i.e. $F(x) = \int_a^x f(y)dy + k$ for some constant $k \in \mathbb{R}$.

2.1.4.13 Definition – Indefinite integral

Define the indefinite integral as the opposite of the derivative, i.e. for a function $f: \mathbb{R} \to \mathbb{R}$, the indefinite integral is the antiderivatives for the function f. This is usually written for some constant $c \in \mathbb{R}$ as

$$\int f(x)\, dx = F(x) + c$$

i.e. $\int f(x)\, dx = F(x) + c = \int_a^x f(y)dy + k$.

2.1.4.14 Proposition – Integration by substitution

Let $g: \mathbb{R} \to \mathbb{R}$ be continuously differentiable on the interval $[a, b]$ and let $f: \mathbb{R} \to \mathbb{R}$ be continuous on both $[a, b]$ and $[g(a), g(b)]$ and such that f has an antiderivative on the interval $[a, b]$. Then

$$\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy.$$

Proof: Let F be an antiderivative of f, then $H(x) = F(g(x))$ is an antiderivative of the continuous function $h(x) = f(g(x))g'(x)$, thus

$$\int_a^b f(g(x))g'(x)dx = H(b) - H(a) = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(y)dy.$$

Another way to consider this is to write $y = g(x)$, this makes $\frac{dy}{dx} = \frac{dg(x)}{dx} = g'(x)$, hence $dy = g'(x)dx$. Thus

$$\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$$

asserting the desired result.▪

2.2 Didactical theory

Didactics is a theory and practical application of teaching and learning. The theory of didactic learning methods focuses on the knowledge that students possess and how to improve it, hence it provides students with the required theoretical knowledge, where a teacher acts as a guide and a resource for students. By means of various didactical theories, it is possible to structure such learning environments and best further the students' knowledge. The following is a description of a part of the anthropological theory of didactics, as well as a foundation of theory for the analysis of this study.

2.2.1 Anthropological Theory of Didactics

Throughout the 1980s and 1990s, Yves Chevallard ("Yves Chevallard (English) | A.R.D.M.," n.d.) made his footprint in the history of the didactical work. His research is epistemological and institutional and he is the founder of the Anthropological Theory of Didactics (ATD). This theory deals with transpositions of knowledge and focuses in part on where the knowledge comes from and to whom it is situated. In high school, mathematical objects created by mathematicians are not the ones being taught, so the mathematical knowledge produced outside school is adapted, often several times, before it is accepted for teaching. As such ATD aims to describe and explain this transformation of knowledge, distinguishing academic knowledge produced by mathematicians, knowledge to be taught, as defined by the educational system, knowledge taught by the professor and knowledge learnt by students. These mathematical and didactical organizations are determined together and by each other. In other words, they are co-determined, as Chevallard puts it. Below is a figure which enables an overview of the 9 levels of co-determination.

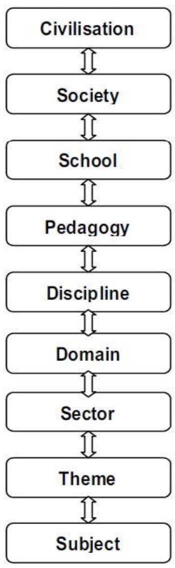


Figure 2.2.1
Shows the 9 levels of didactic co-determination, as proposed by Chevallard, related to the components of MO's. It also includes modelling of praxeologies by MO's.

The didactical transposition of knowledge is often referred to as "the travel of knowledge from source(s) to students". It is a process between different groups of people in different institutions; this could be political authorities, mathematicians and teachers and their associations who chooses what is to be taught. This institutional organization is called the noosphere, which contemplates education. That is to say they set up the limits, and redefine and reorganize the knowledge to make possible or uncertain choices.

As such ATD defines the boundaries within which mathematical education seems to be confined and represents an epistemological approach. ATD also addresses the institutionalized mathematical activities and uses the model of didactic co-determination to address the issues.

2.2.1.1 Mathematical organization (MO)

A mathematical organization can be divided into two blocks: the first block is called the *practical* or *praxis block* and the second is called the *knowledge block*. The practical block is the part of the mathematical organization concerning the *types of tasks* and *techniques* used to solve them. The types of tasks are the proposed problems to be solved in order to explore the new mathematical discipline. Possible examples of tasks are: find the standard part of some hyperreal number, find the standard part of a quotient of infinitesimals (= find the derivative of a function), and find the standard part of an infinite sum of infinitesimals (≈ find the integral). To engage these tasks a technique is used as a method to solve the tasks at hand. Thus techniques encompass both the algorithmic and analytic kinds of mathematics. A specific technique can cover a number of tasks.

The knowledge block likewise consists of two parts, *technology* and *theory*. Technology is the environment wherein the techniques are justified, which is expressed in the word itself: techn (as in technique) and ology (as in study of…). Lastly the theory is what underlies the technology and gives rise to the deepest and most thorough understanding of the techniques and technology used. The four concepts: types of tasks, techniques, technology and theory constitute a *mathematical organization* (MO) or praxeology. The word praxeology is a mixture of praxis and logos which makes sense, seeing as the two first concepts, types of tasks and techniques, is a practical (praxis) block and the technology and theory is the knowledge (logos) block. With this in mind the MO or a praxeology is often written as a four letter tuple, $[T,\tau,\theta,\Theta]$, where T is a set of tasks and τ are the technique(s) used to solve the tasks. The letter θ represents the technology and lastly the theory is represented by the letter Θ.

With the tuple $[T,\tau,\theta,\Theta]$, different types of praxeologies or MO's can be made, the simplest (smallest) praxeology is the punctual praxeology, which describes a praxeology with only a single type of tasks, i.e. if all the tasks T are of the form: determine the derivative of a polynomial function, the praxeology would be called a punctual praxeology. If a praxeology is unified by the technology θ, i.e. the praxeology encompasses a variety of tasks where the technique(s) can all be explained through the same technology, then the praxeology will be called a local MO. An example of a local MO could be to determine the derivative a function given its analytic expressions. In this case the tuple representing the praxeology would contain an index on the types of tasks and techniques, i.e. $[T_i,\tau_i,\theta,\Theta]$. The index is there to represent the different types of tasks and techniques contained in the local MO. A regional MO is a praxeology which encompasses more than one local MO, i.e. the technology for the different local MO's can be different but the theory is the same for the local MO's. An example of this could be infinitesimal calculus, with differential calculus as one local MO and integral calculus as another local MO.

2.2.1.2 Didactical organization

The anthropological theory of didactics has a way of describing the creation of an organization of knowledge; in this study, the organizations at hand are all mathematical, as such the theory of a *didactical organization* (DO) will be explained based on an MO. The way that ATD describes the creation of an MO is what is called a didactical organization and consists of six different moments, each of which is important to generate a meaningful MO. A didactical organization is partly dependent on the MO but at the same time the didactical organization certainly also has an impact on the MO actually taught. With this in mind the 6 moments of a DO concerning an MO= $[T_i, \tau_i, \theta, \Theta]$ are:

1. First encounter with a task from a type of task T_i. As such, the first encounter is the first time a problem of a given type T_i is posed to the individual(s), who is supposed to create the MO.
2. Exploration of the type of task T_i and establishment and elaboration of the technique τ_i. This moment entails for the individual(s) constructing the MO at hand to come up with a way of answering the type of task and explaining the technique, such that others would be able to replicate the technique.
3. Constitution of the technology θ and theory Θ concerning the technique τ_i. As such this moment concerns the theory and technology the technique is subordinate to. This moment, can be difficult to pin point to only happening once, since the introduction of another type of task τ_j might constitute a different knowledge block even if both τ_i and τ_j are part of the same MO.
4. Technical work on the technique τ_i is made in order to decipher when exactly the technique can be used and to what extent. Note that the word "technical" ensures that the work to be done has to be based on the technological and theoretical level of the MO as developed in the moment of constitution.
5. Institutionalization connects the MO with other local MO's, which share the same knowledge block $[\theta, \Theta]$. In this didactical moment the individual(s), who created the MO that the DO describes, establishes a connection to previously constructed MO's. By doing this the newly generated MO is settled among other MO's, which enables a specification of the boundaries and possible ways in which the newly created MO can be used.
6. Evaluation is a moment where the constructed MO is evaluated as to how well the created MO solves the tasks T_i at hand in comparison to other local MO's found in the institutionalization.

It is clear that the sixth moment works very closely together with the fifth.

A passage on this particular subject can be found in "Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools" (BarbÉ et al., 2005)

"It is clear that a 'complete' realisation of the six moments of the didactic process must give rise to the creation of a MO that goes beyond the simple resolution of a single mathematical task. It leads to the creation (or re-creation) of at least the first main elements of a local MO, structured around a technological discourse."

2.2.2 What to be taught

In order to establish what to be taught, the didactical transposition is used. The didactical transposition explains the connection between the *scholarly knowledge*, the *knowledge to be taught* and the *actually taught knowledge* for a specific discipline, such as mathematics. The scholarly knowledge is then the knowledge conducted by researchers; as such the mathematics taught at universities is closely related to the scholarly knowledge. The curricula and exams, created by the government, and the textbooks constitute the knowledge to be taught, which is dependent on the scholarly knowledge. Actually taught knowledge is the knowledge obtained by the students, i.e. how the students are taught in the classroom and their responses, in the form of questions asked during class and homework (assignments). Actually taught knowledge is dependent on the knowledge to be taught and by extension the scholarly knowledge. A model upon which to base the planned teaching and a model used when analyzing the actually taught knowledge is established by the scholarly knowledge and the knowledge to be taught, which is called a *reference MO*. Illustrated below on a figure.

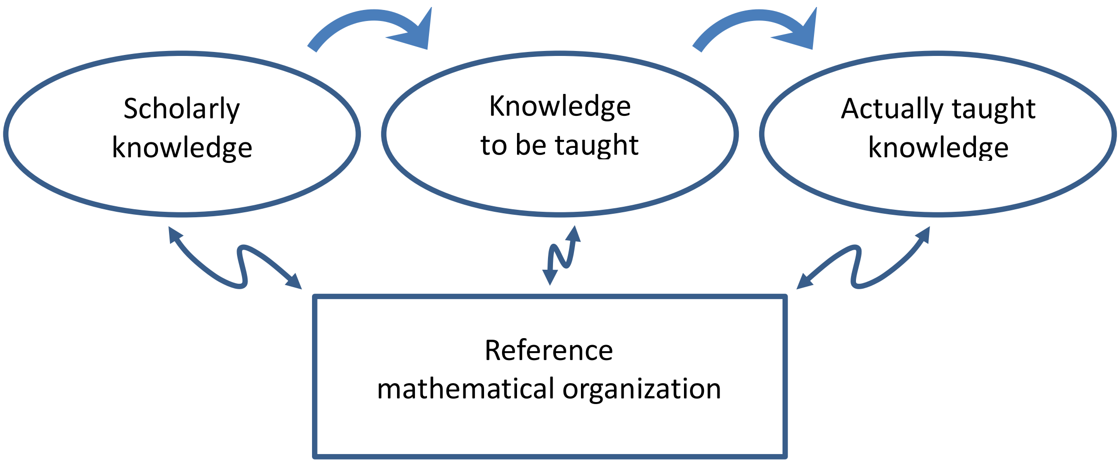


Figure 2.2.2

In order to decipher what knowledge to be taught, when it concerns the teaching of infinitesimal calculus in high school, a reference (regional) MO for the mathematics involved is needed, both as a foundation for establishing the teaching material and as a way to analyze the teaching. This section is divided into 3 subsections, the first one being the hyperreal MO, which concerns the teaching of the hyperreal numbers.

The section about the hyperreal MO is void of theory supporting the construction of the hyperreal numbers; as such the reference MO and knowledge to be taught when considering the hyperreal numbers are close to being identical. This choice, which might seem unfulfilling is made because the construction of the (hyper)real numbers is not a part of the curriculum in high school. As such a reference MO which includes the construction would make it overly complicated to condense the reference MO into the sought knowledge to be taught.

The second subsection called the differential MO (DMO) concerns the teaching of differential calculus and the third subsection called the integral MO (IMO) concerns the teaching of integral calculus. These sections include a reference MO for the analysis and based on this, a description of what to be taught.

2.2.2.1 Hyperreal MO

In order to determine what mathematical knowledge from the non-standard analysis is needed, to teach infinitesimal calculus in high school, the government-given curriculum for A-level math (Ministry of education, 2013) was scrutinized, the textbook they normally used was also looked in (Clausen, Schomacker, & Tolnø, 2006), but since it is not based on the same definitions as the nonstandard ones, some of the problems seemed odd. This made the curriculum the foremost object used to decipher the knowledge to be taught. The curriculum doesn't specify how to introduce neither the differential quotient nor the integral, hence this opens up a different approach to infinitesimal calculus, namely the nonstandard approach. Since the hyperreal numbers is not part of the normal curriculum in high school, an MO about the hyperreal numbers and the connection to real numbers is needed. In high school the construction of the real numbers is not part of the curriculum, which enables the introduction to the hyperreal numbers in the same way as the real numbers: as an intuitive extension of the integer or rational numbers. With this in mind the mathematical knowledge to be taught, regarding the hyperreal numbers and the connection to the real numbers in this situation, can be boiled down to the following question:

What is needed from NSA to define the differential quotient and the integral in high school, where the construction of the real/hyperreal numbers are not part of the curriculum?

- An intuitive understanding of the hyperreal numbers ${}^*\mathbb{R}$, akin the understanding of the real numbers.
    - Infinitesimals, and how to operate with them.
    - Infinite numbers, and how to operate with them.
- The connection between the hyperreal and real numbers, i.e.
    - The standard part of a hyperreal number.
    - An understanding of what the star operation of a function is, as in a way to extend the domain and range of a function $f\colon \mathbb{R} \to \mathbb{R}$ to ${}^*f\colon {}^*\mathbb{R} \to {}^*\mathbb{R}$.
- Continuity (in a point)

In the guide provided by the Ministry of Education (Ministry of education, 2010), the notion of continuity is a concept which should be included, but not given separate treatment. At some point, the ε-δ definition for continuity in high school was a part of the curriculum, but has since been degraded from being mandatory to optional. Even though the definition of continuity was removed from the curriculum, the notion of continuity is still something that has to be covered, to some extent. The nonstandard definition of continuity opens up the possibility to include the definition again, since it is very easy to translate the phrase "a function is continuous if the graph, of the function, does not jump" into "the difference in function value, when the difference in the variable is infinitesimal, is again infinitesimal". With this in mind, continuity was made part of the things to be taught, when teaching the hyperreal numbers.

The hyperreal MO (HMO) can be described by a set of tasks and a list of exercises. The tasks and techniques make up the practical part of the praxeology, as such the practical part of an MO is more or less fully explained through its tasks.

Types of tasks in HMO	HMO= $[\mathrm{T}, \tau, \theta,]$
HT_0	In what set of numbers does a number x belong?
HT_1	Find $st(a)$ when $a \in {}^*\mathbb{R}$?
HT_2	What is ${}^*f(x + dx)$ when dx is an infinitesimal and f a real function?
HT_3	Find $st\left(a \cdot {}^*f(x)\right)$ when $a, x \in {}^*\mathbb{R}$
HT_4	Check if the real function f is continuous (at a point)

With these tasks, an overview of the technological and theoretical level of HMO can be fashioned through a set of questions:

1. What is an infinitesimal?
2. What is an infinite number?
3. How does a number line including infinitesimals and infinite numbers, i.e. ${}^*\mathbb{R}$, look like?
4. How does one operate with the new found quantities?
5. What does it mean to take the standard part?
6. How does one give meaning to a real function evaluated in a hyperreal number?
7. What is continuity (of a real function in a point)?

In order to answer these questions in a scholarly way, the construction of the hyperreal/real numbers would be needed, but since this is not part of the curriculum, the answers to these questions use an intuitive understanding of infinitesimals and infinite numbers. Thus a possible answer to the first question could be, that a positive infinitesimal is a number which is smaller than any given positive real number. Thus, even if the technological level of HMO, θ, is not founded in the scholarly knowledge, it is still present, though it only appears as an intuitive and logical understanding of the hyperreal numbers. Most of the theoretic level of HMO is replaced with intuition, when introducing the hyperreal numbers to the students in high school. In this way, HMO can be described by a tuple, $[\mathrm{T_i}, \tau_j, \theta, \ ]$. It should be noted that this is no different than the way the high school students are using the real numbers, thus this introduction to the hyperreal numbers does not violate the curriculum in any way.

2.2.2.2 *Differential MO*

In order to establish the reference MO a consideration of the NSA definition of the differential quotient as seen in section 2.1.4.2 constitutes the technological and theoretical part of the DMO but in addition it also operates as a technique, though the only task supporting this technique would be to define it. As such when referring to the differential quotient as a technique it will be appointed the term *hypothetical technique*. This term is introduced in order to distinguish it from the regular techniques which are used to describe how to answer the types of tasks in a given MO. The hypothetical technique of the differential quotient makes it possible to elaborate on it in the sense of establishing the rules of differentiation.

Types of tasks in DMO	
DT_0	Define the differential quotient.
DT_1	Find the derivative, $f'(x)$, for a function, $f(x)$.
DT_2	Determine the condition of monotony for a given function, $f(x)$.
DT_3	Find the tangent of a function through a specific point, $\left(x, f(x)\right)$.
DT_4	Determine if $\frac{d}{dx}f(x)$ exists for a given function, $f(x)$.

Marked in grey is the constitutive task of DT_0, since it is not a type of task.

A number of types of tasks that fall under DT_1 are listed below:

DT_{11}: Find the of a graph in a point
DT_{111}: Find the slope of a power function in a point
DT_{112}: Find the slope of an exponential function in a point

DT_{121}: Find the derivative of a power function
DT_{122}: Find the derivative of an exponential function

Other than these types of tasks there exists a list of commonly used questions which elaborate the hypothetical technique of the differential quotient, with appropriate assumptions on the functions and constants below.

1. Prove that $\frac{d}{dx}\left(k \cdot f(x)\right) = k \cdot \frac{d}{dx}\left(f(x)\right)$
2. Prove that $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$
3. Prove that $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$
4. Prove that $\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$

With the reference MO for differential calculus established as described above, the description of what to be taught can be found by scrutinizing the curriculum (Ministry of education, 2013). In this curriculum the following things are described as being necessary for teaching this specific part of calculus in high school:

- Definition of the differential quotient, including growth rate and marginal considerations
- Derivatives of the elementary functions (linear, exponential, power, polynomial and logarithmic functions, cosine and sine)

- Rules for calculating the derivative of $f+g$, $f-g$, $k \cdot f$ and $f \circ g$
- Deduce some selected differential quotients

As such the reference MO and the curriculum coincide, except for the elaboration of the hypothetical technique of the differential quotient in regards to for which functions the derivative exists. Furthermore the reference MO is based on the scholarly knowledge described in section 2.1, hence another difference between this and the knowledge to be taught is the theory of (hyper)real numbers. Of course the knowledge to be taught covers a much smaller range of functions than the scholarly knowledge.

2.2.2.3 Integral MO

In order to establish the reference MO a consideration of the NSA definition of the integral as seen in section 0 constitutes the technological and theoretical part of the IMO. When the integral is seen as finding the area between a function and the first axis over an interval, it operates as a hypothetical technique, since it is impossible to calculate an infinite sum by adding the terms. This makes it possible to elaborate on the hypothetical technique, the integral, in the sense of establishing the rules of integration.

Types of tasks in IMO	
IT_0	Define the integral.
IT_1	Calculate $\int_a^b f(x)\,dx$ for a given function, $f(x)$.
IT_2	Find the antiderivative, $F(x)$, for a given function, $f(x)$.
IT_3	Determine if $\int_a^b f(x)\,dx$ exists for a given function, $f(x)$.

Marked in grey is the constitutive task of IT_0, since it is not a type of task.

A number of types of tasks that fall under the types of tasks are listed below:

IT_{11}: Calculate $\int_a^b f(x)\,dx$ when $f(x)$ is linear
IT_{12}: Calculate $\int_a^b f(x)\,dx$ when $f(x)$ has an antidereivative.
IT_{13}: Calculate $\int_a^b f(x)\,dx$ when $f(x)$ has no antiderivative.

IT_{21}: Calculate $\int_a^x f(x)\,dx$ for a given function $f(x)$.
IT_{22}: Find the antiderivative, $F(x)$, for a power function, $f(x)$
IT_{22}: Find the antiderivative, $F(x)$, for an exponential function, $f(x)$
IT_{23}: Determine $\int f(x)\,dx$ for a given function $f(x)$.

Other than these types of tasks there exists a list of commonly used questions which elaborate the hypothetical technique of the integral, with appropriate assumptions on the functions and constants below.

1. Prove that $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$
2. Prove that $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. Prove that $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
4. Prove that $\int_a^a f(x)\,dx = 0$
5. Prove that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

6. Prove that $\frac{d}{dx}\left(\int_a^x f(x)\,dx\right) = f(x)$

With the reference MO for integral calculus established as described above, the description of what to be taught can be found by scrutinizing the curriculum (Ministry of education, 2013).

- Definition of the indefinite and definite integral
- Antiderivative of the elementary functions (linear, exponential, power, polynomial and logarithmic functions, cosine and sine)
- Rules for calculating the antiderivative of $f + g$, $f - g$, $k \cdot f$ and integration by substitution
- Mention the correlation between the area function and the antiderivative
- Disc method (calculating the volume of a solid of revolution)

By considering this list the knowledge to be taught is seen to differ from the IMO in a few places. The first difference, that might sting the eye, is that any mentioning of existence of an integral is void. The choice for this by the government could be that they thought the techniques to establish such an answer would be too difficult. On top of that the elementary functions, which (almost) are the only functions the students work with in high school, are all integrable on a bounded interval. Another difference is that there is no distinct mentioning of a proof of the fundamental theorem of calculus.

2.2.3 Didactical reasons for or against using NSA

This section will establish some reasons for and against the use of NSA to introduce infinitesimal calculus.

As it was hinted in the Motivation section, teaching analysis in high school has its obstacles, one of which is thoroughly analyzed in the text ”Didactic restrictions on teachers practice – the case of limits of functions in Spanish high schools”, by (BarbÉ et al., 2005). In this text, the picture painted is, that the teaching of limits in high school lacks a technological/theoretical part. The scholarly knowledge, which covers the theoretical block, consists in most part of the completeness of the real numbers, which is based on the construction of the real numbers and is not something that need be taught. The limit-operation is normally (in high school) used as a “tool” to introduce the terms continuity, differentiability and integrability as seen in (Winsløw, 2013). When using NSA, the limit-operation is not needed to define the aforementioned terms. What “tool” is then used, when using NSA, and does it suffer from the same lack of technological/theoretical support in high school? The standard part springs to mind as the obvious substitute for the limit-operation, which will be considered as the “tool”. In order to understand the standard part, the hyperreal numbers, and their connection to the real numbers, are needed. The theoretical block of the standard part is covered by the theoretical block of HMO. This block was expressed by the seven questions listed in section 2.2.2.1, all of which can be answered with intuition and logical deduction, when the construction of the numbers is not needed. As such, a version of the theoretical block of the standard part can be obtained by the students, whereas the theoretical block for the limit-operation needs the completeness of the real numbers, to be logically sound. E.g. when determining the rules for calculating with differentials the limit operation and the rules for when a product of limits equals the limit of the product are used or when determining the rules for calculating with integrals the limit operation and the rules for when a sum of limits equals the limit of the sum are used. These rules can be explained by the completeness of the real numbers but as this is difficult to grasp and even more so to prove, the high school students come up short when trying to prove the rules for calculating with differentials/integrals. When the hyperreal numbers are intuitively understood the standard part is a logical extension. By logical deduction the rules about the standard part can be extracted, thus enabling the students to prove the rules for calculating with differentials/integrals with a set of rules they understand and they are able to argue why the rules apply.

To explain the standard part, it is the operation that takes a hyperreal number to the nearest real number. The limit-operation, on the other hand, is a bit more difficult to explain. Intuitively, the limit-operation can be seen as a way of working with quantities of variable magnitude. Even with this intuitive understanding, the limit-operation still seems incomprehensible, in that mathematics in high school is something static but with this, mathematics becomes something dynamic. Normally mathematic operations are done with constants and variables, meaning that they are fixed or something that can be chosen to be fixed. The limit-operation introduces something which is neither constant, because it is not fixed, nor is it a variable, since it cannot be chosen to be fixed, i.e. limit-operations introduce a different type of unfixed mathematics. Another understanding of the limit-operation can be obtained by ε-δ arguments, which comes down to double-logical deduction. Thus, either an unfixed approach is needed or an approach using something which is beyond the normal high school students’ understanding: the ε-δ arguments.

In order to understand the differential quotient, consider the differentiable function $f: \mathbb{R} \to \mathbb{R}$. The limit-operation allows for the following definition of the derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Remembering that division by 0 is not allowed, then $\Delta x \neq 0$. After some algebraic operations then $\Delta x = 0$ and the differential quotient is obtained, but this approach can confuse even the brightest of student. The students normally understand that division by zero is not doable, but when the same thing is both zero and not zero, the students get confused.

The differential quotient, as introduced by NSA, allows for the following definition, where dx is an infinitesimal.

$$f'(x) = st\left(\frac{{}^*f(x + dx) - f(x)}{dx}\right).$$

Remembering that division by 0 is not allowed, then $dx \neq 0$. Taking the standard part of the quotient is most easily obtained by doing algebraic operations first. The star operation is another operation which needs to be introduced in order for the definition to be mathematically rigorous. In this way one could say that two operations are needed instead of one, but the two operations are simple in comparison to the limit-operation. Thus, the star operation and the standard part become the new candidate for the "tool" to introduce the terms continuity, differentiability, and integrability. It would seem like a mouthful to teach students another set of numbers, especially when it is not necessarily something they will ever meet again. But, as explained, this introduction enables the teaching of infinitesimal calculus, needing neither the unfixed quantity, introduced with the limit-operation, nor the intrinsic ways of ε-δ arguments.

When teaching the integral, the most common introduction to this in high school is to use the area between the graph of a function and the first axis, as an intuitive approach to the integral. With the use of limit-operations, the integral is sometimes introduced as the common limit of the upper and lower Riemann sums. This way of introducing the integral is normally only used on the highest level of math in high school. On the second highest level of math in high school, the students are introduced to the integral through the indefinite integral, which is defined as the opposite operation of differentiation, thus only the highest level math students are able to understand, that the integral of a discontinuous function can be found (Clausen et al., 2006). When using NSA to introduce the integral, the use of limit operation is again circumvented. This allows the students to use the same intuitive approach to the infinite sum as when operating with the quotient of infinitesimals, defining the differential quotient. When considering the things that make the integral difficult to understand, the limit-operation might not seem like the most problematic one. Other problems that can arise when introducing the integral are:

1. The notion of partition of intervals in a mathematical rigorous way, including indexing.
2. The ability to write a sum of n number of terms, where n is an (infinite) integer.

Since these two problems are the same for every approach it neither encourages nor discourages the use of NSA. When the integrals are introduced using the limit-operation, determining the rules for calculating with integrals draws on the techniques for operating with the limits but as specified before the limit-operation lacks (most of) its theoretical block, i.e. the rules for calculating with integrals end up being based on a lacking foundation (Winsløw, 2013). Sadly in both ways of defining the integral, the way to compute an infinite sum is not something that any human can do, thus no shortcuts are created. The need to identify the definite integral with variable endpoints as the inverse of finding the derivative is still needed to compute integrals of functions that are not linear. When introducing integration by substitution, the notion $\frac{du}{dx} = g'$, when $u = g(x)$ is recognized as the inner function, which makes it possible to isolate $dx = \frac{du}{g'}$ in a mathematical correct way, since the quotient $\frac{du}{dx}$ is just a quotient of 2 infinitesimals. This further illustrates some of the intuitive understandings used when introducing infinitesimal calculus with NSA.

When using NSA a possible problem could arise for the students who were to study mathematics after high school, i.e. on a scientifically higher level. The problem lies in when the remaining intuitive reasoning is abolished, i.e. when starting to use the axiomatic system as a basis for mathematics. When using the axiomatic system the numbers need to be constructed in a rigorous way. As seen in section 2.1, the construction of the hyperreal numbers is somewhat more difficult than the construction of the real numbers. Another problem consists in the availability of help outside of class, although this problem might be considered less significant, if infinitesimal calculus using NSA really is that much easier to learn.

One of the more profound differences in teaching infinitesimal calculus by NSA is, that the "tool" to establish the definitions of the differential quotient and integral are shifted, from the commonly (at least in Denmark) use of limit operations, to the use of a new set of numbers and its connection to the real numbers. This makes it possible to introduce problems that only concern how the "tool" functions, whereas questions as to why limit operations function like they do (all) hinges on the completeness of the real numbers, which is hard to understand intuitively for the students.

As mentioned before a pre-university school in Geneva has used NSA for a period of time, the article "Nonstandard analysis at pre-university level: Naive magnitude analysis" (O'Donovan & Kimber, 2006) contains their view on the matter as they experience it.

2.2.3.1 Side effects

Here are some other moments where NSA is applicable.

Besides infinitesimal calculus, hyperreal numbers can be used to describe phenomena where the real numbers come up short. An example could be the last distance between to colliding objects, which can be described with a hyperreal number, namely an infinitesimal.

People who acknowledge $1 = 0{,}99\ldots$ as being clear as daylight are a rare sight. Using NSA one would find that $1 - 0{,}99\ldots = dx$ is an infinitesimal when $0{,}99\ldots$ is considered a hyperreal number. Taking the standard part on both sides makes it clear, that if the two numbers are both seen as real numbers then $st(1 - 0{,}99\ldots) = st(dx) \Leftrightarrow st(1) - st(0{,}99\ldots) = 1 - 0{,}99\ldots = 0$, i.e. the real numbers 1 and $0{,}99\ldots$ are equal.

3 Concrete implementation of the knowledge to be taught

There goes a lot of work into executing a course of teaching on any level. In teaching high school students, a transposition of knowledge is needed. Other than the teacher, high schoolers' primary source of knowledge is the textbooks provided by the school; hence, the plan was to provide them with some reading material. Seeing that the curriculum for high school seldom changes, the textbooks already published rarely change either. Particularly "new" theories for sectors or themes are almost never considered to be implemented. With the nonstandard analysis being introduced in the 1960s, this approach has not been established in any form in the curriculum, hence no textbook that covers this approach in Danish exists, and it has to be in Danish by the high school regulative. In accordance to ATD the students are to study the works of others when acquiring new knowledge if needed. In this study most of the taught material would be acquired through intuition provoked by the teacher during class, hence the need of a textbook material especially emerged in the event where students would be absent. In this case the students could not be expected to obtain the knowledge they missed by being sick, or in other ways being unable to attend class. Even more crucial would be if the students were to study for a potential exam, both oral and written. These were the main reasons for constructing compendiums for the students, describing everything that was gone through in class.

It made perfect sense to start writing the compendiums alongside the planning of the teaching, in that the compendiums should describe the planned teaching. It became evident though, that creating the compendiums before commencing the teaching was too comprehensive with the time available. In getting to know the students, the compendiums could be shaped in a way they would better understand and be written in the same tone, as was used in class. This approach also made it possible to include specific moments that occurred during class, like special headings for paragraphs in the compendiums, taken from the students' input. An extra exercise for the students to Worksheet 04.11 (Appendix 1.2), on finding the slope for a function, was to make a heading for the worksheet and one of the groups came up with *"Tangentman and 2point finds Miss a"*, as a reference to finding the slope a for a linear function, using two points, which is also the slope of the tangent. Also, sections could be rewritten as to match the exact thing that made the students understand something and others could be skipped completely, because the students would get the understanding faster than expected. This also sat well with the intention of giving the students the compendiums gradually (bit by bit), since then the incorporation of the students' contributions could be used to adapt the compendiums to the level of understanding the students possessed.

Since the best way of introducing the hyperreal numbers for students in high school, is intuitively through previously known sets, like the real numbers, the very first section is on number sets in general, to make sure the students were familiar and comfortable with this. Due to time restrictions and reflections on what actually takes advantage of NSA, certain subjects were left more or less untouched. As such, the compendiums simply provide the curriculum-intended knowledge to be taught using NSA and consequently leave subjects that can still be taught later, without NSA, even by other teachers.

Thus, three compendiums were developed: *Hypertal og standardstjerner*, *Differentialregning*, and *Integralregning* on the set of hyperreal numbers, differential calculus, and integral calculus respectively. Essentially, the compendiums are shaped the same way. They consist of both an algebraic and analytic way of describing the subject at hand, as well as an introduction in various forms. Every chapter contains the constitutive definitions, theorems and proofs for each subject, with relevant examples. An excerpt from the compendium *Hypertal og standardstjerner* is seen below.

1.3.1 Kontinuitet

Definition 1.3.2 - Kontinuitet

> En funktion $f(x)$ er kontinuert i et punkt $(x_0, f(x_0))$, hvis en infinitesimal ændring i x-værdien giver en infinitesimal ændring i y-værdien (funktionsværdien). Dvs. $f(x)$ er kontinuert i $(x_0, f(x_0))$, hvis
> $$\Delta y = {}^*f(x_0 + dx) - f(x_0)$$
> er infinitesimal.

Hvis en funktion siges at være kontinuert (der udelades altså i hvilket punkt), så er den kontinuert i hele dens definitionsmængde.

Den besøgende alien udbryder: "Hvad f. sker der for den der stjerne der er foran det ene f?"
Den kloge lærer: "Det er rigtigt, der er noget med den der stjerne, er der nogen der her nogle idéer til hvad den kunne betyde?"
Alien: "Bipbopbip! Nu har jeg lige kigget på alle tallene i definitionsmængden for funktionen f, og der står der altså ikke noget om at $x_0 + dx$ ligger der i."
Den kloge lærer: "Korrekt, funktioner er (normalt) kun defineret for reelle tal, der er altså ingen infinitesimale eller uendelige tal i definitionsmængden til funktioner. Dette er grunden til stjernen er de stjernen udvider altså funktionens definitionsmængde til at indeholde de hyperreelle tal."

1.3.1 Continuity

Definition 1.3.2 - Continuity

> A function $f(x)$ is continuous in a point $(x_0, f(x_0))$, if an infinitesimal change in the x-value makes an infinitesimal change in the y-value (function value). I.e. $f(x)$ is continuous in $(x_0, f(x_0))$, if
> $$\Delta y = {}^*f(x_0 + dx) - f(x_0)$$
> is infinitesimal.

If a function is said to be continuous (in that no specific point is given), then it is continuous in its entire domain.

The visiting alien exclaims: "What the f. happens for that star in front of one of the f's?"
The smart teacher: "That's correct; there is something about that star, anyone's got any ideas as to what could mean?"
Alien: "Bipbopbip! I have just looked through all the numbers in the domain for the function f, and it does not say anything about $x_0 + dx$ is in it."
The smart teacher: "Correct, functions are (normally) only defined for real numbers, hence there are no infinitesimal or infinite numbers in the domain of functions. This is the reason for the star to be there, thu the star extends the domain of the function to include the hyperreal numbers."

Excerpt 3.1

Excerpt 3.1 is an example of a rigorous definition of continuity, with a touch of the tone from the classroom. Here is also one of the many visits from the alien, which is used throughout the three compendiums. The alien was used as a way of asking questions the students might ask, in the way the students might ask them and also to add a dash of humor. With the continuation of the same alien from start to end, a story unfolded, which was to help the students engage the compendiums with enthusiasm and not just see it as a chore.

All chapters, with the exception of one, end with tasks related to the specific chapter, e.g. exercises and proofs of rules of calculations, and may include variations of exercises from previous chapters. If no separate worksheets were prepared for a lesson, these would make a substitute worksheet for the subject at hand, hence letting them complete the exercises during class, for the most part, and thus practicing and strengthening their skills for written exams. Subsequently, in each compendium, there is an overall list of results for the exercises. The compendiums *Differentialregning* and *Integralregning* include a recap of the highlights of the entire compendium, particularly containing a list of functions, with their corresponding derivatives and antiderivatives respectively. Said lists encompass a few more functions (a^x, $\ln(x)$, $\sin(x)$, $\cos(x)$) than actually taught, because even though they weren't familiarized with these types of functions yet, it was just to complete the lists in regards to the curriculum.

4 Presentation of observations

4.1.1 Context

The observational context has not been specified but it is needed to understand the context for which the teaching and observations were carried out. The teaching and observations was carried out in a Danish high school, but the Danish high school is not a very specific term; in Denmark exists a number of different High schools which have different government-given curriculums. In this study the high school used to make the observations is the most common high school called gymnasium. The gymnasium is the high school in Denmark which enables the students to be accepted for the largest number of different university educations. In this regard, the students attending the gymnasium have very different intensions as to what the high school diploma has to be used for. The research was carried out in a larger than most gymnasiums, *Roskilde Katedralskole*, this high school has approximately 1400 students, from which a class of 23 first year students with A-level math was available for 2 months, corresponding to 30 lessons of 65 minutes each. A-level math is the highest level of mathematics a student can attain in high school, there are again a number of ways to attain this level. The class used for observations was a class that chose to have mathematics at the highest level from day one. To this end some of the students could be surprised by how “different” the math is, since they had no previous knowledge of what and how mathematics are taught at high school. This being said the class chose to have the highest level of math, which usually means that the students are, at least to some extent, interested in science.

The class had previously been taught the entire curriculum regarding triangles, linear- and exponential functions and descriptive statistics, plus various rules for calculating with powers and logarithms. Before the study commenced the class had graced the topic of power functions. According to the policy of the school differential- and integral calculus is part of the curriculum to be taught on their second year of high school, which makes sense in terms of the types of functions with which the students are familiarized before learning how to derive and integrate them. In addition, because of how the educational system is put together, graduating from public school does not always yield the concept of a function. Thus, “what is a function” is an isolated course of teaching in high school, which takes some getting used to, mainly through the teaching of different types of functions. The class had previously had 2 other teachers which both said that the class consisted of students with greater than average level of knowledge regarding mathematics. The first of these teachers were the one who taught them during their introductory course of mathematics, which takes up the first half year and is the same for every level of math. After finishing this introductory course the students was appointed a different teacher, this is the teacher who allowed the use of the class for this study.

Seeing as one of the authors of this thesis already had a position as teacher on Roskilde Katedralskole the possibility of “borrowing” a class from this particular school was significantly enhanced. In addition to teaching the class this study is about, the teacher had two other classes with third year students with A-level math and had been a teacher for the last six years, both at Roskilde Katedralskole and another gymnasium, teaching math and Spanish whilst simultaneously studying at the University of Copenhagen, on and off since 2006. The other author of this thesis had the opportunity for a first time experience teaching at the school, having been studying both math and physics at the University of Copenhagen since 2008.

4.1.2 Overall planning

The overall planning of the teaching is based on the MO's described in the previous section. The order of teaching, with the hyperreal numbers first, then differential and lastly integral calculus, was chosen almost before the thesis was started, since doing it in another way is hard to justify. One could ask why the hyperreal numbers were taught as a separate subject and not just as a part of doing infinitesimal calculus. The reason for doing it as a separate thing is that the hyperreal numbers are seldom something the students have heard of before, justified by only a small part of mathematicians knowing what NSA is and even fewer use it for teaching. Another reason is that in order to make use of the knowledge based on the hyperreal numbers as arguments for proofs and definitions in the infinitesimal calculus a thorough understanding of the hyperreal numbers is needed. In planning the course of teaching, the fact that there is not any Danish high school textbooks for neither differential- nor integral calculus using NSA, was addressed by composing three compendiums specifically for teaching these subjects. The compendiums were written during the period of teaching, while discovering what the students needed to understand to meet the requirements stated in the curriculum and to what extend the mathematics had to be explained for the students to be able to understand it. As such, the compendiums were created to suit this specific class hence another class might need more or less guiding during the teaching of this material.

Every lesson was planned separate from the next and seldom before the end of the previous lesson. Worksheets were made for (almost) every lesson, which the students had to complete before the end of the lesson. This was to get them to use what had been discussed in plenum, in groups or by themselves.

The lessons were planned using the words and explanations of the students, both from the worksheets, which were, more often than not, collected after each ended lesson, but also from what was said in the classroom.

Other than the lessons, four hand-in assignments were prepared for the students. They included primarily exercises in the subject up to the point of how far the class had progressed. The students were to hand in the assignments approximately once every other week. These assignments were to be done outside of the lessons for the most part, even though the regulative on the high school in question forced some of the lessons to be used to do the hand-in assignments. The assignments were used to check if the students had the required skills and to establish what holes, if any, the students had. Furthermore the assignments were used as a set of exercises for the students to hone their techniques. When grading the papers the teachers read through every assignment and when a mistake was discovered the teacher posed some clarifying questions.

4.1.3 Observational methodology

In order to take notes of the observations done during the teaching process, a certain way to gather data-material had to be established. Since two teachers were present in every lesson, the decision to let one of them take field notes on a laptop during the lessons was made. Seeing that the teachers themselves planned the teaching, it made it easier to take field notes in the form of a transcript (marked as Logbooks in the digital appendix), in that the observer already knew which questions the teacher (of that lesson) would be asking; a method used, when what the class did was orchestrated in plenum. When the students worked in groups, or by themselves, both teachers acted as teachers, which might not be the most objective way of gathering observations. The reason for this approach, though, was that both teachers had to operate as the main teacher at some point, which would have made it difficult for the students to let one of them operate as a third party observer, when they also had the understanding that both were their teachers. A positive attribute of this approach is that both teachers were able to shape the teaching and in this way the students became familiarized with both teachers. After the students had done group- or individual work, the teachers would share thoughts of what they each observed to further the teachers' understanding of the students' abilities. The observations done by the teachers, and the answers given by the students, were then compared in order to see if the teachers' understanding of the students' abilities were on par with the students' level of work.

Since field notes do not always generate one hundred percent objective accounts of the teaching surveyed, another method of observations was also installed. Audio-recordings of every lesson were made in order to compare the field notes and audio, in case of dubious objectivity. These audio recordings could also be used as a secondary tool, to show some of the students' understanding and abilities in a much more objective manor, than the field notes.

Worksheets, and especially hand ins, were collected and used as a tool to assess the students' progress and development of the taught MO. On a few occasions, video recordings were made; this was when the students would do proofs on the blackboard. These videos served two objectives: firstly, the students could use it as an example of how the oral examination would unfold and secondly, the video acts as a third way to observe the students' work. This type of observation is especially useful when students do work at the blackboard where both the students' explanations and writings can be captured.

Since these observations generate a rather large and cumbersome set of data, it had to be organized in a way, which did not remove too many of the essential points observed, but still condensed the data in such a way that an analysis can be conducted. This approach was done in what could be called a preliminary organization of the teaching observed; these organized sets of data can be found in the Digital appendix.

4.1.4 Method of analysis

The analysis of the three MO's unfold in the following way:

- Firstly; the gathered data was organized in tables as seen in the appendix 1, which is organized in 5 columns: S*hort description of the mathematics involved*, *didactical moment*, *main player*, *mathematical objects involved*, and *didactical activities.* This acts as both a first analysis and an organization of the data in order to further analyze the teaching process.
- Secondly; based on the organized data an analysis table, using ATD, was conducted in appendix 2. The table consists of 6 columns: *Lesson number*, *type of problem*, *mathematical technique*, *technological theoretical elements*, *didactic moment(s)*, and *elements of the didactical techniques.*
- Thirdly; a description of the findings in the analysis table will be presented and elaborated.
- Fourthly; a discussion of the teaching process will be executed.
- Fifthly; some concluding remarks of the teaching process will be established.

With the analysis of the three MO's a general discussion of them is done as seen in section 4.5.

4.2 Hyperreal numbers

The teaching of the hyperreal numbers was based on the HMO as presented in section 2.2.2.1. The HMO formulates what is needed in order to use the hyperreal numbers to introduce infinitesimal calculus but it does not give any inclination of how to introduce this new set of numbers. The following section describes how the planned teaching process was developed from the already established knowledge to be taught and an analysis of said teaching.

4.2.1 Planned teaching

In order to introduce the hyperreal numbers the concept of numbers was brought up as an introductory subject, which included all the normally used sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. This subject is another part of the curriculum in the Danish high school, thus making it advantageous to build on something that already had to be taught (Ministry of education, 2013). The general idea behind teaching the hyperreal numbers was never to make a construction of the numbers but as already explained an intuitive approach had to be found. As such, the sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and the intuitive inclination that infinitely small and large quantities exists, made the foundation for how to introduce the hyper real numbers. To this end a set of activities were made which would end up with an intuitive creation of the hyperreal numbers as a set. With this kind of introduction the standard part of a hyperreal number would follow as a logical connection between the real and hyperreal numbers. The star operation was reduced to what was needed in order to write up correct mathematical statements, i.e. it was introduced as a way to extend the domain of a function such that ${}^*f(x + dx)$ makes sense, and such that the value could be found as the function evaluated in a quantity of 2 parts.

In order to introduce the notion of infinitesimals and infinite numbers, the intuition were used. To this end an infinitesimal was introduced by giving the students the task of letting their phone drop, and consider the speed of the phone during its fall, especially what the first non-zero speed was. The answer to this question would be the beginning of the intuitive understanding of what an infinitesimal is. To build on this intuitive understanding the students were to make a definition of the infinitely small (speed).

In order to use the intuition to introduce the infinitely large quantities, the hunger that only teens have in the afternoon was used. This was introduced in the last lesson of the day, i.e. from 14:45-15:50. The students would surely be hungry, hence the plan was to open a bag of potato chips and ask exactly how much the students wanted the chips, to try and get them to say infinitely much (a commonly used Danish way to describe something really desirable). With this intuitive understanding, the students were to make a definition of what the infinitely large (yearning) was.

With the definitions of the infinitesimals and the infinite numbers the students were then to produce a number line including the new found quantities, i.e. the hyper real numbers.

With this introduction of the hyperreal numbers the notion of continuity could be (re)created by considering the wording of continuity: “Continuity is something that can be drawn without lifting the drawing device”, and how close two points on a graph should be if one was told the function was continuous. To be sure the students would get the right definition with this line of questioning, sub-questions were added: “given a point, x, what is the point on the graph corresponding to that?” and “what is the difference in function value for a function, which can be drawn without lifting the drawing device, when the difference in the variable is infinitesimal?”.

In order to give an overview of the teaching process of the hyperreal numbers the following table has been produced. It consists of 4 columns the first of which indicates the lesson number, date and time period, the second indicates the mathematics to be taught, the third a short description of how the mathematics were to be taught and lastly what part of the planned teaching was actually taught.

Lesson #	Mathematics to be taught	Planned teaching	Executed teaching In lesson #
1 03.30 (14:45-15:50)	Numbers, i.e. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ Real number line.	Whole class discussion of numbers and the classification of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. By asking the students what numbers are and how they should be organized.	**1**
	Infinitesimals Infinite numbers.	An assignment introduce infinitesimals: Drop your phone and find the first nonzero speed. Students write down a definition of infinitesimals. Infinite numbers introduced as a measure for their craving for the chips brought. Students write down a definition of an infinite number.	
	Number line including infinitesimal and infinite numbers.	In which set of numbers, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, should the new quantities be put? Then how should this new set look like? Draw a number line including infinitesimals and infinite numbers.	
2 04.05 (08:10-09:15)	Repetition of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and definition of infinitesimals.	The repetition was done in plenum while the teacher writes on the blackboard.	**2** The Lesson ends with the students doing the first 5 exercises in compendium about hyperreal numbers.
	Number line with infinitesimals	Draw a number line with infinitesimals as an exercise to the students in small groups (of order 2-4). Ends with a drawing of a number line including infinitesimals on the blackboard.	
	Addition, subtraction, multiplication and division with infinitesimals.	Plenum, with every operation given as a question to the students. Lastly the students are asked to draw infinite numbers $1/dx$, $1/dx$ +1, on the number line with infinitesimals.	
	Standard part	With the operations for infinitesimals the students are asked to find the real numbers closest to $7 + dx$, $4 - dx$ and dx. Teacher defines the operation as the standard part, ending with the question, what is $st\left(\frac{1}{dx}\right)$?	

	Function, domain of a function.	Plenum questions; What is a function, can you give examples, What is the domain of a function? Teacher writes/draws answers on the blackboard. Ending with the definition of a function as a connection between two variables.	
	Continuity of a function in a point.	Plenum questions: What is continuity of a function? If the graph of the function can be drawn without lifting the pencil, then what is the smallest distance between 2 points on the graph? Remember the difference in function value, as $\Delta y = f(x + \Delta x) - f(x)$?	
	Star operation	Is $x + dx$ part of the domain for a normal function? The star operation is introduced as a way to extend the domain and range of the function to.	
3 04.06 (10:40-11:45)	(converted) How to operate with hyperreal numbers and standard parts.	They had to do exercise 6-15 at home which were presented at the blackboard by the students, if time permits it, they can ask questions regarding the first hand in assignment.	**3** (time did not permit questions for the hand in)
	Continuity	The teacher gives an example of how to check if a function is continuous in a point.	
4 04.08 (08:10-09:15)	Function evaluation in a quantity of 2 parts or hyperreal number, standard part, star operation.	They do exercise 16-20 from compendium about hyperreals.	**4**
	Determine which numbers are in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Def. of infinitesimal and infinite numbers. Standard part of hyperreal numbers. Equation solving. Continuity	Time to finish up working on the hand ins. The teacher can answer relevant questions to the problems in the hand in.	
5 04.13 (14:45-15:50)	(converted) Determine which numbers are in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Def. of infinitesimal and infinite numbers. Standard part of hyperreal numbers.	10 minutes to read through the comments to their hand in, then Plenum discussion of every exercise in the hand in.	**5**

	Equation solving. Continuity		

While planning the teaching, the challenge of producing relevant and intriguing tasks to introduce a new set of numbers became evident, since quantity is mostly a theoretical part of a praxeology. This is not a challenge which is solely evident when teaching the hyperreal numbers, the same problématique can be found in introducing the real numbers when only the rational numbers are known. The problématique can be further enlightened by the fact that every kind of mathematics is built on some concept of quantity, be it magnitudes, ratios of magnitudes as Euclid (Heiberg et al., 2008) described them in his book 5 or numbers as perceived by later mathematicians. To this end teaching mathematics without quantity is like speaking without words. When imposed with the problem described above, the first task that came to mind was to ask the students to construct the hyperreal numbers. This idea was quickly abandoned, since the students of high school should never construct neither real nor hyperreal numbers. This limitation made the available constitutive tasks diminish from a rather large lake to a puddle with murky water, meaning, all the mathematics involved in constructing a new set of numbers vanished and left was only exercises similar to ordinary calculations with quantities, such as addition, subtraction, multiplication and division. The constitutive tasks left were to define infinitesimals and infinite numbers.

4.2.2 Analysis of teaching the hyperreal numbers.

A list of the types of problems found during the teaching process will provide an overview of the practical part of the HMO that was actually taught.

The sub-indices represent a sub-problem of the problem with the same index. The constants a, b and k are real, the variables x and y are real, dx and dy are infinitesimals, and α and β are hyperreal numbers.

$\boldsymbol{HP_1}$: Specify into which sets, infinitesimals and infinite, numbers belong.

$\boldsymbol{HP_2}$: Determine if a given function, $f(x)$, is continuous.
$HP_{2,1}$: Given a graph of a function determine if the function is continuous.

$\boldsymbol{HP_3}$: Find $k \cdot st(a \cdot x + b \cdot dx + c \cdot dy)$
$HP_{3,1}$: Find $st\big((a + b \cdot dx)(c + k \cdot dy)\big)$
$HP_{3,2}$: Find $st\big((a + b \cdot dx)\big)st\big((c + k \cdot dy)\big)$
$HP_{3,3}$: Find $st\left(a \cdot dx \cdot \frac{1}{dy}\right)$
$HP_{3,4}$: Find $st(a \cdot dx)st\left(\frac{1}{dx}\right)$
$HP_{3,5}$: Solve the equation and find the real number closest to x, where the equation both contains x and dx

$\boldsymbol{HP_4}$: Find the hyperreal function value ${}^*f(a + kdx)$ for a given real function

$\boldsymbol{HP_5}$: What is $st\big({}^*f(x + dx)\big)$ for a real function?
$HP_{5,1}$: What is $st\big({}^*f(x)\big)$?
$HP_{5,2}$: Let f be a continuous real function, what is $st\big({}^*f(x + dx)\big)$?

These problems can be classified into the types of tasks from the HMO, which are presented in the table below.

Types of tasks in HMO	HMO= $[\mathrm{T}, \tau, \theta,]$
HT_0	In what set of numbers does a number x belong? $\boldsymbol{HP_1}$
HT_1	Find $st(a)$ when $a \in {}^*\mathbb{R}$? $\boldsymbol{HP_3}$
HT_2	What is ${}^*f(x + dx)$ when dx is an infinitesimal and f a real function? $\boldsymbol{HP_4}$
HT_3	Find $st\big(a \cdot {}^*f(x)\big)$ when $a, x \in {}^*\mathbb{R}$ $\boldsymbol{HP_5}$
HT_4	Check if the real function f is continuous (at a point) $\boldsymbol{HP_2}$

In the analysis table (appendix 2.1) different parts of the practical block belonging to HMO can be observed. An example of this, is the first encounter with HT_1, when the problem HP_3 was introduced.

An exploratory didactic moment happens when the students establish the technique of envisioning a hyperreal number as a sum of a real number and an infinitesimal.

This constitutes the technological/theoretical part of operating within the real and hyperreal numbers. Through working with HP_3, and its sub problems, the students do technical work on the already established technique. This enables the students to get an intuitive understanding of the connection between the real and hyperreal numbers, through the standard part. Thereby constructing the following techniques to answer problems of type HT_1:

$H\tau_{1,1}$:	Recognize the hyperreal number as a real number plus an infinitesimal, i.e. $st(\alpha) = st(x + dx) = x$.
$H\tau_{1,2}$:	If the hyperreal number can be seen as a sum of two finite hyperreal numbers then one can take the standard part of each of these numbers, i.e. $st(\alpha + \beta) = st(\alpha) + st(\beta)$.
$H\tau_{1,3}$:	If the hyperreal number can be seen as a product of two finite hyperreal numbers then one can take the standard part of each of these numbers, i.e. $st(\alpha \cdot \beta) = st(\alpha) \cdot st(\beta)$.

With this overview, the practical blocks of the actually taught MO and the MO to be taught, HMO, are very close to one another.

The outcome of the taught MO is in part for the students to be able to do infinitesimal calculus via NSA. Another outcome is for the students to be able to determine if a real function is continuous, which also serves as an example of the institutionalization, and partly the moment of evaluation. When the students use the following variant of the nonstandard definition of continuity, $st({}^*f(x + dx) - f(x)) = 0$ for every infinitesimal dx, to check if a function is continuous, it is obvious that the theoretical part of the punctual MO's in HMO overlap with their previous knowledge about functions.

One could even argue that by introducing a set, then subsets of said set become more tangible, i.e. by introducing the hyperreal numbers the real numbers are better understood. Since the theory is mostly intuitive, the technological level comes down to logical deductions. Based purely on intuition, mathematical statements can be produced, which enables further mathematical deduction. Take the standard part as an example. The students' definition of this operation is to find the real number closest to the hyperreal value, which leads to the fact, that the standard part of an infinite number is meaningless. This can be extended to establishing that if the standard part of a fraction with an infinitesimal in its denominator exists, then the numerator has to be an infinitesimal as well, i.e. a differentiable function is continuous. It could be argued that the technological level is not on par with how every high school does it, but this is part of the curricular restrictions, since some high schools might construct the real numbers. Furthermore, the notion of the subsets of the hyperreal numbers, ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, which correspond to the subsets of the real numbers, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, is not thoroughly treated. In the end, the teaching of the hyperreal numbers were enough to satisfy the students' curiosity as to how the intuitive understanding of the world can be explained and sufficient to be used as a foundation for the infinitesimal calculus to be taught.

4.2.2.1 The didactic processes observed while teaching HMO

First encounter with HMO as observed is when the students intuitively understand, that the first non-zero speed of their dropped phone is less than any real value, i.e. the lack of quantities to describe the things the intuition can perceive.

This first encounter can practically be seen as the obvious choice. The infinitesimals are better suited to be intuitively perceived than the infinite numbers, which is the only other possibility of first encounter with HMO, when not constructing the hyperreal numbers. The intuition makes it possible to discover something that is not zero, yet less than any positive number writable, whereas imagining something that is greater than anything writeable is more difficult to visualize, since the "end" of the real numbers is not very well-defined.

The infinitesimals are used more than the infinite numbers, when introducing infinitesimal calculus; even when the integral is defined as an infinite sum it is an infinite sum of infinitesimals. Historically the infinitesimals were used in a longer period of time than the infinite numbers, which can be seen in Cauchy's Cours d'analyse (Bradley & Sandifer, 2009). Cauchy describes the infinitesimals as sequences with limit zero but his infinite numbers only consist of plus or minus infinity (more or less the same way infinity is described in standard analysis).

An exploratory moment concerning the HMO, in particular the standard part, can be said to happen when the students figure out that the technique for this is to remove infinitesimal(s); they are virtually zero!

The constitution of the knowledge block can be seen to happen when the students apply the operations, plus, minus, multiplication and division to the notion of infinitesimals. I.e. the constitution can be said to happen when the students develop the concept of the real numbers including infinitesimals, thus constructing the hyperreal numbers intuitively.

By applying what can be considered technical work on the technique for the standard part the students develop what has been denoted $H\tau_{1,2}$ and $H\tau_{1,3}$.

While the intuitive construction of the hyperreal numbers is a superstructure of the real numbers, all the regular sets of numbers, including the real numbers, are still able to institutionalize the HMO as a set of numbers. The set of hyperreal numbers, ${}^*\mathbb{R}$, is governed by the same rules that apply to all the other sets, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, which can make a moment of evaluation possible. This is seen to be true in the sense that the operations with the hyperreal numbers do not allow for operations that normally cannot be executed.

4.2.2.2 Discussion (Hyperreal numbers)

With this introduction to the hyperreal numbers the students acquire theoretical knowledge, but so far the students' only reason for learning this subject is to be able to describe what their intuition describes. The NSA way of defining continuity is another example of a reason for learning this subject, but the students do not seem to appreciate the definition on the same level as teachers and other scholars of mathematic might do. The students simply lack the reason for why the characteristics of a continuous function is so desirable, they are rarely going to use the continuity of a function to prove something general for continuous functions. The students encounter nigh on only the elementary functions (linear, exponential, power, polynomial and logarithmic functions, cosine, and sine), which is why the classification of continuous functions seems almost superfluous, since most students' perception of a function is gained only by working on the elementary functions.

When introducing the infinitesimals, a way to make the taught MO more edible, could be to let the students do more work on them before trying to get them to define them mathematically. This could make some of the properties of the infinitesimals easier to accept and digest. One possible way to introduce this would be to let the students add all the starting speeds of their dropped phones to see if it would end up being a real speed. In the observations when the students were asked what $20dx$ is, after their preliminary

definitions, the answer, that it was infinitely small, came so abruptly that the approach used by letting the students define the infinitesimal before operating on it might not have made a difference.
With the relative small number of different types of techniques to solve tasks where the standard part needs to be found, a more thorough routinization of the techniques, $H\tau_{1,2}$, $H\tau_{1,3}$, would have been preferable. Finding the standard part as explained before is a purely intuitive thing, but the techniques are something more; this point should have been made clearer for the students. Especially since these are the primary tools to prove rules of calculation within infinitesimal calculus. Teaching the hyperreal numbers in general was very simple when done in the intuitive way. This made it easy to develop the techniques, but just as easy to forget the importance of. The number of tasks underlining the importance of the techniques were not sufficient to establish the sought knowledge regarding the techniques, $H\tau_{1,2}$, $H\tau_{1,3}$.
In conclusion the hyperreal numbers is not challenging to teach when introducing it through intuition. The observed taught MO and the HMO presented in the previous section are more or less equal.
When looking at the hyperreal numbers as a tool to introduce infinitesimal calculus, it can be concluded that the theory is well understood in this high school class.

4.3 Differential calculus (Jonas Kyhnæb)

The teaching of differential calculus was based on the DMO as presented in section 2.2.2.2. The DMO formulates more than what is needed in order to fulfill the required curriculum for differential calculus but it does not give any inclination of how to introduce this subject. The following section describes how the planned teaching process was developed from the already established knowledge to be taught.

4.3.1 Planned teaching

Leading up to differential calculus is typically a course of different types of functions, inevitably covering linear functions. This was the case for the class to teach as well. Furthermore, this was one of the last courses they went through, the only exception being a course on exponential functions. Hence, the students had fairly recently worked a lot with the 2-point formula, the 2-point formula being the formula for finding the slope of a linear function through two given points on the graph. As such, this was used as a stepping stone to the new knowledge, namely the slope in a point. To this end, two worksheets were prepared, both of which were exercises with graphs. The first worksheet was on finding the secant and the other on finding the tangent. Even though the slope in a single point wouldn't immediately make sense for the students, they should get the intuitive understanding of this through these worksheets. In this way, the more technical work of finding the differential quotient would become more comprehensible. The students were of course well-suited for finding the slope through two points and thus the further implementation of the meaning of the slope in a point was, to get the students to establish an understanding of how to find the best two points for the straight line to go through. Another worksheet was prepared for this, which would lead them to the definition of the differential quotient. A big part of this was for the students to not draw any graphs at all but simply use their knowledge about straight lines and that obtained in HMO. In doing this, the students find the definition of the differential quotient themselves, based on previously established praxeologies. This led to the possibility of creating several general problems in finding the differential quotient, beginning with a series of problems with a hidden agenda. Alongside finding the differential quotient, given different values, the students should find the function values for the same values. This was both for the students to better get familiarized with finding function values and to see the connection between functions which added together give a new function. The point of this being for the students to recognize the "pattern" and use it on the differential quotients as well, to get an informal introduction to the differential rule for summation.

To further link the work the students would have done on graphs and the algebraic calculations on finding the differential quotient a worksheet on the conditions of monotony was created. This worksheet contained a single question, simply to say what applies to the original function, when given the derived function. Here the students could use the knowledge already obtained, as well as the internet, since no information was given beforehand, not even what conditions of monotony means. The worksheet was constructed in such a way, though, that if the students needed help, they would be able to get hints in the form of an example in different variations eventually leading to a generalization and hence the answer to the original question posed on the worksheet.

Due to the amount of converted lessons that needed to be incurred, one of the hand-ins were to be used as a kind of worksheet (Hand-in #2). This would be a way of making sure the students would get the intended knowledge, namely because they were to hand it in and the teacher could observe and help at the same time. This converted lesson was to be put in somewhere where they could work on finding the general derivative for a linear function, specifically a constant term, and polynomial functions. As such, the entire first of the two pages that constituted the hand-in were on finding exactly these. Also, the students were to give a proper explanation of the difference between a slope and a differential quotient.

With the definition of the differential quotient, the students would be able to prove the rules for differentiation. Again, a worksheet was created, which they should be able to do with this self-achieved definition and the knowledge from HMO. With a certain kind of time issue in mind, the rules to be proved through this worksheet should be the product rule and the chain rule along with how to find the derivative of $\frac{1}{x}$ and $\sqrt{x}$ and also the proof of how to find the equation of the tangent. Thus, the decision of excluding the quotient rule (the derivative of the quotient of two functions) was made. This was as much due to a time pressing concern as it was the process of the proof being irrelevant, meaning the method is the same for standard analysis as for non-standard analysis. This worksheet should also lead up to the students proving the aforementioned in front of each other, by presenting one of them in groups. This as a way to get each group to completely grasp the proof at hand and then show to the rest of the class, instead of halfheartedly going through them all, which could be the case given the amount of time available. In extension, the students could get new kinds of problems, containing the above. In order to better routinize the students with these types of problems, particularly the chain rule, the derivative for the function e^x was introduced, including its proof.

The table below is to give an overview of the planned teaching.

<table>
<tr><th>Lesson #</th><th>Mathematics to be taught</th><th>Planned teaching</th><th>Executed teaching in lesson #</th></tr>
<tr><td>1
04.08
(09:25-10:30)</td><td>The slope of a graph in a point</td><td>Through the graphs of arbitrary functions and Worksheet 04.08 - Tangent, the students should establish an understanding of the slope of a graph in a point</td><td>1</td></tr>
<tr><td>2+3
04.11
(08:10-10:30)</td><td>The secant and the differential quotient</td><td>With the use of the hyperreal numbers and functions and the 2-point formula on an infinitesimal interval, the students should get to the definition of the differential quotient, by working on Worksheet 04.11</td><td>2+3</td></tr>
<tr><td rowspan="2">4
04.13
(12:15-13:20)
5
04.19
(08:10-09:15)</td><td rowspan="2">The differential quotient</td><td>Routinization by solving exercises in the compendium</td><td>4</td></tr>
<tr><td>The students should go through the exercises on the blackboard, one by one</td><td>5</td></tr>
<tr><td>6
04.20
(10:40-11:45)</td><td>Conditions of monotony</td><td>Through a question about the conditions of monotony for a function, when given the derived function, the students should give an answer by using any and all aids, including the internet and hints from the teacher</td><td>6</td></tr>
</table>

<table>
<tr><td>7
04.21
(14:45-
15:50)</td><td>(converted)
Work with hand in 2</td><td>Finding and generalizing the derivative of the linear function and polynomial function and solving text-based exercises</td><td>7</td></tr>
<tr><td>8
04.25
(08:10-
09:15)</td><td>Conditions of monotony</td><td rowspan="2">Routinization by solving exercises</td><td rowspan="2">8+9</td></tr>
<tr><td>9
04.25
(09:25-
10:30)</td><td>(converted)</td></tr>
<tr><td>10
04.26
(10:40-
11:45)
11
04.27
(14:45-
15:50)</td><td>The derivative of $f(x) = \frac{1}{x}$ and $f(x) = \sqrt{x}$, the equation for the tangent and the proofs for the product rule and the chain rule</td><td>Group work with Worksheet 04.26</td><td>10+11</td></tr>
<tr><td>12
05.03
(08:10-
09:15)</td><td></td><td>The students should present their product from lessons ten and eleven in front of the class; every member should say something</td><td>12</td></tr>
<tr><td>13
05.03
(09:25-
10:30)</td><td>A summary of what have been taught in differential calculus</td><td>The students should fill out worksheet 05.03. In the end, the students’ filled-out worksheets should be compared to generate a final summarization sheet</td><td>13</td></tr>
</table>

4.3.2 Analysis of teaching differential calculus

To give an outline of the practical part of the DMO actually taught, is here a list of the types of problems found during the teaching process. These problems, as well as a multitude of sub-problems, are as seen in the analysis table (appendix 2.2). The sub-indices represent the sub-problem(s) of the problem with the same index. Those depicted in bold are problems to which there is a general technique to solving the exercises subject to this problem. Sub-problems also have a general technique but this might differ from the general technique for the bold problems.

$\boldsymbol{DP_1}$: Find the slope of a given graph in a point by drawing a tangent

$\boldsymbol{DP_2}$: Find an algebraic expression for finding the slope for a function in a point

$\boldsymbol{DP_3}$: Find the equation of a tangent

$\boldsymbol{DP_4}$: Find $\frac{d}{dx}ax^n$

$DP_{4,1}$: Find $\frac{d}{dx}(ax+b)$

$\boldsymbol{DP_5}$: Find $\frac{d}{dx}e^x$

$\boldsymbol{DP_6}$: Show the rules for calculating with differentials

$DP_{6,1}$: Let $h(x) = k \cdot f(x)$. Find $h'(x)$

$DP_{6,2}$: Let $h(x) = f(x) \pm g(x)$. Find $h'(x)$

$DP_{6,3}$: Let $h(x) = f(x) \cdot g(x)$. Find $h'(x)$

$DP_{6,4}$: Let $h(x) = f\big(g(x)\big)$. Find $h'(x)$

$\boldsymbol{DP_7}$: What can be said about the original function, $f(x)$, given the derived function, $f'(x)$?

$DP_{7,1}$: How to find the conditions of monotony for a function, $f(x)$, given its graph

$\boldsymbol{DP_8}$: What have you learned about differential calculus?

The problem $\boldsymbol{DP_8}$ is more a task of reproducing all the techniques already established, than it is a type of problem, but is listed as such, since it simply covers all. All other problems are classified below, into the types of tasks from DMO.

Types of tasks in DMO	
DT_0	Define the differential quotient.
DT_1	Find the derivative, $f'(x)$, for a function, $f(x)$ $\boldsymbol{DP_1}$, $\boldsymbol{DP_2}$, $\boldsymbol{DP_4}$, $\boldsymbol{DP_5}$
DT_2	Determine the condition of monotony for a given function, $f(x)$ $\boldsymbol{DP_7}$
DT_3	Find the tangent of a function, $f(x)$, through a specific point, $\big(x_0, f(x_0)\big)$ $\boldsymbol{DP_3}$
DT_4	Determine if $\frac{d}{dx}f(x)$ exists for a given function, $f(x)$

$\boldsymbol{DP_6}$ is not present in the table since this problem is what can be considered as the technical work done on the hypothetical technique of the differential quotient.

The following is a short description of how two of the more important techniques were developed through the looking glass of ATD.

The first encounter with differential calculus was, for the students, the problem $\boldsymbol{DP_1}$. When given this problem they enter an exploratory state, wherein they can find a graphical technique for finding the slope in a point, namely by drawing a tangent for a graph through two points with an infinitesimal difference and then finding the slope for this straight line. Thus, a constitution of the theory comes with the knowledge of linear functions already established. With the technique of finding the slope in a point being only graphical, new techniques are needed to calculate the slope for a function in a point. As such, another constitution of the theory happens, which includes a previously established praxeology on the concept of a function and the recently obtained knowledge of hyperreal numbers, whence they can establish the differential quotient. Hence, a second technique is acquired for finding the slope in a point. Further development of the differential quotient occurs when the students do technical work on the hypothetical technique the differential quotient can be viewed as to establish the rules of differentiation, $\boldsymbol{DP_6}$. Thus, the students further constitute the knowledge block of DMO. The following shows the main techniques that can be found in the actually taught MO. The remaining techniques, which are more obvious from the problems posed, can be found in the analysis table (appendix 2.2).

$D\tau_{11}$:	Use the graph for a function and the formula for finding the slope for a straight line through two points to calculate slope of the function in a point
$D\tau_{12}$:	Derive functions of the type $f(x) = x^n$ using $f'(x) = n \cdot x^{n-1}$

Taking into account the reasons for $D\tau_{11}$ being the way of introduction to differential calculus, as described in planned teaching, not much exploration nor routinization of this was necessary. Using a previously established praxeology on how to find the slope of a straight line given two points, many students saw it as being remarkably intuitive and just more practice in finding the slope for a straight line.

With this overview, the practical blocks of the actually taught MO and the MO to be taught, DMO, are very close to one another. As seen in the finished hand-ins and on the filled-out worksheet the students managed to understand and use the intended knowledge, with only DT_4 not being explicitly covered by any type of problem. The students realized a part of this themselves through a reference to the HMO about taking the standard part of an infinite number, the part being that the numerator has to be an infinitesimal.

The main problem of teaching differential calculus was finding the derivative of a function. From the very beginning of the teaching course, the goal has been to find the derivatives of different types of functions and to find techniques for finding exactly these. As will be described in the following this started with trying to establish the definition of the differential and use this as a technique to establish techniques which enables finding derivatives of a class of functions.

4.3.2.1 The didactic processes observed while teaching differential calculus

The first encounter with DMO is when the students realized that it is possible to find a slope in a single point, instead of two. This happened when going through an example where to begin with they should find somewhere on a graph where it was the steepest, even though they had not before seen other graphs than linear and exponential, which led to an interval of the graph wherein the graph was at least steeper than other places. This was at first narrowed down to a smaller interval but then a student quickly discovered that, quote “there must be an infinitely small section on the graph where it is steepest”, hence the best interval would be that of infinitesimal length. As such, the first encounter with DMO uses HMO, which further justifies the reason for choosing such an example as an introduction to DMO.

One of the exploratory moments concerning DMO happened when the students established the technique $D\tau_{11}$, hence working with $\boldsymbol{DP_1}$. Given only the graphical representation of a function, this enables the students to construct the technique using the knowledge of the mathematical object, a tangent, and how to find the slope of a straight line. When the students combine these two, they find the slope for a function in a point. Though this technique is usable in solving but a few problems, to let the students do exploratory work like this, with a purely graphical depiction of a function, it gave them a rather relaxed sight on the matter. Seeing that it was so closely related to previously established knowledge, very little was needed from the knowledge block of the DMO.

The exploratory moment of the hypothetical technique happened when the students explored different techniques for $\boldsymbol{DP_2}$. Here, they all ended up finding the same technique, namely that of finding the slope through two points infinitely close to each other, which is almost the differential quotient. The students were all posed the same question, which differed from $\boldsymbol{DP_1}$ by excluding the possibility of drawing any graphs. Hence, they were to find the slope for a function, $f(x)$, in a point, without drawing anything! (see appendix 1.2 worksheet 04.11) The students, knowing from their first encounter it had something to do with a straight line, sought to find the slope by using the 2-point formula on a point and the point that was only an infinitesimal away. The technological knowledge necessary to explain what a slope through a hyperreal point is, does not exist in the MO, so the students rely on the intuitive understanding of the slope going through a point infinitely close to another. To construct the new knowledge the students also used the mathematical object functions. With the hints given (see appendix 1.2 worksheet 04.11) along with the posed question, about the slope, they had little trouble using the established knowledge to get to $\frac{{}^*f(a+dx)-f(a)}{dx}$, only a few forgetting about the star. Every group of students needed help with the last step before getting to the actual definition of the differential quotient, namely taking the standard part. Some students only needed reminding that the question was about finding the slope in a (single) point, others that using the differential quotient could result in a hyperreal number. Thus, the students reached the definition of the differential quotient themselves.

With this definition established, a constitution of the knowledge for DMO happened. An important point is that when the students worked with $\boldsymbol{DP_6}$ they did technical work on this hypothetical technique and thus further constituting the knowledge. With the techniques from HMO they did technical work on the differential quotient as a technique, this by using the established rules for the standard part. Hence working with $\boldsymbol{DP_6}$ contains two didactical moments, namely the constitution of technology for DMO and technical work with the hypothetical technique. The technical work done can be seen in the description of teaching (appendix 1.2) whence the students established the rules for differentiation. Concerning the chain rule, one of the students even went through the proof by saying the inner function could be seen as a variable and saw $st\left(\frac{{}^*f({}^*g(x+dx))-f(g(x))}{{}^*g(x+dx)-g(x)}\right) = f'(g(x))$ as being evident.

In addition, the differential quotient was used for establishing other techniques for finding the derivative of functions, which can also be seen in the description of teaching (appendix 1.2). Here the students found the technique to solve problems of the type $\boldsymbol{DP_4}$ by recognizing a system, through inserting polynomial functions (of one term) of power 1, 2 and 3. Hence, some of the technical work on finding the derivative for $f(x) = x^n$ was done by routinization. The technique for finding the slope in a point by drawing a tangent and finding its slope is also a constitutional moment, be it a smaller one, since it is a punctual MO where most of the knowledge block is known.

Further technical work was done on the technique for finding the derivative of a polynomial function, namely by solving problems of the type $\boldsymbol{DP_4}$, as described above. The students elaborated and expanded the technique $D\tau_{12}$ to also apply to $\frac{1}{x}$ and $\sqrt{x}$. This in turn led to the students being able to derive functions with terms including variables with rational exponents.

The moment of institutionalization was when the students made the connection between the constituted knowledge, being the derivative, and the knowledge about how to find a slope in a point, through tangents to graphs in a coordinate system. Another important part of the institutionalization is the concept of a function and variable. As such, the DMO belongs with the praxeologies about coordinate systems and functions and using functions as a variable.

4.3.2.2 Discussion (differential calculus)

With the introduction to differential calculus being built almost wholly on the knowledge of functions, specifically linear functions, made it rather comprehensible for the students, since this was the second to last subject they had gone through. This made the students more comfortable heading into a new subject in feeling that they actually knew the answers to the questions that was supposedly going to lead them to new knowledge. As such, this became a motivational factor and even more explicitly with the obtained knowledge from HMO, seeing that this had only very recently been taught to them. Hence, if they did not know the exact answer to a question, at least they knew in which sector they should move around, meaning they could narrow down a list of possible answers significantly. In the case of finding the best interval on a graph for drawing a tangent, described as the first encounter in the analysis above, it could be argued that the student only answered the way s/he did, since it was the obvious answer subsequent to the build-up in questions. Regardless, the students were on board and thus the decision was made to skip the entire part about secants. This seemed as the best decision at the time and even more so later when several students got confused when reading the compendium on the matter (see appendix 3 section 3.2). In any case, introducing the subject with a graph describing the battery life for a phone made the students' intuitive understanding of the graph more comprehensible than if it had just been a graph about nothing. Hence, this gave them a chance to answer questions, even if they had difficulties decoding the information the graph would give.

The thing that could have intimidated the students by this approach was the graph being an arbitrarily drawn decreasing one, in that the students had only previously seen graphs for linear and exponential functions. This did not prove to be a problem, though, until getting to do various exercises leading up to the summation rule: if $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$ (see appendix 3 page 21). When finding the slope in different points, one peculiar thing stood out. One student who got annoyed with taking so many steps before reaching a result became convinced there had to be a simpler way. The student argued that it would not make sense to go through so much work for every new value and/or every new function. What even further motivated the student was that the teacher overlooked the student's work and the student asked if the final result was correct. When the teacher used a couple of seconds to calculate the slope in the point and then said yes, the goal for the student was to solve the exercise without writing anything from that point. This resulted in the student realizing that the infinitesimals of higher power than 1 would "disappear" when taking the standard part in the end and as such could be ignored. This is partly the procedure for the proof of the derivative of the function $f(x) = x^n$.

Needless to say, some students created their own challenges after mastering the more elementary, like finding function values given different variable-values, which was the most common problem for the students. Many students struggled with this, but finally realizing the connection between the values, made them almost forget how hard it was and how much time it took them to find the function values which in turn gave the students a personal victory. Here it became evident, though, that the students needed practice in finding function values, which then had to make up many of the future exercises. While finding function values might not seem like it should be the biggest concern, it was a very demotivating problem for the students, in that if they had trouble finding function values how could they learn things on what they felt was a much higher level. This is what shaped the course of conditions of monotony.

As seen in the planned teaching (4.3.1) the course on conditions of monotony was designed for the students to link graphs with algebraic calculations, but also became a way of practicing finding functions values. The conditions of monotony could have been skipped completely, without losing any significant NSA points, but was assessed to be useful for the abovementioned reasons in spite of the potential time problem. Hence, the duration of this was prolonged as to help the students be less "afraid" of inputting values in various functions. This decision was made during the teaching process and was weighed as being worth it for the students, to hone the skill for use in both differential and integral calculus.

Even with the many exercises in finding function values the students had done, it was just this, insert *values* and find function *values*. As such, when proving the rules for calculating with differentials, hence working with functions without inputting number values, the students' skills came up short. This was when doing technical work on the hypothetical technique and thus involving functions as variables. Had the students been more experienced with functions, more than one might have seen that $st\left(\frac{{}^*f\left({}^*g(x+dx)\right)-f\left(g(x)\right)}{{}^*g(x+dx)-g(x)}\right) = f'\left(g(x)\right)$ was the same when considering $g(x)$ as being a variable. The extra challenge of presenting the final product in front of the rest of the class motivated the students to make sure that every step of the proof was fathomed completely. This was done with the aim at practicing for the oral exam. With the hints given, and in the form which they were given, made the challenge all the more "fun", since the students felt like they actually did a proof themselves. It should be noted that these proofs were the first they did algebraically, with no geometrical guidance; the proof for the equation of the tangent being the only exception. This took the students around 15 minutes to prove, from start to finish, meaning from when they got the problem until the teacher came to check on them and they had found the equation for the tangent. No questions, no doubts. In realizing that $a = f'(x_0)$ gave the clear expression that this was extremely well established.

Proving the rules for differentiation might seem odd as a final moment of the teaching process, but the time constraints made it impossible to get through everything related to differential calculus and as such, things like optimization and the amount of routinization were minimized or completely cut. This would normally lay after the proof of the rules, so the students would have all the necessary knowledge to solve the exercises given at a written exam.

To improve the course of teaching one would benefit from letting the students work more with (the concept of) functions, since the majority of complications were related to this. There did not seem to be any trouble with introducing and using NSA, in fact it seemed to be easier for the students to comprehend the parts including this. Thus, the knowledge block established for differential calculus using NSA proved to improve the students understanding. The concerns about whether or not the students answered what they thought the teacher wanted to hear turned out to matter very little. It still takes some realization and in part, an understanding of the problem at hand to even think of a possible answer. That said, the technological and theoretical part of the taught MO took up a lot of the lessons, leaving very little time to do practical work. The time was the foremost restraint when introducing differential calculus. Even though the students understood something one day, they might not remember it the next, especially when not doing much practical work. Taking this into consideration the implementation of DMO seems to be rather successful, with the entire knowledge block covering the entirety of differential calculus.

4.4 Integral calculus

This section covers a short description of the planned teaching as well as a consideration of some of the more important parts of the analysis table. The teaching was based on the MO established in section 2.2.2.3 about what to be taught in integral calculus.

4.4.1 Planned teaching

In order to introduce the integral the commonly used introduction, by finding the area between the function and the first axis over an interval, was chosen. When planning the teaching process for this specific moment a worksheet continuing the story that is present in the compendiums as well, was conducted. This was done in order to make the rather technical work of finding the area between a graph and the first axis more appetizing. When introducing this problem, one could use linear functions, which the students would be able to find the area for, using the geometry they already knew. In this scenario, the intuitive understanding of what the integral is was to be introduced to them before the establishment of what the integral or area functions for linear functions looks like. This was done in order for the students to get familiarized with operating with the integral in a way that is not too technical. As such the students' first task when introducing integral calculus was to figure out a way of describing the area between a second degree polynomial and the first axis. This approach was to ensure that the result of the area were not within the students' comprehension, hence forcing the students to focus more on the "definition" of the integral than on the actual area presented. By comparing the students' results and arguing that the area should be a real number the definition, was to be found.

With the definition of the definite integral a series of problems where the students could find the value of the integral was conducted. As mentioned before, this was to familiarize the students with the new found concept. The problem of not being able to calculate the infinite sum was reduced to finding the area of some geometrical figures the students were able to find. With the definition of the integral and the use of some the more intuitive rules that apply to it, the students were to prove the rules for calculating with integrals. To this end a worksheet was created, which they should be able to do with the definition they produced themselves and the knowledge from HMO. With the integral and the area under a graph as intertwined as this introduction made them, the connection between the area function and the integral with varying endpoint was already established.

The fundamental theorem of calculus was decided to be too difficult a proof for the students to grasp in the rather short time that was available. With this in mind the fundamental theorem was to be stated after a consideration of the infinitesimal difference in the area function, $^*A(x+dx)-A(x)$. This is approximately the infinitesimal multiplied by the function itself, $f(x)dx$. As such an intuitive understanding of the fundamental theorem would have to suffice for the students. The fundamental theorem was to be further backed up by a series of examples originating from the already established definite integrals of linear functions with varying endpoint.

The antiderivative of a function was then to be introduced as the function which when differentiated would give the original function. This would produce a second characteristic, through the use of the fundamental theorem of calculus, of what the antiderivative is namely; a function which only differed from the area function by a constant. The indefinite integral was introduced as an antiderivative which is only determined up to a constant.

The technique, integration by substitution, was introduced as something comparable to the opposite of the chain rule for differentiation, hence it was introduced through integrals which could be solved by recognizing the integrand as the result of the use of the chain rule.

The disc method for calculating cylindrical volumes of objects was not chosen to be taught, since time was a lacking resource, when teaching the integral calculus.

The integral calculus was to be taught over a period of 15 days, accounting for 11 lessons. This resulted in a reduction in the time spent to prepare the teaching and the compendiums, since both had to be finished in the same period of time. This being said, a greater amount of time was used to prepare the compendiums than the actual teaching process, since the compendiums would become more or less the only textbook material they would be able to find using NSA in high school. The production of hand in #4 was, because of the time constraints, changed to be used more or less as a worksheet in order to make sure the students got to do exercises that covered the material needed to be taught, IMO. This choice was based on two considerations, firstly the students became less and less engaged in the teaching the closer they got to the school summer break, thus this change could intensify the students' engagement, since the hand in would be assessed in order to grade the students, secondly this choice also lessened the amount of time used to prepare the teaching by not having to produce worksheets on top of the hand in.

The table below will give an overview of the planned teaching, as with the teaching of the hyperreal numbers.

Lesson #	Mathematics to be taught	Planned teaching	Executed teaching In lesson #
1+2 05.09 (8:10-10:30)	Intuitive understanding of the integral as the area between the graph and the first axis in an interval.	The students work with worksheet 05.09 which consists of one main question of how to express the area between the graph and above the first axis for a second degree polynomial, and some sub questions to help them generate an answer to the main question.	**1+2**
3 05.10 (11:30-11:45)	Definition of the definite integral, and what the boundaries indicate.	A generalization of the infinite sum from the previous lessons, which enables a definition of the definite integral.	**3**
4 05.11 (14:45-15:50)	Definite integrals of linear functions with various endpoints and area functions for said functions.	The students are to work with worksheet 05.11. The worksheet consists of a series of definite integrals of linear functions, which ends with questions of how to find an area function. This allows the students to establish a first technique of how to determine definite integrals of linear functions. In the end the exercise are gone through in plenum.	**4**
5 05.13 (14:45-15:50)	(converted) Definite integrals of linear functions with various endpoints	Further work with the integrals of linear functions	**5+**

6 05.18 (10:40- 11:45)	Rules for calculating with integrals.	By using the definition of the integral in conjunction with the established HMO the student are posed a set of problems concerning the rules of integration in the form of worksheet 05.18. The problems are presented with a list of hints, which enables the students to work by themselves in groups of 3-5 students.	**6**
7 05.20 (08:10- 09:15)	Connection between integral and derivatives, antiderivatives and indefinite integral.	A table is drawn on the blackboard with 3 columns, 1. a column of functions, 2. a column of integrals of the functions and 3. a column of derivatives of the integrated functions. With the table and the graphical consideration of $^{*}\int_a^{x+dx} f(x)\,dx - \int_a^x f(x)\,dx \approx f(x)\cdot dx$ the fundamental theorem of calculus is stated. The antiderivative of $f(x)$ is then a function F, for which $F'(x) = f(x)$. which by the fundamental theorem is the same as $$F(x) = \int_a^x f(x)\,dx + k$$	**7**
8 05.20 (09:25- 10:30)	Work with hand in 4.	Finding indefinite integrals and fixing them through specific points. Ending with using the found antiderivatives to establish definite integrals.	**8+**
9 05.23 (08:10- 09:15)	Integration by substitution.	Plenum questions and the fundamental theorem of calculus produces: $$\int f\big(g(x)\big)\cdot g'(x)dx = F\big(g(x)\big) + c$$ With more questions the following is established $$\int_a^b f\big(g(x)\big)\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$ The students are then set do a couple of problems where this technique is used.	**9**
10 05.23 (09:25- 10:30)	Work with hand in #4.	Determining indefinite integrals where the use of integration by substitution is needed.	**10+**
11 05.24 (13:30- 14:35)	A summarize of what have been taught in integral calculus.	The students work on worksheet 05.24, which is a worksheet containing two blanc lists, one to be filled out with functions and its antiderivatives and a second to be filled out with the rules and techniques for integration. In the end the students' filled out worksheets were to be compared in plenum to generate a final summarization sheet.	**11**

4.4.2 Analysis of teaching integral calculus.

The analysis of the integral calculus will begin with a visualization of the practical block of the knowledge actually taught. The analysis table (appendix 2.3) includes a list of problems and sub problems; a consideration of these problems will make an overview of the actually taught practical part of IMO. The list contains problems in bold and sub problems which are not. The bold problems are problems where there exists some unifying technique to answer the questions. The sub problems also have a general technique to answer those, which may vary from the general technique of the bold problems.

$\boldsymbol{IP_1}$**:** Find the area between the graph of a function and the first axis over a given interval.

$\boldsymbol{IP_2}$: Find $\int_a^b f(x)\,dx$.
$IP_{2,1}$: Find $\int_a^b c\,dx$.
$IP_{2,2}$: Find $\int_a^b kx\,dx$.
$IP_{2,3}$: Find $\int_a^b f(x)\,dx$ when $f(x)$ is an elementary function.

$\boldsymbol{IP_3}$**:** Find an antiderivative of $f(x)$
$IP_{3,1}$: Find $\int_0^x kx + c\,dx$.
$IP_{3,2}$: Find $\int_0^x f(x)\,dx$
$IP_{3,3}$: Find an antiderivative of an elementary function.
$IP_{3,4}$: Find the antiderivative going through a point (x_0, y_0)?

$\boldsymbol{IP_4}$: Show the rules for calculating with integrals.

$\boldsymbol{IP_5}$: Find $\frac{d}{dx} f(x)$.
$IP_{5,1}$: Find $\frac{d}{dx}\int_a^x f(x)\ dx$ for an elementary function $f(x)$.
$IP_{5,2}$: Find $\frac{d}{dx} f(x)$ for an elementary function.
$IP_{5,3}$: What is df?

$\boldsymbol{IP_6}$: Find $\int f(x)\,dx$?

$\boldsymbol{IP_7}$: Find $\int_a^b f\big(g(x)\big)\cdot g'(x)\,dx$.

$\boldsymbol{IP_8}$: What have you learned about integral calculus?

These problems can be classified into the types of tasks which are presented in the IMO. This is done in the following table.

Types of tasks in IMO	
IT_0	Define the integral. $\boldsymbol{IP_1}$
IT_1	Calculate $\int_a^b f(x)\,dx$ for a given real function $f(x)$. $(\boldsymbol{IP_1})$, $\boldsymbol{IP_2}$, $\boldsymbol{IP_7}$
IT_2	Find the antiderivative, $F(x)$, for a given function, $f(x)$. $\boldsymbol{IP_3}$, $\boldsymbol{IP_6}$.
IT_3	Determine if $\int_a^b f(x)\,dx$ exists for a given function, $f(x)$.

As seen in the table the general problem $\boldsymbol{IP_5}$ is not listed. This is because it is (mostly) part of the DMO and not IMO. Furthermore $\boldsymbol{IP_4}$ is not present either since this problem is what can be considered as the technical work done on the hypothetical technique of the integral. The final problem $\boldsymbol{IP_8}$ is not present in the table either since this problem is really no type of task but more a situation for the students to talk about and unify the techniques and rules found while being taught integral calculus.

By looking at this list of problems and how they correspond to the types of tasks from IMO, a first impression of the practical block of what has been taught is comprehensible. In order to do further analysis a short description of how two of the main techniques established in the teaching was developed. This description can be seen as an example of how the analysis table can be deciphered. When the students are posed the problem $\boldsymbol{IP_1}$, the students enter an exploratory moment in order to establish the hypothetical technique to find the area with an infinite sum of rectangles with infinitesimal width. Furthermore a constitution of the theory happens when the students compose the knowledge of functions and the knowledge of the hyperreal numbers to establish what the integral is. With the integral as a hypothetical technique, new techniques in order to calculate the integral have to be established, thus another exploratory moment happens when the students encounter the problems from $IP_{2,1}$ and $IP_{2,2}$. With the techniques for $IP_{2,1}$ and $IP_{2,2}$ another constitution of theory happens which includes geometry from a previously established praxeology. When the students work with $\boldsymbol{IP_4}$ they do technical work on the hypothetical technique of the integral, which further constitutes the knowledge block of IMO. With the knowledge obtained from the punctual MO generated by $IP_{3,1}$ a suggestion of the fundamental theorem of calculus is obtained when the students go through an exploratory moment while working on $IP_{5,1}$. In the exploratory moment of working with $IP_{2,3}$ the students construct a second technique for calculating definite integrals. In the end the following techniques for IT_1 can be found in the actually taught MO

$I\tau_{11}$	Use the graphical representation of the function to recognize the integral as a geometrical figure, and the formulas for the areas of the geometrical figure to calculate the integral.
$I\tau_{12}$	Use the definition of the antiderivative and the rules for calculating with integrals to calculate the integral as the difference in the antiderivative evaluated in the end points of the interval.

It should be noted that a thorough exploration and routinization of $I\tau_{1,2}$ is difficult to find in the actually taught MO; the number of problems where the technique is needed is as little as eight. Since an explanation of how the students get to the actually observed techniques can be found just as well in the analysis table (appendix 2.3) the remainder of the techniques observed will be left out, most of which are easily derived from the problems posed.

As such when considering the observed praxis block in the teaching process the mathematical organization developed is not as close to the described IMO as hoped for. The actually taught MO contains most of the techniques, and most of the knowledge block seen in IMO, but some things are missing. As already explained $I\tau_{12}$ was one of these where it can be argued whether or not the students actually obtained this knowledge when only a quarter of the class independently showed real and correct use of the technique, though almost all of the students did replicate a use of the technique. Another thing to notice is that even though the students determine the rules for calculating with integrals, the small number of problems posed where such rules are used, can make the rules seem inconsequential. Around half of the time was used on the theoretical block, when considering the exploratory moment of establishing the integral as a hypothetical technique part of it.

The main problem observed during the teaching process was for the students to calculate definite integrals. This is understandable since it will allow the antiderivative, and in part indefinite integrals, to be part of the technique to calculate definite integrals.

4.4.2.1 The didactic processes observed while teaching integral calculus

The first encounter with IMO can be considered in two ways. Firstly, the students are posed the problem of determining a way to find the area between (the graph of) a polynomial and the first axis over an interval, i.e. $\boldsymbol{IP_1}$, which can be viewed as the constitutive task IT_0. Secondly, if $\boldsymbol{IP_1}$ is not to be viewed as a type of task, the first encounter would occur when the students calculate the definite integral of linear functions. This in spite of $\boldsymbol{IP_1}$ also including a problem which would be considered a part of IT_1. The students are able to calculate the integral of linear functions using a previously established praxeology, which supports the use of this type of problem as a first encounter.

The exploratory moment concerning the hypothetical technique of the integral happens when the students explore different techniques to answer $\boldsymbol{IP_1}$. In the end the students unify around the technique of the infinite sum of rectangles with infinitesimal width. This technique includes sub techniques, (hyper)finite partition and (hyper)finite sums. Both of these sub techniques are mentioned and used with no real explanation of how they work. The technological knowledge used to explain what exactly a hyperfinite sum entails is not present in the taught MO. The explanation of what the hyperfinite sum is depends on the construction of the hyperreal numbers. By not explaining the hyperfinite sum the students revert to the intuitive understanding of a finite sum. This understanding is not faulty since the hyperfinite sum shares most of the properties of a finite sum, yet it is still infinite. The (hyper)finite partition of the interval was never explained by the teacher, since at least 4 out of 5 of the student groups had come up with this technique by themselves, thus the understanding of the partition relies solely on the students' abilities to generate this mathematical object. It should be noted that several of the students did start out by trying to find the area by finite partition, which would seem as the beginnings of what could be called the technique of numerical integration. Since this technique was never developed through problems supporting the technique, a real elaboration cannot be said to have happened.

The exploratory moments concerning establishment of the technique, $I\tau_{11}$, only use the graphical representation of the integral to enable the use of a geometrical technique to calculate the integrals. This enables the students to construct the technique with a relatively small knowledge block of IMO established. The exploratory moment which establishes $I\tau_{12}$ is almost unnoticeable in the sense, that even though the technological elements needed to establish this technique are presented by the teacher, the technique itself seems to be mentioned as an afterthought. Furthermore the problems where this technique is needed are being posed to them after a lot of work done on indefinite integrals, which does not help the students understand the presented technique any better. On top of this, many of the students misunderstand the first problems posed where this technique is needed. Coupled with the small amount of problems this enables the students to dodge the experience of first encounter and exploratory moment for this technique. The exploratory moment of establishing an antiderivative happens when the students are presented with the table on the blackboard containing the function, area function and the derivative of the area function. As such the technique was established in plenum.

A constitution of the knowledge for IMO happens when the definition of the integral is done. This definition, also viewed as a hypothetical technique, enables a further constitution of knowledge through the technical work done on the integral while working with $\boldsymbol{IP_4}$. The technical work on the integral (as a technique) is done by means of the techniques obtained in HMO for the standard part, projected on to the integral (as a technique). Other smaller constitutional moments can be observed by considering some of the punctual MO's that make up the IMO. Constitution of the theory for the punctual MO of calculating the integral of linear functions through a geometrical approach can be seen as one of these. Another is when the constitution of the theory to enable the construction of $I\tau_{12}$, as seen in the analysis table, is based on the fundamental theorem of calculus (antiderivatives) and some of the rules for the integrals $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ and $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. The rules are determined when the students work on $\boldsymbol{IP_4}$. As seen in the description of teaching integral calculus (appendix 1.3) the students are able to establish the rules which can be seen as techniques in order to rewrite the (hypothetical) integral. It should be noticed that the work done with $\boldsymbol{IP_4}$ can be viewed as two didactical moments at once: a constitution of technology for the IMO and technical work with the hypothetical technique of the integral. The fundamental theorem of calculus is hinted at through a table of differential quotients and area functions and their opposite relation. This table is composed at first only by linear functions, where the students are able to determine the area functions, and thus also the derivative of said area functions. The students are then able to recognize the system as the opposite connection of area functions and derivatives. Based on this, the fundamental theorem of calculus is guessed but the explanation of why it is true escapes the students' understanding when going beyond the intuition that $\int_a^{x+dx} f(x)\,dx - \int_a^x f(x)\,dx \approx f(x) \cdot dx$. As such the technological level supporting $I\tau_{12}$ can be argued to be partly missing.

The technical work done on the techniques obtained during the teaching process concerning other techniques than the hypothetical is not very obvious, though some technical work is done on the technique to calculate the definite integral of a linear function. This enables the students to write a general formula describing how to find the area of a linear function and extend it to a formula which describes the area function. The rule of the sum of intervals (with a common point) for integrals is not used on the techniques discovered for integrals of linear functions, which otherwise would have enabled an earlier elaboration of the technique $I\tau_{12}$. The technical work done for finding the antiderivative is observed after the definition of the antiderivative and the fundamental theorem of calculus. Here the students use the general technique of finding the antiderivative of a power function, $f(x) = x^n$, to establish a number of indefinite integrals, thus doing technical work by routinization. Another profound moment of technical work is when the integration by substitution is presented. This can be seen as technical work done on $I\tau_{12}$.

Moments of institutionalization happen when the students connect the constituted knowledge, attained by defining the integral, to the geometrical knowledge about triangles and rectangles described by functions and the first axis. Another more profound institutionalization happens when the connection between the DMO and IMO is established, where the IMO finds its resting place among the praxeologies concerning functions, coordinate systems, and manipulation with functions as variables.

4.4.2.2 Discussion (integral calculus)

In this section, a consideration of the observed actually taught knowledge, as described in the analysis, will commence.

While the introduction to the integral calculus, that started with something close to the constitutive task IT_0, can be said to have been rather successful, there are some points to be made about the introduction. An introduction of the summation sign and some operations with a finite sum could have been used to introduce the integral in a pace, which better suit high school students. This would also have supported the technique of numerical integration spontaneously used during the exploration concerning $\boldsymbol{IP_1}$. By doing this the students could have also been introduced to the upper and lower sums if need be. By introducing the numerical integration the students would also have been able to calculate integrals of other functions than linear before introducing the fundamental theorem.

When the students work with the geometric technique of establishing integrals of linear functions, some might argue that the technique is too narrow to be worth using the time on. Even though the technique is very limited in its uses the introduction of something the students can achieve and understand rather quickly makes the students more at ease when working with the definite integral.

Before presenting the students with the problems of the rules of integration, an introduction to some of them would make sense. This could be in the form of problems that hint at the rules, which would also back the technical work on the hypothetical technique of the integral with a plausible inclination of the rule before establishing said rule. The general proof, as done by the students, can make it difficult for them to notice in exactly which situations the rules inferred, are beneficial. I.e. With the few number of problems, supporting the rules for calculating with integrals, the students have a hard time recognizing when to use the them.

By defining the antiderivative as $\int_a^x f(x)\,dx + k$, then $I\tau_{12}$ can be suggested before the students are presented with the fundamental theorem of calculus. This would further back the use of the rules of integration as something which is actually used to establish $I\tau_{12}$. As presented in the observed teaching process the establishment of the technique, $I\tau_{12}$, which lacks foundation in the form of number of problems supporting the technique, is done almost entirely through a presentation of technological elements. Even with the antiderivative defined as the opposite of derivation the number of problems where the students had to find an antiderivative did ensure that the students actually established the techniques, presented at the blackboard, themselves.

Moving the definition of the indefinite integral to be the last thing presented to the students would only make a little change in the taught MO. By defining it as an antiderivative, with an unspecified constant added, the indefinite integral is only a little add-on to what the antiderivative is. With this definition the connection between the indefinite integral and HMO is very limited, and it could be done by a teacher with no knowledge of NSA.

In the situation that the teaching of integral calculus did not have to be finalized during the teaching process two candidates for things to remove is integration by substitution and the indefinite integral which is hinted at above. By removing these things from the planned teaching another more appreciable thing could have been included, namely the first part of the fundamental theorem of calculus as presented in section 2.1.4.4. Removing the technique of integration by substitution is a bit of a loss, when considering that isolating dx in the expression of the derivative $\frac{df}{dx}$ is a mathematical correct operation. By removing the indefinite integral the only thing that would change would be for the students to learn what the indefinite integral is, but defining it as an antiderivative with an unspecified constant added, does not have anything to do with NSA. As such, this would not cause problems for any future teacher to introduce it like this. This being said, the proof of the first part of the fundamental theorem of calculus would be doable for the students with the right amount of guidance. The fundamental theorem of calculus, even when reduced to only having proved the first part, would enable the students to have a theoretical block for IMO on par with what is needed in the beginning of a mathematical university study. This is true only because the elementary functions, which is directed by the government, are all continuously differentiable, which makes the first part of the fundamental theorem sufficient.

The main problem leads to two problems which are not addressed in the teaching process: when does a definite integral exist, and is the technique through the use of antiderivative really justified?
Through HMO the definition of the integral is mathematical correct up to the point of the intuitive understanding of the hyperreal and real numbers. When trying to answer when the definite integral exists, the only way for the students to answer this would be to say that the infinite sum has to be finite but other than it being in the compendium, this is never mentioned. This definition is correct, but since the students were never posed with a problem where the infinite sum actually gave an infinite number, the students never constructed a technique of how to check if the infinite sum was finite or not. The justification of the technique $I\tau_{12}$ is another matter of delicacy, the justification boils down to the proof of the fundamental theorem of calculus. The proof of this is only hinted at, making the justification of the technique unsound.

In the general description of the students' work done on the first page of hand in #4 (appendix 1.3) it is seen that only about half of the class used the technique, $I\tau_{11}$, correctly when using it on an integral of a linear function. A first conclusion would be to assert that the technique was not really established by the students. When looking over the students' answers the tipping point of the wrongfully usage of the technique was determined to be that the students thought of $(b-a)^2 = (b^2 - a^2)$ as being true. This was never part of the planned teaching and can thus be said to be another mistake concerning the planned teaching but as described in the context the students were supposed be aware of the hierarchy of mathematical calculations. The hierarchy of mathematical calculations was thought to cover this distinction, but alas!

When the students worked on the rules of integration, this makes another reason that supports the introduction of the summation sign. This showed when some of the students chose to introduce another way of writing the infinite sum. For the students to be able to rewrite the definition, because they thought it tiresome to write the entirety of the sum, they had to understand the definition of the integral thoroughly. This conclusion was supported when their oral explanations of the same problems actually did encompass the entirety of the sum.

In the lesson where the antiderivative was introduced one of the students found the answer to a question posed in plenum of what the antiderivative of x^n was. At first when the question was asked the student did not recognize the rule from the previously established antiderivatives of the functions $f(x) = kx$ and $f(x) = x^2$. The teacher then opted to ask for the rule of differentiating the same expression, which might have helped the student who found the result, since s/he then raised his/her hand and said: "Well I would say $\frac{1}{n}x^{\cdots}$ no, $\frac{1}{n+1}x^{n+1}$." As such even though the students generally did not really grasp the technique IT_{12} the students all familiarized themselves with the technique described above to find the antiderivative of power functions.

With the consideration of the time used working with the technological, and in part the theory level of the MO, it might not come as a surprise that the students are able to explain why a certain technique is valid but have problems when having to produce and use a technique in order to solve a problem. As such, the knowledge block of the taught MO can be said to be better understood than the praxis block.

Considering that most of the things which could be changed to enhance the teaching process are what can be considered situated in the praxis, with the exception of the proposed inclusion of the proof of the fundamental theorem. The parts of the teaching process that can be classified as well executed and understood by the students were mainly the technological and the theoretical part of the taught MO. As such even though the teaching of the integral calculus could have been executed differently and perhaps better, as described above, the thing to take from this is that using the NSA approach does not seem to have any detrimental effects. On the contrary, the use of NSA enables a knowledge block for the integral calculus which is better suited for students who does not need to construct the numbers.

Finally the amount of time that was used to introduce and establish the IMO can be argued to not be in the ballpark of what is usually used on integral calculus in high school. As such it can be said to be a success in itself to establish a knowledge block which can justify (almost) everything in integral calculus.

4.5 Observations of the teaching process

When teaching infinitesimal calculus using NSA certain things became evident during teaching. The importance of the rules of the standard part, particularly the standard part of a product, did not come through as noticeably as needed. The students knew the rule but rarely considered its restraints, leading to an insufficient argumentation in certain proofs. The rule for taking the standard part of the product of hyperreal numbers, could have been further improved to incorporate the case of the standard part of the fraction of hyperreal numbers. While this rule can be easily inferred from the rule for products this was not done, as such the students had problems explaining why they could take the standard part of the nominator and the denominator separately.

The hyperreal extension of the number sets, ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, could be included in the teaching of HMO. This could improve the students' understanding of the hyperreal numbers, ${}^*\mathbb{R}$. As when introducing the subsets for the real numbers, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, a better understanding of what the real numbers is obtained. The same can be said about the hyperreal numbers, when introducing the sets ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$ to the students. This introduction would also facilitate a knowledge which is closer to the scholarly knowledge of the hyperreal numbers, in that the introduction of subsets of a given set makes it possible to describe said set, which will make the set more comprehendible. I.e. the number of different infinitesimals might not seem clear at first, but could be if the students understood the infinite integers, since taking the reciprocal of them will produce an infinitesimal.

The specification of the infinite sum as being hyperfinite, when defining the integral, could have been done with the abovementioned introduction of the hyperreal extension of the natural numbers, ${}^*\mathbb{N}$. With this extra specification the students would be better suited to answer questions about when the integral exists.

A general consideration of the overall teaching as observed would suggest a greater number of problems/exercises to support the techniques. This is especially transparent in the teaching of integral calculus, as such the praxis block could be said to need elaboration. In extension of this it can said that the teaching had a heavy degree of work on the knowledge block, which can humble even the most mathematical intelligent high school student. Since the teachings of the theoretical part of the infinitesimal calculus had to be finalized during the teaching process the workload could not be moved from the knowledge block to the praxis block in the desired degree. In order to try to circumvent this problem a lot of time was spent on making the teaching process as funny as possible. This was done by listening to the students' way of talking and trying to replicate some of the things, which hopefully made the students think it as funny as the creators did. It is always difficult to get an objective opinion about the teachings experienced by the students. That being said one of the students exclaimed "you should write children's books" after reading worksheet 05.09. The students commented several times on the funny features of the hand ins that made the mathematics more edible.

When this study commenced in the spring a class to which infinitesimal calculus could be taught had to be found. Since most of the high schools teach students infinitesimal calculus in the fall of their second year it had to be taught to a class of first year students. As such the actual plan of order for the subjects to be taught was rearranged to introduce infinitesimal calculus much sooner. This came with some limitations. The number of functions the students had at their disposal when starting the teaching process was at a minimum, which made it difficult to see the merits of some of the rules for differentiation and integration, in particular integration by substitution and the chain rule, and possibly product rule, for differentiation.

With the 30 lessons available for the study, 25 of them could be used to introduce new material. It could be argued that teaching differential calculus to this particular class of students might have been possible within this time frame, including even a more thorough practical routinization and thus establishing the entire DMO. As such, teaching both differential and integral calculus on the other hand proved to leave some of DMO and a lot of IMO unfinished.

In the case of a future study, a realistic way of establishing the entire DMO would be to prolong the teaching course with 3 more lessons of 65 minutes each, as described in section 4.1.1. After proving the rules for calculating with differentials, the students would then use 1 lesson for doing exercises in the product rule and the chain rule. They would also need routinization in finding the equation of a tangent through a point and with the possibility of combining this with the aforementioned this would take up 2 lessons. Yet another lesson would be to teach the students about the (everyday) uses for differential calculus particularly with the headline “optimization”.

To better round off integral calculus, 6 more lessons would be a realistic amount of time needed to cover the material in an appreciable way. 1 lesson would be needed to better introduce the rules for calculating with integrals. With the right worksheet, it would be enough for the students to do the proof of the first part of the fundamental theorem of calculus, this taking up 1 more lesson. To establish the rule for finding the antiderivative of a function of the type $f(x) = ax^n$ would take 1 as well. In the end, 3 more lessons would be needed to introduce the disc method and routinize the techniques in IMO.

The need of a textbook was evident. Mainly because of the rules set by the Danish high school regulative, the students needed some reading material in Danish, which led to the creation of the compendiums. Since the hyperreal numbers are not part of the curriculum, the students had no way of getting help from fellow students or outside of the class, such as at home. With the compendiums they at least had a chance to explain the theory with a “textbook” to back it up, hence making it possible for people with no prior knowledge of NSA to get an understanding of the matter. Even though this is true the students would still be able to get help with the more practical techniques, when the theory behind the definitions of differential quotient and integral are not used. Thus the textbook material and the things just explained could potentially lead to outside help with infinitesimal calculus in spite of the different approach.

5 Conclusion

The aim of this thesis was to elucidate reasons for or against nonstandard analysis approach to infinitesimal calculus in high school, hence answering the research questions.

1. What are the reasons for or against the nonstandard approach, to infinitesimal calculus in high school?
2. How can a nonstandard introduction to infinitesimal calculus in high school be developed, in particular how to create textbook material suited for high school students?
3. What results can be observed from a first experiment, implementing such a material?

As described in section 2.2.3 the nonstandard analysis approach is a very intuitive introduction to infinitesimal calculus and can therefore be considered as a(nother) theory that can be used to introduce this. Since it makes it possible to evade the introduction of the limit operation (or ϵ, δ arguments), the problems that come with it are also evaded. With the intuitive introduction of the hyperreal numbers the standard part is a natural extension of the definition of the infinitesimals. Furthermore it is possible to determine, with logical deduction, the rules for the standard part which are needed in order to prove the rules for how to calculate with the differential quotient and integral. Thus, the knowledge block can be established to a greater extend when introducing infinitesimal calculus with nonstandard analysis, as opposed to the limit operation definition.

The analysis showed that an intuitive introduction to the hyperreal numbers is feasible for students in high school. By introducing the hyperreal numbers the students are able to describe what they can perceive in the world around them, in particular the students are able to talk about evanescent quantities in a mathematical coherent way. With this intuitive introduction to the hyperreal numbers the students are able to further indulge their intuition and logical reasoning to (re)produce the infinitesimal calculus, much in the same way as was done when it was first discovered by Newton and Leibniz. Furthermore, the students are able to prove the rules for calculating with differential quotients and integrals which are normally difficult for students in high school. As such the knowledge block is composed of more intuitive and mathematical coherent statements, which in the end makes it easier understood. Also, when intuition and logical deduction is used to do mathematics, practitioners experience a feeling of self-discovery, thus engaging them into further study.

The high school mathematics is a significant step up in difficulty from prior teaching establishments. As such the intuitive and more logical theory lessens the intricacy of one of the most challenging subjects, infinitesimal calculus.

While all of this is true, nonstandard analysis does not make the infinitesimal calculus diminish in intricacy to the point of making it an easy subject. As seen, nonstandard analysis definitely reduces the complexity of infinitesimal calculus, but not to the degree of justifying the introduction of both differential and integral calculus within the timeframe imposed.

Besides the implementation of the compendiums being a requirement it also showed great appreciation amongst the students. With the inclusion of the things the students experienced in class and the expressions they came up with along with a dash of humor and a recurring story throughout, which was only possible with the construction of the compendiums being done during the course of teaching. This combined made the implementation of the compendiums a success.

The only reason for not using nonstandard analysis is the conservatism amongst mathematicians, in that it is not what is normally used. Hence, the problem with nonstandard analysis comes down to the case of students pursuing a higher scientific education, where then the theory is not used. In conclusion, there are plenty advantages for using nonstandard analysis to compensate the downsides of it being uncommon.

In the end, it is called infinitesimal calculus and not ε-δ argument calculus or limit-operation calculus, so why not use the quantities which gave rise to the name?

Bibliography

BarbÉ, J., Bosch, M., Espinoza, L., & GascÓN, J. (2005). Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools. *Educational Studies in Mathematics*, *59*(1–3), 235–268. http://doi.org/10.1007/s10649-005-5889-z

Bradley, R. E., & Sandifer, C. E. (2009). *Cauchy's Cours d'analyse*. New York, NY: Springer New York. Retrieved from http://link.springer.com/10.1007/978-1-4419-0549-9

Clausen, F., Schomacker, G., & Tolnø, J. (2006). *Gyldendals gymnasiematematik. grundbog B [Bind] 2* (1. udgave 3. oplag). københavn: Gyldendalske Boghandel, Nordisk forlag A/S,.

Criticism of non-standard analysis. (2015, October 2). In *Wikipedia, the free encyclopedia*. Retrieved from https://en.wikipedia.org/w/index.php?title=Criticism_of_non-standard_analysis&oldid=683857585

Cutland, N. (Ed.). (1988). An Invitation to nonstandard analysis. In *Nonstandard Analysis and its Applications* (p. 105). Cambridge: Cambridge University Press. Retrieved from http://ebooks.cambridge.org/ref/id/CBO9781139172110

Gray, J. (2015). The Construction of the Real Numbers. In J. Gray, *The Real and the Complex: A History of Analysis in the 19th Century* (pp. 253–258). Cham: Springer International Publishing. Retrieved from http://link.springer.com/10.1007/978-3-319-23715-2_25

Heiberg, J. L., Fitzpatrick, R., & Euclid. (2008). *Euclid's elements of geometry: the Greek text of J.L. Heiberg (1883-1885) : from Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883-1885*. Place of publication not identified: publisher not identified.

Katz, M., & Tall, D. (2011). The tension between intuitive infinitesimals and formal mathematical analysis. *Crossroads In The History Of Mathematics And Mathematics Education*, *12*, 71–89.

Keisler, H. J. (2012). *Elementary calculus: an infinitesimal approach* (3rd ed., Dover ed). Mineola, NY: Dover Publications.

Ministry of education. Læreplanen, Pub. L. No. BEK nr 776 (2013). Retrieved from https://www.retsinformation.dk/Forms/R0710.aspx?id=152507#Bil35

Ministry of education, G. afdelingen. Matematik A - Stx Vejledning / Råd og vink (2010). Retrieved from https://www.uvm.dk/-/media/UVM/Filer/Udd/Gym/PDF10/Vejledninger-til-laereplaner/Stx/100806_vejl_matematik_A_stx.ashx?la=da

O'Donovan, R., & Kimber, J. (2006). Nonstandard analysis at pre-university level: Naive magnitude analysis. *Nigel J. Cultand, Mauro Nasso and David A. Ross (Eds.), Lecture Notes in Logic, 25 Nonstandard Methods and Applications in Mathematics*. Retrieved from http://www.ssrdm.ch/SSRDM/seminaires/NSA_in_school.pdf

Ponstein, J. (2001). *Nonstandard analysis*. Groningen: Faculty of Economics, University of Groningen. Retrieved from http://som.rug.nl/bestanden/ponstein.pdf

Robinson, A. (1996). *Non-standard analysis* (Rev. ed). Princeton, N.J: Princeton University Press.

Stroyan, K. D. (1997). *Mathematical background: Foundations of infinitesimal calculus*. Academic Press. Retrieved from http://www.fulviofrisone.com/attachments/article/175/Analysis%20Calculus.Foundations.Stroyan.182.pdf

Winsløw, C. (2013). Mathematical analysis in high school: a fundamental dilemma. *arXiv Preprint arXiv:1306.2562*, 14.

Yves Chevallard (English) | A.R.D.M. (n.d.). Retrieved August 7, 2016, from http://www.ardm.eu/contenu/yves-chevallard-english

Appendix

1 Description of teaching

This part of the appendix will give a description of what can be observed from the teachings.

1.1 Description of teaching hyperreal numbers

1.1.1 First lesson (03.30)

The first lesson was carried out by Jonas and was mostly a class discussion with the occasional task in small groups (of up to 3). The following table is a way to organize what happened and some of the more interesting answers/results of the tasks. Since the teaching was carried out in Danish the observations given here are translated by the authors of the thesis.

HQ_1: What is the first nonzero speed of your telephone when dropped from your hand?

HQ_2: Give a definition of something infinitely small.

HQ_3: imagine you spend 3 days at school without food; how much would you want potato chips?

HQ_4: Give a definition of something infinitely large.

HQ_5: In what set of numbers should the infinitesimals, dx, and the infinite numbers be put?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Numbers in general		Teacher (T) <-> student	$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$	Whole class discussion, starting with the question posed by the teacher. T: What are numbers?
The infinitely small or infinitesimals	Individual task HQ_1	Student -> teacher	Infinitesimal (dx)	
	HQ_2	Student -> teacher	Infinitesimal (dx) And inequalities (< and >)	Teacher writes some of the partial answers on the blackboard 1. $0 < dx$ 2. $dx < \mathbb{R}$ (infinitely small) 3. 0,00000001… Student (S): You can't write the number… S: $0 < dx < \mathbb{R}$
infinite quantities	HQ_3 HQ_4	Student -> teacher	Infinite number (N) And inequalities (< and >)	Teacher writes some of the partial answers on the blackboard 1. $\mathbb{R}_+ < \infty < N$ 2. Infinity is everything so it can't be bigger than infinity.
Hyperreal numbers	HQ_5	Teacher and students	$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, ({}^*\mathbb{R})$	T: Draws a line on the blackboard and the students fill it out with numbers from $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$. T: where does dx and infinite numbers belong?

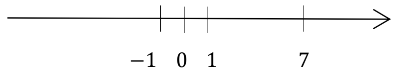
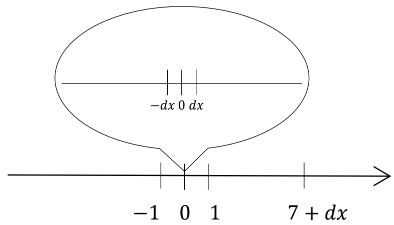
				S: one could make a new group of numbers, which includes infinitesimals and infinite numbers.
Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
The infinitely small or infinitesimals	Institution alization	T	dx	After the teacher and observer does a high five the teacher writes the distance between the hands the moment before they clapped as $\Delta x = x_2 - x_1 = dx$. Calling dx an infinitesimal.

1.1.2 Second lesson (04.05)

The second lesson starts with a short class discussion of what they did last time. Including a number line with the before mentioned sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$.

Again a table will give an overview of the lesson.

HQ_6: Draw a number line which includes infinitesimals.
HQ_7: Which number is infinitely close to $7 + dx$?
HQ_8: What is $1/dx$?
HQ_9: Where does $\pm 1/dx$ belong on the hyperreal line?
HQ_{10}: Give an example of a hyperreal number with no standard part, i.e. find a such that $st(a)$ is meaningless.
HQ_{11}: What is a function?
HQ_{12}: Can you give a general definition of a function?
HQ_{13}: What is continuity?
HQ_{13_1}: What is the distance between 2 consecutive points on the graph?
HQ_{13_2}: What should we write to finish this definition, $dy = f(x) - f(?)$?
HQ_{14}: What is a domain of a function and what kind of number is $f(x + dx)$ (assumes f not constant)?
HQ_{15}: Find $st(x + dx) + st(y + dy)$
HQ_{16}: Find $5 \cdot st(x + dx)$
HQ_{17}: Find $st(5x + 5dx)$
HQ_{18}: Find $st(0 + dx)$
HQ_{19}: Find $st(dx + dy)$

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Real number line		teacher	$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$	A short description of the previous lesson. 
Hyperreal numbers	HQ_6 In small groups.	student	$({}^*\mathbb{R})$	
Part of the hyperreal number line	Institutionalization by whole class discussion	Teacher student	$({}^*\mathbb{R})$	Product of HQ_6: 
Preliminaries for some of the rules of operating with infinitesimals	Institutionalization	Teacher (student)	Sum, multiplication and division	T goes through $dx + dy, 2dx, dx/2$ on the blackboard and S and T argues why they are infinitesimals.
Connection between hyperreal and real line.	First encounter HQ_7 Institutionalization	S T	Standard part: $st: {}^*\mathbb{R} \to \mathbb{R}$	T: The real number a which is infinitely close to a hyperreal number $a + dx$ is called the standard part of $a + dx$, and written as $st(a + dx) = a$.
Infinite numbers And hyperreal number line	(First) encounter HQ_8	S	Division with small numbers (infinitesimals)	T: writes $1/dx$ on the blackboard (S has minor problems with properties of fractions)
	HQ_9 Institutionalization.	S and T	${}^*\mathbb{R}$	S: $\frac{1}{dx}$ should be placed on the other side of the head of the arrow. T writes $\pm\frac{1}{dx}$ on the number line
Infinite numbers and standard part	HQ_{10}	S	${}^*\mathbb{R}$ and $\mathbb{R}$.	S: $st\left(\frac{1}{dx}\right)$ is meaningless.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Functions: Different kinds	HQ_{11}	S	Variables and constants.	S gives examples and T writes on blackboard: $f(x) = ax + b$ $f(x) = b \cdot a^x$ $f(x) = b \cdot x^a$
General definition	HQ_{12}	S		S: For every x-value there is a y-value. T writes it on the blackboard (bb)
Continuity	(First) encounter HQ_{13} HQ_{13_1} HQ_{13_2}	S		S: …there are no jumps on the graph. S: Infinitesimal S: $x + dx$
	Institutionalization	T		A function f is continuous if an infinitesimal increment in the x-value produces an infinitesimal increment in the y-value. That is if $f(x) - f(x + dx) = dy$ is infinitesimal for every infinitesimal dx.
				Hands out the pamphlet about the hyperreal numbers.
Star operation	HQ_{14} Institutionalization	S T	Domain and $^*f(x)$	S: the numbers for which the function makes sense, hyperreal number. T: The star operation extends the domain of functions to include hyperreal numbers.
Standard part and the connection between the real and hyperreal numbers	HQ_{15}-HQ_{19}	S	$st()$, infinitesimals (dx), sum.	T: walks around and helps the S if needed.
End of lesson 2				

1.1.3 Third lesson (converted #1) (04.06)

The third lesson was, due to regulations on the high school we taught at, a lesson where the students are supposed to work on subjects they have already covered, thus the students spent the lesson first going through their homework and then on the exercises in the compendium.

HQ_{20}: Find $st(5dx)$
HQ_{21}: Find $st\left(\frac{1}{dx}\right)$
HQ_{22}: Why is $st\left(5dx \cdot \frac{1}{dx}\right) = 5$
HQ_{23}: Give an example of an infinitesimal which is also a real number
HQ_{24}: Find $st\big((2 + dx)(3 + dy)\big)$
HQ_{25}: Find $st(2 + dx)st(3 + dy)$
HQ_{26}: What can be concluded from the two previous exercises?
HQ_{27}: Find $st(dx)st\left(\frac{1}{dx}\right)$
HQ_{28}: Amend the conclusion from before
HQ_{29}: Find $st((4 + dx)^2)$

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Standard part of different hyperreal numbers.	Institutionalization of HQ_{15} to HQ_{24}. Which are all questions in HT_4 and HT_5	S and T.	st and dx.	10 students go to the blackboard and write their results for the exercises they did at home. After that 10 other students had to argue why the results were correct or incorrect. After 40 minutes the rule of $st(a \cdot b) = st(a)st(b)$ if and only if both a and b are finite numbers are proved and explained by the students with some guiding questions from the teacher.
Continuity	Institutionalization	T	${}^*f(x + dx) - f(x)$	Shows how to prove that a function is continuous with the definition.
Determine which numbers are in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Def. of infinitesimal and infinite numbers. Standard part of hyperreal numbers. Equation solving. Continuity	In groups	S and T	st, dx, $\frac{1}{dx}$	T walks around and helps S if they have questions for hand-in assignment 1.

1.1.4 Fourth lesson (04.08)

In the fourth lesson the students go through the remaining exercises in the compendium.

HQ_{30}: For $f(x) = 3x + 1$, find the hyperreal function value $^*f(1 + dx)$ and $^*f(dx - 2)$.
HQ_{31}: For $g(x) = -x$, find the hyperreal function value $x = 3 - dx$ and $x = 0$.
HQ_{32}: For $h(x) = 2x^3$, find the hyperreal function value $^*h(1 + dx)$ and $^*h(4 + dx)$.
HQ_{33}: What is $st(^*f(x))$?
HQ_{34}: Assume now that f is a continuous real function what is $st(^*f(x + dx))$?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Standard part of different hyperreal numbers.	Institutionalization of HQ_{25} to HQ_{29}. Which are all questions in HT_4 and HT_5	S and T	st, dx, continuity	T walks around and helps S if they have questions for hand-in assignment 1.
Determine which numbers are in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Def. of infinitesimal and infinite numbers. Standard part of hyperreal numbers. Equation solving. Continuity	In groups	S and T	st, dx, $\frac{1}{dx}$	T walks around and helps S if they have questions for hand-in assignment 1.

1.1.5 Hand in #1

The tale of StandardJoe and HyperMick

Problem A

StandardJoe and HyperMick weren't born StandardJoe and HyperMick, one has to earn names like that! StandardJoe and HyperMick that is. Thus, our tale of StandardJoe and HyperMick begins before the standard part of Joe was taken and real-istically seen before Mick was made hyper. Think about it. Back then they were only able to solveexercises like these. Petty..

1. Which of these numbers are natural numbers? $42;\ \ -12\ ;\ 0\ ;\ 1{,}2\ ;\ \frac{1}{2};\ 10000000000000000001$ $\sqrt{8}\ ;\ \frac{6}{1}$
2. Rewrite the following fractions to decimals: $\frac{1}{4};\ \frac{9}{5};\ \frac{5}{6};\ \frac{4}{11}$
3. Rewrite the following fractions to decimals, and find the pattern in the decimals: $\frac{2}{3};\ \frac{1}{6};\ \frac{8}{11};\ \frac{4}{7}$
4. Explain why these numbers are rational numbers: $4{,}825\ ;\ 1{,}3333333\ \ldots\ ;\ 8\ ;\ -3{,}25\ ;\ \frac{3}{4}$
5. Check if $\sqrt{6{,}76}$ is a rational number.
6. Check if the following is true: $3 \in \mathbb{Z}\ ;\ 3 \in \mathbb{Q}\ ;\ -2 \in \mathbb{N}\ ;\ 2 \in \mathbb{R}\ ;\ 0 \in \mathbb{R}$

Problem B

Mick was infinitely close to getting his stribes. He just needed to show he understood what the creed was all about. That's standard.

1. Write a definition of an infinitesimal.
2. Write a definition of an infinitely big number.

Problem C

Solve the following expressions. But how!? Ask Joe. He was the man and was quickly known as StandardJoe. Like, super fast. Insanely fas.. Freakin' psycho f.. I've never seen anyone get a nickname as fast as Joe did! He got it quicker than anything I can even describe.

1. $st(42 + dx)$
2. $st(42 + dy)$
3. $st(42dx)$
4. **4.** $st(42dx + 42)$
5. $st(42dx) + st(42)$
6. $st(42)st(42dx)$
7. **7.** $st\left(42dx \cdot \frac{1}{42dx}\right)$

Problem D

Now they're just messing with Joe. He don't care! Solve the equations and find the real value closest to x.

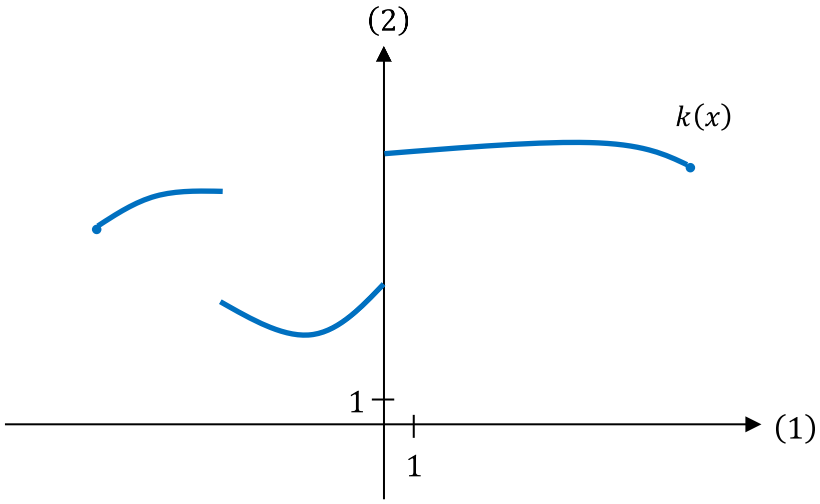
1. $2x + 1 = dx + 5$
2. $4 + 3dx = 3x - 2$
3. $x - 7 = 2x + dx$
4. $2(x + 3) = 8 - dx$
5. $3 - \frac{1}{2}x = 4(2 - dx)$

Problem E

Now they had to prove that they could both keep their names forever. That they didn't suddenly lose them and then get them back. They were to have their names without jumps. Once you go hyper, you.. also.. actually, sometimes go standard.. But not at the same time! So.. Yo.

Determine if the following functions are continuous.

1. $f(x) = ax + b$
2. ~~$g(x) = b \cdot a^x$~~
3. $h(x) = b \cdot x^a$
4.



Good luck with the st(their originstory), Mick'n'Joe.

1.1.6.1 Hand-in #1 notes

Problem A

Generally good understanding of number sets, with a tendency to exclude $\frac{6}{1}$ and include 0 in $\mathbb{N}$.

Problem B

Write a definition of an infinitesimal

Student1: "Infinitesimal is not a constant, since a number can always get smaller" contra "Infinitely big number is on the contrary a number that can always get bigger".

Student2: Something that is infinitely small:
$0 \leq dx < \mathbb{R}_+$
Or if it's negative:
$0 \geq -dx > \mathbb{R}_-$
It cannot be written, since it can always be smaller, thus it is defined as dx.

Write a definition of an infinitely big number

Something, which is bigger than all the real numbers, it can always get bigger.
It is so big that it cannot be written, since one can always put more numbers on it

$$\frac{1}{dx} > \mathbb{R}$$

Or if it's negative:

$$-\frac{1}{dx} < \mathbb{R}$$

Problem C

7. They have difficulties calculating $st\left(42dx \cdot \frac{1}{42dx}\right)$
   The problem is, that they see $st\left(\frac{1}{adx}\right)$ as something not definable, no matter what

Problem D

The only relevant problem in this exercise is notation even though some students really nailed it, which are given below:

Student1:

$$4 + 3dx = 3x - 2$$
$$6 + 3dx = 3x$$
$$\frac{6 + 3dx}{3} = x$$
$$\boldsymbol{x = 2 + dx \approx 2}$$

Student2:

$$2x = 4 + dx$$
$$x = 2 + \frac{1}{2}dx$$

So the real value is $x = 2$

Problem E

Their understanding of continuity

- A graph mustn't contain holes
- Can't lift the pencil
- They use $^*f(x + dx) - f(x)$ and say, that something isn't continuous, if they can't use that
- Not continuous, since the graph is broken several times

1.1.7 Fifth lesson (converted #2) (04.13)

The point of the fifth lesson was to check if all of the students understood the comments, they had gotten in the weekly hand-in about hyperreal numbers, by going through every exercise one by one, with the students giving the answer.

1.2 Description of teaching differential calculus (Jonas Kyhnæb)

The star * marks lessons to which there exists physical material from the students, which can be reviewed. This in the form of filled out worksheets, finished hand-in assignments and/or video recordings (see Digital appendix)

1.2.1 First lesson (04.08)

In this lesson, the teacher refreshed, with the students, the concept of a function. What is a function, what does a function do, what happens when you evaluate a function in a given value and so on. An introduction to the terms “slope” and “tangent” was given through the graphs for different types of known and unknown functions in a coordinate system. The students also got Worksheet 04.08 with exercises about finding the slope in a point on the graph of the function.

DQ_1 What is a function?

DQ_2 How do you find the function value for certain x-values?

DQ_3 How do we get from a function to a graph?

DQ_4 Are the functions, represented by these graphs, continuous?

DQ_5 When, during the day, is the phone being used the most?

DQ_6 Can we tell (exactly) how much the battery is being drained here?

DQ_7 At what specific point during the day, is the phone being used the most?

DQ_8 How do we find out exactly how much battery is being drained at the point of the day where the battery is being drained the most?

DQ_9 Can we find out when the battery is being drained 10% per second?

DQ_{10} When/where can we find out how much battery is being drained?

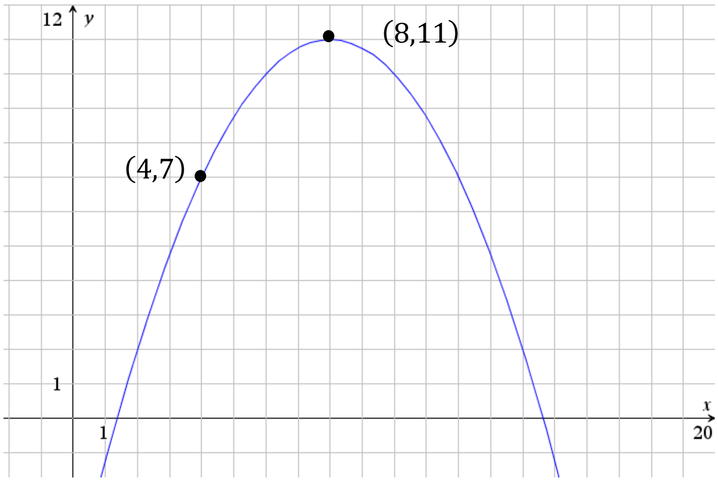
Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
The concept of a function. Continuity	Technical moment with DQ_1 to DQ_4	T and S	$f(x)$, $f(a)$, coordinate system	T asks questions about functions to draw out the students' knowledge and to familiarize them with functions and the correlation between functions and graphs and which functions are continuous
Intervals. Slope of a secant and tangent. Infinitesimal	Institutionalization of DQ_5 to DQ_{10}	T and S	Coordinate system, $a = \frac{y_2 - y_1}{x_2 - x_1}$	T asks indirect questions about the slope of a graph, using an example of a decreasing graph describing a phone battery's drainage during a day. DA_6: It decreases most where the graph is the steepest. S find the secant, without using that word, but "the general slope" between two points on the graph. Gradually S find the slope in a point using tangents in intervals, which at the end is the tangent in an infinitesimal interval. DA_{10}: Everywhere
Slope in a point from a graph. Tangent to a graph. Drawing a graph	In groups	S and T	Coordinate system, $a = \frac{y_2 - y_1}{x_2 - x_1}$	T walks around and helps S if they have questions for Worksheet 04.08, which includes drawing tangents and graphs and finding slopes using the 2-point formula

The students realized that the slope in a point on a decreasing graph is steepest where the graph decreases the most. This by an example of a graph of a phone battery the teacher drew on the blackboard. Here the students concluded relatively quickly that the battery must have been drained the fastest when the graph decreased the most, in a specific interval. Some of the students came up to the blackboard to point and some came up to draw tangents, even before we introduced the term, which resulted in skipping Worksheet 04.08 - Secant and only hand the students Worksheet 04.08 – Tangent. Even though the students had not heard of a polynomial function yet, with this introduction, the students established a graphical understanding of the slope in a point for an arbitrary function.

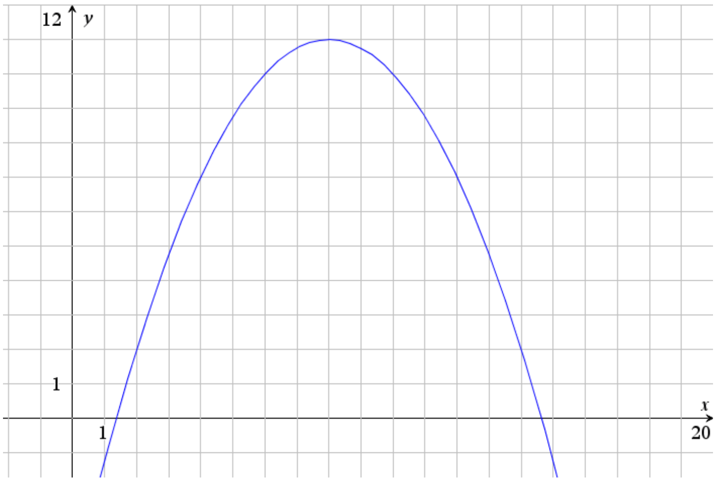
1.2.1.1 Worksheet 04.08 – Secant

On the graph below, 2 points have been marked.

1. Draw a line through the 2 points and find the slope of this line.
2. What can be said of the slope of the line and the slope of the graph between the 2 points?



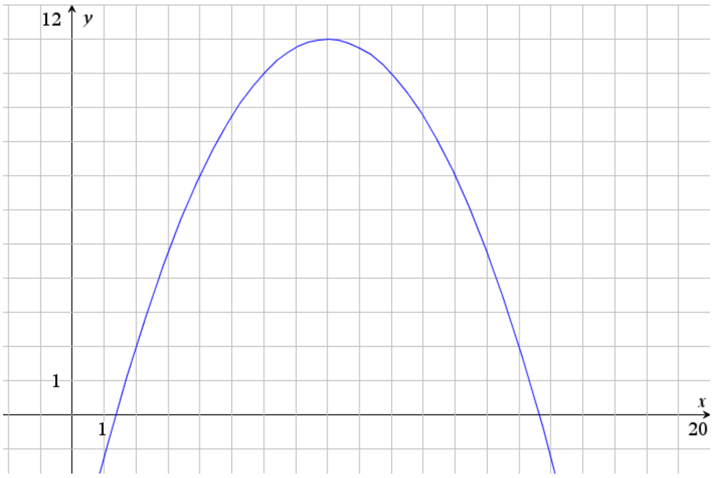
- Plot in 2 other points on the graph below and repeat the steps 1. and 2.



1.2.1.2 Worksheet 04.08 – Tangent

Below is the graph for $f(x) = -\frac{1}{4}x^2 + 4x - 5$.

- Find the slope of the graph in the point $(4,7)$.
- Find the slope of the graph in $(8,11)$.
- Find the slope of the graph in $(12,7)$.



- What is the general procedure for finding the slope of a graph in an arbitrary point?

Draw an arbitrary graph for a continuous function in the coordinate system below.

- Find the slope in an arbitrary point P on the graph.
- Explain your approach from the beginning.

1.2.2 *Second and third lesson (04.11)

The point of this lesson was to get the students to use the 2-point formula on an infinitesimal interval and get the formula for the differential quotient (without actually telling them, that that was what they found). To the students, they established a technique of finding the slope of a graph in a point, by finding the slope of the tangent in that point. The connections were made through Worksheet 04.11 with questions $DQ_{11,i}$.

$DQ_{11,0}$ Find the slope of a function, $f(x)$, in a point, without drawing anything!

$DQ_{11,1}$ Find the general slope of a function, $f(x)$, in the interval $[a; b]$.

$DQ_{11,2}$ What can be said about the slope to a function, $f(x)$, and the slope of the secant (the straight line) through the points $(a, f(a))$ and $(b, f(b))$.

DQ_{12} What did you just do?

DQ_{13} How?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Slope of a secant, tangent and function	Exploratory in groups working with $DQ_{11,i}$	S and T	$a = \frac{y_2 - y_1}{x_2 - x_1}$	T walks around and helps S if they have questions for Worksheet 04.11
Slope in a point	Recap with DQ_{12} to DQ_{13}	T and S	$a = \frac{y_2 - y_1}{x_2 - x_1}$	DA_{12}: Constructed a formula with which we can find the slope of every graph DA_{13}: Using $\frac{y_2 - y_1}{x_2 - x_1}$ by finding two points close to each other like $a + dx$ S writes the definition of a slope in a point $(a, f(a))$ is $st\left(\frac{f(a) -^* f(a+dx)}{a-(a+dx)}\right)$
Differential quotient	Institutionalization	T	$f'(x)$	T tells S that the slope in a point $(x_0, f(x_0))$ on a graph is called the differential quotient and is written $f'(x_0)$

For the students to get the correct understanding of the differential quotient, the secant had to be introduced in whatever small capacity. This only happened through Worksheet 04.11, where it was mentioned as something the students had already worked with, namely a straight line through two points on a graph. The worksheet was mainly created to further expand the students' knowledge, in that this introduced the slope of a function. The students worked on the worksheet in groups, before going through it in class. The build-up for the worksheet is a generating question posed as the first/top one. To help the students, there are sub questions created in such a way that if the students couldn't give an answer to the generating question right away, they could look at the next question. Thus, the last question is a sub question to help with both of the above questions. The students handed in their answers at the end of class and looking at the last sub question. The students all reached the same conclusion (some earlier than others did), which, in different variations, looks like this:

- Finding a slope in a point is only possible by using a second point and the closer the points are to each other, the more precise will the slope of the straight line between the two points then be, compared to that of an arbitrary graph. It would be smart to choose two points that are infinitely close to each other, hence, the points can be dx close to each other.

The next/first sub question was then answered using the statement they had reached. The answers were almost identical, with variants in the choice of what to call the general slope (here S):

- The general slope of a function in the interval $[a; b]$ is $S = \frac{f(b)-f(a)}{b-a}$.

The answer for the generating question was then achieved with the formula just derived from the previous sub question. A generalization for the answers looks like this:

- Let $b = a + dx$ and $f(b) = f(a + dx)$, then the slope in a point $\big(a, f(a)\big)$ will be $st\left(\frac{{}^*f(a+dx)-f(a)}{a+dx-a}\right) = st\left(\frac{{}^*f(a+dx)-f(a)}{dx}\right)$. We take the standard part because we want the real value. And always remember the star when the function is hyperreal.

In conclusion, using the students' words, they were able to use the 2-point formula for straight lines $\left(a = \frac{y_2-y_1}{x_2-x_1}\right)$, since the slope of a tangent has the same slope everywhere hence in every point on the tangent and then clearly also in the point on the graph. These two points had to be found in a small interval of the graph and actually, the smaller the interval, the better the tangent, meaning the more precise the slope in the point. Rather quickly, a student said, that the best slope would occur if we could use the 2-point formula on to points that had an infinitesimal difference. The fact that the difference would then be hyperreal didn't matter, since we could find the real number infinitely close to this hyperreal number by taking the standard part.

This is actually, what led to the decision of making the compendiums gradually, as the students completely skipped the secant-part, since it made so much sense to just pick the second point so they were in an infinitesimal interval. Some students even found it confusing reading the chapter on the secant, because that was not how we did in class nor was it how the student had understood it. On a side note it was also here the headline *Tangentman and 2point find Miss A* came to be.

The teacher summed up the lessons, just to say that the slope in a point $\big(x_0, f(x_0)\big)$ on a graph for a function, could be shortened down to differential quotient, which is written $f'(x_0)$.

1.2.2.1 Worksheet 04.11

Hello humans.

We're gonna give you an exercise now, which you are to make in groups of 3 to 4. You will answer the question below, if necessary by using the sub questions, and write them on this paper, or upload it as a group in Lectio, so that we get it all when you've finished. Thanks!

Question:

- Find the slope of a function, $f(x)$, in a point, without drawing anything!
  - Find the general slope of a function, $f(x)$, in the interval $[a; b]$.
    - What can be said about the slope to a function, $f(x)$, and the slope of the secant (the straight line) through the points $(a, f(a))$ and $(b, f(b))$?

1.2.3 Fourth lesson (04.13)

In the fourth lesson, we made a recap of the last lesson with the formula for the differential quotient, which some of them remembered already. The students were given a number of exercises from the compendium, which were all in finding the differential quotient for different polynomial functions to various values of x.

DQ_{15} to DQ_{19} are exercises 10 to 15 (minus 14) on page 21 in the compendium.

DQ_{14} What did we do last?

DQ_{15} Fill out the table by computing the differential quotient to the function $g(x) = 1$ in the points

x	-3	-1	0	1	4
$g(x)$					
$g'(x)$					

DQ_{16} Fill out the table by computing the differential quotient to the function $h(x) = -4x$ in the points

x	-3	-1	0	1	4
$h(x)$					
$h'(x)$					

DQ_{17} Fill out the table by computing the differential quotient to the function $k(x) = 2x^2$ in the points

x	-3	-1	0	1	4
$k(x)$					
$k'(x)$					

DQ_{18} Fill out the table by computing the differential quotient to the function $l(x) = 2x^2 - 4x + 1$ in the points

x	-3	-1	0	1	4
$l(x)$					
$l'(x)$					

DQ_{19} Let $f(x)$ and $g(x)$ be two arbitrary and differentiable functions. Compose a rule for the differential quotient to $h(x) = f(x) + g(x)$, that is describe $h'(x)$ with $f'(x)$ and $g'(x)$.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Differential quotient	DQ_{14} Technical moment DQ_{15} to DQ_{18}	S and T	$f'(x_0) = st\left(\frac{{}^*f(x_0+dx)-f(x_0)}{dx}\right)$	S dictate how T completes an example of finding the differential quotient for a function $f(x) = x^2$ in a point $(2, f(2))$
				T walks around and helps S if they have questions for the exercises DQ_{15} to DQ_{18} (page 21 in the compendium)
Differential quotient of a sum of two functions	Exploratory DQ_{19}	S	$f(x), f'(x)$	S work in groups to solve the problem

The students got to make exercises in using the definition of the differential quotient, which they had only just established. Hence, the exercises concerned various problems of finding the differential quotient for a function in different points. Another added exercise was the problem of finding the points, when only given the x-value. Furthermore, these exercises were constructed in such a way that the students were to see a pattern in the different functions to which they had to find the differential quotient, thus leading up to finding the differential quotient for $h(x) = f(x) + g(x)$, using $f'(x)$ and $g'(x)$.

1.2.4 Fifth lesson (04.19)

We went through questions DQ_{15} to DQ_{19} on the blackboard.

DQ_{20} How did you solve these exercises?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Differential quotient. Standard part. Star operation. 2-point formula	Technical moment DQ_{20}	S and T	$f'(x_0) = st\left(\frac{^*f(x_0+dx)-f(x_0)}{dx}\right)$	DA_{20}: Use the 2-point formula on an infinitesimal interval
To derive. The derived function	Institutionalization	T	$f'(x)$	T tells S that going from a function to the differential quotient is called to derive the function in a point T tells S that deriving a function gives the derivative of a function

The teacher drew a table on the blackboard containing three different functions, g, h, k, l and their derivatives, g', h', k', l', where $g + h + k = l$ and $g' + h' + k' = l'$. Each function and its derivative was then to be evaluated in the same five x-values. The students filled out the combined table with all the function values and the differential quotients. From start to finish, the student who filled out the box explained how it was done. At this point, the students were acquainted with finding the function values; hence, these were filled out correctly. The interesting part was how they had come to the differential quotient. The process for each function was different:

$g(x) = 1$

I Most of the students used $st\left(\frac{^*f(a+dx)-f(a)}{dx}\right)$, but had trouble finding $^*f(a + dx)$, thinking they should simply add dx to 1, so

$$st\left(\frac{1 + dx - 1}{dx}\right) = st\left(\frac{dx}{dx}\right) = st(1) = 1$$

The way this was solved, was to ask if they remembered what a differential quotient is and how they first found it. This helped them to try to imagine a graph for the function or to draw it and then see that what they had found couldn't be true. Next step was getting the students to see how they had just found the function values and to realize this was clearly how to find $^*f(a + dx)$ as well.

II The latter was exactly how other students found the differential quotient right away and thus circumventing the use of $st\left(\frac{^*f(a+dx)-f(a)}{dx}\right)$ altogether.

III Some students did figure out how to use $st\left(\frac{{}^*f(a+dx)-f(a)}{dx}\right)$ correctly and found the differential quotient in all the points using this formula.

$h(x) = -4x$

All students solved this either by II or III, a few trying to find a way of generalizing the results.

$k(x) = 2x^2$

Each student did as described in III.

One student became convinced, though, that there was a system to it, that they didn't have to go through the entire process of using the formula every time. The student was especially motivated by the prospect of avoiding the many calculations necessary, when having squared expressions with more than one term. The student's reasoning was that there had to be another way, since functions of higher power than 2 must clearly exist and that it would then "take forever". The student quickly realized that all the terms containing dx of power higher than of 1 could be ignored, since they disappear when taking the standard part.

$l(x) = 2x^2 - 4x + 1$

Furthermore, the student realized that all the constant terms cancel each other out just as the terms not containing dx. This at least made finding the differential quotient for polynomial functions of power 2 easier and some of them the student even solved without writing anything down.

A few students, who didn't do it as the one student trying to solve the enigma of differentiating in the head, saw that $l(x)$ consists of the aforesaid functions, so checking if the function values in the different points were the same as adding (or subtracting) the respective function values resulted in the same function value as calculated, was very easy. These students then tried their luck with differential quotients, following the same procedure. They were then told to check some of them by doing as described in III.

The rest of the students did as described in III.

Not all the students had gotten to the exercise about finding the differential quotient for $h(x) = f(x) + g(x)$, using $f'(x)$ and $g'(x)$ but after going through each of the functions like this everybody at least had a suggestion as to how they should go about it. As such, the students all agreed that $h'(x) = f'(x) + g'(x)$.

1.2.5 *Sixth lesson (04.20)

The students worked in groups on what conditions of monotony is for a function, given the derived function. They were even given the opportunity to search the internet, mainly to give a definition of the conditions of monotony. If they needed help, the teacher would give them a hint in the right direction, through the students' knowledge of differential quotients, in the form of $DQ_{21,i}$ in chronological order.

DQ_{21} What can be said about the conditions of monotony of the original function, $(f(x))$, given the derived function $(f'(x))$?

$DQ_{21,1}$ What is conditions of monotony?

$DQ_{21,2}$ Find the conditions of monotony for $f(x)$, when $f'(x) = x + 2$

$DQ_{21,3}$ What can we say about $f(x)$ when $x = -2$?

$DQ_{21,4}$ What can we say about $f(x)$ when $x \leq -2$?

$DQ_{21,5}$ What can we say about $f(x)$ when $x \geq -2$?

$DQ_{21,6}$ What can you say about the conditions of monotony of the original function $(f(x))$, given the derived function $(f'(x))$?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Conditions of monotony. Differential quotient	First encounter, exploratory and technical moment DQ_{21}	S	$f(x), f'(x)$	S work in groups to solve the problem
	Exploratory $DQ_{21,i}$	S and T	$f(x), f'(x)$	T gives S hints one by one if needed, in the form of $DQ_{21,i}$

The lesson was about making a description of what can be said about the conditions of monotony for a function $(f(x))$, given the derived function (DQ_{21}). The students, wanting to make the best description possible, put in a lot of effort in writing this. Seeing as they could search the internet for answers, a lot of them had the same restrictions for when a function is in- or decreasing, namely

$f'(x) > 0$ for all $x \in [a; b] \Rightarrow f$ increasing in $[a; b]$

$f'(x) < 0$ for all $x \in [a; b] \Rightarrow f$ decreasing in $[a; b]$

$f'(x) = 0$ for all $x \in [a; b] \Rightarrow f$ constant in $[a; b]$

Luckily, all the students knew this wouldn't be enough and added self-made explanations as to show they knew what they had written. After class, all the students handed in their descriptions. Some of what they had written as justifications could be:

> When we have the derived function and are told that the slope in an interval is positive, negative or equals zero in more than one point, we'll be able to read off of the graph if it is decreasing, increasing or constant.
>
> All functions have different inequalities, which determines the conditions of monotony.
>
> When given the derived function $\left(f'(x)\right)$ we know everything about the conditions of monotony for the indigenous function $\left(f(x)\right)$.

And also

$$E = mc^2$$

1.2.6 *Seventh lesson (converted #3) (04.21)

The lesson started with the teacher summing up what they did last (*Sixth lesson (04.20)), in regards to conditions of monotony. The teacher gave a definition and an example of how to write it properly. The rest of the lesson the students got to work on Hand-in #2.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Conditions of monotony. Differential quotient	Institutionalization	T	$f(x)$, $f'(x)$, [;]	T explains how to find the conditions of monotony for the function $f(x) = \frac{1}{2}x^2 + 2x - 4$ and how to properly write it with brackets
Derivatives	Technical moment	S and T	$f(x)$, $f'(x)$, coordinate system, $\mathbb{N}$, $\mathbb{Q}$	T walks around and helps S if they have questions for Hand-in #2

The students worked on problems involving finding the derivative of general functions like

- a linear function $f(x) = ax + b$, specifically a constant term
- a polynomial function with an integer as exponent ax^n, $a \in \mathbb{R}$, $n \in \mathbb{N}$, by first using the formula when $n = \{1,2,3\}$ and then giving a substantiated guess for when $n > 3$
- a polynomial function with x^a, $a \in \mathbb{Q}$, by using the results for when the exponent is an integer

These as a way of identifying the rule for $f(x) = ax^n$ then $f'(x) = n \cdot ax^{n-1}$.

They also had to explain the difference between a slope and a differential quotient.

Lastly, the students should practice the newly obtained techniques on a "real life problem" about hens laying eggs and the speed hereof at different times of day.

1.2.7 Hand in #2

This is nog a story of StandardJoe and HyperMick,

Problem A

What's the difference between a slope and a differential quotient?
(Anti?)Hint: Think gosh da.. and write a good explanation, or else..

Problem B

Find out what $f'(x)$ is for the function $f(x) = ax + b$ where a and b are reals. I.e. write an expression for what $f'(x)$ is in terms of a, b og x.

1. In general, what can be said of the derivative of a function, which contains constant terms.
2. Give an example of what $f'(x)$ is for a linear function of your choice.

Problem C

1. What is $f'(x)$ for $f(x) = x$?
2. What is $f'(x)$ for $f(x) = x^2$?
3. What is $f'(x)$ for $f(x) = x^3$?
4. Take a guess at what $f'(x)$ is for $f(x) = x^4$.
5. Give a justified guess on what $f'(x)$ is for $f(x) = x^n$ for $n \in \mathbb{N}$.

Problem D

a is a real number.

1. What is $f'(x)$ er for $f(x) = ax^2$?
2. What is $f'(x)$ for $f(x) = ax^3$?
3. Take a guess at what $f'(x)$ is for $f(x) = ax^4$?
4. Give a justified guess on what $f'(x)$ is for $f(x) = ax^n$ for $n \in \mathbb{N}$.
5. Find the derivative of $a \cdot f(x)$. In English, find out what happens, when you multiply a real number with a function $f(x)$ and the derive it.

Problem E

Give a justified guess on if you can derive $f(x) = x^a$ for $a \in \mathbb{Q}$. Use, if necessary, the results from the exercises above and/or examples.

Problem X

Studies show that when a rooster crows at 0600 A.M. the hens wake up and lay eggs for 12 hours straight. Don't fact check.

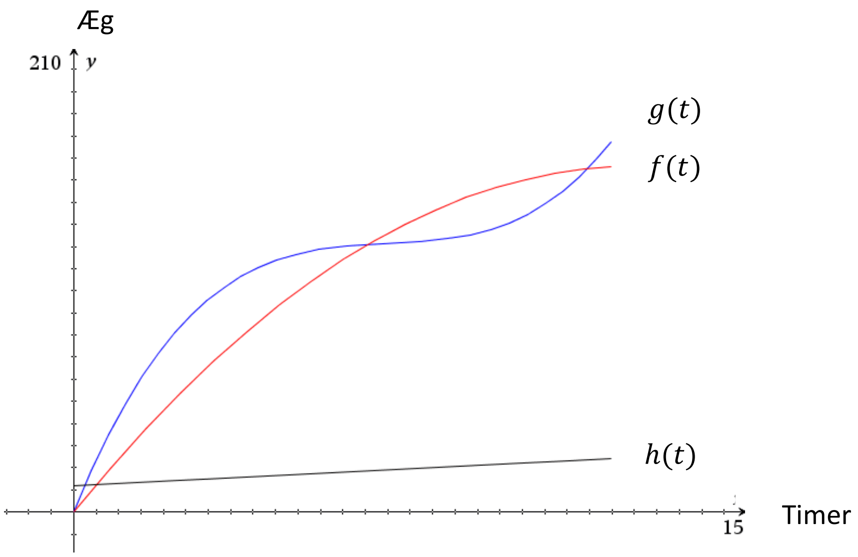
On a certain farm on The Faroe Islands, there is 3 hens, $f$rederegg, $g$geretsae and $h$enriegg. The correlation between the number of eggs the hens lay and the time of day can be described by the functions

$$f(t) = -t^2 + 25t$$

$$g(t) = \frac{1}{3}t^3 - 7t^2 + 50t$$

$$h(t) = t + 12$$

where the values of the functions are the number of eggs the hens have laid at a specific moment t, the number of hours passed since 0600 A.M. The coordinate system below shows the graphs for each of the hens from 0600 A.M. to 0600 P.M.



1. Who has laid the most eggs at 1200 noon?
2. How many eggs does $h$enriegg lay in the first 2 hours? How fast does she lay eggs in the first 2 hours? How fast does she lay eggs at (time) o'clock? Find $h'(t)$. Refleggt on the results.
3. How fast does $f$rederegg, $g$geretsae lay eggs at (time) o'clock?
4. Who lays eggs the fastest at 1200 noon?
5. Eggsplain what the number 12 means in $h(t) = t + 12$.

1.2.7.1 General description of answers to hand-in #2

A: By now, the students had the correct understanding of what the difference between a slope and a differential quotient is; the only problem was, as with all students, to give an adequate explanation.

B: They all found the derivative by using $f'(x) = st\left(\frac{^*f(x+dx)-f(x)}{dx}\right)$.

C+D: The purpose of the exercise was to get the students to generalize the technique for finding the derivative of x^n and ax^n. The table shows how many students actually found the correct technique. The table also shows how many of them actually gave an explanation as to how they had reached that conclusion, as opposed to the students who had only written the technique, whence it was impossible to know if they actually understood what they had written or if they had merely copied an answer. If a student had reached the right conclusion in C, they had done this as well in D.

Exercise C+D	# of students finding the correct technique	# of students not finding the correct technique
With explanation	13	5
Without explanation	5	

More or less the explanation given (in C):

Because that is what happened above and it makes sense, since every time you multiply with an extra $x + dx$ *you get one power higher for* x *and multiplied with one more.*

E: Of the 13 students, who had actually reached the right conclusion in C and D, some of them had ventured a guess here and some got it right.

X: The point of this exercise was to get the students to realize that the slope in a point is actually the speed in that particular point, from looking at the one hen. Not many got this explicitly and therefore had trouble finding the speed for the other two hens. Most of the students thought that the hen had laid 14 eggs, since $h(2) = 2 + 14$. Hence (pun intended) the realization of the hen laying eggs at the speed equal to its slope was lost.

1.2.8 Eighth lesson and ninth lesson (converted #4) (04.25)

The eighth lesson started with the teacher going through an example (DQ_{23}) on how to find values for a function given various values for x, with help from the students and then letting the students draw the graphs of different functions $(DQ_{24,i})$ manually, by finding a number of points the graph runs through and then specify the conditions of monotony. The second lesson the teacher went through $DQ_{24,i}$ with the students and then they got to work with $DQ_{26,i}$ and DQ_{27}.

DQ_{22} How do you draw a graph for a given function, $f(x) = 2x$?

$DQ_{23,1}$ Draw the graphs of the following functions

a. $f(x) = x^2$
b. $g(x) = 2x^2 - 3$
c. $h(x) = -x^2 + 3$
d. $k(x) = x^3 + 2x^2 - 3$

$DQ_{23,2}$ Find the conditions of monotony for the functions above.

DQ_{24} For what do we use conditions of monotony?

$DQ_{25,1}$ Given the function $f(x) = x^2 - 2x + 4$, fill out the table

x	-8	-6	-4	-2	0	2	4	6	8
$f(x)$									
$^*f(x + dx)$									

$DQ_{25,2}$ Identify to which value of x the function most likely has either a maximum or minimum.

$DQ_{25,3}$ Find $^*f(x_0 + dx) - f(x_0)$ for the following values of x_0

x_0	-8	-6	-4	-2	0	2	4	6	8
$^*f(x_0 + dx) - f(x_0)$									

DQ_{26} Determine the extremes of the function (i.e. maxima and/or minima) and make a line of monotony (as in the compendium) for the function. Give a written conclusion at the end. (In case of massive laziness, the table below can be used instead of making a line of monotony)

x					
f'					
f					

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Functions	Technical moment DQ_{22}	T and S	$f(x)$	T goes through DQ_{22} with S
Conditions of monotony. Differential quotient	Technical moment $DQ_{23,i}$ or DQ_{24}	S and T	$f(x)$, $f'(x)$, [;], coordinate system	T walks around and helps S if they have any questions
Functions. Hyperreal functions. Conditions of monotony	Technical moment $DQ_{25,i}$ or DQ_{26}	S and T	$f(x)$, $^*f(x)$, $f'(x)$, [;]	

The students worked some more on the conditions of monotony, to further understand the connection between the graph and the differential quotient. They had a lot of trouble (even) finding the function value for a given x-value, hence the many exercises in this.

1.2.9 *Tenth, eleventh and twelfth lesson (04.26 and 04.27 and 05.03)

In lesson ten and eleven, the students were divided into groups and given one of the questions DQ_{27} to DQ_{31}, with one or more hints on how to get started and/or get through the proof (see Worksheet 04.26). The first 20 minutes of lesson eleven was used by the teacher to go through the proof of how to derive e^x (DQ_{32}).

In the twelfth lesson, the students presented their product from lessons ten and eleven. The requirements for the show was that they should present it on the blackboard by writing and talking and every person in the group should say something.

DQ_{27} Find the derivative of $f(x) = \frac{1}{x}$

DQ_{28} Find the derivative of $f(x) = \sqrt{x}$

DQ_{29} How do you find the equation of the tangent to the function $f(x)$ in the point $\left(x_0, f(x_0)\right)$?

DQ_{30} Let $f(x)$ and $g(x)$ be differentiable functions and let $h(x) = f(x) \cdot g(x)$. Find the derivative of $h(x)$

DQ_{31} Let $f(x)$ and $g(x)$ be differentiable functions and let $h(x) = f\left(g(x)\right)$. Find the derivative to $h(x)$

DQ_{32} What is the derivative of e^x?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Differential quotient. Standard part. Star operation	Technical moment DQ_{27} or DQ_{28}	S and T	$f'(x) = st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$	T walks around and helps S if they have any questions
Equation of a straight line. Tangent	Technical moment DQ_{29}		$f'(x_0), y = ax + b$	
Differential quotient. Standard part. Star operation	Technical moment DQ_{30} or DQ_{31}		$f'(x) = st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$	
Equation of a straight line. Tangent. Differential quotient. Standard part. Star operation	Technical work	S	$f'(x) = st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$, $f'(x_0), y = ax + b$	S present their show
	First encounter with D_2T_2 and D_1T_3			S watch the show
Differential quotient	Institutionalization	T	$f'(x) = st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$	T goes through the proof of how to derive e^x

In the duration of three lessons, the students work independently in two of them and present their finished product orally in front of the rest of the class. The teacher divided the students into groups of 4 or 5.

By walking around in the classroom and observing the students, the process of solving the problems looks like this:

Problem 1 The students knew by then to use the formula $f'(x) = st\left(\frac{^*f(x+dx)-f(x)}{dx}\right)$ and had virtually no problems executing this. The group only asked the teacher one time if they had done it correctly one time and that was a question about the rules of fractions, which they had done correctly though. In addition, they had finished the rest of the proof by then, they simply wanted to know if they could explain the rule properly.

Oral The only part not gone through rigorously enough was the equation

$$st\left(-\frac{1}{x^2 + xdx}\right) = -\frac{1}{x^2}$$

which was only explained with "in taking the standard part xdx disappears".

Problem 2 They had trouble using the first hint because of two reasons, one of which is less important though, namely that there wasn't any a's or b's in finding the derivative of the function. The other reason was that there was no product of two terms being subtracted with the same two terms added. Thus, the students needed help using the hints, since the other hint didn't help them either. They needed to see the connection between the two hints more than perceiving them as two separate clues and then all was fine.

Oral The only part not gone through rigorously enough was the equation

$$st\left(\frac{1}{\sqrt{x+dx}+\sqrt{x}}\right) = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

which was only explained with "in taking the standard part xdx disappears".

Problem 3 From start to finish, this took the students around 15 minutes, meaning from when they got the problem until the teacher came to check on them and they had found the equation for the tangent. No questions, no doubts.

Oral The students first realized, that $a = f'(x_0)$. Then looking at the equation for the straight line, $y = ax + b$, they found that b must be $b = y - ax = f(x_0) - f'(x_0)x_0$ and by putting this intro the equation for the straight line, the equation for the tangent is obtained

$$T = ax + b = f'(x_0)x + f(x_0) - f'(x_0)x = f'(x_0)(x - x_0) + f(x_0).$$

Problem 4 The first of three problems was to connect the hints. Sure, they made sense for the students, but not in the context of the problem. Once this was obtained another problem surfaced, namely how does $-{}^{*}g(x+dx)f(x) + {}^{*}g(x+dx)f(x)$ help, especially if it's 0 anyway? The teacher then placed it in a preferable place and helped the students realize that some of the terms included the same factor and they then figured out how to put this outside the brackets. The last thing was to see the link between what they had and $f'(x) = st\left(\frac{{}^{*}f(x+dx)-f(x)}{dx}\right)$.

Oral The only thing the students didn't mention was why

$$st\left({}^{*}g(x+dx)\right) = g(x)$$

Problem 5 Again, the teacher had to help the students realize how to execute the hint (just the first one). Once this was done, one student in the group saw, that

$$st\left(\frac{{}^{*}f\left({}^{*}g(x+dx)\right) - f\left(g(x)\right)}{{}^{*}g(x+dx) - g(x)}\right) = f'\left(g(x)\right)$$

where the student discovered that it's the same as $f'(x) = st\left(\frac{{}^{*}f(x+dx)-f(x)}{dx}\right)$, but with a change of variable.

Oral The students gave a more elaborate explanation of the equation above, namely that they could see it as using the 2-point formula. The only part not gone through rigorously enough was the equation

$$st\left(\frac{{}^{*}f\left({}^{*}g(x+dx)\right) - f\left(g(x)\right)}{{}^{*}g(x+dx) - g(x)} \cdot \frac{{}^{*}g(x+dx) - g(x)}{dx}\right)$$
$$= st\left(\frac{{}^{*}f\left({}^{*}g(x+dx)\right) - f\left(g(x)\right)}{{}^{*}g(x+dx) - g(x)}\right) \cdot st\left(\frac{{}^{*}g(x+dx) - g(x)}{dx}\right)$$

which was only explained with "by using the rule [for standard parts] we can split them up".

1.2.9.1 Worksheet 04.26

Problem 1

$$f(x) = \frac{1}{x}$$

Find the derivative of $f(x)$☺.
Hint:

Know your rules for fractions… (how does one make a common denominator?)

Problem 2

$$f(x) = \sqrt{x}$$

Find the derivative of $f(x)$☺.
Hints:

1. (mathematics in the days of yore) *the difference between two numbers multiplied with the sum of the same two numbers is the difference between the squared numbers*

$$(a-b)(a+b) = a^2 - b^2$$

2. To multiply with 1 doesn't make a difference, though it makes some fairytale creatures happy to go from the real numbers to the integral numbers.

$$1 = \frac{(a+b)}{(a+b)}$$

Problem 3

How do you find the equation of the tangent to the function $f(x)$ in the point $(x_0, f(x_0))$?
Hints☺:

1. What is a tangent?
2. What does the tangent have to do with the differential quotient?
3. How does one find the slope of a tangent?

Problem 4

Let $f(x)$ and $g(x)$ be differentiable functions and let $h(x) = f(x) \cdot g(x)$. Find the derivative of $h(x)$.
Hints☺:

1. If you owe 2 kr. and someone gives you 2 kr., you have 0 kr.

$$-^{*}g(x+dx)f(x) + {}^{*}g(x+dx)f(x) = 0$$

2. HyperMick and StandardJoe divided a kr. between the two, but that's the same as dividing $a-0$ kr.!

$$\frac{a}{b} = \frac{a-0}{b}$$

Problem 5

Let $f(x)$ and $g(x)$ be differentiable functions and let $h(x) = f\left(g(x)\right)$. Find the derivative to $h(x)$.

Hints:

1. Multiplying with 1 doesn't make a difference, though it makes some mermaids happy to go from unknown to known factors.
$$1 = \frac{dg}{dg} = \frac{{}^*g(x+dx) - g(x)}{{}^*g(x+dx) - g(x)}$$
2. The standard part of a product is the same as the product of the standard parts of each of the factors, as long as the factors are finite.
$$st(a \cdot b) = st(a) \cdot st(b)$$

1.2.10 *Thirteenth lesson (05.03)

The thirteenth lesson was used to summarize what they had done in differential calculus; to this end, (a blank version of) Worksheet 05.03 was prepared and handed out to them. The students then spent around 40 minutes to fill out the worksheet to the best of their abilities in small groups (of order < 5). Afterwards the groups read aloud, to the rest of the class, what they had on their papers, ending with the creation of the following filled out worksheet.

1.2.10.1 Worksheet 05.03

$f(x)$	$f'(x)$
b	0
ax	a
ax^2	$a2x$
ax^n	anx^{n-1}
$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2} = -x^{-2}$
$\sqrt{x} = x^{\frac{1}{2}}$	$-\frac{1}{2\sqrt{x}} = -\frac{1}{2}x^{-\frac{1}{2}}$
e^x	e^x

Rules of calculation	
$h(x) = f(x) \pm g(x)$	$h'(x) = f'(x) \pm g'(x)$
$h(x) = a \cdot f(x)$	$h'(x) = a \cdot f'(x)$
$h(x) = f(x) \cdot g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
$h(x) = f(g(x))$	$h'(x) = g'(x) \cdot f'(g(x))$
$h(x) = \frac{f(x)}{g(x)}$	$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
1	$\frac{dy}{dy}$
$st(a \cdot b)$	$st(a) \cdot st(b)$, hvis hverken a eller b er uendeligt
$(a-b)(a+b)$	$a^2 - b^2$
$h(x) = a \cdot f(x)$	$h'(x) = a \cdot f'(x)$
$\frac{a}{b} = \frac{a-0}{b}$	

- Differential quotient, which is the slope in a point.

$$st\left({}^{*}\frac{f(x+dx) - f(x)}{dx}\right) = f'(x)$$

- Equation of the tangent

$$T_{f(x_0)}(x) = f'(x_0) \cdot x + f(x_0) - f'(x_0)x_0 = f'(x_0)(x - x_0) + f(x_0).$$

- Conditions of monotony, describes if a graph is decreasing or increasing in interval. Between these, the graph is constant.

- Hyperreal numbers, infinitesimals and infinite numbers and the real numbers.

1.2.11 Hand in #3

The sum of composed quotientproducts' difference

Problem All kinds of things

Use the rules for calculating with differentials to derive the following functions

a. $f(x) = e^x$
b. $g(x) = -4x^2 + 4x - 4$
c. $h(x) = 3x^4 + 5x^2 - 16$
d. $i(x) = 2 \cdot \frac{1}{x} - x^{9,9}$
e. $j(x) = -3\sqrt{x} + \frac{2}{x} - x^{6,6}$
f. $k(x) = \frac{1}{x^{-3}} + x^3$
g. $l(x) = x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x^1 - x^0$

Problem Broduct rule

Find the derivative to the following functions, using the rules for calculating with differentials

a. $f(x) = 2x^2 \cdot e^x$
b. $g(x) = 5x \cdot \frac{1}{x}$
c. $h(x) = \frac{1}{x} \cdot \sqrt{x}$
d. $i(x) = 4\sqrt{x} \cdot x^{2,8}$
e. $j(x) = \sqrt{x} \cdot e^x$
f. $k(x) = \frac{e^x}{x}$
g. $l(x) = (e^x - 3^2) \cdot (6x^2 + 6x - 6)$

Problem Composed functions

Derive the following functions by first finding the inner and then the outer function and then using the rules for calculating with differentials

$\left(\text{Given } f\big(g(x)\big) \text{ then the derived function is } f'\big(g(x)\big) \cdot g'(x)\right)$

a. $f(x) = e^{2x+5}$
b. $g(x) = e^{3x^2}$
c. $h(x) = (4x - 7)^7$
d. $i(x) = \sqrt{x^2}$
e. $j(x) = \sqrt{-6x^2 + 4}$
f. $k(x) = \frac{1}{x+9}$
g. $l(x) = \frac{1}{x^4+3x}$

Problem Dat is about conditions of monotony

Find the conditions of monotony for the following functions

a. $f(x) = 3x - 9$
b. $g(x) = 2x^2 + x - 1$
c. $h(x) = -x^2 - 16x + 7$
d. $i(x) = \frac{1}{3}x^3 - 4x + 3.000^{100}$

Problem Equation of the tangent

1.

a. Find the equation of the tangent to the function $f(x) = \frac{1}{x}$ in the point $(3, f(3))$

b. Does an x-value exist for which there is no tangent; if yes, then for which x-value and why is there no tangent to the function in this value?

2. Given a ficticious function, let's call it finction! Finc about it..
$f(x) = 4x^2 - 8x + 6$.

a. Find the equation of the tangent to $f(x)$ in the point $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$

b. Find the equation of the tangent to $f(x)$ in the point $(-1, f(-1))$

Problem Xhausting

Solve 2 of the exercices below, which cannot be the one you solved in groups.

1. Let $f(x) = \frac{1}{x}$, prove that
$$f'(x) = \frac{-1}{x^2}.$$
2. Let $f(x) = \sqrt{x}$, prove that
$$f'(x) = \frac{1}{2\sqrt{x}}.$$

3. Go through the proof for the product rule. I.e. $h(x) = f(x) \cdot g(x)$ then
$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

4. Prove the chain rule (derivative of a composed functions). I.e. $h(x) = f(g(x))$ then
$$h'(x) = g'(x) \cdot f'(g(x)).$$

1.2.11.1 General answers for Hand in #3

When nothing is mentioned about a possible difficulty, the students had no problem with this

Problem A

The students had difficulties with the rules for calculating with powers.

Problem B

A part of the students had miswritten the product rule and thus making mistakes that could have been avoided, had they only used the right rule.

Another part of the students had a hard time just using the product rule correctly in the form of what the one and the other function was.

Problem C

Some of the students had problems recognizing the inner function and others had problems of when to derive which function and what to do with it

Problem D

A small part of the students skipped this problem entirely, but those who did not answered it correctly

Problem E

Trouble concerning calculations with equation solving, not many for the equation of the tangent though. A few students kind of overdid it, by using the definition of the differential quotient to find the derivative of the function. Others again did not even try to give a solution, but skipped the problem entirely.
Some students ventured a (justified) guess for 1.b, very few getting the right solution.

Problem X

Not many students had trouble rewriting the proofs done during class by the other groups.

1.3 Description of teaching integral calculus (Mikkel Mathias Lindahl)

During the beginning of teaching integral calculus one of the students dropped out of the class, which made the number of students go from 23 to 22!

1.3.1 First and second lesson (05.09)

In the beginning of the lesson 15 minutes was used to go through some of the differential calculus done in the previous lessons. Then the students was put in 5 groups and asked to do worksheet 05.09.

IQ_1: Any ideas as to how to find the area of the side of the house?
IQ_2: What is the width of the intervals?
IQ_3: What is the area of rectangles?
IQ_4: How many of these rectangles are needed for the area found by the rectangles to be the same as the sought area?
IQ_5: What should we do with the areas of the infinitely many rectangles?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Find the area between the first axis and the function $f(x) = -x^2 + 4x$. Between 0 and 4. Find the area under a constant function. Can a function be partitioned in intervals such that the function can be viewed as a constant function on the intervals?	First encounter -> exploration	S	$f(x)$, area of rectangle, infinite sum and infinitesimal intervals.	T walks around and helps S if they have questions for worksheet 05.09.
	Constitution IQ_1 to IQ_5	S/T		The lesson ends with a breakdown of what the different groups had come up with. T writes on the BB what the S are saying, ending with the expression for the area between the first axis and the function between 0 and 4 as: $f(x_1)dx + f(x_2)dx + f(x_3)dx + \cdots + f(x_{4/dx})dx$. Where $0 = x_1 < x_2 < \cdots < x_{4/dx} < x_{4/dx} + dx = 4$, is a partition of the interval $[0,4]$.

A list of different answers to the problem was created during the lesson, but with the breakdown of what the students had done in the different groups the students agreed to the same method as written on the blackboard. Approximately 60 % of the students had come up with the indices for the different x-values, the rest of the students had only explained it orally. One student wanted to make the height of the rectangles infinitesimal and the width of them finite, i.e. something that looked more like the Lebesgue integral than the Riemann integral, the student recognized that the area obtained by the different approaches where the same. It should be noted that 3 out of the 5 groups started to do what would have been called numerical integration, by making a finite partition and then calculating the areas of the rectangles.

1.3.1.1 Worksheet (05.09)

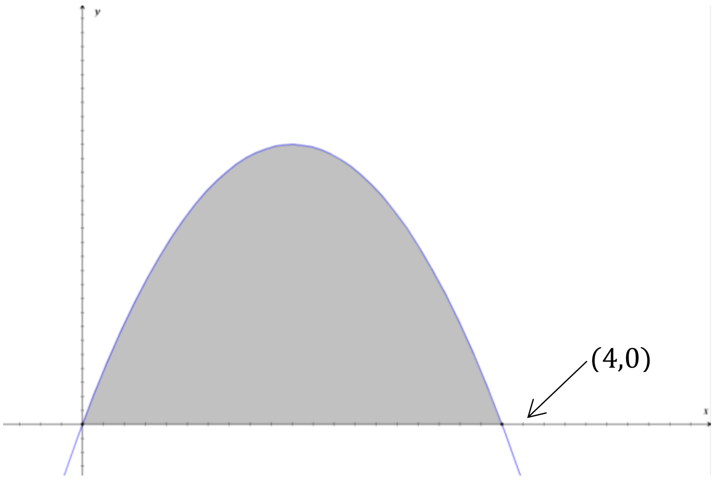
An alien called Laera poses her high school teacher a problem:

StandardJoe in my family we are going to have a party to celebrate my dad, Noitknufmats, he loves both honey and saffron, and as all other aliens; to lick his house to keep it water resistant. By extension I want to cover a side of my dad's house in honey and sprinkle saffron on top of the honey, so that we can lick the house clean during the party together. Since both honey and saffron are in short supply on our planet I don't wish to buy too much of it, hence my question is; If I Have the following function, which describes the height of the house above the surface of the planet,

$$f(x) = -x^2 + 4x,$$

how do I find the Area of the side of the house?

To this end I've brought a picture of my dad's house:



Then the teacher poses some question to help answer the first question:

1. How do you find the area under a constant function?
   - One could try to draw one and find the area between the graph and the first axis on a certain interval $[a, b]$.
2. Can the function be partitioned in intervals such that the function can be viewed as a constant function on the intervals?

When Laera finds the answer to her question one of her friends tells her that it could be problematic with honey and saffron, on the bottommost 2 meters of the house, if the wall are to be used for skull sucking during the party.

Bonus exercises:

1. What area is to be covered in honey and saffron if there should be place for skullsucking?
2. How tall is the house?
3. What is the area of the neighboring house, when the height is described by the function $f(x) = \sqrt{x}$ for $x \in [0,4]$?
4. Write a general expression for the area between a graph (of a function) and the first axis on an interval.

The different groups came up with almost the same sub questions and answers for the generating question in the worksheet. The two questions posed in order to help the students along the way were all answered as a first step to answer the generating question.

1. The area under a constant function can be found in the same way as finding the area of a rectangle, the height times the width, in this case the height would be the function value, and the width would be the length of the interval.
2. Yes, it can be done and in order to view the function as constant in the interval, the intervals have to be of infinitesimal length.

With these two answers the students started to generate a way to find the area. The students started to divide the area into smaller areas, some of them also used triangles in the beginning, the triangles where used to get a better approximation when using a finite partition. When going to the infinite partition the students wrote the following:

The interval had to be infinitesimal, i.e. any of the intervals could be written as $[x, x + dx]$ some of the students also wrote $x_2 - x_1 = dx$ and so on for every other integer (understood as every other set of consecutive integers). With this infinitesimal interval the students then recognized the height of the function as something between $f(x)$ and $f(x + dx)$. One group of students did a very surprising observation; they noticed that it did not matter if they used $f(x)$ or $f(x + dx)$ as long as they used the same for every index. Considering the symmetry of the function they established that the error gained on the first half of the area (until $x = 2$) would be cancelled by the error on the second half.

The majority of the students used the first point in the interval and said the function value in this point corresponds to the height of the rectangle. In this way they found the area of a given rectangle with infinitesimal width as $f(x) \cdot dx$. Since none of the student new of the symbol $\sum$ as a way to write a sum, the students wrote the sum as $f(0) \cdot dx + f(dx) \cdot dx + f(dx + dx)$ ... (yes they forgot the stars!!) but none of them wrote the three dots, they just stopped and then wrote in text that one had to do it an infinite number of times. When the groups got to this point in the answer for the question of finding the area, the teacher asked the groups to figure out how many times this infinitely many had to be done. The first answer to this question was $\frac{1}{dx}$ which prompted another teacher generated question, what is the length of the interval obtained by adding $\frac{1}{dx}$ many intervals of length dx. This led to the conclusion that the number of infinitesimal intervals would be $\frac{4}{dx}$.

Observed (teacher) generated sub problems for the worksheet:

1. How does one divide the area, which need be found, into smaller areas which can be found?
2. Should the smaller areas only be rectangles or also triangles, and why?
3. How does one compute the area of a rectangle?
4. How does one compute the area of a (right) triangle?
5. What is the area of a rectangle with infinitesimal width?
6. How does one partition an interval and how should it be written?
7. **If the length of the infinitesimal intervals is dx then what is the infinite number used to partition the interval?**

1.3.2 Third lesson (05.10)

Most of this lesson was used to determine what the differential of a quotient of functions is through working with worksheet (05.10), and only the last of the lesson were used to introduce the integral and its informal definition obtained in the previous lesson. In hindsight this was not a very good idea as the amount of lessons left for doing integral calculus was already at a minimum.

IQ_6: Is this infinite sum a real or a hyperreal number?

IQ_7: Can something be done to a hyperreal number which ensures it ends up as a real number?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Using an infinite sum of areas of rectangles to define the definite integral.	Recap of previous lesson	T	$^*f(x+dx)$, area of rectangle, infinite sum, $x_1 + x_2 + \cdots + x_n$, infinitesimal intervals and partition	T talks about the house and area from last lesson and rewrites what they had on the blackboard at the end of the previous lesson.
	Institutionalization IQ_6 IQ_7	S/T	$st(a)$, $\int_a^b f(x)\,dx$.	T: when using this approach to talk about the area between the function and the first axis, it is tiresome to write this sum every time, hence we introduce, this weird **S,** $\int_a^b f(x)$, as the standard part of the infinite sum of the area of the rectangles with infinite width and height as the function value. And draws an example of a linear function and explains how to use the integral and what the boundaries are.

1.3.2.1 General description of the answers to the worksheet

The worksheet was answered by the students in groups and with a little help from the teachers the students were able to establish the rule. In the end the rule was presented on the blackboard in plenum.

1.3.2.2 Worksheet (05.10)

Problem

Let $f(x)$ and $g(x)$ be differentiable functions and let $h(x) = \frac{f(x)}{g(x)}$. show that the derivative of $h(x)$ is

$$h'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Hints☺:

3. $\frac{a}{b} = a \cdot \frac{1}{b}$
4. and use the derivative of the following:
    a. $f(x) = \frac{1}{x}$
    b. $h(x) = f(x) \cdot g(x)$
    c. $h(x) = f(g(x))$

1.3.3 Fourth lesson (05.11)

The students were given worksheet (05.11), which included exercises to cover the next part of the planned integral calculus. The last 20 minutes of the lesson was used to go through the exercises in plenum, as to make sure they all had all seen the correct methods, which where needed for the upcoming hand in #4.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Definite integral, of simple functions, and definite integrals with various endpoint, i.e. area functions.	First encounter with problems of type I_1T_1	S	$A(rea) = a \cdot b$, $A(rea) = \frac{1}{2}a \cdot b$.	T walks around and helps S if they don't understand the exercises.

1.3.3.1 General description of the answers to the worksheet

In the beginning they had to think a while in order to understand what the students were supposed to do. After turning paper around they saw the graphs of some of the functions which made them do the connection as to how they should find the area. Most of the students got to finish the worksheet and everyone did at least the first page. With the problems on the worksheet almost self-explanatory a description of the answers will be omitted.

1.3.3.2 Worksheet (05.11)

The definition of the definite integral, for a function $f(x)$ on the interval $[a, b]$, as given in class in the previous lesson, demands a partition of the interval $[a, b]$. The interval is partitioned into infinitely many, $N \in {}^*\mathbb{R}$, parts such that $\frac{1}{N}$ is an infinitesimal. This partition can be written in the following way:

$$a = x_1 < x_2 < x_3 < \cdots < x_N = b.$$

Thus $x_2 - x_1$ is an infinitesimal and $x_5 - x_4$ as well, and so forth. With this in mind the definite integral becomes:

$$\int_a^b f(x)dx = st(f(x_1)dx + f(x_2)dx + f(x_3)dx + \cdots + f(x_{N-1})dx).$$

Determine the following integrals (one could draw the functions which are to be integrated)

Exercise 1	**Exercise 2**	**Exercise 3**
$\int_0^4 1\,dx =$	$\int_2^5 2\,dx =$	$\int_0^6 4\,dx =$
$\int_0^4 2\,dx =$	$\int_3^7 2\,dx =$	$\int_0^6 -4\,dx =$
$\int_0^4 3\,dx =$	$\int_5^7 3\,dx =$	$\int_2^6 -3\,dx =$

4. What is the integral of a general constant function $f(x) = c$?

$$\int_a^b c\,dx.$$

Determine the following integrals (one could draw the functions which are to be integrated)

Exercise 5	**Exercise 6**	**Exercise 7**
$\int_0^4 x\,dx =$	$\int_2^5 2x\,dx =$	$\int_0^6 4x\,dx =$
$\int_0^4 2x\,dx =$	$\int_3^7 2x\,dx =$	$\int_0^6 -4x\,dx =$
$\int_0^4 3x\,dx =$	$\int_5^7 3x\,dx =$	$\int_2^6 -3x\,dx =$

8. What is the area of a general linear function through zero, $f(x) = kx$?

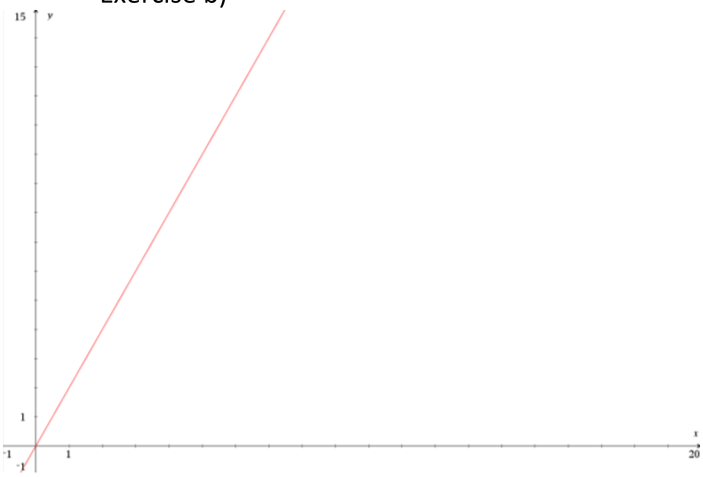
$$\int_a^b kx\,dx.$$

Determine the area under the graph on an interval of the type $[0, x]$ and write an area function, $A(x)$, which describes the area on the interval.

Exercise a)

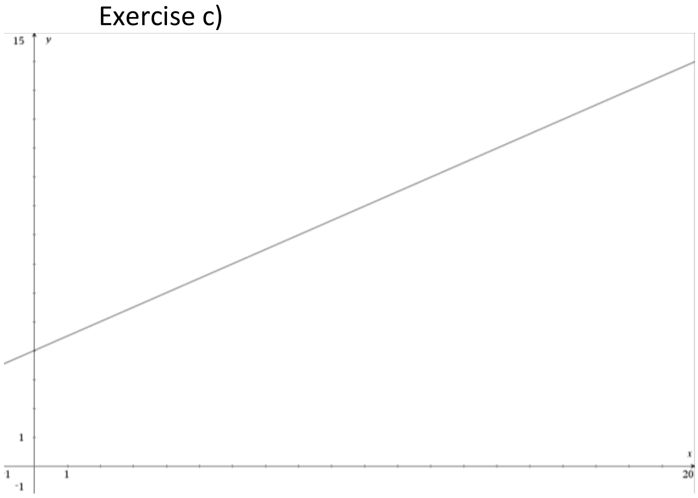
1. Determine the area function for $f(x) = 4$
2. Determine the area function for $f(x) = c$

Exercise b)



1. Determine the area function for funktionen $g(x) = 2x$
2. Determine the area function for $g(x) = kx$

Exercise c)



1. Determine the area function for $h(x) = \frac{1}{2}x + 4$ findes
2. Determine the area function for $h(x) = kx + c$

1.3.4 Fifth lesson (converted #5) (05.13)

They got to spend the lesson doing their Hand-in # 4 which was given to them the very same day. The Hand in was given to them in parts, as it was not fully developed. Most of the students did do the first page in this lesson which all concerned various problems of type $\int_a^b kx + c\,dx$, ending with a general description of what $\int_a^b kx + c\,dx$ is in terms of a, b, c, k and x.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Various problems of type $\int_a^b kx + c\,dx$	Exploration and technical work	S	$\int_a^b kx + c\,dx$	T is present to answer questions if a certain exercise is not understood.

1.3.4.1 General description of answers to hand in #4 page 1.

The techniques developed when doing the previous worksheet should be enough to answer the questions on this page. To this end a table describing the students understanding of the technique can be seen below. The table is generated on basis of the answers the students gave to the exercise on page 1 of hand in #4.

	Number of students who understood the technique	Number of students who misunderstood the technique
Technique to solve the integral of a constant function	21	1
Technique to solve the integral of a linear function.	$12 + (1)$	$9 + (1)$

The parentheses around the 1 is because two of the students used a very geometric understanding of the figure described by the interval, function and first axis, thus obtaining a different technique, where the figure would be described as a triangle plus a rectangle, and these two areas would be added in the end. This technique though correct, might look like a technique which has not been fully routinized.

The 9 students that misunderstood the technique did so in the very same way, all of them seemed to think that $(b-a)^2 = (b^2 - a^2)$. This misunderstanding can be seen as a lack of previous established praxeology or as a misunderstanding by the teachers, since they did not include this as part of the knowledge to be taught. The teachers seemed to think that the hierarchy of mathematical calculations should have covered this aspect.

1.3.5 Hand in #4 page 1

Integration by parts

Exercise Alien

"3, 2, 1 LICK!" –is how this year's Lickeliciousness starts. The day is always on the 7th Fleesday after the Yakis-celebration and this year it is on Octember 35th. The suns are shining and everyone rush out of their houses, to find the perfect house to lick. The rules are simple:

1. The one, who licks the most, is the winner.
2. You are allowed to lick a house which is already being licked, but you can't lick your own house.
   - ∴ In the event of more than one licker, skull sucking is not allowed.
3. When your own house has been licked clean, you can't lick anymore.

For rule number 3, they have a saying: "people(aliens) who live in licked houses shouldn't lick stones". Hahaha! Probably should have been there..

Aera gets to lick 4 houses before her own house is out of the game. What she licked can be described by the following integrals

Noitces' house

$$\int_0^1 2dx$$

Dleif's house

$$\int_1^3 2xdx$$

Tnuoma's house (the T is silent)

$$\int_2^5 (2x+1)dx$$

Ecafrus' house

$$\int_0^4 (-x+4)dx$$

1. Figure out how much Aera has licked on each house before her game is over.
2. Determine if Aera has licked any of the houses by herself.

It was Twolegged Toleg Twoleg who gave Area's house the last lick; literally, it was the only part of her house he got to lick! At the very least, Aera wanted to beat her in the competition. This is why she checks Twolegged Toleg Twoleg's results, which can be described by the integrals below

House 1:

$$\int_0^4 1dx + \int_5^6 1dx$$

House 2:

$$\int_2^7 (2x-4)dx$$

House 3:

$$\int_7^{11} 3dx$$

3. Did Aera beat Twolegged Toleg Twoleg?
4. Find general expressions for licking the different houses above, as shown below.

$$\int_a^b c\,dx \qquad \int_a^b kx\,dx \qquad \int_a^b (kx+c)dx$$

1.3.6 Sixth lesson (05.18)

IQ_8: What is an integral, do you remember?

IQ_9: What does $\int_a^b f(x)\,dx$ mean? Is it a number, a function or a banana?

IQ_{10}: What is $\int_0^x c\,dx$?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Recap of what an integral is.	IQ_8 IQ_9	S/T	$\int_a^b kx\,dx$	T: draws a and b in a coordinate system where a graph of the function $f(x) = kx$ are already drawn.
Recap of some of the exercises from last time.	IQ_{10} Institution alization.	T	$\int_0^x c\,dx$	T: writes the integral on the BB and S answers: $\int_0^x c\,dx = x(c-0)$
Group work with worksheet 05.18	First encounter with I_2T_3		$\int_a^b f(x)\,dx$, and it's definition and the rules for operating with the standard part.	T walks around and helps S if they don't understand the exercises.

1.3.6.1 *Worksheet 05.18*

Exercise 1:

Show that the integral of a sum or difference of functions is equal to the sum or difference of the integral of the functions at hand. I.e. show that

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Exercise 2:

Show that the integral of a constant, $k \in \mathbb{R}$, times a function, $f(x)$, is the same as the constant times the integral of the function. I.e. show that

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

Exercise 3:

Show that the integral from a to b add to the integral from b to c is equal to the integral from a to c, when the integrant is the same throughout. I.e. show that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Exercise 4:

Show that the integral over an interval of type $[a, a]$, thus only containing one real point, is zero. I.e. show that

$$\int_a^a f(x)\,dx = 0$$

Exercise 5:

Show that the definite integral changes sign when the boundaries are switched. I.e show

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Exercise 6:

Write the results of the following integrals and describe what can be said about the derivative of the results.

$\int_0^x c\,dx =$	$\int_0^x kx\,dx =$	$\int_0^x (kx + c)\,dx =$

Hints:

1. remember the definiton of the definite integral (for exercise 1,2,3):

$$\int_a^b f(x)dx = st\left({}^*f\left(a+\frac{dx}{2}\right)dx + {}^*f\left(x_1+\frac{dx}{2}\right)dx + {}^*f\left(x_2+\frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1}+\frac{dx}{2}\right)dx\right),$$

Where $a < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$, is an infinitesimal partition of the interval $[a, b]$ and $N \in {}^*\mathbb{R}$ is an infinitely large integer.

2. $st(a+b) = st(a) + st(b)$, if and only if a and b are finite! (for exercise 1,2,3)
3. Remember that $st(k \cdot a) = k \cdot st(a)$, if k is a real number.
4. Use the answers from 3 and 4 to show 5☺.
5. Remember that the integral can almost be viewed as an area☺ (for exercise 6)

Bonus Exercise 1:

Determine the integral from 0 to x of the function, x^2, i.e. what is

$$\int_0^x x^2\,dx$$

Hints:

1. Believe in what you found in exercise 5.
2. Remember the product rule from differential calculus, if $f(x)$ and $g(x)$ are 2 differentiable functions, and $h(x) = f(x) \cdot g(x)$, then
$$h'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x).$$
3. What is
$$\int_0^x g(x) \cdot f'(x) + f(x) \cdot g'(x)\,dx$$
4. Let $f'(x) = x$ and $g(x) = x$

Bonus Exercise 2:

Determine the area of the side of Noitknufmats house, Laera covered in honey and Saffron. I.e. determine the integral

$$\int_0^4 -x^2 + 4x\,dx.$$

1.3.6.2 Observed answers for Worksheet 05.18

Since the answers generated by the students were lost, here is a general answer to the worksheets as remembered and found in the audio recordings.

1.3.6.2.1 Exercise 1

The students general understanding of the integral as the area between the first axis and the graph, made this exercise intuitively very easy. When "forced" to use the definition of the integral most of the students did not bother with writing the entire sum, they thought it cumbersome. Some students introduced other symbols for the sum, so starting from the right going left they wrote, $st(sum(f)) + st(sum(g)) = st(sum(f) + sum(g)) = st(sum(f + g))$ referring to the techniques established for the standard part. Other students wrote in sentences what amounted to the same thing.

1.3.6.2.2 Exercise 2

The students used some of the same notation as the exercise before or used sentences, if that was what they had done in exercise 1. Finding that it came down to the algebraic operation of multiplying the constant on every term in the sum, and the technique for taking the standard part of a real constant times a hyperreal number. I.e. $k \cdot st(sum(f)) = st(k \cdot sum(f)) = st(sum(k \cdot f))$.

1.3.6.2.3 Exercise 3

This exercise proved to be a little tricky, the students intuitively understood that the equation should hold (concluded from the oral explanations observed) but the students which had exchanged the infinite sum with $sum(f)$ had suddenly lost the boundaries, which were needed to prove this. In the end none of the students did this in a mathematical way, they all reverted to the use of sentences to describe the process. An example of this is the following: "the area is the same if we find the area of the parts and add them or finding the area of all of it".

1.3.6.2.4 Exercise 4

"Finding the area of something with zero width will always be zero" was the general answer to this. Some of the students tried to use a constant function as an example. The students had problems with making a partition of an interval with zero length.

1.3.6.2.5 Exercise 5

They all needed the hint but in the end, most of them found that exchanging c with a in exercise 3, and then using the result from exercise 4, gave the desired result.

1.3.6.2.6 Exercise 6

This exercise enabled some of the students to guess at the connection between the derivative and the definite integral with a variable endpoint.

1.3.6.2.7 Bonus exercise 1

This exercise was only done by a handful of students. The students who did this had already guessed at the connection from exercise 6, so the students who did this exercise really just guessed at a result and had no real way of checking if it were correct (other than asking the teacher).

1.3.7 Seventh lesson (05.20)

Introducing antiderivatives and indefinite integral.

IQ_{11}: Do you know any functions which you also know the integral of?

IQ_{12}: What happens if we differentiate one of the functions in the row with $\int_a^x f(x)\,dx$?

IQ_{13}: What if $f(x) = x^2$, do we then know what $\int_a^x f(x)\,dx$ is?

IQ_{14}: Do we know what $\left(\int_a^x f(x)\,dx\right)'$ for $f(x) = x^2$?

IQ_{15}: What is an antiderivative of $f(x) = x$?

IQ_{15_1}: What if I want the antiderivative to go through the point (0,2)?

IQ_{16}: How many antiderivatives does a function have?

IQ_{17}: What is an antiderivative of the function $f(x) = k$?

IQ_{17_1}: What is the indefinite integral of the function $f(x) = k$?

IQ_{18}: What is an antiderivative of the function $f(x) = kx$?

IQ_{18_1}: What is the indefinite integral of the function $f(x) = kx$?

IQ_{19}: What is an antiderivative of the function $f(x) = x^2$?

IQ_{19_1}: What is the indefinite integral of the function $f(x) = x^2$?

IQ_{20}: What is an antiderivative of the function $f(x) = e^x$?

IQ_{20_1}: What is the indefinite integral of the function $f(x) = e^x$?

IQ_{21}: What is an antiderivative of the function $f(x) = x^n$?

IQ_{21_1}: What is the indefinite integral of the function $f(x) = x^n$?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Recap of what have been done with integrals so far.	Exploration IQ_{11} IQ_{12}	T/S	$\int_a^b kx\,dx$	T: draws a table on the black board with 1. a column of functions, 2. a column of integrals of the functions with variable endpoint and 3. a column of derivatives of the integrated functions. (see table below)
Connection between integral and derivatives	IQ_{13} IQ_{14}	T/S		
	institutionalization	T	$\int_a^b f'(x)\,dx$, $\left(\int_a^x f(x)\,dx\right)'$	T states the fundamental theorem of calculus! With a short consideration of the area function. * $\int_a^{x+dx} f(x)\,dx - \int_a^x f(x)\,dx \approx f(x)dx$
Antiderivative	institutionalization	T	$F'(x) = f(x)$ $F(x)$=antiderivative of $f(x)$.	T presents the definition of the antiderivative as a function F, for which $F'(x) = f(x)$. which by the fundamental theorem is the same as $F(x) = \int_a^x f(x)\,dx + k$
	IQ_{15} IQ_{15_1} IQ_{16}	S (first) encounter, exploratory		
Indefinite integral	Institutionalization	T	$\int f(x)\,dx$	T writes the definition of the indefinite integral as $\int f(x)\,dx = F(x) + c$ For some constant c, and explains why it could be written as $\int f(x)\,dx = \int_a^x f(x)\,dx + k.$
	IQ_{17} to IQ_{21_1} First encounter and exploratory	S		T: writes the students answers on the BB and in the end fills out the rest of the table on the BB.

$f(x)$	$\int_a^x f(x)\,dx$	$\left(\int_a^x f(x)\,dx\right)'$
c	$c(x-a)$	c
kx	$^1/_2\,k(x^2 - a^2)$	kx
x^2	$^1/_3\,(x^3 - a^3)$	x^2
e^x	$e^x - e^a$	e^x
x^n	$\frac{1}{n+1}(x^{n+1} - a^{n+1})$	x^n

1.3.8 Eighth lesson (05.20)

In this lesson they got to spend time on their hand in # 4, which was also used as a kind of worksheet… they should have the techniques to do page 2 of said hand in. most of the students got to do all the indefinite integral and some of the specific antiderivatives by letting them pass through a point.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Various problems of type $\int f(x)\,dx$	Exploratory, constitutional and (technical work)	S	$\int f(x)\,dx$	T is present to answer questions if a certain exercise is not understood.
Determine a specific antiderivative by letting it go through a given point.	First encounter and exploratory.	S	Equation solving.	
Determine the sum of 4 definite integrals written as text only.		S	$\int_a^b f(x)\,dx$	

1.3.8.1 General description of answers to hand in #4 page 2.

Generally speaking the results for this part of the hand in were rather good. The only problems worth mentioning are when the integrand is a product of a sum of power functions (which could be recognized as the derivative of a product of functions). In this part very few remembered to do the multiplication before finding the indefinite integral or even recognized it as the derivative of a product of functions.

In the problem, $\int 6x^2 \cdot e^{2x^3}\,dx$, only a handful of students actually got the idea to look for something which could be differentiated into the integrand using the rule for the derivative of a function composed with another. This problem can be seen as a first encounter with a problem requiring the technique integrations by substitution.

In the problems with finding the specific antiderivative no one seemed to have a problem with figuring out what to do, though a lot of the students complained that the amount of repetitions of the same kind of problems was over the top.

In the last problem of the page, most students did not get, that it was the definite integral of the specific antiderivative found by letting it pass through a point was the one to use as an integrand. As such only a handful of students displayed the use of the technique $\int_a^b f(x)\,dx = F(b) - F(a)$.

1.3.9 Hand in #4 page 2

Exercise Building new houses

After a terrible sun storm, several of the houses were damaged; the houses simply weren't licked enough. It turned out to be the houses that Aera had licked during the contest; you see it was Noitces, Tnuoma, Dleif and Ecafrus who had lost their houses during the storm. The mayor, Noitknufmats, decided that new houses were to be build but this time it should be done as they did in the days of yore. Thus, they had to be built as their ancestors did. Aera was, as her punishment, to find the ancestors' blueprints, which could be found by finding the antiderivatives of the destroyed houses' height-functions for every side.

Noitces' new house can be described by		Tnuoma's new house can be described by	
a1.	$\int 2\,dx$	b1.	$\int 2x\,dx$
a2.	$\int \left(\frac{1}{4}x+2\right)dx$	b2.	$\int (x^2-4x+8)\,dx$
a3.	$\int 3\,dx$	b3.	$\int (-x^3+8)\,dx$
a4.	$\int \left(-\frac{1}{4}x+3\right)dx$	b4.	$\int (-x^2+4x)\,dx$
Dleif's new house can be described by		Ecafrus' new house can be described by	
c1.	$\int (4x^3-3x^2+2x)\,dx$	d1.	$\int e^x\,dx$
c2.	$\int -\frac{1}{x^2}dx$	d2.	$\int \left((2x+8)(2x)+2(x^2)\right)dx$
c3.	$\int \frac{8}{x^2}dx$	d3.	$\int 6x^2\cdot e^{2x^3}\,dx$
c4.	$\int (-4x^3+x^{-4})\,dx$	d4.	$\int \frac{1}{e^x}dx$

1. When Aera had done this tiresome work, she found out that the ancestors had not given thought to the future, in the sense that, in the future there would be string phones between every house. These strings needed some space, which is why Noitknufmats had found the specific points the antiderivatives should go through, in order to make room for the string phones. (The points have been given names, such that they correspond to the names from the previous exercise.)

a1(0,0)	b1(2,9)	c1(2,0)	d1(0,−3)
a2(4,12)	b2(3,15)	c2(1,0)	d2(1,12)
a3(2,4)	b3(2,8)	c3(−8,8)	d3(2,0)
a4(4,8)	b4(3,0)	c4(1,3)	d4(0,−3)

Newly build houses need to be licked thoroughly in order to keep them watertight, which is why Aera was to lick some of Noitces' new house. She licked from 0 to 1 on the first side, from 1 to 2 on the second side, from 2 to 3 on the third side and from 0 to 4 on the last side. How big an area did Aera lick?

1.3.10 Ninth (05.23)

This lesson was used to describe the technique of integration by substitution.

IQ_{22}: What have we used when writing an infinitesimal?

IQ_{22_1}: What other than dx?

IQ_{23}: If I write df, what do you think it is?

IQ_{23_1}: What is $\frac{df}{dx}$?

IQ_{24}: What is $\frac{d}{dx}\left(e^{3x^2}\right)$?

IQ_{24_1}: Then what is $\int 6x \cdot e^{3x^2} dx$?

IQ_{25}: What is $\frac{d}{dx}\Big(F\big(g(x)\big)\Big)$, when $F(x)$ is an antiderivative of $f(x)$?

IQ_{25_1}: Then what is $\int f\big(g(x)\big) \cdot g'(x)dx$?

IQ_{26}: can anyone recognize the inner function in the expression $\int f\big(g(x)\big) \cdot g'(x)dx$?

IQ_{26_1}: What is du if $u = g(x)$?

IQ_{26_2}: How can $\int f\big(g(x)\big) \cdot g'(x)dx$, be written with u and du?

IQ_{27}: Now with boundaries on the integral, what is $\int_a^b f\big(g(x)\big) \cdot g'(x)\, dx$.

IQ_{27_1}: Normally we find the antiderivative of a function and then put the boundaries in instead of the variable but know we use $u = g(x)$ as a variable, hence $\int_a^b f\big(g(x)\big) \cdot g'(x)\, dx$ describes that the variable x runs from a to b but the variable u should not run from a to b. Thus $\int_a^b f\big(g(x)\big) \cdot g'(x)\, dx \neq \int_a^b f(u)\, du$ How do we amend this?

IQ_{28}: Using integration by substitution what is $\int_1^4 \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} dx$?

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
Different ways of writing the derivative	Institutionalization IQ_{22} to IQ_{23_1}	T	dx, df, f'	The S's answer the questions and the T ends up by letting df be defined as $df := f'(x) \cdot dx$.
An introduction to integration by substitution as the opposite of the chain rule for differentiation.	IQ_{24} IQ_{24_1} Exploration and constitution IQ_{25} IQ_{25_1}	T/S	$\frac{d}{dx}, F(x), f(x), \int f(x)\,dx$	T writes what the students are saying on the BB and ends up with the equation $\int f(g(x))g'(x)\,dx = F(g(x)) + c$
Develops the technique used when doing integration by substitution	(first encounter) Exploration -> Constitution IQ_{26} to IQ_{26_2}	T/S	$\frac{d}{dx}, F(x), f(x), \int f(x)\,dx, df$	T writes the students answers to the questions, let $u = g(x)$, then $du = g'(x) \cdot dx$ and then $\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + c = F(g(x)) + c$. The red color indicates that it is not mathematical correct.
Further develops integration by parts to include boundaries.	Technical work IQ_{27} IQ_{27_1}	T/S	$\frac{d}{dx}, F(x), f(x), \int_a^b f(x)\,dx, df$	T writes the students answers to the questions, and ends with the expression $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

1.3.11 Tenth lesson (05.23)

In this lesson they got to spend time on their hand in # 4, which was also used as a kind of worksheet… they should have the technique to do page 3 of said hand in.

Short description of the mathematics involved	Didactical moment	Main player	Mathematical objects involved	Didactical activities
4 problems where the use of integration by substitution is needed.	Exploratory, constitutional and technical work	S	$\frac{d}{dx}, F(x), f(x), \int_a^b f(x)\,dx, df$	After spending almost half the lesson the where the students did not understand the technique the teacher opted to the first and later the second problem on the blackboard in plenum.

With the first two problems having been done on the blackboard the problems where not considered as part of the hand in which where to be graded, this was not explained to the students, since the teachers wanted the students answers to the first two problems to see if the students understood what was done in plenum.

1.3.11.1 General description of answers to hand in #4 page 3

These last four problems really made the students go out of their skins. The students did not really get the technique and had problems reproducing the same steps that were made in plenum. In order to get an overview of this the following table can help

problem	# of students using the technique correct	# of students using the technique incorrect
a)	17	0
b)	16	0
c)	6	9
d)	4	11

As seen in the table not even all the students did the two problems which were done in plenum, supporting the fact that they really did not understand what was going on. The place where most of the students that had done problem c and d got it wrong was that they did nothing or differentiated instead of finding the antiderivative when they had done the substitution.

1.3.11.2 Hand in #4 page 3

Exercise Clone

Being the mayor of Quadrant 23u has its perks. Mayor Noitknufmats has been given the task of testing the first cloning machine of not only Quadrant 23u but of the entire solar system. When they called from the laboratory Aera overheard the string phone conversation between her dad and the lab, so of course she already conspired on how she should use it. There was this yearly math test she had on Februgust 21,5 and could really use a clone who could take her place! She called her mission the method of substitution and started by establishing the expressions for her substitute right away, so that it could look like her as much as possible and then she would swoop in after the test as if nothing had happened.

1. Use the method of substitution on the expressions below, which will, hopefully create a perfect Aera clone.

a. $\displaystyle\int_{1}^{\sqrt{5}} -\frac{2x}{(x^2+3)^2}dx$

b. $\displaystyle\int_{0}^{\frac{1}{2}} (4x^3+2x^2)^{10}\cdot(12x^2+4x)dx$

c. $\displaystyle\int_{0}^{1} \frac{8x^3+14x}{\sqrt{2x^4+7x^2}}dx$

d. $\displaystyle\int_{12}^{\frac{1}{4}} \frac{e^{\frac{1}{x}}}{x^2}dx$

Aera didn't nail it the first time around; it wasn't until she made the third clone, that she had made one that resembled her so much, that she could insert it instead of herself, even though it stuttered a bit.. She named the clone Anti Aera. The first two clones were made because she had used the expressions

$$\int \frac{\cos(\ln(x)+\pi)}{\pi x}dx \quad \text{and} \quad \int \cos(x)\cdot e^{\sin(x)+42}dx$$

but they were too smart, so no one would have believed it was Aera. She sent them to orbit around her planet, but launched them with a little too much speed, so they changed heading to a small unknown planet. They would wander around there for a bit, spreading some good mathematics, until she would need them back one day. They would gain the peoples trust. They would save this planet this.. "Earth".. They would be known as StandardJoe and HyperMick.

1.3.12 Eleventh (and twelfth) lesson

The eleventh lesson was used to summarize what they had done in integral calculus; to this end, (a blank version of) Worksheet 05.24 was prepared and handed out to them. The students then spent around 40 minutes to fill out the worksheet, to the best of their abilities, in small groups (of order < 5). Afterwards the groups read aloud, to the rest of the class, what they had on their papers, ending with the creation of the following filled out worksheet. The twelfth lesson was used for any questions they might have and to say a prober farewell.

1.3.12.1 Worksheet 05.24

$f(x)$	$F(x)$
c	cx
kx	$\frac{1}{2}kx^2$
x^2	$\frac{1}{3}x^3$
x^n	$\frac{1}{n+1}x^{n+1}$
$-\frac{1}{x^2} = -x^{-2}$	$\frac{1}{x} = x^{-1}$
$\frac{1}{2\sqrt{x}}$	$\sqrt{x}$
$\sqrt{x} = x^{1/2}$	$\frac{2}{3}x^{\frac{3}{2}}$
e^x	e^x

Rules for calculating integrals

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du = f\big(g(b)\big) - f\big(g(a)\big)$$

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du = F\big(g(b)\big) - F\big(g(a)\big)$$

2 Analysis tables

This part of the appendix contains the analysis tables for the hyperreal numbers, differential calculus and integral calculus respectively.

2.1 Hyperreal analysis table

The constants a, b and k are real, the variables x and y are real, dx and dy are infinitesimals, and α and β are hyperreal numbers.

Lesson #	Type of problem	Mathematical techniques	Technological theoretical elements	Didactic moment(s)	Elements of the didactical techniques
1 03.30 (14:45-15:50)	$\boldsymbol{HP_1}$: **Specify into which sets of numbers infinitesimals and infinite numbers belong.** $Hp_{1,1}$: Give a definition of an infinitely small quantity. $Hp_{1,2}$: Give a definition of an infinite quantity.	τ_1: Process of exclusion from the known set of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.	ϑ_1: (Informal) Definition of an infinitesimal. ϑ_2: (Informal) Definition of an infinite quantity.	First encounter with HP_1 Institutionalization with the former introduced sets of numbers, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.	Intuition is used to identify an infinitely small quantity as something that is less than any writeable positive number and an infinite quantity as something that is greater than any writeable number.
2 04.05 (08:10-09:15)	$Hp_{1,3}$: Draw a number line with infinitesimals.		ϑ_3: The ordering of the hyperreal numbers	Exploratory (and constitution) for HP_1.	Using two mathematical objects, firstly the definition of infinitesimals and secondly the known real number line, to produce another set of numbers. Using a previously established praxeology on how to operate with real numbers induces how to operate with hyperreal numbers. Identify the finite hyperreal number as a sum of a real number and an infinitesimal. Recognize that an infinite number cannot be written as the sum of a real number and an infinitesimal. Using two mathematical objects, namely the coordinate system and the implicit use of Pythagorean theorem yields the difference in function-
	$Hp_{1,4}$: What is $2 \cdot dx$ and where is it on the number line? $Hp_{1,5}$: What is $\frac{dx}{2}$ and where is it on the number line? $Hp_{1,6}$: What is $dx + dy$ and where is it on the number line?	$\tau_{1,i}$: Using the same technique as when operating with real numbers. $i = \{4,5,6\}$	ϑ_4: How to add, multiply and divide with a mixture of infinitesimals and real numbers.	(partly) Constitution of the theory block for HP_1 and institutionalization by comparing with the theory for real numbers.	
	$\boldsymbol{HP_3}$: **Find** $\boldsymbol{k \cdot st(a \cdot x + b \cdot dx + c \cdot dy)}$. $Hp_{3,1}$: Which real number is infinitely close to $7 + dx$?	$\tau_{3,1}$: Remove the infinitesimal(s); they are virtually 0	ϑ_5: (Informal) Definition of the standard part.	First encounter with HP_3.	
	$Hp_{1,7}$: What is $\frac{1}{dx}$ and where is it on the number line?	$\tau_{1,7}$: Use the rules for dividing by small numbers to establish that $\frac{1}{dx}$ is infinite, placing it beyond the real numbers.	Θ_1: An intuitive construction of the hyperreal numbers as the real numbers plus the infinitesimals and operations between these (ϑ_4)	Further constitution of the theory block for HP_1 and for the entire HMO.	
	$Hp_{4,2}$: Give an example of a hyperreal number with an undefined standard part, i.e. find a such that $st(a)$ is meaningless.	$\tau_{4,2}$: Find a hyperreal number that does not have a real number infinitely close to it			

	HP_2: Determine if a given function, $f(x)$, is continuous. $Hp_{2,1}$: Give examples of functions? $Hp_{2,2}$: What is the distance between 2 consecutive points on a graph of a function which can be drawn without lifting the drawing device? And what is the difference in the function value with an infinitesimal difference in the variable value? $Hp_{2,3}$: Does $f(x+dx)$ make sense when f is a real function? **HP_3: Find $k \cdot st(a \cdot x + b \cdot dx + c \cdot dy)$.** $Hp_{3,2}$: Find $st(x+dx)+st(y+dy)$ $Hp_{3,3}$: Find $5 \cdot st(x+dx)$ $Hp_{3,4}$: Find $st(5x+5dx)$ $Hp_{3,5}$: Find $st(0+dx)$ $Hp_{3,6}$: Find $st(dx+dy)$	$\tau_{2,1}$: Remember what a function is. $\tau_{2,2}$: Deduce that an infinitesimal distance between 2 points makes the difference in both coordinates an infinitesimal. $\tau_{2,3}$: Realize that the value $x+dx$ is not part of the domain. $\tau_{3,i}$: $st(k\alpha+\beta) = k \cdot st(\alpha) + st(\beta)$ When α and β are both finite. $i \in \{2,3,4,5,6\}$	ϑ_6: NSA definition of continuity in a point. ϑ_7: (Informal) Definition of the star operation as the extension of the domain of a function.	First encounter with HP_2. Exploratory (and technical work) on HP_3.	and variable values are infinitesimal. Using previously established praxeology on functions and their domain to determine that $x+dx$ is not part of the domain of a real function.
3 04.06 (10:40-11:45)	$Hp_{3,7}$: Find $st(5dx)$ $Hp_{3,8}$: Find $st\left(\frac{1}{dx}\right)$ $Hp_{3,9}$: Why is $st\left(5dx \cdot \frac{1}{dx}\right) = 5$ $Hp_{3,10}$: Give an example of an infinitesimal which is also a real number $Hp_{3,11}$: Find $st\big((2+dx)(3+dy)\big)$ $Hp_{3,12}$: Find $st(2+dx)st(3+dy)$ $Hp_{3,13}$: What can be concluded from the two previous exercises? $Hp_{3,14}$: Find $st(dx)st\left(\frac{1}{dx}\right)$ $Hp_{3,15}$: Amend the conclusion from $Hp_{3,13}$ $Hp_{3,16}$: Find $st((4+dx)^2)$	$\tau_{3,i}$: $st(\alpha \cdot \beta) = st(\alpha) \cdot st(\beta)$ When α and β are both finite. $i = \{7,8,\dots,16\}$		Exploratory and technical work on HP_3.	Identify the finite hyperreal number as a sum of a real number and an infinitesimal. Recognize that an infinite number cannot be written as the sum of a real number and an infinitesimal. How to operate with hyperreal numbers.
4 04.08 (08:10-09:15)	**HP_4: Find the hyperreal function value $^*f(a+kdx)$ for a given real function** $HP_{4,1}$: For $f(x) = 3x+1$, find the hyperreal function value	$\tau_{4,i}$: Evaluate the hyperreal function as if it were a real function and a real variable. $i = \{1,2,3\}$		First encounter with HP_4.	Using previously established praxeology on functions and their domain and the recently established star- and standard part operation.

Lesson #	Type of problem	Mathematical techniques	Technological theoretical elements	Didactic moment(s)	Elements of the didactical techniques
	${}^*f(1+dx)$ and ${}^*f(dx-2)$. $HP_{4,2}$: For $g(x)=-x$, find the hyperreal function value $x=3-dx$ and $x=0$. $HP_{4,3}$: For $h(x)=2x^3$, find the hyperreal function value ${}^*h(1+dx)$ and ${}^*h(4+dx)$. **HP_5: What is $st({}^*f(x+dx))$ for a real function?** $Hp_{5,1}$: What is $st({}^*f(x))$ for a real function? $Hp_{5,2}$: Assume now that f is a continuous real function what is $st({}^*f(x+dx))$?	$\tau_{5,1}$: Standard part counters the star operation. $\tau_{5,2}$: Standard part counters the star operation and if the function is continuous then it also counters the infinitesimal part of the variable.			

2.2 Differential calculus analysis table (Jonas Kyhnæb)

Lesson #	Type of problem	Mathematical techniques	Technological theoretical elements	Didactic moment(s)	Elements of the didactical techniques
1 04.08 (09:25 - 10:30)	$\boldsymbol{DP_1}$: **Find the slope of a given graph in a point by drawing a tangent** $Dp_{1,1}$: Find an interval where the graph is the steepest $Dp_{1,1,1}$:What is the best interval, to find where the graph is the steepest? $Dp_{1,1,2}$:How to find the exact slope where the graph is the steepest $Dp_{1,2}$: Where on the graph can the slope be found? $Dp_{1,3}$: Find the slope in an arbitrary point on the graph $Dp_{1,4}$: Find the slope for $f(x) = -\frac{1}{4}x^2 + 4x - 5$ in the points $(4,7)$, $(8,11)$ and $(12,7)$	$\tau_{1,1}$: Use the 2-point formula on any two points in the interval $\tau_{1,i}$: Draw a tangent and use the 2-point formula on the tangent $i = \{3; 4\}$	ϑ_1: Informal definition of the tangent as a linear graph which is parallel with the graph in the point	First encounter, exploration with $\boldsymbol{DP_1}$	Using previously established praxeology on how to find the slope of a straight line given 2 points Graph
2+3 04.11 (08:10 - 10:30)	$\boldsymbol{DP_2}$: **Find an algebraic expression for finding the slope for a function in a point** $Dp_{2,1}$: Find the slope of a function, $f(x)$, in a point, without drawing anything $Dp_{2,1,1}$:Find the general slope of a function, $f(x)$, in the interval $[a; b]$ $Dp_{2,1,2}$:What can be said about the slope to a function, $f(x)$, and the slope of the secant (the straight line) through the points $(a, f(a))$ and $(b, f(b))$ $Dp_{2,2}$: Ensure that the slope in the point is a real number	$\tau_{2,1}$: Use the 2-point formula on an infinitesimal interval $\tau_{2,2}$: Standard part is the technique, to ensure that it is a real number	ϑ_2: Definition of the differential quotient as $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$	First encounter, exploration, constitution with $\boldsymbol{DP_2}$. Institutionalization of the theory by using HMO	Using the slope of a linear function and the 2-point formula, together with the notion of the mathematical object: a function, and the hyperreal numbers to construct the new knowledge

4 04.13 (12:15 - 13:20) **5** 04.19 (08:10 - 09:15)	$\boldsymbol{DP_4}$: **Find** $\frac{d}{dx}\boldsymbol{ax^n}$ $\boldsymbol{DP_{4,1}}$:**Find** $\frac{d}{dx}(\boldsymbol{ax+b})$ $Dp_{4,1,1}$:Given $g(x)=1$, find the function value and the differential quotient for $x=\{-3,-1,0,1,4\}$ $Dp_{4,1,2}$:Given $h(x)=-4x$, find the function value and the differential quotient for $x=\{-3,-1,0,1,4\}$ $Dp_{4,2,1}$:Given $k(x)=2x^2$, find the function value and the differential quotient for $x=\{-3,-1,0,1,4\}$ $Dp_{4,2,1}$:Given $l(x)=2x^2-4x+1$, find the function value and the differential quotient for $x=\{-3,-1,0,1,4\}$ $\boldsymbol{DP_6}$ **Show the rules for calculating with differentials** $\boldsymbol{Dp_{6,2}}$: **Find** $\frac{d}{dx}(\boldsymbol{f(x)+g(x)})$	$\tau_{4,1}$: Recognize the differential quotient for a straight line as the slope of the line τ_4: $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$ $\tau_{4,2,1}$:Add the differential quotients from Dp_i, $i=\{3{,}1;3{,}2;4{,}1\}$ for each x separately $\tau_{6,2}$: $st(a+b)=st(a)+st(b)$	ϑ_3: The definition of a derived function	Further constitution of the theory block for DMO. Institutionalization of the theory by using HMO. First encounter, exploration with $\boldsymbol{DP_{6,2}}$	Using a theory block of a previously established praxeology to establish techniques in DMO. Technical work in the form of routinization
6 04.20 (10:40 - 11:45)	$\boldsymbol{DP_7}$: **What can be said about the original function,** $(\boldsymbol{f(x)})$**, given the derived function** $(\boldsymbol{f'(x)})$**?** $Dp_{7,1}$:What can we say about $f(x)$ when $f'(x)>0$? $Dp_{7,2}$:What can we say about $f(x)$ when $f'(x)<0$? $Dp_{7,3}$:What can we say about $f(x)$ when $f'(x)=0$?	τ_7: Find the x-value(s) for which $f'(x)=0$ and determine in which intervals f' is positive or negative	ϑ_4: Describing the relation between a function and its derivative	Technical work with the technique $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$ (τ_4)	Using a theory block of a previously established praxeology about functions and equation solving to establish techniques in DMO

7 04.21 (14:45 - 15:50)	$\boldsymbol{Dp_{4,1}}$: **Find** $\frac{d}{dx}(\boldsymbol{ax+b})$ $\boldsymbol{DP_4}$: **Find** $\frac{d}{dx}\boldsymbol{ax^n}$ $Dp_{4,3}$: Find $f'(x)$ when $f(x)=x$ $Dp_{4,4}$: Find $f'(x)$ when $f(x)=x^2$ $Dp_{4,5}$: Find $f'(x)$ when $f(x)=x^3$ $Dp_{4,6}$: Guess $f'(x)$ when $f(x)=x^4$ $Dp_{4,7}$: Give a justified guess to what $f'(x)$ is when $f(x)=x^n, n\in\mathbb{N}$ $Dp_{4,8}$: Find $f'(x)$ when $f(x)=ax^2, a\in\mathbb{R}$ $Dp_{4,9}$: Find $f'(x)$ when $f(x)=ax^3, a\in\mathbb{R}$ $Dp_{4,10}$:Guess $f'(x)$ when $f(x)=ax^4, a\in\mathbb{R}$ $Dp_{4,11}$:Give a justified guess to what $f'(x)$ is when $f(x)=ax^n, a\in\mathbb{R}$, $n\in\mathbb{N}$ $Dp_{6,1}$:Find $\frac{d}{dx}(a\cdot f(x))$ $Dp_{4,12}$:Given $h(t)=t+12$, $h(t)$ being number of eggs to the time t in hours, how fast does it lay eggs in the first two hours? $Dp_{4,12,1}$:Find $h'(t)$. $Dp_{4,12,2}$:Given $f(t)=-t^2+25t$ and $g(t)=\frac{1}{3}t^3-7t^2+50t$, how fast do these lay eggs at an optional time?	$\tau_{4,1}$: Use τ_4 $\tau_{4,j}$: $(ax^n)'=n\cdot ax^{n-1}$, $n\in\mathbb{N}, a\in\mathbb{R}$ $j=\{6;7;10;11;12{,}2\}$ $\tau_{6,1}$: $st\left(\frac{^*f(x+dx)-f(x)}{dx}\right)$, $st(k\cdot\alpha)=k\cdot st(\alpha)$ $\tau_{4,12}$:2-point formula $\tau_{4,12,1}$:Use τ_4		First encounter, exploration with $\boldsymbol{DP_4}$ and constitution for $\tau_{4,j}$	Recognizing a system for how to differentiate a polynomial function through an iterative process
8 04.25 (08:10 - 09:15) **9** 04.25 (09:25 - 10:30)	$\boldsymbol{DP_7}$: **What can be said about the original function, $(\boldsymbol{f(x)})$, given the derived function $(\boldsymbol{f'(x)})$?** $Dp_{7,4}$:Specify the extremes of the function $f(x)=x^2-2x+4$			Technical work with the technique $st\left(\frac{^*f(x+dx)-f(x)}{dx}\right)$ (τ_4)	

10 04.26 (10:40 - 11:45) **11** 04.27 (14:45 - 15:50) **12** 05.03 (08:10 - 09:15)	$\boldsymbol{DP_4}$: **Find** $\frac{d}{dx}\boldsymbol{ax^n}$ $Dp_{4,14}$: Find the derivative of $f(x) = \frac{1}{x}$ $Dp_{4,15}$: Find the derivative of $f(x) = \sqrt{x}$ $\boldsymbol{DP_6}$: **Show the rules for calculating with differentials** $\boldsymbol{Dp_{6,3}}$: **Let** $\boldsymbol{h(x) = f(x) \cdot g(x)}$**. Find** $\boldsymbol{h'(x)}$ $\boldsymbol{Dp_{6,4}}$: **Let** $\boldsymbol{h(x) = f(g(x))}$**. Find** $\boldsymbol{h'(x)}$ $\boldsymbol{Dp_3}$: **Find the equation of a tangent** $\boldsymbol{Dp_5}$: **Find** $\frac{d}{dx}\boldsymbol{e^x}$	$\tau_{4,i}$: $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$ and do algebraic calculations until the denominator isn't infinitesimal and use $st\left(\frac{a}{b}\right) = \frac{st(a)}{st(b)}$ $i = \{14; 15\}$ $\tau_{6,3}$: $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$, $st(a+b) = st(a) + st(b)$ and use the definition of continuity $\tau_{6,4}$: $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$, $st(a \cdot b) = st(a) \cdot st(b)$ τ_3: Use the differential quotient as the slope in the equation for a straight line τ_5: $st\left(\frac{{}^*f(x+dx)-f(x)}{dx}\right)$ and the definition of e	ϑ_5: A straight line is determined by the slope and a point	Technical work with $\tau_{4,i}$ First encounter, exploration and constitution of the theory for $\boldsymbol{Dp_{6,3}}$ and $\boldsymbol{Dp_{6,4}}$ First encounter and exploration of $\boldsymbol{Dp_3}$ First encounter with $\boldsymbol{Dp_5}$	
13 05.03 (09:25 - 10:30)	$\boldsymbol{DP_8}$: **What have you learned about differential calculus?**				
Lesson #	Type of problem	Mathematical techniques	Technological theoretical elements	Didactic moment(s)	Elements of the didactical techniques

2.3 Integral calculus analysis table (Mikkel Mathias Lindahl)

Lesson #	Type of problem	Mathematical techniques	Technological theoretical elements	Didactic moment(s)	Elements of the didactical techniques
1+2 05.09 (8:10-10:30)	$\boldsymbol{IP_1}$: **Find the area between the graph of a function and the first axis over a given interval.** $Ip_{1,1}$: Find the area between the function $f(x) = -x^2 + 4x$ and the first axis over the interval $[0,4]$. $Ip_{1,2}$: Should the small areas only be rectangles and why? $Ip_{1,3}$: What is the area of a rectangle with infinitesimal width dx and $f(x)$ as its height? $Ip_{1,4}$: How does one partition an interval into infinitesimal intervals? $Ip_{1,5}$: How should an infinite sum be understood?	$\tau_{1,1}$: Partition the interval $[0,4]$ into infinitely many infinitesimal intervals; consider the function as being constant over the infinitesimal interval and recognize the area as the infinite sum of the rectangles area. $\tau_{1,3}$: The area is found in the similar way as a normal rectangle, i.e. the area is $f(x) \cdot dx$.	ϑ_1: Partition of an interval. ϑ_2: Description of a sum of n terms and a sum of (hyper)finite terms. ϑ_3: (Informal) first description of the definite integral.	First encounter, exploration (and constitution of the theory) for IP_1.	Putting together a known technique (of finding the area of a rectangle) and using two new mathematical objects: infinite partition and infinite sums. Both of the newly introduced objects are based on mathematical objects they may or may not have encountered before.
3 05.10 (10:40-11:45)	$Ip_{1,6}$: In what way should one write the area between a function and the first axis over an interval? $Ip_{1,7}$: Ensure that the infinite sum describing the area between a function and the first axis is a real number. $Ip_{1,8}$: Describe the graphical meaning of $\int_1^4 k \cdot x\,dx$.	$\tau_{1,6}$: $\int_a^b f(x)\,dx$. $\tau_{1,7}$: Standard part is the technique, to ensure that it is a real number.	ϑ_4: Definition of the definite integral as the standard part of an infinite sum of areas of rectangles.	Further constitution of the theory block for IMO.	
4 05.11 (14:45-15:50)	$\boldsymbol{IP_2}$: **Find** $\boldsymbol{\int_a^b f(x)\,dx}$ $\boldsymbol{Ip_{2,1}}$: **Find** $\boldsymbol{\int_a^b c\,dx}$ $Ip_{2,1,1}$: Find $\int_0^4 1\,dx$ $Ip_{2,1,2}$: Find $\int_0^4 2\,dx$ $Ip_{2,1,3}$: Find $\int_0^4 2\,dx$ $Ip_{2,1,4}$: Find $\int_2^5 2\,dx$ $Ip_{2,1,5}$: Find $\int_3^7 2\,dx$ $Ip_{2,1,6}$: Find $\int_5^7 3\,dx$ $Ip_{2,1,7}$: Find $\int_0^6 4\,dx$ $Ip_{2,1,8}$: Find $\int_0^6 -4\,dx$ $Ip_{2,1,9}$: Find $\int_2^6 -3\,dx$ $Ip_{2,1}$: Find $\int_a^b c\,dx$ $\boldsymbol{Ip_{2,2}}$: **Find** $\boldsymbol{\int_a^b kx\,dx}$ $Ip_{2,2,1}$: Find $\int_0^4 x\,dx$ $Ip_{2,2,2}$: Find $\int_0^4 2x\,dx$ $Ip_{2,2,3}$: Find $\int_0^4 3x\,dx$ $Ip_{2,2,4}$: Find $\int_2^5 2x\,dx$ $Ip_{2,2,5}$: Find $\int_3^7 2x\,dx$ $Ip_{2,2,6}$: Find $\int_5^7 3x\,dx$	$\tau_{2,1}$: Recognize the rectangle, $c(b-a)$. $\tau_{2,2}$: Recognize the triangle, $\frac{1}{2}k(b^2-a^2)$ $\tau_{3,1}$: Recognize the rectangle(s) and triangle, $\frac{1}{2}kx^2 + cx$		First encounter and exploration with $IP_{2,1}$, $IP_{2,2}$ and $IP_{3,1}$ (Technical work on $\tau_{2,1}$, $\tau_{2,2}$, $\tau_{3,1}$ using a theory block from a previously established praxeology of geometry)	Using a theory block of a previously established praxeology to establish techniques in IMO.

	$Ip_{2,2,7}$: Find $\int_0^6 4x\,dx$ $Ip_{2,2,8}$: Find $\int_0^6 -4x\,dx$ $Ip_{2,2,9}$: Find $\int_2^6 -3x\,dx$ $Ip_{2,2}$: Find $\int_a^b kx\,dx$ **IP_3: Find an antiderivative of $f(x)$** **$Ip_{3,1}$: Find $\int_0^x kx + c\,dx$** $Ip_{3,1,1}$: Find $\int_0^x 4\,dx$ $Ip_{3,1,2}$: Find $\int_0^x c\,dx$ $Ip_{3,1,3}$: Find $\int_0^x 2x\,dx$ $Ip_{3,1,4}$: Find $\int_0^x kx\,dx$ $Ip_{3,1,5}$: Find $\int_0^x \frac{1}{2}x + 4\,dx$ $Ip_{3,1}$: Find $\int_0^x kx + c\,dx$				
5 05.13 (14:45-15:50)	$Ip_{2,1,10}$:Find $\int_0^1 2\,dx$ $Ip_{2,2,10}$:Find $\int_1^3 2x\,dx$ **$Ip_{2,3}$: Find $\int_a^b f(x)\,dx$ when $f(x)$ is an elementary function.** $Ip_{2,3,1}$: Find $\int_2^5 2x + 1\,dx$ $Ip_{2,3,2}$: Find $\int_0^4 -x + 4\,dx$ $Ip_{2,1,11}$:Find $\int_0^4 1\,dx + \int_5^6 1\,dx$ $Ip_{2,1,12}$:Find $\int_7^{11} 3\,dx$ $Ip_{2,3,3}$: Find $\int_2^7 2x - 4\,dx$ $Ip_{2,1}$: Find $\int_a^b c\,dx$ $Ip_{2,2}$: Find $\int_a^b kx\,dx$ $Ip_{2,3,4}$: Find $\int_a^b kx + c\,dx$				
6 05.18 (10:40-11:45)	**IP_4: Show the rules for calculating with integrals.** $IP_{4,1}$: Show that $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ $IP_{4,2}$: show that $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$ $IP_{4,3}$: Show that $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ $IP_{4,4}$: Show that $\int_a^a f(x)\,dx = 0$ $IP_{4,5}$: Show that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	τ_4: Using the definition of the integral, the techniques from HMO, and the use of the rules for (hyper)finite sums (which are never explained), to assert the results. [This is a technique when considering the integral as a technique]	ϑ_5: Using the definition of the integral, the techniques from HMO, and the use of the rules for (hyper)finite sums (which are never explained). [is part of the theoretical block when considering the integral as part of the technology]	Technical work on $\tau_{2,1}$, $\tau_{2,2}$, $\tau_{1,6}$	(Trying to) Bridge the connection between the intuitive understanding of the integral, as an area, and the definition with what rules this infers on the mathematical object the integral is.

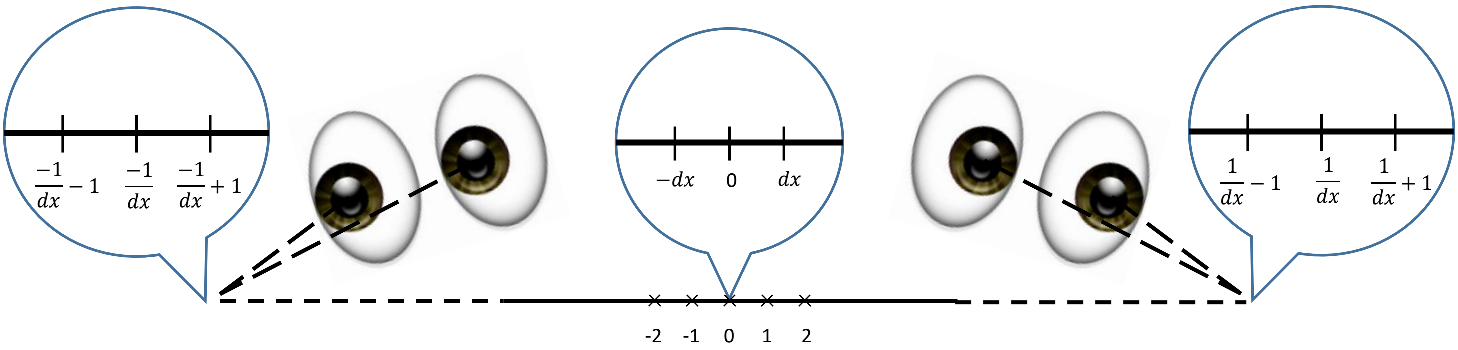
7 05.20 (08:10-09:15)	$Ip_{3,1,2}$: Find $\int_0^x c\,dx$ $Ip_{3,1,4}$: Find $\int_0^x kx\,dx$ $\boldsymbol{IP_5}$: **Find** $\boldsymbol{\frac{d}{dx}f(x)}$ $\boldsymbol{Ip_{5,1}}$: **Find** $\boldsymbol{\frac{d}{dx}\int_a^x f(x)\ dx}$ **for an elementary function** $\boldsymbol{f(x)}$**.** $Ip_{5,1,1}$: Find $\frac{d}{dx}\int_0^x c\,dx$ $Ip_{5,1,2}$: Find $\frac{d}{dx}\int_0^x kx\,dx$ $Ip_{5,1,3}$: Find $\frac{d}{dx}\int_0^x x^2\,dx$ $\boldsymbol{IP_{3,2}}$: **Find** $\boldsymbol{\int_a^x f(x)\,dx}$ $Ip_{3,2,1}$: Find $\int_0^x x^2\,dx$	$\tau_{5,1,i}$: Use the techniques obtained in DMO to establish the results $i = \{1,2\}$ $\tau_{5,1}$: Recognize the integrand as the result, i.e. guess the technique from $Ip_{5,1,1}$ and $Ip_{5,1,2}$. $\tau_{3,2}$: Guess a function for which the derivative is the integrand.	ϑ_6: The fundamental theorem of calculus. $\frac{d}{dx}\int_0^x f(x)\,dx = f(x)$ and $\int_0^x f'(x)\,dx = f(x)$	Further Constitution of the theory for IMO. Technical work on $\tau_{3,2}$. Institutionalization between DMO and IMO.	The statement of the fundamental theorem and the definition of the antiderivative are presented after one another. This is followed by a new definition, ϑ_8, and 3 new types of problems $IP_{3,3}$, $IP_{3,4}$ and $\boldsymbol{IP_6}$.
	$\boldsymbol{Ip_{3,3}}$: **Find an antiderivative of an elementary function.** $Ip_{3,3,1}$: Find an antiderivative of $f(x) = x$. $\boldsymbol{IP_{3,4}}$: **Find the antiderivative going through a point** $\boldsymbol{(x_0, y_0)}$**.** $Ip_{3,4,1}$: Find the antiderivative through the point (0,2)?	$\tau_{3,3}$: Use $\tau_{3,2}$, i.e. check if $F'(x) = f(x)$. $\tau_{3,3,1}$: Use that the area function is an antiderivative $\tau_{3,4,1}$: Use that the point is on the graph, making the correct antiderivative, $F_c(x)$, defined as $F_c(x) = F(x) + c$, such that $c = 2 - F(0)$.	ϑ_7: Definition of the antiderivative, of a function $f(x)$, as a function $F(x)$, such that $F'(x) = f(x)$.	First encounter, (exploration) with $IP_{3,3}$. Constitution of the theory for $IP_{3,3}$. First encounter, (exploration) with $IP_{3,4}$.	
	$Ip_{3,5}$: How many antiderivatives does a function have? $Ip_{3,3,2}$: Find an antiderivative of $f(x) = k$ $\boldsymbol{IP_6}$: **Find** $\boldsymbol{\int f(x)\,dx}$**?** $Ip_{6,1}$: Find $\int k\,dx$? $Ip_{3,3,3}$: Find an antiderivative of $f(x) = kx$ $Ip_{6,2}$: Find $\int kx\,dx$? $Ip_{3,3,4}$: Find an antiderivative of $f(x) = x^2$ $Ip_{6,3}$: Find $\int x^2\,dx$? $Ip_{3,3,5}$: Find an antiderivative of $f(x) = e^x$ $Ip_{6,4}$: Find $\int e^x\,dx$? $Ip_{3,3,6}$: Find an antiderivative of $f(x) = x^n$ $Ip_{6,5}$: Find $\int x^n\,dx$?	$\tau_{3,3,i}$: Use that the area function is an antiderivative $i = \{2,3\}$ $\tau_{3,3,6}$: An antiderivative of x^n is $\frac{1}{n+1}x^{n+1}$. τ_6: Find the antiderivative and add a constant, c, to be fixed later.	ϑ_8: Definition of the indefinite integral, as $\int f(x)\,dx = F(x) + c$	First encounter, (exploration) with $\boldsymbol{IP_6}$**.** Technical work on $\tau_{3,3}$ using the fundamental theorem and the rules for differentiation.	

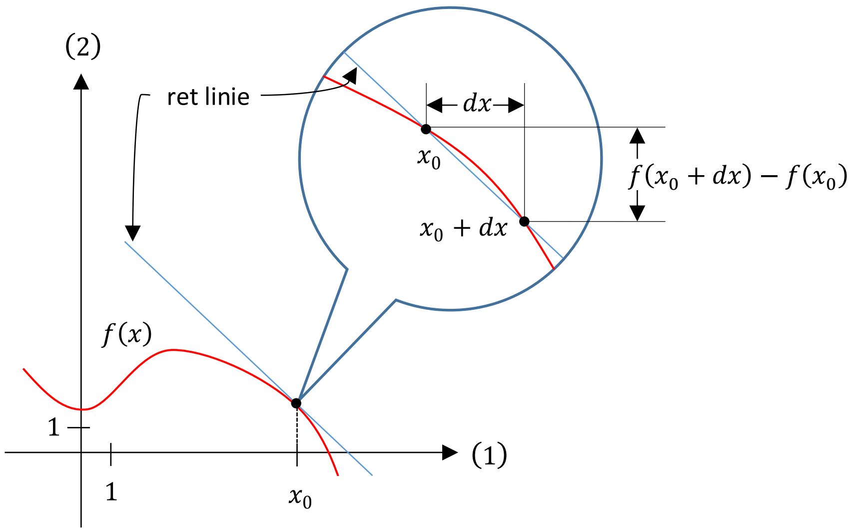
8 05.20 (09:25-10:30)	$Ip_{6,6}$: Find $\int 2\,dx$ $Ip_{6,7}$: Find $\int \left(\frac{1}{4}x+2\right)dx$ $Ip_{6,8}$: Find $\int 3\,dx$ $Ip_{6,9}$: Find $\int \left(-\frac{1}{4}x+3\right)dx$? $Ip_{6,10}$: Find $\int 2x\,dx$? $Ip_{6,11}$: Find $\int (x^2-4x+8)\,dx$? $Ip_{6,12}$:Find $\int (-x^3+8)\,dx$? $Ip_{6,13}$:Find $\int (-x^2+4x)\,dx$? $Ip_{6,14}$:Find $\int (4x^3-3x^2+2x)\,dx$? $Ip_{6,15}$:Find $\int -\frac{1}{x^2}dx$? $Ip_{6,16}$: Find $\int \frac{8}{x^2}dx$? $Ip_{6,17}$: Find $\int (-4x^3+x^{-4})\,dx$? $Ip_{6,18}$: Find $\int e^x\,dx$? $Ip_{6,19}$: Find $\int (2x+8)(2x)+2(x^2)\,dx$? $Ip_{6,20}$: Find $\int 6x^2\cdot e^{2x^3}\,dx$? $Ip_{6,21}$: Find $\int \frac{1}{e^x}dx$? $Ip_{3,4,2}$: For each of the problems $Ip_{6,i}$ where $i=\{4,5,\ldots,21\}$ find a specific antiderivative by letting it go through a specific point (x_0,y_0).	$\tau_{3,4,2}$:Use that the point is on the graph, making the correct antiderivative, $F_c(x)$, defined as $F_c(x)=F(x)+c$, such that $c=y_0-F(x_0)$.		Technical work on $\tau_{3,4}$ in the form of routinization. Technical work on τ_6 in the form of routinization and in specific cases by recognizing the rules for differentiation backwards.	
	$Ip_{2,2,10}$: Find $\int_0^1 2x\,dx$ $Ip_{2,3,5}$: Find $\int_1^2 \frac{1}{8}x^2+2x+2\,dx$ $Ip_{2,3,6}$: Find $\int_2^3 3x-2\,dx$ $Ip_{2,3,7}$: Find $\int_0^4 -\frac{1}{8}x^2+3x-2\,dx$	$\tau_{2,3}$: Use the fundamental theorem of calculus and the technique from $Ip_{4,3}$ to obtain $\int_a^b f(x)\,dx=F(b)-F(a)$.		First encounter and exploration with $IP_{2,3}$.	The problems in $IP_{2,3}$ has to be solved using a technique based on the newly stated fundamental theorem, the definition of an antiderivative and the rule for integration that has been proved in $IP_{5,3}$ but not since used.
9 05.23 (08:10-09:15)	**$Ip_{5,3}$: What is df?** $Ip_{5,3,1}$: What other than dx has been used when writing an infinitesimal? $Ip_{5,2}$: Find $\frac{d}{dx}f(x)$ for an elementary function. $Ip_{5,2,1}$: What is $\frac{d}{dx}\left(e^{3x^2}\right)$? $Ip_{6,22}$:What is $\int 6x\cdot e^{3x^2}dx$?	$\tau_{5,2}$: using $\frac{df}{dx}=f'(x)$ and algebra. $\tau_{6,i}$: Using the fundamental theorem and recognizing the integrand as the previously differentiated expression. $i=\{22,23\}$	ϑ_9: Definition of df as $df=f'(x)\cdot dx$	First encounter, (exploration) with $Ip_{5,2}$. technical work with, $\tau_{6,i}$ establishing τ_7. $i=\{22,23\}$	Combining 3 “new” mathematical objects and one technique, τ_{12}, to create a new technique, τ_{14}. 1: Making use of a function as a variable. 2: Antiderivative. 3: Backwards chain rule for differentiation.

	$Ip_{5,2,2}$: What is $\frac{d}{dx}\left(F\left(g(x)\right)\right)$, when $F(x)$ is an antiderivative of $f(x)$? $Ip_{6,23}$: Find $\int f\left(g(x)\right) \cdot g'(x)dx$? **$IP_7$: Find $\int_a^b f\left(g(x)\right) \cdot g'(x)\, dx$** $Ip_{7,1}$: What is the inner function in the expression $\int f\left(g(x)\right) \cdot g'(x)dx$? $Ip_{5,2,2}$: What is du if $u = g(x)$? $Ip_{7,2}$: Rewrite $\int f\left(g(x)\right) \cdot g'(x)dx$, with $u = g(x)$ and du. $Ip_{7,3}$: Rewrite $\int_a^b f\left(g(x)\right) \cdot g'(x)\, dx$ with $u = g(x)$ and du. IP_7: Find $\int_a^b f\left(g(x)\right) \cdot g'(x)\, dx$ $Ip_{7,4}$: Find $\int_1^4 \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} dx$?	τ_7: Using $\int_a^b f\left(g(x)\right)g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$ where $f(x)$ is a function for which an antiderivative can be found.		First encounter and exploration with **IP_7.**	
10 05.23 (09:25-10:30)	$Ip_{7,5}$: Find $\int_1^{\sqrt{5}} -\frac{2x}{(x^2+3)^2} dx$ $Ip_{7,6}$: Find $\int_0^{\frac{1}{2}} (4x^3 + 2x^2)^{10} \cdot (12x^2 + 4x)\, dx$ $Ip_{7,7}$: Find $\int_0^1 \frac{8x^3+14x}{\sqrt{2x^4+7x^2}} dx$ $Ip_{7,8}$: Find $\int_{12}^{\frac{1}{4}} \frac{e^{\frac{1}{x}}}{x^2} dx$				
11 05.24 (13:30-14:35)	**IP_8: What have you learned about integral calculus?**			Institutionalisation of the punctual MO's in IMO	
Lesson #	Type of problem	Mathematical techniques	Technological theoretical elements	Didactic moment(s)	Elements of the didactical techniques

3 Compendium(s)

INFINITESIMALREGNING





Mikkel Mathias Lindahl Jonas Kyhnæb

KyhDahls Gymnasiematematik

Differentialregning og Integralregning med Ikke-standard Analyse

KyhDahls gymnasiematematik

2. udgave, 1. oplag 2016

Forlagsredaktion: Mikkel M. Lindahl og Jonas Kyhnæb

Grafisk tilrettelæggelse og omslag: KyhDahl A/S, København

Tegninger: Mikkel M. Lindahl og Jonas Kyhnæb

Historie: Mikkel M. Lindahl og Jonas Kyhnæb

Tryk: KyhDahl Press, Roskilde

ISBN N/A

Illustrationsliste:

Forside: Jonas Kyhnæb

Jonas Kyhnæb, s. 1, s. 2, s. 14, s. 15, s. 17, s. 33, s. 61

Mikkel M. Lindahl: s. 26, s. 35, s. 40, s. 41, s. 42, s. 43, s. 44, s. 46, s. 47

Anders: s. 21

Resten er lavet som samarbejde mellem Mikkel M. Lindahl og Jonas Kyhnæb

Trademark(s): Jonas Kyhnæb

www.KyhDahl.dk/uddannelse/matematik/A-niveau/whatever

Indhold

Forord

Denne lærebog dækker de krav til kernestof og supplerende stof, som lærerplanerne stiller til differentialregning og integralregning på A-niveau (stx), bortset fra omdrejningslegemer. Bogen fungerer både som grundbog og arbejdsbog.

Kapitel 1 indeholder supplerende stof inden for infinitesimalregning, i form af ikke-standard analyse. Vi anbefaler at anvende denne tilgang til emnet differentialregning, da kapitlerne 2-7 bygger på netop ikke-standard analyse. Kapitlerne 8-11 om integralregning bygger ligeså på ikke-standard analyse og kapitlerne om differentialregning. Det er således tanken, at bogen skal læses fra ende til anden.

I slutningen af bogen findes øvelser og opgaver med tilhørende facitliste. Der findes ydermere en opsummering af alle kapitler i bogen, efter facitlisten. Opgaverne har varierende kompleksitet.

Der bliver ikke beskrevet nogen it-forløb, hvorfor det er op til læseren/underviseren om et it-værktøj skal tages i brug som et alternativ til en algebraisk eller analytisk fremgangsmåde.

Kapitlet om monotoniforhold er inspireret af WebMatematiks meget fine gennemgang af emnet.

København, juni 2016

Mikkel M. Lindahl *Jonas Kyhnæb*

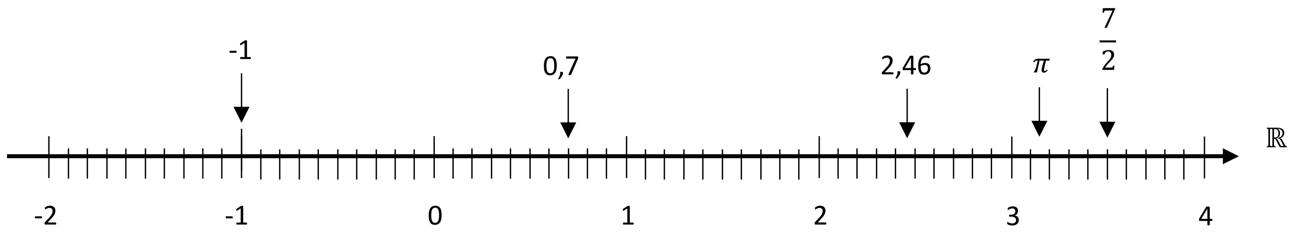
1 Talmængder

I matematik bruger man ofte begreberne ”naturlige tal” (dvs. positive hele tal), ”hele tal”, ”rationale tal” og ”reelle tal”. Derfor har de fået en noget kortere betegnelse.

$\mathbb{N}$ som er mængden af naturlige tal $\{1,2,3, \dots\}$
$\mathbb{Z}$ som er mængden af alle hele tal, dvs. positive, negative og 0 $\{\dots, -3, -2, -1,0,1,2,3, \dots\}$
$\mathbb{Q}$ som er mængden af alle rationale tal, dvs. brøker af heltal og decimaltal med system
$\mathbb{R}$ som er mængden af alle reelle tal, dvs. decimaltal af alle arter

En tallinie viser, hvordan tallene ligger i forhold til hinanden. På tallinien herunder er der tegnet tallene:

$$-1 \qquad 0{,}7 \qquad 2{,}46 \qquad \pi \qquad \text{og} \qquad \frac{7}{2}$$



Figur 1.1

Vi vil udvide de reelle tal til også at inkludere uendeligt små og store tal. Disse tal kalder viiiii..

1.1 Hyperreelle tal!

Et uendeligt lille tal kaldes *en infinitesimal*. Positive infinitesimaler er altså så små, at man ikke kan måle dem og dermed mindre end ethvert positivt reelt tal man kan komme på. Vi vil betegne infinitesimaler dx.

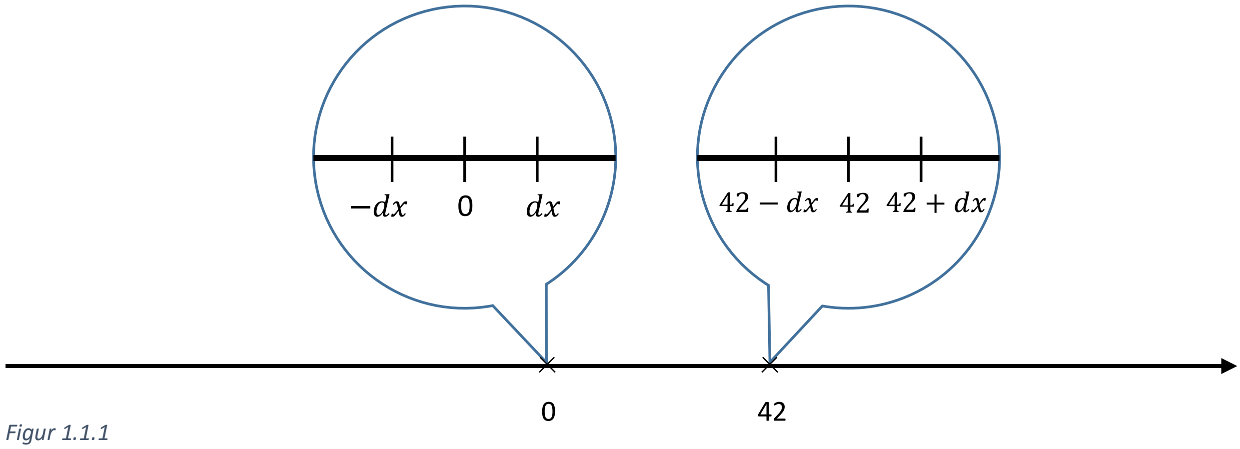
> *Hvorfor dx? Det er da lidt underligt? (siger den irriterende alien der besøgte klassen, netop den dag).*
>
> *Den kloge lærer svarer:*
>
> *Afstanden mellem jeres respektive hænder kan beskrives som afstanden mellem 2 punkter på en tallinie, $x_2 - x_1 = \Delta x$ for $x_2 > x_1$. Når der gives en ”high five” vil afstanden mellem punkterne blive nul, $x_2 - x_1 = \Delta x = 0$ for $x_2 = x_1$. Lige inden hænderne mødes kan man kalde afstanden uendelig lille, $x_2 - x_1 = dx$, hvor afstanden nu betegnes som dx, for ikke at blande dem sammen med de normale afstande Δx.*
> *OBS: Det fungerer også med en ”low five” eller andre berøringer.*

Hvis et uendeligt lille tal dx findes, så må $42 + dx$ også findes. Ligeledes findes $42 - dx$.

For at få en ide om hvad en infinitesimal er, kan følgende ulighed være en hjælp

$$1.000 > 1 > 0{,}0001 > 0{,}000000000001 > 0{,}00000000000000000001 > dx$$

Vi har på figuren nedenfor illustreret den reelle tallinie, med infinitesimaler:



Figur 1.1.1

Det eneste man ikke må dividere med er 0. Hvad sker der, hvis vi dividerer med et uendeligt lille tal? Hvad sker der, hvis vi dividerer med en infinitesimal, for eksempel $\frac{1}{dx}$? En måde at forstå, hvad dette tal egentlig er, kan ske ved at tænke på reglerne for division, altså

Et VILDT lille tal! →

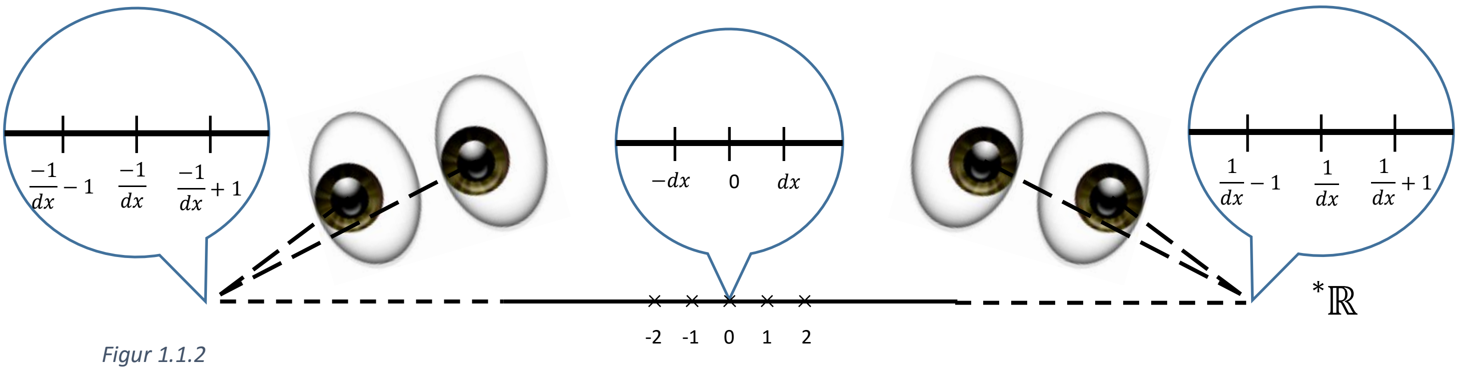
$$\frac{1}{1.000.000} < \frac{1}{1.000} < \frac{1}{10} < \frac{1}{0{,}01} < \frac{1}{0{,}000000001} < \frac{1}{0{,}000000000000000000000001} < \frac{1}{dx}$$

Hvilket er det samme som

$$0{,}000001 < 0{,}001 < 0{,}1 < 100 < 1.000.000.000 < 1.000.000.000.000.000.000.000.000 < \frac{1}{dx}$$

$\frac{1}{dx}$ er altså uendeligt stort! De uendeligt store tal $\frac{1}{dx}$, sammen med alle de reelle tal med de uendeligt små tal dx omkring, kaldes de *hyperreelle tal*. Denne nye talmængde kaldes $^*\mathbb{R}$.

Herunder er der illustreret, hvad der kaldes, den hyperreelle tallinie. Man skal have sine store øjne på, for at kunne se helt ud til de uendelige tal.



Figur 1.1.2

Eksempel 1.1.1

$$-1 < -1+dx < 0 < dx < 1+dx < \frac{1}{dx} < \frac{1}{dx}+1$$

På den hyperreelle tallinie findes derfor

Infinitesimaler:

- Positive tal der er mindre end ethvert forestilleligt positivt (reelt) tal forskelligt fra nul (positiv infinitesimal), altså $0 \leq dx < a$, hvor $a \in \mathbb{R}_+$, altså a er et positivt reelt tal, forskelligt fra 0
- Negative tal der er større end ethvert forestilleligt negativt (reelt) tal forskelligt fra nul (negativ infinitesimal), altså $a < -dx \leq 0$, hvor $a \in \mathbb{R}_-$, altså a er et negativt reelt tal, forskelligt fra 0

Uendelige tal:

- tal der er større end ethvert forestilleligt (reelt) tal (positivt uendeligt tal).
  $\frac{1}{dx} > a$, hvor $a \in \mathbb{R}$
- Tal der er mindre end ethvert forestilleligt (reelt) tal (negativt uendeligt tal).
  $-\frac{1}{dx} < a$, hvor $a \in \mathbb{R}$

Der gælder samme regneregler for $^*\mathbb{R}$, som for $\mathbb{R}$, man kan altså addere, trække fra, multiplicere og dividere. Der findes visse regler, som måske skal nævnes:

1.1.1 Regneregler:

- Lad dx være en infinitesimal, så gælder $dx - dx = 0$, ligesom for alle andre tal
- Lad dx og dy være infinitesimale, så vil $dx + dy$ også være en infinitesimal
- Lad $n \in \mathbb{N}$ være et naturligt tal, så vil $n \cdot dx = dx + dx + \cdots + dx + dx$, også være infinitesimal
- Lad $m \in \mathbb{Z}$ være et heltal, så vil $m \cdot dx$ også være infinitesimal
- Lad $a \in \mathbb{R}$ være et reelt tal, så vil $a \cdot dx$ også være en infinitesimal
- Brøken $\frac{dy}{dx}$ kan være et hvilket som helst hyperreelt tal, det skal altså afgøres fra situation til situation

Eksempel 1.1.2

- Lad dx være en infinitesimal forskellig fra nul, så vil

I.

$$\frac{1}{dx} \cdot 5 \cdot dx = \frac{dx}{dx} \cdot 5 = 1 \cdot 5 = 5$$

II.

$$\frac{dx}{dx^2} - \frac{3}{dx} + 2 = \frac{1}{dx} - \frac{3}{dx} + 2 = -\frac{2}{dx} + 2$$

III.

$$4(3{,}5dx + 2) = 14dx + 8$$

1.2 Standard del

For ethvert endeligt hyperreelt tal, X, findes et reelt tal, x, således at $X - x$ er en infinitesimal, dvs. $X - x = dx$, ved en let jonglering med bogstaverne, ses det at $X = x + dx$. Dermed kan ethvert endeligt hyperreelt tal skrives som summen af et reelt tal og en infinitesimal.
OBS: Vi skriver "**endeligt** hyperreelt tal" fordi der for et **uendeligt** hyperreelt tal A ikke findes et reelt tal a således, at $A - a$ er infinitesimal.

For at kunne tale om denne reelle del af et hyperreelt tal indføres det man kalder *standard delen*. Vi forkorter standard delen med st. Man kan altså tale om at gå fra den hyperreelle tallinie til den reelle tallinie, ved at tage standard delen.

Definition 1.2.1 - Standard del

Standard delen af det hyperreelle tal $a + dx$ er a. Det skrives

$$st(a + dx) = a.$$

Hvis et hyperreelt tal $b \in {}^*\mathbb{R}$ er uendeligt, findes der ikke nogen standard del til dette, dvs.

$$st(b) = \text{eksisterer ikke!!}$$

1.2.1 Regneregler

Lad $a, b \in {}^*\mathbb{R}$, så gælder

- $st(a) \pm st(b) = st(a \pm b)$
- $st(a \cdot b) = st(a) \cdot st(b)$ HVIIIIIIIIIIS a og b er endelige tal (dvs. a og b er ikke uendelige tal)
- $st\left(\frac{a}{b}\right) = \frac{st(a)}{st(b)}$ HVIIIIIIIIIIS a er endelig og b ikke er en infinitesimal

Eksempel 1.2.1

$$st(X) = st(x + dx) = x$$

$$st(42 + dx) = 42.$$

1.3 Funktioner

Husk på hvad en funktion er.

En funktion er en sammenhæng mellem 2 variable, x og y, sådan at der til hver værdi af x hører præcis én værdi af y. Værdien af y kaldes for funktionsværdien af x

Definition 1.3.1 - Funktion

Man kalder x for den uafhængige variabel og y for den afhængige variabel. Det betyder, at man selv vælger værdien af x, mens værdien af y bliver bestemt af funktionen.

Man bruger som regel bogstavet f som betegnelse for en funktion. Man siger, at værdien af y kaldes for funktionsværdien af x for funktionen f og skriver:

$$y = f(x)$$

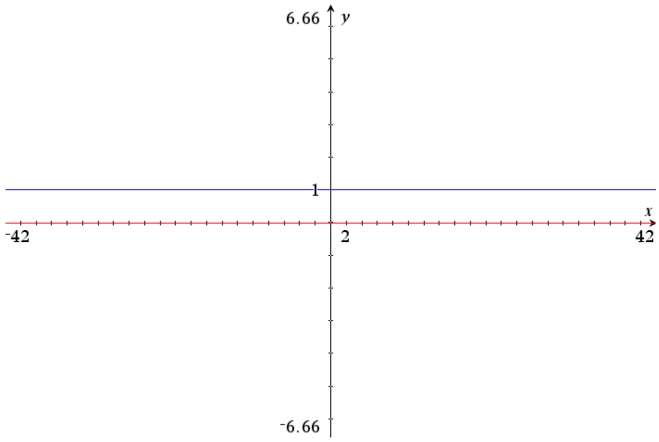
Dette læses "f af x". Hvis man skal bruge flere funktioner, bruges ofte bogstaverne $g, h, k, l, \dots$

Eksempel 1.3.1

$$f(x) = ax + b$$

Eksempel 1.3.2

$$f(x) = \begin{cases} 1 & \text{hvis } x \in \mathbb{Q} \\ 0 & \text{ellers} \end{cases}$$



Eksempel 1.3.2 er måske muligt at forestille sig, men det er ikke muligt at tegne den på en ordentlig måde (se tegning). På grund af disse lidt ”kedelige” funktioner er der udarbejdet nogle grupper af funktioner der er bedre at arbejde med. En af de mere kendte af den salgs grupper af funktioner er dem der hedder *kontinuerte* funktioner. En intuitiv forståelse af hvad en kontinuert funktion er, kan være som følgende:

En kontinuert funktion er en funktion der kan tegnes uden at løfte blyanten/kridtet fra papiret/tavlen.

Med denne intuitive forståelse ses det, at Eksempel 1.3.2 ikke er en kontinuert funktion. Desværre er denne intuitive forståelse ikke en rigtig matematisk definition. For at give en matematisk definition på en kontinuert funktion kan infinitesimale størrelser bruges.

1.3.1 Kontinuitet

Definition 1.3.2 - Kontinuitet

En funktion $f(x)$ er kontinuert i et punkt $(x_0, f(x_0))$, hvis en infinitesimal ændring i x-værdien giver en infinitesimal ændring i y-værdien (funktionsværdien). Dvs. $f(x)$ er kontinuert i $(x_0, f(x_0))$, hvis

$$\Delta y = {}^*f(x_0 + dx) - f(x_0)$$

er infinitesimal.

Hvis en funktion siges at være kontinuert (der udelades altså i hvilket punkt), så er den kontinuert i hele dens definitionsmængde.

Den besøgende alien udbryder: ”Hvad f. *sker der for den der stjerne der er foran det ene* f*?”*
Den kloge lærer: ”Det er rigtigt, der er noget med den der stjerne, er der nogen der her nogle idéer til hvad den kunne betyde?”
Alien: ”Bipbopbip! Nu har jeg lige kigget på alle tallene i definitionsmængden for funktionen f*, og der står der altså ikke noget om at* $x_0 + dx$ *ligger der i.”*
Den kloge lærer: ”Korrekt, funktioner er (normalt) kun defineret for reelle tal, der er altså ingen infinitesimale eller uendelige tal i definitionsmængden til funktioner. Dette er grunden til stjernen er der, stjernen udvider altså funktionens definitionsmængde til at indeholde de hyperreelle tal.”

Nedenfor findes en tekstbogs-version af hvad stjernen betyder.

1.4 Stjerne operation

Med standard delen er det muligt at komme fra den hyperreelle tallinie til den reelle tallinie, men hvad med den anden vej? Et reelt tal, a, er altid også et hyperreelt tal $a + dx$, forstået på den måde at den infinitesimale del dx er 0. Der er derfor ingen grund til at føre et reelt tal til den hyperreelle tallinie, det er der allerede. Stjerne operationen, som fører reelle ting til hyperreelle bruges derfor når der arbejdes med funktioner og intervaller. Intervaller vil først blive behandlet senere.

Hvis en funktion skal evalueres i en hyperreel værdi, $x \in {}^*\mathbb{R}$, skal funktionen udvides til også at gælde for hyperreelle tal, dette skrives således:

$$^*f(x)$$

Dette læses "stjerne f af x". Hvis man skal bruge flere funktioner, bruges $^*g$, $^*h$, $^*k$, $^*l$,...

Eksempel 1.4.1

I. $f(x) = 3x$. Den reelle funktionsværdi for $x = 2$ er $f(2) = 3 \cdot 2 = 6$.
Den hyperreelle funktionsværdi er derfor for $x = 2 + dx$, så
$$^*f(2 + dx) = 3 \cdot (2 + dx) = 3 \cdot 2 + 3 \cdot dx = 6 + 3dx$$
For at komme tilbage til den reelle funktionsværdi tager vi standard delen af den hyperreelle funktionsværdi
$$st\left({}^*f(2 + dx)\right) = st(6 + 3dx) = 6 = f(2)$$
så
$$st\left({}^*f(2 + dx)\right) = f(2).$$
Good to know!

II. $g(x) = x^2$. Den reelle funktionsværdi for $x = 4$ er $g(4) = 4^2 = 16$.
Den hyperreelle funktionsværdi er derfor for $x = 4 + dx$, så
$$^*g(4 + dx) = (4 + dx)^2 = 16 + dx^2 + 2 \cdot 4 \cdot dx = 16 + 8dx + dx^2$$
For at komme tilbage til den reelle funktionsværdi tager vi standard delen af den hyperreelle funktionsværdi
$$st\left({}^*g(4 + dx)\right) = g(4) = 16.$$

#micdrop

1.5 Opgaver

I følgende opgaver er $x \in \mathbb{R}$ og dx og dy er infinitesimale.

1. Løs $st(x + dx) + st(y + dy)$
2. Løs $5 \cdot st(x + dx)$
3. Løs $st(5x + 5dx)$
4. Løs $st(0 + dx)$
5. Løs $st(dx + dy)$
6. Løs $st(5dx)$
7. Løs $st\left(\frac{1}{dx}\right)$
8. Hvorfor er $st\left(5dx \cdot \frac{1}{dx}\right) = 5$?
9. Nævn en infinitesimal der også er reel.
10. Løs $st\big((2 + dx)(3 + dy)\big)$
11. Løs $st(2 + dx)st(3 + dy)$
12. Lav en konklusion på opgave 10 og 11.
13. Løs $st(dx)st\left(\frac{1}{dx}\right)$
14. Præcisér konklusionen fra opgave 12, så den passer til opgave 13
15. Løs $st((4 + dx)^2)$
16. For $f(x) = 3x + 1$, find de hyperreelle funktionsværdier $^*f(1 + dx)$ og $^*f(dx - 2)$.
17. For $g(x) = -x$, find den hyperreelle funktionsværdi for $x = 3 - dx$ og $x = 0$.
18. For $h(x) = 2x^3$, find den reelle funktionsværdi for $^*h(1 + dx)$ og $^*h(4 + dx)$.
19. Hvad er $st(^*f(x))$?

Antag nu at f er en kontinuert funktion (dermed er den kontinuert i alle punkter i dens definitionsmængde)

20. Hvad er $st(^*f(x + dx))$?

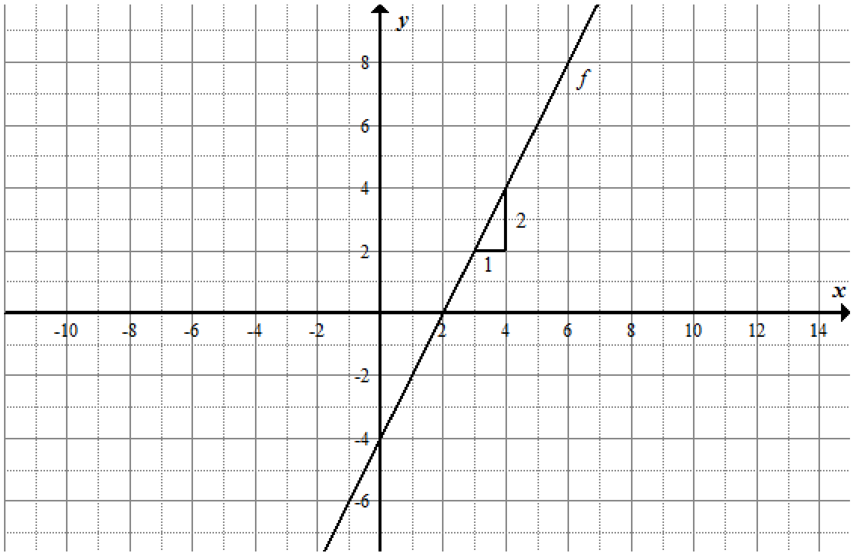
Når alle opgaver i dette kapitel er løst, har man erhvervet sig titlen *standard*, hvilket må sættes foran ens navn.

2 DIFFERENTIALREGNING

Differentialregning handler om funktioners væksthastighed i et bestemt punkt. Altså handler det om at bestemme hældningen af grafen i et bestemt punkt. Vi har lært om hældningskoefficienter for lineære funktioner, men hvad gør vi hvis funktionerne ikke er lineære?

2.1 Hældningskoefficient

En lineær funktion har forskriften $f(x) = ax + b$. Tallet a er hældningskoefficienten og fortæller, hvor meget funktionen vokser eller aftager, når x vokser med 1. Tallet b er y-værdien for grafens skæringspunkt med y-aksen.



Figur 2.1.1

Hældningskoefficienten bestemmes ved at starte i et punkt på grafen, gå 1 til højre og mål lodret op eller ned, indtil man rammer grafen igen, den målte afstand er hældningskoefficienten.
Hældningskoefficienten fortæller, hvor stejl grafen er:

- Hvis hældningskoefficienten er positiv og stor, er grafen voksende og stejl.
- Hvis hældningskoefficienten er positiv og lille, er grafen voksende, men ikke så stejl.
- Hvis hældningskoefficienten er negativ, er grafen aftagende.
- Hvis hældningskoefficienten er 0, er grafen vandret.

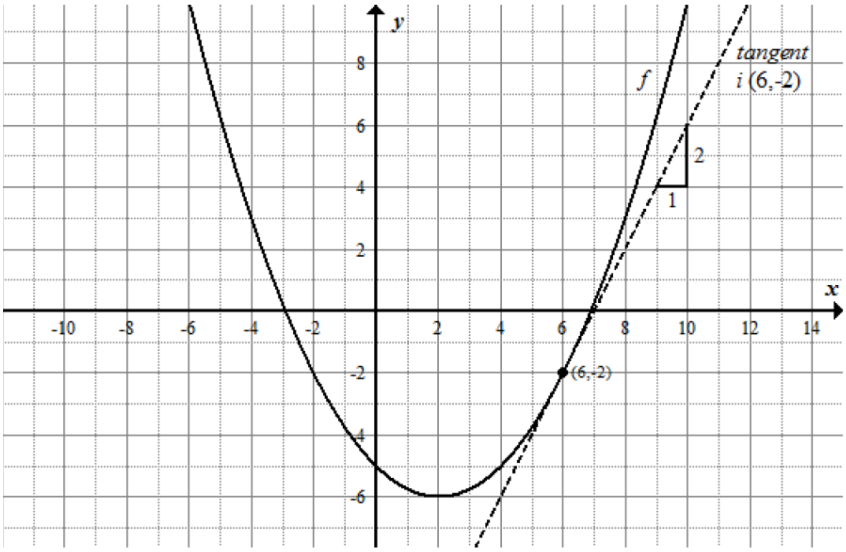
2.2 Tangent

Hvis en funktion ikke er lineær, kan vi godt tale om hældningskoefficient, men først skal vi definere, hvad vi forstår ved hældningskoefficient i det generelle tilfælde.

Først skal det handle om tangenter til grafer. En *tangent* til en graf i et bestemt punkt er en linie, der rører grafen i punktet og er "parallel" med grafen i punktet.

Eksempel 2.2.1

Figuren herunder viser en graf for funktionen $f(x) = 0{,}25x^2 - x - 5$. Der er indtegnet en tangent til grafen i punktet $(6, -2)$:

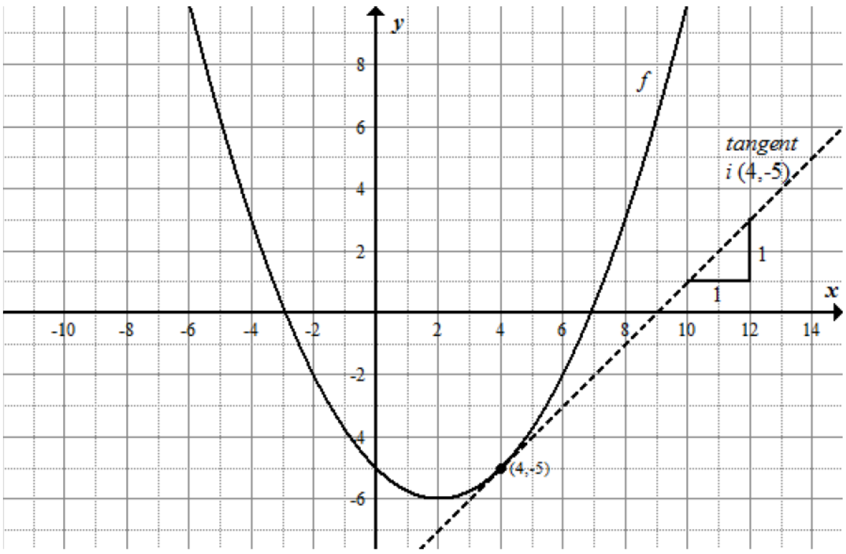


Figur 2.2.1

Tangentens hældningskoefficient er 2 (1 til højre og 2 op). Det svarer til, at grafen også har en hældning på 2 i punktet $(6, -2)$. Grafens hældning ændrer sig fra punkt til punkt.

Eksempel 2.2.2

Figuren herunder viser den samme graf som ovenfor, men her er indtegnet en tangent i punktet $(4, -5)$.



Figur 2.2.2

I punktet $(4, -5)$ er tangentens hældning 1. Det svarer til, at grafen også har en hældning på 1 i punktet $(4, -5)$.

2.3 Hældning i et punkt

Hvis en funktion ikke er lineær, bruger vi ikke ordet ”hældningskoefficient” om hældningen på grafen. SPOILER ALERT! Vi bruger ordet *differentialkvotient*. Vi skal nok give en forklaring på ordet senere.

Differentialkvotienten er grafens hældning i et punkt eller mere præcist: Differentialkvotienten er tangentens hældningskoefficient i et punkt på grafen, og den kan ændre sig fra punkt til punkt på grafen. Det giver følgende definition:

Definition 2.3.1 - Tangent

> Den rette linie gennem punktet $(x_0, f(x_0))$ med hældningskoefficient $a = f'(x_o)$ kaldes kurvens tangent i $(x_0, f(x_0))$.
> $f'(x_o)$ læses f mærke af x nul.

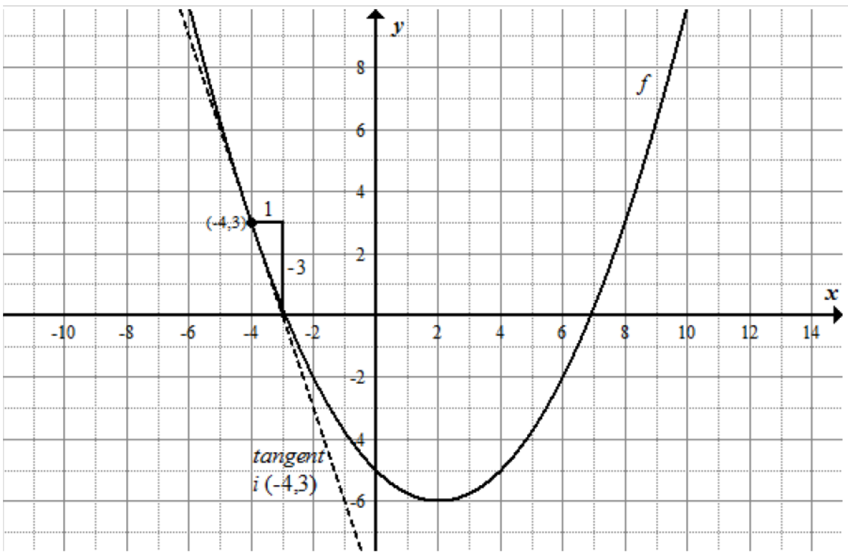
Punktet $(x_0, f(x_0))$ betyder et punkt på grafen for funktionen f, fordi y-værdien på en graf udregnes som $f(x)$.
I Eksempel 2.2.1 er differentialkvotienten 2, når $x = 6$, da tangenten i punktet $(6, -2)$ har hældningskoefficienten 2. Vi kan altså skrive

$$f'(6) = 2.$$

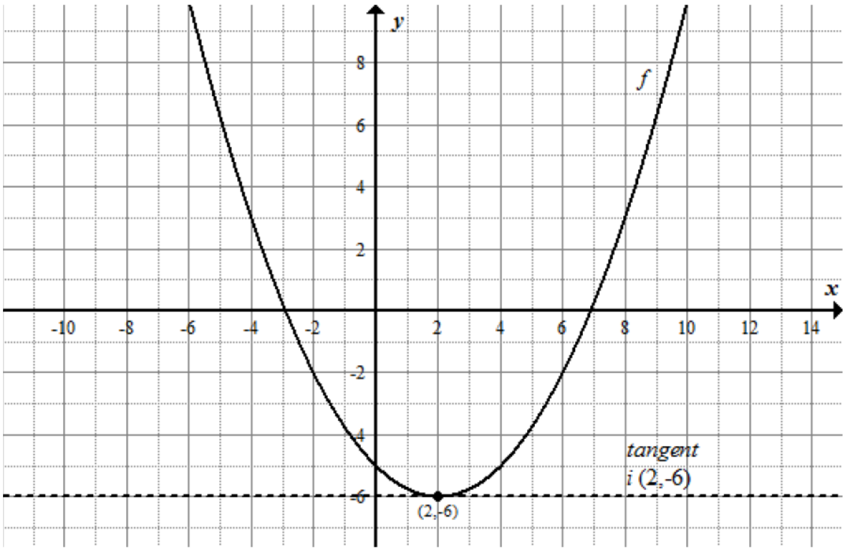
I Eksempel 2.2.2 er differentialkvotienten 1, når $x = 4$, da tangenten i punktet $(4, -5)$ har hældningskoefficienten 1. Vi skriver

$$f'(4) = 1.$$

Lad os bestemme $f'(-4)$ og $f'(2)$ for den samme funktion som før. Vi indtegner tangenterne til grafen i punkterne $(-4,3)$ og $(2, -6)$ og aflæser tangenternes hældningskoefficient



Figur 2.3.1

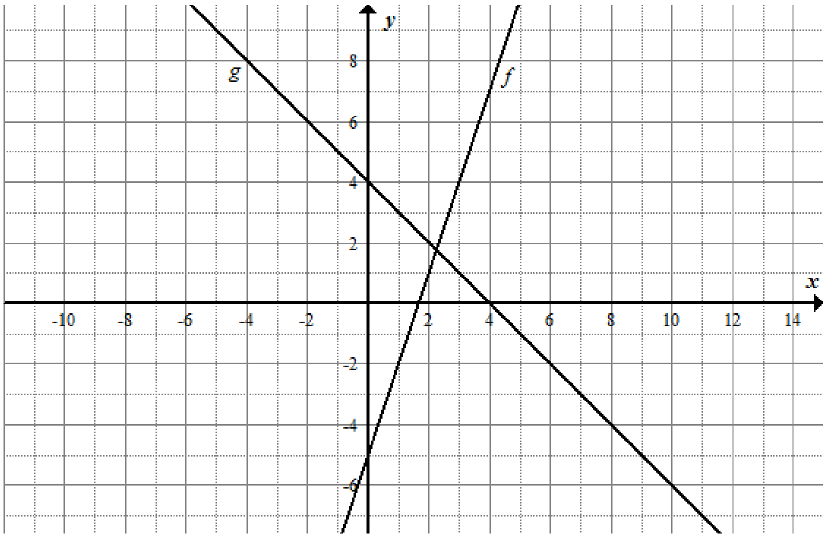


Figur 2.3.2

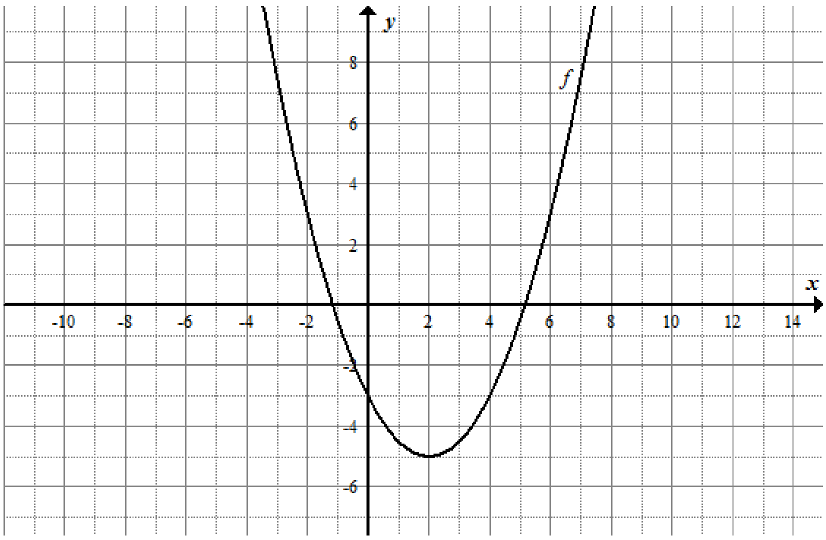
På Figur 2.3.1 er tangentens hældningskoefficient -3, så $f'(-4) = -3$. På Figur 2.3.2 er tangentens hældning 0, så $f'(2) = 0$.

2.4 Opgaver

1. Figuren herunder viser grafer for funktionerne f og g. Bestem hældningskoefficienterne for f og g.



2. Indtegn tangenten til grafen herunder i punktet $(4, -3)$, og bestem tangentens hældningskoefficient.



3. Bestem $f'(4)$ for funktionen i opgave 2.

4. Bestem $f'(-2)$ for funktionen i opgave 2.

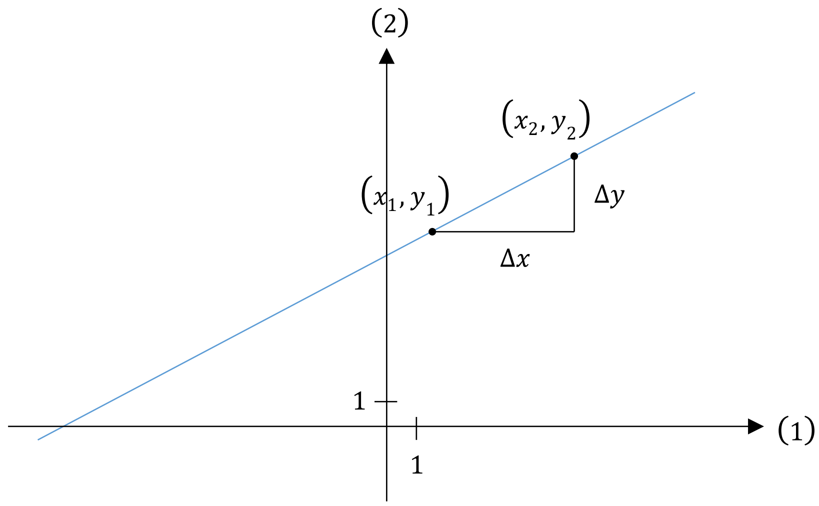
5. Bestem $f(0)$ og $f'(0)$ for funktionen i opgave 2. Hvad er forskellen på $f(0)$ og $f'(0)$?

3 Udvidet 2-punktsformel

Nu har vi set hvordan man kan bestemme hældningskoefficienter og differentialkvotienter grafisk. Det er altid sjovt at kunne beregne ting, især fordi vi har matematik, så inden vi, som lovet, vender tilbage til ordet differentialkvotient, repeterer vi lige hurtigt metoden for at beregne hældningskoefficienten for lineære funktioner. Bagefter angriber vi funktioner generelt. Glæd dig.

3.1 Lineære funktioner

En lineær funktion har forskriften $f(x) = ax + b$, hvor a er hældningskoefficienten, remember? Stærkt (det blev jo også skrevet på side 10 😉). a fortæller altså hvor meget funktionen vokser eller aftager, når x vokser med 1. Hvis vi kender to punkter på grafen, kan vi finde a med to-punkts-formlen. Hvis vi kalder punkterne (x_1, y_1) og (x_2, y_2) kan vi fyre dem ind i et k-syst (koordinatsystem):



Figur 3.1.1

Det ses $\Delta x = x_2 - x_1$ hvilket medfører at $x_2 = x_1 + \Delta x$, på samme måde ses det $\Delta y = y_2 - y_1$ hvilket medfører at $y_2 = y_1 + \Delta y$.
Dermed kan hældningskoefficienten beregnes med formlen

$$a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 + \Delta y - y_1}{x_1 + \Delta x - x_1}.$$

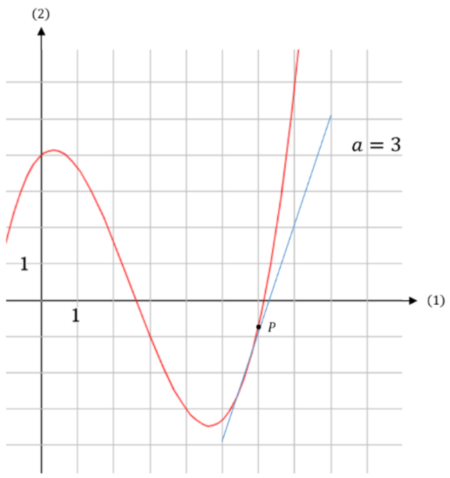
3.2 Sekant

Hvis en funktion ikke er lineær, kan grafens hældning ændre sig fra punkt til punkt.

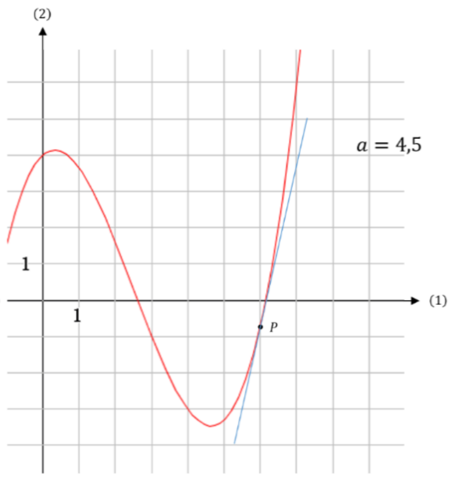
Hvad er hældningen for grafen/en graf i punktet P?

Vi kan bruge vores smukke øjne og udnytte øjemålet og tegne en række linier med forskellige hældninger og derefter vurdere, hvilken linie der passer bedst, ligesom herunder.

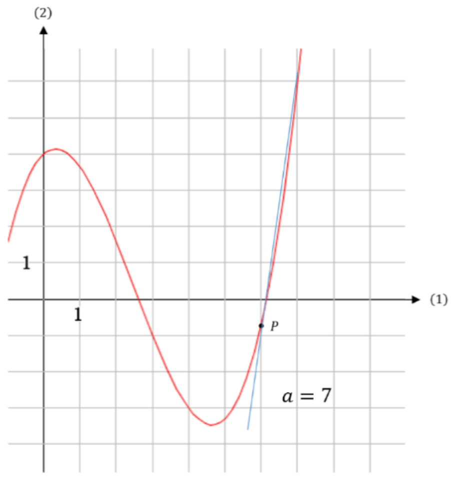
Check de her tre grafer.



Figur 3.2.1



Figur 3.2.2



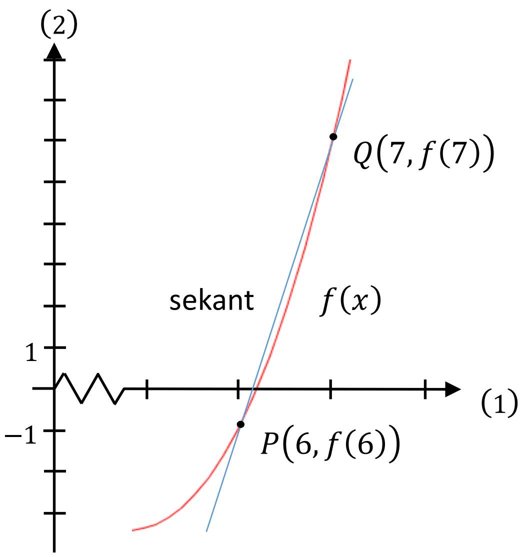
Figur 3.2.3

På Figur 3.2.1 har vi prøvet med en linie med hældning 3. Det er ikke helt vildt lækkert, den er ikke stejl nok. På Figur 3.2.3 har vi smidt en linie ind med hældning 7 og der bliver den lige hakket for stejl. Det ser faktisk ud som om, at linien på Figur 3.2.2 med hældning 4,5 er on fleek. Det vil sige, at i punktet P er værdien 4,5 nok et ret fedt bud på kurvens hældning.

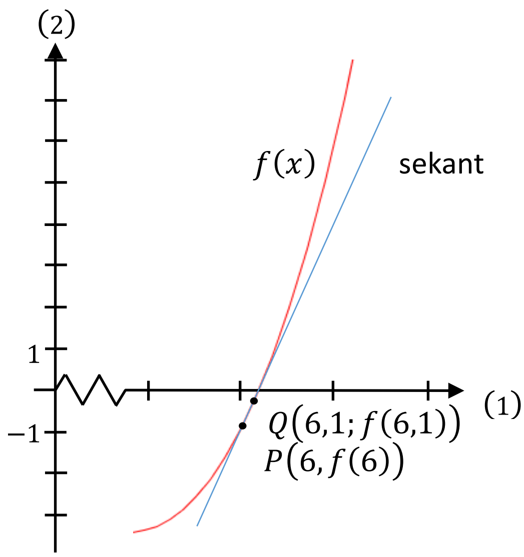
Okay, som sagt er det altid sjovt at kunne beregne ting i matematik, så hvis vi vil beregne stejlheden af den røde graf deroppe lidt mere præcist, kunne det være vi skulle kende forskriften for funktionen. Den røde kurve er graf for funktionen $f(x) = \frac{1}{5}x^3 - \frac{3}{2}x^2 + x + 4$. Punktet P har førstekoordinaten 6. For at kunne beregne hældningen i punktet P, kan vi bruge to-punkts-formlen, men så skal vi bruge et punkt mere.

Dilemmaet er altså at finde en hældning i ét punkt, hvor to-punkts-formlen kræver.. Ja, to punkter. Duh! Så lad os prøve at kigge på to punkter, der bare ligger ret tæt på hinanden.

Kig på grafen ovenfor igen, hvor vi har indtegnet et andet punkt $Q(x_1, y_1)$, og linien gennem P og Q tegnes. Denne linie kaldes en sekant. Vi har lige zoomet lidt ind, så det ikke er så anstrengende for de smukke øjne.



Figur 3.2.4



Figur 3.2.5

På Figur 3.2.4 vælges punktet Q med $x = 7$ dvs. $\Delta x = 1$.

Sekanten får da hældningen

$$a = \frac{f(6+1) - f(6)}{1} = 6{,}9$$

På Figur 3.2.5 vælges punktet Q med $x = 6{,}1$ dvs. $\Delta x = 0{,}1$.

Sekanten får da hældningen

$$a = \frac{f(6+0{,}1) - f(6)}{0{,}1} = 4{,}812.$$

På Figur 3.2.5 er punktet Q så tæt på P at det er svært at se, at sekanten skærer ind over kurven. Vi gider ikke at tegne en figur mere, det tog hjernedødt lang tid, men i stedet vil vi udregne sekanthældningen i et tilfælde mere. Vi skal endnu tættere på P med Q nu. Vi vælger derfor punktet Q med $x = 6{,}001$ dvs. $\Delta x = 0{,}001$. Med to-punkts-formlen får vi

$$a = \frac{f(6+0{,}001) - f(6)}{0{,}001} = 4{,}602.$$

Tabellen herunder viser sammenhængen mellem det x_2 vi vælger for $Q(x_2, f(x_2))$ og hældningskoefficienten, a, for den rette linie der går gennem P og Q

Δx	1	0,1	0,001	0,00001
a	6,9	4,812	4,602	4,60002

Det kunne godt se ud som om, at hældningen i punktet P er cirka 4,6. Det svarer ret godt til resultatet fra Figur 3.2.5.
Vi kunne blive freakin' ved og ved med at vælge Q-punktet endnu tættere på P, altså gøre Δx mindre og mindre og mindre og mindre, men det kunne vi egentlig gøre for evigt.
Forestil dig, eller forestil JER, hvis I sidder og læser højt for hinanden.. Forestil Jer, at værdien for x ændres fra at være 6 til at være $6 + dx$, som er uendeligt tæt på men ikke lig med 6! Så vil den nye værdi for $f(x)$ være ${}^*f(6 + dx)$. På den måde bliver værdien for x ændret med en infinitesimal dx, forskellig fra 0, mens værdien for $f(x)$ vil blive ændret med

$${}^*f(6 + \mathrm{d}x) - f(6).$$

Ved to-punkts-formlen får vi altså, med ændringen i værdien for $f(x)$ og ændringen i værdien for x,

$$\frac{{}^*f(6+dx) - f(6)}{dx}.$$

Dette forhold er netop definitionen på hældningen, altså differentialkvotienten, for $f(x)$, som kommeeeeer her! På næste side..

3.3 Differentialkvotient

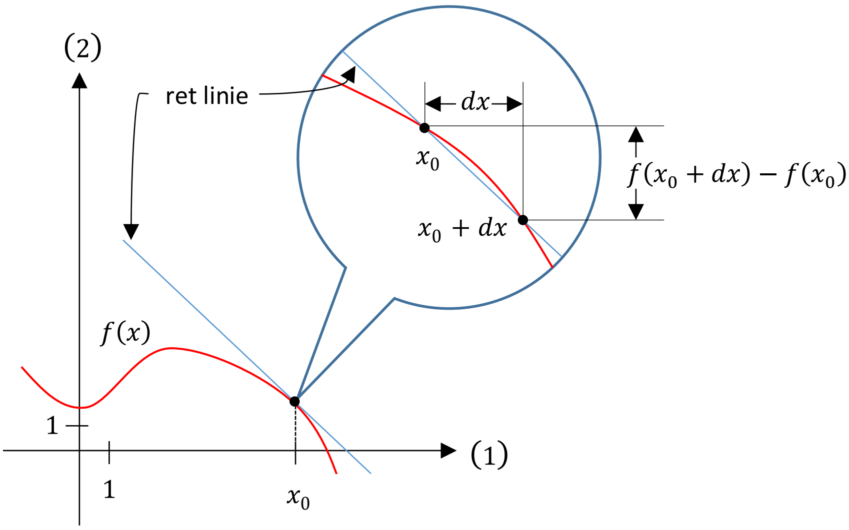
Definition 3.3.1 - Differentialkvotient

Differentialkvotienten for $f(x)$ i x_0 er

$$st\left(\frac{{}^*f(x_0+dx)-f(x_0)}{dx}\right).$$

Hvis differentialkvotienten eksisterer og er den samme for enhver infinitesimal $dx \neq 0$, siges funktionen at være differentiabel i x_0 og værdien benævnes $f'(x_0)$.

Her skal vi bare lige huske på, at når vi arbejder med infinitesimaler, befinder vi os i det hyperreelle univers, det betyder altså, at man kan se infinitesimalerne på den rette linie. Se Figur 3.3.1.



Figur 3.3.1

3.4 3-trins reglen

For at StandardJoe er sikker på, at HyperMick forstår definitionen på differentialkvotienten, udpensler han derfor en regel for hvordan man skal bruge formlen. Ready? Okay, vi finder altså hældningen i et punkt $(x_0, f(x_0))$ sådan her:

1. Find ${}^*f(x_0+dx)-f(x_0)$
2. Divider ovenstående resultat med dx og reducér
3. Tag standard delen af $\frac{{}^*f(x_0+dx)-f(x_0)}{dx}$

HyperMick: ”Stop.. Jeg fatter ikke en dadel af det der..”
StandardJoe: ”Okay, check it”

Eksempel 3.4.1

Find differentialkvotienten til $f(x) = \frac{1}{2}x$ i punktet $(x_0, f(x_0))$.

Trin 1, find ${}^*f(x_0 + dx) - f(x_0)$:

$$ {}^*f(x_0 + dx) - f(x_0) = \frac{1}{2}(x_0 + dx) - \frac{1}{2}x_0 $$

Trin 2, dividér med dx og reducér

$$ \frac{{}^*f(x_0 + dx) - f(x_0)}{dx} = \frac{\frac{1}{2}(x_0 + dx) - \frac{1}{2}x_0}{dx} = \frac{\frac{1}{2}x_0 + \frac{1}{2}dx - \frac{1}{2}x_0}{dx} = \frac{\frac{1}{2}dx}{dx} = \frac{1}{2} $$

Und drei, tag standard delen til $\frac{{}^*f(x_0+dx)-f(x_0)}{dx}$:

$$ st\left(\frac{{}^*f(x_0 + dx) - f(x_0)}{dx}\right) = st\left(\frac{1}{2}\right) = \frac{1}{2} $$

Så har vi fundet differentialkvotienten til $f(x) = \frac{1}{2}x$ i punktet $(x_0, f(x_0))$, altså $st\left(\frac{{}^*f(x_0+dx)-f(x_0)}{dx}\right) =$

$$ f'(x_0) = \frac{1}{2} $$

Eksempel 3.4.2

Find differentialkvotienten til $f(x) = -x^2$ i punktet $(x_0, f(x_0))$.

Step one:

$$ {}^*f(x_0 + dx) - f(x_0) = -(x_0 + dx)^2 - (-x_0^2) $$

Número dos:

$$ \frac{{}^*f(x_0 + dx) - f(x_0)}{dx} = \frac{-(x_0 + dx)^2 - (-x_0^2)}{dx} = \frac{-(x_0^2 + dx^2 + 2 \cdot x_0 \cdot dx) + x_0^2}{dx} $$
$$ = \frac{-x_0^2 - dx^2 - 2x_0 dx + x_0^2}{dx} = \frac{-dx^2 - 2x_0 dx}{dx} = -dx - 2x_0 $$

Og tre:

$$ st\left(\frac{{}^*f(x_0 + dx) - f(x_0)}{dx}\right) = st(-dx - 2x_0) = -2x_0 $$

Så har vi fundet differentialkvotienten til $f(x) = -x^2$ i punktet $(x_0, f(x_0))$

$$ f'(x_0) = -2x_0. $$

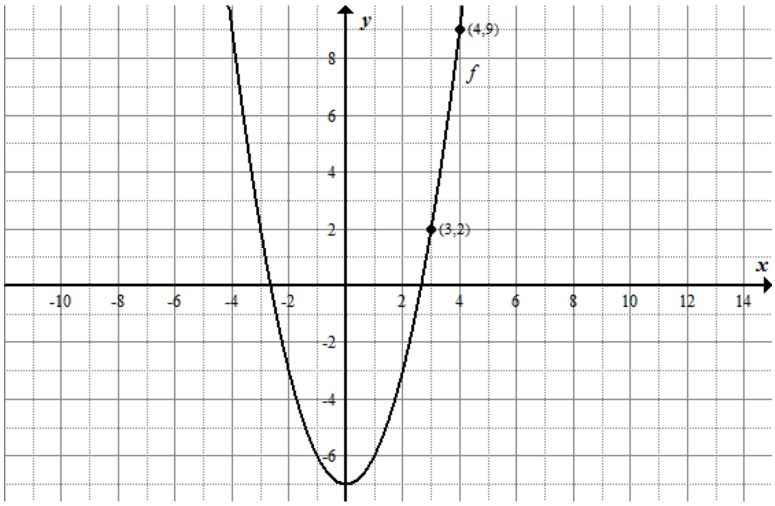
Boom! 

3.5 Opgaver

1. Grafen for en lineær funktion går gennem punkterne $(1,5)$ og $(4,7)$. Beregn differentialkvotienten.

2. Beregn sekantens hældning for grafen for $f(x) = x^2 - 3$ gennem punktet $(2,1)$ og punktet man finder ved at sætte $\Delta x = 0{,}001$ og udregne Δy.

3. Figuren herunder viser grafen for $f(x) = x^2 - 7$. Vis ved udregning, at $f(3) = 2$ og $f(4) = 9$, og beregn sekantens hældning punkterne $(3,2)$ og $(4,9)$.



4. Beregn sekanthældningen gennem punktet $(3,2)$ og punktet, hvor $\Delta x = 0{,}1$.

5. Gentag udregningen, men brug denne $\Delta x = 0{,}001$.

6. Gentag udregningen, men brug $\Delta x = 0{,}00001$.

7. Udfyld et skema som nedenstående ud fra opgave 3, 4, 5 og 6, idet a er sekanthældningen. Hvad bliver hældningen, når Δx kommer tættere og tættere på 0?

Δx	1	0,1	0,001	0,00001
a				

8. Beregn differentialkvotienten i punktet $(3,2)$ ved at sætte $\Delta x = dx$ i funktionen fra opgave 3.

9. Vis, at $f(1) = -6$ for funktionen fra opgave 3, og bestem differentialkvotienten i punktet $(1,-6)$ ligesom i opgave 8.

10. Udfyld skemaet ved at beregne differentialkvotienten til funktionen $g(x) = 1$ i punkterne:

x	-3	-1	0	1	4
$g(x)$					
$g'(x)$					

11. Udfyld skemaet ved at beregne differentialkvotienten til funktionen $h(x) = -4x$ i punkterne:

x	-3	-1	0	1	4
$h(x)$					
$h'(x)$					

12. Udfyld skemaet ved at beregne differentialkvotienten til funktionen $k(x) = 2x^2$ i punkterne:

x	-3	-1	0	1	4
$k(x)$					
$k'(x)$					

13. Udfyld skemaet ved at beregne differentialkvotienten til funktionen $l(x) = 2x^2 - 4x + 1$ i punkterne:

x	-3	-1	0	1	4
$l(x)$					
$l'(x)$					

14. WHAAAAAT!?

15. Lad $f(x)$ og $g(x)$ være to vilkårlige og differentiable funktioner. Opskriv en regel for differentialkvotienten til $h(x) = f(x) + g(x)$, altså beskriv $h'(x)$ ved hjælp af $f'(x)$ og $g'(x)$.
Hint: Find sammenhængen mellem opgaverne 10, 11, 12 og 13.

4 Afledte funktioner

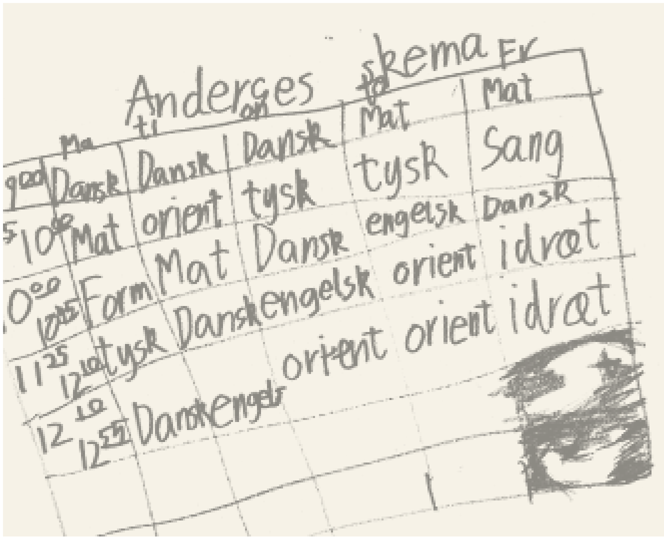
HyperMick syntes det var nederen at skulle køre alle mellemregningerne igennem hver eneste gang han skulle finde differentialkvotienten i et punkt, så ved at join forces har StandardJoe and HyperMick i dette kapitel generaliseret differentialkvotienter til hvad der kaldes *afledte funktioner*. En afledt funktion $f'(x)$ er altså en funktion der beskriver hældningen for funktionen $f(x)$ til enhver x-værdi. SÅDAN! Kapitel slut.

Nej, vi spytter lige et par eksempler. Først skal vi lige have et udtryk for selve fremgangsmåden ”at komme fra en funktion til den afledte funktion”: *At differentiere*.

Use it in a sentence:
StandardJoe: “At differentiere”.

I Eksempel 3.4.2 finder vi differentialkvotienten for en generel x-værdi, x_0, dvs. vi finder frem til en ny funktion, $f'(x_0) = -2x_0$. Denne nye funktion kaldes den afledte funktion til $f(x)$. Da x_0 kan være en hvilken som helst x-værdi, kunne man lige så vel skrive $f'(x) = -2x$.

StandardJoe mente det var nemmere at forstå med et skema:



HyperMick mente det var nemmere at forstå med et skema der havde med emnet at gøre:

(oprindelig) funktion	Operation (det man gør)	Afledt funktion
$f(x) = -x^2$ ———	——— At differentiere ———	——→ $f'(x) = -2x$

Ved at definere $df \coloneqq f'(x)dx$, fås en anden måde at skrive $f'(x)$ på, nemlig $f'(x) = \frac{df}{dx}$ (IMPORTANT AF).

4.1 Den konstante funktion

$f(x) = k \Rightarrow f'(x) = 0$

Her bruges tre-trins reglen til at finde den afledte funktion, $f'(x)$, når den oprindelige funktion, $f(x)$, er en konstant funktion.

$$^*f(x+dx) - f(x) = k - k$$

$$\frac{^*f(x+dx) - f(x)}{dx} = \frac{k-k}{dx} = \frac{0}{dx} = 0$$

$$f'(x) = \frac{df}{dx} = st\left(\frac{^*f(x+dx) - f(x)}{dx}\right) = st(0) = 0$$

4.2 Den lineære funktion

$f(x) = ax + b \Rightarrow f'(x) = a$

Her bruges tre-trins reglen til at finde den afledte funktion, $f'(x)$, når den oprindelige funktion, $f(x)$, er en lineær funktion.

$$^*f(x+dx) - f(x) = a(x+dx) + b - ax - b$$

$$\frac{^*f(x+dx) - f(x)}{dx} = \frac{a(x+dx) + b - ax - b}{dx} = \frac{ax + a \cdot dx - ax}{dx} = \frac{a \cdot dx}{dx} = a$$

$$f'(x) = \frac{df}{dx} = st\left(\frac{^*f(x+dx) - f(x)}{dx}\right) = st(a) = a.$$

4.3 x med en naturlig eksponent

$f(x) = x^n \Rightarrow f'(x) = n \cdot x^{n-1}$

Det bliver lidt cray nu, men prøv lige at følge med. Vi vil gerne differentiere funktionen $f(x) = x^n$. Vi bruger Definition 3.3.1, så

$$f'(x) = st\left(\frac{^*f(x_0+dx) - f(x_0)}{dx}\right) = st\left(\frac{(x+dx)^n - x^n}{dx}\right)$$

De interessante led i tælleren er dem hvor dx bliver ganget på i første potens, dvs. led der ser ud på følgende måde: $x^{n-1} \cdot dx$. Disse led kommer til udtryk ved at forestille sig de n parenteser ganget sammen:

$$\overbrace{(x+dx) \cdot (x+dx) \cdot (x+dx) \cdots (x+dx)}^{n \text{ gange}}.$$

Når disse ganges ud, vil der fremkomme et udtryk på den interessante form for hver parentes der er. Vi ganger først x'et ind på alle x'erne i parenteserne efter, undtagen den sidste parentes, hvor vi ganger med dx.

$$(x+dx) \cdot (x+dx) \cdot (x+dx) \cdots (x+dx).$$

Nu har vi ganget alle de blå tal sammen, det giver

$$x^{n-1} \cdot dx.$$

Når vi ganger x'et ind på alle x'erne i hver parentes efter, men i stedet ganger med dx'et fra den *anden*sidste parentes får vi:

$$(x+dx) \cdot (x+dx) \cdot (x+dx) \cdots (x+dx) \cdot (x+dx)$$

Nu har vi ganget alle de røde tal sammen, det giver også

$$x^{n-1} \cdot dx$$

Når vi ganger x'et ind på alle x'erne i hver parentes efter, men denne gang ganger med dx'et fra den *tredje*sidste parentes får vi:

$$(x+dx) \cdot (x+dx) \cdot (x+dx) \cdots (x+dx) \cdot (x+dx) \cdot (x+dx)$$

og det giver sørme også

$$x^{n-1} \cdot dx.$$

Prøv at gøre det en milliard gange mere.. Eller nej, n gange mere, så får vi

$$f'(x) = st\left(\frac{(x+dx)^n - x^n}{dx}\right) = st\left(\frac{n \cdot x^{n-1} \cdot dx + k}{dx}\right)$$

Hvor k beskriver alle de led der har dx ganget på sig i potenser højere end 1, hvilket medfører $\frac{k}{dx}$ er infinitesimal. Dermed er

$$\frac{d}{dx}(x^n) = st\left(\frac{(x+dx)^n - x^n}{dx}\right) = st\left(\frac{n \cdot x^{n-1} \cdot dx + k}{dx}\right) = st\left(n \cdot x^{n-1} + \frac{k}{dx}\right) = n \cdot x^{n-1}.$$

4.4 1 over x

$f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$

Her bruges tre-trins reglen til at finde den afledte funktion, $f'(x)$, når den oprindelige funktion, $f(x)$, beskriver en hyperbel.

$$^*f(x+dx) - f(x) = \frac{1}{x+dx} - \frac{1}{x} = \frac{x-(x+dx)}{(x+dx)x} = \frac{-dx}{(x+dx)x}$$

$$\frac{^*f(x+dx) - f(x)}{dx} = \frac{\frac{-dx}{(x+dx)x}}{dx} = \frac{-1}{x(x+dx)} = \frac{-1}{x^2+xdx}$$

$$f'(x) = \frac{df}{dx} = st\left(\frac{^*f(x+dx) - f(x)}{dx}\right) = st\left(\frac{-1}{x^2+xdx}\right) = \frac{st(-1)}{st(x^2+xdx)} = \frac{-1}{x^2}.$$

Det orange $=$ kommer fra reglen om standarddelen af en brøk, $st\left(\frac{a}{b}\right) = \frac{st(a)}{st(b)}$ hvor $b = x^2 + xdx$.

4.5 Kvadratroden af x

$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

Her bruges tre-trins reglen til at finde den afledte funktion, $f'(x)$, når den oprindelige funktion, $f(x)$, er en kvadratrodsfunktion. I dette tilfælde skal der dog bruges et lille trick (der ganges med 1..)

$$^*f(x+dx) - f(x) = \sqrt{x+dx} - \sqrt{x}$$

$$\frac{^*f(x+dx) - f(x)}{dx} = \frac{\sqrt{x+dx}-\sqrt{x}}{dx} = \frac{\sqrt{x+dx}-\sqrt{x}}{dx} \cdot 1 = \frac{\sqrt{x+dx}-\sqrt{x}}{dx} \cdot \frac{\sqrt{x+dx}+\sqrt{x}}{\sqrt{x+dx}+\sqrt{x}}$$

$$= \frac{(\sqrt{x+dx}-\sqrt{x})(\sqrt{x+dx}+\sqrt{x})}{dx(\sqrt{x+dx}+\sqrt{x})} = \frac{x+dx-x}{dx(\sqrt{x+dx}+\sqrt{x})}$$

$$= \frac{dx}{dx(\sqrt{x+dx}+\sqrt{x})} = \frac{1}{\sqrt{x+dx}+\sqrt{x}}$$

$$f'(x) = \frac{df}{dx} = st\left(\frac{^*f(x+dx) - f(x)}{dx}\right) = st\left(\frac{1}{\sqrt{x+dx}+\sqrt{x}}\right) = \frac{st(1)}{st(\sqrt{x+dx}+\sqrt{x})}$$

$$= \frac{1}{st(\sqrt{x+dx}) + st(\sqrt{x})} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Det orange $=$ kommer fra reglen om standarddelen af en brøk, hvor $b = \sqrt{x+dx} + \sqrt{x}$.

4.6 x med en reel eksponent

$f(x) = x^q \Rightarrow f'(x) = qx^{q-1}$

Denne afledte funktion gælder for alle reelle tal, $q \in \mathbb{R}$.

Bevis:
Beviset for dette kræver lidt mere end tretrins reglen, her skal der bruges en regneregel for differentiation af sammensatte funktioner, kaldet kædereglen. Endvidere er det korteste bevis baseret på en funktion, $\ln(x)$, og dens afledte funktion, $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$. Således kan man lade

$$f(x) = x^q$$

og

$$g(x) = \ln(x).$$

Kald nu

$$h(x) = g(f(x))$$

så ved brug af kædereglen er

$$h'(x) = f'(x) \cdot g'(f(x)) = f'(x) \cdot \frac{d}{dx}(\ln(f(x))) = f'(x) \cdot \frac{1}{f(x)} = f'(x) \cdot \frac{1}{x^q} = f'(x) \cdot x^{-q}.$$

Samtidig er

$$h'(x) = \Big(g(f(x))\Big)' = (\ln f(x))\,' = (\ln(x^q))' = (\mathrm{q} \cdot \ln(x))' = q \cdot \ln'(\mathrm{x}) = q \cdot \frac{1}{x} = q \cdot x^{-1}$$

Ved det lilla $=$ er der brugt en regneregel der gælder alle logaritmefunktioner, $\ln(x^r) = r \cdot \ln(x)$. Ved det røde $=$ er der brugt reglen for hvordan man differentierer når der er ganget en konstant på og ved det blå $=$ er der brugt hvad den afledte til $\ln(x)$ er.

Nu findes der altså to udtryk der beskriver det samme, endvidere indeholder det ene udtryk $f'(x)$, dermed kan $f'(x)$, isoleres.

$$\begin{aligned} h'(x) &= h'(x) && \Leftrightarrow \\ f'(x) \cdot x^{-q} &= q \cdot x^{-1} && \Leftrightarrow \\ f'(x) &= q \cdot x^{q-1} \end{aligned}$$

Dermed opnås den generelle regel for hvordan man differentierer et udtryk på formen x^q, for alle reelle tal $q \in \mathbb{R}$.

4.7 Eksponentialfunktionen

$f(x) = e^x \Rightarrow f'(x) = e^x$

Her bruges tre-trins reglen til at finde den afledte funktion, $f'(x)$, når den oprindelige funktion, $f(x)$, er en lidt speciel eksponentielfunktion.

$$^*f(x + dx) - f(x) = e^{x+dx} - e^x = e^x \cdot e^{dx} - e^x \cdot 1 = e^x(e^{dx} - 1) = e^x(e^{dx} - e^0)$$

$$\frac{^*f(x + dx) - f(x)}{dx} = \frac{e^x(e^{dx} - e^0)}{dx} = e^x \cdot \frac{e^{dx} - e^0}{dx}$$

$$f'(x) = \frac{df}{dx} = st\left(\frac{{}^*f(x+dx) - f(x)}{dx}\right) = st\left(e^x \cdot \frac{e^{dx} - e^0}{dx}\right) = e^x \cdot st\left(\frac{e^{dx} - e^0}{dx}\right).$$

Hmm.. Her er det lidt svært.. Vi skal altså bruge den afledte for at finde den afledte? Fortvivl ej, der er en redning til vores lille dilemma.

Hvad er det nu $st\left(\frac{e^{dx}-e^0}{dx}\right) = f'(0)$ betyder? Jo, det er hældningen for funktionen $f(x)$ i punktet $\left(0, f(0)\right)$. Vores redning er, at man netop har defineret tallet $e = 2{,}71828182845905\ldots$ som det tal der opfylder, at $st\left(\frac{e^{dx}-e^0}{dx}\right) = 1$. Dermed er

$$f'(x) = \frac{df}{dx} = e^x \cdot st\left(\frac{e^{dx} - e^0}{dx}\right) = e^x \cdot 1 = e^x.$$

Når man har differentieret en funktion $f(x)$ og fundet dens afledte $f'(x)$, kan man altså finde hældningen til grafen for $f(x)$ i et hvilket som helst punkt, ved at indsætte x-værdien i $f'(x)$.

Eksempel 4.7.1

Find hældningen i punktet $\left(1, f(1)\right)$ for funktionen $f(x) = 3x^5 - 6x^2 + 9$.

Whatcha gonna wanna do is, differentier funktionen:

$$f'(x) = 5 \cdot 3x^{5-1} - 2 \cdot 6x^{2-1} + 0 \quad \Leftrightarrow$$

$$f'(x) = 15x^4 - 12x$$

Next up, you should go ahead and indsætte punktet:

$$f'(1) = 15 \cdot 1^4 - 12 \cdot 1 \quad \Leftrightarrow$$

$$f'(1) = 3.$$

Så hældningen for grafen for $f(x)$ i punktet $\left(1, f(1)\right)$ er 3.

Eksempel 4.7.2

Find hældningen i punktet $\left(4, g(4)\right)$ for funktionen $g(x) = \frac{3}{4}x + 2 \cdot \sqrt{x} - \frac{4}{x}$.

Differentiér: $g'(x) = \frac{3}{4} + 2 \cdot \frac{1}{2\sqrt{x}} - \left(-\frac{4}{x^2}\right) \quad \Leftrightarrow$

$$g'(x) = \frac{1}{\sqrt{x}} + \frac{4}{x^2} + \frac{3}{4}$$

Indsæt punkt: $g'(4) = \frac{1}{\sqrt{4}} + \frac{4}{4^2} + \frac{3}{4} = \frac{1}{2} + \frac{1}{4} + \frac{3}{4} \quad \Leftrightarrow$

$$g'(4) = \frac{3}{2}.$$

Så hældningen for grafen for $g(x)$ i punktet $\left(4, g(4)\right)$ er $\frac{3}{2}$.

OPMÆRKSOMHED! KIG PÅ MIG! MIG MIG MIG! Aldrig sæt punktet ind FØR I differentierer! Hverken i dette eller noget alternativt univers. Hvorfor ikke? Food for thought. (Prøv det! Og så aldrig gør det igen.)

4.8 Funktioner uden en afledt funktion

Med ovenstående opremsning af funktioner og hvordan de kan afledes, kunne man være tilbøjelig til at tænke at så kan alle funktioner differentieres i hele deres værdimængde. Dette er desværre ikke tilfældet. Hvornår kan en funktion så afledes, kunne være et spørgsmål det ville være rart at kunne svare på.

Overvej derfor hvad differentialkvotienten for $f(x)$ i x_0 er:

$$st\left(\frac{{}^*f(x_0+dx)-f(x_0)}{dx}\right).$$

Dette er standarddelen af en kvotient, det vides derfor at hvis kvotienten er et uendeligt tal findes der ikke nogen standarddel. Et uendeligt tal kan altid skrives som $\frac{a}{dx}$, for en eller anden infinitesimal dx og hvor a er et endeligt hypperreelt tal. I ovenstående udtryk er der allerede en infinitesimal i nævneren, så hvis der står et endeligt tal i tælleren ville kvotienten blive et uendeligt tal og udtrykket vil derfor ikke eksistere. Derfor skal tælleren være en infinitesimal for at udtrykket giver mening. Dvs. hvis ${}^*f(x_0+dx)-f(x_0)$ ikke er en infinitesimal, så giver udtrykket

$$st\left(\frac{{}^*f(x_0+dx)-f(x_0)}{dx}\right)$$

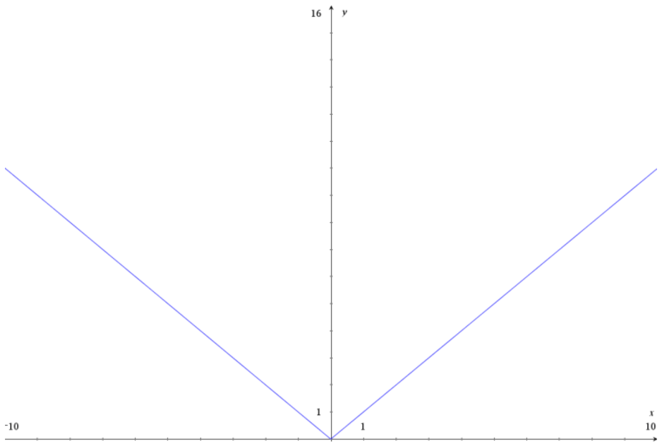
ikke mening, da kvotienten ville være et uendeligt tal. Kravet om at ${}^*f(x_0+dx)-f(x_0)$ skal være en infinitesimal er set før i definitionen af en kontinuert funktion (se 1.3.1).
Dermed er en differentiabel funktion altid kontinuert (husk det, brug det).
Definitionen af differentialkvotienten er mere end bare standarddelen af denne brøk, den indeholder også teksten:

Hvis differentialkvotienten eksisterer og er den samme for enhver infinitesimal $dx \neq 0$, siges funktionen at være differentiabel i x_0 og værdien benævnes $f'(x_0)$.

For at forstå dette er det nødvendigt med et eksempel.

Eksempel 1.3.1

Nedenfor er funktionen $f(x)=\sqrt{x^2}$ afbilledet.



Bestem differentialkvotienten til $f(x)$ en gang hvor den infinitesimale forskydning er positiv og en gang hvor den er negativ i punktet $(0,0)$. dvs. bestem differentialkvotienten med infinitesimalen dx og derefter med infinitesimalen $-dx$.

$$st\left(\frac{{}^*f(0+dx)-f(0)}{dx}\right)=st\left(\frac{\sqrt{dx^2}}{dx}\right)=1$$

$$st\left(\frac{{}^*f(0-dx)-f(0)}{-dx}\right)=st\left(\frac{\sqrt{(-dx)^2}}{-dx}\right)=-1$$

Med teksten der følger med til definitionen af differentialkvotienten er $f(x)=\sqrt{x^2}$ altså ikke differentiabel i punktet $(0,0)$, da differentialkvotienten ikke giver samme værdi for alle infinitesimaler.

4.9 Opgaver

1. Hvad er $f'(x)$ for funktionen $f(x) = 1.000.000.000.000.000.000.000.000$?

2. Differentier $f(x) = 42x + 42$!

3. Bestem $f'(x)$ for funktionen $f(x) = x^7$ ud fra reglen for, hvordan man differentierer $f(x) = x^n$.

4. Bestem på samme måde $f'(x)$ for funktionen $f(x) = x^8$ og find hældningen i punktet $(1, f(1))$, det vil sige find $f'(1)$, som vist i *Eksempel* ***4.7.1*** og *Eksempel* ***4.7.2***.

5. Bestem den afledte af funktionen $f(x) = x^{-5}$ ud fra reglen for, hvordan man differentierer $f(x) = x^q$.

6. Bestem på samme måde $f'(x)$ for funktionen $f(x) = x^{-7}$.

7. Bestem $f'(x)$ for funktionen $f(x) = x^{3,6}$ og find hældningen i punktet $(2, f(2))$

8. Bestem $f'(x)$ for funktionen $f(x) = x^{-0,8}$.

9. Bestem den afledte af funktionen $f(x) = \frac{1}{x} \cdot x$.

10. Differentier funktionen $f(x) = 42 + \frac{1}{x}$ og bestem $f'(-5)$.

11. Find den afledte til funktionen $f(x) = 3e^x$ og bestem $f'(2,5)$ og forklar betydningen af dette.

12. Bestem $f'(x)$ for funktionen $f(x) = 2 \cdot \frac{1}{x} - e^x \cdot 6$.

13. Bestem $f'(x)$ for funktionen $f(x) = -9 \cdot x + 8 - 7x^6 + 5\sqrt{x} - 4 \cdot e^x + 3 \cdot \frac{2}{1 \cdot x} - 0$.

5 Regneregler

5.1 Differentiation med en konstant faktor

Lad $f(x)$ være en differentiabel funktion og lad $a \in \mathbb{R}$ være et reelt tal, endvidere kald $h(x) = af(x)$. Så gælder følgende:

$$h'(x) = a \cdot f'(x)$$

Bevis:
Husk på reglen for hvordan standard delen opfører sig når der ganges med en konstant der ikke er uendelig.

$$h'(x) = st\left(\frac{{}^*f(x+dx) - f(x)}{dx}\right) = st\left(\frac{a{}^*f(x+dx) - af(x)}{dx}\right) = st\left(a \cdot \frac{{}^*f(x+dx) - f(x)}{dx}\right)$$
$$= a \cdot st\left(\frac{{}^*f(x+dx) - f(x)}{dx}\right) = a \cdot f'(x).$$

5.2 Differentiation af en sum

Lad $f(x)$ og $g(x)$ være 2 differentiable funktioner, kald $h(x) = f(x) + g(x)$, så er

$$h'(x) = f'(x) + g'(x)$$

Bevis:

$$h'(x) = \frac{dh}{dx} = st\left(\frac{{}^*h(x+dx) - h(x)}{dx}\right) = st\left(\frac{{}^*f(x+dx) + {}^*g(x+dx) - f(x) - g(x)}{dx}\right)$$
$$= st\left(\frac{{}^*f(x+dx) - f(x)}{dx} + \frac{{}^*g(x+dx) - g(x)}{dx}\right)$$

Da hverken $\frac{{}^*f(x+dx) - f(x)}{dx}$ eller $\frac{{}^*g(x+dx) - g(x)}{dx}$ er uendelige, kan vi bruge reglen fra standard delen ($st(x+y) = st(x) + st(y)$, hvis hverken x eller y er uendelige). Dermed fås:

$$h'(x) = st\left(\frac{{}^*f(x+dx) - f(x)}{dx} + \frac{{}^*g(x+dx) - g(x)}{dx}\right)$$
$$= st\left(\frac{{}^*f(x+dx) - f(x)}{dx}\right) + st\left(\frac{{}^*g(x+dx) - g(x)}{dx}\right) = f'(x) + g'(x).$$

5.3 Differentiation af en differens

Lad $f(x)$ og $g(x)$ være 2 differentiable funktioner, kald $h(x) = f(x) - g(x)$, så er

$$h'(x) = f'(x) - g'(x)$$

Bevis:
Lad $g_1(x) = -g(x)$, da vil $g_1'(x) = -g'(x)$, der er bare ganget konstanten -1 på, dermed fås

$$h'(x) = f'(x) + g_1'(x) = f'(x) - g'(x).$$

5.4 Differentiation af et multiplum af 2 funktioner

Lad $f(x)$ og $g(x)$ være 2 differentiable funktioner, kald $h(x) = f(x) \cdot g(x)$, dermed fås

$$h'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x)$$

Bevis:

$$h'(x) = st\left(\frac{{}^*h(x+dx) - h(x)}{dx}\right) = st\left(\frac{{}^*f(x+dx) \cdot {}^*g(x+dx) - f(x) \cdot g(x)}{dx}\right)$$

Her skal der bruges lidt tricks igen. Vi trækker noget fra, $g(x+dx)f(x)$, og lægger det til igen.

$$\begin{aligned}
h'(x) = st\left(\frac{{}^*f(x+dx){}^*g(x+dx) - f(x) \cdot g(x)}{dx}\right) = \\
&= st\left(\frac{{}^*f(x+dx){}^*g(x+dx) - {}^*g(x+dx)f(x) + {}^*g(x+dx)f(x) - f(x)g(x)}{dx}\right) \\
&= st\left(\frac{{}^*g(x+dx)\left({}^*f(x+dx) - f(x)\right) + f(x)\left({}^*g(x+dx) - g(x)\right)}{dx}\right) \\
&= st\left(\frac{{}^*g(x+dx)\left({}^*f(x+dx) - f(x)\right)}{dx}\right) + st\left(\frac{f(x)\left({}^*g(x+dx) - g(x)\right)}{dx}\right) \\
&= st\left({}^*g(x+dx)\right) \cdot st\left(\frac{\left({}^*f(x+dx) - f(x)\right)}{dx}\right) + f(x) \cdot st\left(\frac{\left({}^*g(x+dx) - g(x)\right)}{dx}\right) \\
&= g(x) \cdot st\left(\frac{\left({}^*f(x+dx) - f(x)\right)}{dx}\right) + f(x) \cdot st\left(\frac{\left({}^*g(x+dx) - g(x)\right)}{dx}\right) \\
&= g(x) \cdot f'(x) + f(x) \cdot g'(x).
\end{aligned}$$

Det grønne = gælder, da $g(x)$ er differentiabel og dermed kontinuert (VIGTIGT!).

5.5 Differentiation af sammensatte funktioner/kæderegel

Lad $f(x)$ og $g(x)$ være 2 differentiable funktioner, kald $h(x) = f\left(g(x)\right)$, dermed fås

$$h'(x) = f'\left(g(x)\right) \cdot g'(x)$$

Bevis:

Overvej først, at ${}^*g(x+dx) - g(x)$ er en infinitesimal, da funktionen er differentiabel, og dermed også kontinuert. Vi kalder ${}^*g(x+dx) - g(x) = dy$, dvs. ${}^*g(x+dx) = g(x) + dy$.

$$\begin{aligned}
h'(x) = st\left(\frac{{}^*h(x+dx) - h(x)}{dx}\right) = st\left(\frac{{}^*f\left({}^*g(x+dx)\right) - f\left(g(x)\right)}{dx}\right) \\
&= st\left(\frac{{}^*f\left(g(x) + {}^*g(x+dx) - g(x)\right) - f\left(g(x)\right)}{dx} \cdot 1\right) \\
&= st\left(\frac{{}^*f\left(g(x) + dy\right) - f\left(g(x)\right)}{dx} \cdot \frac{dy}{dy}\right) \\
&= st\left(\frac{{}^*f\left(g(x) + dy\right) - f\left(g(x)\right)}{dx} \cdot \frac{{}^*g(x+dx) - g(x)}{{}^*g(x+dx) - g(x)}\right) \\
&= st\left(\frac{{}^*f\left(g(x) + dy\right) - f\left(g(x)\right)}{{}^*g(x+dx) - g(x)} \cdot \frac{{}^*g(x+dx) - g(x)}{dx}\right) \\
&= st\left(\frac{{}^*f\left(g(x) + dy\right) - f\left(g(x)\right)}{dy}\right) st\left(\frac{{}^*g(x+dx) - g(x)}{dx}\right) = f'\left(g(x)\right) \cdot g'(x).
\end{aligned}$$

5.6 Differentiation af en kvotient af 2 funktioner

Lad $f(x)$ og $g(x)$ være 2 differentiable funktioner, kald $h(x) = \frac{f(x)}{g(x)}$, dermed fås

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Bevis:

For at vise dette kan man bruge de ovenstående regler. Vi bruger funktionen $p(x) = \frac{1}{x}$ og får, at

$$h(x) = f(x) \cdot \frac{1}{g(x)} = f(x) \cdot p\big(g(x)\big)$$

Her bruges reglen for et multiplum af 2 funktioner og differentiation af sammensat funktion.

$$h'(x) = f'(x) \cdot p\big(g(x)\big) + f(x) \cdot p'\big(g(x)\big) \cdot g'(x) = \frac{f'(x)}{g(x)} - f(x) \cdot \frac{g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Katte brug's t' now? Nix. Kun herre meget! Check it:

Eksempel 5.6.1

To funktioner er givet ved $f(x) = x^3$ og $g(x) = e^x$. Find den afledte til $f(x) \cdot g(x)$.

Okay cool, formlen siger, at $\big(f(x) \cdot g(x)\big)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$, så

$$\big(f(x) \cdot g(x)\big)' = 3x^2 \cdot e^x + x^3 \cdot e^x.$$

Vi kunne også finde den afledte til $\frac{f(x)}{g(x)}$, hvor formlen siger, at $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$, så

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{3x^2 \cdot e^x - x^3 \cdot e^x}{(e^x)^2} = \frac{3x^2 \cdot e^x - x^3 \cdot e^x}{e^{2x}}.$$

Eksempel 5.6.2

En funktion er givet ved $h(x) = \sqrt{3x+4}$. Find $h'(x)$.

Spa! Et godt fif er, at genkende den *indre* og den *ydre* funktion. Den ydre funktion $f(x)$ er $\sqrt{x}$, som vi godt kan differentiere, nemlig $f'(x) = \frac{1}{2\sqrt{x}}$ og den indre funktion $g(x)$ er $3x + 4$, som differentieret giver $g'(x) = 3$. Vi kan altså skrive $h(x)$ som $h(x) = f\big(g(x)\big)$. Så hver gang med ser noget andet/mere end x, der hvor der normalt står x, skal man bruge formlen $\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big) \cdot g'(x)$, så

$$h'(x) = \Big(f\big(g(x)\big)\Big)' = \frac{1}{2\sqrt{3x+4}} \cdot 3.$$

Eksempel 5.6.3

Den virker også her: En funktion er givet ved $h(x) = e^{4x^2-9}$. Find $h'(x)$.

Ydre funktion: $f(x) = e^x$ så $f'(x) = e^x$ og indre funktion: $g(x) = 4x^2 - 9$ så $g'(x) = 8x$ og vi får

$$h'(x) = e^{4x^2-9} \cdot 8x.$$

5.7 Opgaver

1. Givet to funktioner, $f(x) = 4x - 2{,}5$ og $g(x) = x^2$, lad $h(x) = f(x) + g(x)$. Find $h'(x)$.

2. Givet to funktioner, $f(x) = \frac{1}{x}$ og $g(x) = \sqrt{x}$, lad $h(x) = f(x) - g(x)$. Find $h'(x)$.

3. Bestem $h'(x)$ for funktionen $h(x) = -x^3 + 1{,}2e^x$

4. Givet to funktioner, $f(x) = 2\sqrt{x}$ og $g(x) = x^{5{,}9}$, lad $h(x) = f(x) \cdot g(x)$. Find $h'(x)$.

5. Givet to funktioner, $f(x) = -e^x$ og $g(x) = \frac{1}{x}$, lad $h(x) = f(x) \cdot g(x)$. Find $h'(x)$.

6. Bestem $h'(x)$ for funktionen $h(x) = 3x^7 \cdot e^x$

7. Bestem $f'(x)$ for funktionen $f(x) = \frac{3x^7}{e^x}$ og find $f'(-2)$

8. Differentier funktionen $f(x) = \frac{e^x}{\sqrt{x}}$

9. Bestem $g'(4)$ for funktionen $g(x) = \frac{\sqrt{x}}{x^2}$

10. Bestem $f'(x)$ for funktionen $f(x) = e^{5x}$

11. Find den afledte til funktionen $f(x) = \frac{1}{4x^2+2}$ ved at bruge regnereglen for sammensatte funktioner.

12. Find den afledte til funktionen $f(x) = \frac{1}{4x^2+2}$ ved at bruge kvotientreglen.

13. Differentier funktionen $f(x) = 6\sqrt{x^3 - \frac{1}{3}x^2 + \frac{1}{6}x - \frac{1}{12}}$ og bestem $f'(1)$.

14. Bestem $f'(9)$ for funktionen $f(x) = e^{\sqrt{x}}$

15. Differentier funktionen $f(x) = e^{e^x}$

16. Bevis hvad den afledte af $\ln(x)$ er.
    Hints: Lad $h(x) = f\big(g(x)\big) = e^{\ln(x)}$ og isoler $g'(x)$ i udtrykket for $h'(x)$. Husk $e^{\ln(x)} = x$.

17. Vis at den afledte til $f(x) = a^x$ er $f'(x) = \ln(a) \cdot a^x$.
    Hints: Lad $h(x) = f\big(g(x)\big) = \ln(a^x)$ og isoler $g'(x)$ i udtrykket for $h'(x)$. Husk $\ln(a^x) = x \cdot \ln(a)$

6 Monotoniforhold

At bestemme en funktions monotoniforhold svarer til at bestemme i hvilke intervaller, funktionen er voksende, og i hvilke, den er aftagende. Kender man monotoniforholdene, har man en idé om, hvordan grafen ser ud uden man behøver at tegne den. Differentialregning gør det meget lettere at bestemme monotoniforholdene.

Differentialkvotienten i et punkt er jo lig med tangentens hældning i det punkt, så derfor gælder der, at hvis differentialkvotienten er positiv i et punkt, vil tangenthældningen være positiv, og funktionen vil altså være voksende i det punkt. Hvis der er et interval, hvor differentialkvotienten er positiv i alle punkter, så må alle tangenthældningerne altså være positive, og funktionen er derfor voksende på hele intervallet. På samme måde vil et interval med negative differentialkvotienter give et interval, hvor funktionen aftager. Hvis differentialkvotienten er 0 i et interval, betyder det, at tangenthældningen er 0 (tangenten er vandret) og dermed er funktionen konstant på intervallet.

Lad os sammenfatte det

$$f'(x) > 0 \text{ for alle } x \in [a, b] \Rightarrow f \text{ voksende i } [a, b]$$

$$f'(x) < 0 \text{ for alle } x \in [a, b] \Rightarrow f \text{ aftagende i } [a, b]$$

$$f'(x) = 0 \text{ for alle } x \in [a, b] \Rightarrow f \text{ konstant i } [a, b]$$

6.1 Maksimum, minimum og vendetangent

Det første, man gør, når man skal bestemme monotoniforholdene for en funktion, er at differentiere funktionen og sætte den afledte lig med 0. Man løser altså ligningen

$$f'(x) = 0$$

De x-værdier, der løser denne ligning, er dem, hvor tangenten er vandret. Der er tre muligheder for, hvad disse punkter kan være. De kan være makismumspunkter, minimumspunkter eller vendetangentspunkter.

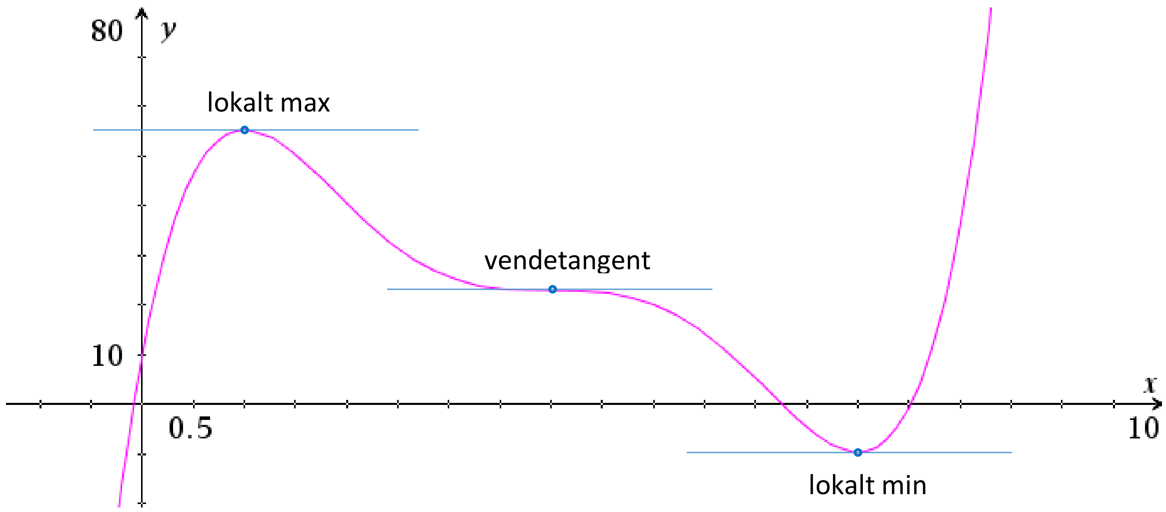
Imellem to punkter, hvor f' er 0 er den enten positiv på hele intervallet eller negativ på hele intervallet. Hvis den skulle skifte mellem at være positiv og negativ ville den jo være nødt til at passere 0.

Altså kan vi undersøge, om f' er positiv eller negativ i intervallerne mellem nulpunkterne ved bare at vælge et tilfældigt punkt i intervallet og se på fortegnet af f' i dette punkt.

Hvis f' er positiv til venstre og negativ til højre for et nulpunkt, så er der tale om et maksimum.

Hvis f' er negativ til venstre og positiv til højre for et nulpunkt, er der tale om et minimum.

Hvis f' har samme fortegn til venstre og højre, er der tale om en vendetangent.



Figur 6.1.1

Eksempel 6.1.1

Vi kunne godt tænke os at bestemme monotoniforholdene for funktionen

$$f(x) = -x^3 - 3x^2 + 2.$$

f er en differentiabel funktion, så vi starter med at differentiere den

$$f'(x) = -3x^{3-1} - 2 \cdot 3x^{2-1} + 0 = -3x^2 - 6x$$

Nu ville det være nice at finde de x-værdier, hvor $f'(x) = 0$.

$$f'(x) = 0 \Leftrightarrow$$

$$-3x^2 - 6x = 0 \Leftrightarrow$$

$$-3x(x + 2) = 0$$

Se lige hvor let det er at finde løsningerne nu. Nulreglen giver, at

$$x = -2 \vee x = 0.$$

I disse to punkter er tangenten altså vandret. Vi undersøger fortegnet for f' i intervallerne mellem dem. Det er nok bare at se på et vilkårligt tal i hvert interval.
Lad os starte med et tal mindre end -2. For eksempel -3! Det er mindre end -2.

$$f'(-3) = -3 \cdot (-3)^2 - 6 \cdot (-3) = -3 \cdot 9 + 6 \cdot 3 = -27 + 18 = -9 < 0.$$

Så ved vi jo, at når x er mindre end -2, er f' mindre end 0, altså

$$x < -2 \Rightarrow f'(x) < 0,$$

og hvad vigtigere er, at f er aftagende når $x \leq -2$, altså

f er aftagende når $x \in]-\infty; -2]$

Så kører vi noget ind der ligger mellem -2 og 0. Vi tager et tilfældigt tal i intervallet. Det kunne være -1:

$$f'(-1) = -3 \cdot (-1)^2 - 6 \cdot (-1) = -3 \cdot 1 + 6 = 3 > 0.$$

Så kan vi skrive

$$x \in [-2; 0] \Rightarrow f'(x) > 0,$$

og sagt på en anden måde,

f er voksende når $x \in [-2; 0]$.

Nu mangler vi kun at se på intervallet, hvor $x$er større end 0. Vi vælger et tilfældigt tal i dette interval. Det kunne jo være noget så cray som 1!

$$f'(1) = -3 \cdot 1^2 - 6 \cdot 1 = -3 - 6 = -3 < 0.$$

Vi kan derfor slutte med at slutte, at

$$x \in [0; \infty[\Rightarrow f'(x) < 0,$$

Og igen på en anden måde:

f er aftagende når $x \in [0; \infty[$.

Okay, så check det ud, monotoniforholdene for f er

f er aftagende i $]-\infty; -2]$ og i $[-2; 0]$.

f er voksende i $[0; \infty[$.

6.2 Monotonilinie

Vi kan tegne resultaterne ind i en monotonilinje.

Man tegner en tallinje. Ovenover den har man x, under den f' og f.

Først tegner man de x-værdier ind, hvor $f' = 0$. Man skriver derfor 0 ud for f' ved disse x-værdier. Dernæst indtegner man fortegnene for f' mellem disse værdier.
Til sidst tegner man pile alt efter, hvad det betyder for f. Under et plus tegner man en pil der går opad mod højre og under et minus tegner man en pil, der går nedad mod højre. Når man har tegnet pilene kan man se, hvad der er lokale maksima og minima, og hvad der er vendetangenter. Her er monotonilinjen tegnet skridt for skridt for eksemplet herover.

x	
f'	
f	

x	-2	0
f'	0	0
f		

x		-2		0	
f'	$-$	0	$+$	0	$-$
f					

x		-2		0	
f'	$-$	0	$+$	0	$-$
f	↘	Lokalt min	↗	Lokalt max	↘

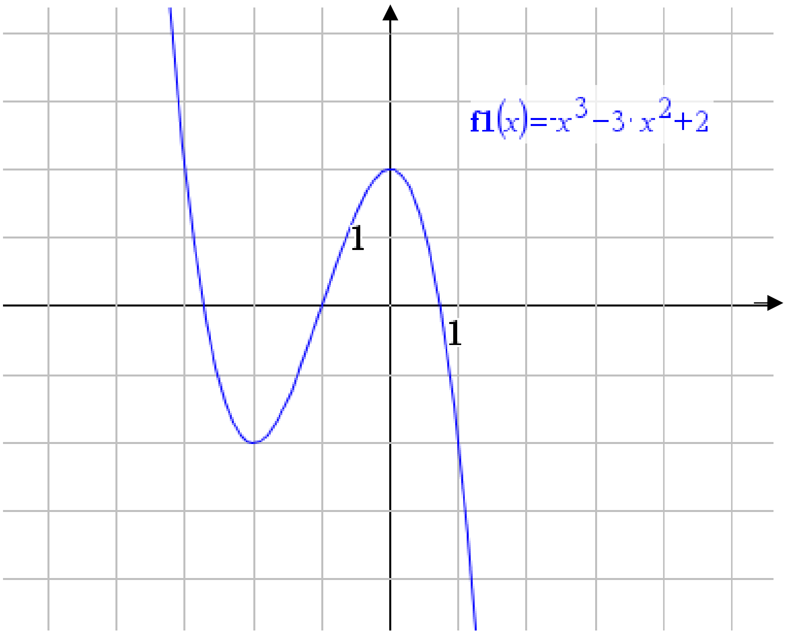
Man skal altid afslutte med at konkludere, hvordan monotoniforholdene er. I dette tilfælde ville man skrive:

f er aftagende i intervallerne $]-\infty; -2]$ og $[0; \infty[$.

f er voksende i intervallet $[-2; 0]$.

f har lokalt minimum i $(-2, f(-2))$ og lokalt maksimum i $(0, f(0))$.

Herunder er f tegnet, så man kan se, at det er det rigtige, man er nået frem til



Figur 6.2.1

Til slut fandt HyperMick og StandardJoe ud af, at alt det der monotoni fis, kunne koges ned til nogle specielle trin der var nødvendige at tage, for at finde monotoniforholdet for any function.

1. Differentier funktionen, altså find den afledte funktion til den givne (ofte givet som $f(x)$).
2. Løs ligningen $f'(x) = 0$, find altså steder hvor $f'(x)$ skærer første-aksen.
3. Bestem fortegnet for $f'(x)$ mellem nulpunkterne.
4. Tegn monotonilinie (den der med pile og plus og minus..)
5. Konkluder med tekst, ligesom der er gjort i eksemplet..

6.3 Opgaver

1. Undersøg om funktionen $f(x) = -x^2 + 4x$ har maksimum eller minimum i $x = 2$.

2. Undersøg om funktionen $f(x) = 2x^3 - 24x$ har maksimum eller minimum i $x = -2$ og $x = 2$.

3. Undersøg om funktionen $f(x) = 3x^2 + 5x - 1$ har et maksimum eller minimum og tegn en monotonilinie.

4. Bestem monotoniforhold for funktionen $f(x) = 5x^2 - 7x$.

5. Find monotoniforhold for funktionen $f(x) = e^x - x$.

6. Bestem monotoniforhold for funktionen $g(x) = -x^3 + 3x + 42$.

7 Tangentens ligning

Som overskriften nok hentyder, har vi her at gøre med en af de ting man kan bruge differentialkvotienten til, nemlig en tangentligning! Det viser sig, at det nogle gange kan være nyttigt at finde ligningen for tangenten til en funktion i et eller andet givet punkt.

Derfor er der en bestemt måde man kan gøre dette på.

Hvis vi skal gøre som der blev gjort i klassen da I fandt tangentens ligning, kan I anskue problemet således:

En tangent er en ret linie der skærer en funktion, $f(x)$, i et enkelt punkt og har samme hældning som funktionen i punktet. Ermergerd.

Dvs. for en differentiabel funktion, $f(x)$, findes forskriften for tangenten, $T(x) = ax + b$, i punktet $\left(x_0, f(x_0)\right)$ på følgende måde:

$$a = f'(x_0)$$

$$b = f(x_0) - f'(x_0)x_0$$

Altså er tangentens ligning:

$$T_{f(x_0)}(x) = ax + b = f'(x_0) \cdot x + f(x_0) - f'(x_0)x_0 = f'(x_0)(x - x_0) + f(x_0).$$

Det bør nok nævnes at man ofte bare skriver $y = f'(x_0)(x - x_0) + f(x_0)$ for tangentens ligning, HyperMick har bare et specielt forhold til udtrykket $T_{f(x_0)}(x)$.

Eksempel 6.3.1

Vi har givet funktionen $f(x) = x^2$ og skal finde tangenten i punktet $\left(4, f(4)\right)$, altså når $x = 4$.

Let's do dis:

Vi kunne gøre som vist ovenfor og finde a først og så b efter, men vi kan også bruge formlen. Der står, at vi skal have fat i $f'(x_0)$, x_0 og $f(x_0)$. Det er i øvrigt ALtid de 3 ting vi skal bruge. Lad os starte blødt og så køre hårdt på:

$x_0 = 4$ (det er pænt meget givet i opgaven, altså førstekoordinaten i punktet.. Se selv!)

$f(x_0)$ er altså at finde $f(4)$, så da $f(x) = x^2$ er $f(x_0) = f(4) = 4^2 = 16$. Yiiir!

$f'(x_0)$ kan først findes når man HAR differentieret, så $f'(x) = 2x$ og $f'(x_0) = f'(4) = 2 \cdot 4 = 8$. Stærkt.

De 3 ting indsættes, så udfra $y = f'(x_0)(x - x_0) + f(x_0)$ vi får

$$y = 8(x - 4) + 16 = 8 \cdot x + 8 \cdot (-4) + 16 = 8x - 32 + 16$$

så

(nu kommer den perfekte konklusion til sådan en opgave)

tangenten til grafen for funktionen $f(x) = x^2$ i punktet $\left(4, f(4)\right)$ er

$$y = 8x - 16.$$

7.1 Opgaver

1. Bestem ligningen for tangenten til grafen for $f(x) = x^3$ når $x = 3$.
2. Find ligningen for tangenten til grafen for $f(x) = 4x^4$ når $x = 2$.
3. Bestem ligningen for tangenten til grafen for $f(x) = 2\sqrt{x}$ når $x = 1$.
4. Find ligningen for tangenten til grafen for $f(x) = e^{2x} - 4x$ når $x = 2$.
5. Bestem ligningen for tangenten til grafen for $f(x) = \frac{2}{x} + 4x^2$ i punktet $\left(2, f(2)\right)$.
6. Find ligningen for tangenten til grafen for $f(x) = 5x^4 - 5e^x$ i punktet $\left(-2, f(-2)\right)$.
7. Hvad hedder hovedstaden i Malaysia?

Når alle opgaver indtil nu er løst, har man erhvervet sig titlen *hyper*, hvilket må sættes foran ens navn.

8 INTEGRALREGNING

8.1 Det bestemte integral

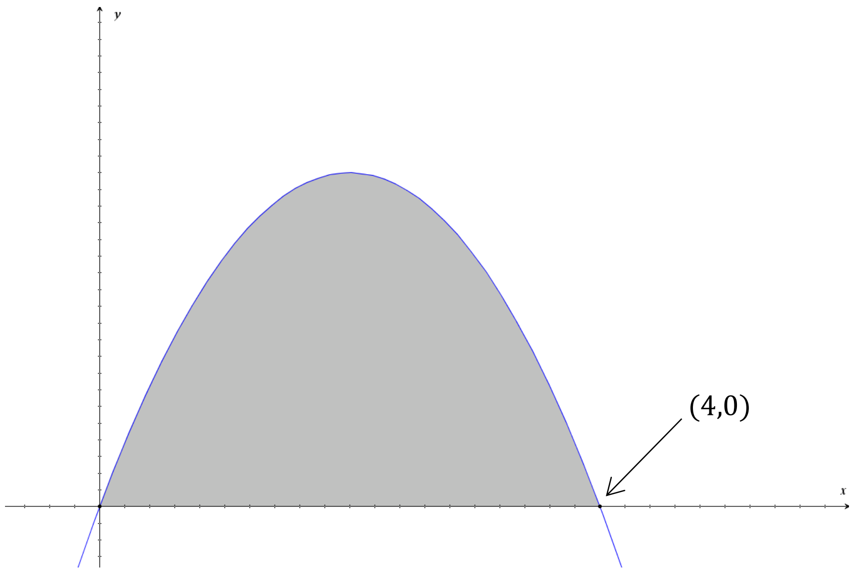
En alien kaldet Laera stiller sin gymnasielærer en opgave:

StandardJoe, I min familie skal vi holde en fest for min far Noitknufmats, han elsker både honning, safran og som alle andre aliens at slikke sit hus for at holde det vandafvisende. I den forlængelse har jeg tænkt mig at smøre en side af min fars hus ind i honning, for derefter at drysse safran ud over det, så vi i fælleskab i løbet af festen kan slikke huset rent. Da både honning og safran er en mangelvare på min planet, ville jeg gerne undgå at købe for meget af det, mit spørgsmål er derfor; hvis jeg har følgende funktion der beskriver hussidens højde over planetens overflade,

$$f(x) = -x^2 + 4x.$$

Hvordan finder jeg så arealet af husets side?

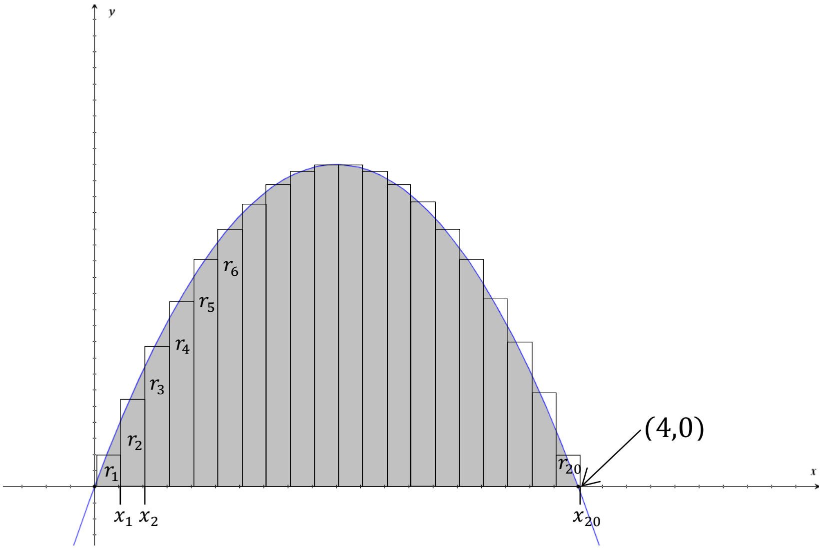
Jeg har i den forbindelse medbragt et billede af min fars hus:



Figur 8.1.1

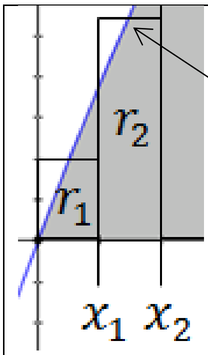
For at besvare dette spørgsmål, må vi først finde ud af hvordan man finder arealer. Et areal af et rektangel er givet som $Areal = højde \cdot bredde$, men hussiden som den er beskrevet og tegnet er ikke et rektangel, derfor må vi se hvad vi kan gøre.

Det vides at hussidens bredde er 4, altså kunne et første estimat af husets areal være højden ganget med bredden, men højden er forskellig for alle x-værdierne, derfor inddeles figuren i nogle rektangler. På denne måde kan arealet af rektanglerne findes og det vil være tæt på at være det samme areal som selve hussiden. På figuren er der indtegnet nogle x-værdier, og rektanglerne er blevet benævnt $r_1, r_2, \ldots r_{20}$.



Figur 8.1.2

Arealet af r_2 kan beskrives som bredden gange højden, lad os derfor kigge lidt nærmere på dette rektangel og dets skæring med funktionen der beskriver hussidens højde.

 $(x_{r_2}, f(x_{r_2}))$ *Figur 8.1.3*	Hvilken værdi er denne x_{r_2}? Lad os sige at denne skæring er præcis i midten af rektanglet, dvs. $x_{r_2} = x_1 + \frac{x_2 - x_1}{2}$. Hvilket giver en højde på r_2 på $f(x_{r_2}) = f\left(x_1 + \frac{x_2 - x_1}{2}\right)$	Derfor vil arealet af r_2, A_{r_2}, være $A_{r_2} = f\left(x_1 + \frac{x_2 - x_1}{2}\right) \cdot (x_2 - x_1)$. På samme måde kan arealet af r_1 findes $A_{r_1} = f\left(0 + \frac{x_1 - 0}{2}\right) \cdot (x_1 - 0)$, og arealet af r_3 findes $A_{r_3} = f\left(x_2 + \frac{x_3 - x_2}{2}\right) \cdot (x_3 - x_2)$. Og for de andre rektangler kan arealet findes på samme måde.

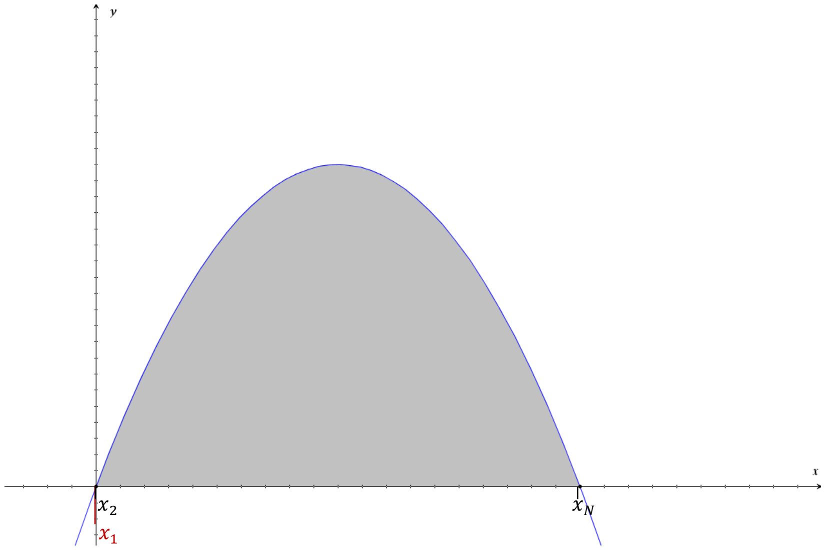
Ved at lægge alle disse rektanglers areal sammen vil et areal der er tæt på hussidens areal altså findes. Vi kan altså skrive at arealet af hussiden, A, er tæt på denne sum:

$$A \approx A_{r_1} + A_{r_2} + A_{r_3} + A_{r_4} + A_{r_5} + A_{r_6} + A_{r_7} + A_{r_8} + A_{r_9} + A_{r_{10}} + A_{r_{11}} + A_{r_{12}} + A_{r_{13}} + A_{r_{14}} + A_{r_{15}} + A_{r_{16}} + A_{r_{17}} + A_{r_{18}} + A_{r_{19}} + A_{r_{20}}$$

for at spare på pladsen skrives dette med nogle prikker, $\cdots$, som forstås ved at fortsætte det system der er at finde, dvs.

$$A \approx A_{r_1} + A_{r_2} + A_{r_3} + \cdots + A_{r_{20}}.$$

Dette areal er altså ret tæt på at være det helt rigtige, men det er stadig ikke det helt rigtige. Ved at dele hussiden op i flere rektangler med smallere bredde, kan man komme tættere på det rigtige areal. Derfor laves der uendeligt mange rektangler. Nu kender vi til infinitesimaler og uendelige tal, så lad os kalde dette uendelige (an)tal for N, altså er N et hyperreelt uendeligt tal. Der indsættes nogle x-værdier, $x_1, x_2, x_3, \dots, x_N$ på førsteaksen (farven er indsat for at vise at punkterne ligger uendeligt tæt).



Figur 8.1.4

For at bestemme arealerne af de uendeligt mange rektangler, $r_1, r_2, \dots, r_N$, skal højden og bredden findes.

Igen tages rektanglet r_2 som eksempel og bredden og højden findes, bredden er $x_2 - x_1 = dx$ og højden er, ligesom ved de 20 rektangler, ${}^*f\left(x_1 + \frac{x_2 - x_1}{2}\right) = {}^*f\left(x_1 + \frac{dx}{2}\right)$ bestemmes på samme måde som ved den endelige inddeling i 20 rektangler. Derfor er

$$A_{r_1} = {}^*f\left(0 + \frac{x_1 - 0}{2}\right) \cdot (x_1 - 0) = {}^*f\left(0 + \frac{dx}{2}\right) dx$$

og

$$A_{r_2} = {}^*f\left(x_1 + \frac{x_2 - x_1}{2}\right) \cdot (x_2 - x_1) = {}^*f\left(x_1 + \frac{dx}{2}\right) dx$$

$$\begin{aligned} A &\approx A_{r_1} + A_{r_2} + A_{r_3} + \dots + A_{r_N} \\ &= {}^*f\left(0 + \frac{dx}{2}\right) dx + {}^*f\left(x_1 + \frac{dx}{2}\right) dx + {}^*f\left(x_2 + \frac{dx}{2}\right) dx + \dots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right) dx \end{aligned}$$

Denne sum er uendeligt tæt på det rigtige areal, men det rigtige areal er jo ikke et hyperreelt tal, derfor tages standard delen til summen, hvilket giver følgende resultat.

$$A = st\left({}^*f\left(0 + \frac{dx}{2}\right) dx + {}^*f\left(x_1 + \frac{dx}{2}\right) dx + {}^*f\left(x_2 + \frac{dx}{2}\right) dx + \dots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right) dx\right)$$

Den kloge lærer, StandardJoe er ikke bare ferm, men MEGET ferm, til at lægge sådanne summer sammen, så han har udregnet dette for os og fik resultatet $A = \frac{32}{3}$. For lang tid siden da HyperMick og StandardJoe endnu ikke var til, fandt man på at denne sum var lidt besværlig at skrive, så man tænkte, lad os skrive det sådan lidt dovent.

$$A = \int_0^4 f(x)dx = \int_0^4 -x^2 + 4x\, dx.$$

Denne sum er uendelig og derfor er det kun StandardJoe der kan lægge alle tallene sammen, vi andre vælger derfor at kalde det noget andet: *det bestemte integral*. Man udtaler det ovenstående som, integralet fra 0 til 4 af $f(x)$.

Denne måde at finde området mellem en funktion og førsteaksen på et interval gælder også for andre funktioner. Hvilket er begrundelsen for følgende definition.

Definition 8.1.1 - Det bestemte integral

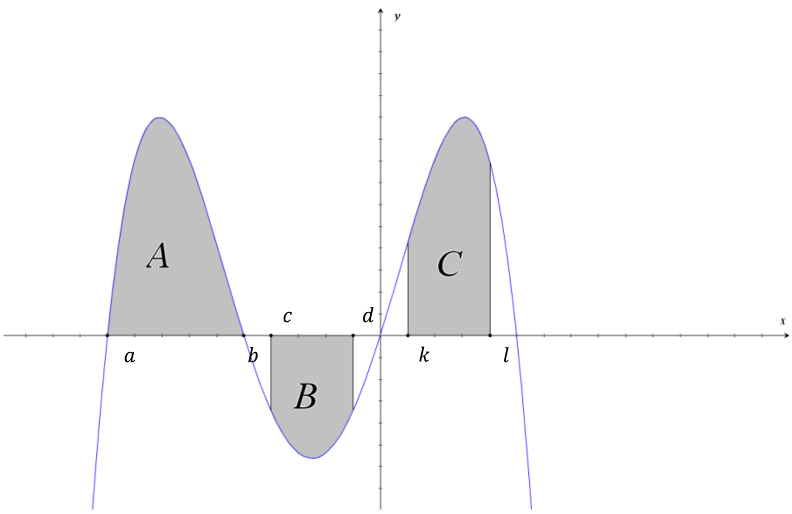
> Det bestemte integral fra a til b af $f(x)$ er defineret som den uendelige sum af infinitesimale rektanglers areal mellem en funktion, $f(x)$, og førsteaksen, på et interval, $[a,b]$. Integralet skrives
>
> $$\int_a^b f(x)dx = st\left({}^*f\left(a+\frac{dx}{2}\right)dx + {}^*f\left(x_1+\frac{dx}{2}\right)dx + {}^*f\left(x_2+\frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1}+\frac{dx}{2}\right)dx\right),$$
>
> hvor $a < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$, er en infinitesimal inddeling af intervallet $[a,b]$ og $N \in {}^*\mathbb{R}$ er et uendeligt tal.

Laera ser straks at, med denne definition så vil integralet af en funktion der ligger under førsteaksen på intervallet $[a,b]$, blive et negativt tal, hvilket også er sandt.

En funktion kaldes integrabel hvis den uendelige sum i definitionen af integralet er et endeligt tal.

Eksempel 8.1.1

Nedenfor ses grafen for en funktion $f(x)$



Figur 8.1.5

Her kan de forskellige områders areal skrives som et integral, dvs.

$$A = \int_a^b f(x)\,dx, \qquad = -\int_c^d f(x)\,dx, \qquad C = \int_k^l f(x)\,dx.$$

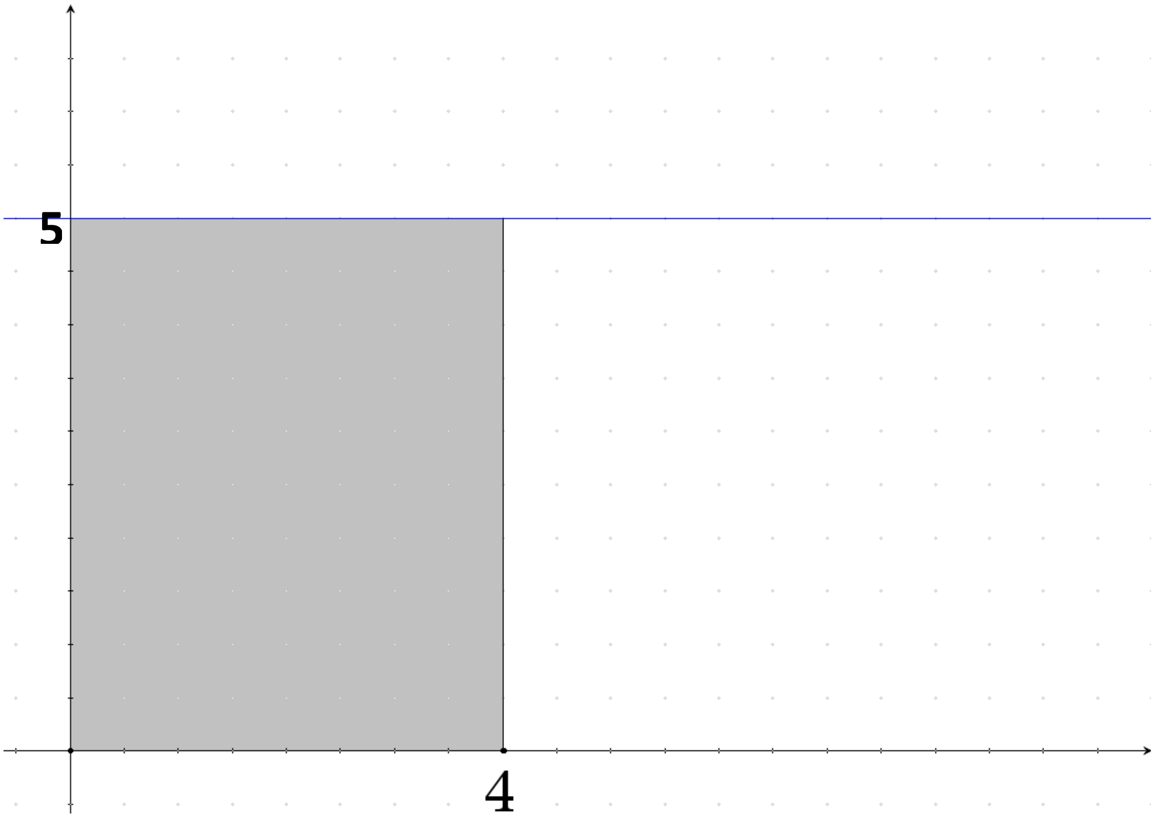
Læg mærke til det røde minus foran integralet der beskriver arealet af B, dette skyldes at området ligger under førsteaksen og derfor bliver integralet negativt, minusset gør derfor at det bliver positivt, hvilket arealer altid er.

8.2 Arealfunktioner

HyperMick og de fleste andre dødelige er ikke i stand til at lægge uendeligt mange tal sammen for at finde resultatet af et integral. Derfor søges der en anden måde at bestemme integralets værdi på. Derfor kigges der på integralet af funktionen $f(x) = 5$ mellem nul og fire. Dvs.

$$\int_0^4 f(x)\,dx = \int_0^4 5\,dx$$

Funktionen $f(x) = 5$ ser således ud i et koordinatsystem



Figur 8.2.1

Det er altså det grå område der søges arealet af.

Det vides at højden er 5, dvs. arealet mellem grafen $f(x)$ og førsteaksen, fra nul til fire er (højde gange bredde)

$$A = 5 \cdot 4 = 20.$$

Dette areal er det samme areal som integralet beskriver, derfor er

$$\int_0^4 f(x)\,dx = \int_0^4 5\,dx = 5 \cdot 4 = 20$$

Kigger vi nu på arealet mellem funktionen, førsteaksen på intervallet $[0{,}8]$, så giver regnestykket

$A = 5 \cdot 8 = 40$, hvilket betyder at integralet fra nul til otte er

$$\int_0^8 f(x)\,dx = \int_0^8 5\,dx = 5 \cdot 8 = 40$$

Hvad sker der egentlig hvis vi kigger på et interval der ikke starter i nul? Lad os kigge på et helt generelt interval af typen $[a, b] \in \mathbb{R}$. Så bliver arealet

$$A = 5 \cdot (b - a)$$

Derfor er integralet fra a til b

$$\int_a^b f(x)\,dx = \int_a^b 5\,dx = 5 \cdot (b - a)$$

Dvs. for enhver konstant funktion, $f(x) = c$, kan vi beskrive arealet under den, på intervallet $[a, b]$, som

$$A = f(x) \cdot (b - a) = c \cdot (b - a)$$

Det ser rimelig besværligt ud i forhold til vi bare ville finde et areal af et rektangel siger den "kloge" elev.

Det er det egentlig også, men vi ville jo gerne prøve at finde arealet af den der aliens husside. Lad os lige opsummere hvad vi har indtil nu.

Så arealet mellem første-aksen og en konstant funktion $f(x) = 5$ over et interval $(0, x)$ kan skrives som en funktion, hvis vi lader endepunktet (b) i intervallet være en variabel (x).

$$A(x) = f(x) \cdot (x - 0) = 5 \cdot x.$$

Her er altså tale om integralet fra nul til x, dvs.

$$A(x) = \int_0^x f(x)\,dx = \int_0^x 5\,dx = 5 \cdot x.$$

8.2.1 Areal mellem en konstant funktion og førsteaksen på intervallet $[\boldsymbol{a}, \boldsymbol{x}]$

For en konstant funktion $f(x) = c \geq 0$, kan arealet udspændt mellem funktionen og førsteaksen på intervallet $[a, x] \in \mathbb{R}$ findes som

$$A(x) = \int_a^x f(x)\,dx = f(x) \cdot (x - a) = c \cdot (x - a).$$

Det kan derfor siges at arealfunktionen der beskriver arealet mellem en funktion og førsteaksen er det samme som integralet hvis den konstante funktion er positiv.

Hvis den konstante funktion er negativ, $f(x) = c \leq 0$, vil arealet fundet med ovenstående formel blive negativt, men der er ingen negative arealer. Derfor er arealfunktionen af en negativ konstant funktion

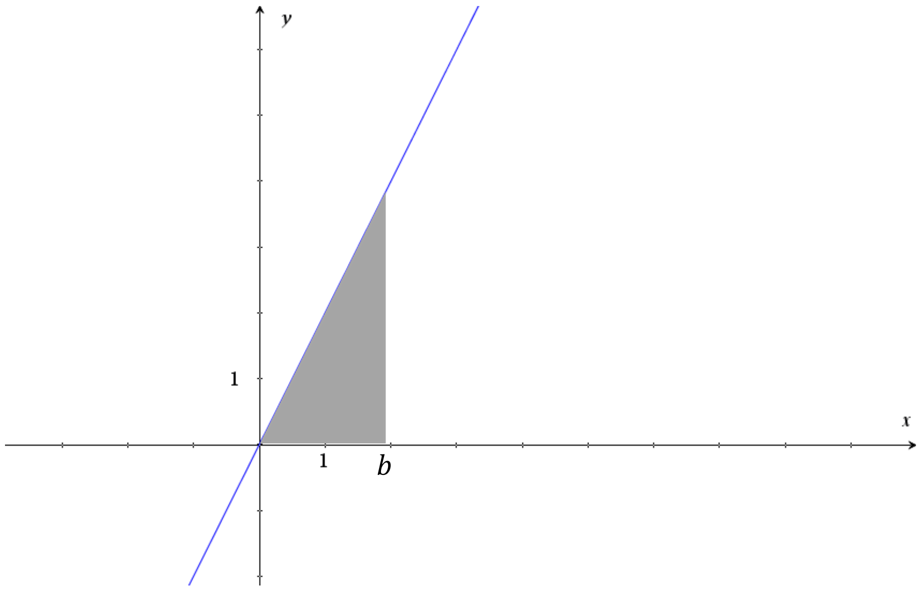
$$A(x) = -\int_a^x f(x)\,dx.$$

På denne måde tages der højde for at integralet af en funktion der ligger under førsteaksen er negativt.

Nu kigges der på en anden funktion, $f(x) = 2x$, arealet mellem grafen for denne funktion og førsteaksen på intervallet $[0, b]$ søges. Da denne funktion ligger over førsteaksen fra nul til b, er der tale om integralet

$$\int_0^b f(x)\,dx = \int_0^b 2x\,dx.$$

Nedenfor er $f(x) = 2x$ tegnet i et koordinatsystem



Figur 8.2.2

Der søges derfor "igen" efter arealet af det grå felt. Vi ved fra trekantsregning at dette areal kan findes som $A = \frac{1}{2} \cdot højde \cdot grundlinie$. I dette tilfælde, skal vi derfor finde højden og grundlinien for at finde arealet af trekanten. Lader vi grundlinien være stykket i bunden af trekanten dvs. længden af intervallet $[0, b]$. Vil $grundlinie = b - 0 = b$. Herefter kan vi lade højden være funktionsværdien i x-værdien b, dvs. $højden = f(b) = 2b$. Nu indsætter vi i formlen for arealet og finder en formel der udtrykker arealet mellem funktion og førsteakse over intervallet $(0, b)$.

$$A = \frac{1}{2} \cdot højde \cdot grundlinie = \frac{1}{2} \cdot f(b) \cdot (b - 0) = \frac{1}{2} \cdot 2b \cdot b = b^2.$$

Derfor er integralet

$$\int_0^b f(x)\, dx = \int_0^b 2x\, dx = \frac{1}{2} \cdot 2b \cdot b = b^2$$

Hvilket giver begrundelsen for følgende

8.2.2 Areal mellem grafen for en lineær funktion og førsteaksen på et interval $[\boldsymbol{a}, \boldsymbol{x}]$

For en lineær funktion, $f(x) = kx$, hvor $k \geq 0$ kan arealet under grafen findes til ethvert interval $[0, x]$, ved følgende arealfunktion

$$A(x) = \int_0^x f(x)\, dx = \frac{1}{2} f(x) \cdot (x - 0) = \frac{k}{2} x^2.$$

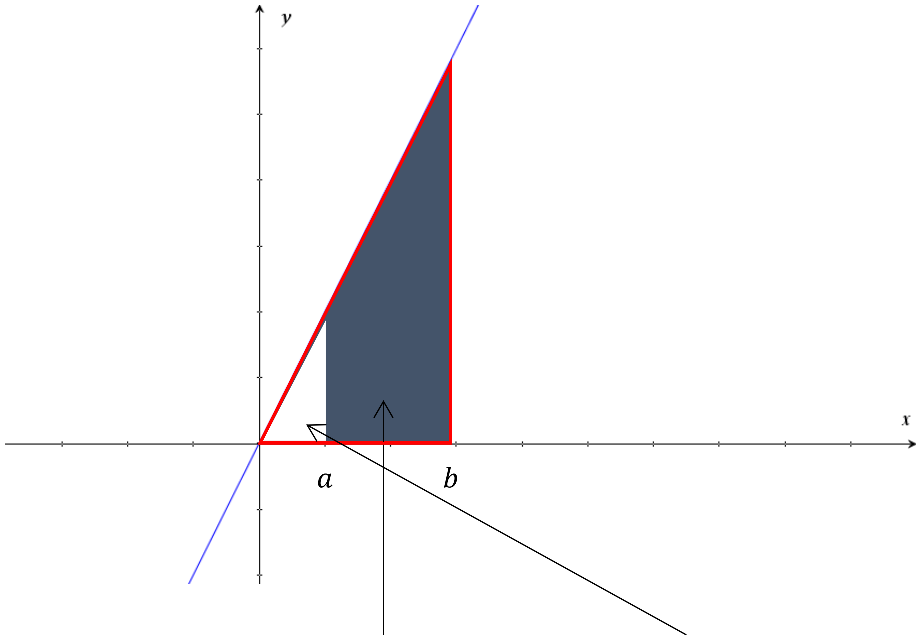
Hvis $k \leq 0$ i ovenstående funktion er integralet negativt, da funktionen derfor ligger under førsteaksen, derfor er arealfunktionen i dette tilfælde

$$A(x) = -\int_0^x f(x)\, dx = -\frac{1}{2} f(x) \cdot (x - 0) = -\frac{k}{2} x^2.$$

på denne måde sikres det at arealet er positivt. ☺

Hvis intervallet ikke starter i nul, bliver det til noget lidt andet. Vi søger derfor integralet af denne type

$$\int_a^b f(x)\,dx = \int_a^b 2x\,dx$$



Figur 8.2.3

Denne figur, kan ses som én stor trekant (det grå område plus den lille hvide trekant), hvor den lille hvide trekant skal trækkes fra igen. Med formlen fra før, kan arealet af de to trekanter findes og trækkes fra hinanden dvs.

$$A = \frac{1}{2}\cdot f(b)\cdot b - \frac{1}{2}\cdot f(a)\cdot a = \frac{1}{2}\cdot 2b\cdot b - \frac{1}{2}\cdot 2a\cdot a = b^2 - a^2$$

Derfor er

$$\int_a^b f(x)\,dx = \int_a^b 2x\,dx = b^2 - a^2$$

Hvis vi kigger på en generel lineær funktion af typen, $f(x) = kx$, findes udtrykket for arealet under grafen på et interval af typen $[a, b]$, på følgende måde

$$A = A(b) - A(a) = \frac{1}{2}f(b)\cdot b - \frac{1}{2}f(a)\cdot a = \frac{1}{2}(kbb - kaa) = \frac{k}{2}(b^2 - a^2)$$

Hvilket medfører at integralet af $f(x) = kx$ fra a til b er

$$\int_a^b f(x)\,dx = \int_a^b kx\,dx = \frac{k}{2}(b^2 - a^2)$$

Ved at sammenholde de ovenstående arealfunktioner, kan arealet under en hvilken som helst lineær funktion findes, der gælder igen at hvis det søgte område ligger over førsteaksen er arealet lig med integralet, hvorimod hvis, området ligger under førsteaksen er arealet lig med minus integralet, for at sørge for at arealet hele tiden er positivt. Dette giver den ophav til den følgende arealfunktion.

8.2.3 Areal mellem førsteaksen og en lineær funktion på et interval $[\boldsymbol{a},\boldsymbol{x}]$

For en lineær funktion, $f(x) = kx + c$ kan arealet under grafen findes til ethvert interval (a, x), hvor $kx + c \geq 0$ i hele intervallet, ved følgende funktion:

$$A(x) = \int_a^x f(x)\,dx = \int_a^x kx + c\,dx = \frac{k}{2}(x^2 - a^2) + c \cdot (x - a).$$

Her bliver det lidt besværligt når man skal sørge for at det er et positivt areal☺. Hvis funktionen er sådan at den er negativ i noget af det interval man søger arealet i og positivt i et andet, må intervallet deles op. Hvis vi siger at den lineære funktion skærer førsteaksen i $x = b$ og den har positiv hældningskoefficient, så bliver arealfunktionen således:

$$A(x) = -\int_a^b f(x)\,dx + \int_b^x f(x)\,dx = -\left(\frac{k}{2}(b^2 - a^2) + c \cdot (b - a)\right) + \frac{k}{2}(x^2 - b^2) + c \cdot (x - b)$$
$$= \frac{k}{2}(x^2 + a^2 - 2b^2) + c \cdot (x - 2b + a)$$

hvis hældningskoefficienten er negativ findes arealfunktionen således:

$$A(x) = \int_a^b f(x)\,dx - \int_b^x f(x)\,dx$$

9 Regneregler

9.1 Sum/differens af funktioner

Integralet af en sum eller differens af to integrable funktioner, $f(x) \pm g(x)$, er lig med en sum eller differens af integralet af de to funktioner. Dvs.

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Bevis:
For at vise dette indsættes definitionen af integralet for den venstre del af ligheden.

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx$$
$$= st\left({}^*f\left(a + \frac{dx}{2}\right)dx \pm {}^*g\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx \pm {}^*g\left(x_1 + \frac{dx}{2}\right)dx + \cdots \right.$$
$$\left. + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx \pm {}^*g\left(x_{N-1} + \frac{dx}{2}\right)dx\right).$$

Dernæst ordnes den uendelige sum, så alle f'erne står først.

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx$$
$$= st\left(\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx\right.\right.$$
$$\left.\left. \pm \left({}^*g\left(a + \frac{dx}{2}\right)dx + {}^*g\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*g\left(x_{N-1} + \frac{dx}{2}\right)dx\right)\right).$$

Da $f(x)$ og $g(x)$ begge er differentiable funktioner er de uendelige summer i ovenstående udtryk endelige. Dvs. vi kan bruge reglen $st(a+b) = st(a) + st(b)$. Derfor bliver

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx$$
$$= st\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx\right)$$
$$\pm st\left({}^*g\left(a + \frac{dx}{2}\right)dx + {}^*g\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*g\left(x_{N-1} + \frac{dx}{2}\right)dx\right)$$
$$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

9.2 Sum af integraler

Integralet fra a til b lagt sammen med integralet fra b til c er lig med integralet fra a til c, når integranden er den samme. Dvs.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Bevis:
For at vise dette indsættes definitionen af det bestemte integral, der skal derfor bruges to uendelige inddelinger af intervaller, en inddeling for intervallet $[a, b]$ og en for intervallet $[b, c]$, lad derfor inddelingen for intervallet $[b, c]$ være

$$x_N = b < x_{N+1} < x_{N+2} < x_{N+3} < \cdots < x_M = c.$$

Så bliver inddelingen af intervallet $[a, c]$

$$x_0 = a < x_1 < x_2 < x_3 < \cdots x_N < x_{N+1} < x_{N+2} < x_{N+3} < \cdots < x_M = c.$$

Hvilket gør at summen af de to integraler kan skrives som

$$\begin{aligned}
&\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \\
&\quad = st\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx\right) \\
&\quad + st\left({}^*f\left(x_N + \frac{dx}{2}\right)dx + {}^*f\left(x_{N+1} + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{M-1} + \frac{dx}{2}\right)dx\right) \\
&\quad = st\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx + {}^*f\left(x_N + \frac{dx}{2}\right)dx \right. \\
&\quad \left. + {}^*f\left(x_{N+1} + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{M-1} + \frac{dx}{2}\right)dx\right) \\
&\quad = st\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{M-1} + \frac{dx}{2}\right)dx\right) = \int_a^c f(x)\,dx.
\end{aligned}$$

9.3 Konstant ganget på

Integralet af en konstant, $k \in \mathbb{R}$, ganget på en funktion, $f(x)$, er det samme som konstanten ganget med integralet af funktionen. Dvs.

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

Bevis:
For at vise dette bruges igen definitionen hvilket giver

$$\begin{aligned}
\int_a^b k \cdot f(x)\,dx &= st\left(k^*f\left(a + \frac{dx}{2}\right)dx + k^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + k^*f\left(x_{N-1} + \frac{dx}{2}\right)dx\right) \\
&= st\left(k\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx\right)\right)
\end{aligned}$$

Da k er et reelt tal kan det komme udenfor standarddelen ($st(k \cdot a) = k \cdot st(a)$). Dermed er

$$\int_a^b k \cdot f(x)\,dx = k \cdot st\left({}^*f\left(a + \frac{dx}{2}\right)dx + {}^*f\left(x_1 + \frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1} + \frac{dx}{2}\right)dx\right) = k \cdot \int_a^b f(x)\,dx.$$

9.4 Integral af et punkt

Integralet over et interval af typen $[a, a]$, altså kun indeholdende et reelt punkt, er nul. Dvs.

$$\int_a^a f(x)\,dx = 0$$

Bevis:
For at vise dette er det egentlig kun en betragtning af intervallet der skal til. Intervallet der integreres over er $[a, a]$, dvs. bredden af dette er $a - a = 0$. Arealet mellem funktionen og førsteaksen på et interval med nul bredde er lig med $f(a) \cdot 0 = 0$, derfor er

$$\int_a^a f(x)\,dx = 0.$$

9.5 Integral fra *a* til *b* eller *b* til *a*

Integralet hvor der er byttet om på grænserne er minus det oprindelige integral. Dvs.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Bevis:
Først bruges regnereglen fra 9.2 og dernæst reglen fra 9.4 til at finde det søgte resultat.

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0.$$

Dvs.

$$\int_a^b f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx - \int_b^a f(x)\,dx = 0 - \int_b^a f(x)\,dx = -\int_b^a f(x)\,dx.$$

9.6 Areal af Noitknufmats' husside

Med areal funktionerne og de bestemte integraler fra det ovenstående opdager Laera at når hun differentierer de bestemte integraler over intervaller af typen $[a, x]$ ender hun op med den oprindelige funktion. Hun opskriver en tabel, som er gengivet her under.

$f(x)$	$\int_a^x f(x)\,dx$	$\left(\int_a^x f(x)\,dx\right)'$
c	$c \cdot (x - a)$	c
kx	$\frac{k}{2}(x^2 - a^2)$	kx

Efter at have lavet denne tabel, kommer hun til at tænke på nogle af reglerne fra differentialregning, her tænker hun især på produktreglen, som siger at for $l(x) = h(x) \cdot g(x)$ så er

$$l'(x) = \big(h(x) \cdot g(x)\big)' = h'(x)g(x) + h(x)g'(x)$$

Med produktreglen og skemaet i tankerne, bliver

$$\int_0^x \big(h'(x)g(x) + h(x)g'(x)\big)\,dx = h(x) \cdot g(x)$$

med dette i tankerne kigger hun på den funktion der beskriver højden af Noitknufmats hus:

$$f(x) = -x^2 + 4x$$

hun sætter x udenfor en parentes og finder:

$$f(x) = x(-x + 4).$$

Her er det hun spørger sin kloge lærer StandardJoe; "hvordan skal jeg komme videre, jeg synes der er en sammenhæng her, men kan ikke helt finde den, kan du hjælpe mig?

StandardJoe, giver hende følgende hints i et skema:

$h(x) =$	$h'(x) = x$
$g(x) = -x + 4$	$g'(x) =$

Med dette lille skema, der ikke er helt udfyldt går Laera i gang med at udfylde det. Og finder at

$$g'(x) = -1$$

og med hjælp fra hendes første skema med den afledte af integralet, finder hun frem til

$$h(x) = \int_0^x h'(x)\,dx = \int_0^x x\,dx = \frac{1}{2}(x^2 - 0^2) = \frac{1}{2}x^2.$$

Laera udfylder skemaet, med rødt (hun havde ikke andet at skrive med)

$h(x) = \frac{1}{2}x^2$	$h'(x) = x$
$g(x) = -x + 4$	$g'(x) = -1$

Laera ser at $f(x) = x(-x+4) = h'(x) \cdot g(x)$, med dette tænker Laera at hun prøver at skrive det ind i formlen for produktreglen, hun finder derfor frem til følgende:

$$\big(h(x) \cdot g(x)\big)' = h'(x)g(x) + h(x)g'(x) = f(x) + h(x)g'(x)$$

Hvilket medfører at

$$h(x) \cdot g(x) = \int_0^x h'(x)g(x) + h(x)g'(x)\,dx = \int_0^x f(x) + h(x)g'(x)\,dx = \int_0^x f(x)\,dx + \int_0^x h(x)g'(x)\,dx.$$

Derfor bliver

$$\int_0^x f(x)\,dx = h(x) \cdot g(x) - \int_0^x h(x)g'(x)\,dx$$

Nu indsætter Laera sine resultater fra skemaet og finder

$$\int_0^x f(x)\,dx = h(x) \cdot g(x) - \int_0^x -\frac{1}{2}x^2(-1)\,dx = h(x) \cdot g(x) + \int_0^x \frac{1}{2}x^2\,dx.$$

Laera mener stadig det ser fjollet ud og sætter derfor udtrykkende for funktionerne, $h(x)$ og $g(x)$ ind.

$$f(x) = \left(\frac{1}{2}x^2(-x+4)\right)' + \frac{1}{2}x^2 = \left(-\frac{1}{2}x^3 + 2x^2\right)' + \frac{1}{2}x^2$$

$$\int_0^x f(x)\,dx = \int_0^x \frac{1}{2}x^2(-x+4)\,dx + \int_0^x \frac{1}{2}x^2\,dx = \int_0^x -\frac{1}{2}x^3 + 2x^2\,dx + \int_0^x \frac{1}{2}x^2\,dx$$

"Ej helt ærligt StandardJoe, det var da slet ikke nogen hjælp med det du gav mig" ☹ siger Laera. Men StandardJoe mener at hun er på rette vej og giver hende endnu et hint:

$$\frac{1}{2}x^2 = -\frac{1}{2}f(x) + 2x.$$

Dette indsætter Laera straks i sit udtryk for integralet af $f(x)$:

$$f(x) = \left(-\frac{1}{2}x^3 + 2x^2\right)' - \frac{1}{2}x^2(-1) = \left(-\frac{1}{2}x^3 + 2x^2\right)' - \frac{1}{2}f(x) + 2x$$

$$\int_0^x f(x)\,dx = -\frac{1}{2}x^3 + 2x^2 + \int_0^x \left(\frac{1}{2}f(x) + 2x\right)dx = -\frac{1}{2}x^3 + 2x^2 + \frac{1}{2}\int_0^x f(x)\,dx + \int_0^x 2x\,dx$$

Ved at ordne dette udtryk kommer Laera frem til:

$$\frac{3}{2}\int_0^x f(x)\,dx = -\frac{1}{2}x^3 + 2x^2 + \int_0^x 2x\,dx,$$

hvilket giver

$$\int_0^x f(x)\,dx = \frac{2}{3}\left(-\frac{1}{2}x^3 + 2x^2\right) + \frac{4}{3}\int_0^x x\,dx$$

men integralet $\int_0^x x\,dx = \frac{1}{2}x^2$, dvs.

$$\int_0^x f(x)\,dx = \frac{2}{3}\left(-\frac{1}{2}x^3 + 2x^2\right) + \frac{4}{3}\left(\frac{1}{2}x^2\right) = \frac{2}{3}\left(-\frac{1}{2}x^3 + 2x^2 + x^2\right) = \frac{2}{3}\left(-\frac{1}{2}x^3 + 3x^2\right)$$
$$= -\frac{1}{3}x^3 + 2x^2.$$

Det ovenstående udtryk beskriver arealet mellem funktionen og førsteaksen, hvor arealerne under førsteaksen vil blive trukket fra. Det er før blevet fastslået at arealet kunne beskrives som

$$\int_0^4 f(x)\,dx,$$

Derfor indsættes tallet fire på x's plads, hvilket vil sige.

$$\int_0^4 f(x)\,dx = \int_0^4 -x^2 + 4x\,dx = -\frac{1}{3}4^3 + 2\cdot 4^2 = -\frac{1}{3}64 + 2\cdot 8 = -\frac{64}{3} + \frac{32\cdot 3}{3} = \frac{-64+96}{3} = \frac{32}{3}.$$

Dette er altså arealet af Noitknufmats husside, hvilket også var det som StandardJoe kom frem til, derfor må det være sandt☺.

Fra det ovenstående og med brug af regnereglerne for integralet, kan følgende udledes

$$\int_0^x f(x)\,dx = \int_0^x -x^2 + 4x\,dx == \int_0^x -x^2\,dx + \int_0^x 4x\,dx = -\int_0^x x^2\,dx + 4\int_0^x x\,dx = -\frac{1}{3}x^3 + 2x^2.$$

Hvilket betyder at integralet

$$\int_0^x x^2\,dx = \frac{1}{3}x^3$$

Der søges efter et resultat af typen

$$\int_a^x x^2\,dx.$$

Dette kan skrives som (ved hjælp af regnereglen for sum af integraler)

$$\int_a^x x^2\,dx = \int_0^x x^2\,dx - \int_0^a x^2\,dx = \frac{1}{3}x^3 - \frac{1}{3}a^3.$$

9.7 Opgaver

1. Vis, at integralet af en sum eller differens af funktioner er lig med en sum eller differens af integralet af de pågældende funktioner. Dvs. vis at

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

2. Vis, at integralet af en konstant, $k \in \mathbb{R}$, ganget på en funktion, $f(x)$, er det samme som konstanten ganget med integralet af funktionen. Dvs. vis at

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

3. Vis, at integralet fra a til b lagt sammen med integralet fra b til c er lig med integralet fra a til c, når integranten er den samme. Dvs. vis følgende.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

4. Vis, at integralet over et interval af typen $[a, a]$, altså kun indeholdende et reelt punkt, er nul. Dvs. vis at

$$\int_a^a f(x)\,dx = 0$$

5. Vis, at det bestemte integral skifter fortegn, når der byttes om på grænserne. Dvs. vis at

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

6. Opskriv formler for følgende integraler og beskriv hvad kan der siges om formlernes afledte funktioner:

$$\int_0^x c\,dx \qquad \int_0^x kx\,dx \qquad \int_0^x (kx + c)\,dx \qquad \int_0^x x^2\,dx$$

10 Integration og differentiation

Indtil videre er der blevet etableret følgende sammenhænge mellem integralet af typen $\int_a^x f(x)\,dx$ og integralets afledte:

$f(x)$	$\int_a^x f(x)\,dx$	$\left(\int_a^x f(x)\,dx\right)'$
c	$c \cdot (x-a)$	c
kx	$\frac{k}{2}(x^2-a^2)$	kx
x^2	$\frac{1}{3}(x^3-a^3)$	x^2

Tabel 10.1

Det kunne, se ud som om der var en sammenhæng mellem den afledte og integralet, og at denne skulle være, de var hinandens modsatte.

10.1 Den afledte af integralet fra a til x

Givet en kontinuert funktion $f(x)$, gælder der

$$\left(\int_a^x f(x)\,dx\right)' = f(x)$$

10.2 Integralet af den afledte funktion fra a til b.

Givet en differentiabel funktion $f(x)$, gælder der at

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Tilsammen kaldes disse 2 sætninger for analysens fundamentalsætning, beviset for denne vil ikke blive givet i dette kompendie. Med analysens fundamentalsætning er sammenhængen mellem differentialregning og integralregning fuldstændigt etableret. Med dette er det derfor muligt at bruge alle de regler man fandt ved differentialregning til at finde de modsatte udtryk, dvs. hvis en integrand (funktionen der skal integreres) kan genkendes som den afledte af en funktion så kan analysens fundamentalsætning bruges til at bestemme integralets værdi. Denne måde at genkende en integrand som en afledt funktion af en anden er faktisk det man ville kalde at finde en stamfunktion.

10.3 Stamfunktion

Definition 10.3.1 – Stamfunktion

> Givet en funktion $f(x)$, så er $F(x)$ en stamfunktion for $f(x)$, hvis
>
> $$F'(x) = f(x).$$

Med denne definition er det muligt at bestemme et bestemt integrals værdi, hvis en stamfunktion til integranden er kendt (se 10.2). Dvs. hvis $F(x)$ er en stamfunktion for $f(x)$, så vil

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

10.4 Integration og sammensatte funktioner

Det vides nu at bestemme stamfunktioner og at differentiere er hinandens modsatte operationer (se 10). Derfor kigges der også på regnereglerne for afledte funktioner. I dette tilfælde tænkes der især på regnereglen for sammensatte funktioner, dvs.

$$\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right)g'(x)$$

Integreres dette udtryk fra a til b fremkommer

$$\int_a^b \left(f\left(g(x)\right)\right)' \, dx = \int_a^b f'\left(g(x)\right)g'(x)\, dx$$

Bruges den anden del af analysens fundamentalsætning (se 10.2) findes følgende udtryk:

$$\int_a^b \left(f\left(g(x)\right)\right)' \, dx = \int_a^b f'\left(g(x)\right)g'(x)\, dx = f\left(g(b)\right) - f\left(g(a)\right).$$

Med denne metode, kan integraler af denne type løses.

Eksempel 10.4.1

Bestem værdien af integralet

$$\int_0^1 e^{2x^3} \cdot 6x^2 \, dx$$

Der søges et udtryk der ligner det der står med blåt, dvs. $f(x)$ og $g(x)$ skal bestemmes således at $f\left(g(x)\right) \cdot g'(x) = e^{2x^3} \cdot 6x^2$. Måden at gøre dette på er derfor at lede efter den indre funktion $g(x)$.

Den indre funktion er $g(x) = 2x^3$, hvilket gør $g'(x) = 2 \cdot 3x^{3-1} = 6x^2$.

Den ydre funktion er $f(x) = e^x$.

Nu er $f\left(g(x)\right) \cdot g'(x) = e^{2x^3} \cdot 6x^2$, som ønsket, derfor er

$$\int_0^1 e^{2x^3} \cdot 6x^2 \, dx = f\left(g(1)\right) - f\left(g(0)\right) = f(2) - f(0) = e^2 - e^0 = e^2 - 1.$$

Hvad sker der hvis nu det ikke helt står som det med blåt? Dvs.

$$\int_a^b f\left(g(x)\right)g'(x)\, dx$$

her kan vi desværre <u>ikke</u> bare se resultatet som en sammensat funktion $f\left(g(x)\right)$. Hvis en funktion opfylder at den afledte er lig med $f(x)$, så ville dette kunne sættes ind i stedet og så ville man kunne bruge reglen fra før. dvs. lad $F(x)$ være således at $F'(x) = f(x)$ (dermed er $F(x)$ en stamfunktion til $f(x)$), så kan det ovenstående integral skrives som

$$\int_a^b f\left(g(x)\right)g'(x)\, dx = \int_a^b F'\left(g(x)\right)g'(x)\, dx.$$

Resultatet af dette kan igen findes ved hjalp af analysens fundamentalsætning (se 10.2), hvilket giver

$$\int_a^b f(g(x))g'(x)\,dx = \int_a^b F'(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

Det grønne udtryk i linien over, kan ses som integralet af $F'(x) = f(x)$ fra $g(a)$ til $g(b)$. Dvs.

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} F'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx.$$

Dermed bliver

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx = F(g(b)) - F(g(a)).$$

Dette er et korrekt udtryk, omen lidt svært at bruge, da både $f(x)$, $g(x)$ og $F(x)$, skal findes inden det kan bruges, derfor er der lavet en slags opskrift, der gør det muligt at gøre det samme som lige er gennemgået, opdelt i forskellige trin.

10.5 Integration ved substitution

..er metoden der bliver brugt for at dele udregningen af integralet op i trin. Det første trin er at genkende den indre funktion, $g(x)$, i det integral der skal bestemmes; altså i

$$\int_a^b f(g(x))g'(x)\,dx.$$

1. Find den indre funktion i integranden og kald den, $u = g(x)$.
2. Udregn $g(a)$ og $g(b)$.
3. Bestem $du = g'(x)dx$.
4. Erstat grænserne a og b med $g(a)$ og $g(b)$ samtidig med **en** af de 2 nedenstående:
    a. Isolér dx i udtrykket for du og indsæt dette i stedet for dx i integralet.
    b. Genkend $g'(x)dx$ i integralet og erstat dette med du.
5. Find en stamfunktion til $f(u)$, hvor u ses som en variabel, og indsæt (de nye) grænser.

Lad os se hvordan det ser ud hvis 1-5 bliver udført på følgende.

Eksempel 10.5.1

Bestem værdien af følgende integral

$$\int_0^1 \frac{e^x + 8x^3}{2\sqrt{e^x + 2x^4}}\,dx$$

1. Den indre funktion findes og kaldes $u = g(x) = e^x + 2x^4$
2. Dernæst udregnes $g(0)$ og $g(1)$, $g(0) = e^0 + 2 \cdot 0^4 = 1$ og $g(1) = e^1 + 2 \cdot 1^4 = e + 2$.
3. Dernæst bestemmes $du = g'(x)dx = (e^x + 2x^4)'dx = (e^x + 8x^3)dx$.
4. Her bruges først metode 4.a.
    a. Ved isolering af dx findes; $dx = \frac{1}{g'(x)}du = \frac{1}{e^x+8x^3}du$. Dvs.

$$\int_0^1 \frac{e^x + 8x^3}{2\sqrt{e^x + 2x^4}} dx = \int_{g(0)}^{g(1)} \frac{e^x + 8x^3}{2\sqrt{u}} \frac{1}{e^x + 8x^3} du = \int_1^{e+2} \frac{1}{2\sqrt{u}} du$$

b. Her genkendes $du = (e^x + 8x^3)dx$ i integralet, dvs.

$$\int_0^1 \frac{e^x + 8x^3}{2\sqrt{e^x + 2x^4}} dx = \int_1^{e+2} \frac{1}{2\sqrt{u}} du$$

5. Nu findes der en stamfunktion til $\frac{1}{2\sqrt{u}}$. Da $\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}}$ vælges stamfunktionen $F(x) = \sqrt{x}$. Dernæst indsættes (de nye) grænser, dvs.

$$\int_0^1 \frac{e^x + 8x^3}{2\sqrt{e^x + 2x^4}} dx = \int_1^{e+2} \frac{1}{2\sqrt{u}} du = \sqrt{e+2} - \sqrt{1} = \sqrt{e+2} - 1.$$

Eksempel 10.5.2

Prøv at køre den lidt mere flydende. Vi skal finde integralet herunder

$$\int_0^{\frac{1}{2}} (4x^3 + 2x^2)^{10} \cdot (12x^2 + 4x) dx$$

dvs. vi skal finde arealet under grafen for funktionen $f(x) = (4x^3 + 2x^2)^{10} \cdot (12x^2 + 4x)$, i intervallet $\left[0; \frac{1}{2}\right]$. Tricket er, at finde et sted i funktionen hvor det havde været lettere, hvis der kun stod x i stedet for alt mulig andet. Det havde for eksempel været en del nemmere hvis $f(x)$ var $x^{10} \cdot (12x^2 + 4x)$, for skulle man ikke sætte $4x^3 + 2x^2$ i tiende! Så vi substituerer (erstatter) $4x^3 + 2x^2$ med en ny variabel, lad os kalde den u, så $g(x) = u = 4x^3 + 2x^2$. Men hvis vi bare smider u ind og regner videre går det lige så stille sindssygt galt – Butterfly effect! Så for at rette op på det, finder vi nye grænser, så de passer med u'et og ikke $x'et$:

$$g(0) = 4 \cdot 0^3 + 2 \cdot 0^2 = 0$$

$$g\left(\frac{1}{2}\right) = 4 \cdot \left(\frac{1}{2}\right)^3 + 2 \cdot \left(\frac{1}{2}\right)^2 = 4 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

Da vi nu gerne vil integrere et udtryk med u, må vi finde du også, ud fra $df = f'dx\ du = (3 \cdot 4x^{3-1} + 2 \cdot 2x^{2-1})dx$ så $du = (12x^2 + 4x)dx$.

Så.. Ind ind ind ind ind ind!

$$\int_0^1 (u)^{10} du.$$

Her har vi en dejlig regel der siger, at når vi integrerer x^a får vi $\frac{1}{a+1}x^{a+1}$, fordi hvis vi differentierer $\frac{1}{a+1}x^{a+1}$ får vi $(a+1) \cdot \frac{1}{a+1}x^{a+1-1} = x^a$. Nooice. Så når vi bruger formlen $\int_a^b f(x)dx = F(b) - F(a)$ fås

$$\int_0^1 (u)^{10} du = \frac{1}{10+1} \cdot 1^{10+1} - \frac{1}{10+1} \cdot 0^{10+1} = \frac{1}{11} \cdot 1^{11} - \frac{1}{11} \cdot 0^{11} = \frac{1}{11} \cdot 1 - 0 = \frac{1}{11}.$$

Nu har vi fundet arealet under grafen og over førsteaksen, som er $\frac{1}{11}$. Prrap!

Eksempel 10.5.3

Vi vil gerne finde arealet mellem grafen for funktionen $f(x) = -\frac{2x}{(x^2+3)^2}$, i intervallet $[1; \sqrt{5}]$, og førsteaksen, så vi skal finde

$$\int_1^{\sqrt{5}} -\frac{2x}{(x^2+3)^2} dx.$$

Det der sker er, at $f(x)$ er en sammensat funktion, så vi identificerer den indre og kalder den u. Søge søge.. Dér! Den indre må være $x^2 + 3$. Vi får altså $g(x) = u = x^2 + 3$ og dermed $du = 2xdx$. Så vi kan indsætte du alle de steder der står $2xdx$. Det gør der umiddelbart kun 1 sted, men inden vi fyrer det ind i udtrykket, skal vi huske at lave grænserne om. Man kan regne dem ud først:

$$g(1) = 1^2 + 3 = 1 + 3 = 4$$

$$g(\sqrt{5}) = \sqrt{5}^2 + 3 = 5 + 3 = 8$$

SÅÅÅÅÅÅÅ!

$$\int_4^8 -\frac{du}{(u)^2} = \int_4^8 -\frac{1}{u^2} du.$$

Vi er imidlertid så hjernedødt heldige, at vi ved, at den afledte til $\frac{1}{x}$ er $-\frac{1}{x^2}$, så stamfunktionen til $-\frac{1}{x^2}$ må være $\frac{1}{x}$. Hvis vi bruger formlen $\int_a^b f(x)dx = F(b) - F(a)$ får vi

$$\int_4^8 -\frac{1}{u^2} du = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}.$$

Så arealet mellem grafen og førsteaksen må være den positive version, altså $\frac{1}{8}$.

10.6 Opgaver

1. Bestem en stamfunktion $F(x)$ til $f(x) = x^3 + 2x$

2. Bestem en stamfunktion $F(x)$ til $f(x) = e^x + 4$

3. Bestem en stamfunktion $F(x)$ til $f(x) = \frac{1}{2\sqrt{x}}$

4. Bestem en stamfunktion $F(x)$ til $f(x) = -\frac{1}{x^2}$

5. Vis, at $F(x) = e^{3x^2}$ er stamfunktion til $f(x) = e^{3x^2} \cdot 6$, ved at differentiere $F(x)$

6. Vis, at $F(x) = \frac{1}{x} + \sqrt{x+42} - e^x$ er stamfunktion til $f(x) = -\frac{1}{x^2} + \frac{1}{2\sqrt{x+42}} - e^x$

7. Vis, at $F(x) = \frac{2}{3}x^{\frac{3}{2}}$ er stamfunktion til $f(x) = \sqrt{x}$

8. Bestem værdien af nedenstående integraler, ved at finde en stamfunktion og indsætte grænserne i

$$\int_1^2 16x^{-2} + 3x^2\,dx \qquad \int_{\frac{1}{2}}^1 24x^2\,dx \qquad \int_{16}^{64} -\frac{1}{\sqrt{x}}dx \qquad \int_4^{16} \sqrt{x} - \frac{x}{16}dx$$

Den sidste kræver nok lommeregner, efter grænserne er sat ind (Challenge: kan godt løses uden)

9. Løs integralet

$$\int_0^{\sqrt{3}} -\frac{4x}{(2x^2+2)^2}dx$$

10. Løs integralet

$$\int_0^{\frac{1}{3}} (3x^3 + 8x^2)^8 \cdot (9x^2 + 16x)dx$$

11. Løs integralet

$$\int_{\frac{1}{4}}^{12} -\frac{e^{\frac{1}{x}}}{x^2}dx$$

12. Løs integralet

$$\int_0^2 \frac{x^3 + \frac{3}{2}x}{\sqrt{\frac{1}{4}x^4 - \frac{3}{4}x^2}}dx$$

11 Integraler uden grænser

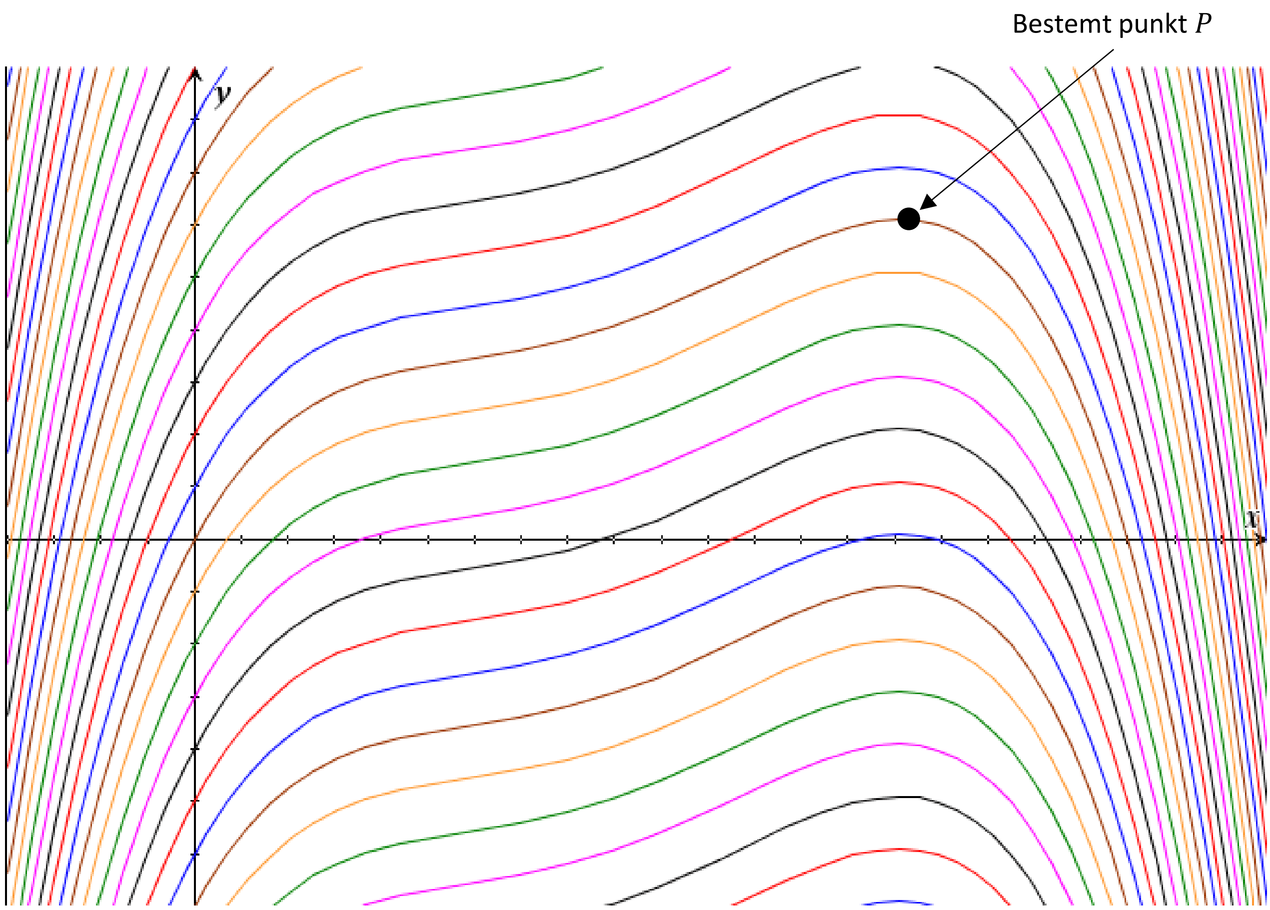
For ikke altid at skulle skrive i ord, at der søges en stamfunktion, er der blevet indført det der kaldes det *ubestemte integral*.

11.1 Det ubestemte integral

Definition 11.1.1 – Ubestemt integral

> Det ubestemte integral er den operation der udføres når stamfunktioner findes, dvs. hvis $F(x)$ er en stamfunktion for $f(x)$, så er det ubestemte integral af $f(x)$ lig med $F(x) + c$, hvor c er en konstant, dette skrives
>
> $$\int f(x)\,dx = F(x) + c$$

Laera er helt sat af.. Hvorfor er der nu en konstant..? Yo, listen up, here's the story, about a little girl, som ikke fattede noget. Konstanten er der fordi sådan som stamfunktioner er defineret, så vil både $F(x) + c$ og $F(x)$ være stamfunktioner, hvis den ene af dem er det! Der er altså uendeligt mange tal man kan sætte ind for c og forskellige værdier giver forskellige grafer. Hvis man får givet et bestemt punkt til gengæld, får man en unik graf, da der kun vil være én af de uendeligt mange grafer der går gennem det bestemte punkt. Se Figur 11.1.1 herunder.



Figur 11.1.1

På Figur 11.1.1 er tegnet en masse stamfunktioner $F(x) + c$ til $f(x)$, men der er kun én eneste graf der går gennem P.

Måske et eksempel ville få det til at gå ned. Da ba dee da ba die.

Eksempel 11.1.1

Find en stamfunktion til $f(x) = 2x$.

Fra Tabel 10.1 kan det ses at man kan vælge $k = 2$ for funktionen af typen kx, for at opnå funktionen $f(x) = 2x$. I den samme figur kan aflæses en stamfunktion

$$g(x) = x^2 - a^2$$

Da HyperMick godt kan lide simple løsninger vælges $a = 0$, for derved at opnå den simpleste stamfunktion til $f(x) = 2x$, kald denne

$$F(x) = x^2.$$

Eksempel 11.1.2

Hvad er det ubestemte integral af funktionen $f(x) = x^2$. Dvs. hvad er

$$\int x^2\,dx.$$

Det ubestemte bestemmes ved at finde alle stamfunktionerne, dette gøres ved at finde frem til den ”simpleste” stamfunktion og dernæst lægge en konstant c til. Fra Tabel 10.1 kan der igen aflæses en stamfunktion, igen sættes $a = 0$ for at opnå den ”simpleste” stamfunktion

$$F(x) = \frac{1}{3}x^3.$$

Dermed bliver det ubestemte integral

$$\int x^2\,dx = F(x) + c = \frac{1}{3}x^3 + c.$$

Eksempel 11.1.3

Bestem det ubestemte integral af funktionen $f(x) = x^a$, hvor $a \in \mathbb{R}$ og $a \neq -1$. Dvs. bestem

$$\int x^a\,dx.$$

For at komme frem til en stamfunktion, kan man opskrive reglen for at differentierer en funktion af denne type. Hvis $g(x) = x^k$, så er $g'(x) = kx^{k-1}$. Der skal altså gøres det modsatte af at differentieres, derfor lægges der 1 til eksponenten. Dermed er et første bud på en stamfunktion $g(x) = x^{a+1}$, den afledte er derfor $g'(x) = (a+1)x^k$. Den endelige stamfunktion til $f(x) = x^a$ er derfor $F(x) = \frac{1}{a+1}x^{a+1}$. Dermed bliver det ubestemte integral

$$\int x^a\,dx = F(x) + c = \frac{1}{a+1}x^{a+1} + c.$$

Eksempel 11.1.4 (det bedste!)

Når man er så heldig, at man skal finde en stamfunktion der går gennem et punkt, skal man have styr på noget ligningsløsning. Let's begin.

Vi starter med en funktion vi vil finde stamfunktionen til, så.. $f(x) = 3x^2 + 4x - 1$. Vi benytter os af det ubestemte integral:

$$\int (3x^2 + 4x - 1)dx = \frac{1}{3} \cdot 3x^3 + \frac{1}{2} \cdot 4x^2 - x + c = x^3 + 2x^2 - x + c.$$

Well, lookie here, lookie here ah what do we have? Vi har nu fundet en af de uendeligt mange stamfunktioner der er til $f(x)$, men vi vil gerne finde den unikke stamfunktion til $f(x)$ som går gennem punktet $(2,16)$, det vil sige, at vi har en x-værdi og en y-værdi, som vi kan sætte ind:

$$16 = 2^3 + 2 \cdot 2^2 - 2 + c \Leftrightarrow$$

$$16 = 8 + 2 \cdot 4 - 2 + c \Leftrightarrow$$

$$16 = 8 + 8 - 2 + c \Leftrightarrow$$

$$16 = 14 + c \Leftrightarrow$$

$$c = 2.$$

Nu kan vi opstille stamfunktionen til f der går gennem punktet $(2,16)$, nemlig $F(x) = x^3 + 2x^2 - x + 2$.

11.2 Opgaver

1. Bestem 2 forskellige stamfunktioner til $f(x) = x^4$

2. Bestem den stamfunktion $F(x)$, til $f(x) = 2x^3$ der går gennem punktet $(2,6)$

3. Bestem den stamfunktion $F(x)$, til $f(x) = \frac{1}{2\sqrt{x}}$ der går gennem punktet $(9,4)$

4. Bestem den stamfunktion $F(x)$, til $f(x) = e^{3x}$ hvor $F(1) = 5$, altså i punktet $(1,5)$

5. Bestem den stamfunktion $F(x)$, til $f(x) = -\frac{8}{x^2}$ hvor $F(2) = 4$

6. Løs integralet
$$\int (5 + 6x^2)dx$$
og find den stamfunktion der går gennem punktet $(2,-16)$

Når alle opgaverne er løst, har man erhvervet sig titlen *alien*, hvilket må sættes foran ens navn. Også bagved, faktisk.. Det lyder også meget frækt!

12 Facitliste

Løsninger til kapitel 1

1. $x + y$
2. $5x$
3. $5x$
4. 0
5. 0
6. 0
7. N/A. Man kan ikke tage standard delen til et uendeligt tal
8. $st\left(5dx \cdot \frac{1}{dx}\right) = st\left(5 \cdot \frac{dx}{dx}\right) = st(5 \cdot 1) = 5$
9. 0
10. 6
11. 6
12. Standard delen til et produkt er det samme som produktet af standard delen til faktorerne
13. N/A. Man kan ikke tage standard delen til et uendeligt tal
14. Reglen fra opg. 12 gælder, hvis ingen faktorer er uendelige tal
15. 16
16. $^*f(1 + dx) = 4 + 3dx$ og $^*f(dx - 2) = -5 + 3dx$
17. $^*g(3 - dx) = -3 + dx$ og $^*g(0) = 0$
18. $st\big(^*h(1 + dx)\big) = 2$ og $st\big(^*h(4 + dx)\big) = 128$
19. $f(x)$
20. $f(x)$

Oooog til kapitel 2

1. 3 og -1
2. 2
3. 2
4. -4
5. $f(0) = -3$ er grafens y-værdi, når $x = 0$, og $f'(0) = -2$ er hældningen for tangenten til grafen i $(0, -3)$

Det her, det' til kapitel 3

1. $\frac{2}{3}$
2. 4,001
3. $f(3) = 3^2 - 7 = 9 - 7 = 2$ og $f(4) = 4^2 - 7 = 16 - 7 = 9$.
   Sekanthældning= 7
4. 6,1
5. 6,001
6. 6,00001
7.

Δx	1	0,1	0,001	0,00001
a	7	6,1	6,001	6,00001

Hældningen bliver 6.

8. $f'(3) = 6$
9. $f(-1) = (-1)^2 - 7 = 1 - 7 = -6$. $f'(1) = 2$

	x	-3	-1	0	1	4
10.	$g(x)$	1	1	1	1	1
	$g'(x)$	0	0	0	0	0
11.	$h(x)$	12	4	0	-4	-16
	$h'(x)$	-4	-4	-4	-4	-4
12.	$k(x)$	18	2	0	2	32
	$k'(x)$	-12	-4	0	4	16
13.	$l(x)$	31	7	1	-1	17
	$l'(x)$	-16	-8	-4	0	12

14. I know, right!? ☺

15. $h'(x) = f'(x) + g'(x)$

Kapitel 4, værsgo

1. 0
2. $f'(x) = 42$
3. $f'(x) = 7x^6$
4. $f'(x) = 8x^7$ og $f'(1) = 8$
5. $f'(x) = -5x^{-6}$
6. $f'(x) = -7x^{-8}$
7. $f'(x) = 3{,}6x^{2{,}6}$ og $f'(2) = 3{,}6 \cdot 2^{2{,}6} \approx 21.826$
8. $f'(x) = -0{,}8x^{-1{,}8}$
9. $f'(x) = 0$
10. $f'(x) = -\frac{1}{x^2}$ og $f'(-5) = -\frac{1}{25}$
11. $f'(x) = 3e^x$ og $f'(2{,}5) = 3 \cdot e^{2{,}5}$ ≈36.548, hvilket er hældningen til funktionen $f(x)$ i punktet $(2{,}5; f(2{,}5))$.
12. $f'(x) = -\frac{2}{x^2} - 6 \cdot e^x$
13. $f'(x) = -9 - 42x^5 - \frac{5}{2\sqrt{x}} - 4e^x - \frac{6}{x^2}$

Kapitel 5, boom

1. $h'(x) = 4 + 2x$
2. $h'(x) = -\frac{1}{x^2} + \frac{1}{2\sqrt{x}}$
3. $h'(x) = -3x^2 + 1{,}2e^x$
4. $h'(x) = -\frac{1}{\sqrt{x}}x^{5{,}9} + 2\sqrt{x} \cdot 5{,}9x^{4{,}9} = -x^{5{,}4} + 11{,}8x^{5{,}4} = 10{,}8x^{5{,}4}$
5. $h'(x) = -e^x \cdot \frac{1}{x} + e^x \cdot \frac{1}{x^2} = e^x(x^{-2} - x^{-1})$
6. $h'(x) = 21x^6e^x + 3x^7e^x$
7. $f'(x) = 21x^6e^{-x} - 3x^7e^{-x} = e^{-x}(21x^6 - 3x^7)$
8. $f'(x) = \frac{e^x}{\sqrt{x}} - e^x 0{,}5x^{-1{,}5} = \frac{e^x}{\sqrt{x}} - \frac{e^x}{2x\sqrt{x}}$
9. $g'(x) = -1{,}5x^{-2{,}5} = -\frac{3}{2}\frac{\sqrt{x}}{x^3}$. Dvs. $g'(4) = -\frac{3}{2}\frac{2}{4^3} = -\frac{3}{4^3} = -\frac{3}{64}$.

10. $f'(x) = 5e^{5x}$
11. $f'(x) = -\frac{8x}{(4x^2+2)^2} = -\frac{8x}{16x^4+4+16x^2}$
12. $f'(x) = -\frac{8x}{(4x^2+2)^2} = -\frac{8x}{16x^4+4+16x^2}$
13. $f'(x) = 6 \cdot \left(3x^2 - \frac{2}{3}x + \frac{1}{6}\right) \cdot \frac{1}{2\sqrt{x^3-\frac{1}{3}x^2+\frac{1}{6}x-\frac{1}{12}}} = \frac{18x^3-4x+1}{2\sqrt{x^3-\frac{1}{3}x^2+\frac{1}{6}x-\frac{1}{12}}}$ dvs.

$$f'(1) = \frac{15}{2\sqrt{\frac{9}{12}}} = \frac{15}{2\sqrt{\frac{3}{4}}} = \frac{15}{\sqrt{3}} = \frac{\sqrt{3}^2 \cdot 5}{\sqrt{3}} = 5 \cdot \sqrt{3}$$

14. $f'(x) = -\frac{e^{\sqrt{x}}}{2\sqrt{x}}$, dvs. $f'(9) = -\frac{e^3}{6} \approx -3{,}3476$
15. $f'(x) = e^x \cdot e^{e^x} = e^{x+e^x}$

Capítulo número 6

1. $f'(x) = -2x + 4$. $f'(0) = 4$ (positiv). $f'(3) = -2$ (negativ)
   $f(x)$ har altså maximum i $x = 2$
2. $f'(x) = 6x^2 - 24$. $f'(-3) = 30$ (positiv). $f'(0) = -24$ (negativ). $f'(3) = 30$ (positiv)
   $f(x)$ har altså maximum i $x = -2$ og minimum i $x = 2$
3. $f(x)$ har minimum $\left(\text{i } x = -\frac{5}{6}\right)$

x		$-\frac{5}{6}$	
f'	$+$	0	$-$
f	↗	max	↘

4. $f'(x) = 10x - 7 = 0 \Leftrightarrow x = \frac{7}{10}$. $f'(0) = -7$. $f'(1) = 3$

x		$\frac{7}{10}$	
f'	$-$	0	$+$
f	↘	min	↗

   $f(x)$ er aftagende i $\left]-\infty; \frac{7}{10}\right]$ og voksende i $\left[\frac{7}{10}; \infty\right[$

5. $f'(x) = e^x - 1 = 0 \Leftrightarrow x = 0$. $f'(-1) = -0{,}63$. $f'(1) = 1{,}72$

x		0	
f'	$-$	0	$+$
f	↘	min	↗

   $f(x)$ er aftagende i $]-\infty; 0]$ og voksende i $[0; \infty[$

6. $g'(x) = -3x^2 + 3 = 0 \Leftrightarrow x = -1 \vee x = 1$. $g'(-2) = -9$. $g'(0) = 3$. $g'(2) = -9$

x		-1		1	
f'	$-$	0	$+$	0	$-$
f	↘	Lokalt min	↗	Lokalt max	↘

   $g(x)$ er aftagende i $]-\infty; -1]$ og i $[-1; 1]$
   $g(x)$ er voksende i $[1; \infty[$

Chapter $9 - 2 = 7$

1. $T_{f(3)} = 27(x-3) + 27 = 27x - 54$
2. $T_{f(2)} = 128(x-2) + 64 = 128x - 192$
3. $T_{f(1)} = (x-1) + 2 = x + 1$
4. $T_{f(2)} = (2e^4 - 4)(x-2) + e^4 - 8 = (2e^4 - 4)x - 3e^4$.
5. $T_{f(2)} = \frac{31}{2}(x-2) + 17 = \frac{31}{2}x - 14$.
6. $T_{f(-2)} = (-160 - 5e^{-2})(x+2) + 80 - 5e^{-2} = (-160 - 5e^{-2})x - 80 - 10e^{-2}$
7. Kuala Lumpur

Kapitel 8 nu!

Psych! 🙈

Chapter 9 homie

1. Brug *- Det bestemte integral*Definition 8.1.1 og at $st(a+b) = st(a) + st(b)$ hvis og kun hvis a og b er endelige
2. Brug Definition 8.1.1 og at $st(a+b) = st(a) + st(b)$ hvis og kun hvis a og b er endelige
3. Brug Definition 8.1.1 og at $st(a+b) = st(a) + st(b)$ hvis og kun hvis a og b er endelige
4. Brug Definition 8.1.1 og tænk på, hvad afstanden mellem a og a er
5. Brug resultaterne fra 3. og 4.
6.
$$\int_0^x c\,dx = cx$$
$$\int_0^x kx\,dx = \frac{1}{2}kx^2$$
$$\int_0^x (kx + c)\,dx = \frac{1}{2}kx^2 + cx$$
$$\int_0^x x^2\,dx = \frac{1}{3}x^3$$

Kapitel 10, zup!

1. $F(x) = \frac{1}{4}x^4 + x^2$
2. $F(x) = e^x + 4x$
3. $F(x) = \sqrt{x}$
4. $F(x) = \frac{1}{x}$
5. $F'(x) = e^{3x^2} \cdot 3 \cdot 2x = f(x)$
6. $F'(x) = -\frac{1}{x^2} + \frac{1}{2\sqrt{x+42}} - e^x = f(x)$
7. $F'(x) = \frac{3}{2} \cdot \frac{2}{3} \cdot x^{\frac{3}{2}-1} = x^{\frac{1}{2}} = \sqrt{x} = f(x)$
8.

$$\int_1^2 16x^{-2} + 3x^2\,dx = 15 \qquad \int_{\frac{1}{2}}^1 24x^2\,dx = 7 \qquad \int_{16}^{64} -\frac{1}{\sqrt{x}}\,dx = -8 \qquad \int_4^{16} \left(\sqrt{x} - \frac{x}{9}\right)dx = 24$$

Så er den her, nummer 11

1. $\frac{1}{5}x^5 + 42$ og $\frac{1}{5}x^5 - 42$
2. $F(x) = \frac{1}{2}x^4 - 2$
3. $F(x) = \sqrt{x} + 1$
4. $F(x) = e^{3x} + 5 - \frac{1}{3}e^3$
5. $F(x) = \frac{1}{x}$
6. $F(x) = 2x^3 + 5x - 42$
7. $\frac{1}{8} - \frac{1}{2} = -\frac{3}{8}$
8. $\frac{1}{9}$
9. $e^{\frac{1}{12}} - e^4$
10. 2

Alle løsningerne til kapitel 12

1.
2.
3.
4.
5.
6.
7.
8.
9.

Facit til resten

Laera er areal bagfra

Noitknufmats er stamfunktion bagfra

Regninger er regninger bagfra

13 Opsummering

13.1 Kapitel 1

De **hyperreelle tal** indeholder alle de reelle tal, infinitesimaler og uendelige tal.

Om en **infinitesimal** dx gælder

- $0 \leq dx < a$, hvor $a \in \mathbb{R}_+$
- $a < -dx \leq 0$, hvor $a \in \mathbb{R}_-$

Om et **uendeligt tal** $\frac{1}{dx}$ gælder

- $\frac{1}{dx} > a$, hvor $a \in \mathbb{R}$
- $-\frac{1}{dx} < a$, hvor $a \in \mathbb{R}$

Mængden af de hyperreelle tal kaldes $^*\mathbb{R}$ og læses "stjerne R".
Der gælder samme regneregler for $^*\mathbb{R}$, som for $\mathbb{R}$, man kan altså addere, trække fra, multiplicere og dividere.
Regneregler for infinitesimalerne dx og dy, hvor $a \in \mathbb{R}$:

- $dx - dx = 0$
- $dx + dy$ er også en infinitesimal
- $a \cdot dx$ er også en infinitesimal
- Brøken $\frac{dy}{dx}$ kan være et hvilket som helst hyperreelt tal, det skal altså afgøres fra situation til situation

Med **standard delen** kommer man fra de hyperreelle tal til de reelle tal. Man finder altså det reelle tal der ligger tættest på det hyperreelle tal.

Regneregler for standard delen, hvor $a, b \in {}^*\mathbb{R}$

- $st(a) \pm st(b) = st(a \pm b)$
- $st(a \cdot b) = st(a) \cdot st(b)$ HVIIIIIIIIIIS a og b er endelige tal

En **kontinuert funktion** er en funktion der kan tegnes uden at løfte blyanten/kridtet fra papiret/tavlen.

Sagt på nice: En funktion $f(x)$ er kontinuert i et punkt x_0, hvis $\Delta y = {}^*f(x_0 + dx) - f(x_0)$ er infinitesimal.

Når en funktion evalueres i en hyperreel værdi, $x \in {}^*\mathbb{R}$, skal funktionen udvides til også at gælde for hyperreelle tal. dette skrives således:

$$^*f(x)$$

Dette læses "stjerne f af x". Hvis man skal bruge flere funktioner, bruges $^*g$, $^*h$, $^*k$, $^*l$,…

13.2 Kapitel 2-7

AFLEDTE FUNKTIONER

$f(x)$	$f'(x) = \frac{d}{dx}\big(f(x)\big)$
b	0
ax	a
ax^2	$2 \cdot ax$
ax^n	anx^{n-1}
$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2} = -x^{-2}$
$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
a^x	$a^x \cdot \ln(a)$

REGNEREGLER

$h(x)$	$h'(x)$
$h(x) = f(x) \pm g(x)$	$h'(x) = f'(x) \pm g'(x)$
$h(x) = a \cdot f(x)$	$h'(x) = a \cdot f'(x)$
$h(x) = f(x) \cdot g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
$h(x) = f(g(x))$	$h'(x) = g'(x) \cdot f'(g(x))$
$h(x) = \frac{f(x)}{g(x)}$	$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

- Differentialkvotient som beskriver hældningen i et punkt:

$$st\left({}^{*}\frac{f(x+dx)-f(x)}{dx}\right) = f'(x) = \frac{d}{dx}\big(f(x)\big)$$

- Tangentens ligning kan skrives på følgende måde til funktionen f i x_0.

$$T_{f(x_0)}(x) = ax + b = f'(x_0) \cdot x + f(x_0) - f'(x_0)x_0 = f'(x_0)(x - x_0) + f(x_0).$$

- Monotoniforhold handler om, at finde maksima og/eller minima for grafen til en funktion $f(x)$ og afgøre hvor grafen er (monotont) voksende og/eller aftagende.

13.3 Kapitel 8-12

STAMFUNKTIONER

$f(x)$	$F(x)$
c	cx
kx	$\frac{1}{2}kx^2$
x^2	$\frac{1}{3}x^3$
x^n	$\frac{1}{n+1}x^{n+1}$
$-\frac{1}{x^2} = -x^{-2}$	$\frac{1}{x} = x^{-1}$
$\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$	$\sqrt{x} = x^{\frac{1}{2}}$
$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{2}{3}x^{\frac{3}{2}}$
e^x	e^x
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$

REGNEREGLER

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du = f\big(g(b)\big) - f\big(g(a)\big)$$

Det ubestemte integral

$$\int f(x)\,dx = F(x) + c$$

- Det bestemte integral fra a til b af $f(x)$ er defineret som den uendelige sum af infinitesimale rektanglers areal mellem en funktion, $f(x)$, og førsteaksen, på et interval, $[a,b]$. Integralet skrives

$$\int_a^b f(x)dx = st\left({}^*f\left(a+\frac{dx}{2}\right)dx + {}^*f\left(x_1+\frac{dx}{2}\right)dx + {}^*f\left(x_2+\frac{dx}{2}\right)dx + \cdots + {}^*f\left(x_{N-1}+\frac{dx}{2}\right)dx\right),$$

  hvor $a < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$, er en infinitesimal inddeling af intervallet $[a,b]$ og $N \in {}^*\mathbb{R}$ er et uendeligt tal. I tilfælde af, at grafen ligger under førsteaksen, fås arealet med negativt fortegn.

- Integration ved substitution handler om, at genkende den indre funktion i integranden og substituere med en ny variabel, u, så man kan integrere noget mere genkendeligt. Husk at lave nye grænser!

”StandardJoe & HyperMick: Infinitesimalregning”
Written by Mikkel M. Lindahl & Jonas Kyhnæb