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The Themes of Trigonometry and Power Functions in Relation to the CAS Tool GeoGebra

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Abstract

In the curricula for the Danish high school as well as for the written examination for mathematics (regardless of the level), it is evident that CAS tools have a significant role in connection with the teaching of mathematics. This thesis investigates the implementation of a CAS tool in a high school and the learning outcome of the students in relation thereto. The teacher's way of organising and exploiting the CAS tool in his/her mathematics teaching is also considered since it has a great influence on the students' work with the tool. The findings of the thesis are based on an observation study of high school teaching performed in the two mathematical themes Trigonometry and Power Functions, in two different classes, having different teachers, where *GeoGebra* (GG) functions as a CAS tool.

For the observation study, four *research questions* (RQs) have been developed with the *Anthropological Theory of Didactics* (ATD) and the notions of *instrumented techniques* and *orchestration* as theoretical framework. Additionally, a *reference praxeological model* (RPM) for both of the themes has been established in order to answer the RQs. Each RPM consists of a profile of the theme, involving scholarly knowledge, and a presentation of the curriculum and teaching material. Finally, an a priori analysis of the CAS tool is included in the RPMs.

The CAS tool, GG, is found to be a versatile tool where the visualisation and dynamic properties are considered a significant strength, in relation to examining several examples of a mathematical object. This thesis finds that it may however be difficult for the students to relate the praxis performed in GG to

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UNIVERSITY OF COPENHAGEN

The Themes of Trigonometry and Power Functions in Relation to the CAS Tool GeoGebra

Master Thesis in Mathematics

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Title: The themes of Trigonometry and Power Functions in relation to the CAS tool GeoGebra

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Department: Department of Science Education, University of Copenhagen

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The CAS tool, GG, is found to be a versatile tool where the visualisation and dynamic properties are considered a significant strength, in relation to examining several examples of a mathematical object. This thesis finds that it may however be difficult for the students to relate the praxis performed in GG to mathematical conceptual knowledge in general. The observed students managed to perform techniques in relation to GG to a greater extent than establishing a knowledge block. In relation to GG related work, both teachers were observed to make use of a projector but, apart from this, their orchestrations were very different with regards to how GG could be approached as a CAS tool.

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For the completion of this thesis, we would like to express our gratitude towards the support from both family, boyfriend/husband and friends when dealing with troubles and challenges as well as proofreading parts of the thesis.

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Acronyms

- GG GeoGebra
- ATD Anthropological Theory of Didactics
- RPM Reference praxeological model
- MO Mathematical organisation
- RQ Research question
- C-L Curriculum - Mathematical level
- TG-L Teacher's guide - Mathematical level
- TT The teacher teaching the Trigonometry course
- TP The teacher teaching the Power Functions course
- TTW The webpage created by the Trigonometry teacher
- PS Participating student (Mac user)
- AS Another student (PC user or no computer)
- T_{ET_n} Type of exam task for Trigonometry where n is an index
- T_{EP_n} Type of exam task for Power Functions where n is an index
- $t_{a,b,c}$ Specific task concerning a: Number related to type of task or abbreviation of content, b: Perspective or View in GG, c: Developer (TT/TP: The teacher teaching the Trigonometry course/ The teacher teaching the Power Functions course)
- $\tau^{i}_{a,b,c}$ Instrumented technique concerning a: Number related to type of task or abbreviation of content, b: Perspective or View in GG, c: Developer (PS: Participating student, TT/TP: The teacher teaching the Trigonometry course/The teacher teaching the Power Functions course, TR/PR: The researcher of the Trigonometry course/ The researcher of the Power Functions course)

1.0 Introduction

The implementation of computer programs[1] functioning as digital tools for doing mathematics in Danish high schools (henceforth referred to as *high schools*) is very current, as it has a significant priority in the curricula. CAS tools should not only function for more complicated symbolic calculations but should also function as support for mathematical conceptualisation and skill learning.[2]

The question has been raised, though, especially by Jessen, Holm and Winsløw (2015), to whether this implementation is favourable, especially when concerning the students' conceptual knowledge of mathematics. Dubinsky and Tall (1991) states, in alignment with this problematique: *Having a computer to perform the algorithms, even to show how those algorithms work is one thing, being able to cope with these concepts meaningfully is another* (Dubinsky & Tall, 1991, p. 7).

Moreover, a Danish high school teacher has expressed that (translated from Danish):

..."as a teacher you have to be idealistic in order to keep up the spirit and teach your students such that they are going to be good at doing mathematics when the students can do just as good at the exam through a teaching focused on CAS-commands and template answers"... (Jessen, Holm, & Winsløw, 2015, p. 13).

Jessen, Holm and Winsløw (2015) hereby put a focus towards the written examination as an influencing factor on the teacher's orchestrations and the students' instrumented work with CAS tools.

The teacher's planning and realisation of the implementation of CAS tools in his/her mathematics teaching has evidently got a great influence on what the students learn through the work with the tools - especially on which praxeologies the students get to establish. Do they only learn how to use the tool, or do they also learn to connect this praxis to relevant corresponding mathematical knowledge and vice versa?

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[1] Such programs cover both CAS tools (understood as programs which can operate with symbolic (algebraic) mathematical expressions), spreadsheets, calculators, graphic calculators and geometric drawing programs (Jessen, Holm, & Winsløw, 2015, p. 5).

[2] To be noted, it is not necessarily the same CAS tool which is used in all high schools.

On this basis, this thesis concerns the study of how the teaching in a high school, in two specific mathematical themes (Trigonometry at A-level and Power functions at B-level) in two separate classes, is performed with respect to the prioritised CAS tool, being *GeoGebra* (GG).[3] Especially, the approach of the students towards the CAS tool and their development of gestures and instrumented techniques and further which type of knowledge they gain in these situations will be studied. This will be analysed in order to determine which observable skills, the students establish with respect to the CAS tool and to which extent the work with the tool contribute to mathematical conceptualisation. The study will be made with reference to the term of praxeologies in the *Anthropological Theory of Didactics* (ATD). Moreover, since the work of the students is influenced by how the teacher choose to implement the CAS tool, the teacher's instrumental orchestrations will be studied, as an inevitable perspective for the praxeologies established by the students.

1.1 Structure of the Thesis

The structure of the thesis is affected by the fact that it concerns an observation study of two mathematical themes. On this note, the thesis consists of a common theory section (2.0) in which both theory concerning implementation of the CAS tool, especially with respect to instrumental genesis and orchestration (section 2.1), as well as theory concerning the notions of a *reference praxeological model* (RPM) and *mathematical praxeologies* (MOs) drawn from ATD (section 2.2). Based on the theory, the four *research questions* (RQs) relating to both of the themes are raised (section 3.0). Thereafter, the RPMs for the two themes are presented, each involving both a profile of the theme based on scholarly knowledge as well as the knowledge to be taught in terms of the curriculum, the written examination on the specific level and teaching material (sections 4.1 and 4.2). Further, two a priori analyses of the CAS tool in relation to the themes are given with a focus on elaborating the main specific features in relation to performing instrumented techniques for solving exercises within each of the themes (section 4.3). In section 5.0, an introduction of the

[3] This is what is chosen as CAS tool in the specific high school and is further elaborated in section 4.3.

observation study is given and in section 6.0, the methodology for both performing the observations and for analysing the empirical data collected is presented. Subsequently, analyses of the empirical data related to each theme are made based on the theory and the RPM (sections 7.1 and 7.2). In the sections 8.1 and 8.2, discussions related to each of the analyses are made and in section 8.3, a common discussion and perspectives is reviewed. Finally, a common conclusion on the results of the thesis is given in section 9.0. A figure sketching the structure of the thesis is presented in appendix section A.1. It should be noted that references to appendix sections henceforth will be denoted by "A.n" (n being a specific number indicating the section), throughout the thesis.
References to webpages in the Bibliography (section 10.0) are shown in brackets with a unique reference inside, e.g. "(Webpage: GG-worksheets, 2016)".
Even though the thesis splits into two RPMs, two analyses of empirical data and two discussions, it is referred to as *one* observation study because of the common RQs. Furthermore, it will be denoted if a section is written by only one researcher. This, in the sense that (A) or (L) is put in prolongation of the title of the section, representing Anne Kathrine Wellendorf Knudsen and Line Steckhahn Sørensen, respectively.

Navigation

Tasks presented in the sections 4.1.2 and 4.2.2, concerning the curriculum and the written examination in relation to each theme, are coloured **green**.
Tasks and techniques presented in the sections 4.3.2 and 4.3.3, concerning the CAS tool in relation to each theme, are coloured **blue**.
Tasks and techniques presented in the sections 7.1 and 7.2, concerning the analyses of the empirical data of the two courses, are coloured **black.**

2.0 Theory

The purpose of this section is to provide a theoretical framework for analysing and discussing the observed teaching courses of this observation study. In particular, the notions of *instrumental genesis* and *instrumental orchestration* as well as the elements of ATD concerning RPMs and MOs will be introduced. This, in order to constitute the RQs sought to be answered.

2.1 Implementation of CAS Tools in Mathematics Teaching

This section is mainly concerned with theory applicable to describe situations in a class room where a CAS Tool is implemented, to be used for both the a priori analysis of the CAS tool (section 4.3) and for the analyses of the empirical data (section 7.0). To begin with, the historical context of implementation of technology[4] in mathematics teaching will be elaborated to provide a framework (section 2.1.1). Thereafter, the notion of a CAS Tool to be applied in this thesis is established (section 2.1.2) and finally, it is elaborated which theoretical terms are chosen to describe both the students' and the teacher's approaches towards the CAS tool (section 2.1.3).

2.1.1 Historical Context of the Use of Technology in Mathematics Teaching

The use of technology in mathematics teaching has been developed and practiced over several years, since the late 1960s, *...when, according to Fey [...], mathematicians and mathematics educators began to feel that computing could have significant effects on the content and emphases of school-level and university-level mathematics* (Drijvers et al., 2010, p. 91).[5] The fascination of the potentials of technology, developed by mathematicians and mathematics educators, was triggered from *...the time of the development of the mainframe computer in 1942, the first four-function calculator in 1967, the microcomputer in 1978, and the graphing calculator in 1985* (Drijvers et al.,

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[4] Throughout this subsection the notion *technology* refers to IT-tools and calculators and is not to be confused with the notion of *technology* presented in ATD (see section 2.2.2).

[5] James T. Fey is an American professor of mathematics education at the University of Maryland and is well known for his work on didactics of mathematics (Webpage: James T. Fey, 2016).

2010, p. 91). From the mid-1980s, ICMI (International Commission on Mathematical Instruction) studies was initiated with the theme of "The Influence of Computers and Informatics on Mathematics and Its Teaching at University and Senior High School Level". Here, conjectures were made around to which extent the learning of mathematics could be supported by the use of computers, and *...activities of exploration and discovery were particularly pointed to* (Drijvers et al., 2010, p. 92). A great amount of optimism existed towards *...the educational potentialities and capabilities of computing technology - such as, visualizing, modeling, and programming - [...] not yet supported by evidence* (Drijvers et al., 2010, p. 92). Alongside to the ICMI studies, where the study papers did not reflect upon theory *...with respect to technology and its use in mathematics education...* (Drijvers et al., 2010, p. 93), such theory was developed during the 1980s. An example of a terminology, describing the possibilities and constraints of the use of technology, are the terms of *white box* and *black box*, introduced by Buchberger[6] around 1990. These terms describe the cases where the students *...are aware of the mathematics they are asking the technology to carry out* (Drijvers et al., 2010, p. 93) (white box approach) and when this is not the case, the use is that of a black box. Throughout the 1990s, further attention was given to the development of the use of technology in the classrooms of mathematics teaching (Drijvers et al., 2010, p. 98). Nowadays, the technology devices have become smaller and handheld and the possession of a calculator as well as a computer is widespread. Furthermore, *...communication has become a more integrated part of technology use: software can be distributed using the Internet, and students can work, collaborate, and communicate with peers and teachers in digital learning environments* (Drijvers et al., 2010, p. 98). It remains however an unanswered question what a beneficial approach, towards obtaining a learning environment involving technology in mathematics teaching, is. Here, *beneficial approach* relates to cases where the potentials of the technology is utilised in an advantageous way. (Drijvers et al., 2010, p. 98).

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[6] Bruno Buchberger is an Austrian professor of Computer Mathematics at Johannes Kepler University in Linz (Webpage: Bruno Buchberger, 2016).

As an end note for this section, the development of the use of technology for learning mathematics in school is noted to be traceable to a significant extent in the teaching plans of mathematics in the high school of today (level A and B). Especially, in the section of organisation ("3. Tilrettelæggelse"), there is a note under the first subsection ("3.1. Didaktiske principper"), concerning didactical principles. Here it says that CAS tools should not only be used for the more complicated symbolic calculations but should also support skill learning and mathematical conceptualisation. Furthermore, there exists a separate subsection called "3.3. It" within in the same section, supporting the fact that IT has a significant priority. (Undervisningsministeriet, 2013a, 2013b).

2.1.2 The Notion of CAS Related to This Study

The classical understanding of CAS is that of a Computer Algebra System, covering the algebraic, symbolic aspects of the computer program as a tool and not immediately potential graphic, dynamic geometric and spreadsheet aspects (Webpage: Computer algebra system, 2016). Nonetheless, in addition to the symbolic, algebraic aspect, both the graphic, dynamic and spreadsheets aspects of a computer program are considered as covered by the term *CAS* within this study. This priority is based on the fact that the observed classes in this study work with the program GG which both functions as a dynamic geometric, a graphical, a spreadsheet and, to some extent, a symbolic, algebraic tool (Misfeldt, 2014, p. 27) (see section 4.3).

Noticeably, there lies a great potential of visualisation when working with this program because of the above mentioned features, especially those of the dynamic geometric- and graphical tool (Artigue, 2005, pp. 252–253). The option of visualisation has the potential of aiding *...more varied access to mathematical problems and concepts* (Lagrange, 2005, p. 74). For instance, the concept of sine with respect to an angle can be visualised to be both a ratio of two sides of a right triangle and a second coordinate of a point on the unit circle. Regarding a power function ($f(x)$), the functional value with respect to a given x-value can be accessed through its graphical representation and not only analytically. Dubinsky and Tall (1991) further propose that visualisation can provide a conceptualisation of the mathematics, since when *...an abstract idea is implemented or represented in a computer, then it is concrete in the* *mind at least in the sense that it exists (electro-magnetically, if not physically)* (Dubinsky & Tall, 1991, p. 5). In continuation thereof, Dubinsky and Tall point out that *...it is generally agreed that ideas are easier to understand when they are made more "concrete" and less "abstract"* (Dubinsky & Tall, 1991, p. 5). Further, when GG is functioning as a dynamic geometric as well as a graphical tool, the visualisation lying herein can help provide an acceptance of the existence of the results visually and not only through algebraic representations (for instance a formula) (Dubinsky & Tall, 1991, p. 10). There are however also performed studies with an alertness towards the risk that the use of computer software may lead to hiding mathematical concepts behind the action made (Confrey et al., 2010, pp. 30–31). For instance, when *sin*, *cos* and *log* are solely considered as being commands related to a CAS tool when solving problems involving these functions - with a black box effect as a result.

Further, there exists the option of making exploratory environments when using a CAS tool like GG for mathematics teaching. Especially, there is the opportunity of studying several examples, *…using varied representations of objects and inductive as well as deductive approaches* (Lagrange, 2005, p. 74). More generally, when students learn through exploration, there is, according to Miller, Lehman and Koedinger[7], a focus *...on stimulating the student's initiative in gaining knowledge about the domain* (Confrey et al., 2010, p. 23).

2.1.3 Managing a CAS Tool

Students' approach

For describing the students' approach towards the CAS tool GG of this study, the notion of *instrumented techniques* (to be denoted τ^i) introduced by Guin and Trouche (2002) is found especially relevant. The separate notion of *techniques* will be further elaborated within the term of *praxeologies* in the theory of ATD, in section 2.2.2.

[7] With reference to (Miller, Lehman, & Koedinger, 1999).

For the definition of an *instrumented technique* to be used in this study, Guin and Trouche (2002) are followed in adopting Lagrange's[8] definition of a technique as being a set of gestures *...built by the subject to accomplish a given task* (Guin & Trouche, 2002, p. 206). The *subject* is considered to be a high school student or teacher in this study. The function of a *gesture* can be understood within the action required of the subject to solve a specific task and noticeably it must not be *...considered in isolation…* (Guin & Trouche, 2002, p. 206) since it depends on human activity. A gesture is more specifically the observable part of human activity (Trouche, 2004, p. 286). As an example, a gesture can be considered as a combination of keystrokes on a symbolic calculator in order to get an output (Trouche, 2005, p. 151). Especially, if the gestures are instrumented, i.e. *…'articulated' to an instrument…* (Guin & Trouche, 2002, p. 206), they constitute an instrumented technique. An example of such a technique (within the theme of Trigonometry) is when the students realise the relationship between a point on the unit circle and cosine and sine to the reference angle by both entering the coordinates of the point and dragging it in GG. Within the theme of Power Functions, an example of an instrumented technique is when the students drag a slider for the value of a in GG (in the functional equation $f(x) = b \cdot x^a$) in order to realise the relation between this value and the look of the graph.

For the notion of an *instrument*, Guin and Trouche (2002)'s understanding (inspired by Verillon and Rabardel[9]) is to be used. Namely that of a psychological construct which can solely be said to exist if it is integrated within the activity of a subject and appropriated for the subject as an individual (Guin & Trouche, 2002, p. 205). More specifically, Verillon and Rabardel distinguish an instrument from an *artifact* in such a way that an *...instrument is what the subject 'builds' from the artifact* (Trouche, 2005, p. 144) whereas an *...artifact is a material or abstract object, aiming to sustain human activity in performing a type of task (a calculator is an artifact, an algorithm for solving quadratic equations is an artifact)…* (Trouche, 2005, p. 144). Furthermore, an artifact

[8] Jean-Baptiste Lagrange is a French professor in Mathematics Education (Webpage: Jean-Baptiste Lagrange, 2016).

[9] With reference to (Verillon & Rabardel, 1995).

has both potentialities and constraints (characteristics) influencing the subject's activity through his/her former working habits and knowledge, in the construction of an instrument; the *instrumental genesis* (Guin & Trouche, 2002, p. 205). The process of *instrumental genesis* consists of the two components *instrumentalization* and *instrumentation* where the former is related to the artifact and the latter is *...related to the organization of the subject's behaviour* (Guin & Trouche, 2002, p. 205). The processes of instrumentalization and instrumentation concern elaboration of schemes, linking concrete gestures and mathematical thinking, and thus these processes are mind related (Guin, Ruthven, & Trouche, 2005, p. 3). The latter aspect of instrumental genesis will not be considered further since this study only concerns the concrete gestures performed in praxis by subjects (being students and teacher) in relation to an artifact (being a computer with GG); i.e. the instrumented techniques. Figure 1, offered by Guin and Trouche (1999), provides an overview of how the artifact, the subject, the instrumental genesis and the instrument are related.

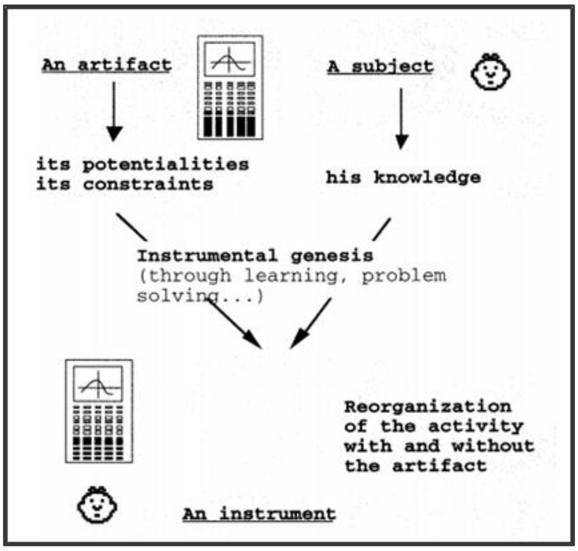

Figure 1 The relation between artifact, subject, instrumental genesis and instrument. (Guin & Trouche, 1999, p. 202).

Teacher's approach

The teacher has a great influence on how the CAS tool is made effective as a learning tool in his/her teaching. Especially, the students' specific work with the CAS tool in correspondence with the mathematical theme is influenced by how the teacher plans and executes the teaching. The term of *instrumental orchestration* is used for describing the teacher's actions towards guiding the individual instrumental geneses in the classroom with the intention of constituting a collective instrumental genesis (Drijvers et al., 2010, p. 112). As for the use of terms, the teacher can be said to function as *...an orchestra conductor rather than a one-man band* (Trouche, 2004, p. 299) with the purpose of making *...the different student instruments playing together...* (Trouche, 2014, p. 4) when working with the artifact. The notion is introduced by

Trouche (2004) in order *...to point out the necessity (for a given institution – a teacher in her/his class, for example) of 'external steering' of students' instrumental genesis* (Trouche, 2004, p. 296). Further, this notion is often put in connection with learning environments involving the computer as an artifact (Drijvers et al., 2010, p. 112) which is exactly the case in the observation study of this thesis.
Regarding the definition of the notion *instrumental orchestration*, Trouche (2004) asserts it to encompass didactic configurations and exploitation modes of these configurations performed by the teacher. The didactic configurations can more specifically be said to refer to *...the layout of the artifacts available in the environment...* (Trouche, 2004, p. 296). An example of an orchestration could be when the teacher sets up a situation where a student gets to present for the whole class how to solve a specific task using the CAS tool GG, via a projector, keyboard and mouse - similar to sherpa-student situation presented by Guin and Trouche (2002).

For analysing situations involving instrumental orchestration, it can be said to act on three levels, inspired by Trouche (2004, p. 297). The *first level* is where the orchestration acts on the artifact itself, for instance on specific possible commands to type in GG in order to construct a triangle or plot the graph of a function. This level is oriented towards instructions in how to use the specific artifact, keeping in mind that such instructions can change relative to the teacher's experience. In a *second level* orchestration, the focus is put on the instrument (or a set of instruments) and more specifically on the instrumented techniques used for solving specific tasks. As an example of a second level orchestration, the example of an orchestration presented in the above sherpa-like situation can be used, since the student gets to share his/her instrumented techniques. A *third level* orchestration is concerned with the subject's relationship with the instrument (or set of instruments) and, more concretely, the subject's reflection on choice of instrumented techniques in his/her approach towards the artifact. This type of orchestration can be made both towards the individual student (subject) or in a collective environment. An example of this type of orchestration could be when the teacher asks the students to compare the presented instrumented techniques (in a second level orchestration) or to reflect on instrumented techniques alternative to the ones presented and to discuss which techniques are the most convenient (either individual or in plenum).

This section ends by noting that *... instrumentation theory is double-layered...* (Drijvers et al., 2010, p. 112) since the teacher, when carrying out teaching involving work with an artifact, him-/herself concurrently develops an instrument from the artifact (i.e. s/he undergoes a process of instrumental genesis as well) (Drijvers et al., 2010, p. 112).

2.2 The Anthropological Theory of Didactics (ATD)

This section is concerned with elaborating the process of the *didactic transposition* (section 2.2.1) which can work as a first framework to be further developed into the terms of praxeologies (section 2.2.2) in order to model the transposed mathematical knowledge. Further, the terms of praxeologies works as to give a perspective on the accomplishment of solving a task through instrumented techniques using a CAS tool involving the questions of *how* to solve the task and *why*.

2.2.1 The Didactic Transposition

The process of didactic transposition works as a model for considering the taught mathematical knowledge in school as being *...in a certain way, an exogenous production, something gathered outside school that is moved - 'transposed' - to a school out of a social need of education and diffusion* (Bosch & Gascón, 2006, p. 53). The taught *mathematical knowledge* includes here the practice as well, i.e. it is both practical and theoretical. Winsløw (2011) states in a wider sense:

> *The whole point of didactic transposition theory is to exhibit and analyse the profound 'changes' which knowledge and practice undergoes as it is "transposed" from one institution to another, or, within the school, from the "official" curriculum to the "implemented" curriculum*
> (Winsløw, 2011, p. 123).

As a final note, before describing the specific steps of the process of didactic transposition, this transposition should be seen as *...“the travel of knowledge from source(s) to students”* (Winsløw, 2011, p. 123).

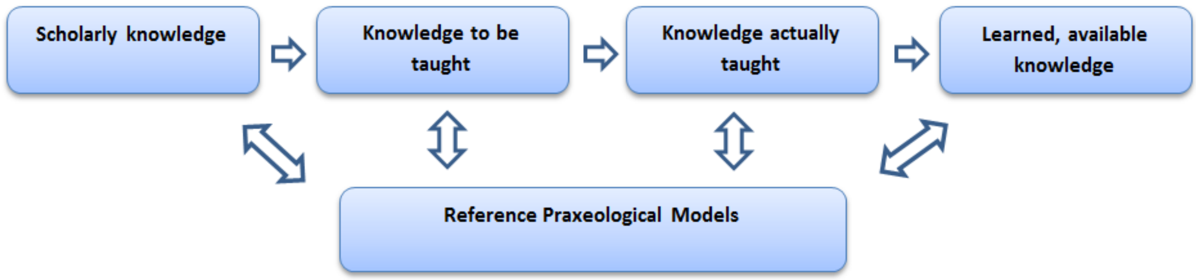

Figure 2 The process of didactic transposition and the position of the researcher in relation thereto.

Figure 2 is made with great inspiration from Bosch and Gascón (2006, p. 57). This, with an awareness of the character of the study of this thesis being observations of praxeologies set up by the teacher and established by the students, why the notion of *praxeological* rather than *epistemological* has been chosen in relation to the reference models. The figure shows the process of didactic transposition which can be said to contain four steps (referring to the four boxes between which the knowledge is transposed, indicated by the three arrows pointing right). The first step is where the *scholarly knowledge* is produced and used by a mathematical community. This influences the *knowledge to be taught* which is based on official programs, textbooks, didactical materials etc. (illustrated through the first arrow). The second arrow indicates the transposition of the *knowledge to be taught* to the *knowledge actually taught* which is what happens in the classroom - i.e. how the specific teacher executes his/her teaching in a specific classroom. The third arrow illustrates the transposition to the *learned, available knowledge* which is the mathematical knowledge learned by the students.

As a final remark for the figure (Figure 2), the *reference praxeological models* (RPMs) *...constitutes the basic theoretical model for the researcher* (Bosch & Gascón, 2006, p. 57). The double arrows indicate a correspondence between the reference praxeological models and *...the empirical data of [...] the mathematical community, the educational system and the classroom* (Bosch & Gascón, 2006, p. 57). In the case of this observation study the RPMs encompass both an analysis of each of the two mathematical themes with respect to scholarly knowledge and knowledge to be taught (sections 4.1 and 4.2) as well as an a priori analysis of the CAS tool GG in relation to each theme (section 4.3).

Now, since this study is anthropological and concerns the teaching of - and especially the learned mathematical knowledge in the classroom, use of the notion of mathematical praxeology/mathematical organisation, proposed by Chevallard within ATD, will be made. This, in order to model mathematical practice considered to include both a material/praxis dimension as well as a theoretical/knowledge dimension (Barbé, Bosch, Espinoza, & Gascón, 2005, pp. 236–237; Bosch & Gascón, 2006, p. 58).

2.2.2 Mathematical Organisations (MOs) and Didactical Organisations (DOs)

ATD proposes that mathematics can be considered as a human activity which can be described in terms of the notion of *mathematical organisation/mathematical praxeology* (MO). Two aspects of mathematical activity are considered when describing an MO, namely a *practical block* and a *knowledge(/theory) block*. The practical block is formed by *types of tasks (T)* and by *techniques* (τ) which are the methods used to solve and carry out the tasks. Noticeably specific tasks will be denoted (t) and specific set of tasks t will be denoted an *exercise*. The knowledge block also consists of two elements, namely *technology* (θ) and *theory* (Θ). The element of technology intends to justify the techniques used, and the theory element *...constitutes a deeper level of justification of practice* (Barbé et al., 2005, p. 237). To sum up, the knowledge block brings the *...mathematical discourse necessary to justify and interpret the practical block* (Barbé et al., 2005, p. 237) and a knowledge block is especially established when the students try to generalise the outcome of their praxis. More generally, these two blocks refer to the two main components of human activity, called *praxis* (practical part) and *logos* (referring to human thinking and reasoning) which are intertwined (Bosch & Gascón, 2006, p. 59). To be noted, an MO can be considered as a quadruple $(T, \tau, \theta, \Theta)$.

Now, in order to argue for Trigonometry and Power Functions both being *themes*, the classification of *MOs* to be either *punctual, local*, or *regional* and especially the relation of MOs to the levels of codetermination, proposed by Chevallard, is needed.
The classifications of MOs are said to be *...of increasing complexity* (Bosch & Gascón, 2006, p. 59). MOs are called punctual *...if they are based around what is considered a unique type of problems...* (Barbé et al., 2005, p. 237). A local MO is a set of punctual

MO's belonging to the same technological discourse, i.e. tasks possible to solve through similar explanations (i.e. the knowledge block (θ, Θ) is common for the punctual MOs constituting a local MO). Similarly, a regional MO can be described as a set of local MO's *...accepting the same theoretical discourse* (Barbé et al., 2005, p. 238) i.e. tasks possible to solve through explanations within the same theory (Θ). In a mathematical set theory sense, it is noted that the relationship between a specific punctual MO and the other types of MOs in some cases can be written as follows:

$$punctual\ MO \subset local\ MO \subset regional\ MO.$$

For instance, when a cathetus[10] (b) and a hypotenuse (c) of a right triangle are given and one has to determine the unknown cathetus (a) (i.e. a punctual MO), then the Pythagorean theorem is applicable as technology within the Synthetic Geometry (see section 4.1.1) (as a regional MO).

It is further noted that it is possible for a punctual MO to exist in several local MOs and that a local MO can exist in several regional MOs. Each of the above mentioned types of MOs can, respectively, be related to the following three lower *levels of determination* (proposed by Chevallard): *subject*, *theme* and *sector*. They are codetermined and one may refer to this relationship as *levels of (didactic) codetermination* (Artigue & Winsløw, 2010, pp. 4–7). The levels of determination are institutional and has been introduced to *...help researchers to identify conditions that go beyond the narrow space of the classroom and the subject that has to be studied in it* (Bosch & Gascón, 2006, p. 61).

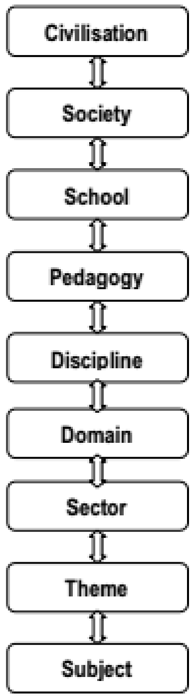

Figure 3 Levels of determination (Bosch & Gascón, 2006, p. 61)

Figure 3 presents the scale of levels of determination, where the upper levels *domain, discipline, pedagogy, school, society* and *civilisation* are of a character going beyond what is happening in the classroom and will not be considered further in this study (Winsløw, 2011). The categorisation of Trigonometry as being a *theme* is made

[10] The notion of a *cathetus* is throughout the thesis used to denote a side of a right triangle not being the hypotenuse.

on the basis that it can be practiced within Analytical Geometry and Synthetic Geometry (to be elaborated further in section 4.1.1) both being considered as sectors since they do not represent the same technological-theoretical discourse. The work with the unit circle is related to the first mentioned and construction of triangles is related to the latter. Noticeably, the *domain* (which can be described to embrace *...a collection of regional organisations involving several theories forming a larger part of the discipline* (Artigue & Winsløw, 2010, p. 5)) embracing the two above mentioned sectors can be said to be Geometry.
Power Functions is considered a *theme* as well, within the sector of Elementary Functions[11] embraced by the domain of Functions[12].

Within the levels of codetermination there noticeably lies the important aspect of the *didactic organisation* (abbreviated DO) concerning the MO's in the classroom. (Artigue & Winsløw, 2010, p. 6). A didactic organisation also called a didactic praxeology, is targeted towards the teaching of a specific MO and can be classified as punctual, local or regional as well (Winsløw, 2011, p. 125). In a more general sense, DOs seek to engage students through MOs in a teaching situation. DOs are performed by the teacher when he/her seeks to help the students perform an MO but can also be performed by the students in the process of studying and learning an MO (Artigue & Winsløw, 2010, p. 6; Barbé et al., 2005, p. 239).
The notion of DOs hereby provides a perspective on the set up of MOs occurring in the classroom - namely that of organising and intermediating the MOs.

On the basis of the presented theory, the following section concerns the established RQs sought to be answered in relation to the observations. The focus will mainly be

---

[11] *Elementary functions* are functions *...built up of a finite combination of constant functions, field operations (addition, multiplication, division, and root extractions--the elementary operations)-- and algebraic, exponential, and logarithmic functions and their inverses under repeated compositions* (Webpage: Elementary Function E. W. Weisstein, 2016).

[12] A *function*, f, is in this thesis understood as a non-multivalued relation from a set A (domain) to a set B (range) *...such that every* $a \in A$ *is uniquely associated with an object* $f(a) \in B$ (Webpage: Function C. Stover & E. W. Weisstein, 2016). It is either a many-to-one or one-to-one but never a one-to-many relation (Webpage: Function C. Stover & E. W. Weisstein, 2016).

put on the taught and learned mathematical knowledge in relation to GG in the two specific courses (concerning Trigonometry and Power Functions).

3.0 Research Questions (RQs)

The RQs related to this observation study concerning both of the themes are formulated to be:

RQ_0: What are the main specific features of the CAS tool, and what specific techniques does it offer for the two mathematical themes?

RQ_1: In the observed teaching; what types of tasks are the students explicitly instructed to solve using the CAS tool? What types of tasks do they spontaneously solve using the CAS-tool?

RQ_2: What mathematical praxeologies do the students establish in their work with the CAS tool (i.e. instrumented techniques, technology and theory based on the tasks identified in RQ_1)? What are the strengths and shortcomings at the level of mathematical theory?

RQ_3: How does the teacher orchestrate students' use of instrumented techniques (i.e. how does s/he undertake instrumental orchestration, e.g. what use of didactical configurations, use of GG worksheets, emphasis of specific command frames, etc.)?

The denotation '0' for **RQ_0** has been selected in this manner since it is to be answered through the a priori analysis of the CAS tool (in section 4.3) in relation to the themes. The **RQ_1-RQ_3** will however be answered through the analyses of the empirical data achieved via the observations in the two specific classes (sections 7.1 and 7.2) as well as the discussion sections related thereto (8.1 and 8.2, respectively).
In relation to **RQ_1**, it is to be noted that *spontaneously* solved tasks by the students is to be understood as cases where the students use techniques in which they are not explicitly instructed by the teacher.

Next, in order to set up a framework of reference mathematical knowledge (concerning both the knowledge block and the praxis block) for answering the above presented RQs, the RPMs for each of the two themes (Trigonometry and Power Functions) is established.

4.0 Reference Praxeological Models (RPMs)

In this section, the RPMs in relation to each of the themes will be established, involving both scholarly knowledge (sections 4.1.1 and 4.2.1) and knowledge to be taught (both in the version of the curriculum (sections 4.1.2 and 4.2.2) and the teaching material (sections 4.1.3 and 4.2.3)).

For the sections on the curriculum connected to each of the themes (4.1.2 and 4.2.2) there are some commonalities (appearing in both A- and B-level curriculum) to be noted before continuing to the RPMs of the themes. First of all, the academic objectives concerning the ability to *manage simple formulas* and to *translate between symbolic and natural language* as well as the ability of *completing simple mathematical reasoning and proofs* are common. Not least, the academic objective of the ability to *apply IT tools for solving given mathematical problems* is common as well. In the core substance of the curricula, both the *order of operations* and *equation solving* are common. Regarding the priority of CAS tools, it is common for both curricula that calculators, IT and mathematical (computer) programs are to be essential aids for both acquisition of mathematical concepts and problem solving. The organisation of the teaching of mathematics is further to be influenced by the fact that the use of the just mentioned aids should be involved in an experimental approach to mathematical subjects and problem solving. (Undervisningsministeriet, 2013a, 2013b). Finally, since the oral examination does not involve the use of a CAS tool, it is the tasks of the written examinations related to the themes which will work as a reference for which types of tasks the students should be able to solve (presented in the sections 4.1.2 and 4.2.2 as well). Noticeably, for both A- and B-level, there exists both a part with aids and a part without aids of the written examination, where the part with aids is dominant and noticeably it is only in this part that the student may use all types of aids including a (classical) CAS tool for performing symbolic manipulation. (Undervisningsministeriet, 2013a, 2013b; section "4.2. Prøveformer"). In both of the sections 4.1.2 and 4.2.2, types of tasks drawn from both the part with aids and the part without aids are however considered (it is only noted when the exam exercises are from the part without aids). This is due to the consideration that the part without aids, as well as the part with aids, indicates which mathematical knowledge (both in relation to the praxis block and to the knowledge block) the students shall learn within the mathematical theme.

4.1 Trigonometry (L)

This section is concerned with the establishment of the RPM connected to the theme of Trigonometry.

4.1.1 A Profile of the Theme

The Greek's need for dealing with computation of angles and distances and circle rotation of the Sun, Moon and planets around the earth led to the development of *Trigonometry*. The Greeks did what no one earlier had done and inspired others in the use of their geometry; they *...geometrized the heavens* (Webpage: Geometry mathematics, 2016).

Hence the Greeks used geometry to develop trigonometry.

Geometry is one of the oldest branches of mathematics and means "Earth measurement". It arose in relation to practical problems and is *...concerned with the shape of individual objects, spatial relationships among various objects...* (Webpage: Geometry mathematics, 2016). The largest contribution to the development of Geometry was given by Euclid[13] in his Euclid's Elements[14].

Geometry I, II, and III

According to a study made by Catherine Houdement and Alain Kuzniak (2003), elementary geometry can be split into three different paradigms, namely natural geometry (*Geometry I*), natural axiomatic geometry (*Geometry II*) and formalist axiomatic geometry (*Geometry III*) (Houdement & Kuzniak, 2003, p. 1).

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[13] Euclid was a great Greek mathematician born around 330 B.C (Webpage: Euclid, 2016)

[14] It is a mathematical and geometric treatise consisting of thirteen books written by Euclid in 300 BC. The book is a collection of definitions, postulates (axioms), propositions (theorems and constructions), and mathematical proofs of the propositions (Webpage: Euclid's "Elements," 2016).

Geometry I is closely related to reality in such a way that it is based on concrete figures. The validity of this Geometry is very sensitive since *...immediate perception, experiment and deduction act on material objects* (Houdement & Kuzniak, 2003, p. 4) when deductions are made. Conclusions are therefore solely based on manipulations of the figures. Dynamic proofs are fully valid and accepted in Geometry I (Houdement & Kuzniak, 2003, p. 4). As an example, if an angle (which will be further elaborated) of a triangle is to be determined then the use of a protractor for measuring the angle and give an answer, is considered enough. The notions of trigonometric functions like cosine and sine are not defined at this point. Even though, drawing a solution as a final answer to a trigonometric problem (without the use of any formulas) nowadays is considered imprecise, the situation is about to change with the implementation of the digital tools such as GG in high school. Digital tools namely offer some *...precise, practical and intellectually challenging ways to deal with triangle problems without the use of the trigonometric functions* (Misfeldt, 2014, p. 31; translated from Danish). Examples of this type of contribution will be presented in the section on a priori analysis of GG in relation to Trigonometry (section 4.3.2).

Geometry II is based on an axiomatic system in order to hypothetical deduce something desired from the figure. Here, the axiomatic system is used for the purpose of manipulating the material figure. This makes the validation of this geometry more rigorous and formal than Geometry I since it is possible to reach more general conclusions. As an example, when given two similar triangles, one can measure the ratio of two corresponding side lengths to be a constant equal to the ratio of another pair of corresponding sides of the same pair of triangles regardless of which pair of sides are chosen - see Figure 4. By measuring the corresponding side lengths of the two

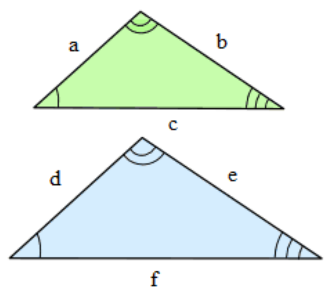

Figure 4 Two similar triangles with congruent angles.
With inspiration from (Webpage: Similar Triangles, 2016)

triangles with congruent angles the scale factor can be expressed algebraically[15] as

$$k = \frac{a}{d} = \frac{b}{e} = \frac{c}{f} \quad (1)$$

In Geometry III *...the umbilical cord is cut between reality and axiomatic...* (Houdement & Kuzniak, 2003, p. 4) meaning that the axiomatic system has no relation to the reality and is completely separated from manipulating with figures. Within this Geometry it is only possible to draw conclusions on the basis of the axiomatic system (Houdement & Kuzniak, 2003, p. 4). This Geometry is very Euclidian since in the mathematics of Euclid there are no examples, just definitions, axioms, theorems and proofs. However, it is worth mentioning that Euclid did not use any symbols, only actual language, which means that there are no numbers (apart from some positive integers) and no units of measurements such as cubits or degrees (Katz, 2009, pp. 51–52). However, Geometry III is not prioritised in the high schools (see section 4.1.2), hence it will not be considered further in this thesis.

In high school, some of the mathematical conceptual difficulties can lie within the transition from Geometry I to II such as the transition from perceiving an angle as a space between two lines, probably measured with a protractor, to define the angle measure in Analytical Geometry[16] as a rotation from one ray to another ray. This is to be elaborated further down. Likewise, it can be just as challenging for the students to link algebra to the figure - for instance to comprehend the expression of the scale factor based on figures (similar triangles).

What is Trigonometry?

So, what exactly is Trigonometry? Trigonometry is a branch of mathematics dealing with: *The study of angles and of the angular relationships of planar and three-dimensional figures...* (Webpage: Trigonometry E. W. Weisstein, 2016). Noticeably, the first part of the word trigonometry, *trigon*, means “triangle” and the second part;

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[15] An algebraic method is within this thesis understood to be performed when one deals and operates with letters and symbols in order to represent numbers and values (Webpage: Algebra, 2016).

[16] *Analytical Geometry* is also known as coordinate geometry or Cartesian Geometry which is the study of geometry using a coordinate system(Webpage: Analytic Geometry E. W. Weisstein, 2016).

metron means “a measure” in Greek (Webpage: What Is Trigonometry?, 2016). To understand what a *triangle* is, the definition of a *polygon* is needed which is a plane figure

> *...that is bounded by a finite chain of straight line segments closing in a loop to form a closed chain or circuit. These segments are called its edges or sides, and the points where two edges meet are the polygon's vertices (singular: vertex) or corners.* (Webpage: Polygon, 2016).

Hence, an *n-gon* has *n* sides and *n* vertices, i.e. a triangle is a 3-gon since a triangle per definition is *...a closed plane figure having three sides and three angles* (Webpage: Triangle, 2016). However, this definition does not say much about what an angle exactly is. This will be elaborated further down.

Before going further it is to be noted, that there exist seven different types of triangles, namely: 1) *Isosceles triangles* (having two sides of equal length), 2) *Equilateral triangles* (where all three sides are of equal length), 3) *Scalene triangles* (where none of the three sides are of equal length), 4) *Obtuse triangles* (having one interior angle over 90°), 5) *Acute triangle* (where one of the interior angles is less than 90°), 6) *Equiangular triangles* (where all the interior angles are equal) and 7) *Right triangles* (where one of its interior angles is equal to 90°) (Webpage: Triangle definition and properties, 2016). The triangles of the types 1)-6) are henceforth denoted *arbitrary triangles*.

Three contexts in connection with Trigonometry

In high school, there are three different and often disjoint presented approaches to the introduction of Trigonometry, namely the *Triangle context*, the *Unit circle context* and the *Function context*. In the Triangle context, which is embraced by the sector of Synthetic Geometry[17], sine, cosine and tangent are defined by the ratios of sides in a right triangle. Hence, trigonometry in relation to this context is solely a toolbox for solving problems concerning triangles in plane geometry. In the Unit circle context,

[17] *Synthetic Geometry ...is the study of geometry without the use of coordinates...* (Webpage: Synthetic geometry, 2016).

within Analytical Geometry, cosine and sine with respect to an angle are respectively the x and y coordinate to a point on the unit circle. In the Function context sine and cosine possess the properties of a function such as range, domain, zeroes, periodicity. (Winsløw, 2016, p. 4).

The risk of the incoherent presentation of the above mentioned contexts is very much consistent with Felix Klein's Plan A[18], which *...divides the total field into a series of mutually separated parts and attempts to develop each part for itself with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields* (Winsløw, 2016, p. 2). Whereas Klein’s Plan B *...involves a more holistic approach which emphasises and exploits connections between different sectors* (Winsløw, 2016, p. 2) and seeks to see mathematical science as a connected whole. The three contexts are presented in Figure 5 below.

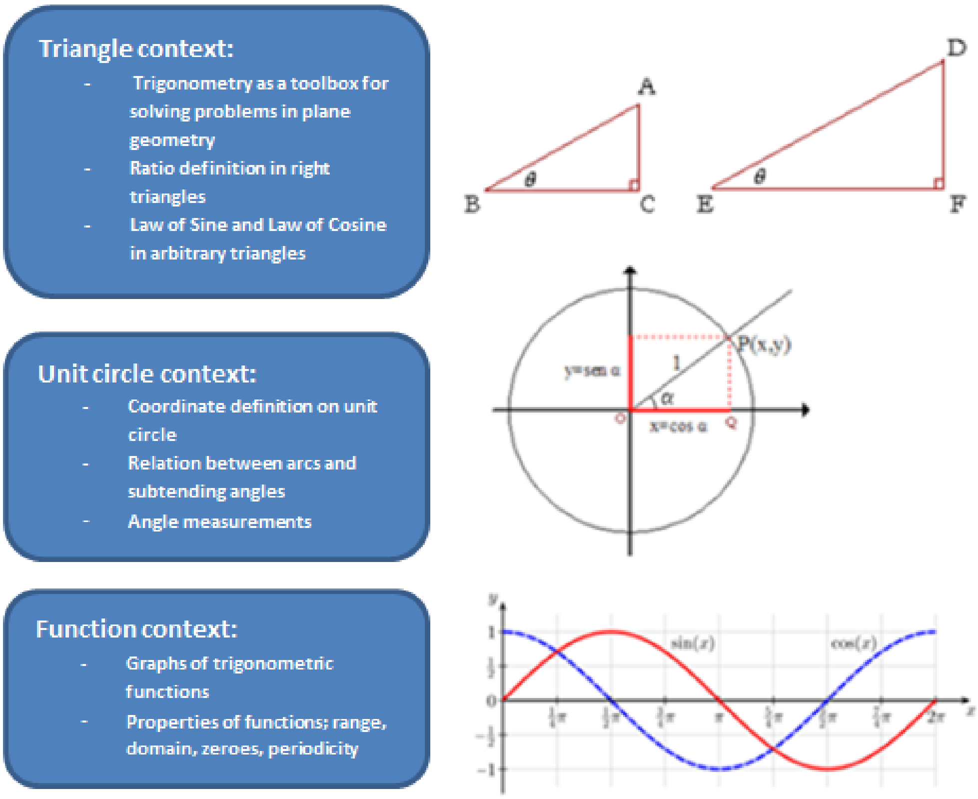

Figure 5
The three contexts; the *Triangle context,* the *Unit circle context* and the *Function context*, in relation to Trigonometry.

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[18] Felix Klein was a German mathematician who lived from 1849-1925 (Webpage: Klein biography, 2016). Klein discovered that two Plans (Plan A and Plan B) can be identified when a new subject (here understood as a theme) is to be developed in high school (Winsløw, 2016, p. 2).

Regarding the Function context, there exist six trigonometric functions being cosine, sine, tangent and their reciprocal functions[19], cotangent, secant and cosecant. The trigonometric functions *...are used in obtaining unknown angles and distances from known or measured angles in geometric figures* (Webpage: Trigonometry, 2016) and are especially connected to triangles and the unit circle. The functions used in the present time can be traced back to 320 to 550 CE and the first publication of the abbreviations *cos*, *sin* and *tan* are from the 16th century, made by Albert Girard[20]. Nowadays, the use of trigonometry exists in fields as chemistry, architecture, meteorology, engineering and astronomy biology, just to mention a few (Sultan & Artzt, 2011, p. 513). It is noticeably only the functions of sine and cosine which are to be considered within the theme of Trigonometry in this thesis, since tangent is not involved in the work with GG (and is only introduced very briefly by TT). Moreover, the function context is not considered further since this aspect is not prioritised on the observed level of high school (see section 4.1.3).

One of the obstacles in relation to the contexts is especially whether the students are capable of seeing the connection between the triangle context and the unit circle context on the level of the mathematical theory. Do they know why sine respectively cosine to an angle are relationships between specific sides of the right triangle when being introduced to sine and cosine in relation to the unit circle or do sine and cosine just function as tools that appears to be magical when calculating angles and sides in triangles? Within this transition, the notion of an angle is expanded to go beyond what meets the eye, which is to be elaborated in the following.

---

[19] x and $\frac{1}{x}$ for $x \neq 0$ are *reciprocal functions* since their product is 1 (Webpage: Reciprokfunktionen, 2016)

[20] Albert Girard (1595-1632) is a French mathematician (Webpage: Albert Girard biography, 2016).

The notion of an angle

The triangle context:

The notion of an angle is very complex and can be considered as much more than just the space between two lines, which continuously will be discovered throughout this section. The word *angle* stems from the Latin word "Angulus" which means "a little bending". It is seen in the English word "ankle" which is the bend between the leg and the foot (Webpage: What Is Angle?, 2016). Within plane geometry, an angle is the measure of the space between two rays with a common vertex. The rays can be called the initial side respectively the terminal side in order to describe the measure of the angle, as seen on Figure 6 (Sultan & Artzt, 2011, p. 524).

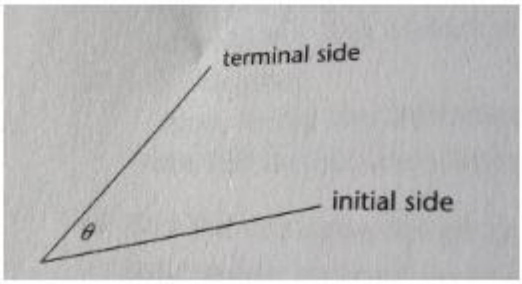


Figure 6 A positive angle, $\boldsymbol{\theta}$, between two rays. (Sultan & Artzt, 2011, p. 524)

Thus, the angle can be seen as the space between the two sides and it can also be seen as the rotation from the initial side to the terminal side.

Now, considering the angle of a triangle, it is noted that it can be both interior and exterior. The sum of the interior angles is 180° (the straight line), π radians, two right angles (cf. the Triangle Postulate) or a half turn of a circle. What exactly is meant by radians of a circle will also be further examined in the subsection below; The unit circle context.

The *Triangle Postulate* is equal to the Parallel Postulate which states that: *Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line, no matter how far they are extended* (Webpage: Parallel Postulate E. W. Weisstein, 2016). This postulate is equivalent to Euclid's fifth postulate which, in two-dimensional geometry, states the following (see Figure 7):

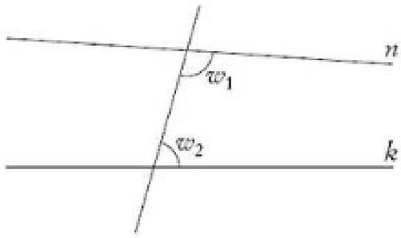


Figure 7 Two line segments *n* and *k* intersect each other since the angles $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ are less than two right angles (Webpage: Parallelaksiomet, 2016).

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles (Webpage: Parallel postulate, 2016).

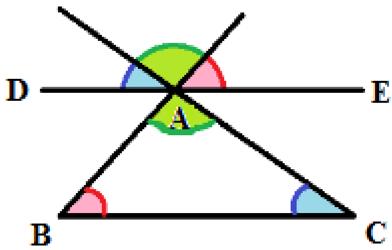


Figure 8 The angle sum in a triangle. With inspiration from (Webpage: Trekanter og vinkler (Matematik C), 2016).

On Figure 8, it is seen how the Parallel Postulate is used in the proof of the theorem saying that *...the sum of the angles of a triangle is 180 degrees* (Sultan & Artzt, 2011, p. 687). It is clearly seen that the sum of the three angles (red, green and blue) on the top of the line segment DE constitutes an angle equal to 180° since a straight angle has 180°. It is also noted that through the point A, located outside the line BC, line DE is drawn which is parallel to BC according to the Parallel Postulate. The blue angle at C and the other blue angle are equal and the red angle at B and the other red angle are equal because DE and BC are parallel to each other. This is due to the fact that *...when two parallel lines are cut by a transversal, the alternate interior angles are equal* (Sultan & Artzt, 2011, p. 687). The two green angles are apexes to the same point A which makes them of equal size. This means that the angles of the triangle ABC are equal to the angles on top of the line DE, i.e. 180°. (Sultan & Artzt, 2011, p. 687).

A triangle noticeably has interior angles as well as exterior angles. The measure of exterior angles of a triangle *...is greater than either of the measures of the remote interior angles* (Webpage: Exterior angle theorem, 2016), according to the exterior angle theorem (Proposition 1.16 in Euclid's Elements).

In the triangle context, sine with respect to an angle and cosine with respect to an angle are defined as $\sin(\theta) = \frac{opposite\ cathetus}{hypotenuse}$ and $\cos(\theta) = \frac{adjacent\ cathetus}{hypotenuse}$, for right triangles, where $0\,° \leq \theta \leq 90\,°$. For arbitrary triangles sine to an angle can be found by the Law of Sine:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \qquad (2)$$

Further, cosine to an angle can be found by the Law of Cosine:

$$\cos(A) = \frac{b^2+c^2-a^2}{2bc}, \cos(B) = \frac{a^2+c^2-b^2}{2ac} \text{ and } \cos(C) = \frac{a^2+b^2-c^2}{2ab} \quad (3)$$

Noticeably, a, b and c are the sides and A, B and C are the angles of the triangle in both of the presented laws and $0° \leq \theta \leq 180$ °, where θ is one of the angles (A, B and C) of the triangle.

Within Analytic Geometry, sine and cosine (considered as functions of a variable x) are however defined on the entire $\mathbb{R}$, meaning that the output also can attain all real values which will be elaborated further in the subsection below.

The unit circle context:

In Analytic Geometry, the space between two sides can be considered as a rotation from the initial side to the terminal side and if the rotation is counter clockwise, the angle is said to be positive. Conversely, if the rotation is clockwise, the angle is negative (see Figure 9). In fact, the following trigonometric relationships hold which can be seen from Figure 9:

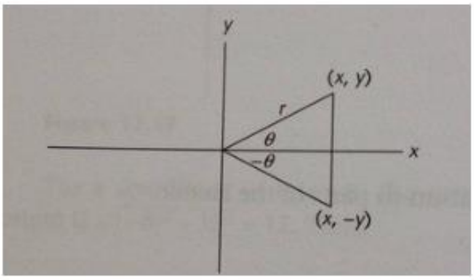


Figure 9 Clockwise versus counter clockwise rotation and the sign of an angle $\boldsymbol{\theta}$ having the x-axis as the initial side.
(Sultan & Artzt, 2011, p. 528)

$$\cos(\theta) = \cos(-\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$

These relationships will also hold in any other quadrant since θ and $-\theta$ always are reflections of each other around the x-axis. In Figure 10, both the angles θ and $180 - \theta$ are presented and it follows immediately that

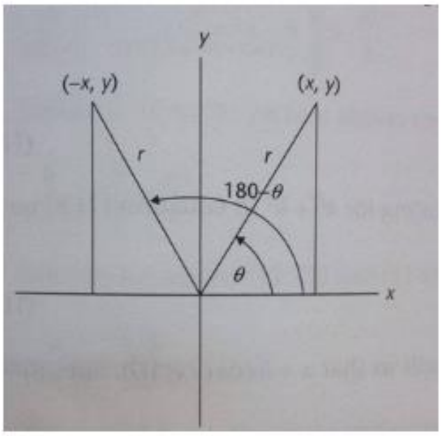


Figure 10 Coordinates with respect to $\boldsymbol{\theta}$ and $\mathbf{180} - \boldsymbol{\theta}$.
(Sultan & Artzt, 2011, p. 529)

$$\cos(180 - \theta) = -\cos(\theta)$$
$$\sin(180 - \theta) = \sin(\theta)$$

Hereby, it is seen how it is possible, within this context, for an angle to be less than zero degrees which is not possible in the triangle context.
The Figure 11 below presents an example of an angle being more than 180° as was used as an upper limit of an angle in the triangle context. Here, the angle is 800° since there are two full counter clockwise rotations of 360° and 80° in addition.

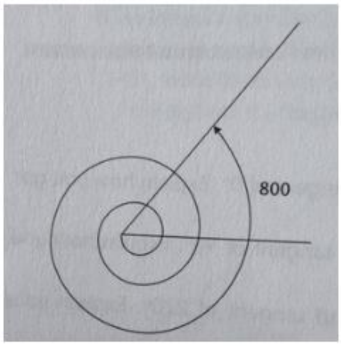


Figure 11 An angle of **800**°. (Sultan & Artzt, 2011, p. 524)

The above presented trigonometric relationships and the option for an angle to be more than 360° may be an aspect which the students find challenging.

When considering angles in a circle, one may talk of both a central angle and an inscribed angle[21], but the latter will not be considered further on. A central angle (θ) has its vertex in the centre of the circle, illustrated in Figure 12 below, and it has the same degree measure as the arc subtended by the central angle which is the arc AB with length s as also shown on the figure (Sultan & Artzt, 2011, p. 180).
This means that the arc subtended by θ is defined to also be θ. When thinking about it, this actually does make sense since a full rotation of the circle is 360° which makes it possible to consecutive have 360 central angles of 1°. This leads to the circumference of the circle being divided in 360 parts which makes the number of central angles equal to the number of parts of the circle, as they are both 360. From this, it follows that *...any arc will have the same number of degrees as its central angle* (Sultan & Artzt, 2011, p. 180).

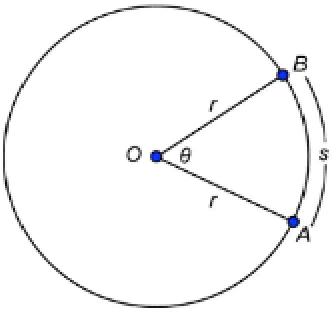


Figure 12 A central angle of a circle. (Webpage: Degree and Radian Measure, 2016)

[21] If the angle is not central but inscribed, it has its vertex on the circumference of the circle and its initial side and terminal side terminates on the circumference of circle (as chords of the circle). An angle inscribed in a circle is measured by $\frac{1}{2}$ of its degree arc (Sultan & Artzt, 2011, p. 181).

Like there exist several units for describing length, such as meter and feet, and temperature, such as Fahrenheit and degree Celsius, degrees and radians both function as units for describing angles. When dealing with the unit circle which has its center in origo $O = (0,0)$ and radius 1, it is more preferable to use radians instead of degrees. When considering Figure 12, a *radian* of a central angle (θ) can be defined to be the ratio of the distance along the circumference of the circle (arc length), *s*, and the circle radius, *r* as

$$\theta = \frac{arc\ length}{radius} = \frac{s}{r} \text{ (Sultan \& Artzt, 2011, p. 537).}$$

The arc length of the entire circle is 2π since the radius in the unit circle is 1 which makes the circumference of the circle,

$$C = 2\pi r = 2\pi \cdot 1 = 2\pi, \text{ where } \pi = \frac{circumference\ of\ the\ circle}{diameter\ of\ the\ circle}.$$

This information gives the possibility of converting angle measures between degrees and radians. A central angle is known to be of 360° and has a radian measure of 2π, i.e. $360° = 2\pi\ radians$. When dividing this by 2, the outcome is $180° = \pi\ radians$ which makes $1\ radian = \frac{180}{\pi}\ degrees$ and $1\ degree\ = \frac{\pi}{180}\ radians$. Hence, this is the way to convert from radians to degrees and vice versa. To sum up, when converting from degrees to radians; multiply by $\frac{\pi}{180}$ radians and when converting from radians to degrees; multiply by $\frac{180}{\pi}$ degrees. (Sultan & Artzt, 2011, p. 180).

Radians is to prefer when describing angles since *...it makes certain formulas in calculus much simpler* (Sultan & Artzt, 2011, p. 538) such as finding the derivative of $\cos(x)$. If *x* is measured in degrees, the derivative of $\cos(x)$ is $-\frac{\pi}{180} \cdot \sin(x)$ whereas, if *x* is measured in radians, the derivative of $\cos(x)$ is $-\sin(x)$ which is a much simpler expression and easier to work with prospectively. This is, however, not the only reason to use radians. Radians are noticeably dimensionless quantities which makes it possible to use them in all contexts regardless of the unit one work with when measuring (for instance *cm*, *m* etc.). This is due to the way radians are defined, namely as the ratio of the arc length and the radius which cancel out the units of measurement. (Sultan & Artzt, 2011, p. 538; Webpage: Radian, 2016). This means that angles can be described through real numbers and when sine or cosine is evaluated on such input it can also attain any real number, ℝ. However, there may lie a conceptual challenge as to understand the fact that the measure of an angle corresponds to the *length* of an arc and not just to a degree measure. Even at university level the notion of an angle remains somewhat a mystery, especially to think of angles as numbers and where those numbers come from (Winsløw, 2016, pp. 8–9). In continuation hereof, the remarkable notions of *sine* and *cosine* are to be considered - what exactly do they mean?

Sine comes from the Latin word "sinus" and means "bowstring" from the Sanskrit word "jiva". Figure 13 below gives an explanation of this, namely that sine to an angle is half of the corresponding "bowstring" to which the bow (CAE on the figure) spans - i.e. CE on the figure; which may look like a bowstring in relation to CAE. In the figure, this means that sine to the angle v_1 is the line segment CD. The sine of v_1 is the ratio of CD the radius of the circle, OA, thus $\sin(v_1) = \frac{CD}{OA}$.

Originally cosine was written co.sine; an abbreviation for sine of the complement. Revisiting Figure 13, it is observed that cosine of the angle v_1 is the sine of $\angle OBA$ which is v_2 (the complementary angle[22]). Because of the Parallel Postulate, $\angle OBA$ and $\angle OCD$ are of equal size, hence $\angle COD'$ are also of the same size. So, $\cos(\angle AOB) = \cos(v_1) = \sin(\angle OBA) = \sin(v_2)$ which gives $\cos(v_1) = \frac{CD'}{OA}$.

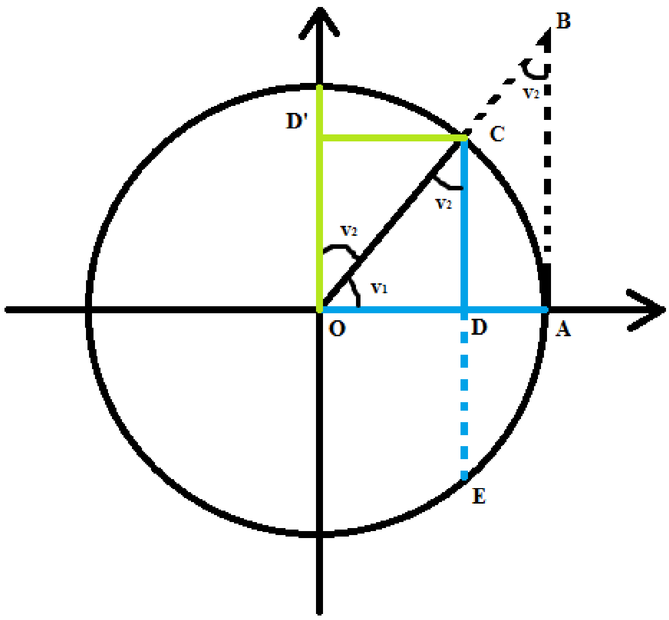


Figure 13 An illustration of cosine and sine with respect to v_1.

[22] A *complementary angle* to an angle (v) is of a size which, when added to *v*, gives 90 ° or $\frac{\pi}{2}$ radians (Webpage: Complementary Angles E. W. Weisstein, 2016).

(Webpage: Latin Origins of Trig Functions, 2016). Hereby, it is seen that angles and their measures are related to sine and cosine just as it applies in the triangle context.

4.1.2 The Curriculum in Relation to the Theme

This section will present an overview of what the students should know how to do in the high school concerning what is especially relevant in relation to the theme of Trigonometry. This overview will be based on the *curricula* for Mathematics A-level (C-A) and B-level (C-B) Stx (Undervisningsministeriet, 2013a, 2013b) as well as a trial curriculum for Mathematics A-level (C-TA) (Digital eksamen i matematik A - forsøgslæreplan, 2009) which is noted to be an expanded version of the C-A, i.e. C-A is included in C-TA why this program will be referred to henceforth. The choice of including C-TA is due to the fact that *the teacher teaching the Trigonometry course* (TT) uses this as curriculum. As a supplement, the *teacher's guide* (TG) for A-level and B-level (respectively TG-A and TG-B: (Matematik A - stx vejledning, 2010; Matematik B - stx vejledning, 2010)) and what is expected of the students for the written examinations (A-level) in relation to Trigonometry will be presented. The choice of including both the C-B and the TG-B, despite of the class having mathematics on A-level, is due to the fact that the class is of first grade of high school, and that the teaching material (see section 4.1.3) does not extend to more than B-level at this point.

The *core substance* (section 2.2 of C-A and C-B) in relation to the theme is the same for both of the EPs, namely:

(1) Ratio calculations in similar triangles and trigonometric calculations in arbitrary triangles.

However, the *academic objectives* (section 2.1 of C-A and C-B), i.e. what the student must have achieved academically, at the two levels are slightly different from each other. On B-level the student must be able to:

(2) Establish and explain geometric models and solve geometric problems. What exactly is meant by this is further elaborated in TG-B. Here, concepts like altitude, medians and angle bisecting lines are mentioned in order for the students to explain geometric models and solve geometric problems. Further, TG-B states that students should have the ability to handle and implement trigonometry including knowledge of the Law of Sine and the Law of Cosine in order for the students to reach the academic objective.

Regarding the academic objectives to be found in C-TA, the students should, besides from establishing geometric models and solving geometric problems (like on B-level), further be able to:

(3) Establish and explore geometric models and solve geometric problems both by means of a dynamic geometry program and on the basis of calculations concerning triangles as well as providing an analytical description of geometric figures in a coordinate system and utilise this to answer given theoretical and practical problems.

(4) Use CAS-tools to explore and solve given mathematical problems.

Noticeably, the academic objective of giving analytical descriptions is only partly covered in the teaching course by TT, namely in relation to the use of the unit circle and therefore the further aspects of the objective covering all three years of the A-level mathematics will not be considered in the following (where the references are C-B, C-TA, TG-A).

TG-A gives the same detailed description of what the students should be able to as for B-level. However, TG-A also states that the students need to be able to determine angles between lines.

In the following, a list of types of tasks (T) has been established on the basis of exam exercises for the written examination related to the theme of Trigonometry (and are denoted T_{ET}). The explored exam exercises can be referred to the written examinations from 2013-2016[23] - more specifically six exam assignments. The types of tasks have been established by exploring specific tasks (t) in the given exam exercises which can be solved by applying the same technique.

$\boldsymbol{T_{ET_1}}$: Given an arbitrary triangle, where two sides and the angle between them are known. Determine the unknown side.

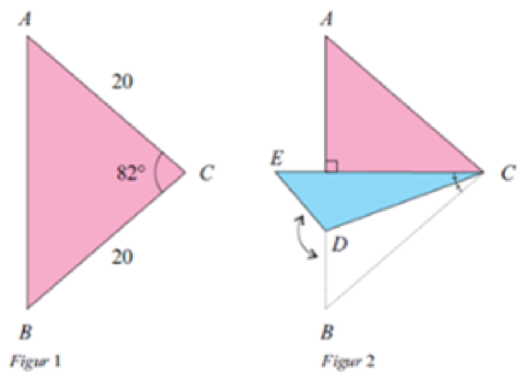
---

[23] Noticeably the written examinations from earlier years are not publicly accessible why there is made no explicit references thereto, besides from the specific examples of exercises and tasks from some of the written examinations.

T_{ET_2}: Given an arbitrary triangle where all three sides are known. Determine one of the angles.

T_{ET_3}: Given an arbitrary triangle. Two angles and the side between them are known. Determine one of the unknown sides.

In Figure 14 an exam exercise with tasks of the types T_{ET_1}, T_{ET_2} and T_{ET_3} is presented.



A piece of carton has the form of an isosceles triangle where the legs are 20 cm as shown on the figure 1.

a) Determine $|AB|$, and determine the angle A.

The carton is folded such that it looks like a sailboat as shown on figure 2. The carton is folded around a line CD such that B is transferred to E.

b) Determine $|DC|$, and determine the area of triangle CDE.

Figure 14 Exercise from 2013 (August) involving tasks of type T_{ET_1} and T_{ET_2} (in exercise *a)*) and T_{ET_3} (in exercise b)).

An interesting example of an exam exercise involving a task of the type T_{ET_1} is presented in Figure 15 and involves the construction of a median as well.

Given a triangle ABC where $\angle A = 80°$, $a = 12$ and $b = 11$

a) Determine $\angle B$ and $\angle C$.

b) Determine the length of the median m_a.

Figure 15 Exercise from exam in 2014 (May) where *b*) relates to T_{ET_1} when one has constructed the median.

T_{ET_4}: Given an arbitrary triangle where two sides and the intermediate angle are known. Determine the area of the triangle.

In Figure 16 an exam task of type T_{ET_4} is presented.

In triangle ABC $\angle A = 62°$, $c = 5$ and $b = 7$

b) Determine the area of ABC

Figure 16 Exam task from 2015 (August) - modified in order to not present the whole exercise involving a part *a)* - of the type T_{ET_4}.

T_{ET_5}: Given two similar triangles, triangle 1 and triangle 2. Two sides are known in triangle 1 and one side, corresponding to one of the sides in triangle 1, are known in triangle 2. Determine the side of triangle 2 which corresponds to one of the sides which is known in triangle 1.

In Figure 17 an exam task related to the task of type T_{ET_5} is presented.

On the figure is shown a model of a sails ADE. It is informed that DE and BC are parallel and $|AE|$=10, $|AC|$=4, and $|BC|$=2.

Determine $|DE|$.

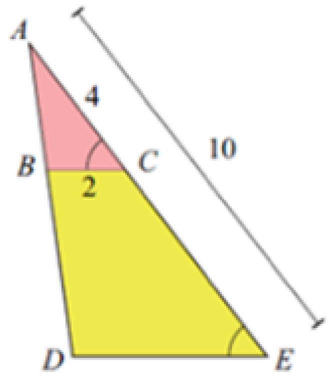


Figure 17 Exam task from 2014 (May) (the part without aids) including a task of type T_{ET_5}.

Noticeably, these exam tasks only require the establishment of a practical block by the students (as a minimum) since there are no questions directly relating to a technological discourse. Regarding the techniques applicable for solving the types of tasks presented in the above, they will be briefly sketched in the following whereas section 4.3.2 presents more detailed (instrumented) techniques therefor.
The students should be able to apply the Law of Cosine (point (2)) in order to answer both T_{ET_1} and T_{ET_2}. For T_{ET_1} one should isolate the unknown side in one of the formulas (3) in section 4.1.1 (subsection: The notion of an angle) and insert the

known sides and angle before calculating. For T_{ET_2} one should solve with respect to the angle to be found after having inserted the known values of the three sides in one of the formulas (3) in section 4.1.1 as well. The Law of Sine (point (2)) is to be applied in order to answer T_{ET_3}, after having determined the last and unknown angle (knowing that the sum of angles is 180° in a triangle). That is, the formula (2) in section 4.1.1 (subsection: The notion of an angle) is to be used by isolating an unknown side, and thereafter the known values for sides and angles are inserted in the formula, before calculating. T_{ET_4} is to be solved based on the student's ability to handle and implement trigonometry (point (2)), which in this case is the area formula $\frac{1}{2}bc\sin(A) = \frac{1}{2}ca\sin(B) = \frac{1}{2}ab\sin(C)$ (Webpage: Triangle Area, 2016). The known sides and angle are inserted in the formula, before calculating. The exercise presented in Figure 15 can noticeably only be solved by knowing how to construct a median (this should be well known for the students since they are already familiar with the term in elementary school (Beck, Kaas, Fink, Danmark, & Styrelsen for Evaluering og Kvalitetsudvikling af Grundskolen, 2010, p. 22)). To answer T_{ET_5}, the students need to know about ratio calculations in similar triangles (point (1)). More specifically, the ratios presented in (1) in section 4.1.1 (subsection: Geometry I, II, and III) are to be applied where the known sides of the similar triangles are inserted and the side to be determined is isolated, followed by calculation. Similar triangles and scaling are neither unknown concepts since it is also knowledge gained from elementary school (Webpage: Fælles mål 2009 (historiske) Folkeskolen, 2016, p. 60 accessed by selecting "Matematik" and then via the link "Matematik (Fælles Mål 2009)(pdf)"). Noticeably, trigonometric calculations in arbitrary triangles (point (1)) is recurring throughout the methods used to solve the tasks given in the written examinations. As an endnote, exam exercises/tasks involving tasks of the types T_{ET_1} - T_{ET_5} may require that the students can use dynamic geometry programs or CAS-tools (classic) in order to solve the tasks (point (3) and (4)).

4.1.3 Teaching Material

In this section, the content of trigonometry on C and B level in the teaching material will be presented since they are intertwined at the point of high school current for the students of this observation study. TT uses two different web pages as teaching material, namely webmatematik.dk (Webpage: Gratis online hjælp - Webmatematik, 2016) and a webpage created by TT (TTW), of which the latter is created by the teacher and contains supplementary materials as well as exercises for the lessons in mathematics[24]. The web page is primarily directed towards the students studying on the observed high school. To be noted, all the students of this class has a formula collection as well; (Dejgaard & Schomacker, 2007) which will not be described here.

Webmatematik.dk is a web page providing free online help in mathematics. On the page mathematics on C, B and A level (stx) can be selected by clicking on the different levels. This click will then lead to a presented overview of the mathematical content at the specific level. On C level by clicking on Trigonometry ("Trigonometri"), it will be presented by yet another overview which is a list of the following headlines: Triangles and angles, Similar triangles, Isosceles and equilateral triangles, Right triangles, Cosine and Sine (defined with respect to the unit circle), Tangent (defined with respect to the unit circle), Cosine, Sine and Tangent in relation to the right triangle. Noticeably, it seems that this web page presents Trigonometry as a *domain* (and not a *theme* like in section 2.2) where each of the above mentioned are considered *themes* which therefore will be done throughout this section (4.1.3).
By clicking on each of the themes, an explanatory text will appear including illustrations. At the bottom of the page, a video is to be found in most cases, in which the theme is presented orally (the same applies for B-level). Hence, Trigonometry at this level deals with cosine, sine and tangent in relation to right triangles, thus the application of the formulas:

$$\cos(v) = \frac{adjacent\ cathetus}{hypotenuse}, \sin(v) = \frac{opposite\ cathetus}{hypotenuse} \text{ and } \tan(v) = \frac{opposite\ cathetus}{adjacent\ cathetus},$$

where *v* denotes an angle. Noticeably, measures of angles are described only in degrees at this level. The web page thus presents trigonometry both within the Synthetic Geometry and the Analytic Geometry and makes an effort for presenting the

[24] In order to keep TT anonymous throughout the thesis no direct reference is made to TTW.

transition between the two. On B-level, the overview of themes under the domain of Trigonometry are:
The Basics (contains information concerning the definitions of cosine, sine and tangent and their relation to the right triangle at C-level), The Law of Cosine, The Law of Sine, the Law of Sine in obtuse triangles, The Area formula, and The basic relation between sine and cosine[25] (including a brief presentation of the unit circle). Here, Trigonometry therefore also contains the arbitrary triangles in relation to the definitions of cosine, sine and tangent (as opposed to C-level), and angles still remain measured in degrees at this level.
What is common for each level is that few tasks are available, however these tasks are not used by TT. TT uses tasks on TTW.

On TTW, tasks can be found within the themes for both levels. The tasks are oriented towards GG, such that the dynamically aspect of GG is required to solve the specific task. A few examples hereof are listed below.

$\boldsymbol{t_{TTW_1}}$: Given the constructed unit circle and the directional point P=(Px,Py), where Px=cos(v) and Py=sin(v). Drag the point P and answer, what is cos(45)?
(Task 6. a. in TTW 1, A.2).

$\boldsymbol{t_{TTW_2}}$: Given the constructed unit circle and the directional point P=(Px,Py), where Px=cos(v) and Py=sin(v). Drag the point P and answer, what is sin(75)?
(Task 6. b. in TTW 1, A.2).

If the students have difficulties in answering t_{TTW_1} and t_{TTW_2} from their own construction work in GG there is a GG worksheet which has all the required information to help the students solve the tasks. This worksheet is described in the a priori analysis in connection with Trigonometry (section 4.3.2). Some other examples of tasks on TTW are tasks related to the actual construction work, such as:

[25] The basic relation is $\cos^2(v) + \sin^2(v) = 1$ (Webpage: Grundrelationen (Matematik B, Trigonometri), 2016).

t_{TTW_3}: Construct a triangle ABC, where a = 9, b = 7 and c = 5. You may only use 8 features of GG (presented in A.2 as TTW 4).
(Task in TTW 2, A.2).

t_{TTW_4}: Construct a triangle ABC, where A = 30°, b = 7 and a = 5. You may only use 8 features of GG (presented in A.2 as TTW 4).
(Task in TTW 3, A.2)[26].

t_{TTW_5}: Draw an arbitrary triangle in GG.

t_{TTW_6}: Construct the median to each of the three sides of the arbitrary triangle constructed in t_{TTW_5}.

(The tasks t_{TTW_5} and t_{TTW_6} are connected and presented in TTW 5, A.2)

A video presentation of the construction work in GG is provided, in relation to both t_{TTW_3}, t_{TTW_4}, t_{TTW_5} and t_{TTW_6}. This is the case for almost all construction tasks on TTW.
The specific tasks just mentioned can noticeably be put in relation to the earlier presented types of tasks T_{ET} from the written examinations since the exam tasks can be solved through techniques involving gestures represented through t_{TTW_3} - t_{TTW_6}. An example of a set of gestures leading up to solving t_{TTW_4} and hence solving some of the exam tasks involving unknown sides and angles will be given in section 4.3.2. Furthermore, the TTW offers GG worksheets (see section 4.3.1), primarily click proofs, which are described in the a priori CAS analysis (section 4.3.2). On the webpage click proofs of the Pythagorean theorem as well as for the area formulas can be found. TTW also offers manuals which describes in detail how a specific object must be constructed. An example of such a manual can be found in the A.2 (TTW 6) which is a description of how to construct a unit circle as well as a directional point

[26] The instrumented technique for solving t_{TTW_4} is examined in section 4.3.2 concerning the a priori analysis of the CAS tool.

and a directional angle. The instructed techniques are equal to the ones presented in section 4.3.2, namely $\tau^i_{UC,AP,TR}$, $\tau^i_{DPDA,AP,TR}$.
As a final note, the GG tasks related to Trigonometry, offered by TTW, provide the option of putting algebra in relation to constructed figures (Geometry II).
To sum up, this webpage provides an insight to the program of GG through tasks, GG worksheets and videos. By combining the TTW and the webmatematik.dk the teacher gives the students the possibility of establishing both a practical block, where the techniques used are mainly instrumented since they are related to GG, and a knowledge block, especially through click proofs on TTW and the knowledge gained from the more detailed explanations of the themes of Trigonometry on webmatematik.dk.

4.2 Power Functions (A)

This section is concerned with the establishment of the RPM connected to the theme of Power Functions.

4.2.1 A Profile of the Theme

The definition of a power function

A *power function* can be defined as a function of the following form:

$$f(x) = bx^a, \text{ where } b, a \in \mathbb{R}\backslash\{0\}$$

(Yoshiwara & Yoshiwara, 2007, p. 257), where *x* denotes the *base* and *a* denotes the *exponent* of the power (x^a), and *b* denotes the *constant of proportionality*.
This form will be supposed throughout this section where nothing else is stated. Noticeably, the exponent (here denoted a) is sometimes allowed to be 0 as well (as in the teaching material for instance (see section 4.2.3)) which just means that constant functions, of the form $f(x) = b$ for $b \in \mathbb{R}\backslash\{0\}$, are included in the definition of a power function (Webpage: Power Functions, 2016). Further, it is common to restrict the constant of proportionality (b) to be greater than zero (also done in the teaching material presented in section 4.2.3) which seems to be a convention rather than a necessity for the function to be well defined. As for the domain of the power function, the teaching material (section 4.2.3) seems to restrict *x* to be greater than zero.

However, the way the power function is presented by Yoshiwara & Yoshiwara (2007, pp. 233-320 (Chapter 3)), there are more options for the domain than the restriction mentioned here which are to be elaborated in the following.

The domain of a power function

For exploring the domain of a power function, the justification of the fact that the exponent of a power can attain any real value - corresponding to the *expanded notion of a power*, referred to in the curriculum (see section 4.2.2) - is to be elaborated. Noticeably, it is often in relation to the domain of the exponential function with a positive real base (see definition in the subsection; *The power function in relation to other elementary functions*), that this expansion of the notion of a power is justified (Bremigan, Bremigan, & Lorch, 2011, pp. 434–435; Eureka Math™, 2015, p. 75). This justification can be divided into the following three steps (when denoting the base of the power *x* and the exponent of the power *a*):

1) $a \in \mathbb{Z}$ [27]
2) $a \in \mathbb{Q}$[28]
3) $a \in \mathbb{R}$[29]

In the following, these steps will be elaborated further.

1) Integer Exponents of Powers (x^a with $a \in \mathbb{Z}$):

The possibility of *positive* integer exponents of powers ($x^a, a \in \mathbb{Z}_+$) can intuitively be argued for, since *a* in this case just express how many times the base *x* occurs as a factor in an expression;

$$x^a = x \cdot x \cdots \cdot x \ (a \text{ times})$$

[27] $\mathbb{Z}$ denotes the set of integers where an integer is either a positive or a negative whole number (including zero); i.e. it can be expressed without any fractional component (Webpage: Integer, 2016).

[28] $\mathbb{Q}$ denotes the field of rational numbers being numbers *...that can be expressed as a fraction 'p/q' where 'p' and 'q' are integers and 'q≠0'* (Webpage: Rational Number E. W. Weisstein, 2016).

[29] $\mathbb{R}$ denotes the field of real numbers consisting of rational and irrational numbers(Webpage: Real Number E. W. Weisstein, 2016). An irrational number *...is a real number that cannot be represented as' $\frac{p}{q}$' for any integers 'p' and 'q' with 'q≠ 0'* (Eureka Math™, 2015, p. 75).

(Yoshiwara & Yoshiwara, 2007, p. 255).

This applies for all $x \in \mathbb{R}$, since any real number can be multiplied by itself a times for any $a \in \mathbb{Z}_+$ and still be a real number (since a (field) property of $\mathbb{R}$ is that it is closed under multiplication (Webpage: Field, 2016)).

Regarding *negative* integer exponents ($a \in \mathbb{Z}_-$), the following applies

$$x^{-a} = \frac{1}{x^a}, \ a \in \mathbb{Z}_+ \text{ and } x \neq 0 \tag{1}$$

Noticeably, there may here lie a conceptual challenge as to understand that a power raised to a negative power is not itself negative;

$$x^{-a} \neq -x^a.$$

Furthermore, the negative exponent only applies to the base, *x*, and not to a factor of the power, if any, i.e.;

$$kx^{-a} \neq \frac{1}{kx^a} \quad \text{but} \quad kx^{-a} = k\frac{1}{x^a}$$

(Yoshiwara & Yoshiwara, 2007, p. 256).

The expression (1) can however readily be argued for by noting a pattern when moving from left to right in the following list;

$$x^1 = x, \qquad x^2 = x \cdot x, \qquad x^3 = x^2 \cdot x, \qquad x^4 = x^3 \cdot x, \qquad \cdots$$

namely, that the exponent increases by 1 every time one factor *x* is added.

Additionally, when moving from right to left, the exponent noticeably decreases by 1 every time one factor *x* is removed, i.e. every time a division by *x* is made. Continuing dividing by *x* each time the exponent decreases by 1, the list can be expanded to the following:

$$\cdots, x^{-3} = \frac{1}{x^3}, x^{-2} = \frac{1}{x^2}, x^{-1} = \frac{1}{x^1}, x^0 = 1, x^1 = x, x^2 = x \cdot x, x^3 = x^2 \cdot x, \cdots$$

Here, *x* clearly cannot be zero.

Now, it can from the list just above be deduced that powers with *negative* integer exponents are unit fractions[30] with powers of *x* in the denominator having exponents equal to the numerical value of the negative integer exponents, i.e.:

$$x^{-a} = \frac{1}{x^a}, \quad \text{for } x \neq 0, a \in Z_+.$$

Furthermore, for the specific case where the exponent equals zero, the list above reveals that any non-zero number raised to the 0th power equals 1;

$$x^0 = 1, \qquad \text{for } x \neq 0.$$

(Yoshiwara & Yoshiwara, 2007, pp. 255–256).
Hereby, integer exponents of a power have been justified.

Regarding the domain of power functions *f* with integer exponents a, it can be summed up to be $\mathbb{R}$ when $a > 0$ and $\mathbb{R}\backslash\{0\}$ for $a \leq 0$, i.e. $\mathbb{R}\backslash\{0\}$ for a being an arbitrary integer.

2) Rational Exponents of Powers (x^a with $a \in \mathbb{Q}$)

Rational exponents of powers are less intuitive than integer exponents. With the conception of a power having integer exponents, it can be a challenge to understand what is meant by an exponent of the form $\frac{1}{n}$ and more generally $\frac{m}{n}$ for m, n integers with $n \neq 0$. Here, the concept of a root becomes relevant.
The nth root of a real number $x \geq 0$ is defined to be the number which, multiplied by itself *n* times (*n* being any integer ≥ 2), is equal to x (denoted $\sqrt[n]{x}$ [31] and called a *radical* with *radicand x* and *index n*). Especially, the following applies for the *n*th root of a positive real number *x*:

$$\sqrt[n]{x} = x^{\frac{1}{n}} \quad \text{for } n \geq 2 \in \mathbb{Z} \text{ and } x \geq 0 \qquad (2)$$

(Yoshiwara & Yoshiwara, 2007, pp. 268–269).

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[30] Unit fractions are fractions with 1 in the numerator, i.e. on the form $\frac{1}{n}$, *n* a non-zero integer (Webpage: Unit Fraction E. W. Weisstein, 2016).

[31] The symbol $\sqrt[n]{x}$ refer to the positive root, whereas the negative root is written as $-\sqrt[n]{x}$ (Yoshiwara & Yoshiwara, 2007, p. 268, p. 275).

For the realisation of (2), the third law of exponents (L_3) in the subsection on *Laws of exponents* further below) is usable, since it leads to the following equality:

$$\left(x^{\frac{1}{n}}\right)^n = x^{\frac{1}{n}\cdot n} = x.$$

A common misunderstanding to be noted, when considering a power with exponent $\frac{1}{n}$ ($n \geq 2 \in \mathbb{Z}$), is that $\frac{1}{n}$ is not the factor of the base;

$$x^{\frac{1}{n}} \neq \frac{1}{n}x$$

(Yoshiwara & Yoshiwara, 2007, p. 269).

Now, for cases where the exponent is on the form $\frac{m}{n}$ where
$m, n \in \mathbb{Z}$, $n \neq 0$ and $m \neq 1$, such powers can be written in the two following forms involving an *n*th root:

$$x^{\frac{m}{n}} = \begin{cases} \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m \\ (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m} \end{cases}, \text{for } x > 0, n \neq 0$$

(Yoshiwara & Yoshiwara, 2007, pp. 282–283).
Noteworthy, the third law of exponents (L_3) in the subsection on *Laws of exponents* further below) (as well as the field property of multiplicative commutativity of elements in $\mathbb{Q}$, whereby $\frac{m}{n} = \frac{1}{n} \cdot m = m \cdot \frac{1}{n}$ (Webpage: Field Axioms E. W. Weisstein, 2016)) has been applied in the above presented forms of $x^{\frac{m}{n}}$.
Hence, $x^{\frac{m}{n}}$ can either be described as "the *n*th root of *x* raised to the *m*th power" or as "the *n*th root of the *m*th power of *x*".

Regarding negative rational exponents (i.e. of the form $\frac{m}{n}$ with *n* or *m* being a negative integer), the existence of such exponents can be realised through a similar approach as the one performed in the reasoning related to the existence of negative integral exponents. This is because of the possible denotation $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m$ where *m* is an integer, whereby the same reasoning as that performed for integer exponents can be applied - as long as it is possible to extract the *n*th root of *x* (to be elaborated further below). For *n* or *m* negative it therefore also applies that:

$$x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \frac{1}{\left(\sqrt[n]{x}\right)^{-m}} = \frac{1}{x^{-\frac{m}{n}}} \quad (3)$$

Noticeably, the domain of powers with rational exponents $\left(\frac{m}{n}\right)$ depends on whether the denominator (n) of the exponent is even or odd and whether the base of the power is positive or negative. This is due to the fact that the existence of real-valued nth roots depends on whether the index, n, is even or odd and whether the radicand is positive or negative. Namely, an even root of a negative number is not a real but a complex number[32]. An odd root can, however, be evaluated for both a positive and a negative real number as radicand. (Yoshiwara & Yoshiwara, 2007, p. 275). Power functions having a power term of the form $x^{\frac{m}{n}}, n \geq 2$, where $\frac{m}{n}$ is positive, accordingly have different domains depending on whether n is even or odd; respectively $\mathbb{R}_+ \cup \{0\}$ and $\mathbb{R}$. Furthermore, zero is not a part of the domain in either case whenever the rational exponent, $\frac{m}{n}$, is negative, since that would mean a denominator equal to zero in the last expression of (3). Hence $\mathbb{R}_+ \backslash \{0\}$ is the domain covering all x^a, $a \in \mathbb{Q}$.

As an endnote for rational exponents of powers, they might as well be represented as decimal numbers rather than fractions (Yoshiwara & Yoshiwara, 2007, p. 269, p. 285). There may lie a conceptual challenge in interpreting a decimal number to be a fraction with integral nominator and denominator, i.e. a rational number.

3. Real exponents of powers (x^a with $a \in \mathbb{R}$)

Now, for the comprehension of a power with an exponent being neither integral nor rational but irrational, it is more challenging to give meaning to such exponents. The justification thereof can be made through an experimenting as well as through a more formal approach.

The experimenting approach requires a calculator and the procedure is to use specific examples of how an irrational number (for instance $\sqrt{2}$) can be approached through a sequence of rational numbers (expressed by decimal numbers). This, in order to

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[32] A complex number can (for $a, b \in \mathbb{R}$) be written in the form $a + bi$, where $i = \sqrt{-1}$ (Yoshiwara & Yoshiwara, 2007, pp. 588–590).

deduce that any real (and especially irrational) number can be approached through a sequence of rational numbers. Then, by considering a power with a specific positive base and a specific irrational exponent (for instance $2^{\sqrt{2}}$) this kind of power can be defined to be a limit - namely the limit of the sequence of powers having the same base and exponents equal to the elements of the sequence of rational numbers (decimal numbers) with the irrational exponent as a limit. As an example, the case of justifying $2^{\sqrt{2}}$ can take place as follows. First, it is realised (by use of calculator) that $\sqrt{2}$ can be approached through a sequence of rational numbers;

$$\{a_n\} \coloneqq \{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, \cdots\}$$

and that these numbers can be written as decimal fractions[33], i.e.

$$\{a_n\} = \left\{\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \cdots\right\}$$

Then, 2 is raised to each of these rational numbers, where it is possible to rewrite the powers to radicals, for instance $2^{1.4} = 2^{\frac{14}{10}} = \sqrt[10]{2^{14}}$, and it is realised that $2^{\sqrt{2}}$ can be defined as being the limit of the values 2^{a_n}, i.e. $\lim_{n\to\infty} 2^{a_n} = 2^{\sqrt{2}}$ (Eureka Math™, 2015, p. 78). As an alternative, the existence of the irrational power under consideration can be realised through squeezing it between rational powers with the same base, using inequality signs, and then adding more and more decimals (known from using the calculator). For instance, by starting out by writing: $2^1 < 2^{\sqrt{2}} < 2^2$, followed by: $2^{1.4} < 2^{\sqrt{2}} < 2^{1.5}$ and so on, in order to acknowledge that the irrational power (here $2^{\sqrt{2}}$) is the only number which for certain is *...contained in every interval...* (Eureka Math™, 2015, p. 80).

The more formal approach is to prove that, for $x > 0$ and $a \in \mathbb{R}$ and $\{a_n\}$ being a sequence of rational numbers converging to $a \in \mathbb{R}$, the sequence $\{x^{a_n}\}$ is convergent with limit x^a (*existence*). Further, if $\{s_n\}$ is also a sequence of rational numbers converging to a, then it can be proven that $\lim_{n\to\infty} x^{s_n} = \lim_{n\to\infty} x^{a_n}$ (*uniqueness*). Both the

33 A decimal fraction is a fraction of the form $\frac{m}{10^n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$ (Webpage: Definition of Decimal Fraction, 2016).

existence and the uniqueness proof are noticeably not straightforward to perform and are based on completeness of the real numbers, being possible to express as limits of convergent (Cauchy) sequences[34] of rational numbers regardless of which real number should be expressed (Webpage: Cauchy sequence, 2016).
More details on the formal approach can be found in (Bremigan et al., 2011, pp. 434–435).

As for the domain of the base x of powers x^a, with a irrational, it is the same as that of bases of rational powers. This is due to the fact that x^a is defined to be a limit of a sequence of rational powers ($\{x^{a_n}\}$) whereby the base can only attain the values possible for the base of a rational power. Hence, the domain of the power function x^a, for a an arbitrary irrational number, is $\mathbb{R}_+\backslash\{0\}$.

An endnote for this review of the possible exponents of a power (*1), 2)* and *3)* above), is that a restriction of the base *x,* which covers all the cases, is $x > 0$ (or $x \in \mathbb{R}_+\backslash\{0\}$). This explains why it is common to make such a restriction of the domain of a power function when defining it (for instance (implicitly) in the teaching material related to this study (section 4.2.3)) since it applies to every type of real exponent.
Next, the power function will be put in relation to three other elementary types of functions (represented in the curriculum of high school mathematics - see section 4.2.2) in order to see what this type of function has to offer compared to the others.

The power function in relation to other elementary functions

Along with the three other elementary functions of exponential, linear and logarithmic functions, the power function represents a basic type of covariation of the variables. Here, *covariation* is understood to involve a coordination of movement from x_m to x_{m+1} and movement from y_m to y_{m+1} (for a function $y = f(x)$) inspired by Confrey & Smith (1994, p. 33). The covariation of the variables of a power function can be described as follows: Whenever the independent variable is multiplied by a specific value ($k > 0 \in \mathbb{R}$) then the dependent variable is multiplied by a specific

[34] A *Cauchy sequence* is a sequence of elements in which the terms *...become 'arbitrarily close to each other' as the sequence progresses* (Webpage: Cauchy sequence, 2016).

value as well (namely k^a). This covariation property will henceforth be denoted $(\cdot/\cdot)$, where the first sign represents the operation between the independent variable and the specific constant, k, and the second sign; the corresponding operation between the dependent variable and the specific constant, k^a. Noticeably, this property of the power function is also referred to as the "percent/percent property" of power growth, since the factors k and k^a each can be expressed as $p = 1 + r$ for some $r > -1$ being the rate of growth/decay (Clausen, Schomacker, & Tolnø, 2013b, pp. 76–79, p. 100). The percent/percent property (henceforth denoted (%/%)) will continuously occur as an aspect of the $(\cdot/\cdot)$ covariation throughout the thesis.
The covariation $(\cdot/\cdot)$, characterising the power function, differs from those characterising the three other types of functions just mentioned - see Table 1.

	Linear	Exponential	Logarithm	Power
x	$+$	$+$	$\cdot$	$\cdot$
$f(x)$	$+$	$\cdot$	$+$	$\cdot$

Table 1 The covariation of the variables of the four types of functions; Linear, Exponential, Logarithmic and Power functions.

The *exponential function* can be defined as:

$$f(x) = ba^x, \text{ where } a > 0 \text{ and } a \neq 1, b > 0 \text{ and } x \in \mathbb{R},$$

where a is called the base of the function.
Noticeably, the restriction of $a \neq 1$ is not always made (for instance not in the definition of the textbook (Clausen et al., 2013b, p. 84) presented in section 4.2.3). This simply relates to the choice of considering the constant function $f(x) = b, b > 0$) an exponential function or not. (Yoshiwara & Yoshiwara, 2007, pp. 340–341). The covariation of the variables of this function is denoted $(+/\cdot)$, meaning that whenever a specific value (k) is added to the independent variable then the dependent variable is multiplied by a specific value (namely a^k when the function is defined as above) - see Table 1.

For the *logarithmic function* (base a), which is defined as the inverse function[35] of the exponential function of the same base and as being of the form:

$$f(x) = log_a(x), \text{ where } a > 0 \text{ and } a \neq 1 \text{ and } x > 0$$ [36]

(where $y = log_a(x) \Leftrightarrow x = a^y$), the covariation of the variables is $(\cdot/+)$. This, since whenever the independent variable is multiplied by a specific value (k) then a specific value ($log_a(k)$) is added to the independent variable - see Table 1 (Yoshiwara & Yoshiwara, 2007, p. 422, p. 425).

Finally, for *linear functions*, defined to be of the form:

$$f(x) = ax + b, \text{ where } a, b \in \mathbb{R} \text{ and } x \in \mathbb{R},$$

where a is called the *slope* (or *gradient*), the covariation of the variables is $(+/+)$, since whenever a specific value (k) is added to *x*, then a specific value (ak) is added to *f(x)* - see Table 1 (Yoshiwara & Yoshiwara, 2007, pp. 77–78).

There may lie a conceptual challenge as to understand the covariation $(\cdot/\cdot)$ characterising power functions, since it may not seem as intuitive as the case where adding something to the independent variable leads to a change in the dependent variable - which is the case for both linear and exponential functions; being the two types to which the Danish students have already been introduced in primary school (mainly linear functions) (Webpage: Tal og algebra - 7. - 9. klasse - Matematik, 2016).

The power function in relation to the exponential function

One of the first things to capture one's attention, when considering the algebraic expression of a power function, is that it looks very similar to that of an exponential function. The difference lies in whether the base is a variable and the exponent is a

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[35] An inverse function of a function *f*, with domain A and range B, is a function g with domain B and range A satisfying $f(g(b)) = b$ for all $b \in B$ as well as $g(f(a)) = a$ for all $a \in A$ (Webpage: Inverse E. W. Weisstein, 2016).

[36] Noticeably, the base (a) is often chosen to be 10 (the *common logarithm*; denoted log) or Euler's number e (the *natural logarithm*; denoted ln)(Webpage: Logarithm, 2016). Furthermore, the reason for the restriction $a \neq 1$ can be related to the exponential function of base a not being a constant function (see definition of exponential function).

constant (power function) or if it is the other way around (exponential function). The two function types may locally look like each other in their graphic representations as well, but the characters of their growths are very different. Cases where there are especially significant similarities to be found between the two functions, are when the exponent for the power function and the base of the exponential function is the same positive integer. This is because of the similar domain in such cases (namely ℝ) (see the subsection on *The domain of a power function* above) as well as the fact that both of the graphic representations have the property of getting steeper for $x > 0$ as the exponent of the power function, respectively the base of the exponential function, increases (Webpage: Power Functions and Exponential Functions, 2016). Further, the locations of the graphs where the graphic courses look alike are within the same quadrant of the coordinate system (namely the first), which is not the case for negative exponents of the power function compared to exponential functions with bases greater than 0 and smaller than 1 (where both functions are decreasing) - see for instance the graphs of x^{-3} and 0.05^x in Figure 18.

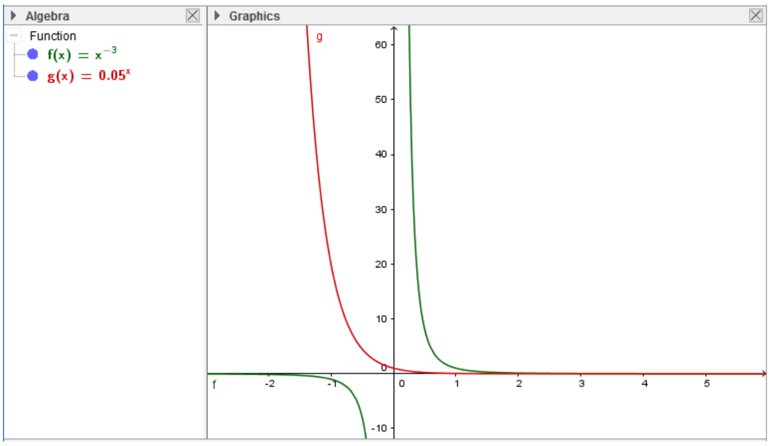

Figure 18 Plot of $\boldsymbol{f(x) = x^{-3}}$ and $\boldsymbol{g(x) = 0.05^x}$ made in GG. Here it is seen that the courses of the two graphs look alike for the part of the exponential graph located in the second quadrant whereas the similar part of the power graph is located in the first quadrant.

As an example of a couple of respectively an exponential function and a power function with the same positive integer as base respectively exponent are $\boldsymbol{2^x}$ and $\boldsymbol{x^2}$ whose graphical representations are presented in Figure 19.

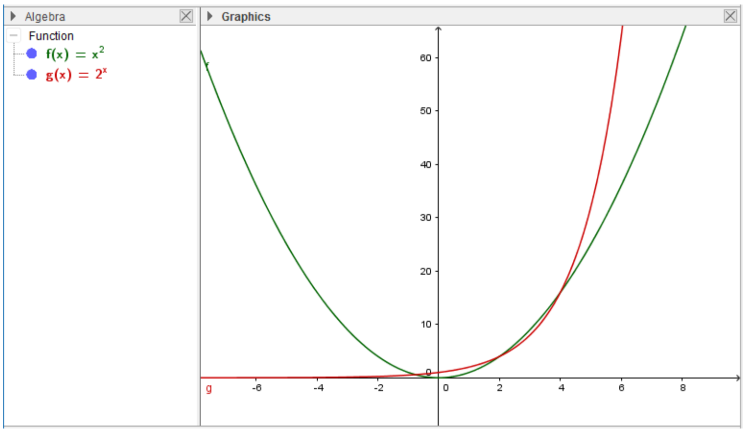

Figure 19 Plot of $\boldsymbol{f(x) = x^2}$ and $\boldsymbol{g(x) = 2^x}$ made in GG.

The functions are clearly similar with respect to their domain ($\mathbb{R}$) but also in the way that, for $x > 0$, they both approach infinity for increasing values of x. There are however important differences of the two functions. The intersections of the graphs and the second axis differ in the way that it is $(0,0)$ for any power function whereas it is $(0, b)$ for an exponential function of the form introduced in the definition earlier (in the case of x^2 it is $(0,1)$). Furthermore, exponential functions always have the x-axis as horizontal asymptote, whereas power functions with positive exponents do not have any asymptotes. Additionally, in this example the following applies; *The exponential function approaches zero while the power function approaches the [...] positive infinity [...] when x approaches negative infinity* (Webpage: Power Functions and Exponential Functions, 2016). Finally, 2^x dominates x^2 for sufficiently large x which can be seen where the graph of the exponential function gets significantly steeper per x-unit than the power function (Figure 19). This holds for any positive integer and actually also any positive real number working as base respectively exponent for the exponential and the power function (Webpage: Power Functions and Exponential Functions, 2016).

As an end note for this comparison, there is a clear difference between the graphic representations for a power and an exponential function when the domain of the power function is restricted to be $x > 0$, since the graph of the exponential function runs in both the first and the second quadrant where the graph of the power function only runs in the first quadrant. Further, another aspect of the challenge in distinguishing the two types of functions is whether one understands what it means to raise a number to a power and not confuses the order of the base and the exponent.

The power function in relation to the linear and the logarithmic function

Regarding the two other types of functions presented in Table 1, the power function has a close relation to linear functions when the exponent is equal to 1 since in such cases, it is in fact a linear function of the form $f(x) = bx, b \in \mathbb{R}\backslash\{0\}$. It may be confusing to realise that it is okay for a power function to be linear of this form - especially when it is represented graphically.

The logarithmic function (being the inverse to the exponential function) has a relevant role when manipulating with the algebraic expression of power functions, since it can function for pulling down exponents. The second of the following three properties of logarithms (base $a > 0$) for $x, y > 0$ is relevant for such actions:

a) $log_a(x \cdot y) = log_a(x) + log_a(y)$
b) $log_a\,(x^k) = k \cdot log_a(x)$
c) $log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$

(Yoshiwara & Yoshiwara, 2007, p. 366).

It may however be a challenge not just to consider the logarithmic function (and the properties thereof) as a tool for solving equations involving powers, rather than being a function.

An endnote, for this review of the power function in relation to other elementary functions, is that it also has a significant relation to polynomials since it is in fact a polynomial (being of the form $f(x) = bx^a$)) when a is a positive integer (Webpage: Polynomials, 2016). Noticeably, polynomials have not yet been formally introduced to the students of the Power Function course, since *the teacher teaching the Power Functions course* (TP) follows the order of mathematical themes as they are presented in the textbook where Polynomials are introduced in the subsequent chapter of that introducing Power Functions (see section 4.2.3).

Laws of exponents

The properties *a)* and *c)* of logarithms (introduced just above) noticeably look similar to the following two laws of exponents, for $m, n \in \mathbb{R}$:

$L_1)\ a^m \cdot a^n = a^{m+n}$

$L_2)\ \frac{a^m}{a^n} = a^{m-n}$

(Yoshiwara & Yoshiwara, 2007, p. 311).

The algebraic structure of these laws can be difficult to remember and one may misunderstand $L_1)$ and write $a^m \cdot a^n = a^{m \cdot n}$ or $a^{m+n} = a^m + a^n$, when simplifying or expanding powers with the same base, which is not valid, or think that it holds for powers with different bases, but $a^m \cdot b^n \neq (a \cdot b)^{m+n}$ (Yoshiwara & Yoshiwara, 2007, p. 743). Similarly, such misjudgements can be made for the operations / and $-$ involved in $L_2)$. Furthermore, there exist the following three laws of exponents (for $b \neq 0$):

$L_3)\ (a^m)^n = a^{mn}$,

$L_4)\ (a \cdot b)^n = a^n \cdot b^n$ and

$L_5)\ \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

(Yoshiwara & Yoshiwara, 2007, p. 311).

It may be a challenge to distinguish between $L_3)$ and $L_1)$ when working with powers with the same base, i.e. to distinguish between the following expressions; $a^m \cdot a^n$ and $(a^m)^n$ and remembering when to add respectively multiply the exponents when simplifying (Yoshiwara & Yoshiwara, 2007, p. 745). Similarly, such challenges, in distinguishing the laws from each other, may occur between $L_2)$ and $L_5)$. Regarding $L_4)$, there may lie a challenge in distinguishing between expressions of the form $a \cdot b^n$ and $(a \cdot b)^n$ where the law only applies for the latter as well as in not applying the law on sums; $(a + b)^n \neq a^n + b^n$ (Yoshiwara & Yoshiwara, 2007, p. 746).

The laws can however quite readily be realised by thinking of the exponent as a positive integer (see subsection on *The domain of a power function* further above) and writing what the left side of the equalities means, for instance for $L_5)$:

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \cdots \cdot \frac{a}{b}\ (n\ times) = \frac{a \cdot a \cdot \cdots \cdot a\ (n\ times)}{b \cdot b \cdot \cdots \cdot b\ (n\ times)} = \frac{a^n}{b^n}.$$

Applications of the power function

This section ends with a consideration of the application oriented rationale of the theme of Power Functions. First of all, the power function is applicable for describing areas and volumes of geometrical objects such as cubes and spheres. For instance, the surface area (y) of a cube with side length x can be expressed as $y = 6x^2$ (Webpage: The Power Function, 2016). Another relevant example is the study of relations between body size to different biological parameters, called *allometry*. Such relations are namely often described through equations of the form of power functions (Webpage: Allometry, 2016). For instance, the relation between the metabolic rate[37] and the mass of an animal introduced by Max Kleiber (in 1932) can be expressed as a power function (Yoshiwara & Yoshiwara, 2007, pp. 233–234) . Noticeably, when introduced to a power model, there lies a significant task of interpreting the algebraic expression to be that of a power function and not just having a focus on it being a formula for calculating some real variable, such as the surface area of a cube (given a side length) or the metabolic rate of an animal (given its weight).

Another aspect of applications of power functions to real variables are the cases where one is given a set of data (on two real variables) where the relation between the variables of the data set is to be fitted into a power function equation - through regression[38]. To be noted power regression is often performed as linear regression of the transformed functional equation appearing when evaluating the logarithm of each side; $\log(f(x)) = \log(b) + a \cdot \log(x)$.

Without going into further details of the type of regression, it is most common for the linear regression to be made through the method of Least Squares which seeks to minimize *...the sum of the squares of the offsets ("the residuals") of the points from the curve* (Webpage: Least Squares Fitting, 2016) - where the here-mentioned curve is the bid on the best-fitting curve. (Webpage: Linear Regression E. W. Weisstein, 2016;

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[37] *Metabolic rate* denotes the *...body's energy expenditure per unit of time* (Webpage: Metabolic rate, 2016).

[38] *Regression* is to be understood as a *...method for fitting a curve (not necessarily a straight line) through a set of points using some goodness-of-fit criterion* (Webpage: Regression E. W. Weisstein, 2016).

Webpage: The Power Function, 2016). Noticeably, the quality of the least squares linear regression can be assessed through evaluating the correlation coefficient (denoted r^2) - where a correlation coefficient equal to 1 indicates a perfect fit (Webpage: Correlation Coefficient, 2016). The aspect of assessing and choosing a type of regression will not be further elaborated[39], since it is not relevant for the Power Function course of this observation study where the students are only required to be able to solve tasks asking for fitting the given data to a power model - hence, not for assessing the type of regression (see section 4.2.2).

4.2.2 The Curriculum in Relation to the Theme

This section is concerned with presenting the mathematical knowledge to be taught and learned in high school in relation to the theme of Power Functions at the level current for the class being taught in this theme of the observation study. More specifically, the attention is oriented towards the *curriculum* for B-level mathematics for high school (C-B) (Undervisningsministeriet, 2013b), which serves as curriculum for the students in the class being taught in the theme. Parts of C-B, as well as supporting parts of the *teacher's guide* for C-B (TG-B) (Matematik B - stx vejledning, 2010), related to the theme will be singled out and associated with the types of tasks occurring in the written examination.

Now, the parts of the core substance (section 2.2 of C-B) in which the theme of Power Functions is clearly involved can be listed as follows:

(1) The expanded notion of a power[40],

(2) Equation solving through both analytic and graphic methods[41] and by using IT-tools,

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[39] For further discussion on this matter, see Jeanette Kjølbæk's master thesis (Webpage: One-dimensional regression in high school Jeanette Kjølbæk, 2016).

[40] The *expanded notion of a power* refers to the case where the notion of a power is expanded from only being considered as positive integers (Webpage: Det udvidede potensbegreb, 2016).

[41] *Analytic* methods refer to classical manipulation with the algebraic representation of an equation whereas *graphic* methods refer to drawing/plotting the graph of the equation in a coordinate system and reading off the solution (Webpage: Ligningsbegrebet, 2016).

(3) Formula expressions for describing direct and inverse proportionality and for expressing power relations between variables,

(4) The notion of $f(x)$,

(5) Characteristic properties of the power function (along with linear functions, polynomials, exponential- and logarithmic functions) involving the meaning of the constants in the functional equation, in relation to the look of the graph, as well as the relation between the percentage rate of growth for the dependent and the independent variable and the characteristic properties of the graphical course of the function (TG-B, p. 16),

(6) Application of regression, and fundamental properties of mathematical models and modelling[42] in general.

The above presented core substance is represented in the exercises for the written examination related to the theme of Power Functions which is to be elaborated in the following. Based on the written examinations from 2007-2015[43], where T_{EP_6} is the only one of the types of tasks which can be found in the part of the examination without aids, the following list of represented types of tasks (denoted T_{EP}) has been made and the figures Figure 20-Figure 23 present specific examples thereof (translated from Danish[44]):

$\boldsymbol{T_{EP_1}}$: Given an equation of a power function, $f(x)$, and x_0. Determine $f(x_0)$.

$\boldsymbol{T_{EP_2}}$: Given an equation of a power function, $f(x)$, and $f(x_0)$. Determine x_0.

---

[42] *Mathematical modelling* is understood as cases where mathematics is brought into play and used for treating cases outside mathematics itself, for instance setting up a model of power growth in order to describe and predict a specific development (TG-B p. 27). Noticeably *modelling* and *regression* are strongly related as mathematical concepts - when performing regression, a model is made for the data.

[43] Noticeably the written examinations from earlier years are not publicly accessible why there is made no explicit references thereto, besides from the specific examples of exercises and tasks from some of the written examinations presented in the figures of this section.

[44] Noticeably, the notation of decimal numbers has also been translated from the use of "," to ".".

T_{EP_3}: Given a power function $f(x)$(or y). Determine the percentage change of $f(x)$(or y), given a percentage change $(r \cdot 100)$ of x.

For Fiddler Crabs, the male has an over dimensional claw which among other is used for territorial battles. The relation, between the weight K of the over dimensional claw (measured in mg) and the weight M of the entire crab (measured in mg), for a specific type of very small Fiddler Crabs is given by

$$K = 0.036 \cdot M^{1.356}$$

a) Determine the weight of the claw if the weight if the entire crab is 400 mg and determine the weight of the entire crab if the weight of the claw is 53 mg.

b) By how many percent does the weight of the claw increase when the weight of the crab increases by 30 %?

Figure 20 Exam exercise from 2015 (May 28th) involving tasks of type T_{EP_1} and T_{EP_2} (in part a)) and T_{EP_3}(part b)).

T_{EP_4}: The function $f(x) = b \cdot x^a$ (or $y = b \cdot x^a$) satisfies $f(x_1) = y_1$ (or $b \cdot x_1^a = y_1$) and $f(x_2) = y_2$ (or $b \cdot x_2^a = y_2$). Determine an equation (or the constants a and b) for f (or the relation between x and y).

The function $f(x) = b \cdot x^a$ satisfies $f(2) = 3$ and $f(4) = 7$.

a) Determine an equation for f.

Figure 21 Exam task from 2012 (May 25th) of type T_{EP_4}.

T_{EP_5}: Given a set of data for two variables. It is assumed that the relation between the variables can be expressed as a power function equation, $f(x) = b \cdot x^a$. Use the data to determine a functional equation for f (or the constants a and b).

The table presents correlated values of height and volume of a particular series of decanters.

Height (cm)	5	10	15	20	25	30
Volume (cm^3)	20	157	530	1256	2453	4239

In a model, it is assumed that the relation between height and volume can be described by a function of the type

$$V(x) = b \cdot x^a,$$

where $V(x)$ denotes the volume (measured in cm^3) and x denotes the height (measured in cm).

a) Use the data of the table to determine a functional equation for V.

b) Determine the height of a decanter which is able to contain 2000 cm^3.

c) Determine by how many percent the volume of a decanter changes when the height is increased by 10 %.

Figure 22 Exam exercise from 2012 (August 15th) involving tasks of type T_{EP_5} (part a)), T_{EP_2} (part b)) and T_{EP_3} (part c)).

T_{EP_6}: Two variables, x and y, are inverse proportional. Given a table presenting one pair of values of these variables in relation to each other as well as another value of x; x_0, where the related value of y; y_0, is not presented. Fill out the rest of the table.

Two variables x and y are inverse proportional.

x	10	20
y	4	

Fill out the rest of the table

Figure 23 Exam task (in the part without aids) from 2014 (May 27th) of the type T_{EP_6}.

In prolongation of the latter presented type of task (T_{EP_6}), concerning inverse proportionality, Figure 24 presents an exercise involving both direct and inverse proportionality and may provide a perspective for which types of problems involving proportionality the students should be able to solve.

> The following is given for three variables x, y and z:
>
> z is direct proportional to y by the factor of proportionality 3,
> x and y are inversely proportional and
> y is 10 when x is $\frac{1}{2}$.
>
> a) Express z by x.

Figure 24 Exam exercise from 2007 (August 6th), in which both direct and inverse proportionality are represented (point (3) of the core substance), where equation solving is needed in order to solve the task of expressing *z* by *x* (i.e. point (2) of the core substance is represented as well).

Noticeably, the task of type T_{EP_6} (Figure 23) and the exercise in Figure 24 have not occurred since 2008 and 2007 respectively, whereas the rest of the presented types of tasks in the list above are more common.
An important thing to be noted in the formulations of T_{EP_1} - T_{EP_6} is the fact that none of these involve any explicit questions for explanations involving an establishment of a knowledge block, since the main terms used are *determine*, *fill out* and *express*.

Now, analytic techniques for the types of tasks T_{EP_1} - T_{EP_6} will be sketched and put in relation to the listed parts of the core substance apart from the points (1) and (4), if any relation thereto. The elimination of these two points for this review is due to the fact that they are represented implicitly in many tasks related to the theme of power functions. Point (1) is namely represented whenever a power function has an exponent different from a positive integer. Further, the point (4) is represented whenever a functional equation is involved in the task. Before sketching analytic techniques for the types of tasks T_{EP_1} - T_{EP_6}, it should be noted that a more explicit review of techniques will be made in section 4.3.3 - namely instrumented techniques related to GG.
Tasks of type T_{EP_1} and T_{EP_2} can be solved by inserting respectively x_0 or $f(x_0)$ in the functional equation for $f(x)$ and subsequently calculating either the right side of the equation or solving the equation with respect to x_0. Both T_{EP_1} and T_{EP_2} clearly represent point (2) of the core substance.
As for tasks of type T_{EP_3}, these can be solved through realising that *x* is multiplied by

$k = 1 + r$. Thereafter, one should calculate k raised to the ath power (k^a), subtract 1 from the result and multiply it by 100. (See the (%/%) property in section 4.2.1; in the subsection *The power function in relation to other elementary functions*). Both the points (2), (4) and (5) of the core substance are represented in connection with T_{EP_3}. Tasks of type T_{EP_4} can be solved by inserting the coordinates of the two given points in the formula for a (presented in section 4.2.3 below as expression (4)) and calculate it. Thereafter one should insert the coordinates of one of the given points and the calculated value of a in the formula for b (presented in section 4.2.3 below as expression (5)). If asked for the functional equation one should finally write the equation for $f(x)$ (or (y)) with the values of a and b inserted. The points (2), (3), and to some extent (5) of the core substance are represented in relation hereto. The latter point, since the technique for these types of tasks may involve using some of the properties of the logarithmic function (see section 4.2.1 above) for establishing the formula for a.

Regarding tasks of type T_{EP_5}, especially point (6) but also point (3) of the core substance are represented. To solve a task of type T_{EP_5}, one can plot the points of the set of data in a coordinate system and fit the data to the equation of a power through regression. Noticeably, a CAS tool/IT-tool is very convenient for performing regression (see instrumented techniques in GG in section 4.3.3).

Now, for the type of tasks T_{EP_6}, it especially represents point (3) but also point (2) of the core substance. Such a task can be solved by inserting the given pairs of values of x and y in the equation for inverse proportionality between x and y; $x \cdot y = k$, to calculate the constant of proportionality k (inverse proportionality is presented in section 4.2.3 below). Then x_0 and k are to be inserted in the equation for inverse proportionality and one should then solve the equation with respect to the related value of y (y_0). For the exam exercise in Figure 24 one should, besides from the equation for inverse proportionality, noticeably also be able to apply the equation for direct proportionality between two variables x and y; $y = k \cdot x$ (also presented in section 4.2.3 below).

4.2.3 The Teaching Material

The teaching material, used by TP, is the textbook *Gyldendals Gymnasiematematik Grundbog B1* (Clausen et al., 2013b) and its associated exercise book (Clausen, Schomacker, & Tolnø, 2013a). The existing exercises, concerning power functions, in the exercise book will however not be considered further. This, since the main types of tasks for the students to be able to solve are the types drawn from the written examinations, introduced in the section just above (4.2.2). It should however be noted that in the first section of the exercise book, called *Tools* (translated from Danish), the laws of exponents are presented. This, only for positive integer exponents to begin with. Subsequently, exponents equal to zero, negative integers and rational numbers are defined under the category of the *expanded notion of a power*. The case of irrational exponents is finally mentioned as an option, noting that the explanation of how such powers are defined will not be made. (Clausen et al., 2013a, p. 15).

The textbook introduces the power function implicitly to begin with, in the subsection 2.2 introducing examples of relations between variables, through examples such as the circumference and the area of a square with side length x. The examples of this subsection involves, noticeably, many linear relationships without an added constant in the expression (i.e. also power functions, but often just considered linear or as proportionality relations). (Clausen et al., 2013b, pp. 50–55). The theme of Power Functions is formally introduced after the introduction of Linear Functions and Linear Growth, Calculations with Percentages, Logarithms and Exponential Functions and Exponential Growth and within the chapter before the chapter (Chapter 2) concerning Polynomials (Chapter 3) (Clausen et al., 2013b). More specifically, the section on *Power Functions and Power Growth* (2.8) starts out by defining the power function to be a function of the form $f(x) = x^a$, introduced immediately after the definition of a power relation to be of the type $y = b \cdot x^a$. It is then noted that $b > 0$ and that a does not have to be a positive number - actually $a \in \mathbb{R}$ is supposed, though not explicitly, but the lack of restrictions on the domain of a and the options of $a < 0$, $a = 0$, $0 < a < 1$ and $1 < a$, throughout the section, indicates it. (Clausen et al., 2013b, p. 94). Noticeably, nothing is explicitly stated about the domain of *f* in this definition, but there seems to prevail an assumption that *x* is strictly greater than zero, from how the graph types and calculations, involving x-values in denominators and x-values being independent variables of the logarithmic function (having $\mathbb{R}_+$ as domain - see section 4.2.1, subsection: *The power function in relation to other elementary functions*), are presented throughout the section (Clausen et al., 2013b, pp. 94–101). The definition is followed by a set of examples introducing power relations of variables (Clausen et al., 2013b, pp. 94–95).

Then, power regression is introduced through two examples, the first of which concerns a data set of two variables, presented in a table. The task is of the type T_{EP_5}. The second example concerns power regression based on only two pairs of related values of variables. The two examples present the answer, of which equation that can be said to describe the power relation - both its algebraic and graphic representation - but techniques in relation thereto are not introduced. (Clausen et al., 2013b, pp. 95–96). To be noted, there is however a separate section elaborating on how to do regression (section 2.10, pp. 104-110), which opens by referring to the use of a tool (e.g. CAS) to do regression. Through examples, it is here introduced how one should approach tasks of regression without any explicit introduction to the instrumented techniques required, only a reference to using a (unspecified) CAS tool for such tasks.

Back to the section on *Power Functions* (2.8), some properties of these functions are presented, in terms of the different look and conditions of monotony of the graph of $f(x) = x^a$, depending on the value of a, presented through a figure (Figure 25). Further, which asymptotes there may be connected to the graph is presented and finally determination of the values of a and b in the equation $y = b \cdot x^a$ knowing two points on the graph.

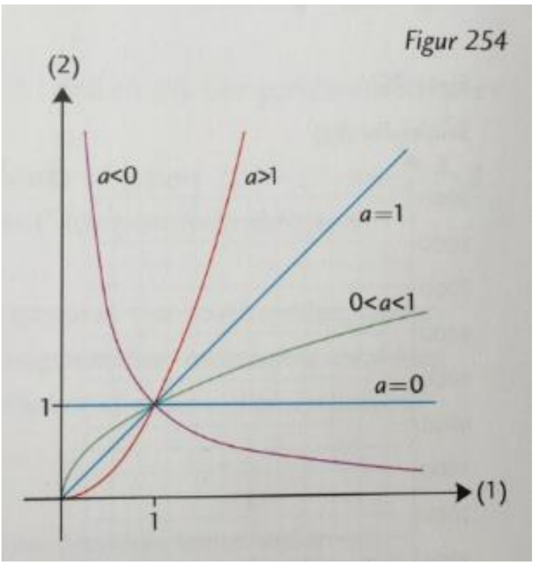

Figure 25 The figure introducing the different looks of the graph of a power function $f(x) = x^a$ from the textbook (Clausen et al., 2013b, p. 96).

The formulas of a and b when knowing two points on the graph of the power function f is introduced through a theorem - henceforth denoted Thm_{ab}. Thm_{ab} states:

Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the graph of $y = f(x) = b \cdot x^a$, the numbers a and b can be calculated as follows:

First calculation: $$a = \frac{\log\left(\frac{y_2}{y_1}\right)}{\log\left(\frac{x_2}{x_1}\right)} \text{ or } a = \frac{\ln\left(\frac{y_2}{y_1}\right)}{\ln\left(\frac{x_2}{x_1}\right)}, \quad (4)$$

Second calculation: $$b = \frac{y_1}{x_1^a} \quad (5)$$

(Clausen et al., 2013b, p. 97).

The proof of Thm_{ab} is presented immediately after but without any explanations for the involved equation solving, i.e. the knowledge block of the praxeology of the proof is not elaborated. Furthermore, no explicit restrictions are made as to not having zero in the denominators of the fractions involved in the theorem, nor is it noted that the logarithm only can be evaluated for positive input, which must mean that there is an implicit comprehension that the domain of the power function is $\mathbb{R}_+$. (Clausen et al., 2013b, pp. 97–98). Thereafter, three examples of tasks related to Thm_{ab} are presented; one of the type T_{EP_4} followed by one where a and a point on the graph is known and b has to be determined and finally one of a more simple case where the functional equation can be determined without a tool for calculation (Clausen et al., 2013b, pp. 98–99) (Clausen et al., 2013b, pp. 98–99).

A *power growth* is subsequently introduced to denote a relation between two variables which can be described by a power function, followed by a theorem introducing what is characteristic for a power growth of the form $y = f(x) = b \cdot x^a$, namely the covariation $(\cdot/\cdot)$ (see section 4.2.1). This is followed by a proof which mainly consists of techniques. There are noticeably no explanations for these techniques (Clausen et al., 2013a, p. 100). Then the covariation $(\cdot/\cdot)$ is reformulated into a language involving a (specific) percent change of the independent variable implying a specific percent change of the dependent variable (the (%/%) property). Subsequently, a couple of examples of tasks involving this property are introduced. Both examples involve tasks of the type T_{EP_3}. The second example further introduces a task of determining the percentage change of x, given a percentage change of $f(x)$

(the reverse version of T_{EP_3}). As a separate section from the one concerning power functions and -growth, just presented, is the one introducing *Proportionality and Inverse Proportionality* (2.9). Here, proportionality and inverse proportionality are presented as two important relations of variables as follows;

(Direct) proportionality: $y = b \cdot x$ or $\frac{y}{x} = b$, and

Inverse proportionality: $y = b \cdot \frac{1}{x}$ or $y = b \cdot x^{-1}$ or $y = \frac{b}{x}$

(Clausen et al., 2013b, p. 101).

Thence, examples of graphs of the two types of relations are introduced with a domain being all of $\mathbb{R}$. There is noticeably no reference to the theme of Power Functions in this section. Hereby, there is no explicit motivation for introducing the two types of relation in prolongation of the theme of Power Functions. This section ends by presenting some examples of both (direct) proportionality and inverse proportionality, related to reality. Only through these examples, the term of the *factor of proportionality* (also denoted the *scale factor*) is implemented. (Clausen et al., 2013b, pp. 101–103).

4.3 A Priori Analysis of the CAS Tool

In order to establish the RPM related to the two themes further, this section presents the main specific features of the CAS tool (section 4.3.1), GG, as well as the CAS tool in relation to the two themes (sections 4.3.2 and 4.3.3). More specifically, this section is concerned with **RQ$_0$** where the gestures and instrumented techniques possible to make in connection with the two mathematical themes in GG are considered, after some main specific features of the program have been presented. Where no other reference is made, this analysis is based on a review made during the course of this study with input from YouTube[45] videos, TTW and manuals for GG (Webpage: GeoGebra (da), 2016; Webpage: GeoGebra (en), 2016). The analysis is made with reservations to the fact that there may be several different approaches to be made for using commands in GG in order to solve a specific task (i.e. there can be more than

[45] Link: www.youtube.com (last visited 12.07.2016).

one instrumented technique/collection of gestures for solving a specific task) and also to the full potential of the single command in GG which may not be discovered. The focus is put on the gestures concerning clicks with the computer mouse or touch pad and not on the several keyboard shortcuts which are possible to make in GG[46]. This priority has been made since the clicking procedure are the same regardless of the computer being PC or Mac[47]. Another reason for focusing on the gestures, used with a computer mouse or touchpad, is that using shortcuts requires a rather advanced user of GG and both the teacher's and the students of the two classes observed have very little experience with - and knowledge about GG (see section 5.0). As a final note, the analysis is based on the language English (US) since it is more suitable for the description of the functions and commands in English. It is however easy to translate the commands into Danish (more or less directly) - if necessary, since some (English) commands work in the same way in both the Danish and English setting.

4.3.1 Main Specific Features

In this section, general gestures and notions of importance in relation to GG, found relevant for both themes, will be presented before linking instrumented techniques to the two themes in the two a priori analyses of the themes. The instrumented techniques will be presented through specific examples of tasks related to GG.
GG is overall *...dynamic mathematics software that joins geometry, algebra and calculus [...] developed for learning and teaching mathematics in schools* (Webpage: GeoGebra (da), 2016). It exists in both a computer program version ("GeoGebra 5.0 Desktop") and in an app version ("GeoGebra 5.0 Web and Tablet App"). The two versions have *...minor differences in terms of use and interface design* (Webpage:

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[46] Keyboard shortcuts possible to make in GG can be found on the webpage (Webpage: Keyboard Shortcuts, 2016).

[47] To be noted here, the research in this analysis is performed on PCs whereas the observed students are Mac users (see section 5.0).

GeoGebra (en), 2016). This analysis is noted to be based on the desktop version, used for Windows.[48]

The user interface of GG can be described roughly through the terms of *Views* and *Perspectives* where the Perspectives are constituted by a collection of Views. There are different possible Views to pursue in GG where the *Algebra View*, the *Graphics View*, the *CAS View* and the *Spreadsheet View* are considered especially suitable for the two mathematical themes of this study[49]. These Views are linked dynamically, meaning *...that if you modify an object in any of the Views, its representations in the other Views automatically adapt to these changes if possible* (Webpage: Views, 2016). This relation between the Views will be clarified through the specific examples of working with GG in relation to the two themes, presented in the a priori analyses below.

The Algebra View displays algebraic representations of mathematical objects which *...can be entered directly using the (virtual) keyboard (e.g. coordinates of points, equations)* (Webpage: Views, 2016) in an *Input Bar* placed at the bottom of the GG interface (see Figure 26).
The Input Bar more specifically *...allows you to directly create and redefine mathematical objects in the Algebra View by entering or*

Figure 26 The interface of GG when opening the program (the Algebra Perspective). The green box marks the Menubar, the red box marks the Graphics View Toolbar and the purple box; the undo and redo buttons in GG.

[48] The program GG can be downloaded for free for both Windows, Mac OS X and Linux, for instance from (Webpage: GeoGebra - Download, 2016).

[49] Additionally, there is a *3D Graphics View* and a *Probability Calculator View* which are not considered suitable for the two themes at this level of high school (Webpage: GeoGebra (en), 2016).

modifying their algebraic representations (e.g. values, coordinates, equations) (Webpage: Input Bar, 2016).

In the Graphics View it is possible to construct mathematical objects, either by using mouse or touch pad in relation to its *Toolbar* (to be explained later). These objects can be *…changed dynamically afterwards* (Webpage: Views, 2016).

The Spreadsheet View offers an interface for working with data and exploring statistical concepts and in the CAS View it is possible to make *...numerical and symbolic computations* (Webpage: Views, 2016).
Before going further, it is noted that, throughout the analysis, when referring to "click", "select" or "choose" gestures, it means using the computer mouse or the touchpad.

Turning towards Figure 26, it shows the interface as it looks when opening the program, where the Algebra View and the Graphics View are open - this is called the *Algebra Perspective* (Webpage: Perspectives, 2016). The Perspective can be changed - e.g. to *Spreadsheet Perspective* or *CAS Perspective* (see Figure 26 where the box with "Perspectives" is available in the *Sidebar*[50] of the GG-document). The choice of Perspective depends *...on the mathematics you want to use GeoGebra for...* (Webpage: GeoGebra (en), 2016). Each of the possible Perspectives display *...those Views and other interface components most relevant for the corresponding field of mathematics* (Webpage: GeoGebra (en), 2016). Noticeably, no matter which Perspective one chooses as a starting point, it is always possible to add or remove Views to this Perspective, depending on what one finds relevant when working with a specific mathematical content. The addition of Views is possible through the *Menubar* (marked by a green box on Figure 26 and explained further below) by clicking "View" and the shutdown of a View can be done by clicking the cross sign in the upper right corner of the View (see Figure 26). The Menubar contains the menus: File, Edit, View, Options, Tools, Window and Help. Through these menus it is, besides the opportunity of adding Views, possible both to save, open and share documents, to customise the

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[50] An elaboration of the Sidebar is given on the webpage https://www.geogebra.org/manual/en/Sidebar (last visited 01.08.2016).

toolbar and search for help for how to approach the tool. It applies for all Views that it is possible to correct mistakes step-by-step by using the undo and redo buttons, respectively the arrow turning left and the arrow turning right which is marked by the purple box in Figure 26.

An important option is the one of editing the preferences for each opened View in the GG-document. This option is available in a Settings Dialog (called *Preferences* - see Figure 27) through the Options menu (in the Menubar) by choosing "Advanced" (having a sprocket wheel icon) or by choosing the small sprocket wheel icon to be found underneath the redo button (see Figure 26) and then selecting "Advanced".

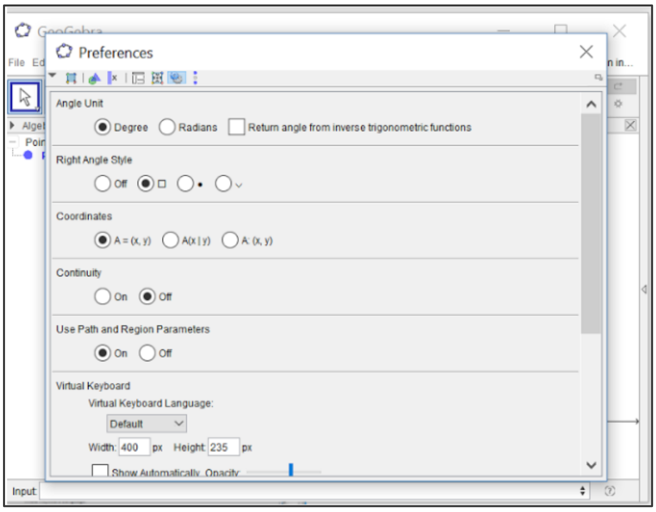

Figure 27 The Settings Dialog *Preferences*, called for in a GG-document (where a point has been plotted in the Algebra Perspective behind the dialog).

As seen on Figure 27, the Settings Dialog is *...divided into different sections, which are shown depending on active objects and Views: 'Properties', 'Graphics', 'CAS', 'Spreadsheet', 'Layout', 'Defaults', and 'Advanced'* (Webpage: Settings Dialog, 2016) and there is an Algebra section as well.

Within the here mentioned sections, it is possible to change the settings in the GG-document - both local and global settings. For instance, changing the colour of an object (in the *Properties* section) respectively changing the angle unit between radians and degrees in the Algebra Perspective (in the *Advanced* section). (Webpage: Settings Dialog, 2016). In relation to the two themes, there are some more options to be noted within the Settings Dialog. In the Graphics section, it is possible to remove or add both axes and grid to the Graphics View. In the CAS section, one may both set a timeout for the CAS calculations and choose how rational exponents should be presented (as roots or not).

Now, before going further to the features of the Views in GG applicable for both themes, it should be noted that decimal numbers are typed using a dot (".") and not a

comma (“,”) (like Danes would do it), regardless of the language chosen for the document (through the Options menu). The number of decimals is set to be two when opening a new GG document but this can be changed through the Options menu as well through clicking on “Rounding”.
In the following, the general aspects (found relevant for both themes) of the Algebra Perspective (the Graphics View and the Algebra View) and thence the CAS View are analysed. Since the Spreadsheet View is only relevant for the theme of Power Functions, it will only be explored in relation thereto in section 4.3.3.

The Algebra Perspective

In Figure 26 it is seen that there is a *Toolbar* which corresponds to the chosen Views - in this case, the Algebra and the Graphics View (marked by a red box in the figure). The Toolbar *...contains a selection of Tools and range of Commands as well as Predefined Functions and Operators that allow you to create dynamic constructions with different representations of mathematical objects* (Webpage: GeoGebra (en), 2016). In order to describe the Graphics View Toolbar, it is worth noting that every *...icon in the Toolbar represents a Toolbox that contains a selection of related construction Tools* (Webpage: Graphics View, 2016). From left to right in the red box in Figure 26, the following Toolboxes are seen to constitute the Graphics View Toolbar: Movements Tools, Point Tools, Line Tools, Special Line Tools, Polygon Tools, Circle and Arc Tools, Conic Section Tools, Measurement Tools, Transformation Tools, Special Objects Tools, Action Object Tools and General Tools. Here, it is worth mentioning that the symbol of each of the Toolboxes (the default construction Tool) can be changed *...on the little arrow in the lower right corner of the default Tools* (Webpage: Tools, 2016). In the two a priori analyses of the mathematical themes (sections 4.3.2 and 4.3.3) there will be an elaboration of the Tools found relevant in relation to each theme.

Besides from using the Toolbar for constructing objects in the Graphics View, the Style Bar (appearing when clicking on the downward facing arrow in the tab of the Graphics View (marked by a purple box on Figure 28)), also has some options to offer. It is worth mentioning that the options of the Style Bar depend on the choice of Tool in the Toolbar since the Style Bar corresponds to the chosen construction Tool.

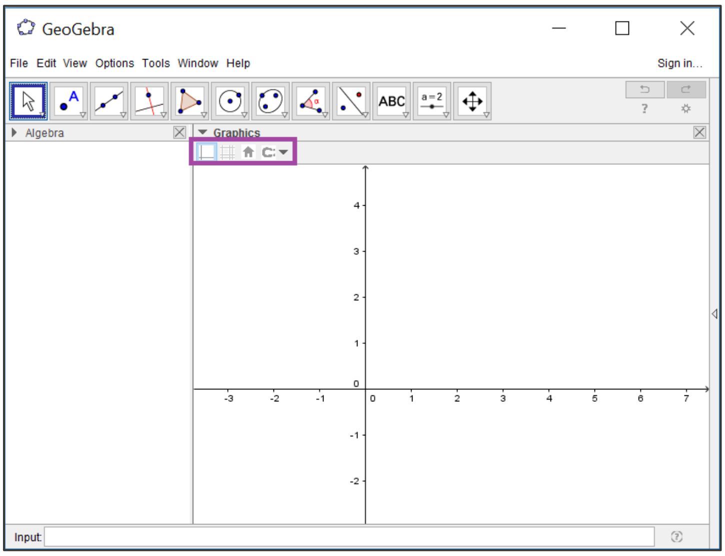

Figure 28 The Style Bar of the Graphics View corresponding to the Move Tool (marked by a purple box).

Via the Style Bar, it is for instance possible to show or hide both the coordinate axes and the grid which is done by respectively using the first and the second option (with respectively axes and grid as icons).

Now, some prospects of the Algebra View are to be considered. In Figure 29, the Algebra View has been selected and it is seen that the exact same Toolbar is available in this View as for the Graphics View. It is however not possible to construct anything to be manipulated with dynamically within the Algebra View, as it more serves as an informant for what is going on in the Graphics View, so the Toolbar is not used. The Input Bar (as shortly presented earlier) is instead used to construct or redefine the mathematical objects (e.g. point coordinates and functional equations) to be put in the Algebra View. Noticeably, you have to *...press Enter after typing algebraic input into the Input Bar* (Webpage: Input Bar, 2016). Furthermore, the input entered in the Input Bar will automatically be given a name if there is not asserted any. Symbolic manipulations are not possible

Figure 29 The Input Bar where the alpha box is checked, presenting mathematical symbols as options to put in the input text.

(for this, the CAS View is applicable). Besides from *x*, *y* and *z* (leading to functions), you can choose any letter to name inputs not being functions. When placing the cursor in the Input Bar, an Alpha box appears far right, containing mathematical symbols, including Greek letters and operators such as the degree sign and less/greater than or equal to signs, as shown in Figure 29.

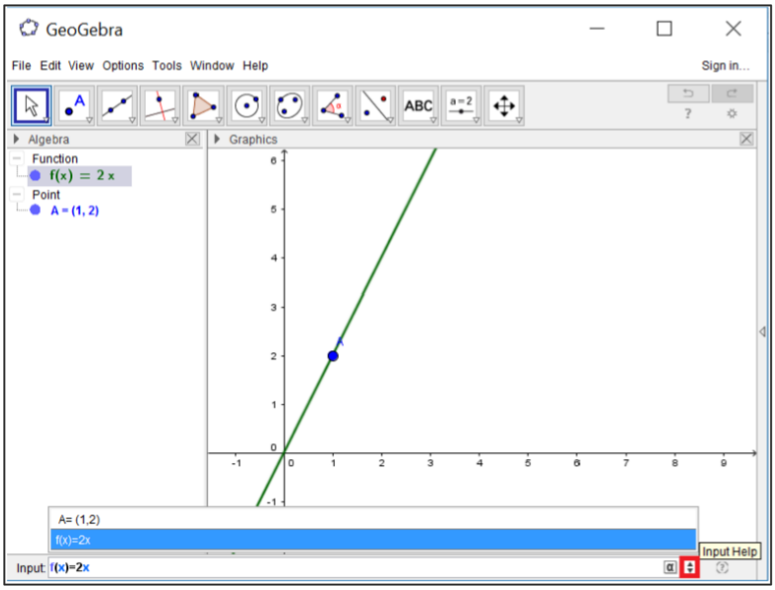

Figure 30 Here, the up and down arrow button (marked by the red box) has been checked whereby the history pops up as bars in a box. Further, the cursor has been put on the Input Help icon.

Further, there is an up and down arrow button offering the Input History in the Input Bar when having made some inputs in the document, where you can *...navigate through prior input step by step* (Webpage: Input Bar, 2016) and choose the input from earlier which is desired to work with (see Figure 30). As also seen on Figure 30, there is a little question mark surrounded by an open circle far right in the bottom of the GG view, next to the Input Bar, offering Input Help for what is desired as an output. Furthermore, Figure 30 reveals how the output of the typed input through the Input Bar is organised in categories (here; "Function" and "Point") and the blue bullets in front of the objects indicate that they are represented in the Graphics View - when clicking on a bullet, it will turn white and the corresponding representation of the object in the Graphics View will disappear.

The CAS View

In Figure 31 it is seen that there is another Toolbar when working in the CAS View. This Toolbar *...provides a range of CAS Tools that allow you to evaluate input and perform calculations* (Webpage: CAS View, 2016).

From left to right in the red box on Figure 31 the following Tools are represented; Evaluation Tools (consisting of three tools marked by the green box), Calculation Tools (marked by the purple box), Analysis Tools (marked by the orange box) and lastly, the General Tools. Four tools will mainly be used in the analyses of the CAS View in relation to the two themes, the first of which is called the Evaluate Tool (marked by a blue box in Figure 31) through which it is possible evaluate an expression in the CAS View by using the Enter Key (Webpage: Evaluate Tool, 2016). The second tool to be used is the Numeric Tool (represented by a "≈" -symbol; the second from the left in the green box in Figure 31), relevant for getting numerical approximations of expressions in the CAS View, also by using the Enter Key (Webpage: Numeric Tool, 2016). The Solve and the Solve Numerically Tool are the third and the fourth of the tools to be used, represented by (respectively) a "$x =$"-symbol and a "$x \approx$"-symbol (the fourth and the fifth in the purple box in Figure 31). Through these two tools, it is possible solve equations in the CAS View. The function of these tools will be further elaborated in the a priori analyses of the two themes in relation to GG.

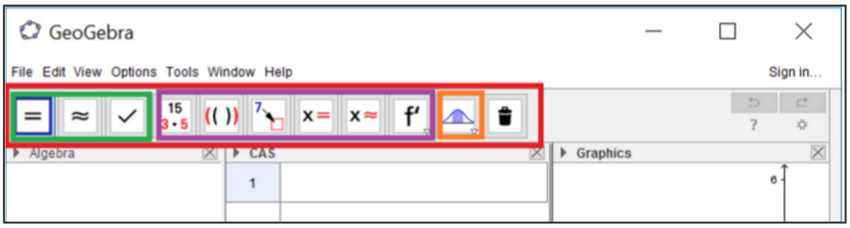

Figure 31 The Algebra Perspective of GG with added CAS View and boxes constituting the Toolbar connected thereto.

Furthermore, the CAS View offers cells where it is possible to enter the input to be evaluated. As Figure 32 shows, the cells consist of an Input Field (described further below) at the top of the cell and an output display at the bottom of the cell. This is clearly seen in cell 1, where "3x^2=12" is the input written in the Input Field and the output (achieved by clicking on the Solve Tool) is displayed just below next to "Solve:", namely "{x=-2, x=2}". Cell 2 is empty and ready for a new input to be typed, just by clicking on the blank bar which is also called the Input Field.

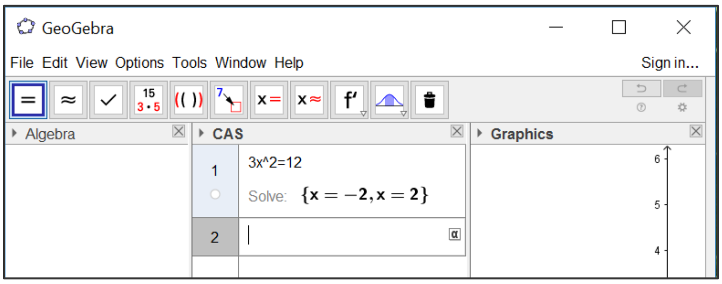

Figure 32 In cell 1 of the CAS View (added to the Algebra Perspective) both an input and an output is displayed (respectively the first and the second line of cell 1).

The *Input Field* almost functions as the Input Bar (connected to the Algebra View) but there are some differences between the two which will now be explained. In the Input Field, it is possible to work with variables which are not assigned any values which has been observed not to be possible in the Input Bar. If values are to be assigned, it is necessary to type ":=" in the Input Field since "=" is used for equations whereas there is no distinctions between "=" and ":=" in the Input Bar. When working with multiplication it should be marked explicitly in the Input Field. As an example, it is possible to type both "a(b+c)" and "a*(b+c)" in the Input Bar but only "a*(b+c)" is valid in the Input Field. Noticeably, a white bullet can be seen underneath number 1 and in front of the output display in the first cell on Figure 32. By clicking on this bullet, the output will be represented in the other Views, if possible. When doing so, the colour of the bullet turns from white into blue.

GG worksheets

As a final note for this review of the main features of GG, it is possible to make dynamic worksheets in GG and share them with other GG users through GeoGebraTube (Webpage: GG-worksheets, 2016). *GeoGebraTube* is an online service where everybody has access to shared worksheets worked out for different mathematical contents[51]. As this type of work with GG is categorised as being suitable for advanced users and especially relevant for teacher material, such worksheets are not designed in the analyses of GG in relation to the themes (Webpage: GeoGebra (en), 2016). Selected examples of existing worksheets from GeoGebraTube (referred to as *GG worksheets* or *worksheets*), and what these have to offer in connection with tasks within the two mathematical themes, will however be presented. Danish worksheets will here be used as examples, since both the teachers and the students are Danish. Quotations from the GG worksheets will nonetheless be made in English (directly translated from the Danish sentences).

[51] Noteworthy, it is possible to download these worksheets so that they can be accessed offline as well as online.

Next, the a priori analyses of GG in relation to each of the themes of Trigonometry and Power Functions will be made (in the sections 4.3.2 and 4.3.3, respectively). These analyses will be based on the Algebra Perspective since it, first of all, is the one proposed by GG when opening a new document and, secondly, it offers an interface appropriate for both of the mathematical themes. The analyses are built up around solving given tasks in GG. The (instrumented) techniques will be presented in such a way that arrows pointing right (→) are used in in order to separate the (sets of) gestures to be made in GG as well as steps of reading off values from the View(s) in GG. Furthermore, a common denotation for the techniques is that of applying an abbreviation of the View or Perspective in which the technique is performed. Here, the *Algebra Perspective* is abbreviated AP, the *CAS View*; CV and the *Spreadsheet View*; SV. Further, the developer, being the *researcher*, is abbreviated TR for the researcher of the Trigonometry course and PR for the researcher of the Power Functions course.

4.3.2 The CAS Tool in Relation to Trigonometry (L)

This section is concerned with an examination of the potentials of GG in relation to Trigonometry. More specifically the *potentials* refer to the gestures and (instrumented) techniques possible to make in GG concerning tasks within the theme. The structure of this analysis is inspired by the structure of the review of the main specific features in section 4.3.1 just above. That means, first an examination of the gestures and techniques (τ^i) possible to make within the Algebra Perspective (i.e. the Algebra View and the Graphics View), then within the CAS View and finally, three chosen worksheets, related to Trigonometry, are presented.

The Algebra Perspective

First, some possible instrumented techniques to construct triangles in GG, both *right triangles* (RT) and *arbitrary triangles* (AT), are presented. It should be noted that arbitrary triangles are involved in all the exam exercises and tasks presented in section 4.1.2 whereas right triangles are not directly related to the presented exam exercises and tasks (even though an arbitrary triangle always can be divided into right triangles). The construction of a right triangle in GG can be done by adding the grid in the Graphics View such that it makes it easier to make collinear points. Constructing the right triangle is done by:

$\tau^{i}_{RT_1,AP,TR}$: Click on the Polygon Tool → Place and combine the three points on the grid such that two of the sides of the polygon are constructed by collinear points (The start point is also the point which should be clicked on at the end in order to close the polygon)

The outcome of this technique is shown on the Figure 33. Noticeably, the right angle has been measured through applying the Angle Tool (to be elaborated further later in this passage). Another way of constructing a right triangle is done by:

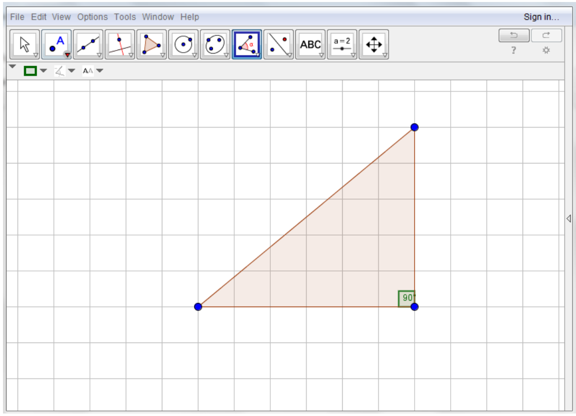

Figure 33 The outcome of $\tau^{i}_{RT_1,AP,TR}$. Further, the gesture of measuring an angle has been performed in order to check that one angle is right (**90**°).

$\tau^{i}_{RT2,AP,TR}$: Click on the Segment Tool → Place the points in the Graphics View (automatically named A and B) → Choose Perpendicular Line Tool → Select the line segment and a point (the straight line is created through the point and is perpendicular to the line segment) → Choose the Intersect Tool (the intersection point can be created in two ways:
• *Selecting two objects creates all intersection points (if possible),*
• *Directly clicking on an intersection of the two objects creates only this 'single intersection point'* (Webpage: Intersect Tool, 2016)) → Click on the line segment and the perpendicular line (intersection point is created (automatically named D)) → Click on the Polygon Tool and combine the points A, C and D.

The outcome of $\tau^{i}_{RT2,AP,TR}$ is shown on Figure 34.

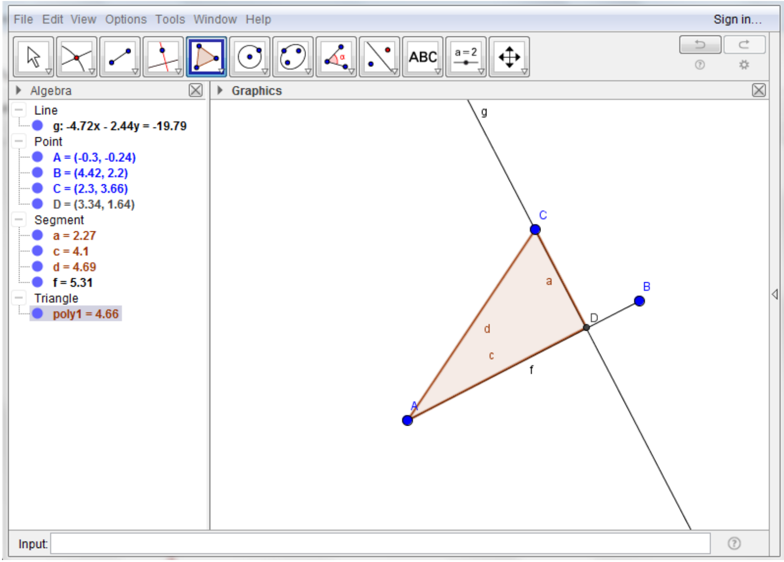

Figure 34 Shows the right triangle *ACD* where it is to be noted that point *D* is in a different shape than the other points. This is due to the fact *D* is fixed to the perpendicular line *g* and the line segment *AB* which means that the point can only be moved so that it follows the line segment *AB*.

A third possible way to construct a right triangle is:

$\tau^{i}_{RT3,AP,TR}$: Click on Segment Tool → Place the points (*A* and *B*) in the Graphics View → Select Angle with Given Size Tool → Click on *A* and *B* and type 90 ° into the appearing input box (clockwise) → Combine the points *A, B* and *A'* via the Polygon Tool.

The input box mentioned in the fourth step of $\tau^{i}_{RT3,AP,TR}$ is illustrated in Figure 35 and the outcome of $\tau^{i}_{RT3,AP,TR}$ is shown in Figure 36.

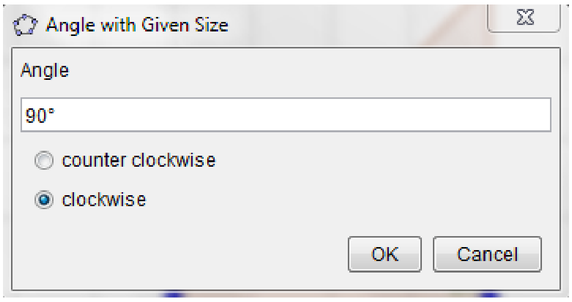

Figure 35 Here, the appearing input box is shown where an angle of **90** ° is entered and is chosen to be constructed "clockwise".

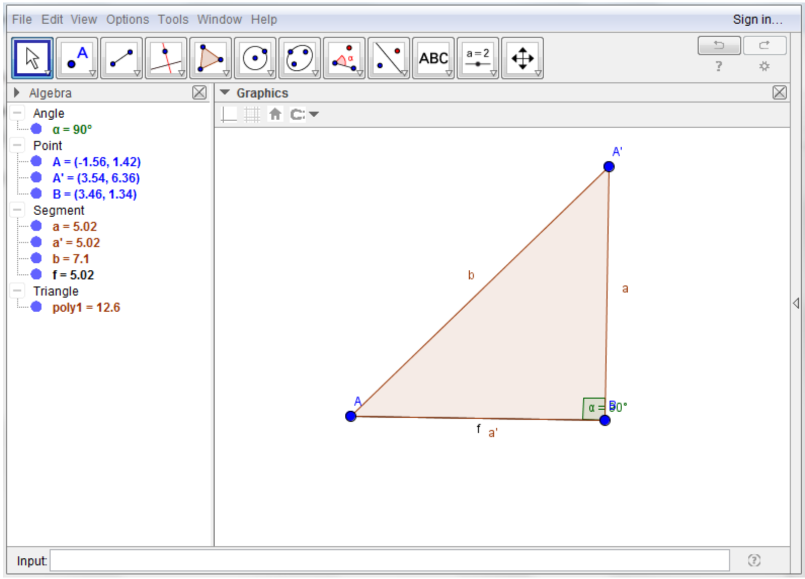

Figure 36 The outcome of $\tau^{i}_{RT3,AP,TR}$ in the form of the right triangle *ABA'*.

With respect to $\tau^{i}_{RT3,AP,TR}$ (and Figure 36) it should be noted that GG automatically place *A'* as an auxiliary point for where the last vertex may be placed. Further, GG automatically gives the line segment *AB* the name *f* and when the polygon is drawn GG name the sides *a,b* and *a'*. It is however possible to rename the vertices, the angles, the line segments and the sides under "Option properties". In the Algebra View, it is possible to see the values for *f, a, b*, and *a'* and the angle measure for α and the coordinates for the vertices. Further, the area (related to the type of task T_{ET_4}) of the 3-gon is shown in the Algebra View as "poly1", which in this case is 12.6.
Additionally, $\tau^{i}_{RT3,AP,TR}$ is an instrumented technique which can also be used for constructing arbitrary triangles. However, none of the angles are here 90°. The technique in relation to constructing an arbitrary triangle will however not be exemplified since the procedure is the same as that of $\tau^{i}_{RT3,AP,TR}$.
Instead, another technique will be presented for constructing arbitrary triangles. This is a possible technique for solving T_{ET_1} and the technique presented below will be based on τ_{TTW_4} where $A = 30°, b = 7$ og $c = 5$ in triangle ABC.

$\tau^{i}_{AT,AP,TR}$: Select Segment with Given Length Tool → Click on the point that is chosen to be the starting point of the segment and type the desired length of the segment (i.e. 5) in the appearing window. The line segment *AB* is hereby constructed. → Click on Angle with Given Size Tool →

Click on the leg point B and the point A and type 30° into the input box (counter clockwise) (GG automatically place an auxiliary point (B') on the basis of the constructed angle → Click on Ray Tool → Select two points A and B' (Ray is constructed from A and through B') → Select the Circle with Center and Radius Tool → Select the center point (in this case; A) and then type the radius of the circle equal to 7 (since $b = 7$ in triangle ABC is given) in the text field of the appearing window (now the circle and the ray intersects each other illustrated by Figure 37) → Choose the Intersect Tool → Click on the ray and the circle (the point C is hereby constructed) → Click on the Polygon Tool and combine A, B and C **[**Hide the unnecessary objects (i.e. the circle and the ray) in the Algebra View by clicking on the related blue bullets (turning into white)**]** → Read the side length of a from the Algebra View in order to answer the task.

The gestures marked by **[ ]** are not necessary for answering the task but leads to a representation of the relevant objects (being the points and sides of the triangle ABC) in the Graphics View - which makes it simpler to consider the triangle ABC.

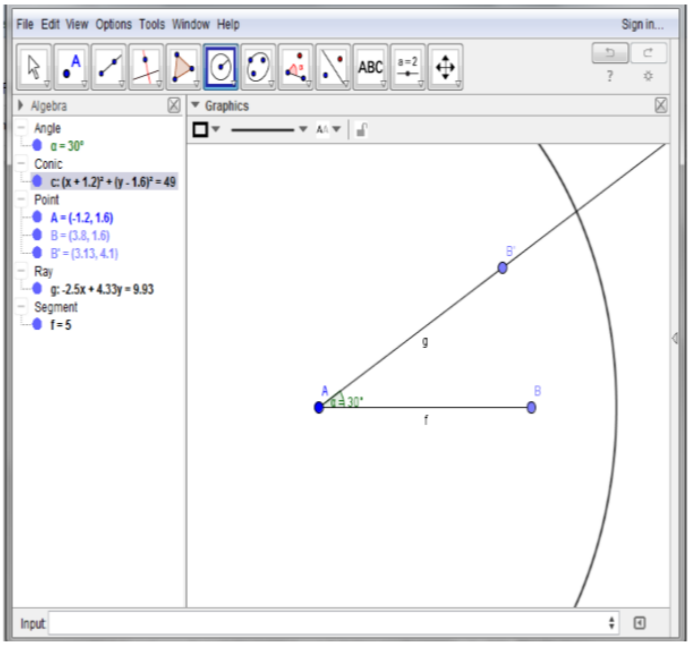

Figure 37 The part of $\tau^{i}_{AT,AP,TR}$ where the circle and the ray intersect each other.

On Figure 38, the final construction of an arbitrary triangle using this technique is seen.

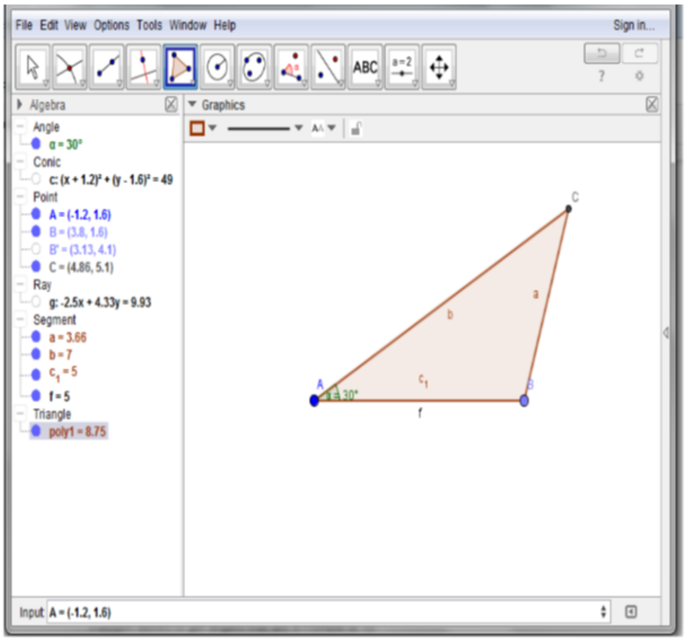

Figure 38 The outcome of $\tau^{i}_{AT,AP,TR}$ in the form of triangle *ABC*. It is clearly seen in the Algebra View that $\boldsymbol{a} = \mathbf{3.66}$, which gives the answer to the exam question $\boldsymbol{T_{ET_1}}$.

T_{ET_2} - T_{ET_5} can be answered using the same gestures as in $\tau^{i}_{AT,AP,TR}$ just in a different order (and with other measures of sides and angles). However, to determine unknown angles (especially relevant for T_{ET_2}), it is good to know the Angle Tool as well, since it will measure the desired angle. To construct such an unknown angle, there are (at least) three options after having clicked on the Angle Tool:

$\boldsymbol{\tau^{i}_{Angle1,AP,TR}}$: Click on three points where the second point selected becomes the vertex of the angle.

$\boldsymbol{\tau^{i}_{Angle2,AP,TR}}$: Click on two line segments and create an angle between the two.

$\boldsymbol{\tau^{i}_{Angle3,AP,TR}}$: Click on the constructed polygon.

For the latter technique, the angle measure depends on whether one connects the points of the polygon clockwise or counter clockwise as seen on Figure 39.

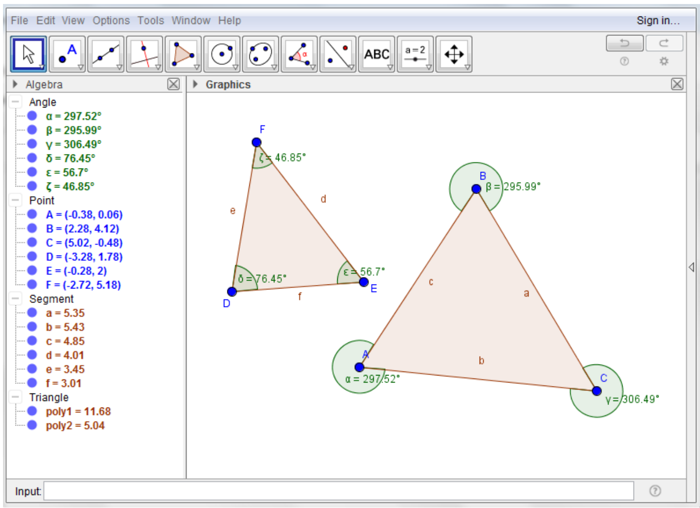

Figure 39 The angles are interior in triangle *DEF* since it is constructed counter clockwise whereas triangle *ABC* has exterior angles since the points of the triangle are connected clockwise.

In GG, it is also possible to construct altitudes, medians, perpendicular bisectors and angle bisectors, which are helpful tools in order to answer exercises like the one presented in Figure 15 (b)). The median can be constructed by the following technique:

$\boldsymbol{\tau^{i}_{Median,AP,TR}}$: Click on the Polygon Tool and combine the points of the triangle → Click on Midpoint or Center Tool → Click on a side in the triangle whereby the midpoint (here named D) of that side is constructed → Click on Segment Tool → Click on the vertex point (here named A) located opposite to D and click on D (whereby the median is constructed and named f).

The outcome of this technique is illustrated in Figure 40**.**

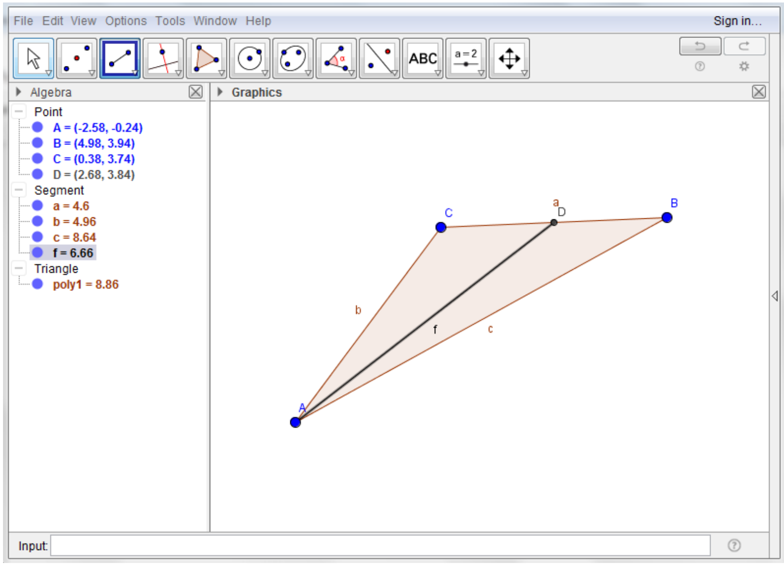

Figure 40 The outcome of $\tau^{i}_{Median,AP,TR}$. Here, the length of the median, named f by GG, is seen in the Algebra View to be $\boldsymbol{f = 6.66}$.

Noticeably, $\tau^{i}_{Median,AP,TR}$ is a very simple technique for solving the second part of the exercise in Figure 15 compared to the one where the median is constructed without the use of GG followed by using the Law of Cosine or the Law of Sine. The latter technique requires first of all knowing how to construct a median by hand (using paper and pencil) and involvement of the Laws in order to determine its length.

To construct the altitude of an arbitrary triangle the following technique can be applied:

$\boldsymbol{\tau^{i}_{Altitude,AP,TR}}$: Click on the Polygon Tool and combine the points of the triangle → Click on the Perpendicular Line Tool → Select a point (vertex) and the line segment opposite to the point (the line f is constructed) → Click on the Intersect Tool to construct the intersection point F between line f and line segment AB → Click on the Segment Tool and combine point F and point C (the line segment g is constructed and is the altitude of triangle ABC).

See Figure 41 for the outcome of $\tau^{i}_{Altitude,AP,TR}$ and note that the length of the constructed altitude g is presented in the Algebra View.

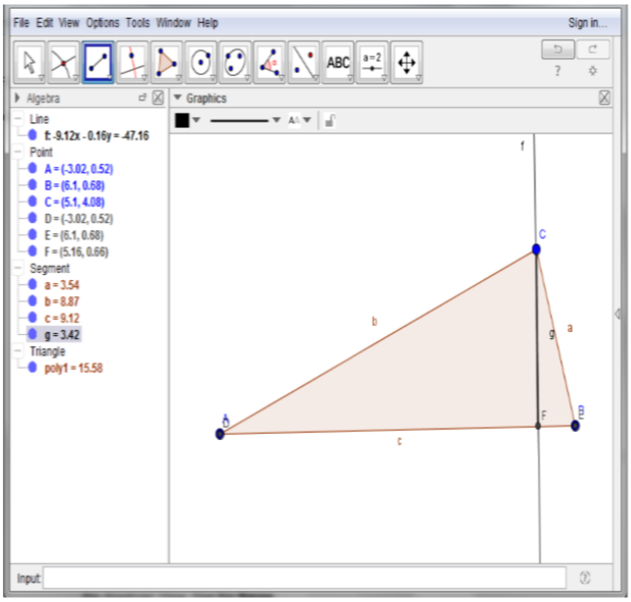

Figure 41 The outcome of $\tau^{i}_{Altitude,AP,TR}$ in the form of the altitude g in triangle ABC. The length of g can be read from the Algebra View to be $\boldsymbol{g = 3.42}$.

An interesting aspect of GG is that it is a dynamic tool, which becomes very useful when working with similar triangles. The thing about constructing similar triangles in GG is that it is possible to make the angles and the sides of the triangles interdependent. The interdependence leads to the option of dragging the points of one of the triangles such that the shape of the other triangle will be changed simultaneously. Hereby, the angles of the other triangle remain identical to the angles of the triangle in which the point(s) have been dragged.

For the construction of the first triangle (ST_1) of two *similar triangles* (ST), the following technique can be applied:

$\boldsymbol{\tau^{i}_{ST_1,AP,TR}}$: Click on the Polygon Tool and combine the points of the triangle (*A, B* and *C*) → Use the Angle Tool to construct all of the three interior angles of triangle *ABC* (renamed *aA*, *aB* and *aC* via Object Properties).

To construct a similar triangle (ST_2) to the one just constructed the following technique can be performed:

$\boldsymbol{\tau^{i}_{ST_2,AP,TR}}$: Click on Segment Tool → Place the points in the Graphics View → Rename the points through Object Properties to *A1* and *B1* and the line segment to *c1* → Choose Angle with Given Size Tool → Click on *B1* and *A1* and type "aA" into the input box (counter clockwise) → Click on *A1* and *B1* and type "aB" into the input box (clockwise) → Select Ray Tool → Select two points, *A1* and the auxiliary point → Click on *B1* and the auxiliary point (now, two intersecting Rays have been constructed) → Click on the Intersect Tool → Select the two Rays and the intersecting point is constructed. This is renamed *C1* via Object Properties → Click on the Polygon Tool and combine the points (*A1, B1*, and *C1*) of the triangle **[**→ Remove all the unnecessary auxiliary lines and points.**]**

The Figure 42 shows the outcome of both $\tau^{i}_{ST_1,AP,TR}$ and $\tau^{i}_{ST_2,AP,TR}$.

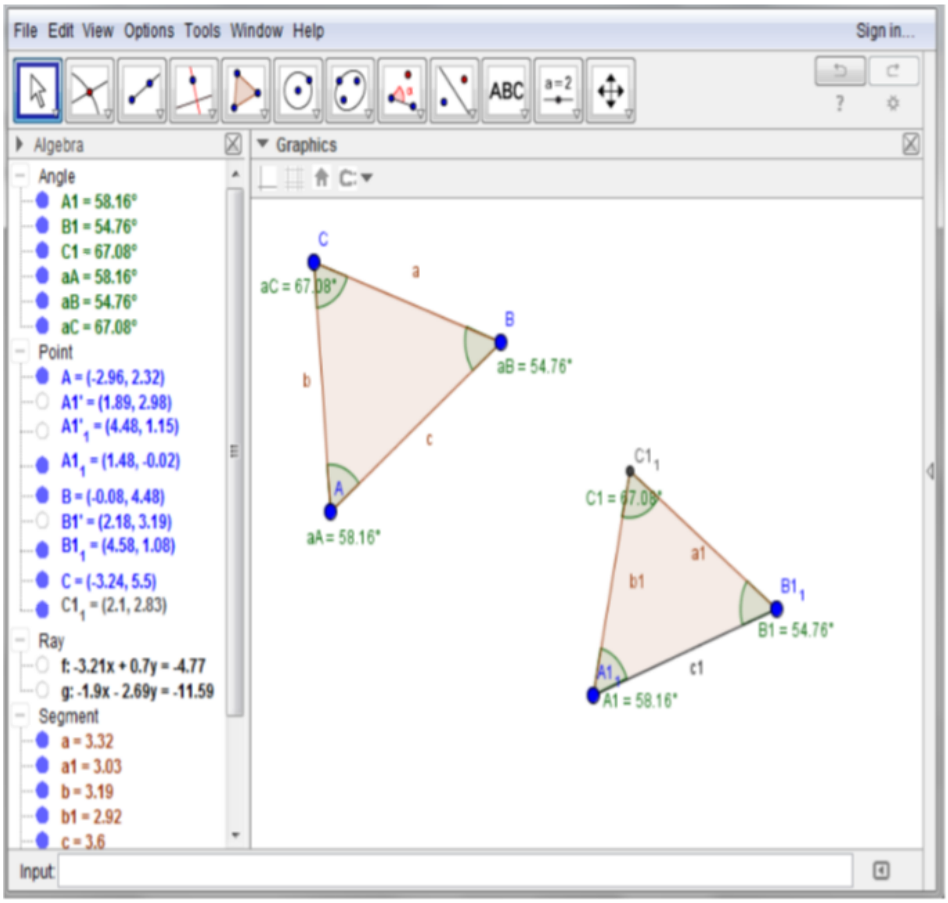

Figure 42 Here, two similar triangles, triangle *ABC* and $\boldsymbol{A1_1B1_1C1_1}$. The reason for the lowered 1 is that GG distinguishes between points and angles of the triangle $\boldsymbol{A1_1B1_1C1_1}$ when the angles have been named *A1, B1* and *C1* as well as the points.

After having performed the techniques $\tau^{i}_{ST_1,AP,TR}$ and $\tau^{i}_{ST_2,AP,TR}$, it is possible to drag the points of the first constructed triangle, triangle *ABC*, via the Move Tool. Here, the other triangle ($A1_1B1_1C1_1$) will change simultaneously with triangle *ABC*, in the sense that the angles are ever congruent. For instance, when dragging the point *B* of the first triangle, the other triangle will adapt to the new settings, see Figure 43.

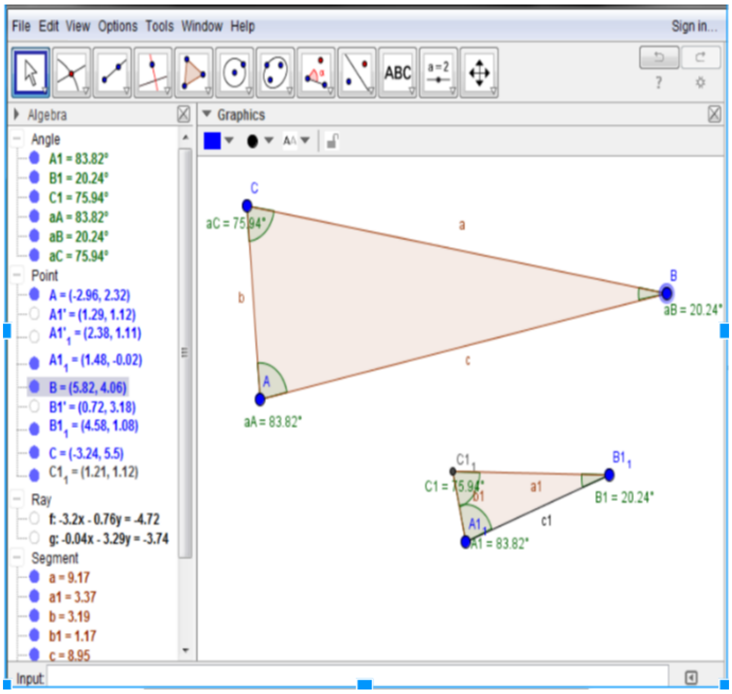

Figure 43 The gesture of dragging the point *B* in triangle *ABC* of Figure 42 has been performed here. It is seen that the angles of the two triangles are congruent and the corresponding sides are similar. Triangle *ABC* is a scaled version of triangle $\boldsymbol{A1_1B1_1C1_1}$.

As an endnote (regarding Figure 43), by dragging in triangle $A1_1B1_1C1_1$ it is not possible to change the size of the angles since they are depending on the angle measure of triangle *ABC* (constructed first). It is however possible to make the length of the sides of the triangle $A1_1B1_1C1_1$ larger or smaller, hence a scaled version of itself and additionally, it is possible to rotate it. This by use of the Move Tool .

To answer tasks of type T_{ET_5} it is possible to drag in the two triangles such that their angles and sides matches the numbers stated in the specific task if possible - this is however not straightforward. GG will then give the lengths of the unknown sides which can be read from the Algebra View. If it is not possible to do it this way, the scale factor can be determined by typing "k=a/a1" or "k=b/b1" or "k=c/c1" in the Input field (depending on which similar sides are known) and then calculate the unknown side in the CAS View (this View will be elaborated further down).

Now, the construction of the unit circle is considered. It is almost unavoidable not to talk about the unit circle when dealing with Trigonometry. Therefore, it is also necessary to know how to construct this in GG. The unit circle has its centre in origo $O = (0,0)$ and radius 1 (already mentioned in the unit circle context section 4.1.1). The technique used to construct the *unit circle* (UC) can be as follows:

$\tau^{i}_{UC,AP,TR}$: Type "O=(0,0)" in the Input Bar and press Enter → Click on Circle with Center and Radius Tool → Select O as the centre point and enter the radius (which in the unit circle is 1) in the appearing dialog window.

A directional point P and a directional angle aP[52], can be added to the construction of the unit circle by the use of this technique (denoted DPDA for Directional Point and Directional Angle):

$\tau^{i}_{DPDA,AP,TR}$: Click on the Point on Object Tool (which creates a point fixed to the unit circle) → Select the unit circle as the object to which the point is fixed to → Name the point on the unit circle (the directional point) P via Object Properties → Select the Ray Tool and create a Ray through O and P → Select Angle Tool to construct the directional angle → Click on the positive side of the x-axis and the ray → name the angle aP via Object properties.

The outcome of the two techniques presented just above is seen in Figure 44.

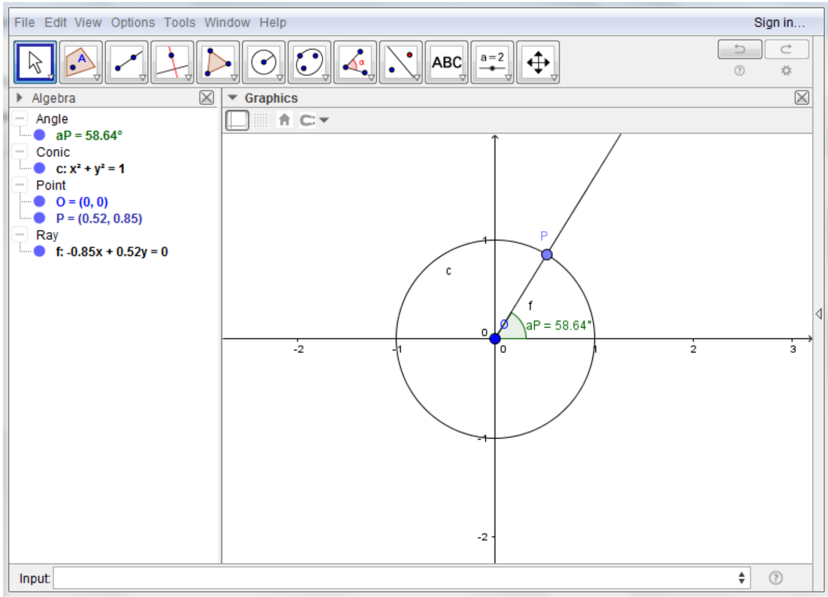

Figure 44 Here, the outcomes of $\tau^{i}_{UC,AP,TR}$ and $\tau^{i}_{DPDA,AP,TR}$ are shown in the form of a unit circle respectively the directional point (P) and the directional angle (aP).

[52] The directional angle emerges from Origo and has its right angle-leg on the positive part of the x-axis and the left angle-leg is where the leg intersects the circle through a point (the directional point).

The point, *P,* in Figure 44 can be moved along the circle via the Move Tool .
When performing the gesture of dragging the point *P*, the angle *aP* automatically follows and changes its measure.
To clarify the notion of cosine and sine in relation to GG, it is possible define "Px=cos(aP)" and "Py=sin(aP)" in the Input Bar and cosine and sine becomes respectively the *x*-coordinate and the *y*-coordinate to the point *P*. Then it is possible to see a relation between the directional angle and cosine and sine. Further it is on this basis possible to solve tasks like t_{TTW_1} and t_{TTW_2}. On Figure 45, the values for both cosine and sine when the directional angle is 58° can be seen, namely 0.53 and 0.85, respectively.

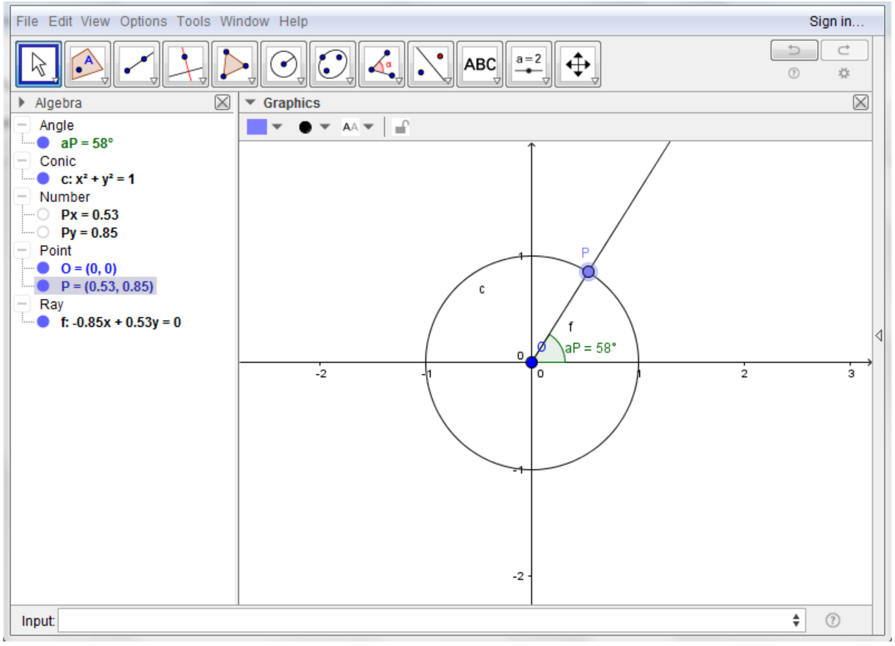

Figure 45 Here, the values of cosine and sine with respect to the angle of **58°** is shown in the Algebra View as respectively "Px" and "Py". Hereby it can be realised that the coordinates of *P* are actually (***Px***, ***Py***).

As an endnote to the review of the Algebra Perspective, it is worth mentioning that there exist boxes with explanations when moving the computer mouse over a Toolbox. These boxes work as a help tool when making constructions in GG. In the Figure 46 a box is illustrated by holding the mouse over the Angle Tool.

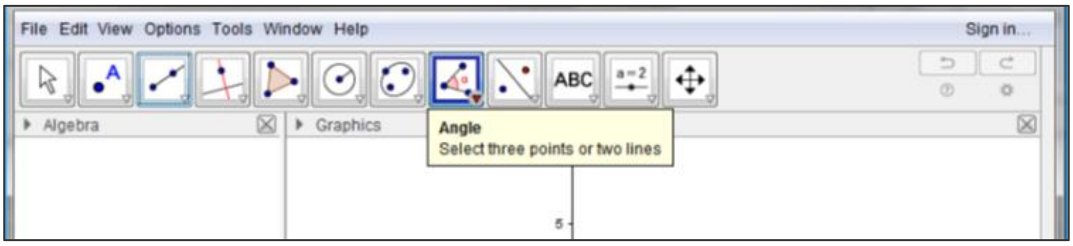

Figure 46 Here, a box helping to construct an angle via the Angle Tool is shown. In the box there is written *Angle - Select three points or two lines*.

The CAS View

GG also offers a CAS View where for instance tasks like t_{TTW_1} and t_{TTW_2} can be solved by use of a classical CAS tool.

When using the CAS View in connection with Trigonometry one has to make sure whether the result shall be given in terms of radians or degrees. If the result is wished to be in degrees, an addition of the degree symbol (°) is needed. This symbol can be made by entering the keyboard shortcut 'Alt' + 'O' (for PC) or 'Ctrl' + 'O' (for Mac OS) or by using the Alpha symbol in the Input Field and selecting the degree symbol. It should be specified that *...GeoGebra does all internal calculations in radians. The degree symbol (°) is nothing but the constant π/180 used to convert degree into radians* (Webpage: Numbers and Angles, 2016).

Hence, the technique of solving cos(45°) (inspired by t_{TTW_1}) in the CAS View is:

$\tau^i_{1,CV,TR}$: Enter "cos(45°)" in the Input Field → Click on the Evaluate Tool [=] → Click on Numeric Tool [≈] to get a numerical approximation.

The outcome of the above presented technique is shown in Figure 47.

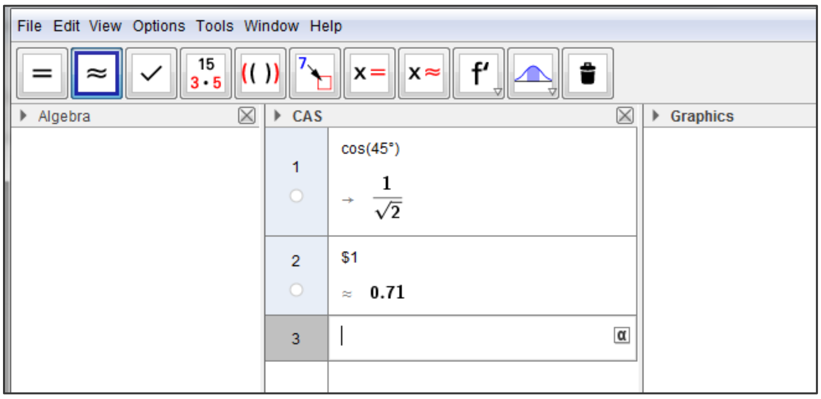

Figure 47 The outcome of $\tau^i_{1,CV,TR}$. Cell 1 shows the result when clicking on the Evaluate Tool and cell 2 shows the result after clicking on the Numeric Tool.

It is also possible to solve equations in the CAS View. E.g. if $\sin(x°) = \frac{1}{2}$ is considered, the following technique can be used for finding the unknown angle:

$\tau^{i}_{2,CV,TR}$: Type "sin(x°)=1/2" in the Input Field (in cell 1) → Click on Solve Tool → Copy the output from cell 1 to be input of cell 2 -> Click on Numerical Tool.

The outcome of this technique is shown in Figure 48.

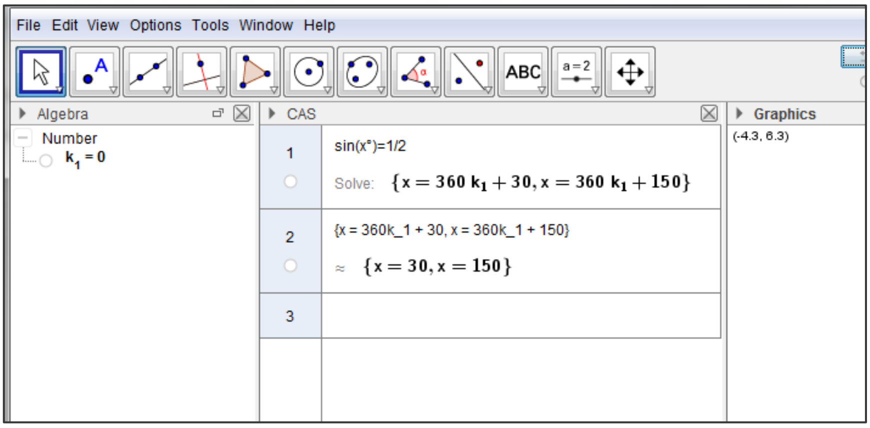

Figure 48 The outcome of $\tau^{i}_{2,CV,TR}$ is shown here. Here, the Solve Tool has been used in cell 1 and the Numerical Tool has been used in cell 2. The output of cell 1 is "x = $\mathbf{360}\ \boldsymbol{k_1} + \mathbf{30}$, x = = $\mathbf{360}\ \boldsymbol{k_1} + \mathbf{150}$" due to the periodicity of sine. The output of cell 2 is "x = **30**, x = **150**".

In connection with $\tau^{i}_{2,CV,TR}$, it should be mentioned that if the Solve Numerically Tool is selected instead of the Solve Tool, the output will be "x = 30".

When applying the Law of Sine or the Law of Cosine, certain reservations must be taken concerning calculations in the CAS View. For example, an angle must be guessed when working with the Laws. This is slightly easier to explain through an example. The example is illustrated in Figure 49 and is based on the task of the type T_{ET_2} from the exam exercise in Figure 14 where the Law of Cosine is needed in order to solve the task.

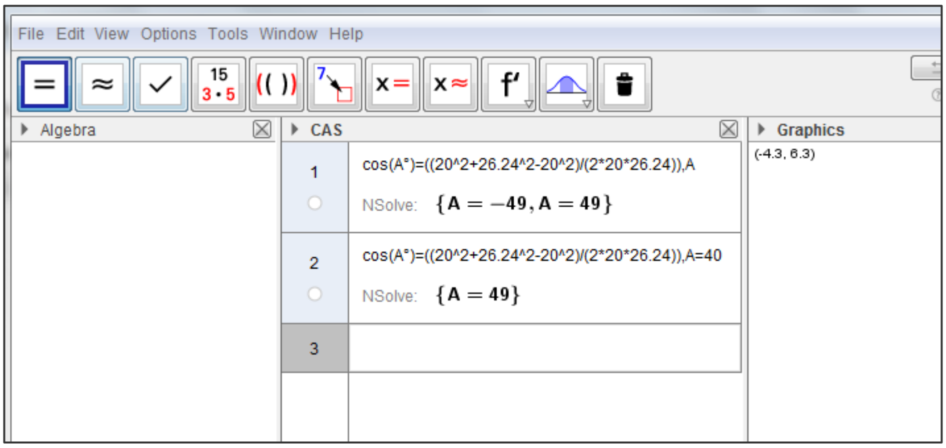

Figure 49 Equation solving when applying the Law of Cosine to solve a specific task of type T_{ET_2} (from Figure 14) in the CAS View. More specifically, the outcome of the use of the techniques $\tau^i_{3,CV,TR}$ and $\tau^i_{4,CV,TR}$.

The NSolve command, also known as Solve Numerically Tool *Attempts (numerically) to find a solution of the equation for the given unknown variable* (Webpage: NSolve Command, 2016) and the command is noted only to work for continuous functions[53]. On Figure 49 the following techniques is used in cell 1 and respectively cell 2:

$\tau^i_{3,CV,TR}$: Enter the equation "cos(A°)=((20^2+26.24^2-20^2)/(2*20*26.24)),A" → Click on the Solve Numerically Tool.

$\tau^i_{4,CV,TR}$: Enter the equation "cos(A°)=((20^2+26.24^2- 20^2)/(2*20*26.24)),A=40" → Click on the Solve Numerically Tool.

The techniques $\tau^i_{3,CV,TR}$ and $\tau^i_{4,CV,TR}$ differ from each other by the guessed angle ($A = 40$). When adding this angle, the output only relates to the triangle context (where angles cannot be negative and the sum of the angles is equal to 180°) which is noted to be the case in the exam tasks T_{ET_1} - T_{ET_5}. Noticeably, it is only possible to guess an angle ∈ (0°, 180°). Further, when solving equations where the Laws are applied and where the unknown variable is a side length of a triangle, it is not required that the length of the side has to be guessed.

---

[53] A function f is *continuous* in a point $a \in D_f$ (the domain of f) if $\forall \varepsilon > 0\ \exists\ \delta > 0$: for $x \in D_f$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$ (Lindstrøm, 2006, p. 212). The functions $\sin(x)$ and $\cos(x)$ are well known to be continuous for all $x \in \mathbb{R}$.

Tasks of type T_{ET_5} (as the one presented in Figure 17) can also be determined within the CAS View. A technique, which can be used hereto, is:

$\boldsymbol{\tau^{i}_{5,CV,TR}}$: Type “10/4” in the Input Field of cell 1 → Click on the Numerical Tool ≈ → Type “2*2.5” (where 2.5 is the output from cell 1) in cell 2 → Click on the Numerical Tool ≈.

The outcome of this technique can be seen in Figure 50.

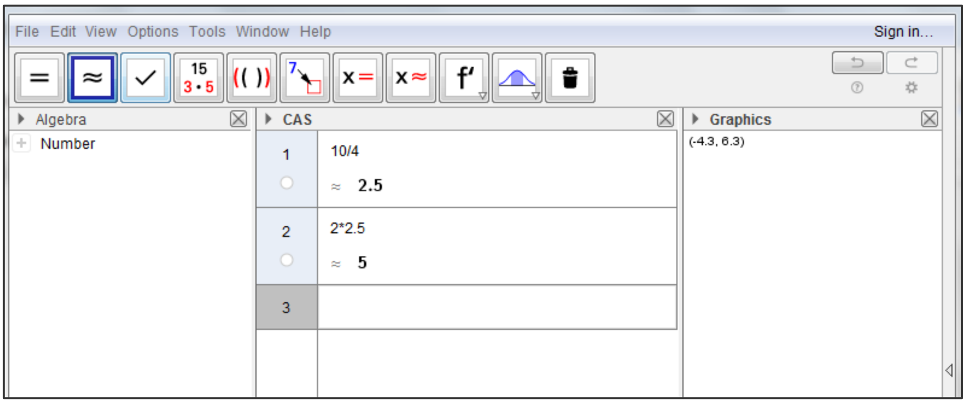

Figure 50 The outcome of the technique $\boldsymbol{\tau^{i}_{5,CV,TR}}$. In cell 1, the scale factor $\boldsymbol{k = \frac{AE}{AC}}$ is calculated and in cell 2, $\boldsymbol{DE = BC \cdot k}$ is calculated.

GG worksheets in relation to Trigonometry

Finally, three relevant GG worksheets in connection to Trigonometry will be presented in this section. The three worksheets involve different aspects of GG, some are dynamical some requires clicking in order to get the correct answers. The first worksheet to be presented is the “Triangle calculator” seen in Figure 51. This worksheet can be found on: http://tube.geogebra.org/m/2248515 (last visited 20.06.2016).

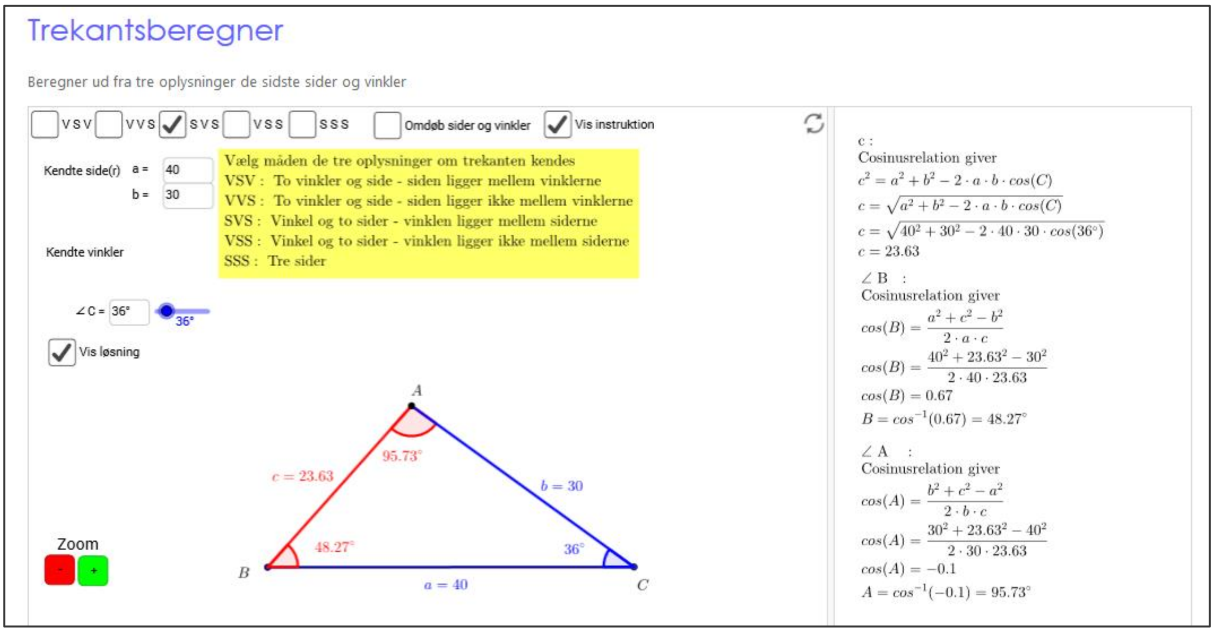

Figure 51 Here, two sides, *a* and *b*, and the intermediate angle *C* are known in triangle *ABC*, which are to be entered as input in the visible boxes. The worksheet offers a yellow instruction box as well as the possibility to rename the angles and sides and it provides a drawing of the triangle as well. Moreover, the worksheet gives the information of all the algebraic manipulations. Here the Law of Cosine is used.

This worksheet is a fully automated problem solver in the sense that it just gives the answers and the calculations behind them when entering the information of given values of side lengths and angles. The worksheet only represents the practical block of solving tasks related to angles and side lengths of triangles (for instance of the type T_{ET_1}, T_{ET_2} or T_{ET_3}) in the sense that it both serves as, provides and performs the technique. Hereby, the worksheet does not necessarily require a knowledge block established by the user.

Another worksheet is concerned with the proof of the Pythagorean Theorem. This worksheet is illustrated in the figure below (Figure 52) and can be found on the link: https://www.geogebra.org/m/219527 (last visited 20.06.2016).

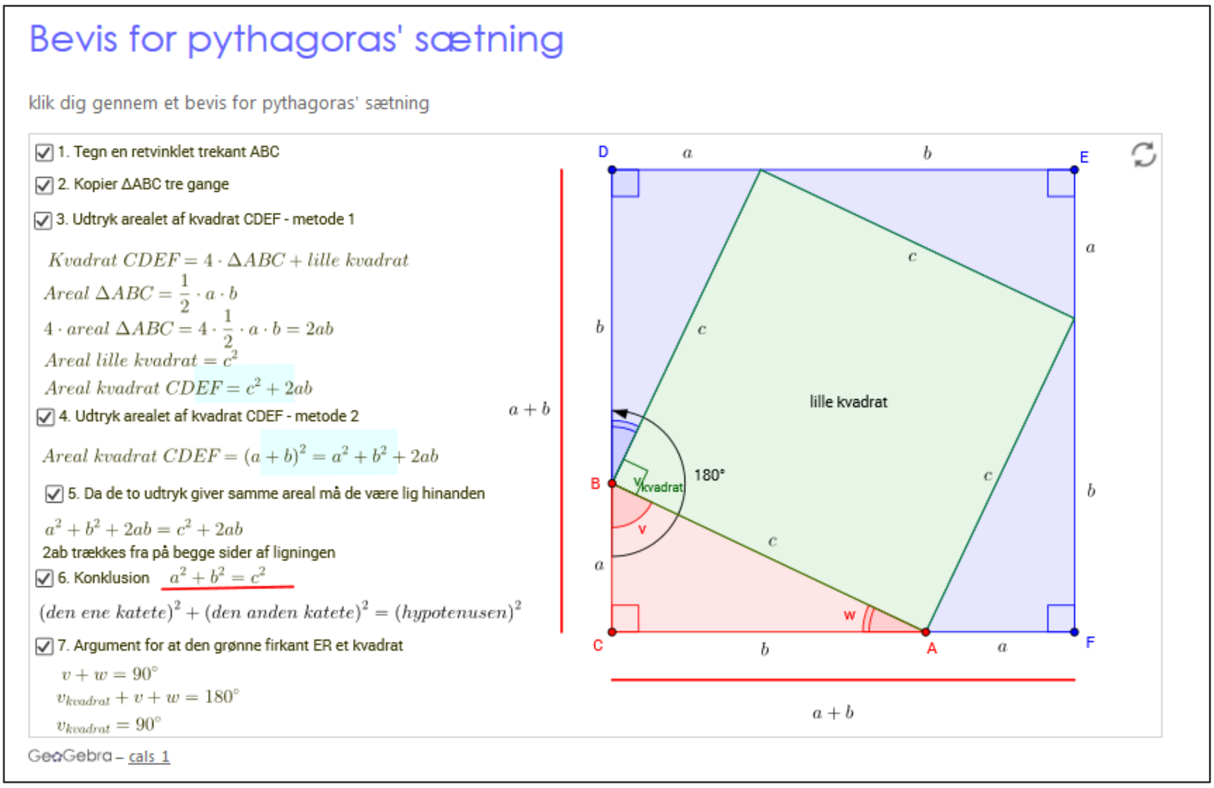

Figure 52 This worksheet is a review of the Pythagorean Theorem through a Click Proof (i.e. hints for the steps of the proof can be called for through clicks). Here, all boxes from 1-7 are marked such that algebraic gestures appears as an output of the click gesture. Along with marking the boxes, the figure to the left appears which gives the students a visualisation of the proof.

This worksheet provides a review of the proof. For each check mark the worksheet gives more and more information needed to finish the proof. The information given is in terms of algebraic gestures and illustrations. Even though the worksheet is concerned with a proof, it is primarily concerned with the practical block of performing the proof since it consists of click steps presenting algebraic gestures without explanations. However, there are technological elements in point 5.

The next and last worksheet to be considered presents sine and cosine by using the unit circle (see Figure 53). The worksheet can be found by the link: https://tube.geogebra.org/m/49281 (last visited 21.06.2016).

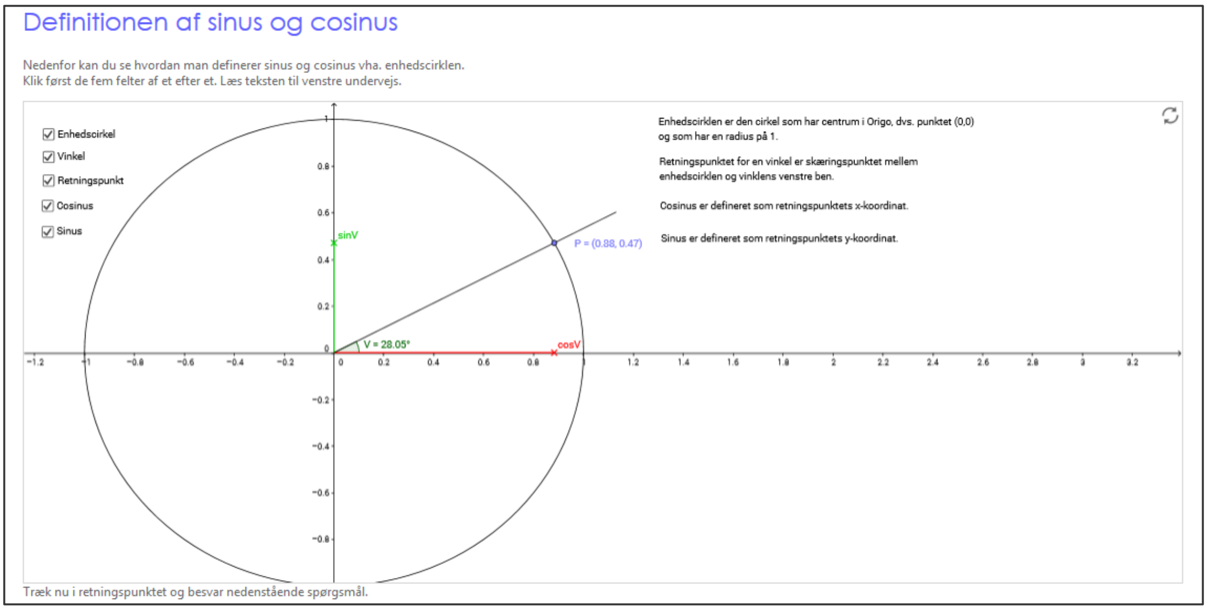

Figure 53 This worksheet is dynamical in the sense that *P* can be dragged. It also contains boxes that can be checked, in order to see illustrations of the unit circle, the (directional) angle (*V*), the directional point (*P*), Cosine or Sine. To the right, some explanatory comments related to the checked boxes are presented.

The worksheet can be used as a supplement to t_{TTW_1} and t_{TTW_2} in section 4.1.3 in relation to the construction work as well as to solving the tasks. By using this worksheet, the students do not have to construct the unit circle from scratch. The unit circle and related features are constructed by clicking on the five boxes (Unit circle, Angle, Directional point, Cosine and Sine) that appears in the upper left corner (see Figure 53). When the boxes are checked, illustrations and explanations related to the boxes appear. For instance, if the box of Sine has been marked, a green arrow appears in the illustration as well as the explanatory comment which in this case is: *Sine is defined as the y-coordinate to the directional point* (the fourth comment presented to the right in Figure 53). The point *P* can be dragged such that it is possible to see how the directional angle and sine and cosine with respect to this angle are related. Besides from the praxis of dragging the point *P,* the worksheet contains explanatory comments associated to the unit circle and its features. Hereby, the user can establish a knowledge block in relation to tasks like t_{TTW_1} and t_{TTW_2}.

As an endnote for this review of GG worksheets, the design of all of the three presented worksheets links algebra to figures (i.e. Geometry II - see section 4.1.1).

4.3.3 The CAS Tool in Relation to Power Functions (A)

This a priori analysis is based on the Views in GG immediately found relevant for working with problems related to the theme of Power Functions and especially tasks of the types T_{EP_1} - T_{EP_6}. The Perspective and Views to be considered here are: The Algebra Perspective, The CAS View and The Spreadsheet View. Each View offers specific opportunities for solving tasks which will be presented here where the techniques are numbered in direct relation to the types of tasks (T_{EP_1} - T_{EP_6}). Finally, four worksheets related to the theme will be explored.

The Algebra Perspective

Through the Algebra Perspective it is possible to solve several of the task types T_{EP_1} - T_{EP_6} which will be elaborated here.

First of all, tasks of type T_{EP_1} can be solved through the following technique, concerning the determination of the coordinates of the intersection point between the line $x = x_0$ and the graph of the given power function f:

$\boldsymbol{\tau^{i}_{1,AP,PR}}$: Type the functional equation for *f* in the Input Bar as "f(x)=b*x^a" where *a* and *b* are given values → Press Enter → Type "x=x_0" where x_0 is a given value → Press Enter **[**→ Choose the Move Graphics View Tool → Drag the *x*- and/or *y*-axis until both the line $x = x_0$ and the graph of *f* are visible in the Graphics View**]** → Choose the Intersect Tool → Click on both the graph of the function *f* and the line $x = x_0$ in the Graphics View (i.e. the intersection) → Read off the second coordinate of the intersection point from the Algebra View representation of the point.

The gesture of clicking on both the graph of f and the line $x = x_0$ is noticeably only possible if the intersection between the line $x = x_0$ and the graph of f is visible in the Graphics View, or else the gestures marked by **[ ]** becomes relevant to perform in order to achieve visibility of the intersection. See Figure 54 for an example of the outcome of $\tau^{i}_{1,AP,PR}$ - performed on the specific task of type T_{EP_1} presented in Figure 20.

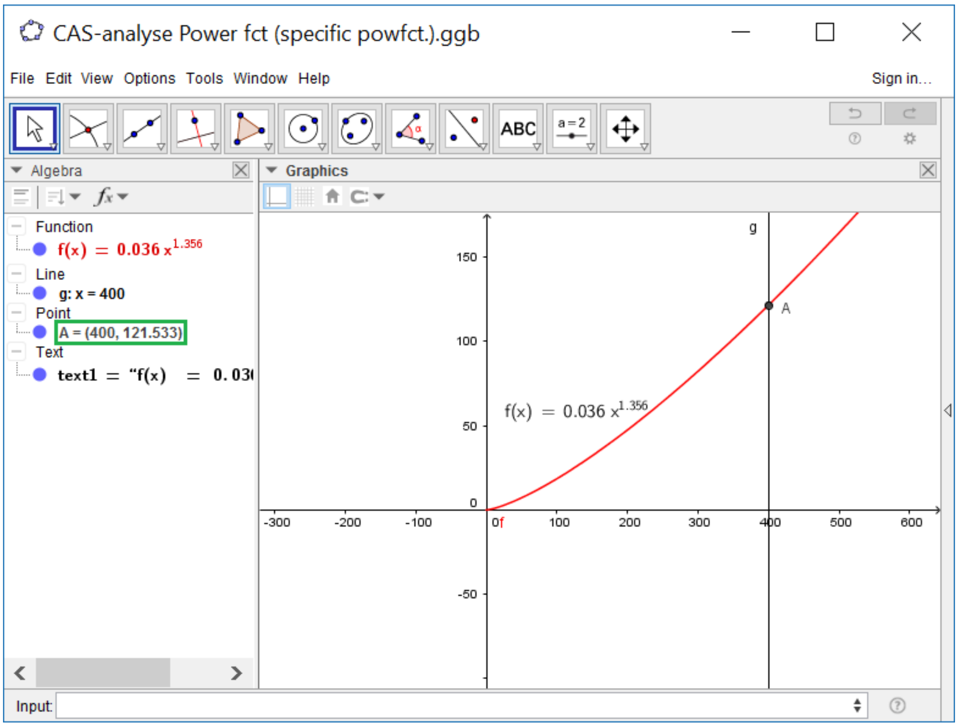

Figure 54 The outcome of $\boldsymbol{\tau^i_{1,AP,PR}}$ performed on the specific task of type $\boldsymbol{T_{EP_1}}$ from Figure 20. The line $\boldsymbol{x = 400}$ has been named g by GG and the intersection point has been named A. The coordinates of A in the Algebra View (marked by a green box) reveal that $\boldsymbol{f(400) = 121.533}$.

An almost similar technique (as $\tau^i_{1,AP,PR}$) can be applied for tasks of type T_{EP_2} - the only difference is that one should plot the line $y = f(x_0)$ and read off the first coordinate of the intersection point, since $f(x_0)$ is given and x_0 is to be determined:

$\boldsymbol{\tau^i_{2,AP,PR}}$: Type the functional equation for f in the Input Bar as "f(x)=b*x^a" where a and b are given values → Press Enter → Type "y=$f(x_0)$" where $f(x_0)$ is a given value → Press Enter **[**→ Choose the Move Graphics View Tool → Drag the x- and/or y-axis until both the line $y = f(x_0)$ and the graph of the f are visible in the Graphics View**]** → Choose the Intersect Tool → Click on both the graph of the function f and the line $y = f(x_0)$ in the Graphics View (i.e. on the intersection) → Read off the first coordinate of the intersection point from the Algebra View representation of the point.

In the cases of solving tasks of the types T_{EP_1} and T_{EP_2}, the option of adding a grid to the Graphics View through the Style Bar may provide help for supporting the determination of the intersection point between the line and the graph - in the sense that it can clarify which second coordinate the intersection point has, before asking for it through the Intersect Tool  (see Figure 55).

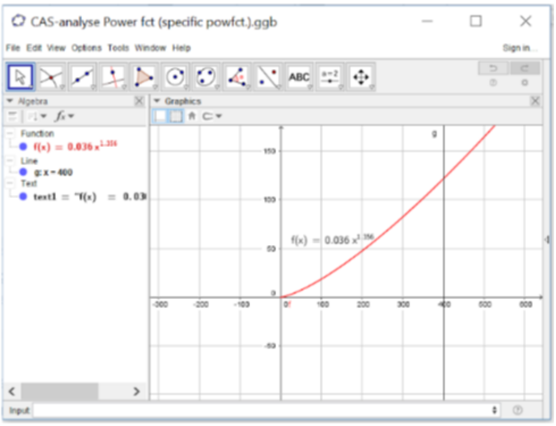

Figure 55 A grid has here been added to the Graphics View for the example presented in Figure 54.

Regarding tasks of the type T_{EP_3}, such tasks are possible to solve within the Algebra Perspective as well. It is namely possible through an intersection point procedure where one starts out by choosing a value of *x* to which it is found easy to add or subtract the given amount of percentage describing the change of *x*. For instance, in relation to the specific task of type T_{EP_3} presented in Figure 20, one can choose to plot the lines $x = 10$ and $x = 13$ in the Graphics View along with the graph of the given power function, *f* (where the relation between the *x*-values of 10 and 13 is an increase of 30 %; the given increase of *x*). Thereafter, one can mark off the two intersection points between each of the lines and the graph of f through the same procedure as used for marking off the intersection points in $\tau^i_{1,AP,PR}$ and $\tau^i_{2,AP,PR}$. The second coordinates of the two intersection points (i.e. $f(10)$ and $f(13)$) can now be read off from the algebraic representations of the points in the Algebra View (see Figure 56).

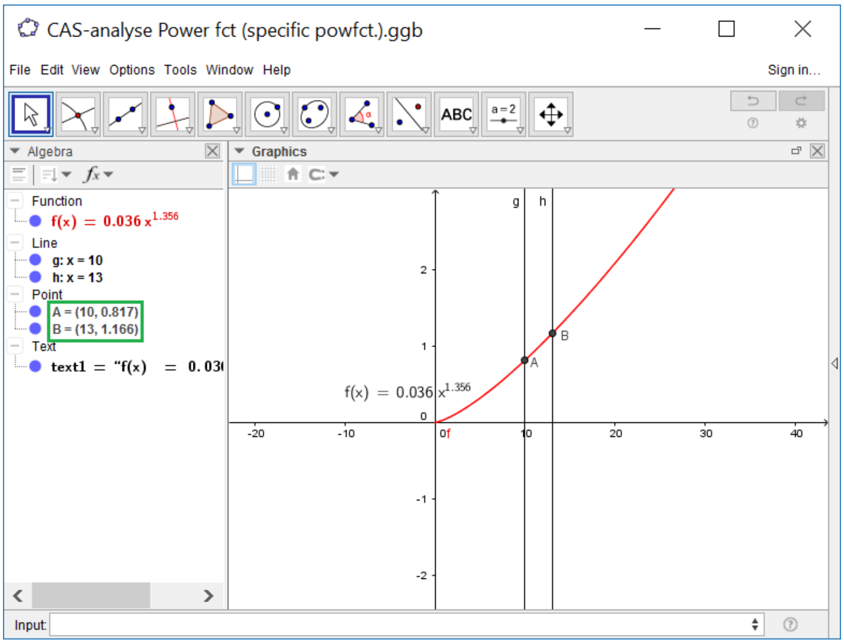

Figure 56 Here, the outcome of the gestures in relation to the example of a task of type T_{EP_3} from Figure 20 is presented. The lines $x = 10$ and $x = 13$ are (respectively) named *g* and *h* by GG and the intersection points between each of the lines and the graph of *f* are named *A* and *B* (respectively). Finally, the second coordinates of *A* and *B* can be read from the Algebra View (namely as 0.817 and 1.166 respectively - see the green box).

After having performed the gestures presented through the example above, the percentage increase between the function values can be calculated by use of a calculator - for instance the CAS View of GG (to be introduced further down). Hence, the following technique for solving tasks of type T_{EP_3} can be applied in the Algebra Perspective:

$\tau^{i}_{3,AP,PR}$: Type the functional equation of f in the Input Bar as "f(x)=b*x^a" where *a* and *b* are given values → Press Enter → Type "x=x_1" where x_1 is a specific value to which it is easy to add or subtract the given amount of percent by which *x* should change → Press Enter → Type "x=x_2" where x_2 is the specific value gotten when adding or subtracting the given amount of percent by which *x* should change → Press Enter **[**→ Choose Move Graphics View Tool → Drag the *x*- and/or *y*-axis until both the line $x = x_1$, the line $x = x_2$ and the graph of *f* are visible in the Graphics View**]** → Choose Intersect Tool → Click on both the graph of the function *f* and the line $x = x_1$ in the Graphics View (i.e. on the intersection)→ Choose Intersect Tool → Click on both the graph of the function f and the line $x = x_2$ in the Graphics View (i.e. on the intersection) → Read off the second coordinates of the two intersection points from the Algebra View representation of the points ($f(x_1)$ and $f(x_2)$) → Calculate (($f(x_2)$ / $f(x_1)$) -1)*100.

The part of $\tau^{i}_{3,AP,PR}$ marked by **[ ]** is only necessary when the two lines and the graph of f are not all visible in the Graphics View. To be noted, such a procedure can also be used for solving the opposite task of T_{EP_3}, namely that of determining the percentage change of the value x, given a percentage change of $f(x)$. One only plots the lines $y = y_1$ and $y = y_2$ where the values y_1 and y_2 are chosen such that y_2 is the value of y_1 changed by the given amount of percentage.

For tasks of the type T_{EP_4}, it is possible to solve such tasks via the Algebra View of GG, following the analytic technique sketched in section 4.2.2:

$\boldsymbol{\tau^{i}_{4,AP,PR}}$: Type "a=log($y_2/y_1$)/log($x_2/x_1$)" in the Input Bar where y_2, y_1, x_2 and x_1 are given values → Press Enter → Type "b=y_1/(x_1^a)" in the Input Bar where y_1 and x_1 are given values, and a is remembered by GG to be the number just calculated through the Input Bar → Press Enter → Read off the values of a and b from the Algebra View and write down the functional equation for the power function; $f(x)(\text{or } y) = b \cdot x^a$ if requested.

Here, one may further want to plot the given points "(x_1, y_1)" and "(x_2, y_2)" along with the graph of the power function f (or y) in order to verify the calculated values of a and b. This can be done through the Input Bar by typing each point as marked by the quotation marks just above and further by typing "f(x)=b*x^a" (or "y=b*x^a") (where each gesture is followed by Enter) where GG noticeably has remembered the calculated values of a and b. Figure 57 presents an example of the outcome of $\tau^{i}_{4,AP,PR}$ supported by the gestures just presented, when solving the specific task of type T_{EP_4} presented in Figure 21.

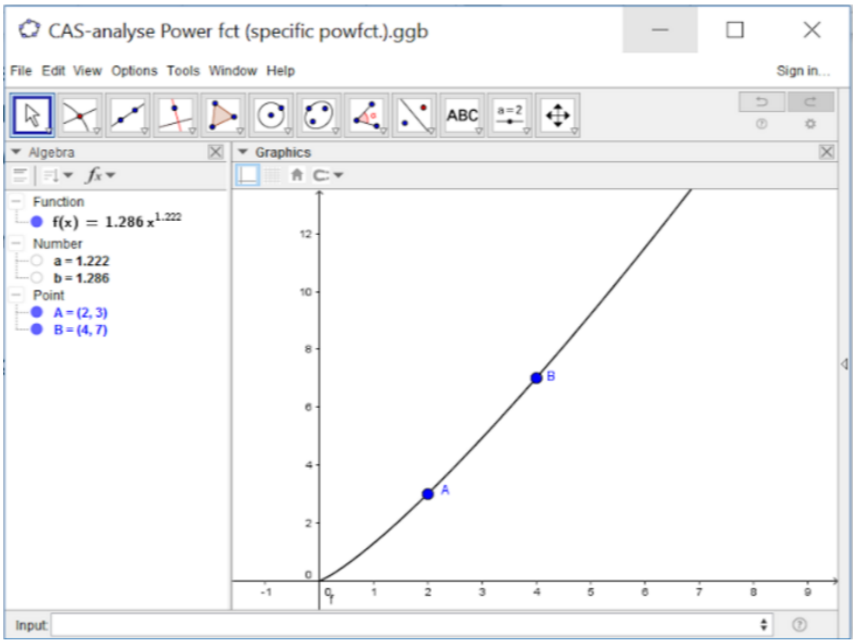

Figure 57 The outcome of $\tau^{i}_{4,AP,PR}$ performed on the specific task of type T_{EP_4} from Figure 21 are the values of *a* and *b* in the Algebra View, namely $\boldsymbol{a = 1.222}$ and $\boldsymbol{b = 1.286}$. Further the graph of "f(x)=b*x^a" along with the two given points (named *A* and *B*) have been plotted in order to check that the points lie on the graph.

Even though the Spreadsheet View (see subsection on this View further below) of GG may seem more immediate to use when solving tasks of the type T_{EP_5}, it is possible to solve such tasks in the Algebra Perspective as well. The following technique is namely possible to apply:

$\boldsymbol{\tau^{i}_{5,AP,PR}}$: Type each of the given points from the set of data; "(x_i,y_i)", each followed by Enter, in the Input Bar (each point is then given a name in the form of a capital letter by GG) → Type a list of the points in the Input Bar: "{A,B,C,...}" when the points are named *A, B, C,...* → Press Enter → Type "FitPow[list1]" in the Input Bar, if the list of points is named *list1*[54] → Press Enter → Read off the functional equation of the power function which has appeared both in the Graphics and the Algebra View.

Figure 58 shows an example of the outcome of the technique $\tau^{i}_{5,AP,PR}$, performed on the example of a task of type T_{EP_5} from Figure 22.

54 This is what GG will call it if it is the first list made in the document - one can read off the name of the list in the Algebra View.

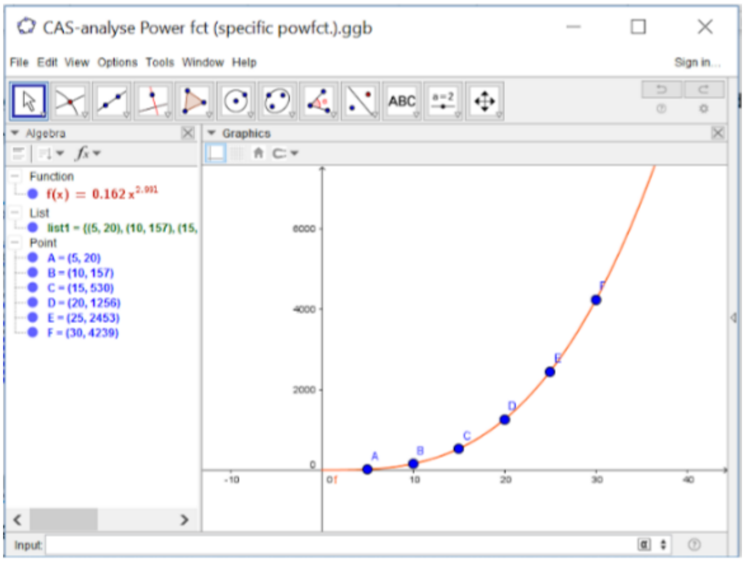

Figure 58 The outcome of $\tau^{i}_{5,AP,PR}$ performed on the example of Figure 22 are the six points *A-F* (named by GG), both represented in the Algebra and in the Graphics View, the list of the points (list 1) in the Algebra View and finally, the equation of the power function $\boldsymbol{f(x) = 0.162x^{2.991}}$ fitted for the data to be read from the Algebra View.

Now, before considering the options in the CAS View of GG, an important option in the Algebra Perspective in relation to the theme should be presented. Namely, the option of constructing a less specific power function (with domain $x > 0$), through the following technique:

$\tau^{i}_{Sliders\ ab,AP,PR}$: Type "f(x)=b*x^a,x>0" in the Input Bar → Press Enter → Click "Create Sliders" in the "Create Slider(s)" view popping up on the screen.

Through $\tau^{i}_{Sliders\ ab,AP,PR}$, sliders for a and b appear in the Graphics View with both values equal to 1 at first, along with the graph for the function "f(x)=1 x , (x>0)" - see Figure 59. Moreover, the functional equation appears as "Function" and the values for both *a* and *b* as "Number" in the Algebra View. To begin with, both the slider for *b* and that for *a* span over the interval of $[-5{,}5]$ with jumps of 0.05. The specifications for both interval and

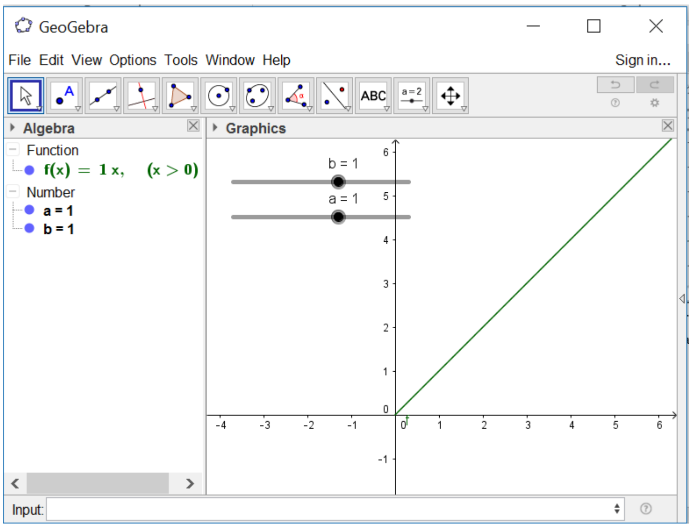

Figure 59 The outcome of $\tau^{i}_{Sliders\ ab,AP,PR}$ in the form of the two sliders for values of *a* and *b* and the equation and graph of the power function $\boldsymbol{f(x) = b \cdot x^{a}, x > 0}$ in respectively the Algebra and the Graphics View.

increment of the sliders can be changed (especially relevant for the b values when $b > 0$ is assumed for the function f to be a power function - see section 4.2.1). This can be done by double clicking on the slider whereby a view with "Preferences" pops up and here the "Min." and "Max" values can be entered for the slider (under the "Slider" tab). As for the function of the sliders, the mouse can be used to click on one of them in the Graphics View and drag the point (being either a or b in this case) to the right or to the left (in order to change its value). Note that, when changing at least one of the a or b values on the sliders, it leads to a change in both the functional equation of f in the Algebra View and in the look of the graph in the Graphics View. Thereby, it is possible to get to see several different types of graphs of a power function $f(x) = b \cdot x^a$ (, $x > 0$) dynamically.

The CAS View

Through the CAS View, it is possible to execute both symbolic and numeric calculations in relation to the Power Functions theme where input and output are made within the same View as opposed to the Input Bar in relation to the Algebra View. For instance, tasks of the type T_{EP_1} can be solved through the following technique in this View:

$\tau^i_{1.1,CV,PR}$: Type the right side of the functional equation of f with the value x_0 inserted, as "b*x_0^a" where a and b are given values, in an Input Field → Choose the Evaluate Tool [=] → Choose the Numeric Tool [≈] to get the numerical approximation → Read off the result from the last output.

Figure 60 presents an example of the outcome of $\tau^i_{1.1,CV,PR}$, performed on the specific task of type T_{EP_1} in Figure 20.

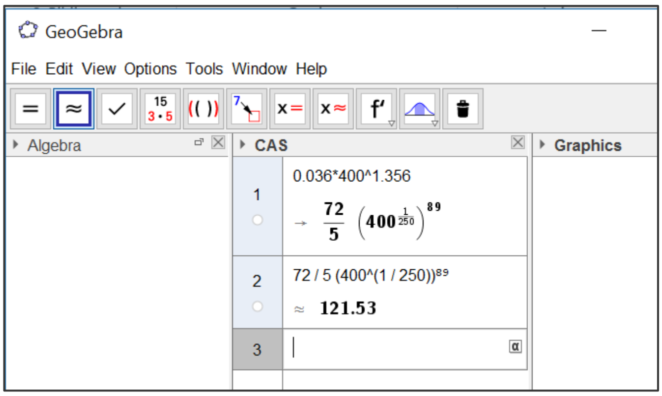

Figure 60 The outcome of $\tau^i_{1.1,CV,PR}$ performed on the task of type T_{EP_1} in Figure 20. In cell 1, the result when using the Evaluate Tool is presented and in cell 2, the Numeric Tool has been used.

There is however also the possibility of defining the power function *f* in the CAS View (or in the Algebra View - see the previous section) and then calculate $f(x_0)$ in the CAS View in order to solve T_{EP_1}:

$\boldsymbol{\tau^{i}_{1.2,CV,PR}}$: Type "f(x):=b*x^a", where *a* and *b* are given values, in an Input Field → Press Enter → Type "f(x_0)", where x_0 is a given value, in the next Input Field → Choose the Evaluate Tool [=] → Choose the Numeric Tool [≈] to get the numerical approximation → Read off the result from the last output.

See Figure 61 for the outcome of $\tau^{i}_{1.2,CV,PR}$ performed on the task of type T_{EP_1} in Figure 20.

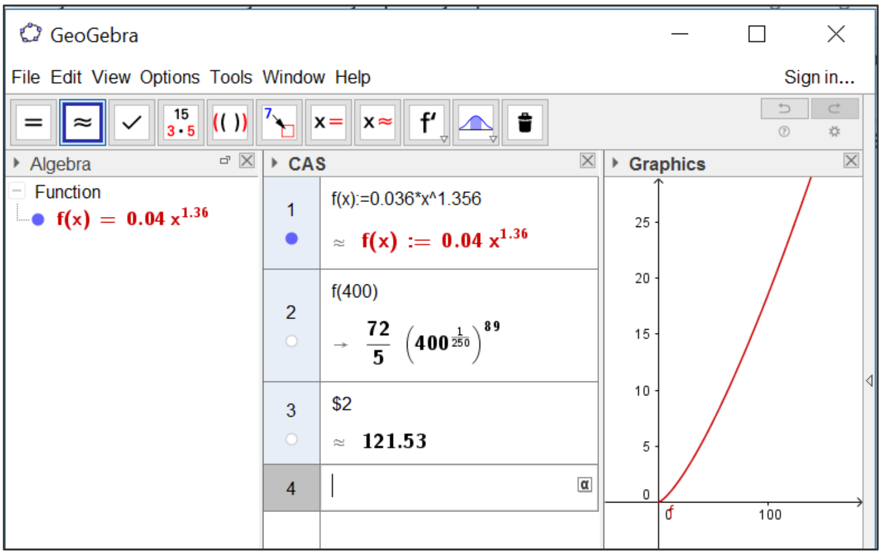

Figure 61 The outcome of $\boldsymbol{\tau^{i}_{1.2,CV,PR}}$ performed on the task of type $\boldsymbol{T_{EP_1}}$ in Figure 20. Noticeably, the power function *f(x)* can be represented in both the CAS, the Algebra and the Graphics View when having defined it by ":=" in the CAS View (through the (blue) bullet).

As for tasks of type T_{EP_2}, these are possible to solve as well, through the following technique in the CAS View:

$\boldsymbol{\tau^{i}_{2.1,CV,PR}}$: Type "f($x_0$) = b*x^a", where $f(x_0)$, *b* and *a* are given values → Choose the Solve Numerically Tool [x≈] to get a numerical approximation → Read off the result from the last output.

It should here be noted that the Solve Numerically Tool (performing the NSolve command) can only be used on continuous functions as well as piecewise-defined functions (Webpage: NSolve Command, 2016; Webpage: Solve Command, 2016). Figure 62 presents the outcome of $\tau^{i}_{2.1,CV,PR}$ performed on the task of type T_{EP_2} from Figure 20.

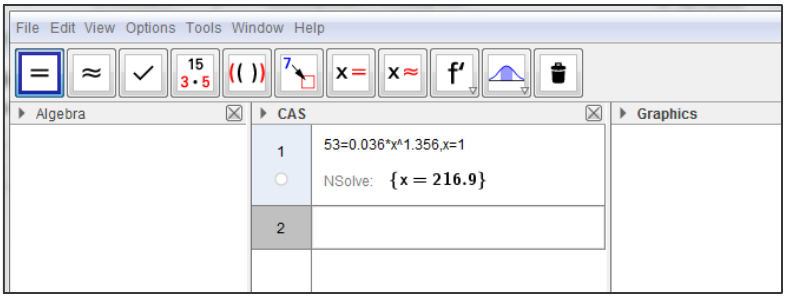

Figure 62 The outcome of $\tau^{i}_{2.1,CV,PR}$ performed on the task of type T_{EP_2} from Figure 20 is presented in cell 1 of the CAS View.

Noticeably, GG is not necessarily able to solve with respect to x when one chooses the Solve Tool instead of the Solve Numerically Tool, which is seen in Figure 63 where this has been practiced on the specific example of a task of type T_{EP_2} task from Figure 20. This can be explained by the fact that the function in the equation is only piecewise-defined (with domain $x > 0$) whereby the Solve command cannot be applied and NSolve is needed as command in order to get a result (Webpage: Solve Command, 2016).

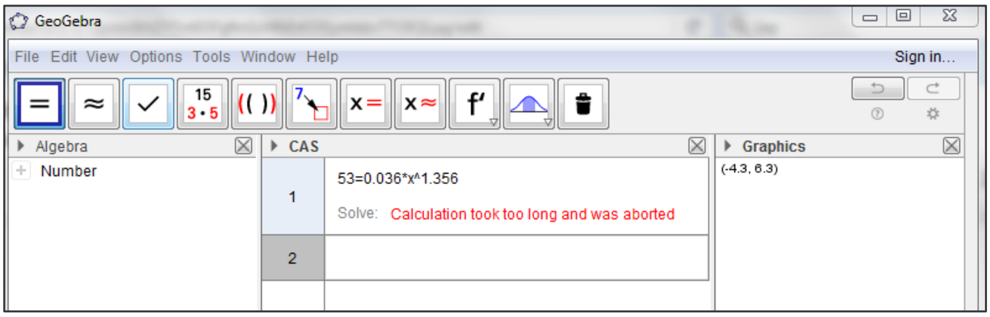

Figure 63 The output of using the Solve Tool for solving the task of type T_{EP_2} task from Figure 20 in the CAS View is seen to be the message: "Calculation took too long and was aborted".

Like for T_{EP_1} tasks, there is also the possibility of defining the power function f in the CAS View (or in the Algebra View - see the previous section) and then solve $f(x) = f(x_0)$ (numerically) in the CAS View for T_{EP_2} tasks:

$\boldsymbol{\tau^i_{2.2,CV,PR}}$: Type "f(x):=b*x^a", where a and b are given values, in an Input Field → Press Enter → Type "f(x)=f(x_0)", where $f(x_0)$ is a given value, in the next Input Field → Choose the Solve Numerically Tool → Read off the result from the last output.

Here, the same applies for the use of the Solve Tool as for the technique $\tau^i_{2.1,CV,PR}$.

Now, for tasks of the type T_{EP_3}, the following technique is applicable in the CAS View, with references to the sketched analytic technique in section 4.2.2:

$\boldsymbol{\tau^i_{3,CV,PR}}$: Type "(1+($r \cdot 100$)/100)^a", where ($r \cdot 100$) and a are given values, in an Input Field → Choose the Numeric Tool → Click on the output (copies it to the next Input Field) → Type "-1" next to the input (/the copied output) → Choose the Numeric Tool → Click on the decimal output (copies it to the next Input Field) → Type "*100" next to the input (/the copied output) → Choose the Numeric Tool → Read off the result from the last output.

Since it is a specific number describing the percentage change of $f(x)$, given a percentage change of x (namely ($r \cdot 100$)) which is to be determined in T_{EP_3} tasks, the Numeric Tool is used in $\tau^i_{3,CV,PR}$ instead of the Evaluate Tool since the Evaluate Tool leads to the exact evaluation as an output instead of a decimal number.
See Figure 64 for an example of the outcome of $\tau^i_{3,CV,PR}$ performed on the specific example of a task of type T_{EP_3} in Figure 20.

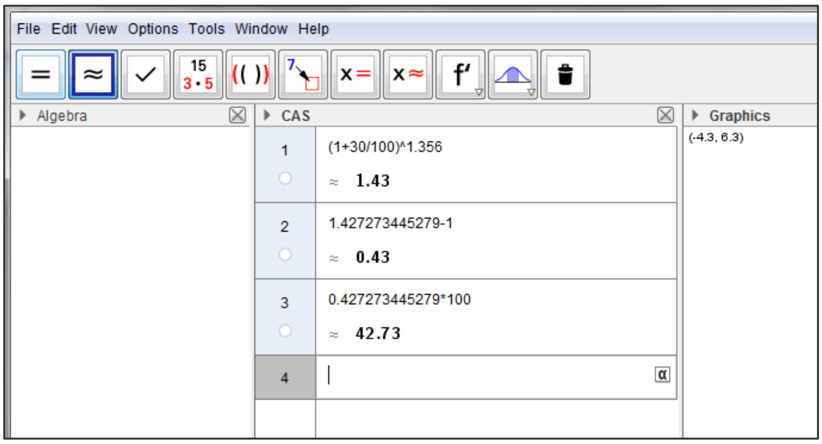

Figure 64 The outcome of $\tau^i_{3,CV,PR}$ performed on the task of type T_{EP_3} in Figure 20. The Numeric Tool is used in all of the three cells.

Regarding tasks of type T_{EP_4}, they can be solved through the following technique in the CAS View of GG:

$\boldsymbol{\tau^i_{4,CV,PR}}$: Type "a:=log($y_2/y_1$)/log($x_2/x_1$)", where y_2, y_1, x_2 and x_1 are given values, in the Input Field → Choose the Numeric Tool → Type "b:=$y_1/(x_1$^a)", where y_1 and x_1 are given values, and a is remembered by GG to be the number just calculated through the Input Field above, in the next Input Field → Choose the Numeric Tool → Read off the value of a and b from the outputs and write down the functional equation for the power function; $f(x)/y = b \cdot x^a$ if requested.

Noticeably the typing "b:=" is not necessary for the technique $\tau^i_{4,CV,PR}$ to work as a technique but it provides an overview when reading off the values of a (also defined by "a:=") and b in the outputs of the CAS View. Further, it makes it possible henceforth to write b instead of the specific number evaluated in GG, where it will be remembered to be of the evaluated value. Figure 65 presents an example of the outcome of $\tau^i_{4,CV,PR}$ performed on the task of type T_{EP_4} presented in Figure 21.

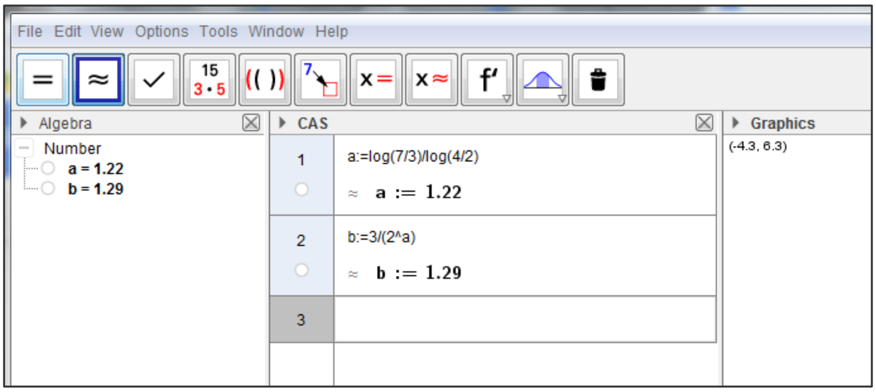

Figure 65 The outcome of $\boldsymbol{\tau^i_{4,CV,PR}}$ performed on the task of type $\boldsymbol{T_{EP_4}}$ in Figure 21 where the Numeric Tool has been used in both cell 1 and cell 2. If requested, the functional equation can be written as $\boldsymbol{f(x)/y = 1.29 \cdot x^{1.22}}$.

Now, regarding tasks of the type T_{EP_6}, such tasks can be solved through the following technique (with references to the analytic techniques sketched in section 4.2.2):

$\boldsymbol{\tau^i_{6,CV,PR}}$: Type "k:=x*y", where x and y are the given set of values, in an Input Field → [Choose the Evaluate Tool] / [Choose the Numeric Tool] →

Type "k/x_0" in the next Input Field where x_0 is the given value for x and k is remembered by GG to be the number just calculated through the Input Field above → **[**Choose the Evaluate Tool **]** / **[**Choose the Numeric Tool **]** → Read off the result from the last output.

Only one of the parts in **[ ]** is here to be chosen, depending on what type of output is wanted (either an exact evaluation or a numerical approximation). In Figure 66 the outcome of $\tau^{i}_{6,CV,PR}$ is presented in relation to the task of type T_{EP_6} in Figure 23.

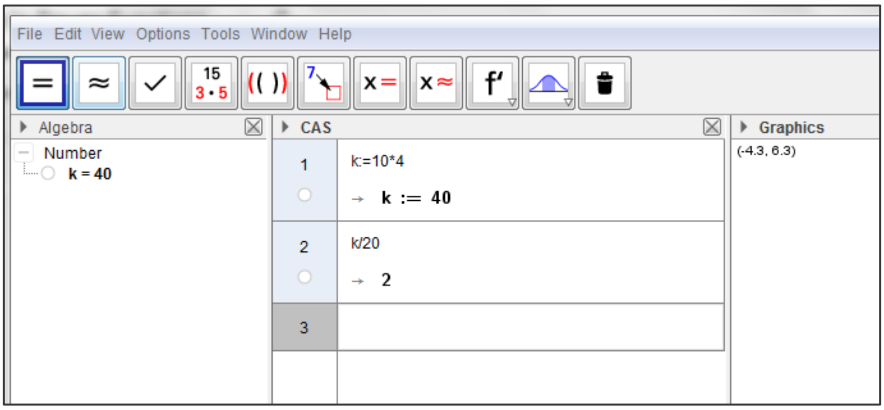

Figure 66 The outcome of $\boldsymbol{\tau^{i}_{6,CV,PR}}$ performed on the task of type $\boldsymbol{T_{EP_6}}$ from Figure 23. The constant of proportionality (k) is defined and calculated in cell 1 and GG remembers the value of k in cell 2 when calculating $\boldsymbol{y_0}$.

As an endnote for this review of the CAS View, it does not appear ideal to solve tasks of the type T_{EP_5} within this View. The Spreadsheet View is however especially ideal for this type of task which will be elaborated in the following.

The Spreadsheet View

The Spreadsheet View is very suitable for solving tasks of power regression or modelling (i.e. of the type T_{EP_5}). Without going further into details with the content of the Spreadsheet Toolbar[55], two possible techniques will here be presented for solving tasks of type T_{EP_5} - one in which the Spreadsheet View interacts with the Algebra and the Graphics View ($\tau^{i}_{5.1,SV,PR}$) and one solely made within the Spreadsheet View ($\tau^{i}_{5.2,SV,PR}$):

[55] An elaboration of the tools of the Spreadsheet Toolbar can be found on the link: https://www.geogebra.org/manual/en/Spreadsheet_Tools (last visited 15.07.2016).

$\boldsymbol{\tau^{i}_{5.1,SV,PR}}$: Type the given values for the independent variable in a column (A) of the spreadsheet → Type the given values for the dependent variable in a column (B) just to the right of column A of the spreadsheet such that the related values of the two variables are presented directly next to each other → Mark the two columns where they are filled out (using either mouse or keyboard) → Right click on the marked area → Choose "Create" and click on "List of points" → Type "FitPow[list1]" in the Input Bar of the Algebra Perspective, if the list of points is named list1[56] → Press Enter → Read off the functional equation of the power function which has appeared both in the Graphics and the Algebra View.

Figure 67 presents the outcome of the technique $\tau^{i}_{5.1,SV,PR}$ performed on the example of a task of type T_{EP_5} from Figure 22.

[56] "list1" is what GG will call the list if it is the first undefined list made in the document. One can read off the name of the list in the Algebra View.

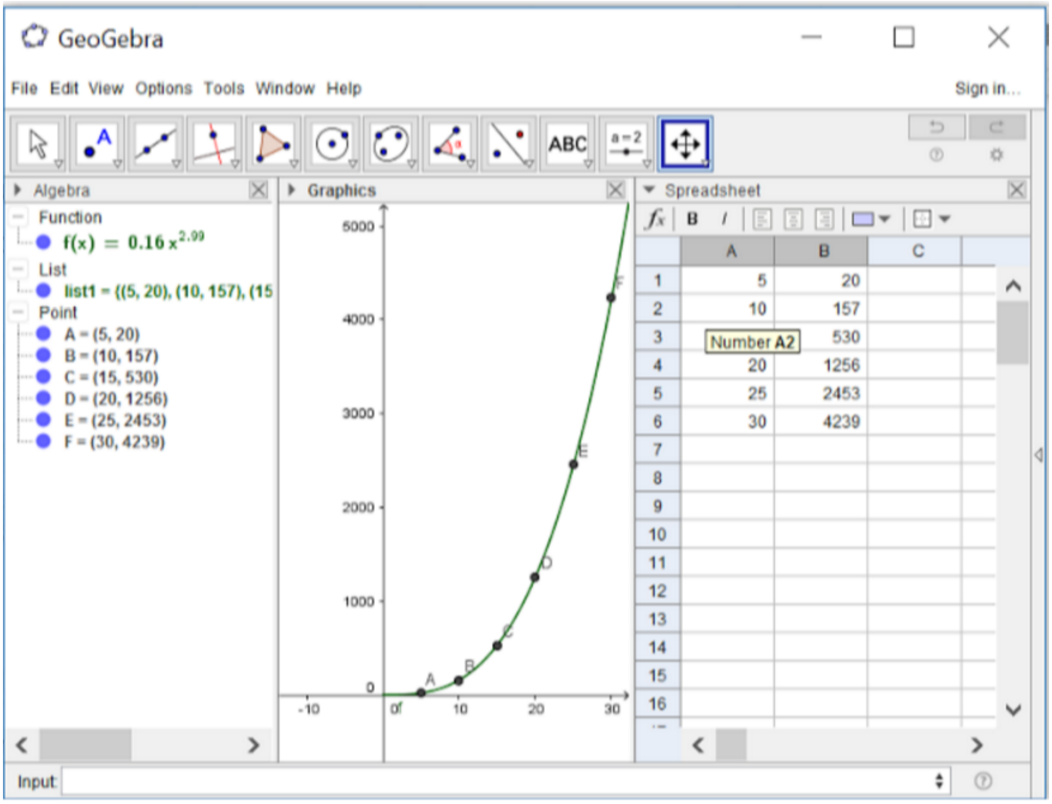

Figure 67 The outcome of $\tau^{i}_{5.1,SV,PR}$ performed on the task of type T_{EP_5} in Figure 22. The functional equation can be read from the Algebra View to be $f(x) = 0.16x^{2.99}$. Furthermore, both *f* and the list of points are noted to be represented in both the Algebra and the Graphics View.

$\tau^{i}_{5.2,SV,PR}$: Type the given values for the independent variable in a column (A) of the spreadsheet → Type the given values for the dependent variable in a column (B) just to the right of column A of the spreadsheet such that the related values of the two variables are presented directly next to each other → Mark the two columns where they are filled out (using either mouse or keyboard) → Choose the "Two Variable Regression Analysis Tool"  in the Data Analysis Tools menu of the Spreadsheet View Toolbar → Click "Analyze" in the appearing "Data Analysis" view → Click on the bar under "Regression Model" and choose "Power" in the pop down menu just appeared → Read off the functional equation of the power function which has appeared under the plot of the points and the fitted graph in the "Data Analysis" view.

An example of the outcome of $\tau^{i}_{5.2,SV,PR}$ is presented in Figure 68 where the technique is performed on the task of type T_{EP_5} in Figure 22.

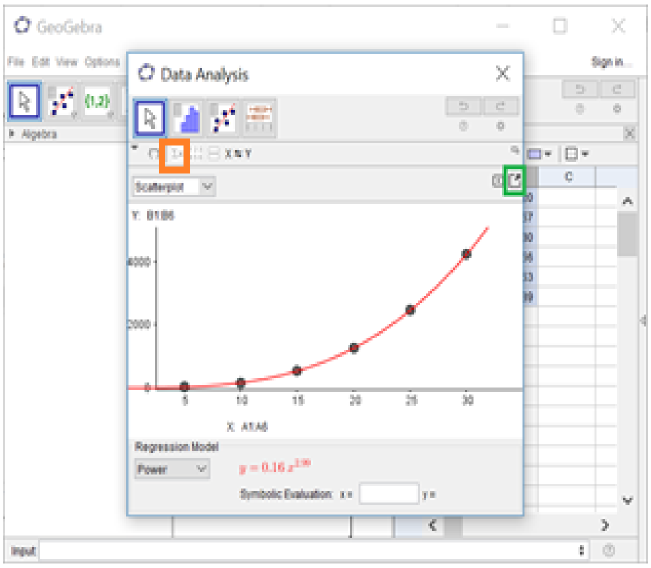

Figure 68 The outcome of $\boldsymbol{\tau^{i}_{5.2,SV,PR}}$ performed on the task of type $\boldsymbol{T_{EP_5}}$ from Figure 22. The functional equation can be read to be $\boldsymbol{y = 0.16x^{2.99}}$ underneath the plot in red text. The green box marks where to click in order to "Copy to Graphics View" whereby the functional equation and the data points are both represented in the Graphics and in the Algebra View when closing the "Data Analysis" view. Clicking on the icon marked by the orange box leads to a presentation of some statistics related to the performed regression - especially the $\boldsymbol{r^2}$ value can be accessed here.

Some further notes to be made, on options in relation to the "Data Analysis" view, are the following (related to Figure 68):

- Evaluating the *y*-value, given an *x*-value (i.e. solving tasks of type T_{EP_1}) by typing a given *x*-value in the bar in front of "Symbolic Evaluation: x=" followed by clicking Enter - this technique will henceforth be denoted $\boldsymbol{\tau^{i}_{1,SV,PR}}$[57], and

- Copying the output of the "Data Analysis" view to the Graphics View by clicking on the icon marked by a green box in Figure 68 whereby the outcome of $\tau^{i}_{5.2,SV,PR}$ can be related to the Algebra and the Graphics View - similar to the outcome of $\tau^{i}_{5.1,SV,PR}$. Furthermore, it is possible to get to see the correlation coefficient (r^2) for assessing the performed regression through clicking on the icon marked by the orange box in Figure 68.

GG worksheets in relation to Power Functions

Some examples of dynamic worksheets made in GG which could be relevant for the theme of Power Functions and its introduction for the high school students, will here

[57] This technique is noticeably only experienced to be possible to perform after having performed $\tau^{i}_{5.2,SV,PR}$.

be presented briefly. These worksheets have all, where nothing else is stated, been found through searching for "potensfunktion" on http://tube.geogebra.org/ (last visited 15.07.2016).

The first example (found on https://www.geogebra.org/m/2719557 (last visited 15.07.2016)) concerns the impact of the value of the exponent a of a power function $f(x) = b \cdot x^a$ on the look of the graph of a power function - see Figure 69. Here, the (multiple choice) task is to choose the right answer for the domain of a in the drop down menu appearing when clicking on the bar in which a statement for a's domain is presented. Thereafter it is possible to check the chosen answer by clicking on the bar "Check mit svar". Finally, it is possible to ask for a new graph by clicking on the bar "Ny graf". Noticeably, this worksheet does not ask for nor does it give any explanations for the relation between the domain of a and the look of the graph of f. It is however noted that, in order to click on the right answer for the domain of a, the user has to have some knowledge of the relation between this domain and the look of the graph.

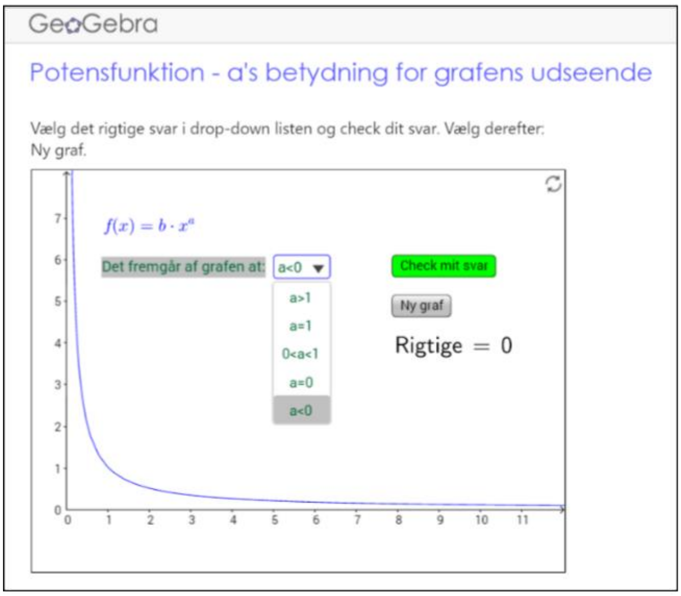

Figure 69 The worksheet named *Power function - a's impact on the look of the graph* (translated from Danish). The drop down menu presenting possible domains of a is presented and the right answer has been chosen - in this case $a < 0$ - whereby clicking the bar called "Check mit svar" makes it turn green (otherwise it will turn red).

The next example (found on https://www.geogebra.org/material/simple/id/2837783 (last visited 15.07.2016)) is that of a worksheet presenting a click proof of Thm_{ab} (see section 4.2.3) presented in Figure 70 where *click* refers to the possibility of clicking your way through helping hints and right answers for the steps of the proof. There are eight steps concerning the techniques required for performing the proof. These steps are presented in a summary underneath the theorem in the left column of two, whereas help to each step of the proof can be obtained through marking off boxes for either hints or right answers in the right column. As an example of: A step → A hint for this step → A right answer for the step, the first set of these is formulated as follows (translated from Danish):

1. The correlated values of x and y from the points P and Q are inserted into the equation $y = b \cdot x^a$ → *In this way, you get two equations* → $y_2 = b \cdot x_2^a$ *and* $y_1 = b \cdot x_1^a$.

This worksheet sets the stage for the user to get to learn the analytic techniques for carrying out the proof of Thm_{ab}. Explanations are however not presented nor requested from the worksheet. Noticeably, there lies a giant pitfall in the case where the user of the worksheet does not think for him-/herself when reading each of the given steps in the summary of the proof and just go clicking for both hint and upshot with a certain amount of impatience because of the availability of these answers. This pitfall relates to the lack of an establishment of a knowledge block when solving the task.

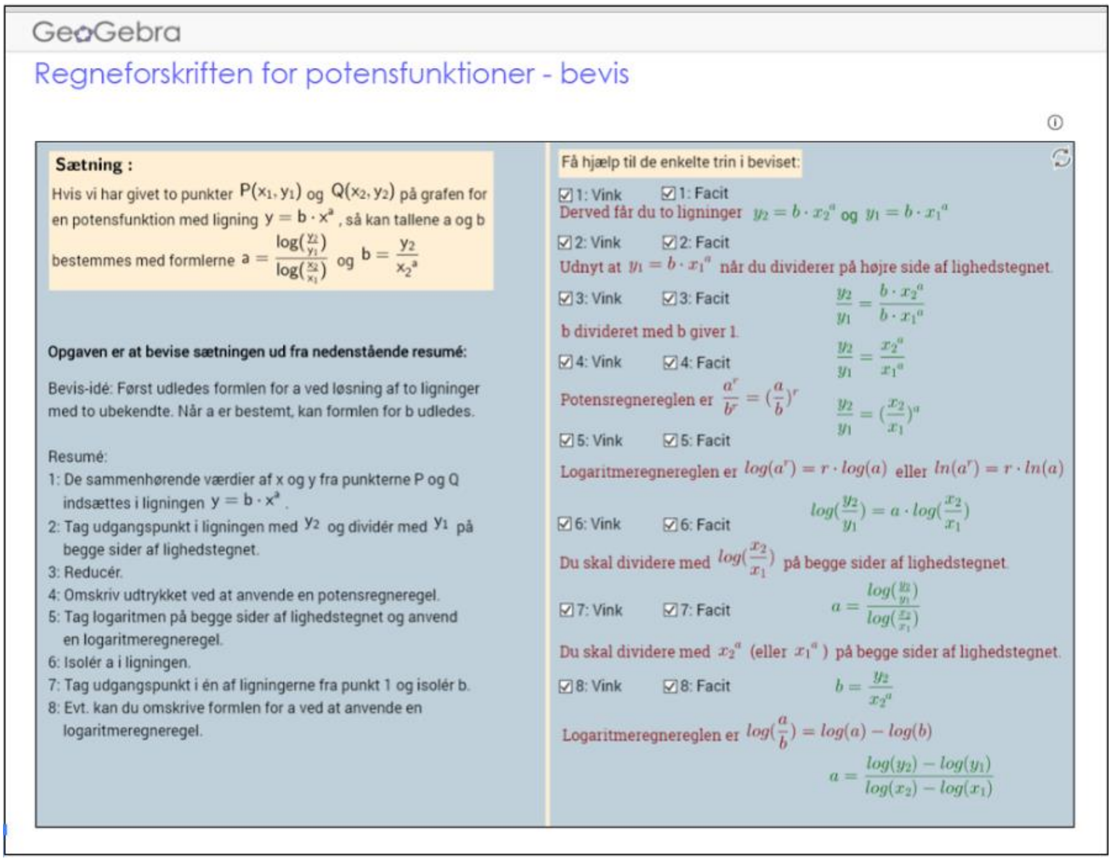

Figure 70 GG worksheet concerning the proof of $\boldsymbol{Thm_{ab}}$. The red sentences of the right column are hints for the steps of the proof (elicited by marking off the “Vink” boxes) whereas the green sentences are right answers for the steps of the proof (elicited by marking off the “Facit” boxes).

Another example of a worksheet is the one presented in Figure 71 with the headline *What is power growth* (translated from Danish). This worksheet can be found through searching for “potensvækst” on (Webpage: GG-worksheets, 2016)[58]. It especially concerns the (%/%) property (see section 4.2.3) in the way that it presents four exercises related to dragging sliders representing both the value of a, the value of b and the percentage change of *x* of the power function of the form $y = bx^a$.

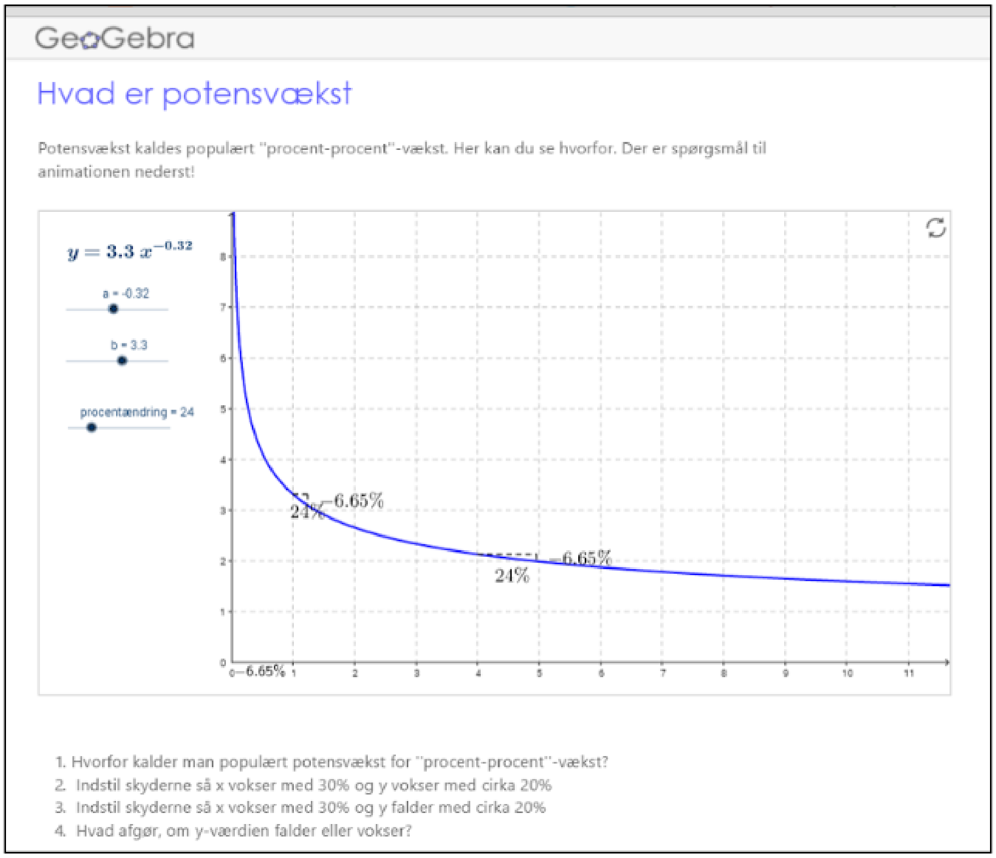

Figure 71 Example of worksheet on power growth where both the *a* and the *b* value of the power function $\boldsymbol{y = bx^a}$ can be changed via sliders. Especially the percentage change (of *x*, though not stated) can be changed via a slider. In a coordinate system, the power function is represented as a graph and it is indicated two places on the graph that when the *x*-value increases by a percent equal to that of the slider (just presented) then *y* decreases (in this case) by a certain percent.

The four exercises of this worksheet can be translated as follows (from Danish):

1. Why does one in a popular phrase call power growth “percent-percent” growth?
2. Set the sliders such that x increases by 30% and y increases by approx. 20 %

[58] Direct link: https://tube.geogebra.org/m/143911 (last visited 15.07.2016).

3. Set the sliders such that x increases by 30% and y decreases by approx. 20 %
4. What determines whether the y-value decreases or increases?

The exercises 1. and 4. calls for some explanation, based on the information given in the worksheet and dragging the sliders whereas the exercises 2. and 3. are more technique-oriented since they involve dragging the sliders in order to achieve a specific (%/%) relation between the variables.

The worksheet presented in Figure 72 presents the final example of a worksheet in this review. It is named *Direct and inverse proportional relations* (translated from Danish) and has been found through searching for “Ligefrem” on (Webpage: GG-worksheets, 2016)[59]. The worksheet concerns (direct or inverse) proportionality and presents the task of filling out the table of data on the related variables *x* and *y* (with significant similarities to the task type T_{EP_6} and the type similar to this, only concerning direct instead of inverse proportionality). The user can from the text underneath the table read whether the variables are inverse (“omvendt”) or direct (“ligefrem”) proportional, which is noted to be a relation of the form y·x=k or y=k·x in the text. The user is then required to know how to determine the constant of proportionality, *k*, and use it for determining the lacking values of the table. This intention of the worksheet is indicated by the ordering of the boxes possible to mark off in order to get answers for, first of all, the constant of proportionality and thereafter the filled out table. Further the last box from above offers the possibility of seeing the graphical representation of the proportional relation between the two variables, indicating a suggestion for the user to contemplate the look of the graph. This worksheet does hereby not present any techniques for calculating either *k* or the *x*- and *y*-values lacking in the table. It does however guide the steps to be taken for solving the given task, through the ordering of the boxes possible to mark off for right answers - especially the first two since the outcome of marking off the third box (being the graph) is not crucial for solving the task.

[59] Direct link: https://tube.geogebra.org/m/1338399 (last visited 15.07.2016).

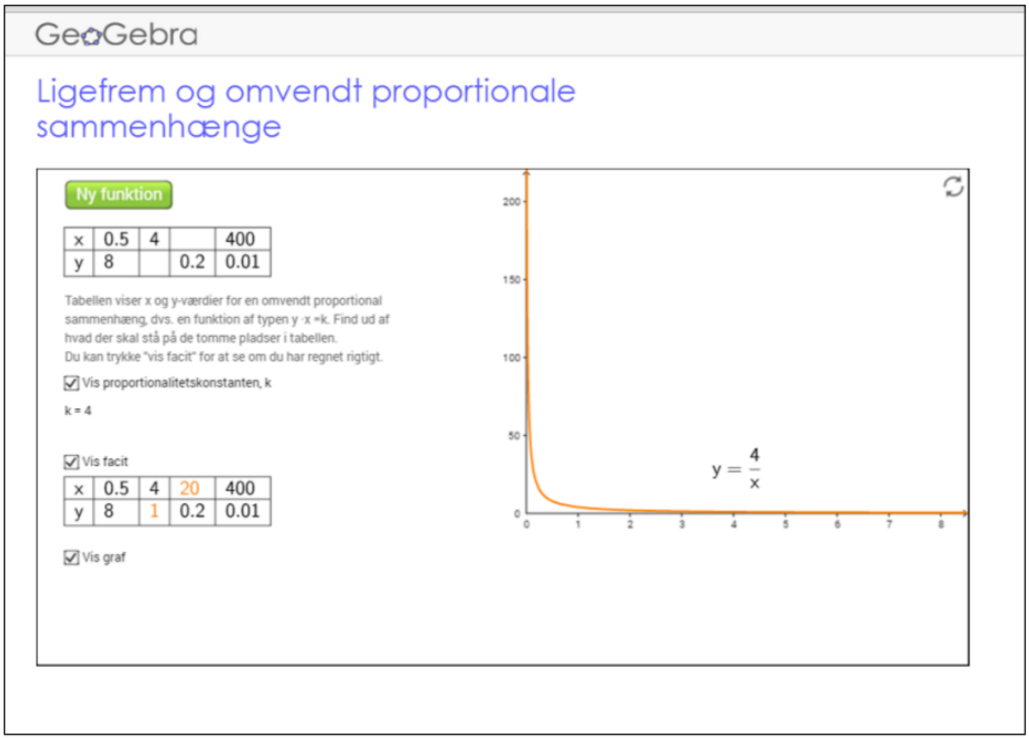

Figure 72 An example of a worksheet on direct and inverse proportionality offering the possibilities of seeing the right answer for the constant of proportionality, the right answer for the filled out table and finally, the graph of the relation between the variables through marking off small boxes. These boxes have all been marked off here. When having completed one task, one can click on "Ny funktion" (the green bar) and a new unfilled table of related *x* and *y* values appears.

5.0 Introduction to the Observation Study

Two different Danish high school teachers in mathematics, who taught in the two themes at the same high school, were involved in the observation study. The observations were performed in the period from Marts 30th to April 29th 2016. More specifically, the one of the teachers (TT) taught Trigonometry on A-level at first grade of high school and the other teacher (TP) taught Power Functions on B-level also at first grade. 10 modules were observed in the Trigonometry course and eight modules in the Power Functions course where each module was of 90 minutes' duration. Both of the teachers used the computer program GG as a CAS tool. GG is noted to be new as a CAS tool to both the teachers and the students since GG has just been implemented on the specific high school this school year. In addition, it should be noted that it was not specified which type of calculator the students must use when doing algebraically represented mathematical problems (the classical understanding of CAS). Noticeably, it is optional in what way the teacher chooses to implement the CAS tool as long as it is consistent with the requirements of the curriculum (see the sections 4.1.2 and 4.2.2). The observations have been performed on students using Mac computers (called the "participating students" (PSs); see section 6.0 for an elaboration hereof). In prolongation of this it should be noted that TT is a Mac user whereas TP uses a PC. Throughout the thesis, anonymity of both the teachers and the students is guaranteed, with respect to not letting them feel judged in any way (see Acronyms p. xi).

In accordance with the RQs presented in section 3.0, a method for collecting the empirical data has been developed and will be clarified in the following section.

6.0 Methodology

In this section, the methodology, for the observations on the high school lessons of the two courses of Trigonometry and Power Functions as well as for the collected empirical data in order to answer **RQ1-RQ3**, is outlined.

The empirical data can be listed as follows:

(1) Tasks proposed by the teacher in relation to GG
(2) Screen recordings
(3) Field notes
(4) Photographs
(5) Take-home group assignments

For the analyses of the two observed courses (see section 7.1 and 7.2), this data will be used to an extent allowing an overview, for what happened in the courses of the themes in relation to the RQs, to be established.

The data contributing to answering **RQ1** constitutes relevant parts from both data (1), data (2), data (3) and data (5). Regarding **RQ2**, relevant parts, also from data (2) and data (3), as well as from data (4) and data (5) are contributing.
The tasks (T) proposed by the teacher in relation to GG (data (1)) has been collected through mail correspondences with each teacher. This data offers an insight into the praxeologies, expected for the students to establish, by the teacher. Selected parts of this data will be analysed using the terminology of MOs (see section 2.2.2) in the sections 7.1 and 7.2.

Data (2) includes corresponding audio recordings from the computers of some of the students - the PSs - when they do work involving GG. Noticeably, the selection of PSs has differed from module to module in order to ensure observations of different approaches to GG, depending on the academic and technical level of the PSs.
The recordings (screen/audio) has provided access to observe the instrumented techniques (τ^i) (both instructed and spontaneously developed) used by the PSs on the screen. Further the recordings have provided access to the oral formulations and explanations (i.e. the knowledge block of the MO if any) for the specific mathematical

tasks the PSs seek to solve. This, with reservations to the lack of access to the typing on the keyboard and thereby any potential shortcuts (also noted in section 4.3). Noticeably, notes made by the PSs (i.e. potential written considerations) on the computer (for instance in Word) in addition to their work in GG is also captured on the screen recordings. The recordings require an installed program on the computer. In general, Mac computers have such a program installed for certain, namely QuickTime Player[60], whereas PCs do not. It may therefore take a separate action for the students with PCs to find and download such a program and it was discovered that it was not necessarily possible to make a full screen recording with unlimited recording time (and not only a couple of minutes) in the programs targeted at PCs. Therefore, the group of PSs was narrowed down to be Mac users.

This data has been collected via USB plugs since it was found favourable compared to uploads or mail-correspondences via the Internet, because of the time-consuming aspect connected to the speed of uploads. Furthermore, the aspect of ensuring anonymity of both the teachers and the students is not guaranteed when using the Internet, thus the choice of USB plugs. This choice was made with reservations to the fact that it also consumes some time (up to two minutes for a 10-minute screen recording file) to save the files on a USB[61], but not as much as when uploading to the Internet. With this in mind, the PSs were asked to save their recordings every 10th minute and start a new recording immediately after, when working in GG. In order to keep track on the succession of the video files, the PSs were further asked to number the files “1”, “2”, “3”,... and so on, depending on the amount of files. The compromise of around 10 minutes’ duration for one screen recording has been adopted such that the mathematical work of the PSs was not interrupted to such an extent where the PSs could not focus on solving the mathematical tasks since they would have to wait for the files to be saved. In addition, unnecessarily disturbance of the students, by more frequently instructing them to save the files, was sought avoided.

[60] For further information and download, follow this link: http://www.apple.com/quicktime/what-is/ (last visited 29.07.2016).

[61] Before initiating the observations, the write speed of each of the involved USB plugs was tested, in order to be certain that it would not take longer than two minutes to save 10-minute video files.

When the audio recordings are transcribed as well as when referring to specific situations, the following abbreviations will be used, for the sake of anonymity of both teacher and students:

TT: Trigonometry teacher
TP: Power Functions teacher
PS_1, PS_2,…: Participating students, i.e. Mac users with a USB plug
AS_1, AS_2,…: Other students participating in dialogue

The transcribed parts of the audio recordings will be translated more or less directly from Danish into English with reservations to the fact that some phrases may be difficult to translate directly into English in order to maintain the meaning. Therefore, the spoken words of the students and the teacher may be interpreted (van Nes, Abma, Jonsson, & Deeg, 2010). The screen recordings have been named "Module number, USB number, File number". A list of the recordings is presented in A.3.

Further, for taking field notes (data 3), a scheme was developed with the RQs in mind (see A.4). Both researchers took field notes throughout both courses to support the validity of the notes taken. They will be used both for given a holistic picture of the actions, of both the students and the teacher, in the lessons as well as for providing supporting comments for the screen recording data. When observing the PSs - being at most three students per researcher - the researchers were placed just behind them. This allowed each researcher to maintain an overview when writing the notes and keeping an eye on what is not immediately traceable from the screen recordings. The researchers attempted however to keep some distance, in order not to disturb the work of the students. Noticeably, the positioning of the researchers in the classroom was further affected by the teacher's organisation of the students' work. When the students worked in groups, each researcher observed one group, preferably with two-three Mac users in the group. This, in order to follow the process more intensely rather than seeking to cover several groups each, at the same time, which can result in a collection of screen recordings with little or no supportive field notes. When the students worked individually, the researchers positioned themselves such that they followed a maximum of three Mac users each, sitting near to each other.

As a supplement for the collection of screen recordings, photographs of any handwritten notes made by the PSs in connection with their work in GG is also functioning as empirical data (data (4)). This with the assumption that it could happen that the PSs would write down their considerations and answers using pencil and paper next to their work in GG. Hereby, data (4) may give access to supplementary technology (θ) and theory (Θ) practised by the PSs. When found relevant such specific photographs will be referred to in the analyses of the empirical data (sections 7.1 and 7.2).

Furthermore, collected answers of take-home group assignments (data (5)) may work as a perspective, when relevant for work with GG, on the learning outcome for the students in form of praxeologies (both instructed by the teacher and developed by the students) in relation to GG.

The data contributing to answering **RQ_3** is mainly data (3), but also relevant parts of data (4). The part of the field notes (data (3)), oriented solely on the praxis of the teacher and how s/he orchestrates the work with GG, was made from a free corner of the classroom, in order for the researcher to not stand out. In the situations where the teacher interacted with the students in their work with GG, notes were taken from the positions near to the PSs. Additionally, photographs of the teacher's work with GG (taken later in the progress of the study[62]) can give a supportive impression of the teacher's exploitation modes of GG work in the lessons.

An endnote for the methodology presented here is that the empirical data will be used to an extent found appropriate for the analyses of the specific courses in relation to **RQ_1-RQ_3** which will be elaborated in the following (sections 7.1 and 7.2).

[62] Only done in connection with the Trigonometry course, since the observations of the Power Function course were finished when a relevance of such photographs was experienced.

7.0 Analyses of the Empirical Data

Here, the analyses of the empirical data collected through the two courses will be presented. More specifically, these analyses are based on the data (1)-(5) presented in the above section (6.0) on the methodology in order to answer **RQ_1-RQ_3**. A common aspect of the two analyses is that the content of the courses in relation to GG work will be listed in tables (Table 2 and Table 3) and herein put in relation to the exam tasks and exercises presented in the sections 4.1.2 and 4.2.2. The latter can further be put in relation to the overall motivation of the study (see section 1.0), lying in the written examination being an overshadowing goal, for which ATD has been chosen as a theoretical framework.
To be noted, whenever the term *explanation* is used in relation to the students' praxis, it refers to a technological-theoretical discourse, i.e. an establishment of a knowledge block.

7.1 The Trigonometry Course (L)

7.1.1 Outline of the Course

The observed course of Trigonometry is based on 10 modules, where the observations were made during the initial modules of the course which means that it was not observed to the end. Primarily part 1 (concerning right and similar triangles) but also a very tiny part of part 2 (concerning the area formula) out of three parts of an *assignment of the theme*, working as an outline of the modules, provides the framework of the observed course (see A.5; "1.del" and "2.del"). In Table 2 below, an overview of the observation study in Trigonometry is presented.

Module no.	Content in relation to GG	Relation to exam exercises
1	Construction of triangles - altitudes, medians, angle bisecting and perpendicular bisector	Exercise presented in Figure 15
2	Construction of triangles - altitudes, medians, angle bisecting and perpendicular bisector	Exercise presented in Figure 15
3	Construction of triangles - similar triangles	T_{ET_5}
4	Construction of triangles - similar triangles	T_{ET_5}
5	GG worksheets concerning the proof of the Pythagorean Theorem via sliders and a click proof (see A.6.1 and Figure 52, respectively)	No direct relation to exam exercises. Applicable in relation to tasks involving right triangles.
6	Construction of the unit circle including the definition of cosine and sine (herein lies working with the directional point and directional angle)	No direct relation to exam exercises
7	Work with take-home group assignments (Part 1 ("1.del"), exercises (a)-(l) in A.5)	Exercise presented in Figure 15 and T_{ET_5}
8	Work with cosine and sine in relation to the unit circle (see Figure 53) and the proof of cosine and sine in right triangles via GG worksheet (click proof) (see A.6.2)	No direct relation to exam exercises
9	Work with definitions of sine and cosine through GG worksheet (see Figure 53) as well as in relation to right triangles	No direct relation to exam exercises
10	Work with sine and cosine in relation to right triangles. Work with the proof of the area formula in arbitrary triangles via GG worksheet (click proof) (see A.6.3)	Area formula relates to T_{ET_4}

Table 2 From left to right column; The number of the observed modules, a brief overview of what was taught and learned in the respective module with respect to GG, and the last column represents the taught and learned knowledge in relation to the exam exercises presented in section 4.1.2.

Note that the area formula was only briefly introduced in module 10 and was to be further examined in the modules to come.

From Table 2 it is shown that TT introduced the definition of cosine and sine in the unit circle context first (module 6, 8, and 9) and then later introduced them in the triangle context (module 9 and 10).

7.1.2 A General Overview of the Teacher's Orchestrations in Relation to GG

The following section explains the overall teacher orchestrations in relation to GG throughout the modules. TT always started the modules by presenting the structure of the specific module, possibly homework, academic goals, and the exercises of the day on a projector to all the students in the class. TT highlighted the exercises of the day on a projector, which were always related to GG either through direct construction work or GG worksheets, and left the screen like this for the rest of the module unless something was going to be presented on the projector such as the CAS View of GG. By highlighting these points, the students were always aware of what to do and which exercises to solve. This is shown in Figure 73.

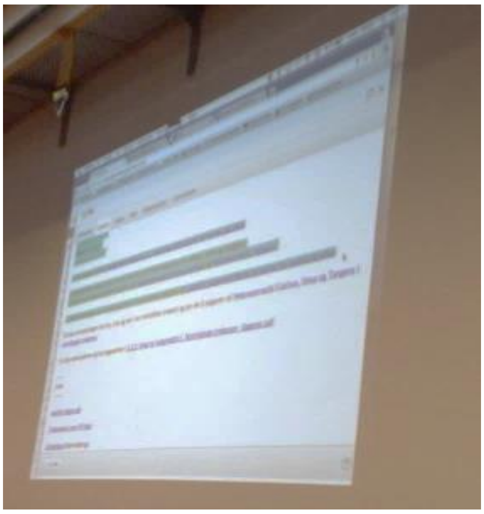

Figure 73 TTs use of projector in the introduction to a module. As shown, TT highlighted what was to be examined.

It is worth mentioning that TT also made use of the blackboard but not in relation to GG. The situations where TT found it necessary to use the blackboard are for instance when reviewing technological-theoretical elements or homework of the day or when setting an exercise in the beginning of the module (not necessary related to Trigonometry) which the students should be able to solve without using any aids.

In general, there was a major focus on the use of GG throughout the modules. This is not surprising since most of the tasks instructed by TT are to be found on TTW where all the tasks are oriented towards GG, as earlier mentioned (in section 4.1.3). Some examples of tasks have already been presented, as t_{TTW_1}- t_{TTW_6}, but in the following, some relevant exercises and tasks for this analysis will be presented as well. These will be presented in connection with the instrumented techniques (τ^i) used by the students as well as potential explanations in relation to the instructed exercises and tasks. As also previously mentioned (in section 4.1.3), TT used GG worksheets when introducing proofs like the Pythagorean Theorem and the formulas of cosine and sine in right triangles (see Figure 52 and A.6.2) in an attempt to have the students to establish a knowledge block.

TT wants for the students to be familiar with GG since it is a tool, they will be using throughout their entire study time. This is one of the reasons why TT established an exploratory environment where the students were allowed to explore on their own and gain knowledge of the different gestures and instrumented techniques that GG has to offer. Another reason is that TT learns GG simultaneously with the students and thereby discovers the potentials of and possible obstacles with GG.

It is clear from section 4.1.3 that no book was used in the teaching of the course. On that basis, TT expected the students to take individual notes in which they for instance could explain what they had observed in GG. [63] In that manner, TT encouraged the students to establish a knowledge block. Whether the students were capable of establishing such a block will be explored throughout this section.

TT let the students work on their own computers when working in GG, also when it came to group work. Hence, all of the students got to practise gestures and techniques in GG. In the modules 5, 7, 8, and 10, TT let the students work in groups. In these modules, they were working with GG worksheets (especially click proofs) and the take-home group assignment where it is preferable to have someone to talk to in order to justify and explain the results (often via algebraic expressions) and the way to reach them.

In two of the modules, TT instructed the students to position themselves in such a way that their backs were turned to the middle of the classroom such that TT could better keep an eye on what was going on their computer screens. Another reason was because it was easier for TT to get around in the classroom in order to help the students. TT circulated among the students when they worked on their own, whether it was individually or in groups, in order to help and to keep track of the students' work.

Consistently for all modules, TT (in interaction with the students) made a recap in the last five minutes of the module of what the students were supposed to have learned in that particular module. This very often ended with TT having to explicitly ask them

[63] These statements on TT's methods related to GG work and the teaching material were expressed by TT him-/herself in connection with an introduction meeting with TT.

to share what they have gained from their work with GG, since many of the students only referred to what was learned outside of the work with GG (through blackboard and projector presentations by TT). Hereby a distinction of the mathematics learned in GG and the mathematics learned outside of GG was (seemingly unintentionally) made.
Next, a presentation, of the mathematical praxeologies established by the students in relation to GG on the basis of the instructed exercises and tasks by TT, will be given.

7.1.3 The MOs Instructed by the Teacher and Established by the Students

This section will examine the gestures and instrumented techniques used by the students in order to answer the specific exercises and tasks related to GG, determine the students' ability to establish a knowledge block, and examine the common observed characteristics related to GG which includes an example of the challenge in making the variables dependent of one another.

All the exercises instructed by TT either require constructions of arbitrary triangles (especially in the modules 1-4) or constructions of right triangles (primarily in the modules 9-10). A task related to the need of constructing an arbitrary triangle (AT) could be the second task (2.) of the instructed exercise (1.1) (see A.6.4), since it explicitly asks for a drawing of an arbitrary triangle. The technique used by all of the PSs to answer this task was as follows:

$\boldsymbol{\tau^{i}_{AT,AP,PS}}$: Click on the Polygon Tool → Click three different places in Graphics View to make three corners → Connect the points so that the last click ends at the same place where the first point was placed.

The above technique has emerged on the basis of the video belonging to exercise 1.1 concerning an examination of the altitude where it is required to draw an arbitrary triangle. Such a video can noticeably provide shared knowledge, concerning the instrumented techniques to be performed in GG, to the students. This is a didactical configuration made by TT and can be equated with a second level orchestration, since the video presents a combination of gestures forming a technique.

The exercises concerning right triangles (RT) have no video nor a detailed manual associated with the exercise (see section 4.1.3), which may lead to more individual and explorative techniques for the students to develop. Hence the techniques may become more spontaneous since there is no given instruction of how to solve the exercises. Some exercises concerning right triangles instructed by TT are listed here:

$\boldsymbol{exercise_{RT_1,AP,TT}}$: Given a right triangle ABC, where angle C is right, and $a = 6$, $b = 8$, and $c = 10$. Construct the triangle and measure the angles ("Øvelse 3.50" a) in A.6.5).

$\boldsymbol{exercise_{RT_2,AP,TT}}$: In triangle ABC, angle $A = 90°$, angle $B = 63°$ and $b = 7$. Construct the triangle, and measure the sides and the last angle unknown ("Øvelse 3.51" b) in A.6.6).

The different observed spontaneously developed τ^i for constructing right triangles in GG in order to solve these exercises are listed below:

$\boldsymbol{\tau^i_{RT_1,AP,PS}}$: Click on the Segment with Given Length Tool → Click on the Perpendicular Line Tool → Click on the point and the line on which the perpendicular line should be perpendicular to → Click on the Segment with Given Length Tool → Click on the Move Tool such that the line segment just constructed can be moved such that it lies on top of the perpendicular line → Click on the Segment with Given Length Tool → Click on the Move Tool such that all the line segments touches each other → Click on the Polygon Tool to combine the points **[**→ Remove all lines and points that are not to be used.**]**

To be noted, the part marked by **[ ]** is not a necessity for solving the exercise (this is also the case in the technique below; $\tau^i_{RT_2,AP,PS}$). Further, note that the technique just described can be applied to exercises similar to $exercise_{RT_1,AP,TT}$.

$\boldsymbol{\tau^i_{RT_2,AP,PS}}$: Click on the Segment with Given Length Tool → Click on the Perpendicular Line Tool → Click on the point and the line on which the perpendicular line should be perpendicular to → Click on the Angle with Given Size Tool (clockwise) → Click on the angle leg point and the angle vertex point → Click on the Ray Tool → Click on the Intersect Tool → Click on the Ray and the Perpendicular Line (creates intersecting point) → Click on the Polygon Tool to combine the points **[**→ Remove all lines and points that are not to be used.**]**

This technique can be related to exercises similar to $exercise_{RT_2,AP,TT}$.

The last technique developed by the students to be presented here does not relate to the exercises instructed ($exercise_{RT_1,AP,TT}$ and $exercise_{RT_2,AP,TT}$) but is just an attempt to construct a right triangle with no given length or sides and is in a sense related to $\tau^{i}_{RT_1,AP,TR}$:

$\boldsymbol{\tau^{i}_{RT_3,AP,PS}}$: Click on the Polygon Tool → Click three different places in the Graphics View and try to let two of the three points be collinear and combine them such that the first point clicked is also the last one.

To be noted, in relation to this technique, the outcome is not guaranteed to be a right triangle whereas the use of the grid in $\tau^{i}_{RT_1,AP,TR}$ makes it certain.

There were even some PSs that discovered a gesture in GG in relation to $exercise_{RT_1,AP,TT}$ which was to click on the Distance or Length Tool which puts the length measure onto the triangles sides in the Graphics View as shown on Figure 74. This gesture is noteworthy not explicitly instructed by TT. From the above techniques and gestures, concerning right triangles, it becomes clear that there are different paths to the same goal. It is also possible to put together some of the different gestures used in the techniques in a different order and still reach the same outcome. In fact, none of these techniques just presented are precisely the same as $\tau^{i}_{RT_1,AP,TR}$, $\tau^{i}_{RT_2,AP,TR}$ and $\tau^{i}_{RT_3,AP,TR}$ which demonstrates that GG is a versatile tool where not only one technique is applicable for solving one specific exercise. The more explorative techniques established by the students are based on

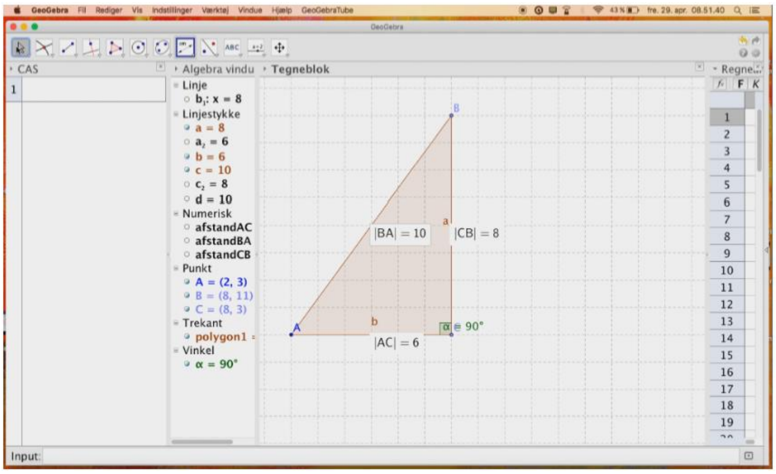

Figure 74 A right triangle where the Distance or Length Tool is used. Hereby, GG presents the side lengths in the Graphics View where $|\boldsymbol{AC}| = \mathbf{6}$, $|\boldsymbol{CB}| = \mathbf{8}$, and $|\boldsymbol{BA}| = \mathbf{10}$.

TT's more open-formulated exercises where there is no exact technique instructed, for instance in the shape of a detailed manual or a video or by TT him-/herself. This way, the students do not for certain develop the same techniques which gives them the possibility to personalise their techniques in such a way that the individual student choose the ones most intuitive to perform. This can be equated with a third level orchestration since the students can reflect upon the techniques chosen to be used.
There are more examples of different techniques solving the same task throughout the modules, for instance the construction of a perpendicular bisector (PB) (see exercise 1.4 task 3. in A.6.7). Here, two different types of techniques are discovered by some PSs:

$\boldsymbol{\tau^{i}_{PB,AP,PS_1}}$: Click on the Polygon Tool to combine the points → Click on the Midpoint or Center Tool → Select the line segment and the midpoint of that line will be placed → Click on the Perpendicular Line Tool → Click on the line where the midpoint is placed and the midpoint (the perpendicular line is constructed).

$\boldsymbol{\tau^{i}_{PB,AP,PS_2}}$: Click on the Polygon Tool to combine the points → Click on the Perpendicular Bisector Tool → Select the line to which the Perpendicular Bisector should belong.

Actually, TT was not aware of the existence of the latter technique, which again indicates that there are many facets of GG which almost makes it impossible for TT to be familiar with all the gestures possible to perform in GG.

One very important feature, which GG offers and which TT is well aware of, is the feature where the figures constructed can be accessed dynamically. To make a figure dynamic and not static, one must make the variables interdependent. To make the variables dependent of each other requires a consistency with the notions. There is a huge importance attached to an awareness of the use of uppercase and lowercase letters, which became evident for both TT and the students, especially in modules 3 and 6 concerning construction of similar triangles (ST) and the unit circle (UC), respectively. Some of the tasks instructed by TT in module 3 were as follows:

$\boldsymbol{t_{ST_1,AP,TT}}$: Construct two similar triangles (see A.6.8 for the manual of this task; exercise 3.1 "Opgave 1").

$\boldsymbol{t_{ST_2,AP,TT}}$: Given two similar triangles constructed in GG. Try to imagine, that you have to calculate one of the side lengths. How can it be determined if the other five side lengths are known? (From exercise 3.1, "Opgave 2", in A.6.8).

This task is followed by hints such as writing "ka=a1/a" in the Input Bar in GG. Noticeably, $t_{ST_2,AP,TT}$ and the hints given in connection thereto has a significant relation to solving tasks of type T_{ET_5}. This, since $t_{ST_2,AP,TT}$ is solved by realising the need of the scale factor to determine the unknown side length.
A task instructed in module 6 reads:

$\boldsymbol{t_{UC,AP,TT}}$: Given the constructed unit circle. Drag the directional point *P*. What is the connection between *Px*, *Py* and the coordinates to the point *P*? (See A.2; TTW 1 task 3.).

The technique used by all of the PSs for $t_{ST_1,AP,TT}$ was the same and is actually a combination of the two techniques $\tau^i_{ST_1,AP,TR}$ and $\tau^i_{ST_2,AP,TR}$. This is maybe due to the fact that detailed manuals are given in connection with the exercises in which the tasks occur. The aspect of the provided manuals constitutes a second level orchestration made by TT. For the construction of the unit circle and the construction of the directional point and angle, the techniques used were $\tau^i_{UC,AP,TR}$ and $\tau^i_{DPDA,AP,TR}$, respectively - again due to the given manuals. However, many of the PSs did not follow or read the manuals carefully since they were not able to make the point *P* on the unit circle dynamic. This was due to the fact that many named the directional angle "V" but when entering the point "P=(Px, Py)=(cos(V), sin(V))" they used "v" as the angle instead of "V". Hereby, GG automatically created a slider since "v" was not defined. This may reveal that they were not yet aware of cosine and sine with respect to the angle being the coordinates to the point *P* on the unit circle. Nor could they construct two interdependent triangles because of the lack of awareness on how to type the defined angles. It is again very important here to be consistent with the notation.

Figure 75 below shows a screenshot of the work of a PS in GG who has been able to construct two interdependent and similar triangles *ABA'* and *vAvBvC* in GG.

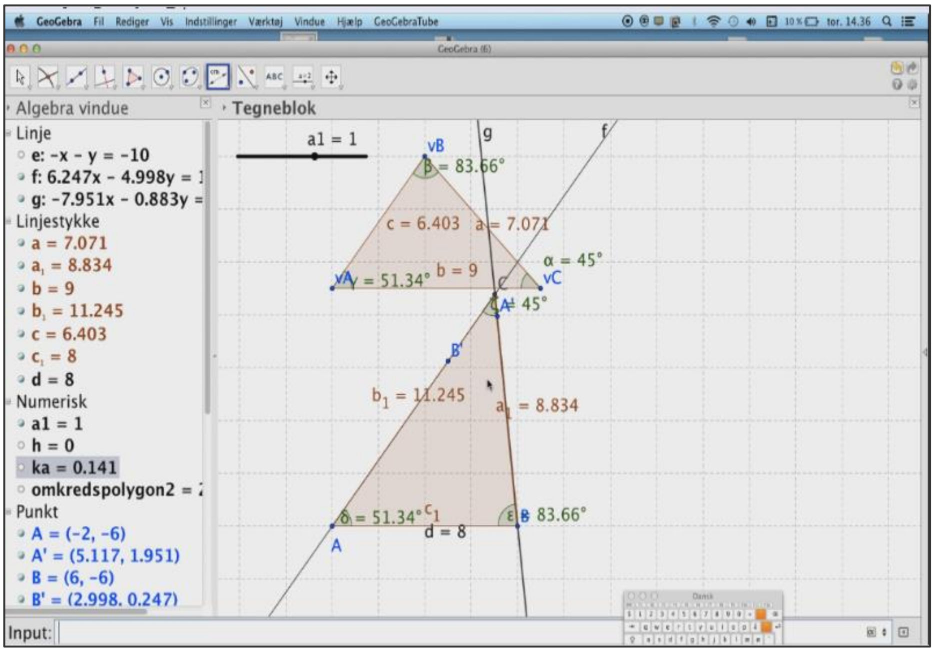

Figure 75 Two similar triangles, triangle *ABA'* and triangle *vAvBvC* have been constructed here, where the sides of triangle *ABA'* are $\boldsymbol{a_1}$, $\boldsymbol{b_1}$, and $\boldsymbol{c_1}$ and the sides of triangle *vAvBvC* are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. The triangles are shown in the Algebra Perspective where a slider for the value *a1* also has been constructed by GG.

In an attempt to solve $t_{ST_2,AP,TT}$, the PS entered "ka=a1/a" in the Input Bar in GG and GG consequently created a slider for *a1* since it was not defined. The PS did not understand the use of the slider and abandoned the attempt to solve the task. What the PS should have written was "a_1" and not "a1" because of the way the sides are defined in the triangles. The PS however never became aware of this, and merely followed the hint without realising that the hint was to use the ratio of two corresponding sides in order to determine the scale factor. This again shows the importance of the consistency of the notation. This raises the question of whether the need of a focus on such consistency in the praxis related to GG may remove focus from the mathematical conceptual understanding. This will be discussed in section 8.1. Thus this PS did not make use of GG to calculate the scale factor since it was challenging for the PS to link algebra to the figure constructed in GG (with references to Geometry II). In prolongation of this problematique, it noticeably seemed easier for this PS to perform solely algebraic techniques in relation to solving a task similar to $t_{ST_2,AP,TT}$ later in the same module (see A.6.9).

Due to the notation problematique, TT chose to make an oral recap of five minutes, where this important feature was to be shared with every student in the classroom.

Many of the PSs, especially in the beginning of the course, actually had a hard time using GG, both dynamically but also for explaining their observations of what they had just seen or constructed, where the latter is one of the long term goals for the teaching planned by TT. Some of the PSs rather used the formulas for cosine and sine connected to right triangles regardless of being presented to a unit circle in GG (or in a GG worksheet) or having constructed a right triangle in GG. In the just mentioned cases, the students were respectively supposed to drag the directional point on the unit circle and read off the measured angles and/or side lengths of the right triangle constructed, instead of the using the formulas. Especially, this reveals a struggle with managing the relation between the analytic and the synthetic geometry.

In the following passage, some examples of the students struggling when using the GG dynamically or making conclusions on the basis of something constructed in GG will be presented. The first example is when the students had to solve the instructed task of constructing medians in GG and explain the observed (TTW 5 task 3. in A.2).

This task noticeably relates to the exercise presented in Figure 15.

A PS called for TT's help in order to answer this task. This need for help arose after the PS had constructed the triangle and the median by applying $\tau^{i}_{Median,AP,TR}$ as well as using www.google.com in order to try to understand the notion of a median. Below, a transcript excerpt from USB Blå, file 2, Module 2 (see A.3.1) is presented. TT wants, in this situation, to hear the PS's explanation of a median:

PS: The median cuts in the middle.

TT: Yes, in the middle of what?

PS: Of the triangle.

TT: Of the triangle? Where is the middle of the triangle? I don't think it is what you mean?

PS: In the middle of… in the middle of… the baseline.

TT: Yes, that you can call it… Or maybe we should define it as.. We have an angle and it cuts the opposite side.. and it is the opposite side we look at… It is the middle of that side not the middle of the triangle.. it is the middle of the side.

PS: Yes.

TT: Write it down.

The PS hereafter opens Word and write down: *A line which cuts the line on the side opposite to the angle* (USB Blå file 3 Module 2; see A.3.1). As mentioned, TT here tried to get the students to establish a knowledge block in the terms of the instructed task having technological characteristics when asking "explain the observed". Here, it is however seen that the PS was not entirely capable of describing a median since the written description is not precise enough for it to concern a median.

Another example that should be pointed out is from module 8 where the students worked with the GG worksheet presented in Figure 53. Here, the students had to find cosine and sine with respect to a given angle by dragging the point *P*. Instead, the PSs used the formulas for cosine and sine in connection with right triangles. Hence, sine and cosine were just used as a toolbox for solving trigonometric problems. In addition, the following transcript excerpt from USB 6 file 1 Module 10 (see A.3.1) point out the exact same (thus using formulas instead of GG dynamically), where a PS and another student (AS) talk to each other about solving $exercise_{RT_1,AP,TT}$:

PS: What was it that we have to calculate?

AS: I used the Law of Sine but you probably shouldn't since we are not allowed to do that.

PS: Okay.

Obviously, the AS knows that it is wrong to use the formulas but does it anyway, which also shows a clear example of the students feeling insecure in using GG at that point. Many of the PSs found it hard to understand the unit circle context and preferred the triangle context which could be because it is well known as a context from the elementary school (Webpage: Fælles mål 2009 (historiske) Folkeskolen, 2016, p. 60). However, even though the PS and the AS have the dialogue above in module 10, it is worth mentioning that as the modules progressed, most of the PSs became more comfortable in performing gestures and techniques in GG, also dynamic ones, especially after the take-home group assignment (in module 7) which is to be elaborated further down.

The next example concerns student work with exercise 4.2.2 (TTW 1 in A.2) where the students had to find cosine to a specific angle as well as sine to a given angle by dragging the directional point *P* on a unit circle, just as in t_{TTW_1} and t_{TTW_2}. Hence, the students were encouraged to work with cosine and sine in relation to the unit circle.

The following transcript excerpt is from USB 6, file 4, module 6 (see A.3.1) and indicates that the two students AS_1 and AS_2 have a hard time letting go of cosine and sine in relation to the triangle context:

AS_1: Cos to an angle is the relationship between the adjacent side and the hypotenuse, right?

AS_2: Yes.

AS_1: This means, if the adjacent side has length 5 and the hypotenuse…

AS_2: Well, it is not the intention.. It is not something definitely, you just have it on you calculator…

AS_1: But, isn't it possible to calculate it in this way?

AS_2: I don't know… If you take the ratio of two sides you just get a constant… you know the scale factor.

AS_1: Yes yes exactly.

AS_2: So, cosine is just something you enter on your calculator.

AS_1: Yes.

The students try to put into words their understanding of the notions (i.e. they try to establish a knowledge block) but the statements above show a clear example of a black box effect since cosine is just a button on the calculator (according to AS_2). What this also shows is that AS_1 perceives cosine to an angle in relation to the triangle context and not the unit circle context. Overall, it seems however problematic for these two students to manage the definition of cosine and sine with respect to an angle, whether in the triangle context or in the unit circle context.
TT made a recap in the last five minutes of this module where the students contributed with terms, such as directional point and directional angle, as well as the importance of the notation in GG by being consistent in using the same letter for the directional angle. In this recap it was not explicitly mentioned by TT that no formulas were allowed to be applied at this point but TT repeatedly stated that the students must use GG dynamically.

One common observed gesture/technique used by all of the PSs is the Angle Tool  in relation to triangles, for instance in relation to $exercise_{RT_2,AP,TT}$ or $t_{ST_1,AP,TT}$. The challenge in using this tool is that one must know which angle leg to pick first, in order to get an interior angle which is in the students' interest as they are working with triangles. Almost all the PSs had a hard time figuring this out. As shown on Figure 76 the angle α is exterior and not interior where the latter was the PS' intention. Angle α was obtained through the following technique (similar to $\tau^{i}_{Angle2,AP,TR}$):

$\boldsymbol{\tau^{i}_{\alpha,AP,PS}}$: Click on the Angle Tool and select to segments → Click on line segment AB and then BC.

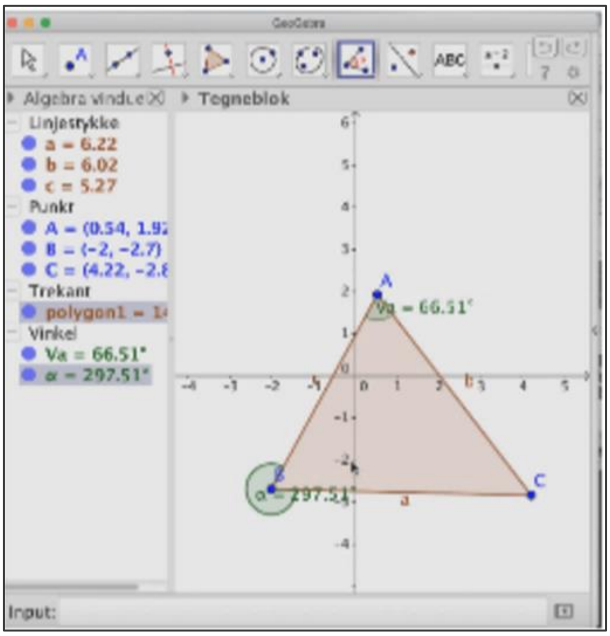

Figure 76 Here, an arbitrary triangle ABC has been constructed by a PS where the $\boldsymbol{\alpha}$ is an exterior angle and $\boldsymbol{V\alpha}$ is an interior angle.

Here, GG is programmed to interpret AB to be the initial leg which was not the intention. Many of the PSs however found out which leg to start with in order to get an interior angle merely through what seems to be clicks on the legs in a random order until the angle is interior. The question is here, whether they learn from this in general or if they only see the clicking on the right leg as a first as an important feature in GG for constructing interior angles.
There was a brief recap of the angle notation where it was described by TT in interaction with the students that the interior angle C of a triangle ABC (drawn on the blackboard) is the same as angle $\angle BCA$, where BC and AC are the initial and the terminal leg, respectively, and the exterior angle C is the same as angle $\angle ACB$ where AC is the initial leg and BC is the terminal leg. Hereby, TT tried to make the students aware of the importance of the order of the angle legs. In prolongation of this, there was a brief sum up where it was stated that an angle in a triangle $\in (0°, 180°)$.
It has further been observed that most of the PSs used the legs of the triangle to construct angles, a few used the three vertices ($\tau^{i}_{Angle1,AP,TR}$) but no one discovered $\tau^{i}_{Angle3,AP,TR}$.

As mentioned earlier, TT made a great deal out of letting the students explain and justify their results and what they have observed in GG. Examples thereof have occurred throughout this section, for instance when a PS should write down how to describe a median in Word. Here, some more examples are to be presented.
The first example is based upon one of the only exercises instructed by TT that actually required the formulas of sine and cosine in relation to right triangles:

$\boldsymbol{exercise_{RT_3,AP/CV,TT}}$: Given a right triangle ABC where angle C is right and angle $B = 77°$ and the cathetus $a = 13$. Determine the unknown sides and angles through calculations and check the results by construction and measurements ("Øvelse 3.47" in A.6.10).

For solving the above exercise, several PSs used Word or paper and pencil to write down the formulas and rewrite them in order to solve the exercise, using a classical CAS tool of own choice (including GG). Within this praxis, the PSs managed to explain their calculations to a great extent. As an example of such a praxis in relation to $exercise_{RT_3,AP/CV,TT}$, one PS used Word for the notes (see A.6.11) and started out by calculating the unknown angle A by using that the sum of the angles in a triangle is 180° and isolating A in this equation. Next, the PS calculated the unknown side c via the formula $\cos(A)\ [= \frac{adjacent\ cathetus}{hypotenuse}] = \frac{13}{c}$ (without writing the part marked by [ ] though) and isolating c herein. Finally, the PS calculated b by using the Pythagorean Theorem. Throughout this PS' notes it is clear that the PS knows how to manipulate with the formulas. The PS more specifically makes an effort as to establish a knowledge block, in the sense that the PS explains the calculations through written sentences by referring to the angle sum of a triangle and the Pythagorean Theorem. Noticeably, this PS used the CAS View of GG when calculating both of the unknown sides. However, the PS calculated in radians without realising it, even though TT had, at this point of the course, instructed the importance of using the degree symbol (°) - this will be elaborated further down.

The great importance that lies in the students being able to use correct terms when working with a task or an exercise is expressed through some of the observed dialogues that TT had with the students while circulating around in the classroom to help the students, a part of the teacher orchestration. For instance, when having a conversation with two students, a PS and an AS, in relation to the task $t_{UC,AP,TT}$. Here, TT made an effort as to get the students to deduce, from working with the unit circle, that cosine to the *directional angle* is the *x*-coordinate of the *directional* point and sine to the *directional angle* is the *y*-coordinate of the *directional* point. Here TT really tries to get the students to use the correct terms in a coherent manner - for instance by noting that it is not enough just to say "the directional point" but the students should say "the directional point on the unit circle" (from USB 6, file 4, module 6 (see A.3.1)). Another example is the dialogue between TT and a PS (from USB 8, file 1, module 1) where the PS tries to explain the definition of an altitude. The PS has asked TT for help after seeing the video that belongs to the task (exercise 1.1; see A.6.4).

PS:	How do I construct an altitude?
TT:	You can get help from the video or try to explore the Menu. What is an altitude? Have you looked it up in the formula collection? Tell me, what is an altitude of a triangle?
PS:	It is from the apex... or the top of the triangle and onto the baseline.
TT:	How onto the baseline? There are many ways it can fall down to the baseline?
PS:	[Points with the computer mouse from *A* to the line between *C* and *B*] straight down.
TT:	What do you mean when you are saying straight down?
PS:	Just straight. What is it you want me to say?
TT:	Well, it is not quite the correct terms you use. Straight is not precise enough. Straight it is, but it just means that it is not skewed.
PS:	What is it called when it goes like this [the PS draws a straight vertical line in the air]
TT:	You mean perpendicular?
PS:	Aaaarh... It must be perpendicular to the baseline.
TT:	Yes, isn't it what is special about altitudes?
PS:	Yes.

Subsequently, TT tells the PS to go to the Menubar and select something that can make a perpendicular line. At no point does TT tell the PS how to construct the altitude in GG and wants the PS to think about the techniques possible to use from what the PS now knows about an altitude. However, TT gives this student knowledge to the boxes in GG that describes how the specific tool works (these boxes are also described in section 4.3.2). Noticeably, this gesture within using a helping box was not to be summed up by TT. The way of letting the students find the gestures in order

to solve a given task on their own was, as mentioned before, a deliberate choice by TT. From the above dialogue, it is (again) seen that TT wants the students to use the exact terms. This had a high priority throughout all the modules.
To be noted, after having had the above dialogue with TT, this PS dragged in the constructed triangle with three altitudes in GG and wrote down (in Word) that *the altitudes of the triangle always intersect* (USB 8, file 1, module 1). This shows the PS' ability to use GG dynamically and reflect upon it.
Many of the exercises and tasks had an associated video attached just as the task that lead to the dialogue just above. In connection hereto, it was also observed that many of the PSs just followed such a video parallel to their own constructions and did not reflect upon what they had just constructed or just watched in the video, as the situation leading to the above dialogue was a clear example of. In general, the aspect of the observed student approach to the videos may be an example of a black box effect since the students may blindly perform the gestures in GG without reflecting upon them.
Thus, TT tried to establish praxeologies of both a practical block and a knowledge block for the students through the instructed exercises, but they are not always in accordance with what praxeologies the PSs actually established.

Now, some of the praxeologies established when the students spoke together in groups will be reviewed. Here, some of the students established a knowledge block by explaining to the other classmates of the group how to solve the given task. For instance:

$\boldsymbol{t_{Cosine\ to\ angle, Worksheet, TT}}$: Argue for the formula $\cos(v) = -\cos(180 - v)$ from the figure (exercise 6, task 10.; see A.6.12).

This task is associated with a GG worksheet which illustrates a unit circle and two points on the unit circle with respect to angle v and $180 - v$ (see Figure 1 in A.6.3). By dragging the points, the respective angle follows. The following conversation between a PS and an AS, in connection to the task, takes place (from USB 6, file 3, module 10 (see A.3.1)):

AS: I know why. We are on the negative side of the x-axis.

PS: What is negative? [points to the angle $180 - v$]

AS: Cosine to v is equal to minus cosine to 180 minus v because when it [the point] is on this side [points to the point in the second quadrant] then it [the x-coordinate] is minus cosine.

The AS here try to explain in words what is observed in the dynamic GG worksheet. The AS has clearly understood that cosine with respect to the angle is the first coordinate to the directional point and was in general very accurate, by using terms as *directional point* and *directional angle* in the work with the PS as a partner. In connection with this presented worksheet, it is seen that the students got the possibility of explaining what they observed in the unit circle context. It should be noted that this took place in the very last observed module where the students may have become more comfortable in considering cosine and sine in relation to the unit circle.

As mentioned, TT also used click proofs in order to give the students the possibility of establishing a knowledge block, for instance the click proof of the formulas of sine and cosine in relation to the right triangle (A.6.2). Here the students could click their way through the steps of constructing a unit circle, two similar right triangles and their interrelationship. Along with the clicks, an illustration appears, which could make it easier for the students to grasp the proof. However, in the work with these GG worksheets, the students did not necessarily need to explain or reflect on why and how it was so, since they could just click their way through the proof and write off. This was also the observed in connection with the above mentioned click proof where an example is seen in the notes of a PS (see A.6.13) which were merely written off from the worksheet. Hereby, the worksheet may have become a black box rather than the intended kick starter for the establishment of a knowledge block. It should be noted that this worksheet is extremely important in the way that it connects the unit circle context with the triangle context. This, in the sense that it provides insight to why the formulas for sine and cosine with respect to an angle in the triangle context are as they are. One can ask oneself if the students realised this?

In connection with the implementation of click proofs in the course in general, it should further be noted that TT only sometimes performed recaps on such worksheets.

Until now, there has been a very large focus on the construction work and the dynamic aspect of GG but, as briefly mentioned earlier and also in section 4.3.2, there is a classical CAS tool in GG (the CAS View) which now will be reviewed for its implementation in the course.
The CAS View in GG was introduced by TT in module 9 on the projector in order to create a common knowledge to the gestures and techniques to be used in this View of GG, thus both a first and a second level of orchestration. There was a need for TT's common presentation since there are several reservations one must make when working with the CAS View in GG as a classical CAS tool. Many of them are already described in 4.3.2. A picture of TT's presentation is shown on Figure 77.

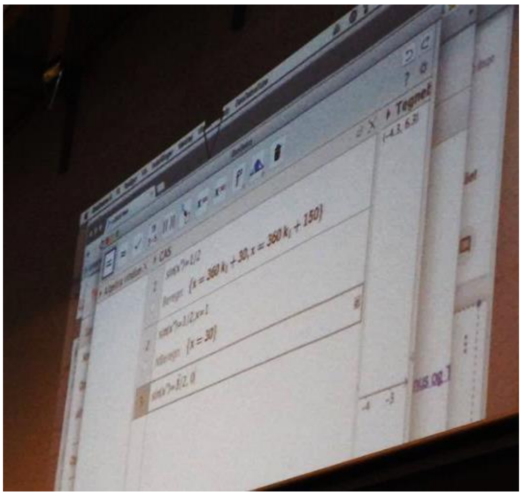

Figure 77 A picture of TT's introduction to the CAS View in GG to the students on a projector. On the projector, cell 1 of the CAS View can be seen, where TT has written "sin(x°)=1/2" in the Input Field and then clicked on the Solve Tool and then the output is shown to be "{x=360k_1+30, x=360k_1+150}". In cell2, TT has used the Numerically Solve Tool on the same input where the output is "{x=30}".

The presentation in Figure 77 shows elements of a first level orchestration included in the second level orchestration since TT has presented a really important gesture, namely the typing of the degree symbol ("Ctrl+O" on a Mac). Otherwise, GG will calculate in radians which is a term, the students will not be introduced to at this level of high school. TT noticeably did not mention this effect of not typing the degree symbol but merely put an emphasis on typing it. Another important gesture that was introduced in this orchestration is the typing of "sqrt" when calculating square roots and furthermore, the difference between using the techniques Solve and NSolve when making calculations. Even though this detailed introduction was made by TT, some of the PSs were not comfortable in using the CAS tool yet. Many forgot the degree symbol and calculated in radians, without noticing it, and they had trouble distinguishing between NSolve and Solve as well. If the output was complicated, like the output in cell 1 on Figure 77, they simply just tried using a different gesture (browsing between the four tools presented in section 4.3.1; The CAS View). Many of the PSs did not use the classical CAS tool of GG and chose to operate with handheld or computer calculators instead, despite of the introduction made by TT. However, it is important to keep in mind that this is the first time that both TT and the students use GG in relation to Trigonometry, so it seems understandable that they do not feel as confident yet in choosing this as a CAS tool in general.

Although many of the students found it difficult to use GG, it has a huge impact in the sense that the students can visualise the problem that they are faced with. In the light of this feature of GG, TT gave the students a take-home group assignment which included the use of GG dynamically (e.g. geometric constructions) and an oral presentation of the chosen tasks in a screen recording. The assignment was based upon the tasks (a)-(l) of the assignment of the theme part 1 ("1.del" in A.5).
The students worked in groups of four to five students but each on their own computer in order to make an individual presentation of a specific task from (a) to (l). It should be noted that only task (a) and (b):

(a) Define concepts to describe triangles, including altitude, angle bisector, median, perpendicular bisectors, apexes and adjacent- and opposite side.

(b) Explain *similar triangles*.
Along the way, you must include the scale factor and theorems that apply for similar triangles.

are represented in the types of exam tasks and exercises presented in section 4.1.2 in the sense that (a) relates to the exercise presented in Figure 15 and (b) relates to T_{ET_5}. All the other tasks from the take-home group assignment deal with right triangles and the definition of cosine and sine in this context, none of which are represented in any of the exam tasks. In this assignment, the students were asked to describe what they saw, to explain whether they worked with a definition, a theorem or a proof, to explain what was happening and to draw conclusions of the tasks. This strongly indicates that TT wanted the student to establish praxeologies of both the practical block and the knowledge block.

When going through the hand-ins, it becomes clear that some of the students did not use the dynamic aspect which GG offers. Instead, some of the students had constructed the triangle in Paint[64] and made use of the computer mouse to point at the specific angle or side they talked about. Furthermore, most of the students lacked justifications and explanations when presenting the answer to the specific task. Hence, these students did not establish quite the same praxeologies as TT had expected them to. However, through this take-home group assignment and hand-ins, a common understanding of what was expected of the students, in connection to their work in GG, as well as a demonstration of which praxeologies the students could manage was established which the students and TT (respectively) could take with them consecutively. The students' approach to the video recordings in the groups was observed to be very individual in the sense that they only practised their own task(s) without communicating much with their group members on the praxeologies.
TT gave feedback to the screen recordings handed in by the students by making comments throughout the video like it is shown in Figure 78.

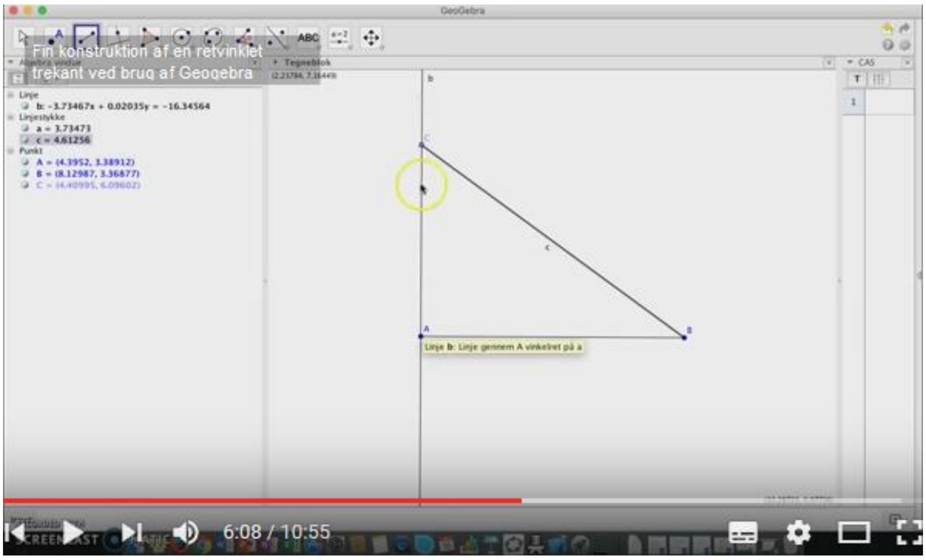

Figure 78 A construction of a right triangle in GG is shown. A student tries here to answer task (d) of the take-home group assignment: *Define concepts to describe the right triangles, including hypotenuse, adjacent and opposite cathetus*. As seen in the upper left corner, there is a grey, transparent, box with TT's comments to the students work in GG; "Fine construction of a right triangle by the use of Geogebra".

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[64] Paint is a standard Windows program offering the option of drawing objects.

The presentation of the student in Figure 78 noticeably shows the correct way of answering the task, since the student has actually used GG in order to construct a right triangle. Hence, there were students who managed to fulfil TT's demands of using GG. The feedback was referred to by TT in module 9 (to be found in a shared folder on YouTube) where the students had the possibility of seeing the videos with feedback before continuing doing the exercises related to GG of the module. Here, only some of the PSs chose to have a look on the feedback video.

As a sum up on this entire review, it was clear throughout the modules, through the formulations of the instructed tasks and the take-home group assignments that TT wanted the students to use the GG dynamically and in this connection write down what they observed through the visualisations. More specifically, the students had the possibility of seeing a lot of examples of the same type of object through GG's dynamic features. However, the gestures for the construction of the objects and the approach to the dynamic features of the objects had not yet been fully established by the observed students through this course. Some of the students had difficulties in connecting algebra to the figures, to cope with the definitions of sine and cosine in the analytic geometry, to construct interior angles, and to make the variables interdependent such that the constructed figures actually became dynamic. However, it seems to be a great visualisation tool at this point of a Trigonometry course, especially through the worksheets where the objects have been constructed a priori, and the students only had to (mainly) drag in a point to deduce something from the observed object. As the modules progressed and especially after the take-home group assignments and the stated expectations in connection thereto, most of the PSs tried to use GG more dynamically.

7.2 The Power Functions Course (A)

7.2.1 Outline of the Course

The Power Function course consisted of eight modules, listed in the Table 3 below presenting the content for which GG and GG worksheets have been involved and which of the types of exam tasks (T_{EP_1}-T_{EP_6}) that can be put in relation to the modules.

Module no.	Content in relation to GG/GG worksheets	Relation to exam tasks
1	- The impact of the constants a and b on the look of the graph of a power function $f(x) = b \cdot x^a$; Exercises related to GG (see A.7 Module 1)	No direct relation to T_{EP_1}- T_{EP_6}
2	- Thm_{ab}; Worksheet 1 with click proof offering hints and right answers (see Figure 70)	Some relation to T_{EP_4}
3	- Calculation of a and b knowing two points on the graph of $y = b \cdot x^a$; Worksheet 2 with exercises with hints and right answers (see A.7 Module 3) - Regression/modelling exercises from the exercise book (Clausen et al., 2013a) (see A.7 Module 3)	T_{EP_4} T_{EP_1}, T_{EP_2}, T_{EP_3} and especially T_{EP_5}
4	- Review (in plenary) of the techniques used in GG for answering the regression/modelling exercises from Module 3; Sherpa-like situation - Power growth and the (%/%) property; Worksheet 3 with graph of power function and sliders (see Figure 71)	T_{EP_1}, T_{EP_2}, T_{EP_3} and especially T_{EP_5} Some relation to T_{EP_3}
5	The $(\cdot/\cdot)$ covariation of variables in power growth $f(x) = b \cdot x^a$; GG-document made by TP working as a worksheet (see A.7 Module 5)	Some relation to T_{EP_3}
6	Proportionality (direct and inverse); Worksheets 4-7; Student work with Worksheet 7 (see Figure 72) with exercises offering access to right answers and graphical representation (see A.7 Module 6 for Worksheets 4-6)	T_{EP_6}
7	- Proportionality (direct and inverse) continued; Worksheet 7 (see Figure 72) - Distinguishing linear, exponential and power functions; Worksheet 8 with multiple choice exercises related to graphic representations (see A.7 Module 7)	T_{EP_6} No direct relation to T_{EP_1}- T_{EP_6}
8	- Distinguishing linear, exponential and power functions continued → Worksheet 8 - Take-home group assignment (see A.8 Module 8)	No direct relation to T_{EP_1}- T_{EP_6} T_{EP_1}, T_{EP_2}, T_{EP_3} and T_{EP_5}

Table 3 Overview of the content related to GG and GG worksheets in the eight modules of the course and its relation to the exam tasks T_{EP_1}- T_{EP_6}.

It is noted that GG or GG worksheets have been involved throughout all eight modules and used for many of the conclusions of the theme - from the impact of the values of a and b on the look of the graph to distinguishing the power function from linear and exponential functions (as Table 3 indicates). Furthermore, it should be noted that TP set up more sessions related to GG worksheets than sessions related to performing techniques in GG itself in this course. GG itself was primarily prioritised in module 1 in connection with the impact of the values of a and b on the look of the graph and in the modules concerning power regression/modelling (Module 3 and 4). For answering **RQ_1-RQ_3**, it is the teacher and student work in relation to GG itself which is of special interest why these sessions will be reviewed as a first. The GG worksheets are, even though they do not necessarily call for performing techniques in GG, an interesting aspect as to how TP has organised the course in relation to GG why the implementation of these will also be reviewed. Further, the take-home group assignment and its relation to GG-use will be analysed. Finally, a sketch of the teacher's orchestrations respectively the student work in relation to GG will be made. When referring to the modules in this analysis, Table 3 may work as to provide both further details on the GG work as well as an overview of the course.

7.2.2 Teacher and Student Work in Relation to GG Itself

In Module 1, TP gave the students a sheet of exercises with some relation to GG (see A.7) just after having introduced the definition of a power function on the blackboard with an emphasis on both $x > 0$ and $b > 0$, followed by a brief review of the concepts of *domain*, *range*, *conditions of monotony* and *asymptotes*. The main technique instructed by TP to be carried out in GG (via these exercises) is the following, which is seen to be an extension of $\tau^{i}_{Sliders\ ab,AP,PR}$:

$\boldsymbol{\tau^{i}_{Sliders\ ab,AP,TP}}$:	Type "f(x)=b*x^a,x>0" in the Input Bar **[**→ Press Enter**]** → Click "Create Sliders" in the "Create Slider(s)" view popping up on the screen → Drag in "f(x)" from the Algebra View to the Graphics View → Right click on the slider for b and choose "Object Properties" → Set the interval of b (in this specific case to min: 0.01 and max: 10).

The gesture marked by **[ ]** is not instructed but necessary. The outcome of $\boldsymbol{\tau^{i}_{Sliders\ ab,AP,TP}}$ is noticeably both the graphic (in the Graphics View) and the algebraic representation (in both the Algebra and Graphics View) of a power function with domain $x > 0$ where a can attain the values $[-5, 5]$ (see subsection on The Algebra Perspective in section 4.3.3) and *b*; the values $[0.01, 10]$. The typing of "x>0" in this technique may lead to a further awareness of the domain of a power function being $\mathbb{R}_{+}$. The fact that the functional equation can be seen in the Graphics View next to the graph via $\boldsymbol{\tau^{i}_{Sliders\ ab,AP,TP}}$ makes it unnecessary to move one's gaze from the Graphics to the Algebra View in order to see that the functional equation changes simultaneously with the look of the graph when dragging the sliders for *a* and/or *b*. Thereafter, a description of what happens when the value of *a* is changed is required as well as a determination of the intervals for *a* where the look of the graph changes significantly. Here, it seems that the praxis of dragging a slider (for *a*) in GG is sought to lead to some explanation for the look of the graph in terms of which intervals of *a* leads to different graphical representations. Next, the technique of plotting a specific point on the graph on the graph of f(x), namely "P(1,f(1))" is instructed;

$\boldsymbol{\tau^{i}_{Plot\ point,AP,TP}}$: Type "P=(1,f(1))" in the Input Bar **[**→ Press Enter**]**.

The gesture marked by **[ ]** is not instructed but necessary. In prolongation of this, it is asked what happens to the point, *P*, when the value of *b* is changed. It is further asked which relation there is between the coordinates of *P* and the value of *b*. Finally, a proof of this relation is requested through use of paper (i.e. not GG) with a hint of putting "x=1" into the functional equation "f(x)=b*x^a". Here, it seems that the praxis of dragging a slider (for *b*) in GG is sought to motivate a description of the relation between *P* and *b* analytically. Especially, the proof of the relation between *P* and *b* can provide some explanation to the praxis, i.e. an establishment of a knowledge block. Further, an exercise of determining *a* and *b* values related to five given graphical representations of power functions (in a figure very similar to Figure 25, besides from the given domains for *a*), by using the sliders in GG, is given. A table is set up as answer field for the *a* and *b* values. Hereby, the praxis of dragging sliders (for both *a* and *b*) are put in relation to determining values of these constants for specific graphical representations - and thereby, it may work as a praxis leading to explaining the specific looks of the graphs. Finally, an exercise of concluding on the domain, range, conditions of monotony and asymptotes for the five given types of power functions, with none instructed GG-work, is given. This exercise may lead to adding some further explanation to the look of the graph of a power function. There is a general priority traceable in these exercises; namely that of putting algebra in relation to the graphical representation of a power function and using the sliders of GG to realise what applies for the values of a and b in relation to the look of the graph. The last half of the module was spent on these exercises where the students worked in groups of two or three, made by TP, and TP circulated and guided the students individually when needed. Regarding the praxeologies of the students performed in relation to the exercises, $\tau^{i}_{Sliders\ ab,AP,TP}$ could be a challenge for some of the students - at least for one of the PSs (PS_1) (and partner (AS_1)) for whom the input became "f(x)=b*xâ,x>0" instead of "f(x)=b*x^a,x>0" leading to a whole other output than the graph of a power function with sliders for b and a - namely that of a constant function with sliders for b and $x\hat{a}$ (see A.8 Module 1 for the outcome). Subsequently, these two students did not have a proper base for answering the following questions since they trusted the output on PS_1's computer instead of the (right) output on AS_1's computer (who closed the lid and put it aside) and they continued doing the exercises on PS_1's computer without AS_1 getting to perform any of the gestures related to the exercises in GG him-/herself. For instance, PS_1 got the possibility of realising the typing of decimal numbers using dots rather than commas, when setting the interval for b, as well as spontaneously realising the difference between typing a lower- and uppercase letter "p" when trying to plot the point $P = (1, f(1))$ - which in the first case leads to the vector[65] as an output and in the second case; a point. Even though PS_1 verbalised the problem in plotting the point to both AS_1 and neighbours, AS_1 did not get to perform the praxis of typing the input necessary for getting a point (and not a vector). Especially, the gestures of dragging the sliders of a and b were only performed by PS_1 while AS_1 merely observed. At one point, they found out that the graphic representations looked different on some other student's computer and then asked

[65] The students have not yet been introduced to vectors at this level of high school.

TP for help. TP immediately announced that there had been a problem in typing the power and related it to Mac computers and called out to ask if anyone knew how to type the input on a Mac. Then another student announced that one should enter 'space' after the '^'-sign whereby a first level orchestration took place. PS_1 and AS_1 then started from the top in order to get the output right on PS_1's computer. They did not determine the intervals for a, for which the graph has different looks, in detail. PS_1 wrote however: *...increases and decreases when 'a' changes from being positive to negative and vice versa...* (A.8 Module 1; Front page of PS_1's notes) about the graph, so something was deduced from dragging the sliders. For the proof of the relation between the point P and b, PS_1 asked TP for help. Here, TP made an effort as to motivate PS_1 into "calculating" by use of symbols instead of PS_1 using the specific set value of b on the slider on PS_1's screen. This seemed challenging to PS_1 who, to begin with, had written down the functional equation with the specific values of both b and a read directly from the sliders. PS_1 did however succeed in writing down the relation analytically in prolongation of having asked TP for help - without any explanations made though (see A.8 Module 1; Back page of PS_1's notes). When reaching the exercise of determining a and b values using sliders in GG, PS_1 initially only dragged the slider for a and let b be equal to a random set value (7.81) for the first couple of functions (see A.8 Module 1; Front page of PS_1's notes). For the next three functions, PS_1 began to drag the slider for b as well but did not seem to understand the precise impact of b on the look of the graph since the only correct answer for b was given when PS_1 heard it from a neighbour and was thereby not reached in dialogue with AS_1 (for *k(x)* representing $0 < a < 1$; see A.8 Module 1; Front page of PS_1's notes). PS_1's answers for a were however made in dialogue with AS_1. These seem solely to have been made based on how the graph looks or curves (and not on the specific points lying on the graph) and the answered values were noticeably within the right domains of a (see A.8 Module 1; Front page of PS_1's notes). For another PS (PS_2)[66] and partner (AS_2), their answers for this exercise were much more precise, especially with respect to the b-values, demonstrating a more successful outcome of the praxis

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[66] Screen recordings of PS_2's work was not successfully made - only field and handwritten student notes.

of dragging the sliders for both a and b in order to determine their values for specific functions based on their graphical representations (see A.8 Module 1; PS_2's notes and AS_2's notes). The answers for the subsequent tasks made by PS_1 in dialogue with AS_1 reveal that it was challenging for PS_1 to conclude on both domain, conditions of monotony and asymptotes in relation to the given graphical representations of the five functions - not least to specify the answers to be related to one or more of the functions. For PS_2, it was however possible to conclude on all these terms more correctly if one assumes that the answers given are related to the function representing $a < 0$ (i.e. $f(x)$) which is not made explicit - since, in that case, it is only the answer of the y-axis being an asymptote which lacks. AS_2 wrote down the conclusions underneath the columns representing each of the functions in the table of the previous exercise - demonstrating an awareness of the answers not necessarily being the same for each of the functions. AS_2's answers concerning asymptotes were the shakiest ones but apart from this, the answers were quite precise. It seems that it was the easiest to conclude on the range for these PSs and ASs. However, none of the students were able to realise that the range of the constant function ($m(x)$) was only the number 2.
TP used the blackboard for the recap of these exercises, in the beginning of the subsequent module (2), to draw graphical representations for power functions as a reaction on the students' input on how the graph looks when $a > 1$, $a = 1$, $0 < a < 1$, $a = 0$, and $a < 0$. Further, a student got to present the proof of the relation between b and P on the blackboard where TP asked for explanations for the techniques, being a bit challenging for the student. Finally, TP and the students had a dialogue on the last tasks where TP used the drawn graph types (on the blackboard), related to the domains of a, as reference point. TP did not at any time in this recap refer to the use of sliders for a and b in GG and there was made no further recap on whether the students had understood how to perform the instructed gestures and techniques in GG of the exercises or if there had occurred any other realisations in connection thereto.

For the modules (3 and 4) concerning regression/modelling of power functions, TP used the projector presenting the view of TP's computer screen, to introduce in plenum how to perform power regression in GG in relation to a specific example[67] where TP controlled the keyboard and mouse; i.e. a second level orchestration. The technique instructed here was $\tau^{i}_{5.2,SV,PR}$. TP subsequently had a brief dialogue with the students on the r^2 value of a regression and what it expresses when a student mentioned it[68]. TP did however not introduce explicitly how to find the r^2 value in GG. The students were then instructed to do some exercises in the exercise book (see A.7 Module 3) each involving a task of the type T_{EP_5} where $\tau^{i}_{5.2,SV,PR}$ is applicable. These exercises furthermore consisted of tasks of the types T_{EP_1}, T_{EP_2} and T_{EP_3} as well as a singular task of plotting a given data set along with the graph of the functional equation derived through power regression of the data - for which no instructions related to GG has been made by TP. The tasks concerning regression were hereby all of type T_{EP_5}, asking for power regression and not an assessment of the regression, which may explain TP's choice of not prioritising the finding of the r^2 value in the plenum dialogue. None of these tasks explicitly call for explanations, and thereby an involvement of a knowledge block, but are very technique minded, as also noted in section 4.2.2.

For these exercises, TP circulated and answered questions, if any, while the students worked together with their neighbours. The observed student work was, possibly as a consequence of the formulations of the tasks, very technique oriented. For all of the PSs, the technique $\tau^{i}_{5.2,SV,PR}$ seemed quite straightforward for them to perform for the tasks of type T_{EP_5}. Noticeably, all of the four PSs (PS_3-PS_6) consciously chose to use dots instead of commas for decimal numbers in GG when typing the data. All of the PSs furthermore found and used the "Copy to Graphics View" icon in the "Data Analysis" View. To be noted, the PSs did however not make an effort as to explain the outcome of $\tau^{i}_{5.2,SV,PR}$ which may cause a black box of GG in relation hereto, not knowing what power regression actually means only how to perform it.

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[67] Concerning the relation between diameter and length of a specific bone of African antelopes (Clausen, Schomacker, & Tolnø, 2013a, pp. 61–62).

[68] The students have noticeably already been introduced to regression and the r^2 value for assessing data to be fitted for an exponential function earlier during this year of high school.

In order to solve the tasks of type T_{EP_1} and T_{EP_2}, two of the PSs (PS5 and PS6) (working together) spontaneously performed $\tau^{i}_{1,SV,PR}$, respectively a technique of entering different *x*-values in the box for this value in the "Data Analysis" View (see Figure 68) followed by clicking Enter, in order to approach the given y-value in the task of type T_{EP_2} until they found themselves sufficiently close to the given y-value whereby the entered x-value was their answer - this technique will henceforth be denoted $\boldsymbol{\tau^{i}_{2,SV,PS}}$. The techniques $\tau^{i}_{1,SV,PR}$ and $\tau^{i}_{2,SV,PS}$ are noted to be neither analytic nor graphic whereby there is a significant risk of a black box effect here, in the sense that the students neither know *how* nor *why* to solve tasks of the types T_{EP_1} and T_{EP_2}. When trying to solve a task of type T_{EP_2} analytically, these two PSs noticeably came to a standstill when wanting to isolate x in the power function equation. The one of them (PS5) tried to use the logarithm for this, instead of root extraction (see A.8 Module 3). Root extraction had noticeably not yet been presented by TP as an analytic gesture for solving power equations with respect to the base of the power at this point of the course. PS3 (neighbour to PS4 who had AS4 as partner) as well as PS4 developed for these exercises, without instructions from TP, other instrumented techniques for solving tasks of type T_{EP_1} and T_{EP_2} in GG. These were graphic techniques performed within the Algebra Perspective, after having copied the function, derived through $\tau^{i}_{5.2,SV,PR}$, to the Graphics View. Since they are very similar, the one used for the tasks of type T_{EP_1} is only presented here:

$\boldsymbol{\tau^{i}_{1,AP,PS}}$: Click on the icon in the Graphics View Style Bar to add a grid to the Graphics View → Click on the Point Tool → Find via the grid the point of the graph of f where $x = x_0$ (approximately) and click on this spot of the graph to create a point (A) → Check if the first coordinate of A is equal to x_0 in the Algebra View (if not; adjust the point in the Graphics View by dragging it) → Read off the second coordinate of the point from the Algebra View.

See A.8 (Module 3) for $\tau^{i}_{2,AP,PS}$ (developed by PS3 for tasks of type T_{EP_2}) as well as screenshots of the outcomes of $\tau^{i}_{1,AP,PS}$ and $\tau^{i}_{2,AP,PS}$. Noticeably, these techniques do not necessarily lead to a precise answer for the tasks of type T_{EP_1} and T_{EP_2}

(respectively). The placement of the point may namely be a bit imprecise since it can be difficult to drag the point on the graph such that it precisely has a specific first or second coordinate when using the Point Tool. The task of type T_{EP_3} was only reached by PS3 (as well as PS4 and AS4) among the PSs of this session. Here the following technique ($\tau^{i}_{3,AP,PS}$) was developed spontaneously by PS3 (in dialogue with PS4 and AS4) within the Algebra Perspective and by use of a calculator outside GG. PS3 did however not succeed in performing the last set of (analytic) gestures necessary for solving the task fully (not necessarily instrumented) before the session ended.

$\boldsymbol{\tau^{i}_{3,AP,PS}}$: Having a specific point (x_0, y_0) plotted on the graph of $f(x)$ in the Graphics View, calculate "$x_0/100 \cdot$ (100·r) + x_0" (denoted x_1 here) on a calculator, where $(100 \cdot r)$ is a given value → Zoom in on the x-axis of the Graphics View to find the value x_1 → Click on the Perpendicular Line Tool → Click on the x-axis at x_1 **[**→ Use the Move Graphics View Tool to zoom out and drag the Graphics View in order to see the intersection between the line $x = x_1$ and the graph of f**]** → Click on the Point Tool → Click on both the graph of the function f and the line $x = x_1$ in the Graphics View to create the intersection point (A) → Read off the second coordinate (y_1) of A from the Algebra View → Determine the percentage increase from y_0 to y_1 (**unfinished gesture**).

The gestures marked by **[ ]** are only performed to an extent making it possible to see the intersection point in the Graphics View (see A.8 Module 3 for a screenshot outcome of $\tau^{i}_{3,AP,PS}$). PS3 was not sure how to determine the percentage increase from y_0 to y_1 analytically. AS4 had the bid of 32 %, but when PS3 tried to write down the required technique in Word in dialogue with AS4, s/he only managed to write down a calculation checking if AS4's bid was right and not to write down a calculation leading to 32 % as a result (see A.8 Module 3 for PS3's notes in Word). There are noticeably many similarities between $\tau^{i}_{3,AP,PS}$ and $\tau^{i}_{3,AP,PR}$. Actually, the last set of gestures of $\tau^{i}_{3,AP,PR}$ can be applied as the "unfinished gesture" of $\tau^{i}_{3,AP,PS}$.

An endnote for this session is that AS4 did not perform any gestures or instrumented techniques in GG but merely assisted and observed PS_4's work on PS_4's computer, using a hand-held calculator.

Now, the only recap, related to GG organised as a second level orchestration by TP of the course, was performed in the subsequent module (4) and concerned the techniques applied for solving the above mentioned exercises of the exercise book. The recap was orchestrated by TP in a way that can be associated to the sherpa-student situation presented earlier (section 2.1.3) since TP let some of the students present their techniques for the tasks of the types T_{EP_5}, T_{EP_1} and T_{EP_2}. The chosen student namely got to use the keyboard and mouse connected to the computer of TP, the screen of which was projected for the whole class to see, whereby it was possible both to share techniques in relation to the given tasks and to get TP's guidance or approval of these. TP asked the chosen student to put into words how s/he performed the gestures in GG. This may have given the other students and TP a better chance for keeping up with the performed technique. Figure 79 illustrates the situation.

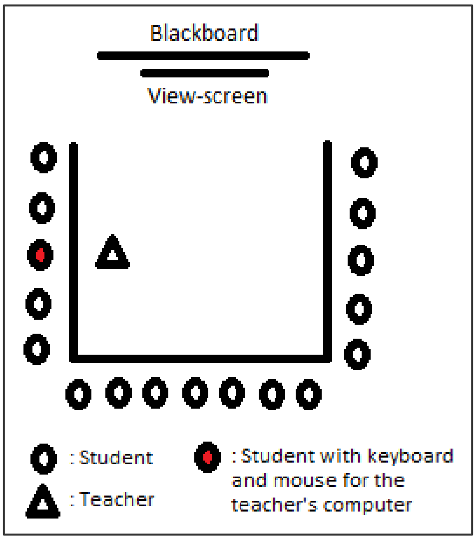

Figure 79 The situation where one (red) student at a time (not necessarily everyone) gets to share his/her techniques in relation to a task concerning regression/modelling of a power function. The teacher is in a dialogue with the students - and especially the chosen student - on what is happening on the view screen.

One student presented how s/he had solved the first task (of type T_{EP_5}) in the exercises through $\tau^{i}_{5.2,SV,PR}$ in GG and put into words which gestures s/he performed leading to a correct outcome. This may have worked as a repetition, of the second level orchestration on this technique made by TP in the previous module, for the other students. Later in the session, TP however encouraged a student to use another program than GG, named WordMat[69], for solving a task of type T_{EP_5}. TP did hereby not propose GG to be a mandatory CAS tool for solving tasks of type T_{EP_5}. When another

[69] WordMat is an add-on to (Microsoft) Word and can be downloaded via this link: http://www.eduap.com/wordmat/ (last visited 29.07.2016).

student at some point got to present how s/he had solved a task of type T_{EP_1}, s/he presented the graphic technique $\tau^i_{1,AP,PR}$ which had not been presented before by either TP or another student in plenum (i.e. it was a spontaneously developed technique). Noticeably, $\tau^i_{1,AP,PR}$ for certain gives a precise answer for the task of type T_{EP_1} as opposed to $\tau^i_{1,AP,PS}$ performed by PS_3 in module 3. To be noted, $\tau^i_{1,AP,PR}$ was not immediately the technique intended by TP who did not seem to have thought of the possibility of performing this set of gestures in GG to solve a task of type T_{EP_1}. TP namely ended this part of the session by noting that one could also have put the given $x = x_0$ into the functional equation and calculated this, but acknowledged that $\tau^i_{1,AP,PR}$ works as well. Hence, it seems here that TP has expected a technique of more analytic character - where the techniques $\tau^i_{1.1,CV,PR}$ or $\tau^i_{1.2,CV,PR}$ for instance are applicable in GG.

The CAS View of GG is however used by a third student, when presenting how to solve a task of type T_{EP_2}. Here, the technique was close to be of the type $\tau^i_{2.1,CV,PR}$, only with the use of the Solve Tool instead of the Solve Numerically Tool. The output was not as expected and can be related to the output presented in Figure 63. When neither TP nor the students were able to see what has gone wrong in the performed technique, they agreed on the chosen student to perform the technique $\tau^i_{2,AP,PR}$ (developed here in plenum) instead. This choice seems to be a reaction to the just presented $\tau^i_{1,AP,PR}$. The CAS View was noticeably not used henceforth in this session for performing (symbolic) calculations.

At some point, TP became the one to control mouse and keyboard for some of the exercises with input from the students, starting with the exercise of plotting a specific data set along with the graph of the power function derived through regression. The regression was here performed in WordMat instead of GG. TP noted however thereafter that it is better to use GG than WordMat for plotting the graph and opened GG again. To be noted, TP verbalised here, among other, that one has to use dots instead of commas when typing decimal numbers in GG; a first level orchestration, confirming what some of the students may already had concluded (at least the PS_3-PS_6). Within this part of the session, TP further asked how to solve another task of the type T_{EP_1}. A student suggested here to put the given value of *x* into the functional equation and calculate it but TP then proposed and tried to perform the graphic technique $\tau^{i}_{1,AP,PR}$ instead, in dialogue with the students. Noticeably, TP hereby learned to perform the technique in GG simultaneously with introducing it to the students as a repetition of the presentation made by the first mentioned student earlier in the session. When TP here at some point got stuck in getting an intersection point as outcome, a student noted that TP had made the point invisible and TP then found out how to make the point visible in order to answer the task. TP did however not verbalise this gesture.

The only task of type T_{EP_3} was, in the end of this recap, noticeably not solved in GG but through an analytic technique proposed by a student, where TP performed it on the blackboard with input from the student, who had calculated the results on a hand-held calculator. Spontaneously developed techniques in relation GG for this task, like $\tau^{i}_{3,AP,PS}$, was hereby not shared.

It should be noted that in general, the CAS View of GG was not used by any of the PSs of this observation study, why it is only the cases where it was presented on the projector, orchestrated by TP, which work as data on how this aspect of GG was approached. The only other situation of the course, where this was the case, was in module 5, where TP actually made an attempt as to introduce some instrumented techniques in the CAS View of GG for solving equations (a second level orchestration). This took place as the first thing immediately after a break, where the students had worked with a GG document (see A.7 Module 5) concerning the covariation property, $(\cdot/\cdot)$, of power functions before the break. TP introduced here briefly, through a couple of examples ($2x = 4$ and $3x + 3 = 5$), the gestures of clicking either [=] or [≈] in the CAS View Toolbar and the difference of the output gained through these two gestures. After this orchestration, TP said that it was time for proving the property $(\cdot/\cdot)$ and used the blackboard with input from the students. Further, analytic techniques, for some specific examples of tasks (of type T_{EP_3} and the opposite to T_{EP_3}; asking for the percentage change of x given the percentage change of $f(x)$) where the $(\cdot/\cdot)$ property is applicable, were performed by TP on the blackboard with input from the students. Here, the students got the possibility of calculating the results using calculators. TP did however not propose GG's CAS View as calculator and the majority of the students were observed to use handheld calculators here (it was difficult to see all computer screens of the students though). The above mentioned GG-document, concerning the $(\cdot/\cdot)$ property, was made by TP and noticeably it functioned more as a worksheet with an exercise of realising the $(\cdot/\cdot)$ property analytically rather than an exercise of performing instrumented techniques in GG. This, since calculations were made for the user within cells of the Spreadsheet View - of *k*x*, *f(k*x)*, *f(x)*, *f(k*x)/f(x)* and k^a - when typing in specific values for x, a, b and k.

As an endnote for this review, tasks of or close to being of the types T_{EP_4} and T_{EP_6} are not at any point in the course solved in GG itself by neither TP nor the PSs and tasks of type T_{EP_3}; only to the extent achieved by PS3, PS4 and AS4.

7.2.3 Teacher and Student Work in Relation to GG Worksheets

The GG worksheets chosen by TP[70] will here be reviewed for their involvement in connection with a significant part of the content of the theme - they are denoted Worksheet 1-8 (see Table 3).

Worksheet 1 of module 2 concerns a click proof of Thm_{ab} and is an interesting example of a GG worksheet with an exercise in which a focus is put on relating knowledge and praxis (also presented in Figure 70). TP put an emphasis (verbally) on the fact that the students should try for themselves (in groups of two) before clicking for hints, and that the clicking for right answers should only take place if all progress in performing the exercises stalled. The PSs did however not manage to perform the steps of the proof without clicking their way through it and they seemed in general very impatient in their approach to the exercise - whenever it was the least challenging, they or a partner clicked for a hint and/or a right answer. Further, their answers did not include any explanations and noticeably, the application of a law of exponents (L_5) in section 4.2.1) as well as a logarithmic rule (denoted *b)* in section 4.2.1) were not performed without clicking for hints and right answers (see A.8 Module 2 for notes of PS7 (Word) and PS8 (handwritten)). PS7 and AS7 actually asked TP for help in connection with the logarithmic rule and TP then suggested that they

[70] The GG worksheets are not developed by TP but selected from (Webpage: GG-worksheets, 2016).

should click for a hint and try to apply it after which TP left (seemingly a bit busy in circulating from group to group). But when the hint was not quite enough, for them to understand how to perform the step, they rapidly clicked for a right answer (on AS_7's PC) while PS_7 wrote down the techniques in Word. The recap of this worksheet was made by TP in the subsequent module (3) where different students were asked to perform the steps of the proof on the blackboard after having had a couple of minutes to reconsider the steps of the proof. In those minutes, some of the PSs had a tendency to mark off both hints and right answers in the worksheet impatiently and maybe a bit frantic. TP put, in the recap by the blackboard, a lot of focus on why the techniques could be performed asking for explanations of the student by the blackboard. Moreover, TP tried to defuse and guide the praxis of performing these techniques whenever the student by the blackboard became confused, letting the student try and explain for him-/herself why it was okay to perform the technique.

Worksheet 2 was introduced immediately after, in order for the students to use the formulas of Thm_{ab} for calculating several exercises with tasks strongly related to T_{EP_4} (see A.7 Module 3). Here, TP introduced (using projector) the options of getting the answers confirmed, when they turned green, as well as the options of clicking for help in some boxes (providing right answers for the techniques to be performed). TP ended this introduction by reminding the students to use the formulas which they had just learned and finally noted that a and b were to be calculated by hand or some place on the computer. TP did hereby not propose GG as calculator. The PSs either used a handheld calculator or a basic calculator on their phone or computer for these exercises which however seemed challenging for some of them to use. One of the PSs (PS_9) did not perform the calculations him-/herself but worked together with two neighbours where only one of them (AS_9) performed the calculations. The option of getting hints was used more in the beginning than in the end of this session by PS_9, whereas PS_{10} and PS_{11} (being neighbours) did not click for hints but tried to perform the techniques on paper (see PS_{10}'s notes in A.8 Module 3).

The Worksheet 3 of module 4 involves sliders and concerns the (%/%) property (see Figure 71). It was introduced by TP on the projector in plenum where TP presented the sliders of a, b and the percentage change (not explicitly noted to be of x-values), through dragging the sliders and asking the class: *What do the percent numbers mean?* Here, a student answered that the *x*-value increases by a specific percent. Then TP asked for the meaning of the other percent value of the worksheet, and another student answered, that it tells by how much *y* increases, confirmed by TP. To some extent, this may be considered as both a first and second level orchestration when considering the dragging of sliders a gesture respectively a technique. The students worked together with their neighbours (approx. two and two) and TP circulated and guided the students in answering the tasks when needed. Here, the PSs dragged the sliders in order to answer the tasks of the exercise - they did this orally and some of them tried to explain their answers to each other by dragging the sliders to demonstrate what happens. Besides from the possibility of exploring the (%/%) property dynamically in a graphics view, the students here got the opportunity to reconsider the meaning of *a* for the look of the graph through the last task - which they did successfully. One PS (PS_{12}) and partner (AS_{12}) even spontaneously investigated the meaning of *b* through dragging this slider when having gone through all four tasks of the exercise - confirming AS_{12}'s suggestion that it is the *y*-value for *x* equal to 1 (having forgotten it from module 1). Noticeably, only PS_{12}, and not AS_{12}, dragged the sliders in this worksheet. AS_{12} did not work by a computer but assisted and observed PS_{12}'s praxis. PS_{12} and AS_{12} (as well as PS_{13}; neighbour to PS_{12}) were a bit unsure why it is called "percent-percent" growth (the first question of the worksheet) and asked TP. TP answered by asking for which type of growth is an exponential growth followed up by asking for by what *y* increases when *x* increases by a fixed number (i.e. referring to the covariation $(+/\cdot)$ of the variables). Thereafter, TP asked them to describe linear growth in a similar way, before asking them to describe power growth (referring to the (%/%) property). It was mainly a conversation between TP and PS_{13} after which PS_{13} told AS_{12} and PS_{12} how it should be understood. Hereby, an establishment of a knowledge block was sought through the first question of the worksheet.

The recap of Worksheet 3 was immediately performed by the blackboard by TP without any recap on the praxis of dragging the sliders to be performed in relation to solve the tasks. TP put instead a great emphasis on the difference between linear, exponential and power functions in relation to the covariation of the variables. In prolongation of this recap, TP however wrote the theorem on the (·/·) property of power growth. Then TP asked the students for the factor by which x is multiplied when it increases by a given percent (50 %), presented on the projector in Worksheet 3. In dialogue with the students, TP here deduced that y is multiplied by 1.50^a where a was read directly from Worksheet 3 (on projector showing TP's computer screen). Then the students got to calculate the increase of y (where the majority used handheld calculators) and TP and the students reached an agreement that this increase agrees with what the percentage increase is given to be in the worksheet (on the projector). Hereby, TP proposed the use of Worksheet 3 to confirm the calculations made through analytic techniques on the (·/·)/(%/%) property - which seems to be an attempt of combining graphic and algebraic representations, confirming the (·/·)/(%/%) property of power growth.

For introducing (direct and inverse) proportionality, TP for the first and only time of the course used (several) GG worksheets (Worksheet 4-7) presented on projector to introduce both technological-theoretical elements, examples and the graphic representation of inverse proportionality in dialogue with the students, instead of using the blackboard as usual. TP however summed up on the introduction by writing up equations for direct and inverse proportionality on the blackboard. Thereafter, Worksheet 7 (presented in Figure 72) providing exercises with strong relation to T_{EP_6} (and the like; concerning direct proportionality instead) was introduced to the students on projector by TP, through asking the students for how to solve the given task. Further, TP marked off the boxes for right answers after receiving bids from the students in order to validate the bids. Before clicking on the "show graph" box, TP asked the students what they expected regarding its slope. Thereafter they went through a new task in the worksheet, concerning the other type of proportionality, before the students got to work with it the last few minutes of the module (6).
They got to work with it again in module 7, after TP had made a recap in dialogue with the students, by the blackboard, on which type of functions inverse and direct relations are - reaching the conclusion of them being power functions.
Thereafter some homework exercises in the exercise book (see A.7 Module 6, Opgave 2080 and Opgave 2081) involving tasks of type T_{EP_6} (and the like) were reviewed in

plenum, before the students got to work with Worksheet 7 again. This, after TP had presented it briefly on the projector and shut it off again whereby the blackboard was visible presenting (among other notes) the formulas of direct and inverse proportionality (written earlier in the module). The students hereby got a significant amount of time to solve a bunch of the same types of tasks through the worksheet, working with neighbour(s). This actually lead to the PSs and their partners trying to put into words which (analytic) techniques in general worked for direct respectively inverse relationships in order to fill out the table. It was however very different to which extent, the PSs and their partners reflected on the look of the graph in the sense of trying to predict the look of the graph (before clicking for it) or even talk about the look of the graph (when having clicked for it).
TP summed up by the blackboard by asking the students to tell how the graphs look like and relate it to the equations for direct respectively inverse proportionality where TP drew the graphs and wrote down the equations. It was easier for them to tell that direct proportionality is a linear function with a constant term equal to zero than describing the graph of inverse proportionality to be a *hyperbola*[71] (verbalised by TP). In the end of this session, one student said that the latter looks like an exponential growth with the *x*- and *y*-axes as asymptotes. TP answered that it is not appropriate to describe it as an exponential growth, and after a break TP announced that the students have learned what they should on the theme of Power Functions.

Hereafter, TP introduced Worksheet 8 (see A.7 Module 7) providing the option of seeing (and distinguishing) graphical representations of either linear, exponential or power functions in a random order (one at a time). Here, TP went through an example of the multiple choice exercises of the worksheet in dialogue with the students before letting them work with it. It was very different from PS to PS how patient they seemed to be when answering the exercises of this worksheet. Some of the PSs conversed with neighbours (PS_{14}, AS_{14} and PS_{15}) about the look of the graph before checking off an answer and put into words how the graphs look for different intervals of a. PS_{16} and partner (AS_{16}) more rapidly went through the exercises, when

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[71] The notion of a *hyperbola* is noticeably not a part of C-B.

one of them had a bid on the answer, without talking much about why it was the answer in relation to the graph. It was easiest for PS_{16} to answer for linear functions and the most difficult to answer for power functions. Subsequently, the students got to play a homemade game (by TP) on the three elementary functions (see A.7 Module 7), on their tables in groups of four where TP circulated.

Worksheet 8 was used in the last module (8) as well, where TP started out by letting the students go through a lot of functions in the worksheet for around 10 minutes before recapping very briefly by asking how many of the students have got the three types of functions under control. Only some of the students lifted their hands. For the PSs of this session it was (still) the power function which was the most challenging of the three types of functions, especially when determining the domain of a. Noticeably, one of the PSs (PS_{17}) got help from a neighbour (AS_{17}) for distinguishing exponential and power function graphs, as AS_{17} noted that the graph of the power function was only in the one (pointing at the first quadrant) whereas the graph of an exponential was in the other as well (referring to the second quadrant). Hereby, AS_{17} had deduced something from the visualisation of the graphs of the different functions in order to distinguish the exponential from the power function. This, not necessarily with an awareness of the definition of the domains to be $\mathbb{R}$ and $\mathbb{R}_+$ respectively. Namely, it seemed that it was primarily the look of the graph and its placement in the coordinate system which was used as explanation for the answers made by the PSs in both sessions concerning Worksheet 8.

7.2.4 Teacher and Student Work in Relation to the Take-Home Assignment

TP ended the course by giving a take-home assignment (see A.7 Module 8) to the students to be handed in in groups of three-four students made by TP. The exercises of this assignment consisted of several exercises (the first four) concerning the algebraic representation of the power function where GG was not immediately called for to be applied but where a great focus was put on establishing a knowledge block through tasks of doing proofs and explanations. Thereafter, three exam related exercises were given, all of which consisted of tasks of type T_{EP_5}. Further, tasks of type T_{EP_3} (and the opposite of T_{EP_3}), T_{EP_1} and T_{EP_2} were represented in some of the exercises.

From the group hand-ins it was not clear which (symbolic) calculator the students had chosen to use when calculating answers to the tasks of type T_{EP_1}, T_{EP_2} and T_{EP_3}. Several of the students noticeably chose to use the formulas for a and b from Thm_{ab} to answer the tasks of type T_{EP_5}, instead of performing the required regression on the data set. However, some of the students did regression and used GG successfully for this (seemingly $\tau^{i}_{5.2,SV,PR}$) - a few even added the r^2 value to their answers, without any explanation connected thereto though. The hand-ins were in general characterised by technical answers without explanations of why it is possible to perform the techniques and some did not even write down the gestures necessary for achieving the answers. Hereby the praxeologies clearly lacked involvement of the knowledge block and not all students demonstrated confidence within the praxis block either.

Now, in prolongation of this review of the GG-related teacher and student work throughout the course, an outline of TP's orchestrations as well as the students' approach to GG-related work in terms of praxeologies will be made.

7.2.5 The Teacher's Orchestrations in Relation to GG

In general, TP initiated GG-work in prolongation of having presented related technological-theoretical elements on the blackboard in dialogue with the students. The projector was often used by TP, either to introduce how to approach a GG-document opened on TP's computer, where gestures were executed in interaction with the students, or to present a GG worksheet and how to approach it - mainly first level orchestrations were made here, but also a few second level orchestrations. TP often instructed the students to work in groups of at least two persons when doing GG-related work and did not instruct them to work individually at any time. Nor were the students at any time explicitly instructed to work by a computer each. Under GG-sessions, TP circulated and helped the students individually in approaching GG/GG worksheets and often asked them to put into words why and how they could perform the techniques. Here, first level orchestrations could occur where TP proposed a gesture or two, without solving the task at hand for them. The blackboard was used as a main presentation tool by TP when doing recaps on GG-work, sometimes supported

by projector presentations but not to the same extent as the introductions to GG-work. In the recaps, TP often wrote on the blackboard which analytic techniques and/or answers, the students had or should have gained through the GG-work with an emphasis on explanations for the performed techniques as well as for the outcome in relation to the general properties of power functions, in dialogue with the students. TP did however not to the same extent sum up on the applicable instrumented techniques in GG for answering the exercises and did thereby not perform many second-level orchestrations. The especially prioritised instrumented techniques to be instructed and shared in the course, by TP, were the one for solving regression/modelling tasks of type T_{EP_5} as well as those for solving tasks of type T_{EP_1} and T_{EP_2}. The CAS View of GG was immediately proposed as an optional tool for (symbolic) calculations but TP did not prioritise it to be mandatory and did neither seem confident herein.
As for TP's choice of exercises within GG itself and within GG worksheets, it varied immensely to which extent, the exercises had a focus on praxis and knowledge - however always with a predominant focus on praxis. In general, the transition between the graphic and algebraic representation of a power function and its properties was often represented in these exercises. In several of the worksheets (2, 7 and 8) it is possible to do several types of the same exercise through a click. These worksheets noticeably involve tasks of type T_{EP_4}, T_{EP_6} (and the like) as well as the exercise of distinguishing the three function types linear, exponential and power functions. Regarding the dynamic aspect of using sliders in GG in order to realise specific properties of the power function, this is also represented in the GG related exercises - first, in relation to GG itself (in Module 1) and later, in relation to Worksheet 3. Hereby the praxis related to Worksheet 3 may work as a recap on the praxis related to GG in Module 1. Finally, there were several worksheets offering hints and right answers for the exercise of the worksheet (1, 2 and 7) which TP introduced only to be approached if one really did not manage to perform a gesture or technique in relation to the exercise. It seemed here that TP saw it as a possibility for the student to get some help when TP was helping another student and thereby not directly available. For the take-home group assignment, TP had not given any

exercises which explicitly called for instrumented techniques in GG, but the exam related exercises in the last part of the assignment did however allow techniques like $\tau^{i}_{5.2,SV,PR}$, $\tau^{i}_{1,AP,PR}$ and $\tau^{i}_{2,AP,PR}$ (shared earlier in the course) and even techniques like $\tau^{i}_{3,AP,PS}$ were possible to perform here for the students.

7.2.6 Student Praxeologies in Relation to GG

In general, the PSs were very technique oriented in their approach to GG-related work and when they verbalised anything on their work, it was often descriptions of the techniques rather than explanations of same. It was challenging for some PSs to relate algebra to the graphical representations obtained in GG - especially in the exercises of using sliders in GG to deduce which impact *a* and *b* have on the graphical representation of a power function. When approaching the GG worksheets offering hints and right answers for which techniques to perform, the PSs were in general more concerned with performing the right techniques and could be quite impatient in connection thereto, without considering why it was possible to perform the techniques. Herein lies a great risk of a black box effect of such worksheets. For the worksheets offering several exercises of the same type, there were noticeably some PSs who managed to try and generalise which (analytic) techniques to apply in relation to the involved tasks (especially in connection with Worksheet 7). It should further be noted that it differed greatly whether the PSs were patient in their approach to GG exercises whether in a worksheet or not. As an example, in relation to the tasks of type T_{EP_1}, some PSs performed $\tau^{i}_{1,SV,PR}$ without performing gestures related to the functional equation - neither graphically nor analytically - whereas other PSs developed graphic techniques in GG's Algebra Perspective instead of using analytic techniques. In general, it seemed that TP's interference was necessary in many of the cases of GG related work in order to get the PSs to establish a knowledge block. As an endnote, regarding the optional use of (symbolic) calculators, it seems that the students were most confident in using handheld calculators (those who owned one) and it could lead to challenges for some of those who chose to use the basic calculator of the computer (or phone). The CAS View of GG was however never used by the PSs as (symbolic) calculator, regardless of the challenges faced.

8.0 Discussion

In this section, the student and teacher work in relation to GG within both of the two courses will be discussed (in the sections 8.1 and 8.2) in order to answer **RQ_1-RQ_3**. Finally, a common discussion concerning the methodology of the collected data of the observation study will be made followed by perspectives for further studies (in section 8.3).

8.1 The Trigonometry Course (L)

In the following, a discussion of the analysed empirical data based on the observations made of the Trigonometry course will be presented in order to answer the RQs.

Based on the analysis, the explorative environment set up by TT for the students in their work with GG can be said to both have potentialities and constraints. The potentialities especially lie in the option for the students to develop their own individual gestures and techniques in GG in their own pace. This flexibility in the pace is among other possible through the videos, manuals and click proofs provided for the exercises. Here, *individual developed* gestures/techniques cover both spontaneously developed and adopted instructed gestures/techniques. As for the constraints, the development of individual gestures and techniques may lead to challenges for the students when working together in groups since the gestures and techniques are not necessarily common. This may be difficult to grasp since the tool is new to them. On the other hand, the individual gestures/techniques developed can be an inspiration for the other students when working in groups. Especially, the demonstration of solving a task through different techniques may lead to a broader conceptual mathematical understanding, in the manner that the different techniques may give a perspective on the mathematical concepts involved in the specific task.

One can raise the question whether the explorative environment set up by TT gave that much space for the students to develop gestures and techniques independently from scratch. This, since almost all of the exercises had a video with instructions on gestures and techniques attached and the PSs used these videos to a great extent whereby these students developed the same instrumented techniques. Also the detailed manuals within the exercises can have had an influence on the students' development of the same instrumented techniques. In addition, it is seen from the analysis that the PSs did not seem confident in the established environment by TT, in the sense that they would rather use formulas than the dynamic features of GG. This may be explained through three factors. The first factor is the fact that GG is newly implemented whereby the students have to get used to dealing with the program and its features - especially the dynamic ones. The second factor is that the use of formulas is a familiar way of dealing with tasks from elementary school for the students whereby this method is projected to solving tasks in high school. The third factor is that the students may be very concerned with the final goal in the form of the written examination.

Regarding the work with GG in general, it seems that TT had the best intentions of intertwining the praxis and the knowledge block but from the observed (very short) recaps it does not seem to be succeeded. This, in the sense that the students separated the learned techniques in GG from the learned outside of GG. For instance, when determining the scale factor on the basis of the construction of two similar triangles in GG, the observed PS did not know how to solve the task in GG but did know how to solve it algebraically. Another example was the student approach to interior angles in GG. Even though it was verbalised, it was challenging for the students to transfer the knowledge of the construction of an angle to GG since they just randomly clicked on the legs or points in GG until the desired angle was shown. In connection hereto, the challenges in using GG, especially the dynamic features, which the students undergo may overshadow the potentialities of gaining conceptual mathematical understanding through the praxis.

The separation, between techniques in GG and mathematical knowledge outside of GG, made by the students could maybe have been avoided by TT making more out of the recaps so that examples like the just mentioned became verbalised to everyone in the class. For this, TT could for instance have established something similar to a sherpa-student situation as an orchestration. According to the features of GG and the techniques to be performed in order to solve the given exercises - for instance making the variables interdependent in order to construct a dynamic object - TT could have made use of the projector, like TT did when presenting the restrictions of the CAS View (like entering the degree symbol (°)). This, in order to secure a shared knowledge on how to approach GG as a tool in general. TT probably did not choose to do several of these orchestrations because of the orchestration made through presenting the gestures and techniques to be performed in GG through video and manuals. There is however a significant shortcoming in connection with this orchestration since the students were not patient in their approach to the videos and manuals which may explain their use of formulas and www.google.dk.

The take-home group assignment may have been considered as a major, enclosing recap by TT in the sense that TT hereby got to both see the students' approach to GG and give feedback. Moreover, the framework for the expectations to the students' praxeologies in relation to GG can be said to have been established through this assignment. According to the learning outcome of the students in relation to this assignment it is however not certain that the students learn from their mistakes when they have a free choice of watching the feedback videos. The aspect of sharing the videos and thereby the techniques performed by all of the students in a common YouTube folder could have contributed to a common knowledge hereto. This was however not exploited since this was not explicitly articulated to be a part of the handling of the videos in the folder by TT.

Regarding TT's focus on the establishment of a knowledge block in relation to GG related exercises, this may be due to the lack of a textbook and the associated focus on the students taking their own notes. This has its strengths in the sense that the students are motivated to reflect upon their mathematical praxis. However, shortcomings can be found in such an orchestration since the students not necessarily write down the right explanations using the right terms - like the case concerning the median presented in the analysis as well as the copy paste procedure to the click proof in a worksheet. Hereby, they may take imprecise and maybe not fully understood notes which work as a reference henceforth if not accessed critically by the student, fellow students, or TT.

According to TT's focus mentioned just above, it is probably a very deliberate choice of TT not to involve a triangle calculator, like the one in the worksheet presented in Figure 51, in the sense that the students get to learn the features of GG and on this basis reflect upon the praxis in order to deduce something in general and gain a conceptual mathematical understanding.

In relation to the fact that this course was not observed until its end, the students' management of the transition between the synthetic and the analytic geometry when considering cosine and sine with respect to an angle in the unit circle context, through constructions in GG and through worksheets, can be discussed. From the analysis, it is seen that a student is fully capable of considering cosine and sine with respect to an angle as the coordinates of the directional point in the unit circle in module 10. The challenges faced by the observed students in general in relation hereto throughout the course may therefore be explained by the course being an initial phase of the Trigonometry course. GG's visualisation properties and dynamic features actually seem to have great potentialities in relation to both considering cosine and sine as being coordinates to a point as well as the transition between the two contexts (see A.6.2).

Regarding the observation of the 10 modules, this part of the whole Trigonometry course is not greatly affected by the written exam as being a final goal (see Table 2). But as seen from the assignment of the theme part 3 ("3.del Vilkårlige trekanter. Opgaveløsning" in A.5), this merely concerns how to solve exercises and tasks in relation to arbitrary triangles which very well could be related to the types of exam tasks T_{ET_1} - T_{ET_5} since they are concerned with arbitrary triangles. This can however only be speculated since this part of the assignment of the theme was not observed.

8.2 The Power Functions Course (A)

Within this section, the discussion of the RQs in relation to the Power Functions course will be reviewed. More specifically, the student work in relation to GG will first be discussed (in section 8.2.1) followed by a discussion of TP's work in relation thereto (in section 8.2.2). The student work will especially be discussed in terms of the mathematical praxeologies established and which tasks they spontaneously solved using GG. TP's work is to be discussed in terms of the tasks instructed in relation GG as well as the orchestrations performed concerning both instrumented techniques and gestures in relation to GG.

8.2.1 Student Work in Relation to GG

Regarding the mathematical praxeologies established by the observed students in relation to GG, it is noted that one thing is to know *how* and another thing is to know *why*. The very technique oriented approach of the PSs to the GG related exercises can be put in relation to this problematique. The aspect of knowing why, herein explaining the techniques and/or generalising the outcome of the praxis, seems in general difficult for the PSs to do without TP's intervention and guidance. This approach may be due to the goal of passing the written examination for the students, since the main part of this examination is to be solved with aids (where GG is an optional tool). More specifically, the priority for the PSs may be just to learn to perform the instrumented techniques in order to solve and answer the tasks for the examination whereas the establishment of a knowledge block is not prioritised to the same extent. It may however also be due to GG being considered as a tool for performing techniques and only techniques - i.e. not as a tool for establishing a knowledge block. For instance, regarding the realisation of the meaning of the value *b* in connection with Worksheet 3, this noticeably requires the students to be curious towards the praxis of the worksheet when dragging the slider for *b* since it is not a part of the exercises to answer this question. Both such curiosity and realisations are however also noted to be greatly influenced by how TP has orchestrated the GG work which will be discussed in section 8.2.2 below.
In connection with this problematique, a question could be raised in relation to the case that the PSs did not prioritise explaining the techniques but rather put a focus on describing the techniques (if even done) - namely whether the students think they *explain* the techniques when merely describing them? However, putting into words *how* to perform the techniques may be an important step towards understanding *why* they can be performed, and for this step, the GG sessions of the course in general seemed quite appropriate.

Now, regarding the example of the rather impatient and uncritical attitude of PS_1 towards performing the commands in GG related to the technique $\tau^{i}_{Sliders\ ab,AP,TP}$ as well as the subsequent exercises, this attitude may have been developed to be a bit more critical for future situations in GG when PS_1 realises the mistake in the typing of the "^" symbol. For students like PS_1, it may in general take some time for them to be confident enough in accessing GG as a CAS tool and thereby become more patient and critical towards the instrumented techniques to be performed.

As for the case of the spontaneously developed techniques (by some of the students) being of graphic character in GG, this reveal an alternative way of solving tasks of types T_{EP_1} and T_{EP_2} as well as T_{EP_3} rather than performing pure analytic techniques. The students performing such graphical techniques may not be able to solve such tasks successfully when GG is not available as a tool. Therefore, GG may have had a positive influence on such students in the sense that they realise how to solve specific tasks in an alternative independently developed manner which works just as well as an analytic technique in order to solve the tasks at hand. An example, of a student showing more confidence in a graphical approach rather than an analytic approach for solving a task in GG, is PS_3 in relation to $\tau^{i}_{3,AP,PS}$.

A general observation is that it seems more demanding for the PSs to consider the algebraic representation of a power function and its properties than considering the look of the graph. The latter is especially possible to do in the GG related exercises. The question is however whether the students in their work with GG are open minded towards GG being able to contribute to more than presenting the graphical representation of a power function and performing graphic techniques for solving tasks. Are they especially open towards making deductions on the relation to the algebraic representation of a power function as well as performing analytic techniques? This seems in general to be a challenge for the PSs of this study - some however try harder than other (e.g. PS_{14} and PS_{15} compared to PS_{16} in their approach to Worksheet 8).

As a final note, regarding the aspect that some of the students did not confront GG but merely assisted and observed on a neighbours work herein, it may be problematic in regards to the learning and acceptance of instrumented techniques to be performed

in GG in relation to the given tasks. Such students may especially not develop any techniques - or at least not to the same extent as the one performing the instrumented technique. This does not immediately make it easier for such students to become more confident in using GG as a CAS tool.
Now since the teacher's orchestration has a great influence on the approach of the students, this will be discussed in the following.

8.2.2 Teacher Work in Relation to GG

Regarding the work of TP in relation to GG, there are without doubt many good intentions relating to the establishment of praxis blocks as well as knowledge blocks to be detected in the implementation of this throughout the course. TP's recaps on GG related work had a great focus on the knowledge to be learned in connection to the content of the theme but the rationale of GG's implementation in connection thereto may have suffered when TP did not relate the knowledge to the concrete gestures or techniques performed in GG or GG worksheets. TP was more concerned on orchestrations relating praxis and knowledge in the introductions to GG exercises rather than in the recaps. The students may however not have had the maturity in the introductions to GG-work as to understand such relations and thereby understand why they should perform the exact techniques or gestures in GG itself or in a GG worksheet. Therefore, recaps with a focus on this may also be beneficial for the rationale of the techniques or gestures as well as a sharing of such. According to the orchestration of the introduction to the technique $\tau^{i}_{5.2,SV,PR}$, made by TP on projector, TP possibly had a great influence here on the students' approach to such types of tasks henceforth. TP's main focus was here put on nothing more than the instrumented technique to be performed in GG without motivating further explanation of the praxis of performing regression. This may explain the lack of explanations in these kinds of tasks within the student work - even in the take-home group assignment where some students chose to note the r^2 value without explaining what it means. The fact that there is no requirement of assessing the regression to be made in T_{EP_5} tasks may explain TP's choice of orchestration in this situation. This, in the sense that TP is aware of the tasks to be solved in the written exam, in relation to power regression, as a main perspective for the learning outcome of the students -

nothing more, nothing less. The fact that many of the exercises set up by TP in relation to GG or GG worksheets throughout the course has a relation to all of the exam tasks T_{EP_1} - T_{EP_6} (see Table 3) further reveals a prevailing focus on the students learning to solve such tasks.

In relation to GG itself, the sherpa-like situation orchestrated by TP in module 4 had a great potential of the students and TP sharing instrumented techniques in GG in relation to the tasks of type T_{EP_1}, T_{EP_2} and T_{EP_5}. This shows to be a success to the extent of performing graphical techniques for solving the tasks of T_{EP_1} and T_{EP_2} as well as students demonstrating a management of $\tau^{i}_{5.2,SV,PR}$. The CAS View of GG may however have suffered in its rationale as a result of this orchestration, which can be explained by the fact that both TP and the students are quite new users of GG. Because of the latter mentioned aspect, a pursuit of repeating such types of orchestrations could likely have been beneficial for both TP and the students in their development of (common) instrumented techniques in relation to GG. Regarding the CAS View of GG, it may however require some more patience of TP to prioritise this as a (symbolic) calculator after the challenges met in the above mentioned orchestration - and therefore also of the students. The fact that some of the PSs seem confused in having the free choice of (symbolic) calculators may however reveal a need for a mandatory (symbolic) calculator. Now, since GG is accessible for everyone in the class as a CAS tool, it seems to be worth giving it a second chance, both by TP and the students. This could lead to more sharing of instrumented techniques in relation hereto which may be beneficial for both TP and the students as to enlarging their confidence herein. An important note to be made in prolongation of this, concerning the goal of teaching the students techniques in GG, is that this goal does not seem prevailing, especially with regards to TP not putting an effort in securing that the students work by a computer each when doing GG related exercises. The choice of not putting an effort in this, may be due to TP not intending to introduce GG as a mandatory CAS tool in general. This intention may further be due to the perspective of giving the students the option of choosing a CAS tool more freely to be the one found most intuitive to approach of the ones available (such as handheld calculators and WordMat, besides from GG). It can however be discussed whether this approach is beneficial or not with regards to the future use and common acceptance of GG as a CAS tool for the mathematics teachings in this class.

8.3 Common Discussion and Perspectives

This section starts out by making a common note on the potentials of the implementation of GG worksheets in mathematics teaching. In spite of the possible shortcomings lying in the worksheets with click proofs being black boxes, it is worth crediting the limited user attention towards specific command frames to be made in GG, in order to create the objects in play within the worksheets. These worksheets therefore allow the user to be more focused on the conceptual mathematical knowledge, connected to the objects and exercises, rather than on which specific command to type or button to press in GG.

Next, the methodology for the collection of the empirical data will be discussed and thereafter, possible perspectives on further studies are presented.

The methodology for the collection of the empirical data

The analyses and discussions on the praxeologies established in relation to GG of the students have been made with reservations to the fact that it was a small group of students that was observed with a maximum of three PSs per researcher. Hereby, the tool GG may have potentialities with respect to the praxeologies established by the students which have not been detected through this study. Further, an observation on more students could have shed light on to which extent, gestures and techniques in relation to GG that were managed by the students in the class.

The reason why only a small group of students was selected for observation was the need to take field notes on the student work - especially in order to keep track on their articulations and gesticulations (e.g. pointing on the screen) in relation to their work and on whom was performing which praxeologies in the group work. The question remains however whether the observed student approach to GG and the computer in general was as natural as wished for, because of the possible acknowledgement of an observing researcher sitting close to them. In relation to this *natural approach* the question is whether the observed students would have initiated the observed gestures and techniques to the same extent if not observed (greater/lesser extent). In relation to this question, the continuous interruption of the researcher requesting the PSs to save and start up a new screen recording every tenth minute may have had a negative impact on the natural use of - and approach to GG. This aspect was however difficult to optimise because of the restrictions in the write speed of the USB plugs and the lack of access to USB plugs with greater writing speed.

Perspectives for further studies

Regarding the very technique oriented approach to GG work of the students, it could be interesting, for further studies, to observe classes where GG is not as new a tool to the students and to the teachers as in the case of this observation study. Hereby, it could be possible to realise whether the students were able to be more keen on establishing a knowledge block in their praxis with GG without the disturbance of just getting to know its features. Also, a more confident approach to GG performed by the teacher would possibly have a great influence on the students' management of and approach to GG.

Furthermore, it could be interesting to observe on teaching in the same theme(s) with GG as a CAS tool on another high school(s) with a different teacher(s). This, in order to gain a perspective on the praxeologies possible to develop for the students as well as the orchestrations possible to perform by the teacher in relation to GG within the specific theme(s).

According to the potentials of GG as a CAS tool in relation to mathematics teaching, it could also be interesting to compare the approach to this tool with the approach to another CAS tool on computer (also being newly implemented), for instance TI-Nspire or Maple[72], at the same level of high school. This, in order to get a perspective on mathematical praxeologies possible for the students to establish in relation to a CAS tool as well as on the teacher's exploitation modes of such a tool in the mathematics teaching.

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[72] For further elaboration and examination hereof, the following links (last visited 21.07.2016) can be pursued: https://education.ti.com/da/danmark/products/ti-nspire-produkt-familien/ti-nspire-cas-elevsoftware/tabs/overview and http://www.maplesoft.com/products/maple/students/.

9.0 Conclusion

From this observation study, it becomes evident that the implementation of a CAS tool like GG in the mathematics teaching of high school may unfold in very different ways. Such an implementation is apparently very dependent of the mathematical theme as well as on the specific teacher's orchestrations thereof. It is however also very dependent on the students' approach to such a tool and especially their openness towards the option that the establishment of the praxeologies, in connection with the tool, can be more than that of a practical block.
The conclusions presented are noted to be based on what has been discovered both a priori and through the observations of the teachings.

Through this study, GG has been noticed to be a versatile CAS tool in the sense that it consists of different Views where significant features, in addition to the classical CAS properties, are the visualisation property as well as the dynamic aspect. These features are considered beneficial in order to observe several examples of the mathematical object to be examined and in order to deduce something general thereof in an explorative manner. Moreover, it is possible to see different representations of the same mathematical object through the different Views selected in a GG document. Furthermore, several types of techniques are possible to perform in GG in order to solve the same task.
Regarding the theme of Trigonometry, GG is especially suitable for construction work of triangles and of the unit circle as well as for accessing the objects dynamically within the Algebra Perspective. For the theme of Power Functions, GG is especially appropriate for providing a graphical representation (with possible dynamic aspects through sliders). Furthermore, the Spreadsheet View is applicable for performing power regression. The CAS View of GG has certain restrictions when performing (symbolic) calculations in relation to both themes.
The aspect of GG worksheets, providing prefabricated exercises involving visual and dynamic constructions, hints and/or right answers concerning algebraic/analytic techniques and/or results, has both its strengths and shortcomings. A definite strength is the presentation of exercises where the objects to be considered are

constructed beforehand. This may namely move the user's focus from the gestures required to construct the objects to the conceptual mathematical knowledge connected to the objects and exercises within the accessed worksheet. A striking shortcoming in relation to the student work with the worksheets has been observed in relation to those providing click proofs, since they may function as black boxes rather than tools for creating a connection between the praxis block and the knowledge block.
A more general shortcoming concerns the very technique oriented approach of the students in GG related work where the students may not be able to relate the outcome of GG (or GG worksheets) to mathematical knowledge in general (i.e. establishing a knowledge block).

An important note to be made in connection with the GG related work observed in the two courses is that TP prioritised the implementation of worksheets to a greater extent than TT. In the Trigonometry course, the instructed exercises for GG related work revealed a greater focus on construction work (and gestures related hereto) and the establishment of a knowledge block in relation to the specific praxis in GG, than in the Power Functions course.
For the observed orchestrations, both of the teachers made use of projector, however in different ways. TP prioritised to demonstrate gestures and techniques in plenum as well as organising a sherpa like situation whereas TT did not prioritise such demonstrations to the same extent. TT however implemented videos and manuals for the GG related work which TP did not. TT primarily used the projector for introducing the exercises from TTW. Both of the teachers introduced the CAS View of GG in a very similar way and this was actually the only situation in which TT demonstrated gestures and techniques on the projector. The CAS View was noticeably not considered a mandatory classical CAS tool in either of the two courses and was briefly prioritised by the teachers.
The recaps of GG related work performed by the teacher were very different in the two courses with respect to the articulation of the praxis performed in relation to GG. TT explicitly did this whereas TP did not.

Another significant difference between the orchestrations performed within the two courses was the design of the take-home group assignment given to the students. The assignment of the Trigonometry course focused solely on the approach to the construction properties and dynamic features of GG as well as the application of these in order to make explanations and deductions. In the assignment of the Power Functions course, no specific references to the application of GG were made but it involved exam exercises for which GG was possible to apply.

Regarding the overall observation on the relation of the GG related work and the exercises and tasks of the written examination in the two courses, this was found to be curiously different. The GG related work in the Trigonometry course did not seem to be focused on the written examination. TT primarily instructed exercises through TTW where there is a significant focus on construction work and dynamic features of GG in relation to right and similar triangles. Moreover, it seems that the focus was put on getting to know how to approach GG rather than getting to know how to solve exam related tasks. However, such management of GG can be considered a foundation for being able to manage the exercises of the written examination. In contrast to these observations, the GG related work of the Power Functions course can be put in relation to the detected types of exam tasks T_{EP_1} - T_{EP_6} to a significant extent.
Overall, the aspect of the students' technique oriented approach to GG calls for added attention of the teacher towards the establishment of a knowledge block in order to connect the praxis performed in GG to the conceptual mathematical knowledge. It is certainly a demanding job for the teacher to implement such a CAS tool without it solely being directed as a tool for the written examination.

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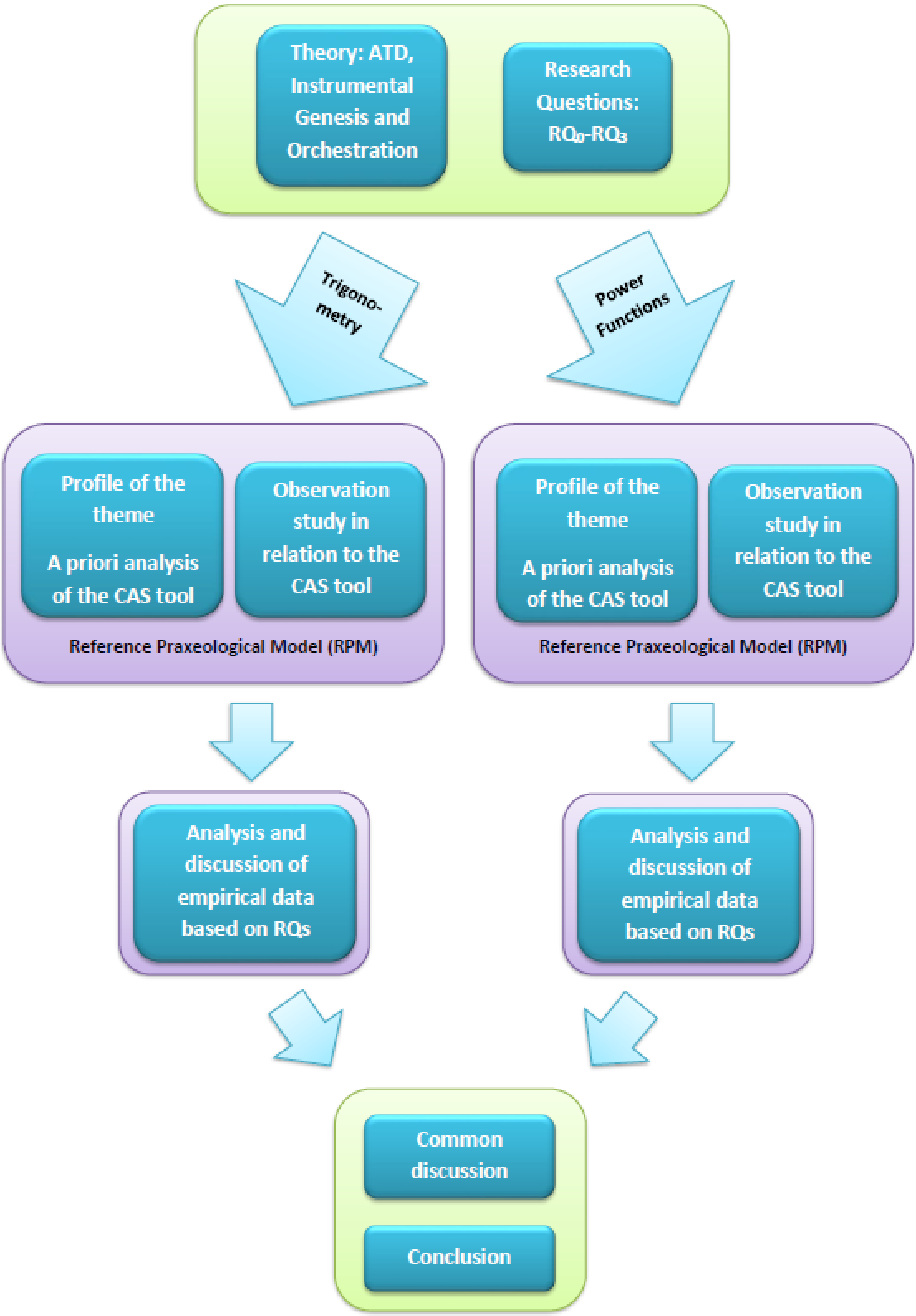
Appendix

A.1 Structure of the Thesis



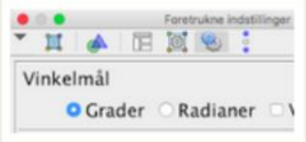
A.2 Teaching material from TTW

TTW 1:

4.2.2: Undersøg definitionen af cosinus og sinus i enhedscirklen

Her skal du arbejde videre i dit GeoGebra-dokument fra opgave 4.2.1: Konstruktion af enhedscirklen.

1. Tjek at GeoGebra regner i grader. Vælg Indstillinger/Avanceret.



2. I input-feltet indtastes værdierne Px=cos(v) og Py=sin(v).
   Begge værdier trækkes fra algebra vinduet ind på tegneblokken.
3. Træk i retningspunktet P. Hvad er sammenhængen mellem Px, Py og koordinatsættet til punktet P?
4. Cosinus er en funktion af vinklen v. Giv et bud på hvordan cosinus er defineret ud fra enhedscirklen.
5. Sinus er en funktion af vinklen v. Giv et bud på hvordan sinus er defineret ud fra enhedscirklen.
6. Træk i retningspunktet P og svar på nedenstående spørgsmål:
   a. Hvad er cos(45)?
   b. Hvad er sin(75)?
   c. Hvad er cos(90)?
   d. Hvad er sin(180)?
   e. Hvad er cos(210)?
   f. Hvad er sin(285)?
   g. Løs ligningen cos(v)=1.
   h. Løs ligningen sin(v)=0.

Hvis du har svært ved at svare på spørgsmålene ud fra dit eget GeoGebra-dokument, kan du få hjælp i nedenstående applet.

To be noted, the $t_{UC,AP,TT}$ can be used to solve task 3.

TTW 2:

2.4: Tre sider

Opgave
Konstruer en trekant, hvor a = 9, b = 7 og c = 5 (alle tre sidelængder).
Du må kun benytte de 8 faciliteter til konstruktion af trekanter i GeoGebra.
Vink:

- Et linjestykke med en given længde.
- Cirkel ud fra centrum og radius.
- Skæringsværktøj.

Hvis du har problemer med at løse opgaven, kan du finde hjælp i videoen.

TTW 3:

2.5: To sider og en ikke mellemliggende vinkel

Opgave
Konstruer en trekant, hvor A = 30°, b = 7 og a = 5 (én vinkel og én hosliggende sidelængde og den modstående sidelængde).
PAS PÅ: Der er noget meget specielt ved dette trekanttilfælde - HVAD ?
Du må kun benytte de 8 faciliteter til konstruktion af trekanter i GeoGebra.
Vink:

- Linjestykke med given længde.
- Cirkel ud fra centrum og radius.
- Vinkel med given størrelse.
- Halvlinje.
- Skæringsværktøj.

Hvis du har problemer med at løse opgaven, kan du finde hjælp i videoen.

Trekant defineret ved to sider og en ikke mellemliggende vinkel

Opgaver

2.1: En side og to hosliggende vinkler
2.2: En side og to vinkler
2.3: To sider og en mellemliggende vinkel
2.4: Tre sider
2.5: To sider og en ikke mellemliggende vinkel

TTW 4:

Konstruktion af trekanter

Når en trekant skal konstrueres på papir, så kan det altid klares ved hjælp af en lineal, en vinkelmåler og en passer. Når vi arbejder med GeoGebra benyttes UDELUKKENDE følgende faciliteter:

- Afsætning af punkt
- Linjestykke med given længde ud fra et punkt
- Halvlinje gennem to punkter (første punkt er startpunkt)
- Afsæt en given vinkel i et punkt
- Cirkel med kendt radius afsat i et punkt
- Skæring mellem to objekter
- Vinkel mellem to kendte linjer (eller 3 punkter) - med udgangspunkt i højre vinkelben
- Afstand mellem to punkter - klik på punkterne (ved aflæsning) (eller vis linjestykkets "Navn og værdi" under Egenskaber (højreklik))

TTW 5:

Trigonometri > 1: Højde, median m.m. >

1.2: Undersøg median

1. Undersøg: Hvad er *medianen* til en side i en trekant?
2. Åbn et nyt dokument i GeoGebra og tegn en tilfældig trekant.
3. Afsæt medianen til hver af de tre sider.
   - Hvad observerer du?
   - Træk i trekantens hjørner. Gælder din observation altid?
   - Hvornår ligger skæringspunktet mellem medianerne inde i trekanten og hvornår ligger det udenfor?

I videoen kan du få hjælp til at konstruere medianen i en trekant.

Median i trekant

Noticeably, the technique $\tau^{i}_{Median,AP,TR}$ can be performed in order to answer task 3. in exercise 1.2.

TTW 6:

4.2.1: Konstruktion af enhedscirklen

I nedenstående opgave skal du arbejde med begreberne: origo, enhedscirkel, retningspunkt og retningsvinkel.

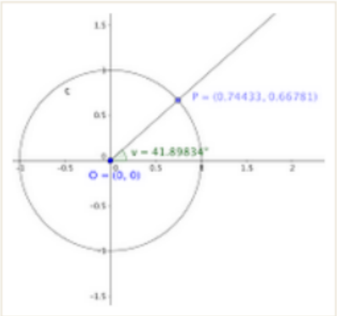
1. Åbn et nyt dokument i GeoGebra.

   I et koordinatsystem kaldes punktet O(0,0) for *Origo*.
2. Skriv O=(0,0) i input-feltet.

   Enhedscirklen er defineret som cirklen med centrum i (0,0) og radius 1.
3. Tegn enhedscirklen ved brug af GeoGebras redskab: "Cirkel ud fra centrum og radius".

   Et punkt på enhedscirklens periferi kaldes for et *retningspunkt*.
4. Ved brug af GeoGebras redskab: "Punkt på objekt" tegnes et punkt på den del af cirkelperiferien som ligger i 1.kvadrant (undgå at lægge punktet på akserne).
5. Omdøb punktet til P og vælg "Vis navn & værdi" for punktet (højreklik og vælg egenskaber).

   Retningsvinklen er den vinkel der ligger i Origo O(0,0), har højre vinkelben på x-aksens positive del og hvor venstre vinkelben skærer enhedscirklen.
6. Ved brug af GeoGebras skæringsværktøj bestemmes cirklens skæringspunkter med x-aksen.
7. Tegn en halvlinje fra Origo og gennem punktet P.
8. Tegn retningsvinklen med GeoGebras vinkel-redskab. Omdøb vinklen til v.
9. Gem GeoGebra-dokumentet - du skal anvende det i opgave 4.2.2: Undersøg definitionen af cosinus og sinus i enhedscirklen.
10. Ud fra din figur skal du nu kunne forklare begreberne: origo, enhedscirkel, retningspunkt og retningsvinkel.



A.3 Screen recordings

For accessing these files, contact Line Sørensen: linesteckhahn@gmai.com or Anne Wellendorf: annewellendorf@gmail.com. To be noted, the screen recordings are in Danish.

A.3.1 Trigonometry course:

Module 1:

USB 1 (1 file)
USB 2 (1 file)
USB 3 (1 file)
USB 7 (1 file)
USB 8 (1 file)

Module 2:

USB Blå (3 files)
USB Gul (3 files)
USB 6 (3 files)
USB 9 (3 files)

Module 3:

USB Blå (4 files)
USB Gul (5 files)
USB 6 (5 files)
USB 9 (3 files)

Module 4:

USB Gul (4 files)
USB 6 (4 files)
USB 7 (4 files)
USB 9 (4 files)

Module 5:

USB Blå (3 files)
USB Gul (3 files)
USB 4 (2 files)
USB 6 (3 files)
USB 7 (2 files)
USB 9 (1 file)

Module 6:

USB Gul (5 files)
USB 6 (5 files)
USB 7 (5 files)

Module 7:

No files

Module 8:

USB Gul (6 files)
USB 6 (4 files)
USB 7 (5 files)

Module 9:

USB Blå (4 files)
USB Gul (3 files)
USB 5 (3 files)
USB 6 (3 files)
USB 7 (3 files)

Module 10:

USB 5 (6 files)
USB 6 (4 files)

A.3.2 Power Function Course

Module 1:

GG related exercises:

- USB 6 (PS_1) (6 files)
- USB 9 (PS_2): Screen recordings failed

Module 2:

Worksheet 1:

- USB 6 (PS_7) (3 files)
- USB 7 (PS_8) (3 files)
- USB 9 (3 files)

Module 3:

Worksheet 1 (reconsidered):

- USB 6 (1 file)
- USB Blå (2 files)
- USB 9 (1 file)

Worksheet 2:

- USB 6 (PS_9) (2 files)
- USB Blå (PS_{10}) (2 files)
- USB 9 (PS_{11}) (neighbor to PS_{10}) (2 files)

Regression/modelling exercises from the exercise book:

- USB Gul (PS_3) (3 files)
- USB 5 (PS_4) (3 files)
- USB Blå (PS_5) (3 files)
- USB 9 (PS_6) (2 files)

Module 4:

Worksheet 3:

- USB Blå: Screen recordings failed
- USB 6 (PS_{12}) (1 file)
- USB Gul (PS_{13}) (1 file)

Module 5:

GG document:

- USB 6 (1 file)
- USB 7 (2 files)
- USB 9 (2 files)
- USB Blå (2 files)

Module 6:

Worksheet 7:

No screen recordings collected here.

Module 7:

Worksheet 7:

- USB 6 (2 files)
- USB Gul (1 file)
- USB Blå (2 files)

Worksheet 8:

- USB 6 (PS_{14}) (1 file)
- (USB Gul) (PS_{15}): Neighbor to PS_{14} (No recordings collected)
- USB Blå (PS_{16}) (1 file)

Module 8:

Worksheet 8:

- USB 7 (PS_{17}) (1 file)
- USB 5 (1 file)
- USB Blå (1 file)

A.4 Scheme for the field notes

Modul nr. + tidsrum	Antal udvalgte elever	Underemne	Individuelt el. gruppe [I. el. G.]	"CAS" [Word el. GeoGebra - W el GG]	Lærerarbejde ift. "CAS" [intro, afrunding, orkestrering]	Elevarbejde ift. "CAS" [instr. teknikker, forklaring, verificering, praxeologier] + USB-numre	Andet

X

A.5 Assignment of the theme of Trigonometry

Temaopgave: Trigonometri

Formålet med forløbet er at få redskaber til at løse geometriske problemer. I skal kunne opstille og udforske geometriske modeller og lave forholdsberegninger i ensvinklede trekanter samt trigonometriske beregninger i vilkårlige trekanter.

En temaopgave har samme tema som undervisningsforløbet og dækker forløbets faglige mål. Ved at få opgaven udleveret ved forløbets start, kan I følge med i og sikre jer at I lærer det der kræves. Besvarelsen af opgaven fungerer som en efterbehandling og sammenskrivning af det, der er foregået i timerne. Temaopgaven vil indgå i et eller flere eksamensspørgsmål til mundtlig eksamen. Det er derfor vigtigt at I formulerer svarene med jeres egne ord, så I selv kan forstå og bruge dem senere hen. Af samme grund bør I skrive kildehenvisninger til alle relevante afsnit i besvarelsen.

1.del af temaopgaven skal forberedes individuelt og fremlægges mundtligt i grupper i en skærmvideo. Beviserne i 2.del evalueres i videoproduktioner med en responsgruppe, mens opgaverne i 3.del laves i en individuel, skriftlig aflevering. Det er en fordel i undervisningen at holde øje med hvilke oplysninger I skal indsamle for at kunne besvare temaopgaven.

1.del: Ensvinklede og retvinklede trekanter

Produktkrav: Skærmvideoer i grupper (1,5 times omlagt elevtid **d.18.april**).
2,5 times elevtid hjemmefra som forberedelse til gruppearbejdet: Nå så meget som muligt af temaopgavens 1.del (deadline **d.14.april**).
Skærmvideo i grupper: Bliv enige om hvilke opgaver fra temaopgavens 1.del I ønsker at demonstrere.

• Hver gruppe skal lave én video på *højst 10 minutter*.
• Alle gruppemedlemmer skal have lige lang taletid på videoerne.
• I skal præsentere jer selv, når hver ny person siger noget (det kan være svært at genkende jeres stemmer).
• Videoerne uploades på holdets YouTube kanal

Det er et krav at i anvender GeoGebra dynamisk (med bevægelse) i skærmvideoen; det kan være jeres egne geometriske konstruktioner, eller en applet. I må ikke lave en skærmoptagelse med en færdig tekst, som er skrevet på forhånd.

➢ Beskriv hvad I ser.
➢ Fortæl om I arbejder med hhv. en definition, en sætning eller et bevis.
➢ Forklar hvad der sker.
➢ Husk til sidst at konkludere på opgaven.

Opgaver:

(a) Definér begreber til beskrivelse af trekanter, herunder højde, vinkelhalveringslinje, median,
midtnormal, topvinkler, samt hosliggende- og modstående side.
(b) Gør rede for *ensvinklede trekanter*.
Undervejs skal du inddrage skalafaktoren og nogle sætninger der gælder for ensvinklede
trekanter.
(c) Giv et eksempel på et geometrisk problem som kan løses ved brug af forholdsberegninger i ensvinklede trekanter.
(d) Definér begreber til beskrivelse af *retvinklede trekanter*, herunder hypotenuse, hosliggende og modstående katete.
(e) Opskriv Pythagoras' sætning og angiv i hvilken type trekanter den kan anvendes. Oversæt Pythagoras' sætning fra symbolholdigt til naturligt sprog. Undervejs skal du anvende ordene: hypotenuse, katete, sum og kvadrat.
(f) Bevis Pythagoras' sætning.
(g) Giv et eksempel på et geometrisk problem som kan løses ved brug af Pythagoras' sætning.
(h) Definér begreberne *cosinus*, *sinus* og *tangens.* Undervejs skal du anvende enhedscirklen.
(i) Opskriv sætningen om cosinus, sinus og tangens i en retvinklet trekant ved brug af ordene:
hypotenuse, modstående- og hosliggende katete.
(j) Bevis sætningen om cosinus, sinus og tangens i en retvinklet trekant.
(k) Giv et eksempel på et geometrisk problem som kan løses ved brug af cosinus, sinus eller tangens i en retvinklet trekant.
(l) Angiv de kilder der er anvendt til besvarelsen af temaopgavens 1.del.

2.del: Vilkårlige trekanter. Sætninger og beviser.

Produktkrav: Videoproduktion af bevisfremlæggelser i grupper med responsgrupper. Klassen formulerer i fællesskab nogle fokuspunkter til videoerne, som responsgrupper efterfølgende giver feedback på. To tilfældige videoer vil blive vist for alle i en efterfølgende time.

Bliv enige om hvilke beviser fra temaopgavens 2.del I ønsker at demonstrere. Hver gruppe skal lave én video på *højst 10 minutter*. Alle gruppemedlemmer skal have lige lang taletid på videoerne.
Videoerne uploades på holdets YouTube kanal.

Videoproduktionen er en træning i at gå til mundtlig eksamen:
• Lav relevante forberedelser inden i optager: figurer og formler.
• Mellemregninger skal skrives og forklares mens I optager. Det kan f.eks. gøres ved en tavle.

Disposition for beviset:
1. Introduktion: Hvad siger sætningen – hvad er det, vi skal komme frem til?
2. Indfør notation. Hvordan navngiver du trekantens sider og vinkler? Tegn og skriv på figuren.
3. Argumentationen som udgør selve beviset. Hvilke kendte sætninger anvendes undervejs? Hvordan udføres de matematiske udregninger (med bogstaver)?

4. Afrunding: Hvordan beviser dette den relevante sætning?

Opgaver:
(m)Definér begreberne *spids* og *stump* vinkel. Angiv nogle egenskaber for cosinus- og sinusfunktionerne, som gælder for to vinkler som tilsammen er 180°.
(n) Opskriv sætningen om beregning af arealet i en vilkårlig trekant ved brug af sinus. Beskriv med ord hvilke oplysninger du skal have om trekanten for at kunne anvende denne formel.
(o) Bevis sætningen om beregning af arealet i en vilkårlig trekant ved brug af sinus.
(p) Giv et eksempel på et geometrisk problem som kan løses ved brug af arealformlen.
(q) Opskriv sinusrelationerne. Beskriv med ord hvilke oplysninger du skal have om trekanten for at kunne anvende denne formel.
(r) Bevis sinusrelationerne.
(s) Giv et eksempel på et geometrisk problem som kan løses ved brug af sinusrelationerne.
(t) Opskriv cosinusrelationerne. Beskriv med ord hvilke oplysninger du skal have om trekanten for at kunne anvende denne formel.
(u) Bevis cosinusrelationerne.
(v) Giv et eksempel på et geometrisk problem som kan løses ved brug af cosinusrelationerne.
(w)Angiv de kilder der er anvendt til besvarelsen af temaopgavens 2.del.

3.del: Vilkårlige trekanter. Opgaveløsning.

Produktkrav: Skriftlig, individuel aflevering med facitliste som Google spørgeskema. Der er 4 timers samlet elevtid og deadline er ved lektionens start **d.10.maj** (uge 19).

(x) Løs opgaverne i dokumentet: [redacted]
(y) Angiv de kilder der er anvendt til besvarelsen af temaopgavens 3.del.

Faglige mål

Faglige mål fra læreplanen:

- Håndtere formler, herunder kunne oversætte mellem symbolholdigt og naturligt sprog, og selvstændigt kunne anvende symbolholdigt sprog til at beskrive variabelsammenhænge og til at løse problemer med matematisk indhold.
- Opstille og udforske geometriske modeller og løse geometriske problemer både ved hjælp af et dynamisk geometriprogram og på grundlag af trekantsberegninger samt kunne give en analytisk beskrivelse af geometriske figurer i koordinatsystemer og udnytte dette til at svare på givne teoretiske og praktiske spørgsmål.
- Redegøre for matematiske ræsonnementer og beviser samt deduktive sider ved opbygningen af matematisk teori.

- Anvende CAS-værktøjer til udforskning af og til løsning af givne matematiske problemer.

Kernestof fra læreplanen:

- Ligningsløsning med analytiske og grafiske metoder og med brug af it-værktøjer.
- Forholdsberegninger i ensvinklede trekanter og trigonometriske beregninger i vilkårlige trekanter.
- Ræsonnement og bevisførelse.

Ved forløbets afslutning skal du kunne følgende:

Skriftligt – med formelsamling

- Anvende Pythagoras' læresætning.
- Foretage beregninger i ensvinklede trekanter.
- Opstille enkle formler og ligninger ud fra en sproglig beskrivelse, dvs. anvende symbolholdigt sprog til at beskrive variabelsammenhænge.
- Anvende begreber til beskrivelse af trekanter, herunder højde, vinkelhalveringslinje, median, midtnormal, topvinkler, hypotenuse, katete, samt hosliggende- og modstående side.
- Anvende formler for vilkårlige trekanter, herunder areal og vinkelsum.

Skriftligt – med CAS

- Bestemme funktionsværdien af cosinus, sinus og tangens til en given vinkel, samt bestemme værdien for de inverse funktioner til cosinus, sinus og tangens.
- Anvende sinusrelationer og cosinusrelationer til at bestemme sider og vinkler i vilkårlige trekanter.
- Anvende arealformlen for i en vilkårlig trekant ved brug af sinus (dvs. uden at kende trekantens højde).
- Tegne skitser af situationer med geometriske problemer ved hjælp af et dynamisk geometriprogram (GeoGebra).
- Udforske og løse geometriske problemer ved hjælp af et dynamisk geometriprogram (GeoGebra).

Mundtligt

- Definere cosinus, sinus og tangens ud fra enhedscirklen.
- Bevise Pythagoras' sætning.
- Bevise sætningen om cosinus, sinus og tangens i en retvinklet trekant.
- Bevise arealformlen i vilkårlige trekanter.
- Bevise sinusrelationerne i vilkårlige trekanter.
- Bevise cosinusrelationerne i vilkårlige trekanter.

Årsprøvespørgsmål (1.g)

Geometri.

Gør rede for udvalgte emner fra projekter og temaopgaver i forløbene: Geometri I-II.Undervejs skal du inddrage:

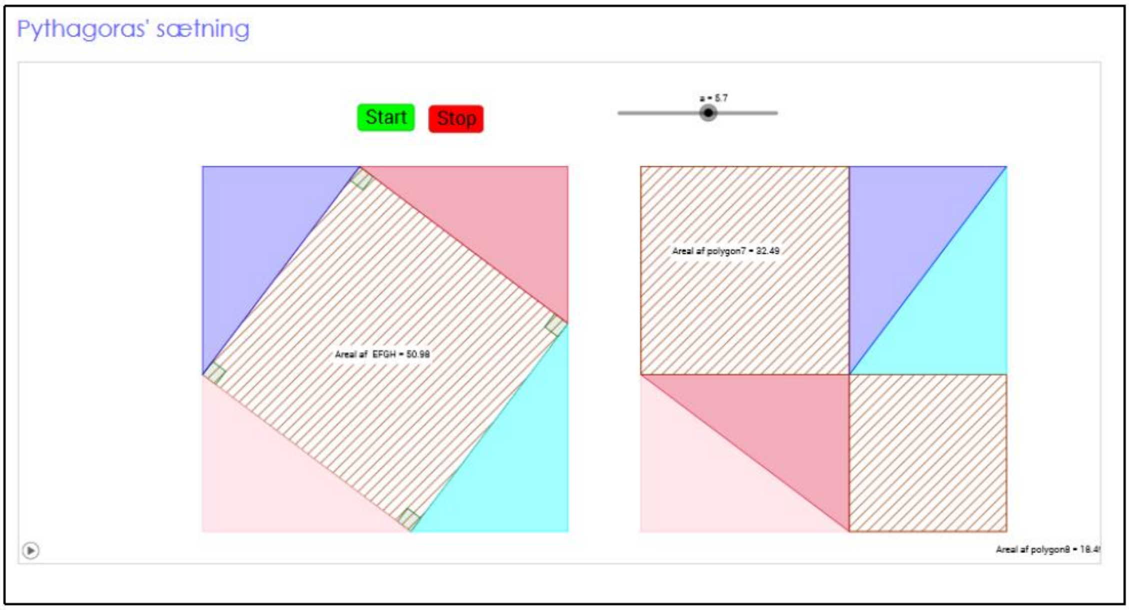
a) Definitionen af cosinus, sinus og tangens ud fra enhedscirklen.
b) Et bevis for sinusrelationerne.
c) Eksempler på anvendelse af sinusrelationerne.

Geometri.
Gør rede for udvalgte emner fra projekter og temaopgaver i forløbene:
Geometri I-II.
Undervejs skal du inddrage:
a) Definitionen af cosinus, sinus og tangens ud fra enhedscirklen.
b) Et bevis for cosinusrelationerne.
c) Eksempler på anvendelse af cosinusrelationerne.

A.6 Trigonometry course. Exercises proposed by TT

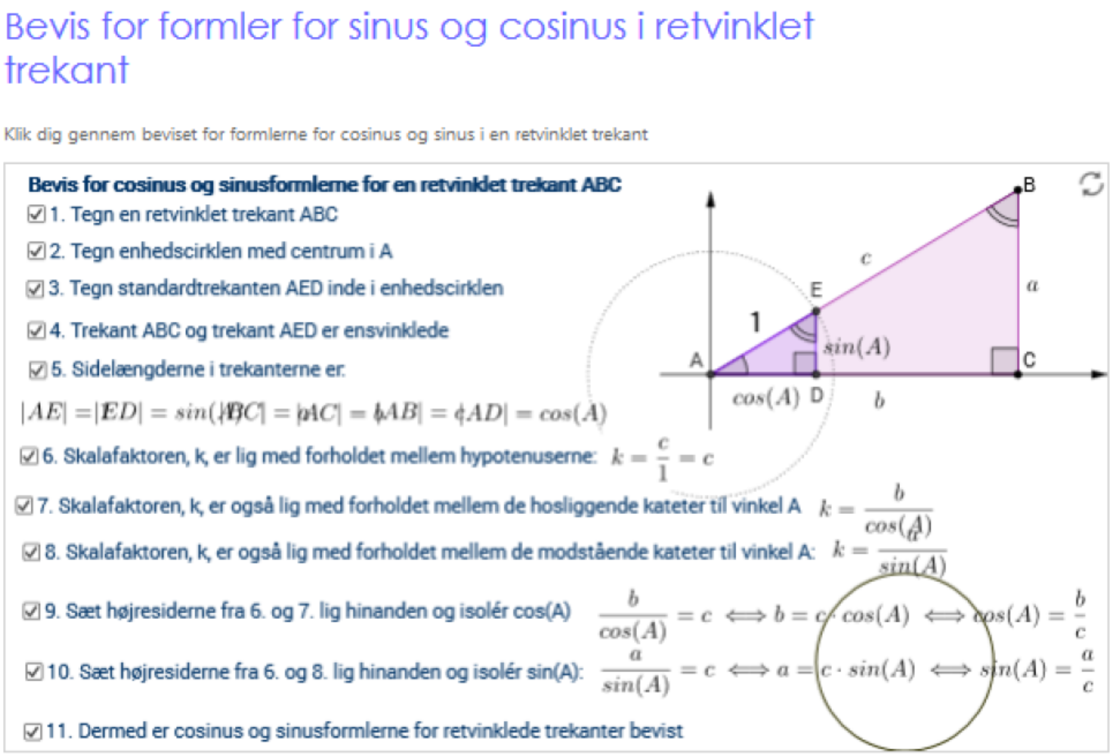
A.6.1

Worksheet concerning the Pythagorean Theorem (dynamically by use of a slider); https://tube.geogebra.org/m/UjZFKMNu:



A.6.2

The proof of cosine and sine in right triangles via GG worksheet (click proof) https://www.geogebra.org/m/238691



A.6.3

Worksheet for the area formula (click proof): https://tube.geogebra.org/m/2297493

Figure 1:

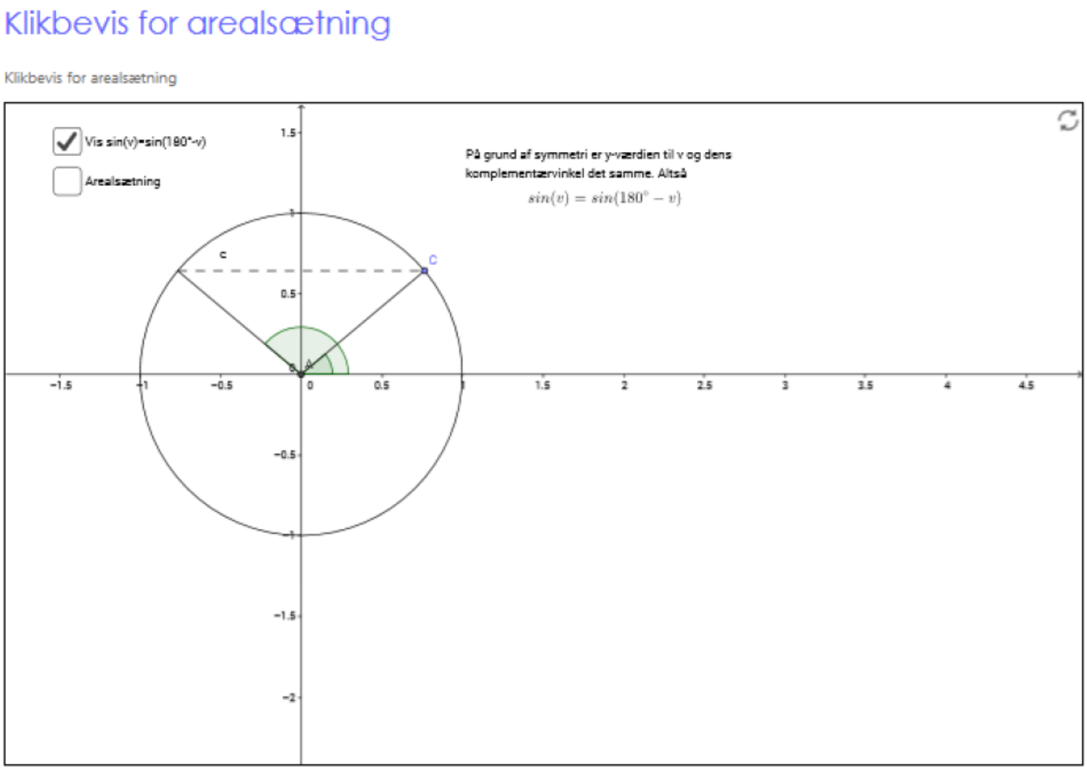
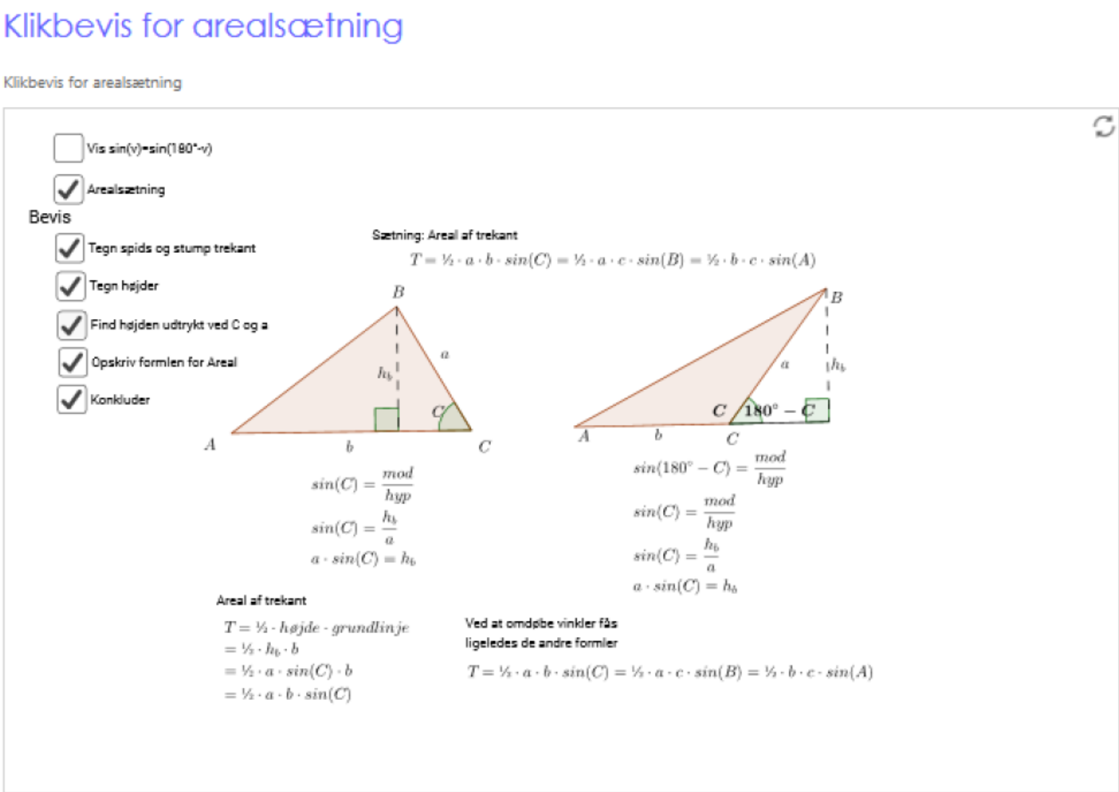


Figure 2:



A.6.4

The exercise 1.1 presenting drawing an arbitrary triangle as a task:

Trigonometri > 1: Højde, median m.m. >

1.1: Undersøg højde

1. Undersøg: Hvad er *højden* til en side i en trekant?
2. Åbn et nyt dokument i GeoGebra og tegn en tilfældig trekant.
3. Afsæt højden til hver af de tre sider.
   - Hvad observerer du?
   - Træk i trekantens hjørner. Gælder din observation altid?
   - Hvornår ligger skæringspunktet mellem højderne inde i trekanten og hvornår ligger det udenfor?

I videoen kan du få hjælp til at konstruere højden i en trekant.

Højde i trekant

Opgaver

1.1: Undersøg højde
1.2: Undersøg median
1.3: Undersøg vinkelhalveringslinje
1.4: Undersøg midtnormal

A.6.5

The exercise $exercise_{RT_1,AP,TT}$:

Øvelse 3.50
I den retvinklede trekant *ABC*, hvor vinkel *C* er ret, gælder der, at $a = 6$, $b = 8$ og $c = 10$.
a) Konstruer trekanten, og mål vinklerne.

A.6.6

The exercise $exercise_{RT_2,AP,TT}$:

Øvelse 3.51
I trekant ABC er $\angle A = 90^\circ$, $\angle B = 63^\circ$ og $b = 7$.
a) Bestem vinkel *C*, og bestem trekantens øvrige sider ved hjælp af formlerne ovenfor.
b) Konstruer trekanten, og mål de resterende sider og den sidste vinkel.

A.6.7

The exercise 1.4 in which the techniques τ^{i}_{PB,AP,PS_1} and τ^{i}_{PB,AP,PS_2} can be performed in order to answer task 3.:

Trigonometri > 1: Højde, median m.m. >

1.4: Undersøg midtnormal

1. Undersøg: Hvad er *midtnormalen* til en side i en trekant?
2. Åbn et nyt dokument i GeoGebra og tegn en tilfældig trekant.
3. Afsæt midtnormalen til hver af de tre sider.
   - Hvad observerer du?
   - Træk i trekantens hjørner. Gælder din observation altid?
   - Hvornår ligger skæringspunktet mellem midtnormalerne inde i trekanten og hvornår ligger det udenfor?

Opgaver

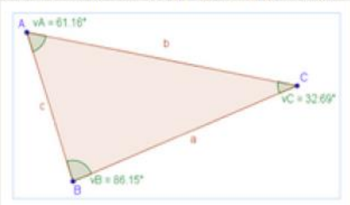
A.6.8

Exercise 3.1 providing a manual for the task $t_{ST_1,AP,TT}$ (in "Opgave 1") and the specific task $t_{ST_2,AP,TT}$ (in "Opgave 2"):

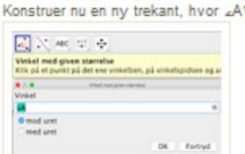
3.1: Undersøg ensvinklede trekanter

Opgave 1

1. Tegn en tilfældig trekant med endepunkterne A, B og C i Geogebra.
2. Marker vinklerne som vist på figuren og giv vinklerne navnene vA, vB og vC.



3. Afsæt tilfældigt to nye punkter A1 og B1 og forbind de to punkter med et linjestykke c1.
4. Konstruer nu en ny trekant, hvor ∠A1 = vA og hvor ∠B1 = vB. Når du laver de nye vinkler, skal du skrive vA henholdsvis vB som vinklens størrelse:



5. Siderne i den nye trekant kaldes a1, b1 og c1.

Du har nu fået konstrueret to *ensvinklede trekanter* (kaldes også ligedannede trekanter). Da to af vinklerne er parvis ens, må den sidste vinkel i de to trekanter jo også være det samme.

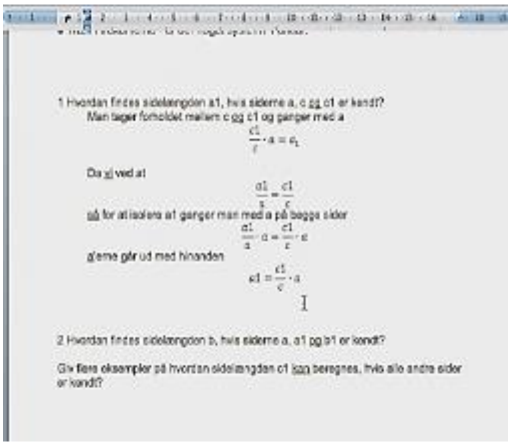
- Hvis du trækker i den første trekants punkter, vil den anden trekant følge med, så vinklerne i den nye trekant forbliver identisk med vinklerne i den første trekant.
- Hvis det ikke virker, så har du konstrueret trekanten forkert. Få hjælp ved at lave opgave 2.1: En side og to hosliggende vinkler

Opgave 2
Forestil dig, at du mangler at beregne en af sidelængderne. Hvordan kan den findes, hvis de øvrige 5 sidelængder er kendte?

Vink:
- Prøv at skrive ka=a1/a i inputfeltet (division af de to sidelængder a1 og a gemmes i ka).
- Træk svaret fra algebravinduet til venstre ind på tegneblokken).
- Definer tilsvarende kb og kc for b- og c-siderne.
- Træk i trekanterne - er der noget system? Forklar.

A.6.9

Notes made by a PS in relation to a task similar to $t_{ST_2,AP,TT}$; How to determine the side length a1, when the sides a, c, and c1 are known?:



1 Hvordan findes sidelængden a1, hvis siderne a, c og c1 er kendt?

Man tager forholdet mellem c og c1 og ganger med a

$$\frac{c1}{c} \cdot a = a_1$$

Da vi ved at

$$\frac{a1}{a} = \frac{c1}{c}$$

så for at isolere a1 ganger man med a på begge sider

$$\frac{a1}{a} \cdot a = \frac{c1}{c} \cdot a$$

a'erne går ud med hinanden

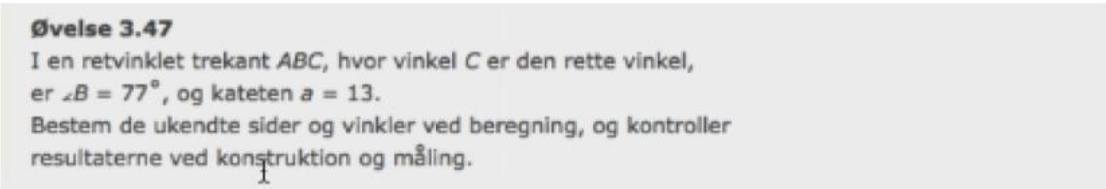
$$a1 = \frac{c1}{c} \cdot a$$

2 Hvordan findes sidelængden b, hvis siderne a, a1 og b1 er kendt?

Giv flere eksempler på hvordan sidelængden c1 kan beregnes, hvis alle andre sider er kendt?

A.6.10

The exercise presented here is $exercise_{RT_3,AP/CV,TT}$:



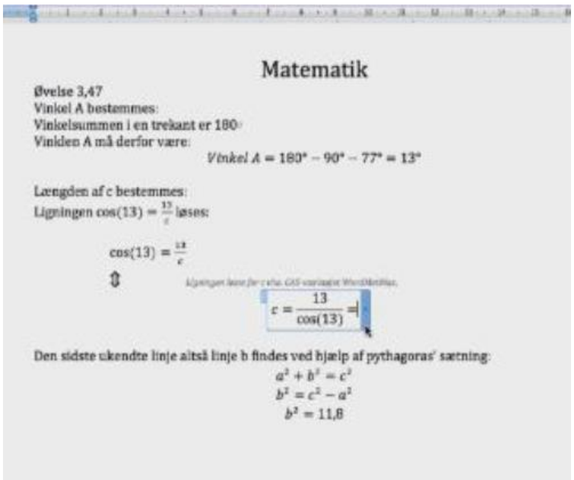
Øvelse 3.47

I en retvinklet trekant *ABC*, hvor vinkel *C* er den rette vinkel, er $\angle B = 77°$, og kateten $a = 13$.

Bestem de ukendte sider og vinkler ved beregning, og kontroller resultaterne ved konstruktion og måling.

A.6.11

Notes made by a PS in relation to “Øvelse 3.47”; $exercise_{RT_3,AP/CV,TT}$:



Matematik

Øvelse 3,47
Vinkel A bestemmes:
Vinkelsummen i en trekant er 180
Vinklen A må derfor være:

$$Vinkel\ A = 180° - 90° - 77° = 13°$$

Længden af c bestemmes:
Ligningen $\cos(13) = \frac{13}{c}$ løses:

$$\cos(13) = \frac{13}{c}$$

⇕

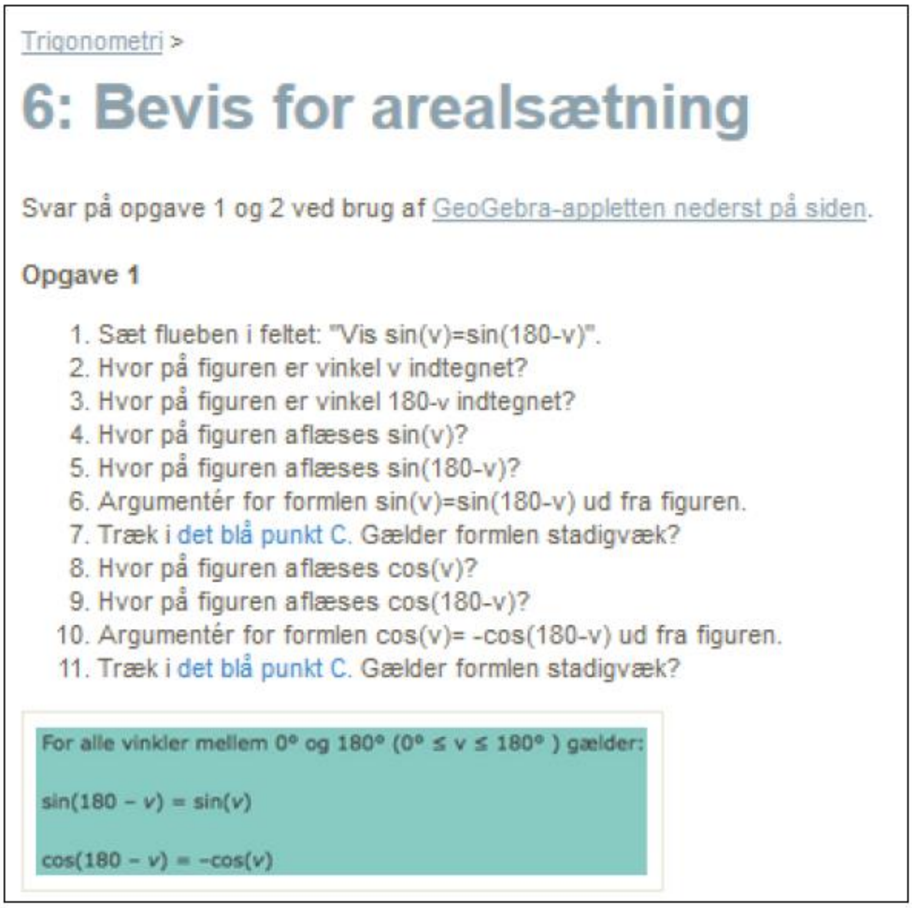
$$c = \frac{13}{\cos(13)}$$

Den sidste ukendte linje altså linje b findes ved hjælp af pythagoras' sætning:

$$a^2 + b^2 = c^2$$
$$b^2 = c^2 - a^2$$
$$b^2 = 11,8$$

A.6.12

Here the task $t_{Cosine\ to\ angle,Worksheet,TT}$ is presented in point 10. Of exercise 6:



Trigonometri >

6: Bevis for arealsætning

Svar på opgave 1 og 2 ved brug af GeoGebra-appletten nederst på siden.

Opgave 1

1. Sæt flueben i feltet: "Vis sin(v)=sin(180-v)".
2. Hvor på figuren er vinkel v indtegnet?
3. Hvor på figuren er vinkel 180-v indtegnet?
4. Hvor på figuren aflæses sin(v)?
5. Hvor på figuren aflæses sin(180-v)?
6. Argumentér for formlen sin(v)=sin(180-v) ud fra figuren.
7. Træk i det blå punkt C. Gælder formlen stadigvæk?
8. Hvor på figuren aflæses cos(v)?
9. Hvor på figuren aflæses cos(180-v)?
10. Argumentér for formlen cos(v)= -cos(180-v) ud fra figuren.
11. Træk i det blå punkt C. Gælder formlen stadigvæk?

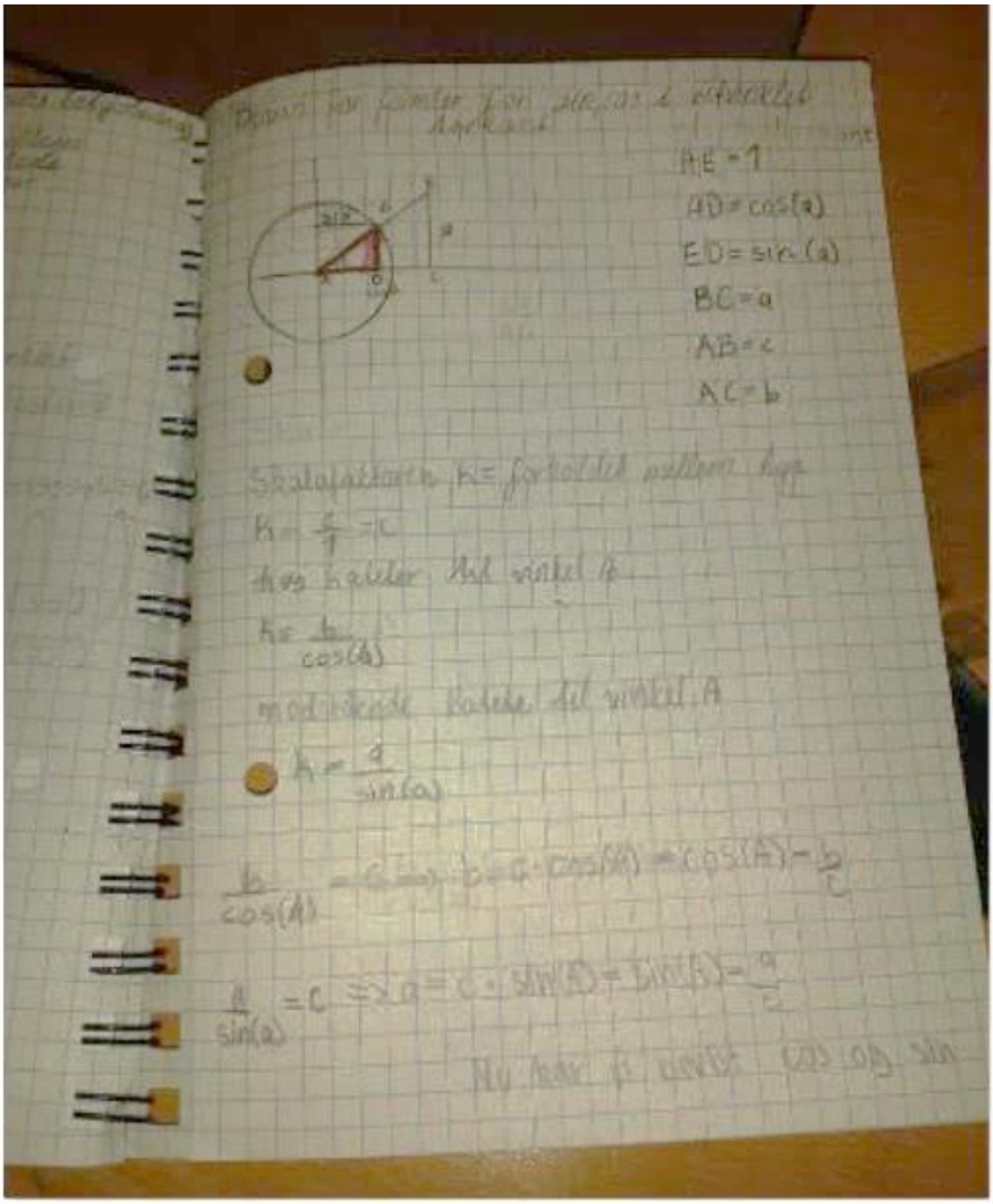
For alle vinkler mellem 0° og 180° (0° ≤ v ≤ 180°) gælder:

$\sin(180 - v) = \sin(v)$

$\cos(180 - v) = -\cos(v)$

A.6.13

Notes of a PS in relation to the worksheet presented in A.6.2:



A.7 Power Functions course. Exercises proposed by TP

Module 1:

UNDERSØGELSE AF POTENSFUNKTIONER VHA. GEOGEBRA

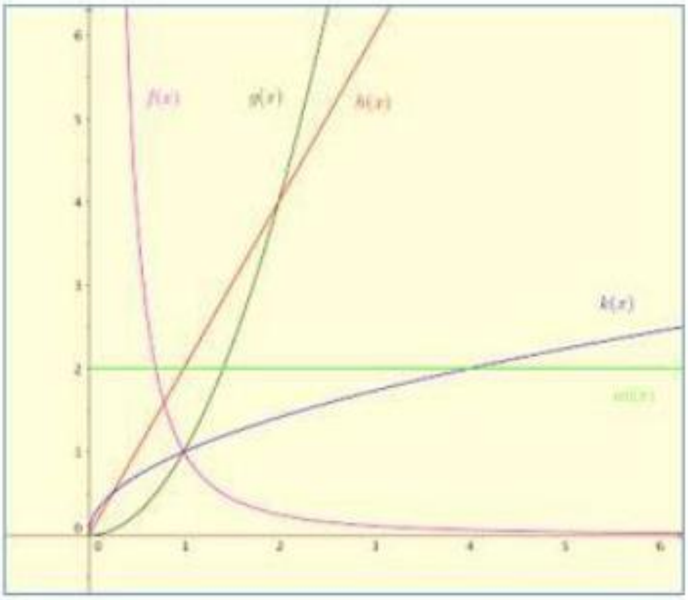
Opret et nyt GeoGebra-dokument.

- Indtast f(x)=b*x^a,x>0 i Input-feltet nederst.
- I dialogboksen vælges: Opret skydere.
- Fra algebra-vinduet skal du trække "f(x)" ind på tegneblokken (graf-vinduet).
- Højreklik på skyderen for b og vælg: egenskaber.

  Sæt intervallet til min: 0.01 og maks: 10.

1. Beskriv hvad der sker når værdien af a ændres.
   - Bestem de intervaller for a-værdien, hvor grafens udseende ændres væsentligt.
2. Indtast punktet P=(1,f(1)) i input-feltet.
   - Hvad sker der med punktet, når værdien af b ændres?
   - Hvilken sammenhæng er der mellem koordinaterne til punktet P og værdien af b?
   - Giv et bevis for sammenhængen mellem koordinaterne til punktet P og værdien af b. I denne opgave skal du anvende et stykke papir - ikke GeoGebra.

     Vink: Sæt x=1 i funktionsforskriften f(x)=b*x^a
3. På nedenstående figur ses grafer for funktionerne f(x), g(x), h(x), k(x) og m(x). Alle fire grafer er fremkommet ved at indsætte værdier for a og b i f(x)=b*x^a.



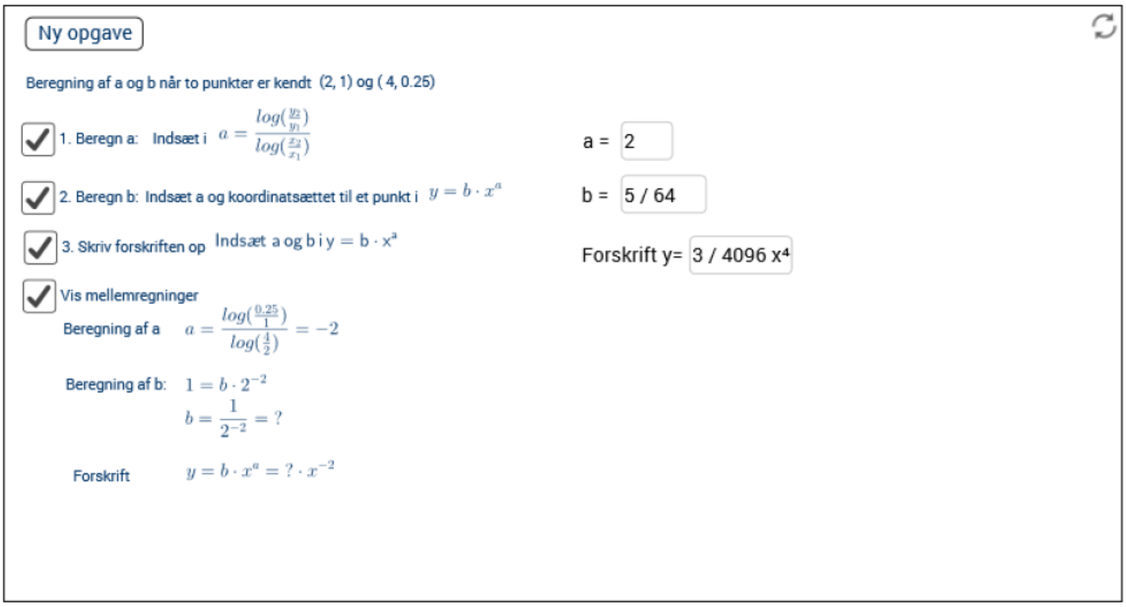
a) Brug Geogebra (med skydere) til at finde kvalificerede gæt på værdien for a og b for hver af de fem potensfunktioner.

Funktion:	f(x)	g(x)	h(x)	k(x)	m(x)
a					
b					

b) Hvad kan du konkludere om hver af de fem potensfunktioner med hensyn til:
- definitionsmængde (anvendte x-værdier)?
- værdimængde (anvendte y-værdier)?
- monotoniforhold (voksende/aftagende/konstant)?
- asymptoter (rette linjer, som grafen "nærmer sig")?

Module 3:

Worksheet 2 with exercises of calculating a and b knowing two points with the possibility of getting to see hints and right answers (intermediate calculations); https://tube.geogebra.org/m/148843:



Regression/modelling exercises from the exercise book (Gyldendals Gymnasiematematik. Arbejdsbog B1)

- **Øvelse 278 (p. 61)**, translated from Danish (including the decimal numbers notation (from using "," to ".")):

  The table presents measures of the time of oscillation *y* seconds for a bullet which is hung up in a cord of *x* metres and it oscillates back and forth. The time of oscillation can be expressed by the formula $y = b \cdot x^a$.

The length of the cord (*x* metres)	0.50	0.75	1.00	1.25	1.50
Time of oscillation (*y* seconds)	1.4	1.7	2.1	2.2	2.5

  a) Determine the numbers *a* and *b* through regression.
  b) What is the time of oscillation if the cord is 2.00 metres?
  c) How long shall the cord be if the time of oscillation is to be 2.0 seconds?

- **Opgave 2046 (p. 84)** translated from Danish:

  The table presents, for a series of windmills, how much power the mill is able to produce. In a model it is expected that the power *P(x)*, measured in kW, can be expressed by the relation $P(x) = b \cdot x^a$, where *x* is the wing diameter, measured in meters.

Wing diameter (m)	20	29	47	86	120
Power (kW)	100	225	660	2500	5000

a) Determine the numbers *a* and *b* through power regression
b) Plot the given points and the graph of the function found in the same coordinate system.
c) Determine the power from a mill with wing diameter of 100 m.
d) What shall be the diameter for the power to be 1800 kW?

- **Opgave 2047** (p. 84) translated from Danish:

For certain trees it is possible to apply a model of the type $m(x) = b \cdot x^a$ for the relation between the diameter of the tree *x* (measured in cm) and the weight of the tree *m(x)* (measured in kg). The table presents some values for this relation.

x (cm)	10	20	30	40	50
m(x) (kg)	16	62	140	250	390

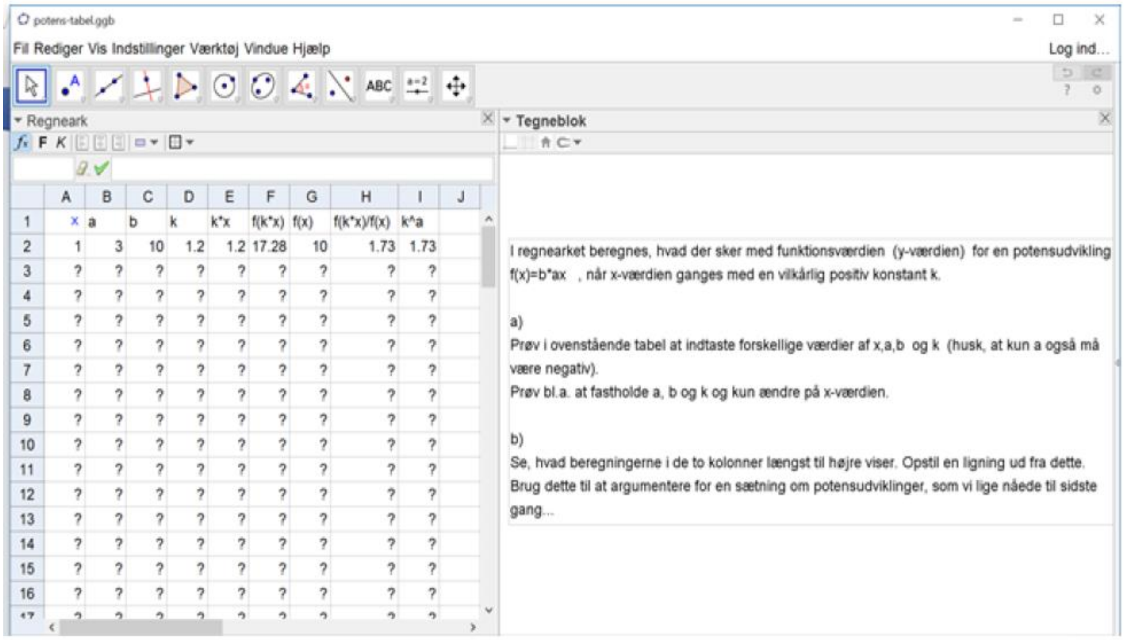
a) Determine the numbers *a* and *b* through regression.

b) Determine the weight of a tree with the diameter 25 cm.

c) Over some years, the diameter of a tree has increased by 15 %. By how many percent has the weight increased?

Module 5:

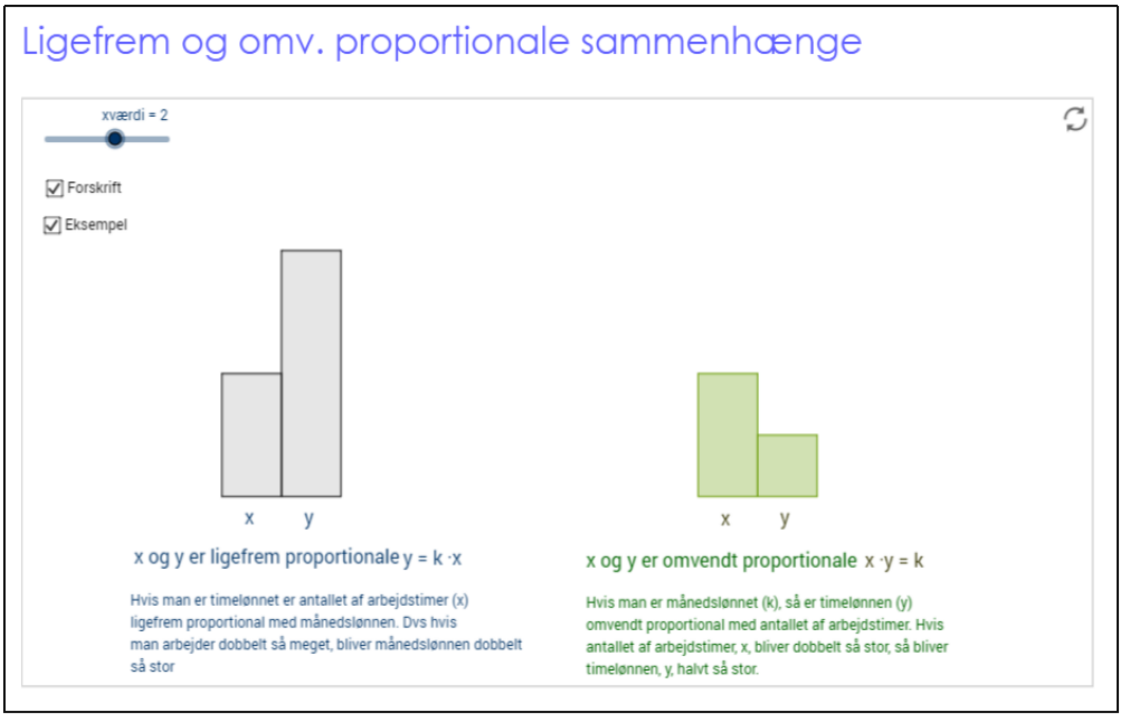
<u>GG document made by TP:</u>



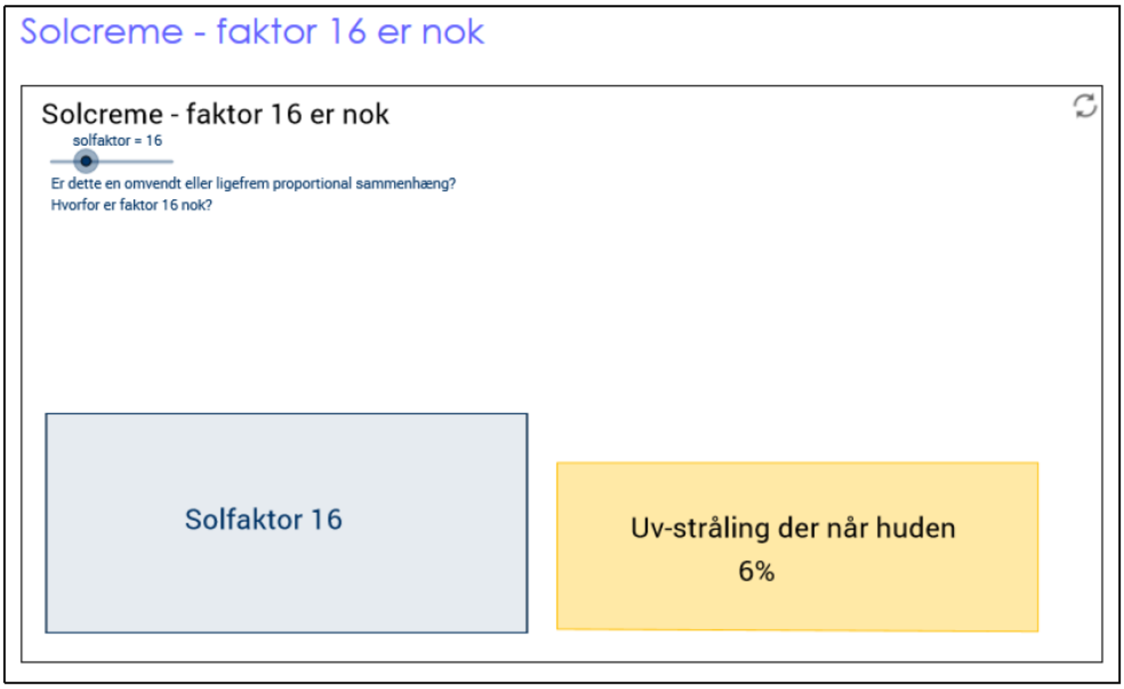
Module 6:

Worksheet 4:

http://tube.geogebra.org/m/XFMRVbgg?doneurl=%2Fsearch%2Fperform%2Fsearch%2Fproportionalitet#material/gHDSpgmf

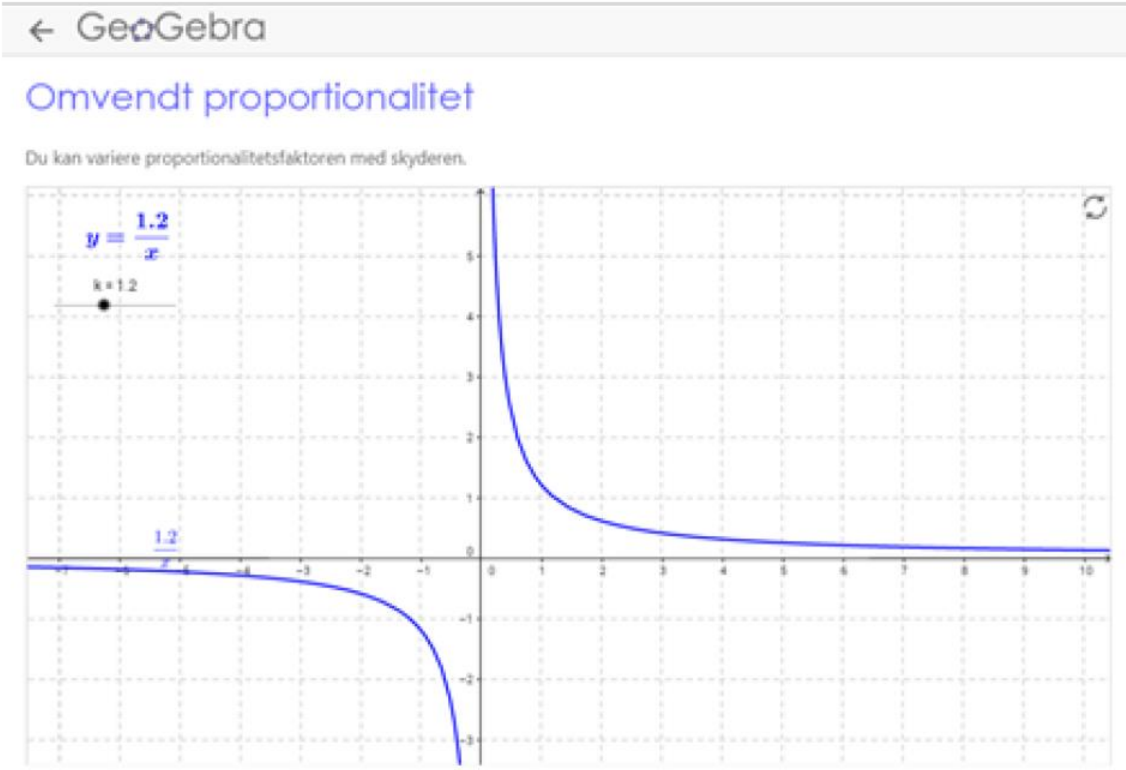


Worksheet 5:
http://tube.geogebra.org/m/XFMRVbgg?doneurl=%2Fsearch%2Fperform%2Fsearch%2Fproportionalitet#material/ArUVcAMy



Worksheet 6:

https://tube.geogebra.org/material/simple/id/79879



Homework: Exercises related to direct and inverse proportionality (from the exercise book; Gyldendals Gymnasiematematik. Arbejdsbog B1 p. 89):

- **Opgave 2080** (translated from Danish):

Two variables *x* and *y* are proportional. Finish the filling of a table as the one below:

x	2	4	
y	3		15

- Opgave 2081 (translated from Danish):

Two variables *x* and *y* are inverse proportional. Finish the filling of a table as the one below:

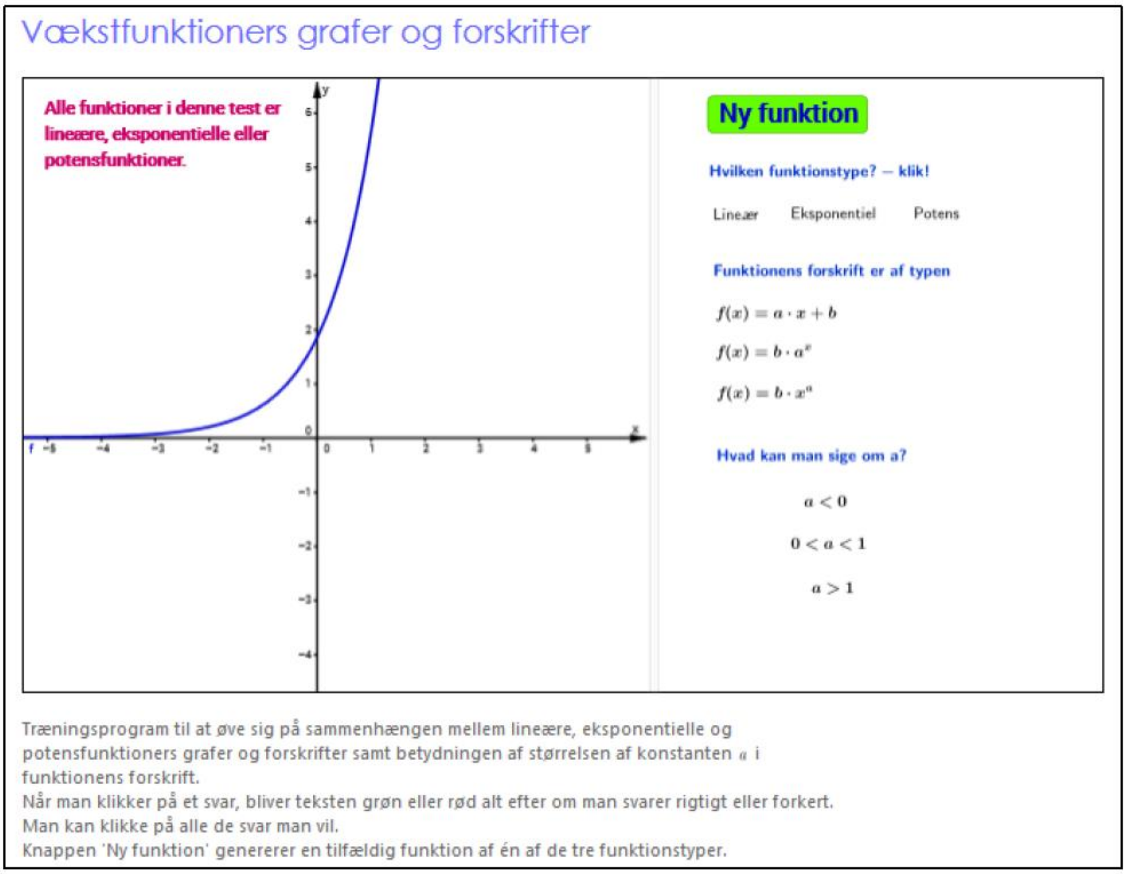
x	2		16
y	24	8	

XXX

Module 7:

Worksheet 8: “Graphs and equations of functions on growth”:

[http://www.geogebra.org/m/3153079:](http://www.geogebra.org/m/3153079)



Homemade game by TP on the three elementary functions:

Den har en hældningskoefficient	Når x vokser med 5, bliver y halveret	Alle y-værdier er positive, men x-værdierne er også negative
Hver gang x vokser med 1, ganges y med 2,3	Forskriften er $f(t) = 23 - t$	Forskriften er $N(v) = 2*v^6$
Når x vokser med 1, vokser y med 3 Den eksisterer ikke, når x<0	Forskriften er $g(x) = 3^2 * x$	Når y aftager med 5%, er x vokset med 1
Når x vokser med 2%, aftager y med 4%	For alle punkter på grafen gælder, at x>0 og y>0.	$f(x+1) = a*f(x)$
Den har en fremskrivningsfaktor	Forskriften er $h(x) = (2x)^3$	$g(x+1) = g(x)-3$
Punktet (2,-4) ligger på grafen	Forskriften er $y = b * a^x$	$f(1,2*x) = 1,2^a * f(x)$
Den har en halveringskonstant	Forskriften er $f(x) = 2,3 * 10^2$	Hver gang x vokser med 3, vokser y med 4
Den går igennem punkterne: (-3,4), (0,8) og (3,16)	Grafen indeholder punkterne (0,1), (1,3), (2,5) og (3,7)	Hver gang x vokser med 5, vokser y med 45%
	Når x vokser med 112, bliver f(x) ganget med 2	Grafen er en ret linje

Spilleregler:

Brikkerne lægges med bagsiden opad på bordet.
Spillerne (man må godt være to sammen!) trækker efter tur en brik, hvis de ikke har en fra en tidligere tur.
Hvis spilleren har en brik, der passer på det felt, man er kommet til, må den lægges:
l betyder "lineær funktion" og giver 1 point
e betyder "eksponentiel udvikling" og giver 2 point
p betyder "potensudvikling" og giver 3 point
Hvis der ikke kan lægges, gives 0 point, og det er den næste spillers tur.
Hvis en brik lægges forkert, fratrækkes det tilsvarende antal point, og brikken skal tages tilbage - hvis nogen kan påvise fejlen!
Når målet er nået, er vinderen den, der har flest point.
Spillet kan evt. gentages…
God fornøjelse!

Module 8:

Take-home group assignement:

Potensvækst

Projektet skal udarbejdes som et gruppearbejde og afleveres senest den 29.4.

Rapporten skal forsynes med en forside, der foruden jeres navn fortæller om rapportens titel og indhold.

Opgaven skal indeholde følgende:

I.

Lav en skematisk oversigt over de tre forskellige slags vækst - ved at udfylde et skema som bilag A.

II.

Bevis for potensvækst formlerne for bestemmelse af a og b ud fra to punkter på grafen.

III.

Bevis betydningen af b i forskriften for potensvækst.

IV.

Bevis sætningen $f(k*x) = k^a * f(x)$ - og forklar betydningen af denne sætning.

V.

Løs følgende opgaver fra STX-B-niveau (om modeller med potensvækst):

5.003, 5.004 og 8.040.

God arbejdslyst!

<u>The formulation of the exercises given in V. above:</u>

5.003 Tabellen viser for 3-slået polyester tovværk sammenhængen mellem tovværkets diameter (målt i mm) og tovets brudstyrke (målt i kg).

Diameter (mm)	4	5	6	8	10	14	16	20	24	26
Brudstyrke (kg)	250	400	600	1000	1550	3200	4000	6000	8600	10000

Det oplyses, at brudstyrken $f(x)$ (kg) som funktion af diameteren x (mm) er en potensfunktion.

a) Benyt tabellen til at bestemme $f(x)$.

b) Benyt den fundne forskrift for $f(x)$ til at bestemme, hvor mange gange så stor diameteren skal være, hvis brudstyrken skal fordobles.

5.004 Dugongs, også kaldet Søkøer, er havdyr, som kan blive omkring 3 meter lange, og som har en levetid på 50-60 år. Tabellen viser sammenhængen mellem søkøers længde (målt i meter) og deres alder (målt i år).

Alder	1,5	2,5	5,0	7,0	9,5	10,0	13,0	17,0	22,5	29,0
Længde	1,97	2,02	2,15	2,35	2,39	2,41	2,47	2,56	2,70	2,72

Kilde: Marsh, H. R. (1980). Age determination of the dugong in Northern Australia and its biological implications.

I det følgende antages, at søkøers længde som funktion af alderen med tilnærmelse er en funktion af typen $f(x) = b \cdot x^a$, hvor x er alderen, og $f(x)$ er længden.

a) Bestem tallene a og b, og opskriv en forskrift for funktionen f.

b) Bestem ved hjælp af f, længden af en søko, der er 8 år gammel, og bestem alderen på en søko, som har en længde på 2,25 meter.

8.040:

Opgave 8 Tabellen viser sammenhængen mellem tætheden af fiskebiomasse M (målt i kg pr. ha) og fosforkoncentration x (målt i µg fosfor pr. liter) i en bestemt sø.

Fosforkoncentration (µg fosfor pr. liter)	40	48	61	74	83	90
Tæthed af fiskebiomasse (kg pr. ha)	200	219	247	272	288	300

Det oplyses, at denne sammenhæng kan beskrives ved $M = b \cdot x^a$.

a) Bestem tallene a og b.

b) Bestem, hvor mange procent M vokser med, når x vokser med 50%.

Bilag A:

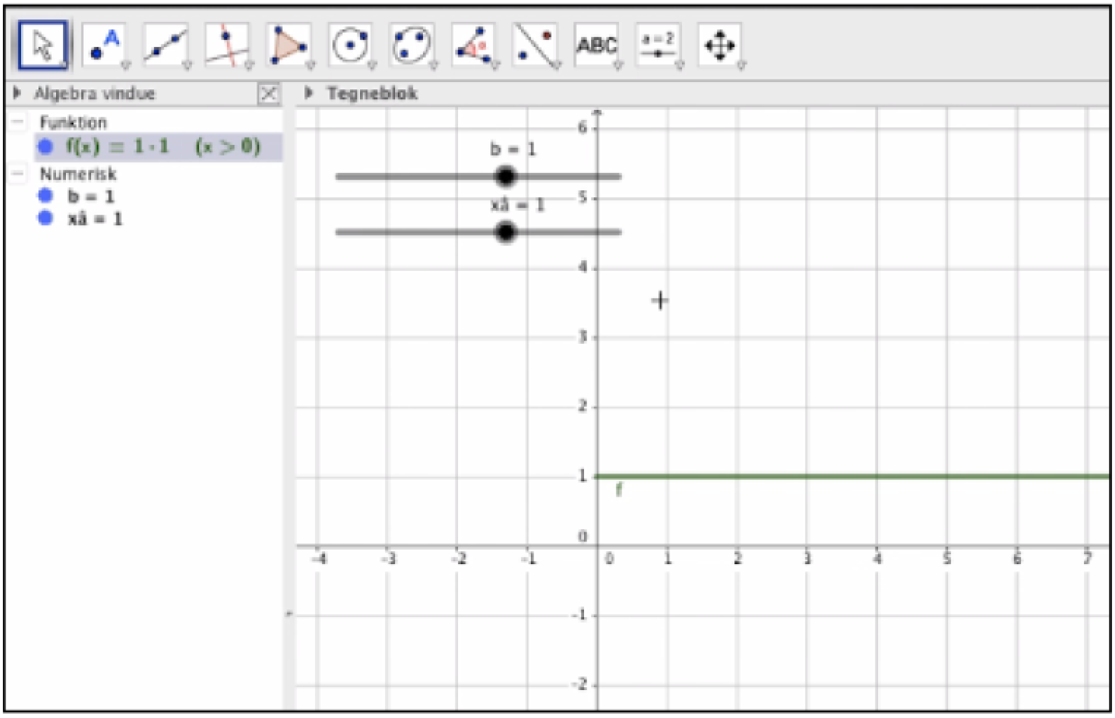
	Lineær udvikling	**Eksponentiel udvikling**	**Potensudvikling**
Regneforskrift			
Beregning af konstanterne i regneforskriften ud fra to punkter			

Hvad angiver a i forskriften?			Skal ikke udfyldes (jf. opg. IV)
Hvad angiver b I forskriften?			
Hvad afgør, om udviklingen er voksende eller aftagende?			

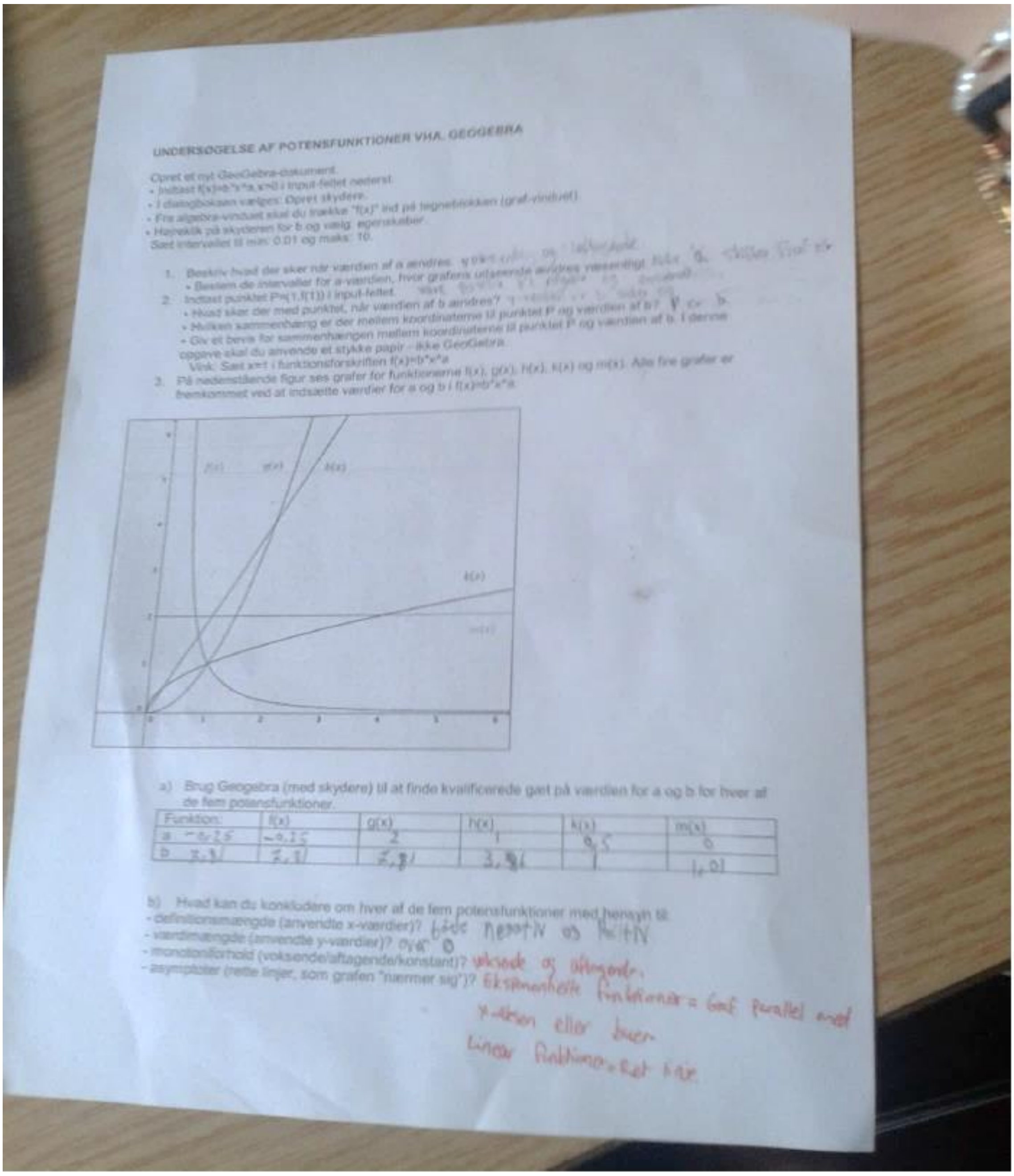
A.8 Power Functions course. Student notes and outcome of screen recordings

Module 1:

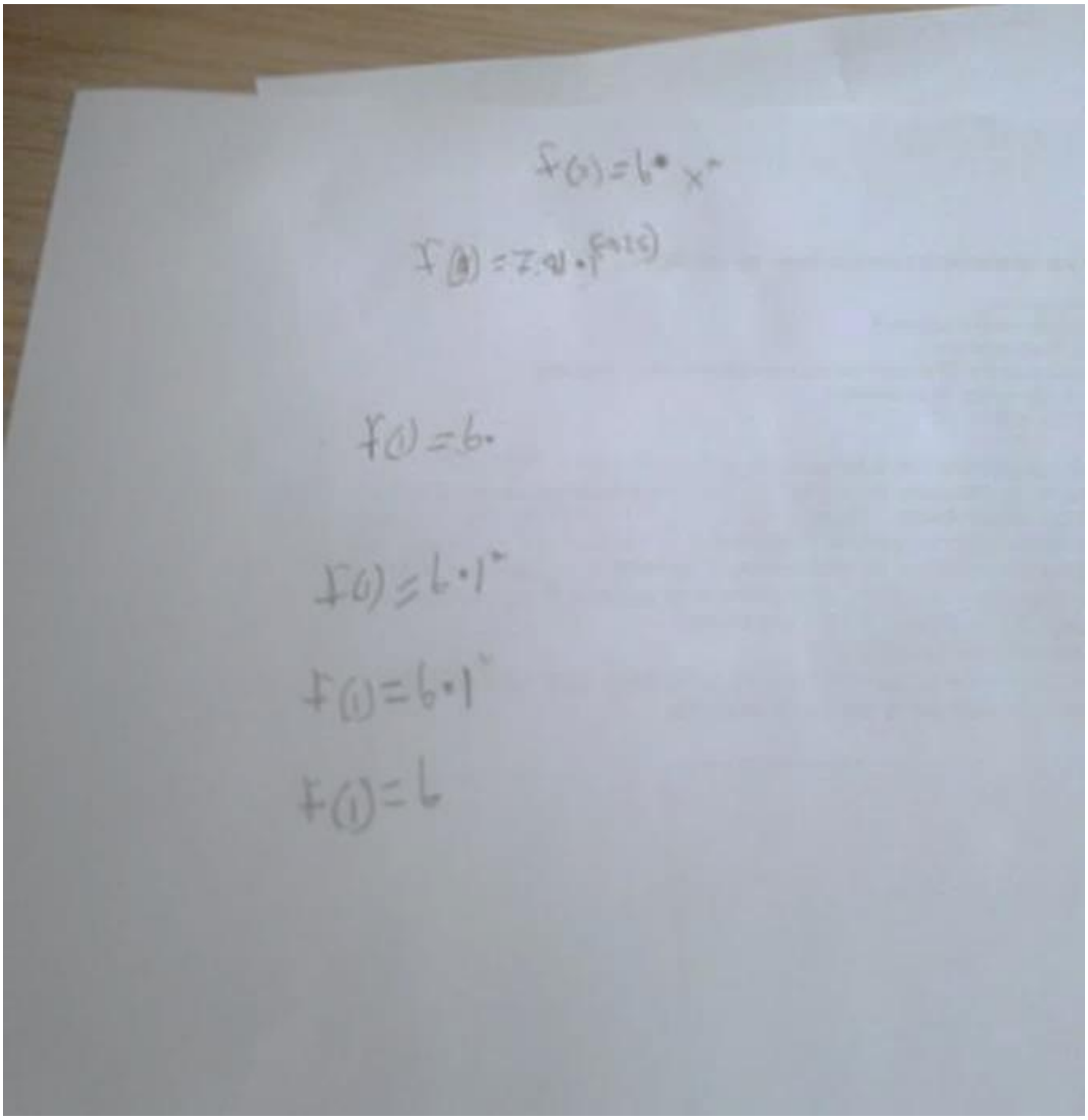
Screenshot of the outcome of the technique performed by PS_1 when trying to perform $\tau^{i}_{Sliders\ ab,AP,TP}$:



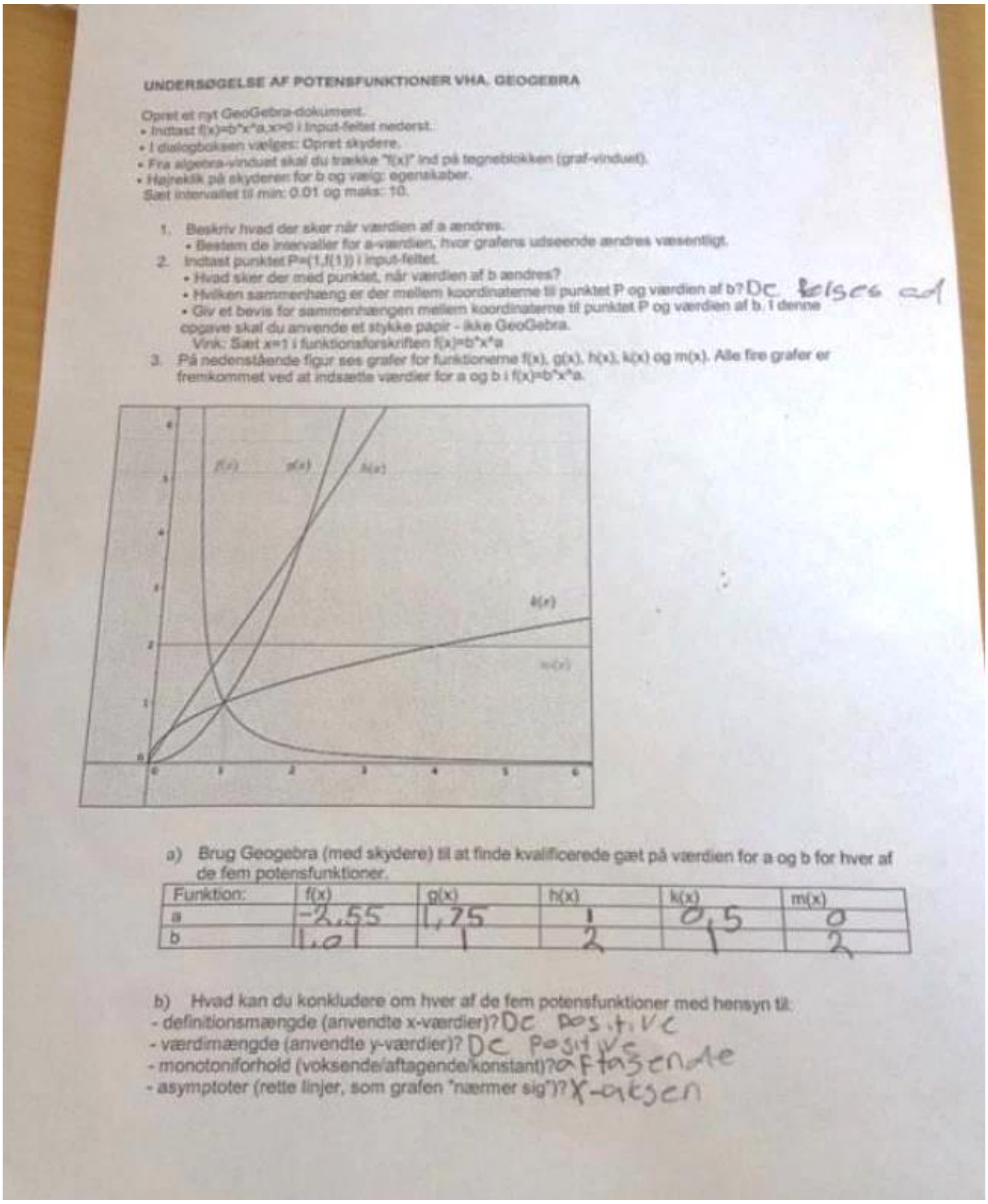
Front page of PS_1's notes:



Back page of PS_1's notes:



<u>PS$_2$'s notes:</u>



UNDERSØGELSE AF POTENSFUNKTIONER VHA. GEOGEBRA

Opret et nyt GeoGebra-dokument.
- Indtast f(x)=b*x^a,x>0 i Input-feltet nederst.
- I dialogboksen vælges: Opret skydere.
- Fra algebra-vinduet skal du trække "f(x)" ind på tegneblokken (graf-vinduet).
- Højreklik på skyderen for b og vælg: egenskaber.

Sæt intervallet til min: 0.01 og maks: 10.

1. Beskriv hvad der sker når værdien af a ændres.
   - Bestem de intervaller for a-værdien, hvor grafens udseende ændres væsentligt.
2. Indtast punktet P=(1,f(1)) i input-feltet.
   - Hvad sker der med punktet, når værdien af b ændres?
   - Hvilken sammenhæng er der mellem koordinaterne til punktet P og værdien af b? De løses ad
   - Giv et bevis for sammenhængen mellem koordinaterne til punktet P og værdien af b. I denne opgave skal du anvende et stykke papir - ikke GeoGebra.

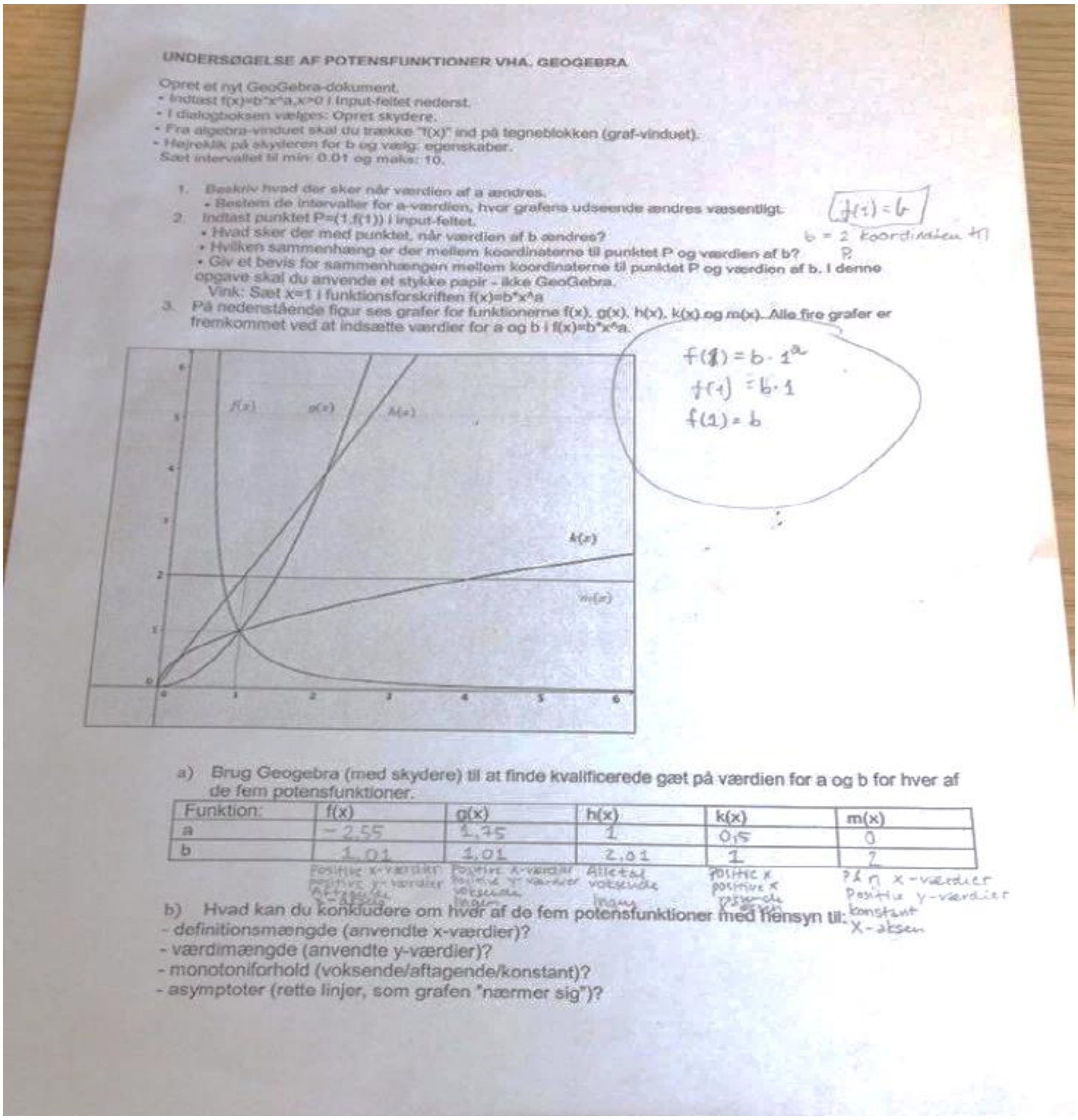
     Vink: Sæt x=1 i funktionsforskriften f(x)=b*x^a
3. På nedenstående figur ses grafer for funktionerne f(x), g(x), h(x), k(x) og m(x). Alle fire grafer er fremkommet ved at indsætte værdier for a og b i f(x)=b*x^a.

a) Brug Geogebra (med skydere) til at finde kvalificerede gæt på værdien for a og b for hver af de fem potensfunktioner.

Funktion:	f(x)	g(x)	h(x)	k(x)	m(x)
a	-2,55	1,75	1	0,5	0
b	1,01	1	2	1	2

b) Hvad kan du konkludere om hver af de fem potensfunktioner med hensyn til:
- definitionsmængde (anvendte x-værdier)? De positive
- værdimængde (anvendte y-værdier)? De positive
- monotoniforhold (voksende/aftagende/konstant)? aftagende
- asymptoter (rette linjer, som grafen "nærmer sig")? X-aksen

AS$_2$'s notes:



UNDERSØGELSE AF POTENSFUNKTIONER VHA. GEOGEBRA

Opret et nyt GeoGebra-dokument.
- Indtast f(x)=b*x^a,x>0 i Input-feltet nederst.
- I dialogboksen vælges: Opret skydere.
- Fra algebra-vinduet skal du trække "f(x)" ind på tegneblokken (graf-vinduet).
- Højreklik på skyderen for b og vælg: egenskaber.

Sæt intervallet til min: 0.01 og maks: 10.

1. Beskriv hvad der sker når værdien af a ændres.
   - Bestem de intervaller for a-værdien, hvor grafens udseende ændres væsentligt.
2. Indtast punktet P=(1,f(1)) i input-feltet.
   - Hvad sker der med punktet, når værdien af b ændres?
   - Hvilken sammenhæng er der mellem koordinaterne til punktet P og værdien af b?
   - Giv et bevis for sammenhængen mellem koordinaterne til punktet P og værdien af b. I denne opgave skal du anvende et stykke papir - ikke GeoGebra.

   Vink: Sæt x=1 i funktionsforskriften f(x)=b*x^a
3. På nedenstående figur ses grafer for funktionerne f(x), g(x), h(x), k(x) og m(x). Alle fire grafer er fremkommet ved at indsætte værdier for a og b i f(x)=b*x^a.

$f(1) = b$

b = 2 koordinaten til P.

$f(1) = b \cdot 1^a$

$f(1) = b \cdot 1$

$f(1) = b$

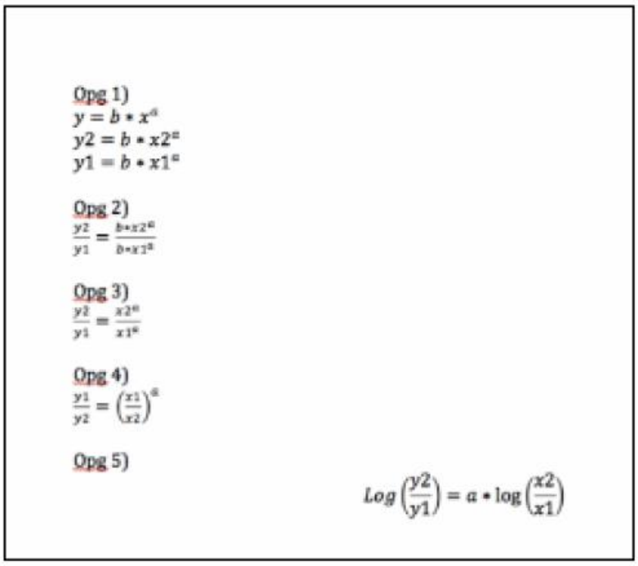
a) Brug Geogebra (med skydere) til at finde kvalificerede gæt på værdien for a og b for hver af de fem potensfunktioner.

Funktion:	f(x)	g(x)	h(x)	k(x)	m(x)
a	−2,55	1,75	1	0,5	0
b	1,01	1,01	2,01	1	2

b) Hvad kan du konkludere om hver af de fem potensfunktioner med hensyn til:
- definitionsmængde (anvendte x-værdier)?
- værdimængde (anvendte y-værdier)?
- monotoniforhold (voksende/aftagende/konstant)?
- asymptoter (rette linjer, som grafen "nærmer sig")?

Module 2:

PS7’s notes in Word for **Worksheet 1**:


Opg 1)
$y = b * x^a$
$y2 = b * x2^a$
$y1 = b * x1^a$

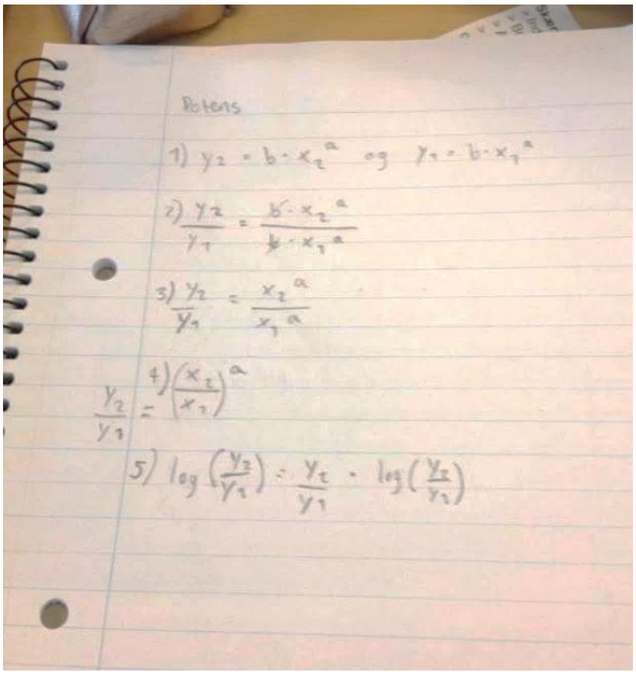
Opg 2)
$\frac{y2}{y1} = \frac{b*x2^a}{b*x1^a}$

Opg 3)
$\frac{y2}{y1} = \frac{x2^a}{x1^a}$

Opg 4)
$\frac{y1}{y2} = \left(\frac{x1}{x2}\right)^a$

Opg 5)
$Log\left(\frac{y2}{y1}\right) = a * \log\left(\frac{x2}{x1}\right)$

PS8’s notes for **Worksheet 1**:


Potens

1) $y_2 = b \cdot x_2^a$ og $y_1 = b \cdot x_1^a$

2) $\frac{y_2}{y_1} = \frac{b \cdot x_2^a}{b \cdot x_1^a}$

3) $\frac{y_2}{y_1} = \frac{x_2^a}{x_1^a}$

4) $\frac{y_2}{y_1} = \left(\frac{x_2}{x_1}\right)^a$

5) $\log\left(\frac{y_2}{y_1}\right) = \frac{y_2}{y_1} \cdot \log\left(\frac{y_2}{y_1}\right)$

Module 3:

PS5's notes in Word:

Potensfunktioner- opgaver:

Side 61- øvelse 278

a)

$g(x) = 2{,}01x^{0{,}52}$

b) y=2,89

c) $2 = 2{,}01x^{0{,}52}$

$\frac{2}{2{,}01} = x^{0{,}52}$ <=> 0,995

$\log(0{,}995) = 0{,}52 \cdot \log(x)$

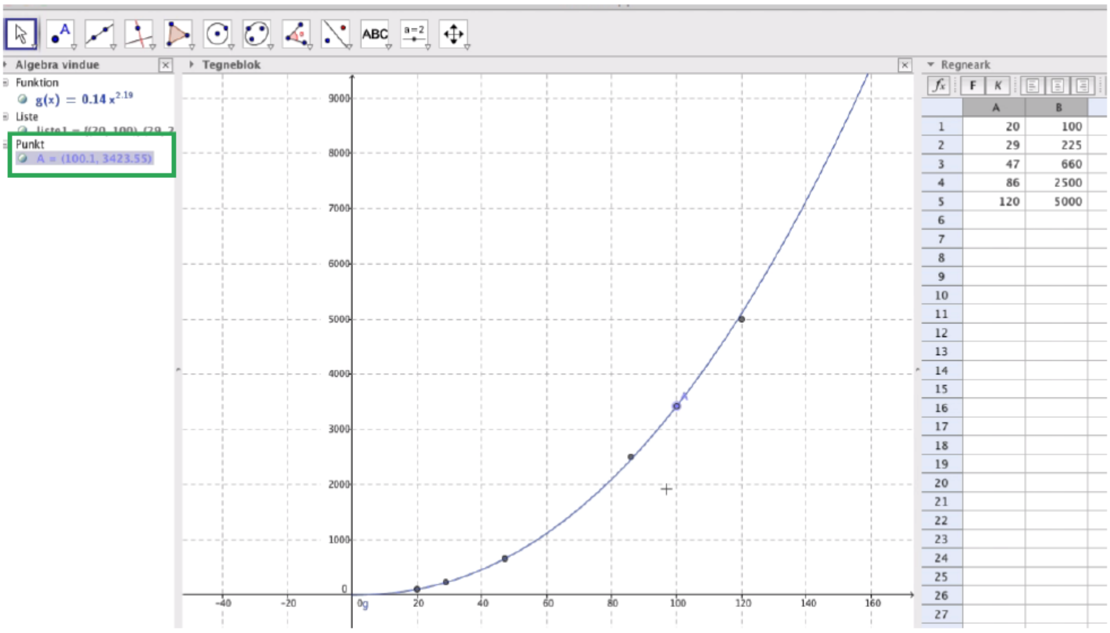
$x = 10^{0{,}004}$

$x = 0{,}991$

The technique for solving tasks of type T_{EP_2} developed by PS3:

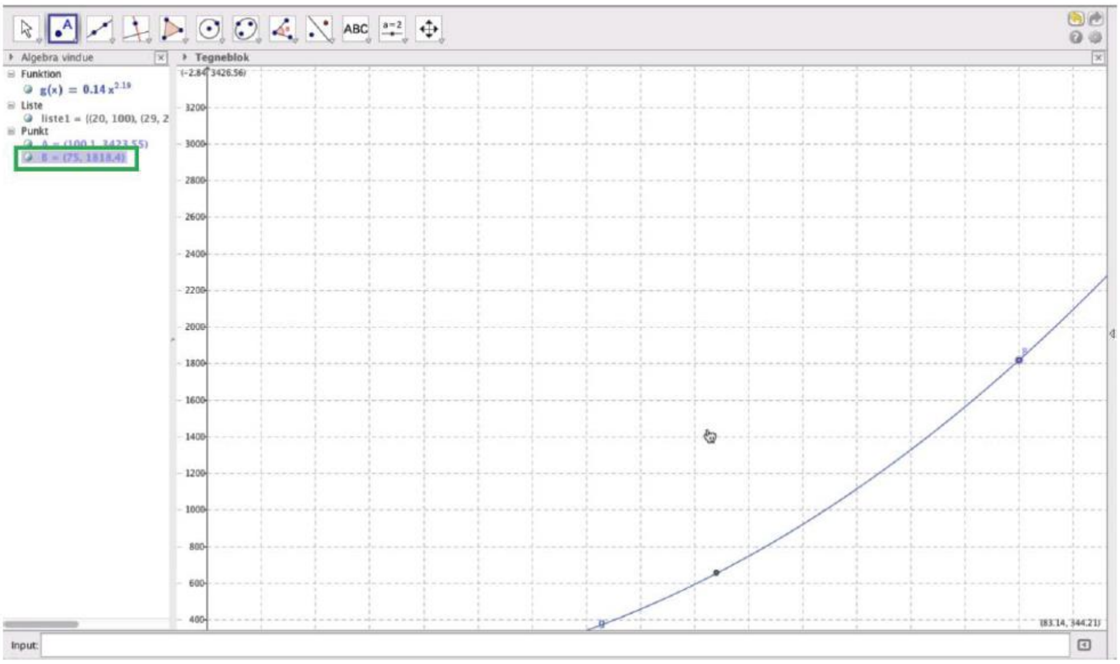
$\tau^{i}_{2,AP,PS}$: Click on the icon in the Graphics View Style Bar to add a grid to the Graphics View → Click on the Point Tool → Find via the grid the point of the graph of f where $f(x) = f(x_0)$ (approximately) and click on this spot of the graph to create a point (A) → Check if the second coordinate of A is equal to $f(x_0)$ in the Algebra View (if not; adjust the point in the Graphics View by dragging it) → Read off the first coordinate of the point in the Algebra View.

Screenshot of the outcome of $\boldsymbol{\tau}^{\boldsymbol{i}}_{\mathbf{1},\boldsymbol{AP},\boldsymbol{PS}}$:



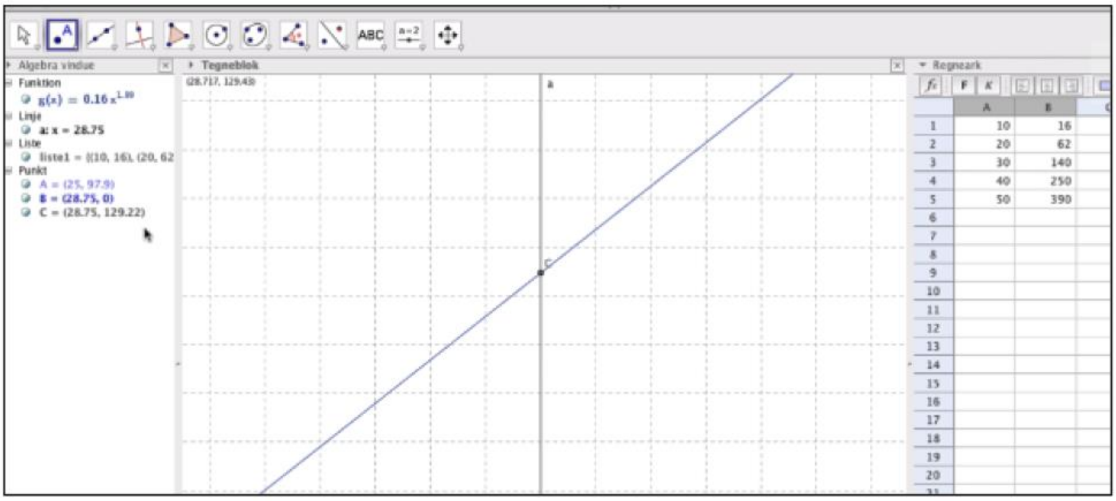
Here, the given value of x_0 is 100.

Screenshot of the outcome of $\boldsymbol{\tau}^{\boldsymbol{i}}_{\mathbf{2},\boldsymbol{AP},\boldsymbol{PS}}$:



Here, the given value of $f(x_0)$ is 1800.

Screenshot of the outcome of $\boldsymbol{\tau^{i}_{3,AP,PS}}$:



Here, point $C = (x_1, y_1) = (28.75, 129.22)$ (i.e the point called A in $\tau^{i}_{3,AP,PS}$) and $A = (x_0, y_0) = (25\ , 97.9)$.

PS3's notes for $\tau^{i}_{3,AP,PS}$ in Word:

c)

$$\frac{98}{100} \cdot 32 = 31{,}36 + 98 = 129{,}36$$

PS$_{10}$'s notes for **Worksheet 2**: